SIAM J. CONTROL OPTIM.
Vol. 48, No. 2, pp. 688–718

FIRST- AND SECOND-ORDER OPTIMALITY CONDITIONS FOR A CLASS OF OPTIMAL CONTROL PROBLEMS WITH QUASILINEAR ELLIPTIC EQUATIONS*

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Abstract. A class of optimal control problems for quasilinear elliptic equations is considered, where the coefficients of the elliptic differential operator depend on the state function. First- and second-order optimality conditions are discussed for an associated control-constrained optimal control problem. Main emphasis is laid on second-order sufficient optimality conditions. To this aim, the regularity of the solutions to the state equation and its linearization is studied in detail and the Pontryagin maximum principle is derived. One of the main difficulties is the nonmonotone character of the state equation.

Key words. optimal control, distributed control, quasilinear elliptic equation, Pontryagin maximum principle, second-order optimality conditions

AMS subject classifications. 49J20, 35J65

DOI. 10.1137/080720048

1. Introduction.

In this paper, we consider optimal control problems for a quasilinear elliptic equation of the type

$$\begin{cases} -\mathrm{div}\,[a(x,y(x))\,\nabla y(x)] + f(x,y(x)) = u(x) & \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad y(x) = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.1}$$

Equations of this type occur, for instance, in models of heat conduction, where the heat conductivity a depends on the spatial coordinate x and on the temperature y. For instance, the heat conductivity of carbon steel depends on the temperature and also on the alloying additions contained; cf. Bejan [2]. If the different alloys of steel are distributed smoothly in the domain, then $a = a(x, y)$ should depend in a sufficiently smooth way on (x, y). Similarly, the heat conductivity depends on (x, y) in the growth of silicon carbide bulk single crystals; see Klein et al. [22].

If a is independent of x, then the well-known Kirchhoff transformation is helpful to solve (1.1) uniquely. Also in the more general case $a = a(x, y)$, a Kirchhoff-type transformation can be applied. Here, we may define $b(x, y) := \int_0^y a(x, z)dz$ and set $\theta(x) := b(x, y(x))$. Under this transformation, we obtain a semilinear equation of the type $-\Delta\,\theta + \mathrm{div}\,[(\nabla_x b)(x, b^{-1}(x, \theta))] + f(x, b^{-1}(x, \theta)) = u$. We thank an anonymous referee for this hint. However, b should at least be Lipschitz with respect to x and, due to the new divergence term, the analysis of this equation is certainly not easy, too. We believe that the direct discussion of the quasilinear equation is not more difficult. Moreover, the form (1.1) seems to be more directly accessible to a numerical solution.

*Received by the editors April 2, 2008; accepted for publication (in revised form) October 31, 2008; published electronically February 13, 2009.
http://www.siam.org/journals/sicon/48-2/72004.html

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In the case $a = a(x, y)$, in spite of the nonmonotone character of the equation (1.1), there exists a celebrated comparison principle proved by Douglas, Dupont, and Serrin [16] that leads to the uniqueness of a solution of (1.1); for a more recent paper, extending this result the reader is referred to Křížek and Liu [23]. We will use the approach of [23] to deduce that (1.1) is well posed under less restrictive assumptions than those considered by the previous authors.

For other classes of quasilinear equations, in particular for equations in which a depends on the gradient of y, we refer the reader to, for instance, Lions [24] and Nečas [27].

As far as optimization is concerned, there exists a rich literature on the optimal control of semilinear elliptic and parabolic equations. For instance, the Pontryagin principle was discussed for different elliptic problems in [5], [4], [1], while the parabolic case was investigated in [6] and [29]. Problems with quasilinear equations with nonlinearity of gradient type were considered by [7], [8], [11], and [12]. This list on first-order necessary optimality conditions is by far not exhaustive. However, to our knowledge, the difficult issue of second-order conditions for problems with quasilinear equations has not yet been studied.

There is some recent progress in the case of semilinear equations. Quite a number of contributions to second-order necessary and/or sufficient optimality conditions were published for problems with such equations. We mention only [3], [14], and the state-constrained case in [10], [15], [28].

Surprisingly, the important state equation (1.1) has not yet been investigated in the context of optimal control. Our paper is the first step towards a corresponding numerical analysis. We are convinced that our analysis can also be extended to other quasilinear equations or associated systems, since the main difficulties are already inherent in (1.1).

First-order optimality conditions are needed to deduce regularity properties of optimal controls as an important prerequisite for all further investigations. The second-order analysis is a key tool for the numerical analysis of nonlinear optimal control problems. As in the minimization of a function $f : \mathbb{R} \to \mathbb{R}$, second-order sufficient conditions are commonly assumed to guarantee stability of locally optimal controls with respect to perturbations of the problem. For instance, an approximation of the PDEs by finite elements is a typical perturbation of a control problem. Associated error estimates for local solutions of the FEM-approximated optimal control problem are based on second-order sufficiency. Likewise, the standard assumption for the convergence of higher order numerical optimization algorithms such as SQP-type methods is a second-order sufficient condition at the local solution to which the method should converge.

A review on important applications of optimal control theory to problems in engineering and medical science shows that in most of the cases the underlying PDEs are quasilinear. Although our equation has a particular type, our problem might serve as a model case for the numerical analysis of optimal control problems with more general quasilinear equations or systems.

The theory of optimality conditions of associated control problems is the main issue of our paper, which is organized as follows:

First, we discuss the well-posedness of this equation in different spaces. Next, the differentiability properties of the control-to-state mapping are investigated. Based on these results, the Pontryagin maximum principle is derived. Moreover, second-order necessary and sufficient optimality conditions are established.

Notation. By $B_X(x, r)$ we denote the open ball in a normed space X with radius r centered at x, and by $\bar{B}_X(x, r)$ we denote its closure. In some formulas, the partial derivative $\partial/\partial x_j$ is sometimes abbreviated by ∂_j. By c (without index), generic constants are denoted. Moreover, $\langle \cdot\, , \cdot\rangle$ stands for the pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

2. Study of the quasilinear equation.

2.1. Existence, uniqueness, and regularity of solutions. The proof of the existence and uniqueness of a solution of (1.1) relies on the following assumptions:

(A1) $\Omega \subset \mathbb{R}^n$ is an open bounded set with a Lipschitz boundary Γ.

(A2) The functions $a : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory, f is monotone nondecreasing with respect to the second variable for almost all $x \in \Omega$, and

$$\exists \alpha_0 > 0 \text{ such that } a(x, y) \geq \alpha_0 \text{ for a.e. } x \in \Omega \text{ and } \forall\, y \in \mathbb{R}. \tag{2.1}$$

The function $a(\cdot, 0)$ belongs to $L^\infty(\Omega)$, and for any $M > 0$ there exist a constant $C_M > 0$ and a function $\phi_M \in L^q(\Omega)$, with $q \geq pn/(n+p)$ and $n < p$, such that for all $|y|, |y_i| \leq M$

$$\begin{aligned} |a(x, y_2) - a(x, y_1)| &\leq C_M |y_2 - y_1| \text{ and} \\ |f(x, y)| &\leq \phi_M(x) \text{ for a.e. } x \in \Omega. \end{aligned} \tag{2.2}$$

In the rest of the paper q and $p \in (n, +\infty)$ will be fixed. Let us remark that $q \geq pn/(n+p) > n/2$.

Example 2.1. The following equation satisfies our assumptions if we assume ϕ_0, $\phi_1 \in L^\infty(\Omega)$, $\phi_0(x) \geq \alpha_0 > 0$ a.e. in Ω, $\phi_1(x) \geq 0$ a.e. in Ω, and $1 \leq m \in \mathbb{N}$:

$$\begin{cases} -\text{div}\left[(\phi_0(x) + y^{2m}(x))\, \nabla y(x)\right] + \phi_1(x) \exp(y(x)) = u(x) & \text{in } \Omega, \\ y(x) = 0 & \text{on } \Gamma. \end{cases}$$

THEOREM 2.2. *Under the assumptions* (A1) *and* (A2), *for any element* $u \in W^{-1,p}(\Omega)$ *problem* (1.1) *has a unique solution* $y_u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. *Moreover there exists* $\mu \in (0, 1)$ *independent of* u *such that* $y_u \in C^\mu(\bar{\Omega})$ *and for any bounded set* $U \subset W^{-1,p}(\Omega)$

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C^\mu(\bar{\Omega})} \leq C_U \quad \forall u \in U \tag{2.3}$$

for some constant $C_U > 0$. *Finally, if* $u_k \to u$ *in* $W^{-1,p}(\Omega)$, *then* $y_{u_k} \to y_u$ *in* $H_0^1(\Omega) \cap C^\mu(\bar{\Omega})$.

Proof. Existence of a solution. Depending on $M > 0$, we introduce the truncated function a_M by

$$a_M(x, y) = \begin{cases} a(x, y), & |y| \leq M, \\ a(x, +M), & y > +M, \\ a(x, -M), & y < -M. \end{cases}$$

In the same way, we define the truncation f_M of f. Let us prove that the equation

$$\begin{cases} -\text{div}\,[a_M(x, y)\, \nabla y] + f_M(x, y) = u \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma \end{cases} \tag{2.4}$$

admits at least one solution $y \in H_0^1(\Omega)$. We define, for fixed $u \in W^{-1,p}(\Omega)$ and $M > 0$, a mapping $F : L^2(\Omega) \to L^2(\Omega)$ by $F(z) = y$, where $y \in H_0^1(\Omega)$ is the unique solution to

$$\begin{cases} -\text{div}\,[a_M(x,z)\,\nabla y] + f_M(x,z) = u \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad y = 0 \text{ on } \Gamma. \end{cases} \tag{2.5}$$

Thanks to assumption (A2), (2.2), we have

$$|f_M(x,z)| \le \phi_M(x)$$

and $\phi_M \in L^q(\Omega) \subset H^{-1}(\Omega)$. Therefore, (2.5) is a linear equation and $u - f_M(\cdot, z)$ belongs to $H^{-1}(\Omega)$; hence (2.5) admits a unique solution $y_M \in H_0^1(\Omega)$ and F is well defined. It can be shown by standard arguments invoking in particular the compact injection of $H^1(\Omega)$ in $L^2(\Omega)$ that F is continuous. Furthermore, we have

$$\|y_M\|_{H^1(\Omega)} \le \frac{1}{\alpha_0}\left(\|u\|_{H^{-1}(\Omega)} + \|\phi_M\|_{H^{-1}(\Omega)}\right). \tag{2.6}$$

Using this estimate and the fact that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, it is easy to apply the Schauder theorem to prove the existence of a fixed point $y_M \in H_0^1(\Omega)$ of F. Obviously, y_M is a solution of (2.4).

Since $q \ge np/(n+p)$ we have that $L^q(\Omega) \subset W^{-1,p}(\Omega)$. Now an application of the Stampacchia truncation method yields

$$\|y_M\|_{L^\infty(\Omega)} \le c_\infty \|u - f(\cdot, 0)\|_{W^{-1,p}(\Omega)}, \tag{2.7}$$

where c_∞ depends only on the coercivity constant α_0 given in (2.1) but neither on $\|a_M(\cdot, y_M)\|_{L^\infty(\Omega)}$ nor on $f_M(\cdot, y_M)$. For the idea of this method, the reader is referred to Stampacchia [30] or to the exposition for semilinear elliptic equations in Tröltzsch [31, Theorem 7.3]. By taking

$$M \ge c_\infty \|u - f(\cdot, 0)\|_{W^{-1,p}(\Omega)},$$

(2.7) implies that $a_M(x, y_M(x)) = a(x, y_M(x))$ and $f_M(x, y_M(x)) = f(x, y_M(x))$ for a.e. $x \in \Omega$, and therefore $y_M \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.1). The Hölder regularity follows as usual; see, for instance, Gilbarg and Trudinger [19, Theorem 8.29]. Inequality (2.3) follows from (2.6), (2.7), and the estimates in [19, Theorem 8.29]. Finally, the convergence property can be deduced from (2.3) easily once the uniqueness is proved.

Uniqueness of a solution. Here we follow the method by Křížek and Liu [23]. Let us assume that $y_i \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $i = 1, 2$, are two solutions of (1.1). The regularity results proved above imply that $y_i \in C(\bar{\Omega})$, $i = 1, 2$. Let us define the open sets

$$\Omega_0 = \{x \in \Omega : y_2(x) - y_1(x) > 0\}$$

and for every $\varepsilon > 0$

$$\Omega_\varepsilon = \{x \in \Omega : y_2(x) - y_1(x) > \varepsilon\}.$$

Now we take $z_\varepsilon(x) = \min\{\varepsilon, (y_2(x) - y_1(x))^+\}$, which belongs to $H_0^1(\Omega)$ and $|z_\varepsilon| \le \varepsilon$. Multiplying the equations corresponding to y_i by z_ε and doing the usual integration by parts we get

$$\int_\Omega \{a(x, y_i)\nabla y_i \cdot \nabla z_\varepsilon + f(x, y_i) z_\varepsilon\}\, dx = \langle u, z_\varepsilon \rangle, \quad i = 1, 2.$$

By subtracting both equations, using the monotonicity of f, (2.1) and (2.2) and the fact that $\nabla z_\varepsilon(x) = 0$ for a.a. $x \notin \Omega_0\setminus\Omega_\varepsilon$ and in view of $\nabla z_\varepsilon = \nabla(y_2-y_1)^+ = \nabla(y_2-y_1)$ a.e. in $\Omega_0 \setminus \Omega_\varepsilon$ we get

$$\alpha_0\|\nabla z_\varepsilon\|^2_{L^2(\Omega)} \le \int_\Omega \{a(x,y_2)|\nabla z_\varepsilon|^2 + [f(x,y_2) - f(x,y_1)]z_\varepsilon\}dx$$
$$= \int_\Omega \{a(x,y_2)\nabla(y_2-y_1)\cdot\nabla z_\varepsilon + [f(x,y_2) - f(x,y_1)]z_\varepsilon\}dx$$

and, invoking the weak formulation of the equation for y_1,

$$= \int_\Omega [a(x,y_1)\nabla y_1 - a(x,y_2)\nabla y_1]\cdot\nabla z_\varepsilon\,dx$$
$$= \int_{\Omega_0\setminus\Omega_\varepsilon} [a(x,y_1)\nabla y_1 - a(x,y_2)\nabla y_1]\cdot\nabla z_\varepsilon\,dx$$
$$\le C_M\|y_2-y_1\|_{L^\infty(\Omega_0\setminus\Omega_\varepsilon)}\|\nabla y_1\|_{L^2(\Omega_0\setminus\Omega_\varepsilon)}\|\nabla z_\varepsilon\|_{L^2(\Omega_0\setminus\Omega_\varepsilon)}$$
$$\le C_M\varepsilon\|\nabla y_1\|_{L^2(\Omega_0\setminus\Omega_\varepsilon)}\|\nabla z_\varepsilon\|_{L^2(\Omega_0\setminus\Omega_\varepsilon)}.$$

From this inequality, along with Friedrich's inequality, we get

$$\|z_\varepsilon\|_{L^2(\Omega)} \le C\|\nabla z_\varepsilon\|_{L^2(\Omega)} \le C'\varepsilon\|\nabla y_1\|_{L^2(\Omega_0\setminus\Omega_\varepsilon)}. \tag{2.8}$$

Now by $\lim_{\varepsilon\downarrow 0}|\Omega_0\setminus\Omega_\varepsilon| = 0$ and (2.8) we deduce

$$|\Omega_\varepsilon| = \varepsilon^{-2}\int_{\Omega_\varepsilon}\varepsilon^2 \le \varepsilon^{-2}\int_\Omega |z_\varepsilon|^2 \le C''\|\nabla y_1\|^2_{L^2(\Omega_0\setminus\Omega_\varepsilon)} \to 0,$$

which implies that $|\Omega_0| = \lim_{\varepsilon\to 0}|\Omega_\varepsilon| = 0$ and hence $y_2 \le y_1$. In the same way, we prove that $y_1 \le y_2$ □

As in this theorem, throughout our paper, the solutions of PDEs are defined as weak solutions.

Remark 2.3. Let us remark that the Lipschitz property of a with respect to y assumed in (A2) was necessary only to prove the uniqueness of a solution of (1.1), but it was not needed to establish existence and regularity. We can get multiple solutions of (1.1) if the Lipschitz property (2.2) fails; see Hlaváček, Křížek, and Malý [21] for a one-dimensional example.

By assuming more regularity on a, f, Γ, and u, we can obtain higher regularity of the solutions of (1.1).

Theorem 2.4. *Let us suppose that* (A1) *and* (A2) *hold. We also assume that* $a : \bar\Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ *is continuous and* Γ *is of class* C^1. *Then, for any* $u \in W^{-1,p}(\Omega)$, (1.1) *has a unique solution* $y_u \in W_0^{1,p}(\Omega)$. *Moreover, for any bounded set* $U \subset W^{-1,p}(\Omega)$, *there exists a constant* $C_U > 0$ *such that*

$$\|y_u\|_{W_0^{1,p}(\Omega)} \le C_U \quad \forall u \in U. \tag{2.9}$$

If $u_k \to u$ *in* $W^{-1,p}(\Omega)$, *then* $y_{u_k} \to y_u$ *strongly in* $W_0^{1,p}(\Omega)$.

The proof of this theorem follows from Theorem 2.2 and the $W^{1,p}(\Omega)$-regularity results for linear elliptic equations; see Giaquinta [18, Chap. 4, p. 73] or Morrey [25, pp. 156–157]. It is enough to remark that the function $\hat a(x) = a(x, y_u(x))$ is continuous in $\bar\Omega$ and $u - f(\cdot, y_u)$ belongs to $W^{-1,p}(\Omega)$.

Let us state some additional assumptions leading to $W^{2,q}(\Omega)$-regularity for the solutions of (1.1).

(A3) For all $M > 0$, there exists a constant $c_M > 0$ such that the following local Lipschitz property is satisfied:

$$|a(x_1, y_1) - a(x_2, y_2)| \leq c_M \ \{|x_1 - x_2| + |y_1 - y_2|\} \tag{2.10}$$

for all $x_i \in \bar{\Omega}$, $y_i \in [-M, M]$, $i = 1, 2$.

THEOREM 2.5. *Under the hypotheses* (A1)–(A3) *and assuming that* Γ *is of class* $C^{1,1}$*, for any* $u \in L^q(\Omega)$*,* (1.1) *has one solution* $y_u \in W^{2,q}(\Omega)$*. Moreover, for any bounded set* $U \subset L^q(\Omega)$*, there exists a constant* $C_U > 0$ *such that*

$$\|y_u\|_{W^{2,q}(\Omega)} \leq C_U \quad \forall u \in U. \tag{2.11}$$

Proof. (i) From Sobolev embedding theorems (cf. Nečas [26, Theorem 3.4]), it follows that

$$L^q(\Omega) \hookrightarrow W^{-1,\frac{nq}{n-q}}(\Omega) \ \text{ if } \ 1 < q < n, \tag{2.12}$$

$$L^q(\Omega) \hookrightarrow W^{-1,\infty}(\Omega) \ \text{ if } \ n \leq q < \infty. \tag{2.13}$$

Since $L^q(\Omega) \subset W^{-1,p}(\Omega)$, we can apply Theorem 2.4 to get the existence of at least one solution in $W_0^{1,p}(\Omega)$ for every $1 < p < \infty$ if $q \geq n$, and for $p = \frac{nq}{n-q}$ if $q < n$. We have to prove the $W^{2,q}(\Omega)$-regularity. To this aim, we distinguish between two cases in the proof.

(ii)(a) *Case* $q \geq n$. We have that $y \in W_0^{1,p}(\Omega)$ for any $p < \infty$, in particular in $W_0^{1,2q}(\Omega)$. By using assumption (A3), expanding the divergence term of the PDE (1.1), and dividing by a we find that

$$-\Delta y = \underbrace{\frac{1}{a}}_{L^\infty} \left\{ \underbrace{u - f(\cdot, y)}_{L^q} + \sum_{j=1}^{n} \underbrace{\partial_j a(x, y)}_{L^\infty} \underbrace{\partial_j y}_{L^q} + \underbrace{\frac{\partial a}{\partial y}}_{L^\infty} \underbrace{|\nabla y|^2}_{L^q} \right\}, \tag{2.14}$$

hence the right-hand side of (2.14) is in $L^q(\Omega)$. Notice that $\frac{\partial a}{\partial y} \in L^\infty$ follows from (2.10) and the boundedness of y. The $C^{1,1}$ smoothness of Γ permits us to apply a well-known result by Grisvard [20] on maximal regularity and to get $y \in W^{2,q}(\Omega)$.

(ii)(b) *Case* $n/2 < q < n$. Notice that $y \in W_0^{1,\frac{nq}{n-q}}(\Omega)$. It follows that $|\nabla y|^2 \in L^{\frac{nq}{2(n-q)}}(\Omega)$. A simple calculation confirms that

$$\frac{nq}{2(n-q)} > q, \tag{2.15}$$

since this is equivalent to $q > n/2$, a consequence of our assumption on q. Therefore, it holds that $|\nabla y|^2 \in L^q(\Omega)$ and once again the right-hand side of (2.14) belongs to $L^q(\Omega)$. We apply again the regularity results by Grisvard [20] to obtain $y \in W^{2,q}(\Omega)$. ∎

COROLLARY 2.6. *Suppose that the assumptions of Theorem* 2.5*, except the regularity hypothesis of* Γ*, are satisfied with* $q = 2$*. Then, if* $\Omega \subset \mathbb{R}^n$ *is an open, bounded, and convex set,* $n = 2$ *or* $n = 3$*, there exists one solution of* (1.1)*:* $y \in H^2(\Omega) \cap H_0^1(\Omega)$*.*

Proof. This is a simple extension of Theorem 2.5 for $q = 2$. Notice that we have assumed $n \leq 3$ so that $q > n/2$ is true. The $C^{1,1}$ smoothness of Γ is not needed for convex domains, since maximal regularity holds there; cf. [20]. ∎

2.2. Differentiability of the control-to-state mapping. In order to derive the first- and second-order optimality conditions for the control problem, we need to assume some differentiability of the functions involved in the control problem. In this section, we will analyze the differentiability properties of the states with respect to the control. To this aim, we require the following assumption.

(A4) The functions a and f are of class C^2 with respect to the second variable and, for any number $M > 0$, there exists a constant $D_M > 0$ such that

$$\sum_{j=1}^{2} \left|\frac{\partial^j a}{\partial y^j}(x,y)\right| + \left|\frac{\partial^j f}{\partial y^j}(x,y)\right| \le D_M \quad \text{for a.e. } x \in \Omega \quad \text{and } \forall\ |y| \le M.$$

Now we are going to study the differentiability of the control-to-state mapping. As a first step we study the linearized equation of (1.1) around a solution y_u. The reader should note that the well-posedness of the linearized equation is not obvious because of the linear operator is not monotone.

THEOREM 2.7. *Given $y \in W^{1,p}(\Omega)$ for any $v \in H^{-1}(\Omega)$ the linearized equation*

$$\text{(2.16)} \qquad \begin{cases} -\text{div}\left[a(x,y)\nabla z + \dfrac{\partial a}{\partial y}(x,y) z\, \nabla y\right] + \dfrac{\partial f}{\partial y}(x,y)\, z = v & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma \end{cases}$$

has a unique solution $z_v \in H_0^1(\Omega)$.

Remark 2.8. As a consequence of the open mapping theorem, assuming that (A2) and (A4) hold, we know that the relation $v \mapsto z_v$ defined by (2.16) is an isomorphism between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Indeed, it is enough to note that the linear mapping

$$z \mapsto -\text{div}\left[a(x,y)\nabla z + \frac{\partial a}{\partial y}(x,y) z\, \nabla y\right] + \frac{\partial f}{\partial y}(x,y)\, z$$

is continuous from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. To verify this, we notice first that $a(x,y)$, $\frac{\partial a}{\partial y}(x,y)$, and $\frac{\partial f}{\partial y}(x,y)$ are bounded functions because of our assumptions and the boundedness of y, which follows from the fact that $y \in W_0^{1,p}(\Omega) \subset C(\bar{\Omega})$ for $p > n$. The only delicate point is to check that

$$\frac{\partial a}{\partial y}(\cdot,y) z \nabla y \in L^2(\Omega)^n.$$

This property follows from the Hölder inequality

$$\left(\int_\Omega \left|\frac{\partial a}{\partial y}(\cdot,y) z \nabla y\right|^2 dx\right)^{1/2} \le D_M \|z\|_{L^{\frac{2p}{p-2}}(\Omega)} \|\nabla y\|_{L^p(\Omega)}$$

and the fact that

$$H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega) \subset L^{\frac{2p}{p-2}}(\Omega) \text{ if } n > 2,$$
$$H_0^1(\Omega) \subset L^r(\Omega)\ \forall\ r < \infty \text{ if } n = 2,$$

where we have used that

$$p > n \Rightarrow \frac{2n}{n-2} > \frac{2p}{p-2}.$$

Remark 2.9. The reader can easily check that the proof of Theorem 2.7 can be modified in a very obvious way to state that the equation

$$\begin{cases} -\text{div}\left[a(x,y_1)\nabla z + \dfrac{\partial a}{\partial y}(x,y_2)z\,\nabla y\right] + \dfrac{\partial f}{\partial y}(x,y_3)\,z = v & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma \end{cases}$$

has a unique solution in $z \in H_0^1(\Omega)$ for any elements $y \in W^{1,p}(\Omega)$ and $y_i \in L^\infty(\Omega)$, $i = 1, 2, 3$.

Proof of Theorem 2.7. First we prove the uniqueness and then the existence.

Uniqueness of solution of (2.16). We follow the same approach used to prove the uniqueness of a solution of (1.1). Let us take $v = 0$ and assume that $z \in H_0^1(\Omega)$ is a solution of (2.16); then the goal is to prove that $z = 0$. Thus we define the sets

$$\Omega_0 = \{x \in \Omega : z(x) > 0\} \quad \text{and} \quad \Omega_\varepsilon = \{x \in \Omega : z(x) > \varepsilon\}.$$

Now we set $z_\varepsilon(x) = \min\{\varepsilon, z^+(x)\}$, so that $z_\varepsilon \in H_0^1(\Omega)$, $|z_\varepsilon| \le \varepsilon$, $zz_\varepsilon \ge 0$, $z\nabla z_\varepsilon = z_\varepsilon \nabla z_\varepsilon$, and $\nabla z \cdot \nabla z_\varepsilon = |\nabla z_\varepsilon|^2$. Then multiplying the equation corresponding to z by z_ε and performing an integration by parts we get

$$\int_\Omega \left\{a(x,y)|\nabla z_\varepsilon|^2 + \frac{\partial a}{\partial y}(x,y)z_\varepsilon \nabla y \cdot \nabla z_\varepsilon + \frac{\partial f}{\partial y}(x,y)z_\varepsilon^2\right\} dx = 0;$$

then, by the monotonicity of f and (A2),

$$\begin{aligned} \alpha_0 \|\nabla z_\varepsilon\|^2_{L^2(\Omega)} &\le \int_\Omega \left\{a(x,y)|\nabla z_\varepsilon|^2 + \frac{\partial f}{\partial y}(x,y)z_\varepsilon^2\right\} dx \\ = -\int_\Omega \frac{\partial a}{\partial y}(x,y)z_\varepsilon \nabla y \cdot \nabla z_\varepsilon\, dx &= -\int_{\Omega_0\setminus\Omega_\varepsilon} \frac{\partial a}{\partial y}(x,y)z_\varepsilon \nabla y \cdot \nabla z_\varepsilon\, dx \\ &\le C_M \|\nabla y\|_{L^p(\Omega_0\setminus\Omega_\varepsilon)} \|\nabla z_\varepsilon\|_{L^2(\Omega)}. \end{aligned}$$

From here follows an inequality analogous to (2.8), and continuing the proof in a similar manner, we conclude that $|\Omega_0| = \lim_{\varepsilon\to 0} |\Omega_\varepsilon| = 0$, and therefore $z \le 0$ in Ω. But $-z$ is also a solution of (2.16), so by the same arguments we deduce that $-z \le 0$ in Ω, and therefore $z = 0$.

Existence of solution of (2.16). For every $t \in [0, 1]$ let us consider the equation

$$\text{(2.17)} \qquad \begin{cases} -\text{div}\left[a(x,y)\nabla z + t\dfrac{\partial a}{\partial y}(x,y)z\,\nabla y_u\right] + \dfrac{\partial f}{\partial y}(x,y)\,z = v & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases}$$

For $t = 0$, the resulting linear equation is monotone, and by an obvious application of the Lax–Milgram theorem we know that there exists a unique solution $z_0 \in H_0^1(\Omega)$ for every $v \in H^{-1}(\Omega)$. Let us denote by S the set of points $t \in [0, 1]$ for which (2.17) defines an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. S is not empty because $0 \in S$. Let us denote by t_{max} the supremum of S. We will prove first that $t_{max} \in S$, and then we will see that $t_{max} = 1$, which concludes the proof of existence.

Let us take a sequence $\{t_k\}_{k=1}^\infty \subset S$ such that $t_k \to t_{max}$ when $k \to \infty$ and let us denote by z_k the solutions of (2.17) corresponding to the values t_k. Multiplying the equation of z_k by z_k and integrating by parts, using assumptions (A1) and (A2) we get

$$\alpha_0\|\nabla z_k\|^2_{L^2(\Omega)} \le \int_\Omega \left\{a(x,y)|\nabla z_k|^2 + \frac{\partial f}{\partial y}(x,y)z_k^2\right\} dx$$
$$= \langle v, z_k\rangle - t_k \int_\Omega \frac{\partial a}{\partial y}(x,y) z_k \nabla y \cdot \nabla z_k \, dx$$
$$\le \left(\|v\|_{H^{-1}(\Omega)} + t_k D_M \|\nabla y\|_{L^p(\Omega)} \|z_k\|_{L^{\frac{2p}{p-2}}(\Omega)}\right) \|\nabla z_k\|_{L^2(\Omega)},$$

which implies

$$\|\nabla z_k\|_{L^2(\Omega)} \le C\left(\|v\|_{H^{-1}(\Omega)} + \|z_k\|_{L^{\frac{2p}{p-2}}(\Omega)}\right). \tag{2.18}$$

In principle it seems that there are two possibilities: either $\{z_k\}_{k=1}^\infty$ is bounded in $L^{\frac{2p}{p-2}}(\Omega)$ or it is not. In the first case (2.18) implies that $\{z_k\}_{k=1}^\infty$ is bounded in $H_0^1(\Omega)$; then we can extract a subsequence, denoted in the same way, such that $z_k \rightharpoonup z$ weakly in $H_0^1(\Omega)$ and strongly in $L^{\frac{2p}{p-2}}(\Omega)$ because of the compactness of the embedding $H_0^1(\Omega) \subset L^{\frac{2p}{p-2}}(\Omega)$ for $p > n$. Therefore we can pass to the limit in (2.17), with $t = t_k$, and check that z is a solution of (2.17) for $t = t_{max}$, and therefore $t_{max} \in S$, as we wanted to prove.

Let us see that the second possibility is not actually a correct assumption. Indeed, let us assume that $\|z_k\|_{L^{\frac{2p}{p-2}}(\Omega)} \to \infty$, taking a subsequence if necessary. We define

$$\rho_k = \frac{1}{\|z_k\|_{L^{\frac{2p}{p-2}}(\Omega)}} \to 0 \quad \text{and} \quad \hat{z}_k = \rho_k z_k.$$

Then from (2.18) we deduce

$$\|\nabla \hat{z}_k\|_{L^2(\Omega)} \le C\left(\rho_k\|v\|_{H^{-1}(\Omega)} + \|\hat{z}_k\|_{L^{\frac{2p}{p-2}}(\Omega)}\right) = C\left(\rho_k\|v\|_{H^{-1}(\Omega)} + 1\right). \tag{2.19}$$

Moreover $\hat{z}_k$ satisfies the equation

$$\begin{cases} -\text{div}\left[a(x,y)\nabla \hat{z}_k + t_k \frac{\partial a}{\partial y}(x,y)\hat{z}_k \nabla y\right] + \frac{\partial f}{\partial y}(x,y)\hat{z}_k = \rho_k v & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases} \tag{2.20}$$

From (2.19) we know that we can extract a subsequence, denoted once again in the same way, such that $\hat{z}_k \rightharpoonup \hat{z}$ weakly in $H_0^1(\Omega)$ and strongly in $L^{\frac{2p}{p-2}}(\Omega)$. Then $\|\hat{z}\|_{L^{\frac{2p}{p-2}}(\Omega)} = 1$ and passing to the limit in (2.20) we have that $\hat{z}$ satisfies the equation

$$\begin{cases} -\text{div}\left[a(x,y)\nabla \hat{z} + t_{max} \frac{\partial a}{\partial y}(x,y)\hat{z} \nabla y\right] + \frac{\partial f}{\partial y}(x,y)\hat{z} = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases}$$

But we have already proved the uniqueness of solution of (2.16); the fact of including t_{max} in the equation does not matter for the proof. Therefore $\hat{z} = 0$, which contradicts the fact that its norm in $L^{\frac{2p}{p-2}}(\Omega)$ is one.

Finally we prove that $t_{max} = 1$. If it is false, then let us consider the operators $T_\varepsilon, T_{max} \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ for any $\varepsilon > 0$ with $t_{max} + \varepsilon \le 1$, defined by

$$T_\varepsilon z = -\mathrm{div}\left[a(x,y)\nabla z + (t_{max}+\varepsilon)\frac{\partial a}{\partial y}(x,y)z\,\nabla y\right] + \frac{\partial f}{\partial y}(x,y)z,$$

$$T_{max} z = -\mathrm{div}\left[a(x,y)\nabla z + t_{max}\frac{\partial a}{\partial y}(x,y)z\,\nabla y\right] + \frac{\partial f}{\partial y}(x,y)z.$$

Then we have

$$\|T_\varepsilon - T_{max}\|_{\mathcal{L}(H_0^1(\Omega),H^{-1}(\Omega))} = \sup_{\|z\|_{H_0^1(\Omega)}\le 1} \|(T_\varepsilon - T_{max})z\|_{H^{-1}(\Omega)}$$

$$\le D_M \sup_{\|z\|_{H_0^1(\Omega)}\le 1} \varepsilon \|z\|_{L^{\frac{2p}{p-2}}(\Omega)} \|\nabla y\|_{L^p(\Omega)} \le C\varepsilon.$$

Since T_{max} is an isomorphism, if $C\varepsilon < 1$, then T_ε is also an isomorphism, which contradicts the fact that t_{max} is the supremum of S. ∎

THEOREM 2.10. *Let us suppose that* (A1), (A2), *and* (A4) *hold. We also assume that* $a : \bar\Omega \times \mathbb{R} \mapsto \mathbb{R}$ *is continuous and* Γ *is of class* C^1. *Then the control-to-state mapping* $G : W^{-1,p}(\Omega) \to W_0^{1,p}(\Omega)$, $G(u) = y_u$, *is of class* C^2. *Moreover, for any* $v, v_1, v_2 \in W^{-1,p}(\Omega)$ *the functions* $z_v = G'(u)v$ *and* $z_{v_1,v_2} = G''(u)[v_1, v_2]$ *are the unique solutions in* $W_0^{1,p}(\Omega)$ *of the equations*

$$\begin{cases} -\mathrm{div}\left[a(x,y_u)\nabla z + \dfrac{\partial a}{\partial y}(x,y_u)z\,\nabla y_u\right] + \dfrac{\partial f}{\partial y}(x,y)\,z = v & \text{in } \Omega,\\ z = 0 & \text{on } \Gamma \end{cases} \tag{2.21}$$

and

$$\begin{cases} -\mathrm{div}\left[a(x,y_u)\nabla z + \dfrac{\partial a}{\partial y}(x,y_u)z\nabla y_u\right] + \dfrac{\partial f}{\partial y}(x,y_u)\,z = -\dfrac{\partial^2 f}{\partial y^2}(x,y_u)z_{v_1}z_{v_2} \\ \quad + \mathrm{div}\left[\dfrac{\partial a}{\partial y}(x,y_u)(z_{v_1}\nabla z_{v_2} + \nabla z_{v_1} z_{v_2}) + \dfrac{\partial^2 a}{\partial y^2}(x,y_u)z_{v_1}z_{v_2}\nabla y_u\right] \quad \text{in } \Omega,\\ z = 0 \quad \text{on } \Gamma, \end{cases} \tag{2.22}$$

respectively, where $z_i = G'(u)v_i$, $i = 1, 2$.

Proof. We introduce the mapping $F : W_0^{1,p}(\Omega) \times W^{-1,p}(\Omega) \to W^{-1,p}(\Omega)$ by

$$F(y,u) = -\mathrm{div}\,[a(\cdot,y)\nabla y] + f(\cdot,y) - u.$$

Because of the assumptions (A2) and (A4), it is obvious that F is well defined, of class C^2, and $F(y_u, u) = 0$ for every $u \in W_0^{1,p}(\Omega)$. If we prove that

$$\frac{\partial F}{\partial y}(y_u,u) : W_0^{1,p}(\Omega) \longrightarrow W^{-1,p}(\Omega)$$

is an isomorphism, then we can apply the implicit function theorem to deduce the theorem, getting (2.21) and (2.22) by simple computations. Let us remark that

$$\frac{\partial F}{\partial y}(y_u,u)z = -\mathrm{div}\left[a(x,y_u)\nabla z + \frac{\partial a}{\partial y}(x,y_u)z\,\nabla y_u\right] + \frac{\partial f}{\partial y}(x,y_u)\,z.$$

According to Theorem 2.7, for any $v \in H^{-1}(\Omega)$, there exists a unique element $z \in H_0^1(\Omega)$ such that

$$\frac{\partial F}{\partial y}(y_u, u)z = v.$$

It is enough to prove that $z \in W_0^{1,p}(\Omega)$ if $v \in W^{-1,p}(\Omega) \subset H^{-1}(\Omega)$. More precisely, this means that the unique solution of (2.16) in $H_0^1(\Omega)$ belongs to $W_0^{1,p}(\Omega)$. First of all, let us note that

$$a(\cdot, y_u) \in L^\infty(\Omega),\ \frac{\partial a}{\partial y}(\cdot, y_u)\nabla y_u \in L^p(\Omega)^n,\ \frac{\partial f}{\partial y}(\cdot, y_u) \in L^\infty(\Omega), \quad \text{and} \quad v \in W^{-1,p}(\Omega).$$

Therefore, we can apply a result by Stampacchia [30, Theorem 4.1 and Remark 4.2] about $L^\infty(\Omega)$-estimates of solutions of linear equations to get that $z \in L^\infty(\Omega)$. Now we have that

$$-\text{div}[a(x, y_u)\nabla z] = v + \text{div}\left[\frac{\partial a}{\partial y}(x, y_u)z\nabla y_u\right] - \frac{\partial f}{\partial y}(x, y_u)z \in W^{-1,p}(\Omega)$$

and $x \mapsto a(x, y_u(x))$ is a continuous real-valued function defined in $\bar\Omega$. Finally, as in the proof of Theorem 2.4, we can use the $W_0^{1,p}(\Omega)$-regularity results for linear equations (see [18, Chap. 4, p. 73] or [25, pp. 156–157]) to deduce that $z \in W_0^{1,p}(\Omega)$. ∎

From Theorem 2.5 we know that the states y corresponding to controls $u \in L^q(\Omega)$, with $q > n/2$, can have an extra regularity under certain assumptions. In this situation, a natural question arises. Can we prove a result analogous to Theorem 2.10 with $G : L^q(\Omega) \to W^{2,q}(\Omega)$? The answer is positive if we assume some extra regularity of the function a.

(A5) For all $M > 0$, there exists a constant $d_M > 0$ such that the following inequality is satisfied:

$$\left|\frac{\partial^j a}{\partial y^j}(x_1, y_1) - \frac{\partial^j a}{\partial y^j}(x_2, y_2)\right| \le d_M\ \{|x_1 - x_2| + |y_1 - y_2|\} \tag{2.23}$$

for all $x_i \in \bar\Omega$, $y_i \in [-M, M]$, $i = 1, 2$ and $j = 1, 2$.

THEOREM 2.11. *Suppose that* (A1)–(A5) *hold and* Γ *is of class* $C^{1,1}$. *Then the control-to-state mapping* $G : L^q(\Omega) \to W^{2,q}(\Omega)$, $G(u) = y_u$, *is of class* C^2. *For any* $v, v_1, v_2 \in L^q(\Omega)$, *the functions* $z_v = G'(u)v$ *and* $z_{v_1,v_2} = G''(u)[v_1, v_2]$ *are the unique solutions in* $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ *of* (2.21) *and* (2.22), *respectively.*

Proof. The proof follows the same steps as in the previous theorem, with obvious modifications. Let us note the main differences. This time, the function F is defined by the same expression as above and acts from $(W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)) \times L^q(\Omega)$ to $L^q(\Omega)$. We have to check that F is well defined, and we must determine the first- and second-order derivatives. By using the assumptions (A3)–(A5), we have for $j = 0, 1, 2$ and $y \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ that

$$\begin{aligned}\text{div}\left[\frac{\partial^j a}{\partial y^j}(x, y(x))\nabla y(x)\right] &= \left[\nabla_x \frac{\partial^j a}{\partial y^j}\right](x, y(x)) \cdot \nabla y(x) + \frac{\partial^{j+1} a}{\partial y^{j+1}}(x, y(x))|\nabla y(x)|^2 \\ &\quad + \frac{\partial^j a}{\partial y^j}(x, y(x))\Delta y(x) \in L^q(\Omega).\end{aligned} \tag{2.24}$$

We have used the fact that $(\partial^j a/\partial y^j)$ is Lipschitz in x and y, and therefore differentiable a.e., and that the chain rule is valid in the framework of Sobolev spaces.

On the other hand, (A2) and (A4) imply that

$$\frac{\partial^j f}{\partial y^j}(\cdot, y) \in L^q(\Omega) \quad \text{for} \quad j = 0, 1, 2.$$

From these remarks, it is easy to deduce that F is of class C^2. Let us prove that (2.16) has a unique solution $z \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ for any $v \in L^q(\Omega)$. The uniqueness is an immediate consequence of the uniqueness of solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$. It remains to prove the $W^{2,q}$-regularity. We argue similarly to the proof of Theorem 2.4. From (2.16) we get

$$\begin{aligned} -\Delta z &= \frac{1}{a}\left\{ v + \operatorname{div}\left[\frac{\partial a}{\partial y}(x,\bar{y})\, z\, \nabla\bar{y}\right] - \frac{\partial f}{\partial y}(x,\bar{y})\, z + v \right\} + \nabla_x a \cdot \nabla z + \frac{\partial a}{\partial y} \nabla\bar{y} \cdot \nabla z \\ &= \frac{1}{a}\left\{ v - \frac{\partial f}{\partial y}(x,\bar{y})\, z + \nabla_x \frac{\partial a}{\partial y}\, z \cdot \nabla\bar{y} + \frac{\partial^2 a}{\partial y^2}\, z\, |\nabla\bar{y}|^2 + \frac{\partial a}{\partial y} \nabla z \cdot \nabla\bar{y} + \frac{\partial a}{\partial y}\, z\, \Delta\bar{y} \right\} \\ &\quad + \nabla_x a \cdot \nabla z + \frac{\partial a}{\partial y}\, \nabla z \cdot \nabla\bar{y}. \end{aligned}$$

The right-hand side is an element of $L^q(\Omega)$. To verify this, consider, for instance, the term with the lowest regularity, i.e., the term $\nabla\bar{y} \cdot \nabla z$:

$$\begin{aligned} \left(\int_\Omega |\nabla\bar{y}|^q |\nabla z|^q dx\right)^{\frac{1}{q}} &\le \left(\int_\Omega |\nabla\bar{y}|^n dx\right)^{\frac{1}{n}} \left(\int_\Omega |\nabla z|^{\frac{nq}{n-q}} dx\right)^{\frac{n-q}{nq}} \\ &\le c \left(\int_\Omega |\nabla\bar{y}|^{\frac{nq}{n-q}} dx\right)^{\frac{n-q}{nq}} \|z\|_{W_0^{1,\frac{nq}{n-q}}(\Omega)} \\ &\le c \|\bar{y}\|_{W^{2,q}(\Omega)} \|z\|_{W_0^{1,\frac{nq}{n-q}}(\Omega)}, \end{aligned}$$

where we have used that $z \in W_0^{1,\frac{nq}{n-q}}(\Omega)$, which is a consequence of the embedding $L^q(\Omega) \subset W^{-1,\frac{nq}{n-q}}(\Omega)$ along with Theorem 2.10. Notice that we have assumed $q > n/2$. This inequality is equivalent to $nq/(n-q) > n$ and is also behind the estimate of the integral containing $\nabla\bar{y}$. ☐

Remark 2.12. If $q = 2$, then Theorem 2.11 remains true for $n = 2$ or $n = 3$ if we replace the $C^{1,1}$-regularity of Γ by the convexity of Ω. This is a consequence of the H^2-regularity for the elliptic problems in convex domains; see Grisvard [20].

3. The control problem. Associated to the state equation (1.1), we introduce the control problem

$$\text{(P)} \qquad \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))\, dx, \\ u \in L^\infty(\Omega), \\ \alpha(x) \le u(x) \le \beta(x) \text{ for a.e. } x \in \Omega, \end{cases}$$

where $L : \Omega \times (\mathbb{R} \times \mathbb{R}) \to \mathbb{R}$ is a Carathéodory function, $p > n$, and $\alpha, \beta \in L^\infty(\Omega)$, with $\beta(x) \ge \alpha(x)$ for a.e. $x \in \Omega$. A standard example for the choice of L is the quadratic function

$$L(x, y, u) = (y - y_d(x))^2 + \frac{N}{2}\, u^2,$$

where $y_d \in L^q(\Omega)$ is given fixed.

First of all, we study the existence of a solution for problem (P).

THEOREM 3.1. *Let us assume that* (A1) *and* (A2) *hold. We also suppose that L is convex with respect to u and, for any $M > 0$, there exists a function $\psi_M \in L^1(\Omega)$ such that*

$$|L(x, y, u)| \le \psi_M(x) \quad \text{for a.e. } x \in \Omega \quad \text{and} \quad |y|, |u| \le M.$$

Then (P) *has at least one optimal solution $\bar{u}$.*

Proof. Let $\{u_k\}_{k=1}^\infty \subset L^\infty(\Omega)$ be a minimizing sequence for (P). Since $\{u_k\}_{k=1}^\infty$ is bounded in $L^\infty(\Omega) \subset W^{-1,p}(\Omega)$, Theorem 2.4 implies that $\{y_{u_k}\}_{k=1}^\infty$ is bounded in $W_0^{1,p}(\Omega)$ and, taking a subsequence, denoted in the same way, we get $u_k \rightharpoonup \bar{u}$ weakly* in $L^\infty(\Omega)$, and hence strongly in $W^{-1,p}(\Omega)$. Therefore, $y_{u_k} \to \bar{y}_u$ in $W_0^{1,p}(\Omega)$. Moreover, it is obvious that $\alpha \le \bar{u} \le \beta$, and hence $\bar{u}$ is a feasible control for (P). Let us denote by $\bar{y}$ the state associated to $\bar{u}$. Now we prove that $\bar{u}$ is a solution of (P). It is enough to use the convexity of L with respect to u along with the continuity with respect to (y, u) and the Lebesgue dominated convergence theorem as follows:

$$\begin{aligned} J(\bar{u}) = \int_\Omega L(x, \bar{y}(x), \bar{u}(x))\, dx &\le \liminf_{k\to\infty} \int_\Omega L(x, \bar{y}(x), u_k(x))\, dx \\ &\le \limsup_{k\to\infty} \int_\Omega |L(x, \bar{y}(x), u_k(x)) - L(x, y_{u_k}(x), u_k(x))|\, dx \\ &+ \limsup_{k\to\infty} \int_\Omega L(x, y_{u_k}(x), u_k(x))\, dx = \lim_{k\to\infty} J(u_k) = \inf\,(\mathrm{P}). \qquad \square \end{aligned}$$

Our next goal is to derive the first-order optimality conditions. We get the optimality conditions satisfied by $\bar{u}$ from the standard variational inequality $J'(\bar{u})(u-\bar{u}) \ge 0$ for any feasible control u. To argue in this way, we need the differentiability of J, which requires the differentiability of L with respect to u and y. Since we also wish to derive second-order optimality conditions, we require the existence of the second-order derivatives of L. More precisely, our assumption is the following.

(A6) $L : \Omega \times (\mathbb{R} \times \mathbb{R}) \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the last two variables and, for all $M > 0$, there exist a constant $C_{L,M} > 0$ and functions $\psi_{u,M} \in L^2(\Omega)$ and $\psi_{y,M} \in L^q(\Omega)$, such that

$$\left|\frac{\partial L}{\partial u}(x, y, u)\right| \le \psi_{u,M}(x), \quad \left|\frac{\partial L}{\partial y}(x, y, u)\right| \le \psi_{y,M}(x), \quad \|D^2_{(y,u)} L(x, y, u)\| \le C_{L,M},$$

$$\|D^2_{(y,u)} L(x, y_2, u_2) - D^2_{(y,u)} L(x, y_1, u_1)\| \le C_{L,M}(|y_2 - y_1| + |u_2 - u_1|)$$

for a.e. $x \in \Omega$ and $|y|, |y_i|, |u|, |u_i| \le M$, $i = 1, 2$, where $D^2_{(y,u)} L$ denotes the second derivative of L with respect to (y, u), i.e., the associated Hessian matrix.

By applying the chain rule and introducing the adjoint state as usual, an elementary calculus leads to the following result.

THEOREM 3.2. *Let us assume that $a : \bar{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$ is continuous, Γ is of class C^1, and* (A1), (A2), (A4), *and* (A6) *hold. Then the function $J : L^\infty(\Omega) \to \mathbb{R}$ is of class C^2. Moreover, for every $u, v, v_1, v_2 \in L^\infty(\Omega)$, we have*

$$J'(u)v = \int_\Omega \left(\frac{\partial L}{\partial u}(x, y_u, u) + \varphi_u\right) v\, dx \tag{3.1}$$

and

$$J''(u)v_1v_2 = \int_\Omega \Big\{ \frac{\partial^2 L}{\partial y^2}(x,y_u,u)z_{v_1}z_{v_2} + \frac{\partial^2 L}{\partial y \partial u}(x,y_u,u)(z_{v_1}v_2 + z_{v_2}v_1) \tag{3.2}$$
$$+\frac{\partial^2 L}{\partial u^2}(x,y_u,u)v_1v_2 - \varphi_u \frac{\partial^2 f}{\partial y^2}(x,y_u)z_{v_1}z_{v_2}$$
$$-\nabla\varphi_u \left[\frac{\partial a}{\partial y}(x,y_u)(z_{v_1}\nabla z_{v_2} + \nabla z_{v_1} z_{v_2}) + \frac{\partial^2 a}{\partial y^2}(x,y)z_{v_1}z_{v_2}\nabla y_u\right]\Big\}\, dx,$$

where $\varphi_u \in W_0^{1,p}(\Omega)$ *is the unique solution of the problem*

$$\left\{\begin{array}{r} -\mathrm{div}\,[a(x,y_u)\nabla\varphi] + \dfrac{\partial a}{\partial y}(x,y_u)\nabla y_u \cdot \nabla\varphi + \dfrac{\partial f}{\partial y}(x,y_u)\varphi = \dfrac{\partial L}{\partial y}(x,y_u,u) \ \textit{in } \Omega, \\ \varphi = 0 \ \textit{on } \Gamma, \end{array}\right. \tag{3.3}$$

where $z_{v_i} = G'(u)v_i$ *is the solution of* (2.21) *for* $y = y_u$ *and* $v = v_i$, $i = 1, 2$.

Proof. The only delicate point in the proof of the previous theorem is the existence and uniqueness of a solution of the adjoint state equation (3.3). To prove this, let us consider the linear operator $T \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ given by

$$Tz = -\mathrm{div}\left[a(x,y)\nabla z + \frac{\partial a}{\partial y}(x,y)z\,\nabla y\right] + \frac{\partial f}{\partial y}(x,y)\,z.$$

According to Remark 2.8, T is an isomorphism and its adjoint operator is also an isomorphism $T^* \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ given by

$$T^*\varphi = -\mathrm{div}\,[a(x,y_u)\nabla\varphi] + \frac{\partial a}{\partial y}(x,y_u)\nabla y_u \cdot \nabla\varphi + \frac{\partial f}{\partial y}(x,y_u)\varphi.$$

This is exactly equivalent to the well-posedness of the adjoint equation (3.3) in $H_0^1(\Omega)$. Finally, Theorems 2.2 and 2.4 along with assumption (A6) imply that the adjoint state φ belongs to the space $W_0^{1,p}(\Omega)$, as claimed in the theorem, provided that the term

$$\frac{\partial a}{\partial y}(x,y_u)\nabla y_u \cdot \nabla\varphi$$

belongs to $W^{-1,p}(\Omega)$. Let us prove this fact. Thanks to the boundedness of y_u and the assumption (A4), it is enough to prove that $\nabla y_u \cdot \nabla\varphi \in L^r(\Omega) \subset W^{-1,p}(\Omega)$ holds for some r large enough. By using that $\nabla y_u \in L^p(\Omega)$, $\nabla\varphi \in L^2(\Omega)$ and invoking the Hölder inequality, we get that $\nabla y_u \cdot \nabla\varphi \in L^{2p/(p+2)}(\Omega)$. For $n = 2$, $L^{2p/(p+2)}(\Omega) \subset W^{-1,p}(\Omega)$. Let us consider the case $n > 2$. In this case, we have

$$L^{2p/(p+2)}(\Omega) \subset W^{-1,r}(\Omega), \quad \text{with } r = \frac{2pn}{p(n-2)+2n}.$$

Therefore it turns out that $\varphi \in W_0^{1,\sigma}(\Omega)$, with $\sigma = \min\{p, r\}$. If $\sigma = p$, then the proof is complete. If it is not true, then let us notice that

$$r = 2 + \varepsilon, \quad \text{with } \varepsilon = \frac{4(p-n)}{p(n-2)+2n}.$$

The proof proceeds by induction: For $k \geq 1$, we assume that $\varphi \in W_0^{1,2+k\varepsilon}(\Omega)$ and then we prove that $\varphi \in W_0^{1,\sigma}(\Omega)$, with $\sigma = \min\{p, 2+(k+1)\varepsilon\}$. Consequently, for k large enough, we have that $\sigma = p$. By using the embedding of Sobolev spaces in L^r spaces and after performing some obvious computations, we get that

$$\nabla y_u \in L^p(\Omega) \text{ and } \nabla\varphi \in L^{2+k\varepsilon}(\Omega) \Rightarrow \nabla y_u \cdot \nabla\varphi \in W^{-1,r}(\Omega),$$

with

$$r = \frac{pn(2+k\varepsilon)}{p[n-(2+k\varepsilon)]+(2+k\varepsilon)n}.$$

We have to prove that $r-(2+k\varepsilon) \geq \varepsilon$, which is equivalent to

$$\frac{(p-n)(2+k\varepsilon)^2}{p[n-(2+k\varepsilon)]+(2+k\varepsilon)n} \geq \varepsilon.$$

From the definition of ε, we obtain that the previous inequality is equivalent to

$$(p-n)(2+k\varepsilon)^2 \geq \frac{4(p-n)}{p(n-2)+2n}\{p[n-(2+k\varepsilon)]+(2+k\varepsilon)n\}$$

if and only if

$$[p(n-2)+2n](2+k\varepsilon)^2 \geq 4\{p[n-(2+k\varepsilon)]+(2+k\varepsilon)n\}.$$

Let us set for every $p \geq n$

$$\rho(p) = [p(n-2)+2n](2+k\varepsilon(p))^2,\ \mu(p) = 4\{p[n-(2+k\varepsilon(p))]+(2+k\varepsilon(p))n\}$$

and

$$\varepsilon(p) = \frac{4(p-n)}{p(n-2)+2n}.$$

Using that $\varepsilon(n) = 0$, we get that $\rho(n) = 4n^2 = \mu(n)$. If we prove that $\rho'(p) > \mu'(p)$ for every $p > n$, then the inequality $\rho(p) > \mu(p)$ will be true for all $p > n$ and the proof of the theorem is concluded. Using that $\varepsilon'(p) > 0$ and $\varepsilon(p) > 0$ for $p > n$, we get

$$\rho'(p) = (n-2)(2+k\varepsilon(p))^2 + 2k[p(n-2)+2n](2+k\varepsilon(p))\varepsilon'(p) > 4(n-2)$$

and

$$\begin{aligned}\mu'(p) &= 4(n-2-k\varepsilon(p)) + 4(-kp\varepsilon'(p)+kn\varepsilon'(p))\\ &= 4(n-2) - 4k[\varepsilon(p)+(p-n)\varepsilon'(p)] < 4(n-2),\end{aligned}$$

which leads to the desired result. ∎

Remark 3.3. By using the expression given by (3.2) for $J''(u)$, it is obvious that $J''(u)$ can be extended to a continuous bilinear form $J''(u) : L^2(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$.

By using the inequality $J'(\bar{u})(u-\bar{u}) \geq 0$ and the differentiability of J given by (3.1) and (3.3) we deduce the first-order optimality conditions.

THEOREM 3.4. *Under the assumptions of Theorem* 3.2, *if $\bar u$ is a local minimum of* (P), *then there exists $\bar\varphi \in W_0^{1,p}(\Omega)$ such that*

$$\left\{\begin{array}{r} -\mathrm{div}\,[a(x,\bar y)\nabla\bar\varphi] + \dfrac{\partial a}{\partial y}(x,\bar y)\nabla\bar y\cdot\nabla\bar\varphi + \dfrac{\partial f}{\partial y}(x,\bar y)\bar\varphi = \dfrac{\partial L}{\partial y}(x,\bar y,\bar u) \text{ in } \Omega,\\ \bar\varphi = 0 \text{ on } \Gamma, \end{array}\right. \tag{3.4}$$

$$\int_\Omega \left(\frac{\partial L}{\partial u}(x,\bar y(x),\bar u(x)) + \bar\varphi(x)\right)(u(x)-\bar u(x))\,dx \ge 0 \quad \forall\ \alpha\le u\le\beta, \tag{3.5}$$

where $\bar y$ is the state associated to $\bar u$.

From (3.5) we get as usual

$$\bar u(x) = \left\{\begin{array}{ll} \alpha(x) & \text{if } \bar d(x) > 0,\\ \beta(x) & \text{if } \bar d(x) < 0 \end{array}\right. \quad \text{and} \quad \bar d(x) = \left\{\begin{array}{ll} \ge 0 & \text{if } \bar u(x) = \alpha(x),\\ \le 0 & \text{if } \bar u(x) = \beta(x),\\ = 0 & \text{if } \alpha(x) < \bar u(x) < \beta(x) \end{array}\right. \tag{3.6}$$

for almost all $x\in\Omega$, where

$$\bar d(x) = \frac{\partial L}{\partial u}(x,\bar y(x),\bar u(x)) + \bar\varphi(x). \tag{3.7}$$

We finish this section by studying the regularity of the optimal solutions of (P).

THEOREM 3.5. *Under the assumptions of Theorem* 3.4 *and assuming that*

$$\frac{\partial L}{\partial u} : \bar\Omega\times(\mathbb{R}\times\mathbb{R}) \to \mathbb{R} \textit{ is continuous,} \tag{3.8}$$

$$\exists\Lambda_L > 0 \textit{ such that } \frac{\partial^2 L}{\partial u^2}(x,y,u) \ge \Lambda_L \quad \textit{for a.e. } x\in\Omega \textit{ and } \forall y,u\in\mathbb{R}^2, \tag{3.9}$$

then the equation

$$\frac{\partial L}{\partial u}(x,\bar y(x),t) + \bar\varphi(x) = 0 \tag{3.10}$$

has a unique solution $\bar t = \bar s(x)$ for every $x\in\bar\Omega$. The function $\bar s : \bar\Omega \to \mathbb{R}$ is continuous and is related to $\bar u$ by the formula

$$\bar u(x) = \mathit{Proj}_{[\alpha(x),\beta(x)]}(\bar s(x)) = \max\{\min\{\beta(x),\bar s(x)\},\alpha(x)\}. \tag{3.11}$$

Moreover, if α,β are contained in $C(\bar\Omega)$, then $\bar u$ belongs to $C(\bar\Omega)$, too. Finally, if Γ is $C^{1,1}$, (A3) *holds, $q > n$ is taken in the assumptions* (A2) *and* (A6), *$\alpha,\beta\in C^{0,1}(\bar\Omega)$, and for every $M>0$ there exists a constant $C_{L,M}>0$ such that*

$$\left|\frac{\partial L}{\partial u}(x_2,y,u) - \frac{\partial L}{\partial u}(x_1,y,u)\right| \le C_{L,M}|x_2-x_1| \quad \forall x_i\in\Omega \textit{ and } \forall |y|,|u|\le M, \tag{3.12}$$

then $\bar s,\bar u\in C^{0,1}(\bar\Omega)$.

Proof. Given $x\in\bar\Omega$, let us define the function $g:\mathbb{R}\to\mathbb{R}$ by

$$g(t) = \frac{\partial L}{\partial u}(x,\bar y(x),t) + \bar\varphi(x).$$

Then g is of class C^1 and from (3.9) we know that it is strictly increasing and

$$\lim_{t\to-\infty} g(t) = -\infty \quad \text{and} \quad \lim_{t\to+\infty} g(t) = +\infty.$$

Therefore, there exists a unique element $\bar{t} \in \mathbb{R}$ such that $g(\bar{t}) = 0$.

Taking $\bar{d}$ as defined by (3.7) and using (3.6) along with the strict monotonicity of $(\partial L/\partial u)$ with respect to the third variable, we obtain

$$\begin{cases} \text{if } \bar{d}(x) > 0, \text{ then } \alpha(x) = \bar{u}(x) > \bar{s}(x), \\ \text{if } \bar{d}(x) < 0, \text{ then } \beta(x) = \bar{u}(x) < \bar{s}(x), \\ \text{if } \bar{d}(x) = 0, \text{ then } \bar{u}(x) = \bar{s}(x), \end{cases}$$

which implies (3.11).

Let us prove that $\bar{s}$ is a bounded function. By using the mean value theorem along with (3.8), (3.9), and (3.10), we get

$$\Lambda_L|\bar{s}(x)| \le \left|\frac{\partial L}{\partial u}(x,\bar{y}(x),\bar{s}(x)) - \frac{\partial L}{\partial u}(x,\bar{y}(x),0)\right| = \left|\bar{\varphi}(x) + \frac{\partial L}{\partial u}(x,\bar{y}(x),0)\right|,$$

and hence

$$|\bar{s}(x)| \le \frac{1}{\Lambda_L}\max_{x\in\bar{\Omega}}\left|\bar{\varphi}(x) + \frac{\partial L}{\partial u}(x,\bar{y}(x),0)\right| < \infty.$$

The continuity of $\bar{s}$ at every point $x \in \bar{\Omega}$ follows easily from the continuity of $\bar{y}$ and $(\partial L/\partial u)$ by using the inequality

$$\begin{aligned} \Lambda_L|\bar{s}(x) - \bar{s}(x')| &\le \left|\frac{\partial L}{\partial u}(x',\bar{y}(x'),\bar{s}(x)) - \frac{\partial L}{\partial u}(x',\bar{y}(x'),\bar{s}(x'))\right| \\ &\le |\bar{\varphi}(x') - \bar{\varphi}(x)| + \left|\frac{\partial L}{\partial u}(x',\bar{y}(x'),\bar{s}(x)) - \frac{\partial L}{\partial u}(x,\bar{y}(x),\bar{s}(x))\right|. \end{aligned} \tag{3.13}$$

If $\alpha, \beta \in C(\bar{\Omega})$, then the identity (3.11) and the continuity of $\bar{s}$ imply the continuity of $\bar{u}$ in $\bar{\Omega}$.

Finally, if Γ is $C^{1,1}$ and (A3) and (A6) hold with $q > n$, then $\bar{y}, \bar{\varphi} \in W^{2,q}(\Omega) \subset C^{0,1}(\Omega)$. Then we can get from (3.13), the boundedness of $\bar{s}$, and (3.12) that $\bar{s} \in C^{0,1}(\bar{\Omega})$. Once again, (3.11) allows us to conclude that $\bar{u} \in C^{0,1}(\bar{\Omega})$, assuming that α and β are also Lipschitz in $\bar{\Omega}$. Indeed, it is enough to realize that

$$\begin{aligned} |\bar{u}(x_2) - \bar{u}(x_1)| &\le \max\{|\beta(x_2) - \beta(x_1)|, |\alpha(x_2) - \alpha(x_1)|, |\bar{s}(x_2) - \bar{s}(x_1)|\} \\ &\le \max\{L_\beta, L_\alpha, L_{\bar{s}}\}|x_2 - x_1|, \end{aligned}$$

where L_β, L_α, and $L_{\bar{s}}$ are the Lipschitz constants of α, β, and $\bar{s}$, respectively. ∎

4. Pontryagin's principle. The goal of this section is to derive the Pontryagin principle satisfied by a local solution of (P). We need this principle for our second-order analysis. There is already an extensive list of contributions about Pontryagin's principle, but none of them was devoted to quasilinear equations of nonmonotone type. This lack of monotonicity requires an adaptation of the usual proofs to overcome this difficulty. For this purpose, we will make the following assumption.

(A7) $L : \Omega \times (\mathbb{R} \times \mathbb{R}) \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^1 with respect to the second variable and, for all $M > 0$, there exists a function $\psi_M \in L^q(\Omega)$, with $q \geq pn/(p+n)$, such that

$$\left|\frac{\partial L}{\partial y}(x,y,u)\right| \leq \psi_M(x) \text{ for a.e. } x \in \Omega,\ |u| \leq M, \text{ and } |y| \leq M.$$

Associated with the control problem (P), we define the Hamiltonian as usual by

$$H(x,y,u,\varphi) = L(x,y,u) + \varphi[u - f(x,y)].$$

The Pontryagin principle is formulated as follows.

THEOREM 4.1. *Let $\bar{u}$ be a local solution of* (P). *We assume that $a : \bar{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$ is continuous, Γ is of class C^1, and* (A1), (A2), (A4), *and* (A7) *hold. Then there exists $\bar{\varphi} \in W_0^{1,p}(\Omega)$ satisfying* (3.4) *and*

$$H(x,\bar{y}(x),\bar{u}(x),\bar{\varphi}(x)) = \min_{s \in [\alpha_{\varepsilon_{\bar{u}}}(x), \beta_{\varepsilon_{\bar{u}}}(x)]} H(x,\bar{y}(x),s,\bar{\varphi}(x)) \quad \textit{for a.e. } x \in \Omega, \tag{4.1}$$

where

$$\alpha_{\varepsilon_{\bar{u}}}(x) = \max\{\alpha(x), \bar{u}(x) - \varepsilon_{\bar{u}}\} \quad \textit{and} \quad \beta_{\varepsilon_{\bar{u}}}(x) = \min\{\beta(x), \bar{u}(x) + \varepsilon_{\bar{u}}\},$$

$\varepsilon_{\bar{u}} > 0$ is the radius of the $L^\infty(\Omega)$ ball where J achieves the minimum value at $\bar{u}$ among all feasible controls.

Relation (4.1) is an immediate consequence of (3.5) if L is convex with respect to the third variable, but this assumption is not made in the above theorem. To prove (4.1), we will use the following lemma whose proof can be found in [13, Lemma 4.3].

LEMMA 4.2. *For every $0 < \rho < 1$, there exists a sequence of Lebesgue measurable sets $\{E_k\}_{k=1}^\infty \subset \Omega$ such that*

$$|E_k| = \rho|\Omega| \quad \textit{and} \quad \frac{1}{\rho}\chi_{E_k} \rightharpoonup 1 \quad \textit{in } L^\infty(\Omega) \quad \textit{weakly}^\star, \tag{4.2}$$

where $|\cdot|$ denotes the Lebesgue measure.

PROPOSITION 4.3. *Under the assumptions of Theorem* 4.1, *for any $u \in L^\infty(\Omega)$ there exist a number $0 < \hat{\rho} < 1$ and measurable sets $E_\rho \subset \Omega$, with $|E_\rho| = \rho|\Omega|$ for all $0 < \rho < \hat{\rho}$, that have the following properties: If we define*

$$u_\rho(x) = \begin{cases} \bar{u}(x) & \textit{if } x \in \Omega \setminus E_\rho, \\ u(x) & \textit{if } x \in E_\rho, \end{cases}$$

then

$$y_\rho = \bar{y} + \rho z + r_\rho, \quad \lim_{\rho \searrow 0} \frac{1}{\rho}\|r_\rho\|_{W_0^{1,p}(\Omega)} = 0, \tag{4.3}$$

$$J(u_\rho) = J(\bar{u}) + \rho z^0 + r_\rho^0, \quad \lim_{\rho \searrow 0} \frac{1}{\rho}|r_\rho^0| = 0 \tag{4.4}$$

hold true, where $\bar{y}$ and y_ρ are the states associated to $\bar{u}$ and u_ρ, respectively, z is the unique element of $W_0^{1,p}(\Omega)$ satisfying the linearized equation

$$\operatorname{div}\left[a(x,\bar{y})\nabla z + \frac{\partial a}{\partial y}(x,\bar{y})z\,\nabla\bar{y}\right] + \frac{\partial f}{\partial y}(x,\bar{y})\,z = u - \bar{u} \quad \textit{in } \Omega, \tag{4.5}$$

and

$$z^0 = \int_\Omega \left\{ \frac{\partial L}{\partial y}(x, \bar{y}(x), \bar{u}(x)) z(x) + L(x, \bar{y}(x), u(x)) - L(x, \bar{y}(x), \bar{u}(x)) \right\} dx. \tag{4.6}$$

Proof. Let us define the function $g \in L^1(\Omega)$ by

$$g(x) = L(x, \bar{y}(x), u(x)) - L(x, \bar{y}(x), \bar{u}(x)).$$

Given $\rho \in (0, 1)$, we take a sequence $\{E_k\}_{k=1}^\infty$ as in Lemma 4.2. Since $L^\infty(\Omega)$ is compactly embedded in $W^{-1,p}(\Omega)$, there exists k_ρ such that

$$\left| \int_\Omega \left(1 - \frac{1}{\rho}\chi_{E_k}(x)\right) g(x)\, dx \right| + \left\| \left(1 - \frac{1}{\rho}\chi_{E_k}\right)(u - \bar{u}) \right\|_{W^{-1,p}(\Omega)} < \rho \quad \forall k \geq k_\rho. \tag{4.7}$$

Let us denote $E_\rho = E_{k_\rho}$. Let us introduce $z_\rho = (y_\rho - \bar{y})/\rho$. By subtracting the equations satisfied by y_ρ and $\bar{y}$ and dividing by ρ we get

$$-\text{div}\left[a(x, \bar{y}) \nabla z_\rho + \frac{a(x, y_\rho) - a(x, \bar{y})}{\rho} \nabla y_\rho \right] + \frac{f(x, y_\rho) - f(x, \bar{y})}{\rho} = \frac{u_\rho - \bar{u}}{\rho} \quad \text{in } \Omega.$$

Now setting

$$a_\rho(x) = \int_0^1 \frac{\partial a}{\partial y}(x, \bar{y}(x) + \theta(y_\rho(x) - \bar{y}(x)))\, d\theta,$$
$$f_\rho(x) = \int_0^1 \frac{\partial f}{\partial y}(x, \bar{y}(x) + \theta(y_\rho(x) - \bar{y}(x)))\, d\theta$$

we deduce from the above identity

$$-\text{div}\, [a(x, \bar{y}) \nabla z_\rho + a_\rho(x) z_\rho \nabla y_\rho] + f_\rho(x) z_\rho = \frac{1}{\rho} \chi_{E_\rho}(u - \bar{u}) \quad \text{in } \Omega. \tag{4.8}$$

Let us define $T, T_\rho : W_0^{1,p}(\Omega) \mapsto W^{-1,p}(\Omega)$ by

$$T\xi = -\text{div}\left[a(x, \bar{y}) \nabla \xi + \frac{\partial a}{\partial y}(x, \bar{y}) \xi \nabla \bar{y} \right] + \frac{\partial f}{\partial y}(x, \bar{y}) \xi,$$
$$T_\rho \xi = -\text{div}\, [a(x, \bar{y}) \nabla \xi + a_\rho(x) \xi \nabla y_\rho] + f_\rho(x) \xi.$$

Since $y_\rho \to \bar{y}$ in $W_0^{1,p}(\Omega) \subset C(\bar{\Omega})$, we deduce from our assumptions on a and f that

$$a_\rho(x) \to \frac{\partial a}{\partial y}(x, \bar{y}(x)) \quad \text{and} \quad f_\rho(x) \to \frac{\partial f}{\partial y}(x, \bar{y}(x)) \text{ uniformly in } \bar{\Omega}, \tag{4.9}$$

and consequently

$$\|T_\rho - T\|_{\mathcal{L}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))} \leq C \left\{ \|y_\rho - \bar{y}\|_{W_0^{1,p}(\Omega)} \right.$$
$$\left. + \|a_\rho(x) - \frac{\partial a}{\partial y}(x, \bar{y}(x))\|_{C(\bar{\Omega})} + \|f_\rho(x) - \frac{\partial f}{\partial y}(x, \bar{y}(x))\|_{C(\bar{\Omega})} \right\} \to 0. \tag{4.10}$$

Since T is an isomorphism, by taking $\hat{\rho}$ small enough, we have that T_ρ is also an isomorphism and $T_\rho^{-1} \to T^{-1}$ in $\mathcal{L}(W^{-1,p}(\Omega), W_0^{1,p}(\Omega))$ too. Taking into account (4.7), we obtain

$$\|z - z_\rho\|_{W_0^{1,p}(\Omega)} = \left\| T^{-1}(u - \bar{u}) - T_\rho^{-1}\left[\frac{1}{\rho}\chi_{E_\rho}(u - \bar{u})\right]\right\|_{W_0^{1,p}(\Omega)}$$

$$\leq \left\| T_\rho^{-1}\left[\left(1 - \frac{1}{\rho}\chi_{E_\rho}\right)(u - \bar{u})\right]\right\|_{W_0^{1,p}(\Omega)} + \|(T^{-1} - T_\rho^{-1})(u - \bar{u})\|_{W_0^{1,p}(\Omega)}$$

$$\leq C\left\|\left(1 - \frac{1}{\rho}\chi_{E_\rho}\right)(u - \bar{u})\right\|_{W^{-1,p}(\Omega)}$$

$$+ \|T^{-1} - T_\rho^{-1}\|_{\mathcal{L}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))}\|u - \bar{u}\|_{W^{-1,p}(\Omega)} \to 0.$$

Now it is enough to notice that, by definition of z_ρ and the convergence $z_\rho \to z$ in $W_0^{1,p}(\Omega)$, we have

$$\varepsilon_\rho = \frac{y_\rho - \bar{y}}{\rho} - z \to 0,$$

and hence $y_\rho = \bar{y} + \rho z + \rho\varepsilon_\rho$. By putting $r_\rho = \rho\varepsilon_\rho$ we get (4.3). Finally, let us prove (4.4). Similarly to the definitions of a_ρ and f_ρ, we introduce

$$L_\rho(x) = \int_0^1 \frac{\partial L}{\partial y}(x, \bar{y}(x) + \theta(y_\rho(x) - \bar{y}(x)), u_\rho(x))\, d\theta.$$

Then we have

$$\frac{J(u_\rho) - J(\bar{u})}{\rho} = \int_\Omega \frac{L(x, y_\rho(x), u_\rho(x)) - L(x, \bar{y}(x), \bar{u}(x))}{\rho}\, dx$$

$$= \int_\Omega \frac{L(x, y_\rho(x), u_\rho(x)) - L(x, \bar{y}(x), u_\rho(x))}{\rho}\, dx$$

$$+ \int_\Omega \frac{L(x, \bar{y}(x), u_\rho(x)) - L(x, \bar{y}(x), \bar{u}(x))}{\rho}\, dx$$

$$= \int_\Omega L_\rho(x) z_\rho(x)\, dx + \int_\Omega \frac{1}{\rho}\chi_{E_\rho}(x)[L(x, \bar{y}(x), u(x)) - L(x, \bar{y}(x), \bar{u}(x))]dx$$

$$\to \int_\Omega \frac{\partial L}{\partial y}(x, \bar{y}(x), \bar{u}(x)) z(x)\, dx + \int_\Omega [L(x, \bar{y}(x), u(x)) - L(x, \bar{y}(x), \bar{u}(x))]dx = z^0,$$

which implies (4.4). ∎

Proof of Theorem 4.1. Since $\bar{u}$ is a local solution of (P), there exists $\varepsilon_{\bar{u}} > 0$ such that J achieves the minimum at $\bar{u}$ among all feasible controls of $\bar{B}_{L^\infty(\Omega)}(\bar{u}, \varepsilon_{\bar{u}})$. Let us take $u \in B_{L^\infty(\Omega)}(\bar{u}, \varepsilon_{\bar{u}})$ with $\alpha(x) \leq u(x) \leq \beta(x)$ a.e. $x \in \Omega$. Following Proposition 4.3, we consider the sets $\{E_\rho\}_{\rho > 0}\}$ such that (4.3) and (4.4) hold. Then $u_\rho \in B_{L^\infty(\Omega)}(\bar{u}, \varepsilon_{\bar{u}})$ and therefore (4.4) leads to

$$0 \leq \lim_{\rho \searrow 0} \frac{J(u_\rho) - J(\bar{u})}{\rho} = z^0.$$

By using (4.5) and the adjoint state given by (3.4), we get from the previous inequality after an integration by parts

$$0 \leq \int_\Omega \{\bar\varphi(x)(u(x)-\bar u(x)) + L(x,\bar y(x),u(x)) - L(x,\bar y(x),\bar u(x))\}\,dx$$

$$= \int_\Omega [H(x,\bar y(x),u(x),\bar\varphi(x)) - H(x,\bar y(x),\bar u(x),\bar\varphi(x))]dx. \tag{4.11}$$

Since u is an arbitrary feasible control in the ball $B_{L^\infty(\Omega)}(\bar u,\varepsilon_{\bar u})$, taking into account the definitions of $\alpha_{\varepsilon_{\bar u}}$ and $\beta_{\varepsilon_{\bar u}}$ given in the statement of Theorem 4.1, we deduce from (4.11)

$$\int_\Omega H(x,\bar y(x),\bar u(x),\bar\varphi(x))\,dx = \min_{\alpha_{\varepsilon_{\bar u}}\leq u\leq\beta_{\varepsilon_{\bar u}}} \int_\Omega [H(x,\bar y(x),u(x),\bar\varphi(x))\,dx. \tag{4.12}$$

To conclude the proof, we will show that (4.12) implies (4.1). Let the sequence $\{q_j\}_{j=1}^\infty$ exhaust the rational numbers contained in $[0,1]$. For every j we set $u_j = q_j\alpha_{\varepsilon_{\bar u}} + (1-q_j)\beta_{\varepsilon_{\bar u}}$. Then every function u_j belongs to $L^\infty(\Omega)$ and $\alpha_{\varepsilon_{\bar u}}(x) \leq u_j(x) \leq \beta_{\varepsilon_{\bar u}}(x)$ for every $x\in\Omega$. Now we introduce functions $F_0, F_j : \Omega \mapsto \mathbb{R}$ by

$$F_0(x) = H(x,\bar y(x),\bar u(x),\bar\varphi(x)) \ \text{ and } \ F_j(x) = H(x,\bar y(x),u_j(x),\bar\varphi(x)), \quad j=1,\ldots,\infty.$$

Associated to these integrable functions we introduce the set of Lebesgue regular points E_0 and $\{E_j\}_{j=1}^\infty$, which are known to satisfy $|E_j| = |\Omega|$ for $j=0,1,\ldots,\infty$, and

$$\lim_{r\searrow 0} \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} F_j(x)\,dx = F_j(x_0) \ \ \forall x_0\in E_j,\ j=0,1,\ldots,\infty, \tag{4.13}$$

where $B_r(x_0)$ is the Euclidean ball in $\mathbb{R}^n$ of center x_0 and radius r. Let us set $E = \cap_{j=0}^\infty E_j$. Then it is obvious that $|E| = |\Omega|$ and (4.13) holds for every $x_0\in E$. Given $x_0\in E$ and $r>0$ we define

$$u_{j,r}(x) = \begin{cases} \bar u(x) & \text{if } x\notin B_r(x_0),\\ u_j(x) & \text{if } x\in B_r(x_0),\ j=1,\ldots,\infty. \end{cases}$$

From (4.12) and the above definition we deduce

$$\int_\Omega H(x,\bar y(x),\bar u(x),\bar\varphi(x))\,dx \leq \int_\Omega H(x,\bar y(x),u_{j,r}(x),\bar\varphi(x))\,dx,$$

and therefore

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} H(x,\bar y(x),\bar u(x),\bar\varphi(x))\,dx,$$
$$\leq \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} H(x,\bar y(x),u_j(x),\bar\varphi(x))\,dx,$$

and passing to the limit when $r\to 0$ we get

$$H(x_0,\bar y(x_0),\bar u(x_0),\bar\varphi(x_0)) \leq H(x_0,\bar y(x_0),u_j(x_0),\bar\varphi(x_0)).$$

Since the function $s\to H(x_0,\bar y(x_0),s,\bar\varphi(x_0))$ is continuous and $\{u_j(x_0)\}_{j=1}^\infty$ is dense in $[\alpha_{\varepsilon_{\bar u}}(x_0),\beta_{\varepsilon_{\bar u}}(x_0)]$, we get

$$H(x_0,\bar y(x_0),\bar u(x_0),\bar\varphi(x_0)) \leq H(x_0,\bar y(x_0),s,\bar\varphi(x_0)) \ \ \forall s\in[\alpha_{\varepsilon_{\bar u}}(x_0),\beta_{\varepsilon_{\bar u}}(x_0)].$$

Finally, (4.1) follows from the previous inequality and the fact that x_0 is an arbitrary point of E. ∎

Remark 4.4. If we consider that $\bar{u}$ is a global solution or even a local solution of (P) in the sense of the $L^p(\Omega)$ topology, then (4.1) holds with $\varepsilon_{\bar{u}} = 0$. More precisely

$$H(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)) = \min_{s\in[\alpha(x),\beta(x)]} H(x, \bar{y}(x), s, \bar{\varphi}(x)) \quad \text{for a.e. } x \in \Omega.$$

The proof is the same. The only point we have to address is that the functions u_ρ defined in Proposition 4.3 corresponding to feasible controls u satisfy

$$\|u_p - \bar{u}\|_{L^p(\Omega)} = \left(\int_{E_\rho} |u(x) - \bar{u}(x)|^p \, dx\right)^{1/p} \leq \|u - \bar{u}\|_{L^\infty(\Omega)} |E_\rho|^{1/p}$$
$$\leq \|\beta - \alpha\|_{L^\infty(\Omega)} |\Omega|^{1/p} \rho^{1/p}.$$

Therefore for ρ small enough the functions u_ρ are in the corresponding ball of $L^p(\Omega)$ where $\bar{u}$ is the minimum.

5. Second-order optimality conditions. The goal of this section is to prove first necessary and next sufficient second-order optimality conditions. For it we will assume that (A1), (A2), (A4), and (A6) hold, the function $a : \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and Γ is of class C^1.

If $\bar{u}$ is a feasible control for problem (P) and there exists $\bar{\varphi} \in W_0^{1,p}(\Omega)$ satisfying (3.4) and (3.5), then we introduce the cone of critical directions

$$C_{\bar{u}} = \left\{ h \in L^2(\Omega) : h(x) = \left\{ \begin{array}{ll} \geq 0 & \text{if } \bar{u}(x) = \alpha(x) \\ \leq 0 & \text{if } \bar{u}(x) = \beta(x) \\ = 0 & \text{if } \bar{d}(x) \neq 0 \end{array} \right. \quad \text{for a.e. } x \in \Omega \right\}, \tag{5.1}$$

where $\bar{d}$ is defined by (3.7). In the previous section, we introduced the Hamiltonian H associated to the control problem. It is easy to check that

$$\frac{\partial H}{\partial u}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)) = \bar{d}(x).$$

In what follows, we will use the notation

$$\bar{H}_u(x) = \frac{\partial H}{\partial u}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)) \quad \text{and} \quad \bar{H}_{uu}(x) = \frac{\partial^2 H}{\partial u^2}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)).$$

Now we prove the necessary second-order optimality conditions.

THEOREM 5.1. *Let us assume that $\bar{u}$ is a local solution of* (P). *Then the following inequalities hold:*

$$\left\{ \begin{array}{l} J''(\bar{u})h^2 \geq 0 \quad \forall h \in C_{\bar{u}}, \\ \bar{H}_{uu}(x) \geq 0 \quad \textit{for a.a. } x \textit{ with } \bar{H}_u(x) = 0. \end{array} \right. \tag{5.2}$$

Proof. Let us take $h \in C_{\bar{u}}$ arbitrarily and $0 < \varepsilon < \varepsilon_{\bar{u}}$. Then we define

$$h_\varepsilon(x) = \left\{ \begin{array}{l} 0 \quad \text{if} \quad \alpha(x) < \bar{u}(x) < \alpha(x) + \varepsilon \text{ or } \beta(x) - \varepsilon < \bar{u}(x) < \beta(x), \\ \max\{-\frac{1}{\varepsilon}, \min\{+\frac{1}{\varepsilon}, h(x)\}\} \quad \text{otherwise.} \end{array} \right.$$

It is clear that $h_\varepsilon \in C_{\bar{u}} \cap L^\infty(\Omega)$ and $h_\varepsilon \to h$ in $L^2(\Omega)$. Moreover, we have

$$\alpha(x) \le \bar{u}(x) + th_\varepsilon(x) \le \beta(x) \quad \text{for a.e. } x \in \Omega \quad \text{and } 0 \le t < \varepsilon^2.$$

Therefore, if we define $g_\varepsilon : [0, \varepsilon^2] \longrightarrow \mathbb{R}$ by $g_\varepsilon(t) = J(\bar{u} + th_\varepsilon)$, we have

$$g_\varepsilon(0) = \min_{t \in [0,\varepsilon^2]} g_\varepsilon(t).$$

From our assumptions it is clear that g_ε is a C^2 function. From the fact $h_\varepsilon \in C_{\bar{u}}$ we deduce that

$$g'_\varepsilon(0) = J'(\bar{u})h_\varepsilon = \int_\Omega \bar{H}_u(x)h_\varepsilon(x)\,dx = 0.$$

Now, an elementary calculus and Theorem 3.2 yield

$$\begin{aligned} 0 \le g''_\varepsilon(0) = J''(\bar{u})h_\varepsilon^2 = \int_\Omega \Bigg\{ &\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u})z_{h_\varepsilon}^2 + 2\frac{\partial^2 L}{\partial y \partial u}(x,\bar{y},\bar{u})z_{h_\varepsilon}h_\varepsilon \\ &+\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})h_\varepsilon^2 - \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y})z_{h_\varepsilon}^2 \\ &-\nabla\bar{\varphi}\cdot\left[2\frac{\partial a}{\partial y}(x,\bar{y})z_{h_\varepsilon}\nabla z_{h_\varepsilon} + \frac{\partial^2 a}{\partial y^2}(x,\bar{y})z_{h_\varepsilon}^2\nabla\bar{y}\right]\Bigg\}\,dx, \end{aligned} \tag{5.3}$$

where $z_{h_\varepsilon} \in H_0^1(\Omega)$ is the solution of (2.16) corresponding to $v = h_\varepsilon$. Moreover, the convergence $h_\varepsilon \to h$ in $L^2(\Omega)$ implies that $z_{h_\varepsilon} \to z_h$ in $H_0^1(\Omega)$, where z_h is the solution of (2.16) for $v = h$; see Remark 2.8. Now we estimate the terms of (5.3). Arguing as in Remark 2.8, and taking into account the embedding $H_0^1(\Omega) \subset L^{\frac{2p}{p-2}}(\Omega)$ and assumption (A4), we get

$$\begin{aligned} \int_\Omega \left|\nabla\bar{\varphi}(x)\cdot\frac{\partial a}{\partial y}(x,\bar{y})z_{h_\varepsilon}(x)\nabla z_{h_\varepsilon}(x)\right| dx &\le D_M\|\nabla\bar{\varphi}\|_{L^p(\Omega)}\|z_{h_\varepsilon}\|_{L^{\frac{2p}{p-2}}(\Omega)}\|\nabla z_{h_\varepsilon}\|_{L^2(\Omega)} \\ &\le CD_M\|\bar{\varphi}\|_{W_0^{1,p}(\Omega)}\|z_{h_\varepsilon}(x)\|_{H_0^1(\Omega)}^2. \end{aligned}$$

Analogously we have

$$\begin{aligned} \int_\Omega \left|\nabla\bar{\varphi}(x)\cdot\frac{\partial^2 a}{\partial y^2}(x,\bar{y})z_{h_\varepsilon}^2(x)\nabla\bar{y}(x)\right| dx &\le D_M\|\nabla\bar{\varphi}\|_{L^p(\Omega)}\|z_{h_\varepsilon}\|_{L^{\frac{2p}{p-2}}(\Omega)}^2\|\nabla\bar{y}\|_{L^p(\Omega)} \\ &\le CD_M\|\bar{\varphi}\|_{W_0^{1,p}(\Omega)}\|z_{h_\varepsilon}(x)\|_{H_0^1(\Omega)}^2\|\bar{y}\|_{W_0^{1,p}(\Omega)}. \end{aligned}$$

The rest of the terms in the integral (5.3) are easy to estimate with the help of assumptions (A4) and (A6). Therefore, we can pass to the limit in (5.3) and deduce

$$0 \le \lim_{\varepsilon\to 0} J''(\bar{u})h_\varepsilon^2 = J''(\bar{u})h^2.$$

This proves the first inequality of (5.2). Finally, the second inequality is an obvious consequence of (4.1). Indeed, it is a standard conclusion of (4.1) that

$$\bar{H}_u(x) = \begin{cases} \ge 0 & \text{if } \bar{u}(x) = \alpha(x), \\ \le 0 & \text{if } \bar{u}(x) = \beta(x), \\ = 0 & \text{if } \alpha(x) < \bar{u}(x) < \beta(x) \end{cases} \qquad \text{for a.e. } x \in \Omega$$

and

$$\bar{H}_{uu}(x) \geq 0 \text{ if } \bar{H}_u(x) = 0 \text{ for a.e. } x \in \Omega. \qquad \square$$

Let us consider the Lagrangian function associated to the control problem (P),

$$\mathcal{L} : L^\infty(\Omega) \times W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \longrightarrow \mathbb{R},$$

given by the expression

$$\begin{aligned}\mathcal{L}(u, y, \varphi) &= \mathcal{J}(y, u) + \int_\Omega \{\varphi[u - f(x, y)] - a(x, y)\nabla\varphi \cdot \nabla y\}\, dx \\ &= \int_\Omega \{H(x, y(x), u(x), \varphi(x)) - a(x, y(x))\nabla\varphi(x) \cdot \nabla y(x)\}\, dx,\end{aligned}$$

where we denote

$$\mathcal{J}(y, u) = \int_\Omega L(x, y(x), u(x))\, dx.$$

Defining $\bar{H}_y$, $\bar{H}_{yy}$, and $\bar{H}_{yu}$ similarly to $\bar{H}_u$ and $\bar{H}_{uu}$, after obvious modifications, we can write the first- and second-order derivatives of $\mathcal{L}$ with respect to (y, u) as follows:

$$\begin{aligned}D_{(y,u)}\mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi})(z, h) &= \int_\Omega \left\{\bar{H}_y(x)z(x) + \bar{H}_u(x)h(x)\right\} dx \\ &- \int_\Omega \nabla\bar{\varphi}(x) \cdot \left\{a(x, \bar{y}(x))\nabla z(x) + \frac{\partial a}{\partial y}(x, \bar{y}(x))z(x)\nabla\bar{y}(x)\right\} dx.\end{aligned}$$

If we assume that z is the solution of (2.16) associated to $v = h$, then by using the adjoint state (3.4) we get

$$D_{(y,u)}\mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi})(z, h) = \int_\Omega \bar{H}_u(x)h(x)\, dx. \tag{5.4}$$

Moreover, we find

$$\begin{aligned}D^2_{(y,u)}\mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi})(z, h)^2 &= \int_\Omega \left\{\bar{H}_{yy}(x)z^2(x) + 2\bar{H}_{yu}(x)z(x)h(x) + \bar{H}_{uu}(x)h^2(x)\right\} dx \\ &- \int_\Omega \nabla\bar{\varphi}(x) \cdot \left\{\frac{\partial^2 a}{\partial y^2}(x, \bar{y}(x))z^2(x)\nabla\bar{y}(x) + 2\frac{\partial a}{\partial y}(x, \bar{y}(x))z(x)\nabla z(x)\right\} dx.\end{aligned}$$

Once again if we take z as the solution of (2.16) associated to $v = h$, we deduce from (3.2) that

$$J''(\bar{u})h^2 = D^2_{(y,u)}\mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi})(z, h)^2. \tag{5.5}$$

Therefore the necessary optimality conditions (5.2) can be written as follows:

$$\begin{cases} D^2_{(y,u)}\mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi})(z, h)^2 \geq 0 \ \ \forall (z, h) \in H_0^1(\Omega) \times C_{\bar{u}} \text{ satisfying (2.16)}, \\ \bar{H}_{uu}(x) \geq 0 \text{ if } \bar{H}_u(x) = 0 \quad \text{for a.e. } x \in \Omega. \end{cases} \tag{5.6}$$

We finish this section by establishing the sufficient second-order optimality conditions.

THEOREM 5.2. *Let us assume that $\bar{u}$ is a feasible control for the problem* (P) *and that there exists $\bar{\varphi} \in W_0^{1,p}(\Omega)$ satisfying* (3.4) *and* (3.5)*. If, in addition, there exist $\mu > 0$ and $\tau > 0$ such that*

$$(5.7) \qquad \begin{aligned} &J''(\bar{u})h^2 > 0 \quad \forall h \in C_{\bar{u}} \setminus \{(0,0)\}, \\ &\bar{H}_{uu}(x) \geq \mu \quad \text{if} \quad |\bar{H}_u(x)| \leq \tau \quad \text{for a.e. } x \in \Omega, \end{aligned}$$

then there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$(5.8) \qquad J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \leq J(u)$$

for every feasible control $u \in L^\infty(\Omega)$ for (P) *such that $\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \varepsilon$.*

Remark 5.3. 1. If we compare the first inequality of (5.7) with the analogous inequality of (5.2), we see that the gap is minimal between the necessary and sufficient conditions, as is usual in finite dimensions. However, the second inequality of (5.7) is stronger than the corresponding one of (5.2). This is a consequence of the infinite number of constraints on the control: one constraint for every point of Ω. In general we cannot take $\tau = 0$. The reader is referred to Dunn [17] for a simple example proving the impossibility of taking $\tau = 0$.

2. Let us recall that $\bar{H}_{uu}(x) = (\partial^2 L/\partial u^2)(x, \bar{y}(x), \bar{u}(x))$. Therefore, the second condition of (5.7) is satisfied if we assume that the second derivative of L with respect to u is strictly positive. A standard example is given by the function

$$L(x, y, u) = L_0(x, y) + \frac{N}{2}u^2, \quad \text{with } N > 0.$$

3. The sufficient optimality conditions (5.7) can be written as follows:

$$\begin{aligned} &D^2_{(y,u)}\mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi})(z, h)^2 > 0 \quad \forall (z, h) \in (H_0^1(\Omega) \times C_{\bar{u}}) \setminus \{(0,0)\} \text{ verifying (2.16)}, \\ &\bar{H}_{uu}(x) \geq \mu \quad \text{if} \quad |\bar{H}_u(x)| \leq \tau \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

Once again this is an obvious consequence of (5.5).

Proof.

Step 1*: Preparations.* We will argue by contradiction. Let us assume that there exists a sequence of feasible controls for (P), $\{u_k\}_{k=1}^\infty \subset L^\infty(\Omega)$, such that

$$(5.9) \qquad \|u_k - \bar{u}\|_{L^\infty(\Omega)} < \frac{1}{k} \quad \text{and} \quad J(\bar{u}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Omega)} > J(u_k).$$

Let us define

$$(5.10) \quad y_k = G(u_k) = y_{u_k},\ \bar{y} = G(\bar{u}) = y_{\bar{u}},\ \rho_k = \|u_k - \bar{u}\|_{L^2(\Omega)} \ \text{ and } \ v_k = \frac{1}{\rho_k}(u_k - \bar{u}).$$

Then

$$(5.11) \qquad \lim_{k\to\infty} \|y_k - \bar{y}\|_{W_0^{1,p}(\Omega)} = 0, \quad \lim_{k\to\infty} \rho_k = 0 \quad \text{and} \quad \|v_k\|_{L^2(\Omega)} = 1 \ \ \forall k.$$

By taking a subsequence, if necessary, we can assume that $v_k \rightharpoonup v$ weakly in $L^2(\Omega)$. We will prove that $v \in C_{\bar{u}}$. Next, we will use (5.7). In this process we will need the following result:

$$(5.12) \qquad \lim_{k\to\infty} \frac{1}{\rho_k}(y_k - \bar{y}) = z \quad \text{in} \quad H_0^1(\Omega),$$

where $z \in H_0^1(\Omega)$ is the solution of (2.16) corresponding to the state $\bar{y}$. Let us prove it. We will set $z_k = (y_k - \bar{y})/\rho_k$. By subtracting the state equations satisfied by (y_k, u_k) and $(\bar{y}, \bar{u})$, dividing by ρ_k, and applying the mean value theorem, we get

$$(5.13)\quad -\text{div}\left[a(x,y_k)\nabla z_k + \frac{\partial a}{\partial y}(x,\bar{y}+\theta_k(y_k-\bar{y}))z_k\nabla\bar{y}\right] + \frac{\partial f}{\partial y}(x,\bar{y}+\nu_k(y_k-\bar{y}))z_k = v_k.$$

Taking into account that $z_k \in W_0^{1,p}(\Omega)$, we can multiply (5.13) by z_k and make an integration by parts to get, with the aid of (2.1) and (5.11), that

$$\alpha_0\int_\Omega |\nabla z_k(x)|^2\,dx \le \int_\Omega a(x,y_k)|\nabla z_k(x)|^2\,dx$$
$$= \int_\Omega\left\{v_k z_k - \frac{\partial f}{\partial y}(x,\bar{y}+\nu_k(y_k-\bar{y}))z_k^2 - \frac{\partial a}{\partial y}(x,\bar{y}+\theta_k(y_k-\bar{y}))z_k\nabla z_k\cdot\nabla\bar{y}\right\}dx$$
$$\le \|v_k\|_{L^2(\Omega)}\|z_k\|_{L^2(\Omega)} + C\|z_k\|_{L^{\frac{2p}{p-2}}(\Omega)}\|\nabla\bar{y}\|_{L^2(\Omega)}\|\nabla z_k\|_{L^2(\Omega)}.$$

We have used that the term $-\partial f/\partial y\, z_k^2$ is nonpositive. Therefore,

$$\|\nabla z_k\|_{L^2(\Omega)} \le C\left\{1 + \|z_k\|_{L^{\frac{2p}{p-2}}(\Omega)}\right\}.$$

As in the proof of Theorem 2.7, $\{z_k\}_{k=1}^\infty$ must be bounded in $L^{\frac{2p}{p-2}}(\Omega)$; otherwise we could obtain a nonzero solution of (2.16). Then the above inequality leads to the boundedness of $\{z_k\}_{k=1}^\infty$ in $H_0^1(\Omega)$. Therefore we can extract a subsequence, denoted in the same way, such that $z_k \rightharpoonup z$ weakly in $H_0^1(\Omega)$ and strongly in $L^{\frac{2p}{p-2}}(\Omega)$. Thanks to this convergence and to (5.10), we get the strong convergences in $L^2(\Omega)$:

$$\frac{\partial a}{\partial y}(x,\bar{y}+\theta_k(y_k-\bar{y}))z_k\nabla\bar{y} \to \frac{\partial a}{\partial y}(x,\bar{y})z\nabla\bar{y} \text{ and } \frac{\partial f}{\partial y}(x,\bar{y}+\nu_k(y_k-\bar{y}))z_k \to \frac{\partial f}{\partial y}(x,\bar{y})z.$$

Therefore we can pass to the limit in (5.13) and deduce

$$(5.14)\qquad -\text{div}\left[a(x,\bar{y})\nabla z + \frac{\partial a}{\partial y}(x,\bar{y})z\nabla\bar{y}\right] + \frac{\partial f}{\partial y}(x,\bar{y})z = v.$$

Moreover by using (5.13), (5.14), and the uniform convergence $y_k \to \bar{y}$ it is easy to prove that

$$\int_\Omega a(x,\bar{y})|\nabla z_k|^2\,dx \to \int_\Omega a(x,\bar{y})|\nabla z|^2\,dx.$$

This fact, along with the weak convergence of $\{z_k\}_{k=1}^\infty$ in $H_0^1(\Omega)$, implies the strong convergence $z_k \to z$ in $H_0^1(\Omega)$.

Step 2: $v \in C_{\bar{u}}$. Since $\alpha(x) \le u_k(x) \le \beta(x)$ a.e., we have that $v_k(x) \ge 0$ if $\bar{u}(x) = \alpha(x)$ and $v_k(x) \le 0$ if $\bar{u}(x) = \beta(x)$ a.e. Since the set of functions satisfying these sign conditions is convex and closed in $L^2(\Omega)$, then it is weakly closed, and therefore the weak limit v of $\{v_k\}_{k=1}^\infty$ satisfies the sign condition too. It remains to prove that $v(x) = 0$ for a.a. x such that $\bar{d}(x) \ne 0$. From (5.9), by using the mean value theorem we obtain

$$\frac{\rho_k}{k} = \frac{1}{k\rho_k}\|u_k - \bar u\|^2_{L^2(\Omega)} > \frac{J(u_k) - J(\bar u)}{\rho_k}$$
$$= \int_\Omega \frac{\partial L}{\partial y}(x, \bar y + \theta_k(y_k - \bar y), \bar u + \theta_k(u_k - \bar u)) z_k\, dx$$
$$+ \int_\Omega \frac{\partial L}{\partial u}(x, \bar y + \theta_k(y_k - \bar y), \bar u + \theta_k(u_k - \bar u)) v_k\, dx.$$

Taking limits in both sides of the inequality, using (3.4), (5.14), the already proved convergence $z_k \to z$ in $H_0^1(\Omega)$, and integrating by parts, we get

$$0 \ge \int_\Omega \left\{\frac{\partial L}{\partial y}(x, \bar y, \bar u) z + \frac{\partial L}{\partial u}(x, \bar y, \bar u) v\right\} dx$$
$$= \int_\Omega \left\{\bar\varphi + \frac{\partial L}{\partial u}(x, \bar y, \bar u)\right\} v\, dx = \int_\Omega \bar d(x) v(x)\, dx = \int_\Omega |\bar d(x)||v(x)|\, dx,$$

the last equality being a consequence of proved signs for v and (3.6). The previous inequality implies that $|\bar d(x) v(x)| = 0$ holds a.e., and hence $v(x) = 0$ if $\bar d(x) \neq 0$, as we wanted to prove.

Step 3*:* $v = 0$. The next step consists of proving that v does not satisfy the first condition of (5.7). This will lead to the identity $v = 0$. By using (5.9), the definition of $\mathcal{L}$, and the fact that $(\bar y, \bar u)$ and (y_k, u_k) satisfy the state equation, we get

$$\mathcal{L}(u_k, y_k, \bar\varphi) = \mathcal{J}(y_k, u_k) < \mathcal{J}(\bar y, \bar u) + \frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Omega)}$$
$$= \mathcal{L}(\bar u, \bar y, \bar\varphi) + \frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Omega)}. \tag{5.15}$$

Performing a Taylor expansion up to the second order, we obtain

$$\mathcal{L}(u_k, y_k, \bar\varphi) = \mathcal{L}(\bar u + \rho_k v_k, \bar y + \rho_k z_k, \bar\varphi) = \mathcal{L}(\bar u, \bar y, \bar\varphi) + \rho_k D_{(y,u)}\mathcal{L}(\bar u, \bar y, \bar\varphi)(z_k, v_k)$$
$$+ \frac{\rho_k^2}{2} D^2_{(y,u)}\mathcal{L}(\bar u + \theta_k\rho_k v_k, \bar y + \theta_k\rho_k z_k, \bar\varphi)(z_k, v_k)^2.$$

This equality, along with (5.15) and (5.9), leads to

$$\rho_k D_{(y,u)}\mathcal{L}(\bar u, \bar y, \bar\varphi)(z_k, v_k) + \frac{\rho_k^2}{2} D^2_{(y,u)}\mathcal{L}(w_k, \xi_k, \bar\varphi)(z_k, v_k)^2 < \frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Omega)} \le \frac{\rho_k^2}{k},$$

where we have put $\xi_k = \bar y + \theta_k\rho_k z_k$ and $w_k = \bar u + \theta_k\rho_k v_k$. It is obvious that $\xi_k \to \bar y$ in $W_0^{1,p}(\Omega)$ and $w_k \to \bar u$ in $L^\infty(\Omega)$. Dividing the previous inequality by ρ_k^2 and taking into account the expressions obtained for the derivatives of $\mathcal{L}$, we obtain

$$\frac{1}{\rho_k}\int_\Omega \bar H_u(x) v_k(x)\, dx + \frac{1}{2}\int_\Omega \left\{H^k_{yy}(x) z_k^2(x) + 2H^k_{yu}(x) z_k(x) v_k(x) + H^k_{uu}(x) v_k^2(x)\right\} dx$$
$$-\frac{1}{2}\int_\Omega \left\{\frac{\partial a}{\partial y}(x, \xi_k) z_k \nabla z_k + \frac{\partial^2 a}{\partial y^2}(x, \xi_k) z_k^2 \nabla\xi_k\right\} \nabla\bar\varphi\, dx < \frac{1}{k}, \tag{5.16}$$

where

$$H^k_{yy}(x) = H_{yy}(x, \xi_k(x), w_k(x), \bar\varphi(x)),$$

with analogous definitions for H^k_{uu} and H^k_{yu}. It is easy to check that

$$\begin{cases} (H^k_{yy}(x), H^k_{yu}(x), H^k_{uu}(x)) \to (\bar{H}_{yy}(x), \bar{H}_{yu}(x), \bar{H}_{uu}(x)) \\ |H^k_{yy}(x)| + |H^k_{yu}(x)| + |H^k_{uu}(x)| \le C \end{cases} \quad \text{for a.e. } x \in \Omega$$

for some constant $C < \infty$. We also have the following convergence properties:

$$\begin{cases} \dfrac{\partial^j a}{\partial y^j}(x, \xi_k) z_k \nabla\bar{\varphi} \to \dfrac{\partial^j a}{\partial y^j}(x, \bar{y}) z \nabla\bar{\varphi},\ j = 1, 2, \\ \nabla z_k \longrightarrow \nabla z \ \text{ and } \ z_k \nabla \xi_k \longrightarrow z \nabla \bar{y}. \end{cases} \quad \text{in } L^2(\Omega)^n.$$

Using these properties we can pass to the limit in (5.16) as follows:

$$\limsup_{k\to\infty} \left\{ \frac{1}{\rho_k} \int_\Omega \bar{H}_u(x) v_k(x)\, dx + \frac{1}{2} \int_\Omega H^k_{uu}(x) v_k^2(x)\, dx \right\} + \frac{1}{2} \int_\Omega [\bar{H}_{yy}(x) z^2(x) + 2\bar{H}_{yu}(x) z(x) v(x)]\, dx - \frac{1}{2} \int_\Omega \left\{ \frac{\partial a}{\partial y}(x, \bar{y}) z \nabla z + \frac{\partial^2 a}{\partial y^2}(x, \bar{y}) z^2 \nabla \bar{y} \right\} \nabla \bar{\varphi}\, dx \le 0. \tag{5.17}$$

The rest of the proof is devoted to verifying that the above upper limit is bounded from below by $\frac{1}{2}\int_\Omega \bar{H}_{uu} v_k^2\, dx$. If this is proved, then from (5.17) and (5.5) we deduce that $J''(\bar{u})v^2 = D^2_{(y,u)}\mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi})(z, v)^2 \le 0$. According to (5.7) this is possible only if $v = 0$. The proof of the mentioned lower estimate is quite technical, which makes an important difference with respect to the finite dimension. In our framework the difficulty is due to the fact that we only have a weak convergence $v_k \rightharpoonup v$. To overcome this difficulty we use a convexity argument. In order to achieve this goal the essential tool is the second condition of (5.7).

From (A4) and (A6) we get

$$\|\bar{H}_{uu} - H^k_{uu}\|_{L^\infty(\Omega)} \le C \left\{ \|\bar{y} - y_k\|_{L^\infty(\Omega)} + \|\bar{u} - u_k\|_{L^\infty(\Omega)} \right\} \to 0.$$

Using this property, $\|v_k\|_{L^2(\Omega)} = 1$, and the identity $\bar{H}_u(x) v_k(x) = |\bar{H}_u(x)||v_k(x)|$, we obtain

$$\begin{aligned} &\limsup_{k\to\infty} \left\{ \frac{1}{\rho_k} \int_\Omega \bar{H}_u(x) v_k(x)\, dx + \frac{1}{2} \int_\Omega H^k_{uu}(x) v_k^2(x)\, dx \right\} \\ &= \limsup_{k\to\infty} \left\{ \frac{1}{\rho_k} \int_\Omega |\bar{H}_u(x)||v_k(x)|\, dx + \frac{1}{2} \int_\Omega \bar{H}_{uu}(x) v_k^2(x)\, dx \right\} \\ &\ge \limsup_{k\to\infty} \left\{ \frac{1}{\rho_k} \int_{\{|\bar{H}_u(x)| > \tau\}} \left[|\bar{H}_u(x)||v_k(x)| + \frac{1}{2} \bar{H}_{uu}(x) v_k^2(x) \right] dx \right. \\ &\qquad \left. + \frac{1}{2} \int_{\{|\bar{H}_u(x)| \le \tau\}} \bar{H}_{uu}(x) v_k^2(x)\, dx \right\}, \end{aligned} \tag{5.18}$$

where τ is given by (5.7).

Remembering that $\rho_k\|v_k\|_{L^\infty(\Omega)} = \|u_k - \bar{u}\|_{L^\infty(\Omega)} < 1/k$, we deduce the existence of an integer $k_0 > 0$ such that

$$\frac{\|\bar{H}_{uu}\|_{L^\infty(\Omega)}\rho_k\|v_k\|_{L^\infty(\Omega)}}{\tau} < \frac{\|\bar{H}_{uu}\|_{L^\infty(\Omega)}}{k\tau} < 1 \quad \forall k \geq k_0,$$

and therefore

$$\frac{\tau}{\rho_k}|v_k(x)| \geq \|\bar{H}_{uu}\|_{L^\infty(\Omega)} v_k^2(x) \quad \text{for a.e. } x \in \Omega \quad \forall k \geq k_0.$$

Then we have, with the help of the second condition of (5.7),

$$\begin{aligned}
&\limsup_{k\to\infty}\left\{\frac{1}{\rho_k}\int_{\{|\bar{H}_u|>\tau\}}\left[|\bar{H}_u||v_k| + \frac{1}{2}\bar{H}_{uu}v_k^2\right]dx + \frac{1}{2}\int_{\{|\bar{H}_u|\leq\tau\}}\bar{H}_{uu}v_k^2\,dx\right\}\\
&\geq \limsup_{k\to\infty}\left\{\int_{\{|\bar{H}_u|>\tau\}}\left[\|\bar{H}_{uu}\|_{L^\infty(\Omega)} + \frac{1}{2}\bar{H}_{uu}\right]v_k^2\,dx + \frac{1}{2}\int_{\{|\bar{H}_u|\leq\tau\}}\bar{H}_{uu}v_k^2\,dx\right\}\\
&\qquad\geq \int_{\{|\bar{H}_u|>\tau\}}\left[\|\bar{H}_{uu}\|_{L^\infty(\Omega)} + \frac{1}{2}\bar{H}_{uu}\right]v^2\,dx\\
&\qquad+\frac{1}{2}\int_{\{|\bar{H}_u|\leq\tau\}}\bar{H}_{uu}v^2\,dx \geq \frac{1}{2}\int_\Omega \bar{H}_{uu}v^2\,dx. \qquad (5.19)
\end{aligned}$$

Combining (5.18) and (5.19) we get the sought-after lower estimate.

Step 4: Final contradiction. Using that $\|v_k\|_{L^2(\Omega)} = 1$ along with (5.16), (5.17), (5.18), (5.19), the second condition of (5.7), and the fact that $v = 0$, we deduce

$$\begin{aligned}
0 &\geq \limsup_{k\to\infty}\left\{\int_{\{|\bar{H}_u|>\tau\}}\left[\|\bar{H}_{uu}\|_{L^\infty(\Omega)} + \frac{1}{2}\bar{H}_{uu}\right]v_k^2\,dx + \frac{1}{2}\int_{\{|\bar{H}_u|\leq\tau\}}\bar{H}_{uu}v_k^2\,dx\right\}\\
&\geq \limsup_{k\to\infty}\left\{\frac{\|\bar{H}_{uu}\|_{L^\infty(\Omega)}}{2}\int_{\{|\bar{H}_u|>\tau\}}v_k^2\,dx + \frac{\mu}{2}\int_{\{|\bar{H}_u|\leq\tau\}}v_k^2\,dx\right\}\\
&\geq \frac{\min\{\|\bar{H}_{uu}\|_{L^\infty(\Omega)},\mu\}}{2}\limsup_{k\to\infty}\int_\Omega v_k^2\,dx = \frac{\min\{\|\bar{H}_{uu}\|_{L^\infty(\Omega)},\mu\}}{2} > 0,
\end{aligned}$$

providing the contradiction that we were looking for. ◻

We finish this section by formulating a different version of the sufficient second-order optimality conditions which is equivalent to (5.7); see [9, Theorem 4.4] for the proof of this equivalence. This formulation is very useful for numerical purposes.

THEOREM 5.4. *Let us assume that $\bar{u}$ is a feasible control for problem* (P). *We also assume that there exists $\bar{\varphi} \in W_0^{1,p}(\Omega)$ satisfying* (3.4) *and* (3.5). *Then* (5.7) *holds if and only if there exist $\delta, \sigma > 0$ such that*

$$J''(\bar{u})h^2 \geq \delta\|h\|^2_{L^2(\Omega)} \quad \forall h \in C^\sigma_{\bar{u}}, \qquad (5.20)$$

where

$$C^\sigma_{\bar{u}} = \left\{h \in L^2(\Omega) : h(x) = \left\{\begin{array}{lll} \geq 0 & \text{if } \bar{u}(x) = \alpha(x) & \\ \leq 0 & \text{if } \bar{u}(x) = \beta(x) & \text{for a.e. } x \in \Omega \\ = 0 & \text{if } |\bar{d}(x)| > \sigma & \end{array}\right.\right\}.$$

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