A PARADOX IN THE APPROXIMATION OF DIRICHLET CONTROL PROBLEMS IN CURVED DOMAINS*

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Abstract. In this paper, we study the approximation of a Dirichlet control problem governed by an elliptic equation defined on a curved domain Ω. To solve this problem numerically, it is usually necessary to approximate Ω by a (typically polygonal) new domain Ω_h. The difference between the solutions of both infinite-dimensional control problems, one formulated in Ω and the second in Ω_h, was studied in [E. Casas and J. Sokolowski, *SIAM J. Control Optim.*, 48 (2010), pp. 3746–3780], where an error of order $O(h)$ was proved. In [K. Deckelnick, A. Günther, and M. Hinze, *SIAM J. Control Optim.*, 48 (2009), pp. 2798–2819], the numerical approximation of the problem defined in Ω was considered. The authors used a finite element method such that Ω_h was the polygon formed by the union of all triangles of the mesh of parameter h. They proved an error of order $O(h^{3/2})$ for the difference between continuous and discrete optimal controls. Here we show that the estimate obtained in [E. Casas and J. Sokolowski, *SIAM J. Control Optim.*, 48 (2010), pp. 3746–3780] cannot be improved, which leads to the paradox that the numerical solution is a better approximation of the optimal control than the exact one obtained just by changing the domain from Ω to Ω_h.

Key words. Dirichlet control, error estimates, semilinear elliptic equations, finite elements

AMS subject classifications. 49J20, 35J65

DOI. 10.1137/100794882

1. Introduction. In this paper we are concerned with the approximation of the control problem

$$
\text{(P)} \quad
\begin{cases}
\min J(u) = \displaystyle\int_\Omega L(x, y_u(x))\, dx + \frac{N}{2} \int_\Gamma u^2(x)\, d\sigma(x) \\
\text{subject to } (y_u, u) \in (L^\infty(\Omega) \cap H^{1/2}(\Omega)) \times L^\infty(\Gamma), \\
\alpha \le u(x) \le \beta \quad \text{for a.e. } x \in \Gamma,
\end{cases}
$$

where the state y_u associated to the control u is the solution of the Dirichlet problem

$$
(1.1) \quad
\begin{cases}
-\Delta y + a(x, y) = 0 & \text{in } \Omega, \\
 y = u & \text{on } \Gamma.
\end{cases}
$$

Ω is an open, convex, and bounded subset of $\mathbb{R}^2$ with a C^2 boundary Γ. The boundedness of the control is required to deal with the nonlinearity of the state equation

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and the integrand $L(x, y_u)$, but it is not necessary for linear equations and functionals L with a quadratic polynomial growth in y.

To solve this problem it is necessary to approximate Ω by a simpler domain (typically polygonal) Ω_h with a boundary Γ_h. In a recent paper Casas and Sokolowski [4] studied the influence of the replacement of Ω by Ω_h on the solutions of the control problems. To this aim, a polygonal approximation Ω_h of Ω was considered, h being the maximum length of the sides of the polygon. Then a one-to-one mapping $g_h : \Gamma_h \longrightarrow \Gamma$ was defined and a control problem (P_h) was formulated in Ω_h in a similar way to (P). The convergence of the these approximations was proved in the following sense:

1. For any sequence $\{\bar{u}_h\}_{h>0}$ of solutions of control problems (P_h), the sequence $\{\bar{u}_h \circ g_h^{-1}\}_{h>0} \subset L^2(\Gamma)$ is bounded in $H^{1/2}(\Gamma)$ and any weak limit $\bar{u}$ is a solution of (P).

2. For any strict local minimum of (P), $\bar{u}$, there exists a sequence $\{\bar{u}_h\}_{h>0}$ such that $\bar{u}_h$ is a local solution of (P_h) and $\{\bar{u}_h \circ g_h^{-1}\}_{h>0}$ converges weakly in $H^{1/2}(\Gamma)$ to $\bar{u}$.

Finally, the error $\|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)} \leq Ch$ was proved for local solutions of (P) and (P_h) such that $\bar{u}_h \circ g_h^{-1} \rightharpoonup \bar{u}$ weakly in $H^{1/2}(\Gamma)$.

On the other hand, Deckelnick, Günther, and Hinze [5] studied the problem (P) without the nonlinear term in the state equation, $a(x, y) = -f(x)$, with $f \in L^2(\Omega)$, and taking $L(x, y) = \frac{1}{2}(y - y_d(x))^2$ in the cost functional. Their goal was different: they discretized the control problem by using finite elements associated to a triangulation of the polygonal domain Ω_h. In this case the control problems (P) and (P_h) have a unique solution $\bar{u}$ and $\bar{u}_h$, respectively. Under a nonrestrictive assumption in practice on the triangulation of Ω_h, they proved the error estimate $\|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)} \leq Ch^{3/2}$.

If we compare the results of [4] and [5], the difference is surprising. In [4] the problem (P_h) is the same as the problem (P) except for the change of domain Ω by Ω_h, but there is no discretization of the control problem. In [5] the control problem (P_h) is a discrete problem where Ω has been replaced by Ω_h and the partial differential equation has been discretized so that the states y_u are approximated by piecewise linear functions $y_h(u)$ solving the discrete variational equation. However, in the second case we get a better approximation to $\bar{u}$ than in the first case. The reader could conclude that the error estimates of [4] are not sharp and should be improved. In this paper we provide an example showing that the error estimates of [4] cannot be improved. Nevertheless, as predicted by the theory, the numerical computation on this example confirms the order $h^{3/2}$ for the difference among $\bar{u}$ and the solutions of the discrete problems. The goal of this paper is to show this paradox that reminds us of Babuska's paradox. Indeed, Babuska's paradox concerns the approximation of a simply supported circular plate, uniformly loaded, by a sequence of regular polygonal plates inscribed in the circle, also simply supported and uniformly loaded. It happens that the solutions for the polygons do not converge to that of the circle; see [1]. In the case we are considering in this paper the convergence holds, but it is not so good as the numerical approximation, which is also rather paradoxical.

The plan of the paper is as follows. In the next section we formulate an example of a control problem falling into the framework defined above, and we prove that the estimates obtained in [4] are optimal for this problem. In section 3 we describe the finite element approximation of the example and show the computational results, which confirm the theoretical estimates proved in [5]. Finally, in section 4 we explain why the numerical approach provides a better approximation of $\bar{u}$ than the exact solution of (P_h).

2. The example. In what follows Ω will denote the unit ball of $\mathbb{R}^2$ centered at 0; then Γ is the unit circumference. In this domain we consider the control problem

(P) $$\begin{cases} \min J(u) = \displaystyle\int_\Omega 4y_u(x)\,dx + \frac{1}{2}\int_\Gamma u^2(x)\,d\sigma(x) \\ \text{subject to } (y_u, u) \in (L^2(\Omega) \cap H^{1/2}(\Omega)) \times L^2(\Gamma), \end{cases}$$

where the state y_u associated to the control u is the solution of the Dirichlet problem

(2.1) $$\begin{cases} -\Delta y = 0 & \text{in } \Omega, \\ y = u & \text{on } \Gamma. \end{cases}$$

It is obvious that (P) is strictly convex and has a unique solution $\bar{u}$ that can be characterized by the optimality system

(2.2) $$\begin{cases} -\Delta \bar{y} = 0 & \text{in } \Omega, \\ \bar{y} = \bar{u} & \text{on } \Gamma, \end{cases}$$

(2.3) $$\begin{cases} -\Delta \bar{\varphi} = 4 & \text{in } \Omega, \\ \bar{\varphi} = 0 & \text{on } \Gamma, \end{cases}$$

(2.4) $$\bar{u} = \partial_\nu \bar{\varphi},$$

where $\nu(x)$ denotes the unit outward normal vector to Γ at the point x. For the selected domain we have that $\nu(x) = x$. It is very easy to check that the solution of the above system is given by

(2.5) $$\bar{\varphi}(x) = 1 - |x|^2, \quad \bar{u}(x) = \partial_\nu \bar{\varphi}(x) = -2, \quad \text{and} \quad \bar{y}(x) = -2.$$

Now we define the polygonal domain Ω_h. For every positive integer n we consider the points of Γ,

$$x_j = \left(\cos \frac{2\pi j}{n}, \sin \frac{2\pi j}{n} \right), \quad j = 1, 2, \ldots, n.$$

For convenience, we set $x_{n+1} = x_1$. It is easy to check that the distance between two consecutive points is $h = |x_{j+1} - x_j| = 2\sin\frac{\pi}{n}$. We take Γ_h as the polygonal line joining the knots $\{x_j\}_{j=1}^n$, and Ω_h is the open domain enclosed by Γ_h. In this domain we consider the control problem

(P$_h$) $$\begin{cases} \min J_h(u_h) = \displaystyle\int_{\Omega_h} 4y_{h,u_h}(x)\,dx + \frac{1}{2}\int_{\Gamma_h} u_h^2(x)\,d\sigma_h(x) \\ \text{subject to } (y_{h,u_h}, u_h) \in (L^2(\Omega_h) \cap H^{1/2}(\Omega_h)) \times L^2(\Gamma_h), \end{cases}$$

where the state y_{h,u_h} associated to the control u_h is the solution of the Dirichlet problem

(2.6) $$\begin{cases} -\Delta y = 0 & \text{in } \Omega_h, \\ y = u_h & \text{on } \Gamma_h. \end{cases}$$

The previous example is inspired in another one given by Thomée [8] to prove that the estimates derived by him in the approximation of Dirichlet's problem were sharp. In fact, he considered the adjoint state equation (2.3) as the example of Dirichlet's problem.

Problem (P_h) has a unique solution that is characterized by the system

$$(2.7) \qquad \begin{cases} -\Delta \bar{y}_h = 0 & \text{in } \Omega_h, \\ \quad \bar{y}_h = \bar{u}_h & \text{on } \Gamma_h, \end{cases}$$

$$(2.8) \qquad \begin{cases} -\Delta \bar{\varphi}_h = 4 & \text{in } \Omega_h, \\ \quad \bar{\varphi}_h = 0 & \text{on } \Gamma_h, \end{cases}$$

$$(2.9) \qquad \bar{u}_h = \partial_{\nu_h} \bar{\varphi}_h,$$

where $\nu_h(x)$ is the unit outward normal vector to Γ_h at the point x; hence if $x \in (x_j, x_{j+1})$, then $\nu_h(x) = \frac{x_j + x_{j+1}}{|x_j + x_{j+1}|}$. The solution $\bar{\varphi}_h$ of (2.8) is of class C^1 in $\bar{\Omega}_h$ (see section 4 for more details), and therefore $\partial_{\nu_h} \bar{\varphi}_h(x) = \nabla \bar{\varphi}_h(x) \cdot \nu_h(x)$ is well defined for every $x \in \Gamma_h$, with $x \neq x_j$, $j = 1, \ldots, n$.

To compare the solutions $\bar{u}$ and $\bar{u}_h$ we introduce the mapping $g_h : \Gamma_h \longrightarrow \Gamma$ as follows. For every $1 \leq j \leq n$, $\widehat{x_j x_{j+1}}$ denotes the arc of Γ delimited by the points x_j and x_{j+1}. Then we have that $\Gamma = \cup_{j=1}^n \widehat{x_j x_{j+1}}$ and $\Gamma_h = \cup_{j=1}^n [x_j, x_{j+1}]$. Now we introduce a parametrization of $\widehat{x_j x_{j+1}}$,

$$\psi_j : [0, h] \longrightarrow \widehat{x_j x_{j+1}} \text{ is defined by } \psi_j(t) = x_j + t\tau_j + \phi(t)\nu_j,$$

where $\tau_j = (x_{j+1} - x_j)/h$, ν_j is the restriction of ν_h to the side (x_j, x_{j+1}) of Γ_h, and $\phi : [0, h] \longrightarrow [0, +\infty)$ is chosen such that $\psi_j(t) \in \Gamma$. It is evident that ϕ is uniquely defined. Since Ω is convex and Γ is of class C^2, the following properties hold:
1. ϕ is of class C^2 and $\phi(0) = \phi(h) = 0$.
2. There exists a constant $C_\Gamma > 0$ such that $\phi(t) + h|\phi'(t)| \leq C_\Gamma h^2$ for all $t \in [0, h]$.

Finally, we define

$$g_h : \Gamma_h \longrightarrow \Gamma, \quad g_h|_{[x_j, x_{j+1}]}(x) = g_h|_{[x_j, x_{j+1}]}(x_j + t\tau_j) = x_j + t\tau_j + \phi(t)\nu_j = \psi_j(t).$$

Clearly g_h is one-to-one. We denote by $\tau(x)$ the unit tangent vector to Γ at the point x such that $\{\tau(x), \nu(x)\}$ is a direct reference system in $\mathbb{R}^2$. We can obtain the expressions for these vectors from the given parametrization. If x is a point of the arc $\widehat{x_j x_{j+1}}$, then

$$\tau(x) = \frac{1}{\sqrt{1 + \phi'(t)^2}}(\tau_j + \phi'(t)\nu_j) \text{ and } \nu(x) = \frac{1}{\sqrt{1 + \phi'(t)^2}}(\nu_j - \phi'(t)\tau_j),$$

where $x = x_j + t\tau_j + \phi(t)\nu_j$. From these expressions and the properties of ϕ we deduce that

$$(2.10) \qquad \max\{|\tau(g_h(x)) - \tau_h(x)|, |\nu(g_h(x)) - \nu_h(x)|\} \leq C_\Gamma h \quad \forall x \in \Gamma_h.$$

Given a function $v \in L^1(\Gamma)$, we have

$$\int_\Gamma v(x) \, d\sigma(x) = \sum_{j=1}^n \int_0^h v(\psi_j(t)) \sqrt{1 + \phi'(t)^2} \, dt$$

and

$$\int_{\Gamma_h} v(g_h(x)) \, d\sigma_h(x) = \sum_{j=1}^n \int_0^h v(g_h(x_j + t\tau_j)) \, dt = \sum_{j=1}^n \int_0^h v(\psi_j(t)) \, dt.$$

From these expressions we deduce that

$$(2.11) \qquad \int_{\Gamma_h} |v(g_h(x))| \, d\sigma_h(x) \le \int_{\Gamma} |v(x)| \, d\sigma(x) \quad \forall v \in L^1(\Gamma)$$

and

$$\left| \int_{\Gamma} v(x) \, d\sigma(x) - \int_{\Gamma_h} v(g_h(x)) \, d\sigma_h(x) \right| \le \sum_{j=1}^{n} \int_0^h |v(\psi_j(t))| |1 - \sqrt{1 + \phi'(t)^2}| \, dt$$

$$(2.12) \qquad \le C_\Gamma h^2 \sum_{j=1}^{n} \int_0^h |v(\psi_j(t))| \, dt \le C_\Gamma h^2 \int_{\Gamma} |v(x)| \, d\sigma(x) \quad \forall v \in L^1(\Gamma).$$

We also have

$$(2.13) \qquad \int_{\Gamma} v(x) \, d\sigma(x) = \int_{\Gamma_h} v(g_h(x)) |Dg_h(x) \cdot \tau_h(x)| \, d\sigma_h(x) \quad \forall v \in L^1(\Gamma).$$

The following result is an immediate consequence of [4, Theorem 9.1]

THEOREM 2.1. *Let $\bar{u}$ and $\bar{u}_h$ denote the solutions of problems* (P) *and* (P$_h$); *then there exists a constant $C > 0$, independent of h, such that the following estimate holds:*

$$(2.14) \qquad \|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)} \le Ch.$$

Now we prove that this estimate cannot be improved. To get an underestimate for $\bar{u} - \bar{u}_h \circ g_h^{-1}$ we use (2.4) and (2.9); then

$$\bar{u}(x) - (\bar{u}_h \circ g_h^{-1})(x) = \partial_\nu \bar{\varphi}(x) - (\partial_{\nu_h} \bar{\varphi}_h \circ g_h^{-1})(x)$$

$$= \nabla \bar{\varphi}(x)[\nu(x) - \nu_h(g_h^{-1}(x))] + [\nabla \bar{\varphi}(x) - \nabla \bar{\varphi}(g_h^{-1}(x))]\nu_h(g_h^{-1}(x))$$

$$(2.15) \qquad + [\nabla \bar{\varphi}(g_h^{-1}(x)) - \nabla \bar{\varphi}_h(g_h^{-1}(x))]\nu_h(g_h^{-1}(x)).$$

Using that $\bar{\varphi} = 0$ on Γ and (2.10) we get that

$$|\nabla \bar{\varphi}(x)[\nu(x) - \nu_h(g_h^{-1}(x))]| = |[1 - \nu(x) \cdot \nu_h(g_h^{-1}(x))]\nabla \bar{\varphi}(x)\nu(x)|$$

$$(2.16) \qquad = \frac{1}{2}|\nu(x) - \nu_h(g_h^{-1}(x))|^2 |\partial_\nu \bar{\varphi}(x)| = |\nu(x) - \nu_h(g_h^{-1}(x))|^2 \le C_\Gamma^2 h^2.$$

On the other hand, from the definition of g_h and the properties of ϕ we get

$$|[\nabla \bar{\varphi}(x) - \nabla \bar{\varphi}(g_h^{-1}(x))]\nu_h(g_h^{-1}(x))| = |-2[x - g_h^{-1}(x)]\nu_h(g_h^{-1}(x))|$$

$$(2.17) \qquad = 2\phi(t)|\nu_j \cdot \nu_h(g_h^{-1}(x))| \le 2C_\Gamma h^2.$$

From (2.15), (2.16), and (2.17) we conclude

$$\|\partial_{\nu_h}\bar{\varphi} \circ g_h^{-1} - \partial_{\nu_h}\bar{\varphi}_h \circ g_h^{-1}\|_{L^2(\Gamma)} - (2 + C_\Gamma)C_\Gamma\sqrt{2\pi}h^2 \le \|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)}$$

$$\le \|\partial_{\nu_h}\bar{\varphi} \circ g_h^{-1} - \partial_{\nu_h}\bar{\varphi}_h \circ g_h^{-1}\|_{L^2(\Gamma)} + (2 + C_\Gamma)C_\Gamma\sqrt{2\pi}h^2.$$

Then it is enough to prove the existence of a constant $C > 0$, independent of h, such that

$$\|\partial_{\nu_h}\bar{\varphi} \circ g_h^{-1} - \partial_{\nu_h}\bar{\varphi}_h \circ g_h^{-1}\|_{L^2(\Gamma)} \ge Ch,$$

to conclude that the estimate (2.14) is sharp for the controls. But, according to (2.11), the last inequality holds if

$$(2.18) \qquad \|\partial_{\nu_h}\bar{\varphi} - \partial_{\nu_h}\bar{\varphi}_h\|_{L^2(\Gamma_h)} \geq C'h$$

for some $C' > 0$ independent of h.

Let us prove (2.18). To this end, we set $\phi_h = \bar{\varphi} - \bar{\varphi}_h$; then from (2.3) and (2.8) we get

$$(2.19) \qquad \begin{cases} -\Delta\phi_h = 0 & \text{in } \Omega_h, \\ \phi_h = \bar{\varphi} & \text{on } \Gamma_h. \end{cases}$$

Following Kenig [7, p. 121], $\phi_h \in H^{3/2}(\Omega_h)$ and the following estimates hold:

$$(2.20) \qquad C_1\|\phi_h\|_{H^1(\Gamma_h)} \leq \|\phi_h\|_{H^{3/2}(\Omega_h)} \leq C_1'\|\phi_h\|_{H^1(\Gamma_h)},$$

$$(2.21) \qquad C_2\|\partial_{\nu_h}\phi_h\|_{L^2(\Gamma_h)} \leq \|\phi_h\|_{H^{3/2}(\Omega_h)} \leq C_2'\|\partial_{\nu_h}\phi_h\|_{L^2(\Gamma_h)}.$$

From these inequalities we deduce that

$$\|\partial_{\nu_h}\phi_h\|_{L^2(\Gamma_h)} \geq C\|\phi_h\|_{H^1(\Gamma_h)} = C\|\bar{\varphi}\|_{H^1(\Gamma_h)}.$$

Hence, if we prove that $\|\bar{\varphi}\|_{H^1(\Gamma_h)} \geq Ch$, then (2.18) is concluded. We have that

$$\|\bar{\varphi}\|_{H^1(\Gamma_h)} = \left\{ \|\bar{\varphi}\|_{L^2(\Gamma_h)}^2 + \|\partial_{\tau_h}\bar{\varphi}\|_{L^2(\Gamma_h)}^2 \right\}^{1/2}.$$

From [2, Lemma 1] we know that the $L^2(\Gamma_h)$-norm of $\bar{\varphi}$ is of order h^2. Let us compute the norm of the tangential derivative. Since $\bar{\varphi}(x) = 1 - |x|^2$ and $\tau_h(x) = \frac{1}{h}(x_{j+1} - x_j)$ for $x \in (x_j, x_{j+1})$, we have that $\partial_{\tau_h}\bar{\varphi}(x) = -(2/h)x \cdot (x_{j+1} - x_j)$; then

$$\|\partial_{\tau_h}\bar{\varphi}\|_{L^2(\Gamma_h)}^2 = \sum_{j=1}^{n}\int_{x_j}^{x_{j+1}} \left| \frac{2}{h}x \cdot (x_{j+1} - x_j) \right|^2 d\sigma_h(x)$$

$$= \frac{4}{h}\sum_{j=1}^{n}\int_0^1 |[x_j + t(x_{j+1} - x_j)] \cdot (x_{j+1} - x_j)|^2\, dt = \frac{4}{h}\sum_{j=1}^{n}\int_0^1 |x_j \cdot x_{j+1} - 1 + th^2|^2\, dt.$$

Using that the angle between x_j and x_{j+1} is $2\pi/n$, we get

$$x_j \cdot x_{j+1} = \cos\frac{2\pi}{n} = 1 - 2\sin^2\frac{\pi}{n} = 1 - \frac{h^2}{2}.$$

Therefore,

$$\|\partial_{\tau_h}\bar{\varphi}\|_{L^2(\Gamma_h)}^2 = \frac{4}{h}\sum_{j=1}^{n}\int_0^1 \left| -\frac{h^2}{2} + th^2 \right|^2 dt = 4h^3 n\int_0^1 \left(t - \frac{1}{2} \right)^2 dt = \frac{h^3 n}{3}.$$

Now we observe that

$$h = 2\sin\frac{\pi}{n} \Rightarrow hn \geq \pi \quad \text{for} \quad n \geq 4.$$

Thus, we conclude

$$\|\partial_{\tau_h}\bar{\varphi}\|_{L^2(\Gamma_h)}^2 \geq \frac{\pi h^2}{3}$$

and then

$$\|\bar{\varphi}\|_{H^1(\Gamma_h)} \geq \|\partial_{\tau_h}\bar{\varphi}\|_{L^2(\Gamma_h)} \geq \sqrt{\frac{\pi}{3}}h.$$

3. Finite element approximation of (P). To get an order of convergence $O(h^{3/2})$ for the finite element discretization of (P) we can use a piecewise $O(h^2)$ irregular family of meshes of Ω (see [5, Theorem 5.4]). To get such a family $\mathcal{T}_h$ we consider the regular polygon of vertices $(x_j)_{j=1}^n$ for some fixed $n \geq 4$ and the triangulation formed by the n triangles of vertices $(0,0)$, x_j, x_{j+1}, $j = 1, \ldots, n$. Subsequent meshes are obtained by regular refinement of the initial mesh, taking into account that boundary nodes of Γ_h must lie on Γ. The families obtained in this way are piecewise $O(h^2)$-irregular (see Remark 6.2 in [5]). In Figure 3.1 we have put the starting mesh for an octagon and two subsequent refinements of it.
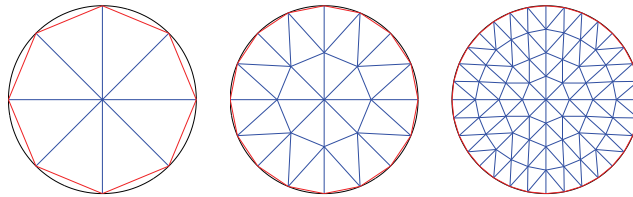

FIG. 3.1. *Family of piecewise $O(h^2)$-irregular meshes of the circle.*

Associated to $\mathcal{T}_h$ we consider the spaces

$$X_h = \{\phi_h \in C(\bar{\Omega}_h) : \phi_h \in P_1(T) \ \forall T \in \mathcal{T}_h\},$$

$X_{h0} = X_h \cap H_0^1(\Omega_h)$, and U_h the restriction to Γ_h of functions in X_h.
We consider the following approximation of (P):

$$(Q_h) \quad \begin{cases} \min J_h(u_h) = \displaystyle\int_{\Omega_h} 4y_h(u_h)(x)\, dx + \frac{1}{2}\int_{\Gamma_h} u_h^2(x)\, d\sigma_h(x) \\[2mm] \text{subject to} \ \ (y_h(u_h), u_h) \in X_{h0} \times U_h, \end{cases}$$

where the discrete state $y_h(u_h) \in X_h$ associated to the control u_h is the unique solution of the following finite-dimensional problem:

$$\text{Find } y_h \in X_h \text{ s.t. } y_h = u_h \text{ on } \Gamma_h \text{ and } \int_{\Omega_h} \nabla y_h \cdot \nabla \phi_h dx = 0 \ \forall \phi_h \in X_{h0}.$$

Problem (Q_h) has a unique solution $\hat{u}_h \in U_h$. Let us denote by $\hat{\varphi}_h \in X_{h0}$ the discrete adjoint state associated to $\hat{u}_h$, characterized by the equation

$$\int_{\Omega_h} \nabla \hat{\varphi}_h \cdot \nabla \phi_h dx = \int_{\Omega_h} 4\phi_h dx \ \forall \phi_h \in X_{h0}.$$

Then we can write

(3.1)
$$\hat{u}_h = \partial_{\nu_h}^h \hat{\varphi}_h,$$

where $\partial_{\nu_h}^h \hat{\varphi}_h \in U_h$ is the so-called discrete normal derivative, described in [3], and characterized by the system

(3.2)
$$\int_{\Gamma_h} \partial_{\nu_h}^h \hat{\varphi}_h \phi_h \, d\sigma_h = \int_{\Omega_h} \nabla \hat{\varphi}_h \cdot \nabla \phi_h dx - \int_{\Omega_h} 4\phi_h dx \ \forall \phi_h \in X_h.$$

TABLE 3.1
Mesh parameters, error, and order of convergence for full discretization. Data for meshes obtained from i refinements of an octagon.

| i	n	h	n_t	n_p	E_h^2	EOC^2	E_h^∞	EOC^∞
0	8	0.765367	8	9	0.3767136	0.000	0.152241	0.000
1	16	0.390181	32	25	0.5895520	-0.665	0.441659	-1.581
2	32	0.196034	128	81	0.2296750	1.370	0.232952	0.929
3	64	0.098135	512	289	0.0881355	1.384	0.128178	0.863
4	128	0.049082	2048	1089	0.0326268	1.434	0.067788	0.919
5	256	0.024543	8192	4225	0.0118212	1.465	0.034916	0.957
6	512	0.012272	32768	16641	0.0042341	1.481	0.017734	0.977
7	1024	0.006136	131072	66049	0.0015073	1.490	0.008940	0.988
8	2048	0.003068	524288	263169	0.0005349	1.495	0.004489	0.994
9	4096	0.001534	2097152	1050625	0.0001895	1.497	0.002250	0.997
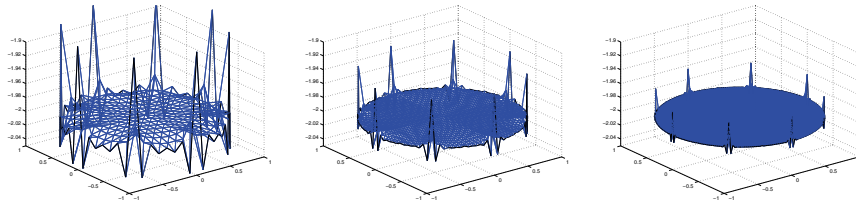
FIG. 3.2. *Approximations of the discrete optimal state and the boundary control.*

Since there are no control constraints, this approach is equivalent to the one given in [5], where the control is not discretized, but it is finally obtained as the pointwise projection of the discrete normal derivative.

We will take as mesh size h the length of one side of Γ_h. For quasi-uniform meshes this is equivalent to the usual choice of the maximum edge size of the triangulation. With these settings, Theorem 5.4 in [5] states that there exists a constant $C > 0$ such that

$$\|\bar{u} \circ g_h - \hat{u}_h\|_{L^2(\Gamma_h)} \leq Ch^{3/2}.$$

Numerical testing confirms this order of convergence. For $p = 2$ or $p = \infty$, the experimental error is given by

$$E_h^p = \|\bar{u} \circ g_h - \hat{u}_h\|_{L^p(\Gamma_h)}$$

and the experimental order of convergence is

$$EOC_{h_1,h_2}^p = \frac{\log(E_{h_1}^p) - \log(E_{h_2}^p)}{\log(h_1) - \log(h_2)}.$$

We obtain the results summarized in Table 3.1.

A picture of the solution for meshes obtained after successive refinements from an octagon is shown in Figure 3.2. Notice that the numerical solution has some needles located at the vertexes of the initial rough mesh. Nevertheless, these deviations are small and the convergence order in $L^\infty(\Gamma_h)$ is linear on this example.

4. Explaining the paradox. The reason for lower accuracy than expected in approximating $\bar{u}$ by $\bar{u}_h$ is found at the vertices x_j of the polygonal boundary Γ_h. Indeed, from (2.8) we deduce that $\bar{\varphi}_h \in W^{2,p_h}(\Omega_h)$ for some $p_h > 2$ depending on

the angles of the polygonal domain Ω_h; see Grisvard [6]. In fact, it holds for any $p_h < 2 + 4/(n-4)$, where n is the number of vertices of Ω_h. The point is that $W^{2,p_h}(\Omega_h) \subset C^1(\bar{\Omega}_h)$. Then, taking into account that $\bar{\varphi}_h = 0$ on Γ_h, we have at every vertex x_j

$$\nabla\bar{\varphi}_h(x_j) \cdot \tau_{j-1} = \nabla\bar{\varphi}_h(x_j) \cdot \tau_j = 0,$$

and therefore $\nabla\bar{\varphi}_h(x_j) = 0$. Hence, $\nabla\bar{\varphi}_h \cdot \nu_h$ is a continuous function on Γ_h if we take the value zero on the vertices x_j. Even more, we have that $\partial_{\nu_h}\bar{\varphi}_h \in W^{1-1/p_h,p_h}(\Gamma_h) \subset C(\Gamma_h)$. Therefore, $\bar{u}_h = \partial_{\nu_h}\bar{\varphi}_h \in W^{1-1/p_h,p_h}(\Gamma_h)$ and $\bar{u}_h(x_j) = \partial_{\nu_h}\bar{\varphi}_h(x_j) = 0$. Thus, the singularities of Γ_h on the vertices x_j force the optimal controls $\bar{u}_h$ to vanish on them. Taking into account that $\bar{u}(x_j) = -2$, we observe a big error between $\bar{u}$ and $\bar{u}_h$ at the vertices. Notice that the number of vertices tends to infinity when $h \to 0$ and $\{x_j\}_{j=1}^n$ becomes dense in Γ. This does not happen if we consider the numerical approximation of $\bar{u}$ on Γ_h. Indeed, the discrete optimal control is given by (3.1), where we use the discrete normal derivative of the discrete adjoint state; see (3.2). This discrete normal derivative does not vanish necessarily at the vertices x_j.

Let us finish by showing the solution $\bar{u}_h$ of (P_h). To compute $\bar{u}_h$ we make a finite element approximation of (P_h). For that purpose we take a quasi-uniform family of triangulations $\mathcal{T}_\rho^h$ of Ω_h (see Figure 4.1). Associated to $\mathcal{T}_\rho^h$ we consider the spaces

$$X_\rho^h = \{\phi_\rho \in C(\bar{\Omega}_h) : \phi_\rho|_T \in P_1(T) \ \forall T \in \mathcal{T}_\rho^h\},$$

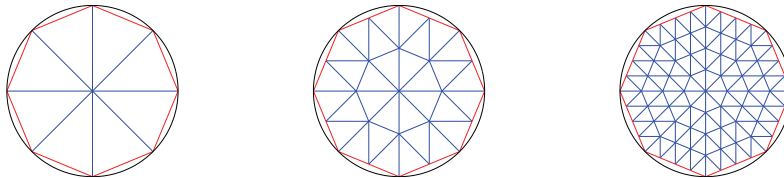$X_{\rho 0}^h = X_\rho \cap H_0^1(\Omega_h)$, and U_ρ^h the restriction to Γ_h of functions in X_ρ^h.

FIG. 4.1. *Triangulations of the octagon for $\rho = h/2^k$, $k = 0, 1, 2$.*

We consider the following approximation of (P_h):

$$(\mathrm{Q}_\rho^h) \quad \begin{cases} \min J_\rho^h(u_\rho^h) = \displaystyle\int_{\Omega_h} 4y_\rho^h(u_\rho^h)(x)\,dx + \frac{1}{2}\int_{\Gamma_h} u_\rho^h(x)^2\,d\sigma_h(x) \\[2mm] \text{subject to} \ \ (y_\rho^h(u_\rho^h), u_\rho^h) \in X_{\rho 0}^h \times U_\rho^h, \end{cases}$$

where the discrete state $y_\rho^h(u_\rho^h) \in X_\rho^h$ associated to the control u_ρ^h is the unique solution of the following finite-dimensional problem:

$$\text{Find } y_\rho^h \in X_\rho^h \text{ s.t. } y_\rho^h = u_\rho^h \text{ on } \Gamma_h \text{ and } \int_{\Omega_h} \nabla y_\rho^h \cdot \nabla\phi_\rho\,dx = 0 \ \forall\phi_\rho \in X_{\rho 0}^h.$$

Problem (Q_ρ^h) has a unique solution $\hat{u}_\rho^h \in U_\rho^h$. Let us denote by $\hat{\varphi}_\rho^h \in X_{\rho 0}^h$ the discrete adjoint state associated to $\hat{u}_\rho^h$, characterized by the equation

$$\int_{\Omega_h} \nabla\hat{\varphi}_\rho^h \cdot \nabla\phi_\rho\,dx = \int_{\Omega_h} 4\phi_\rho\,dx \ \forall\phi_\rho \in X_{\rho 0}^h.$$
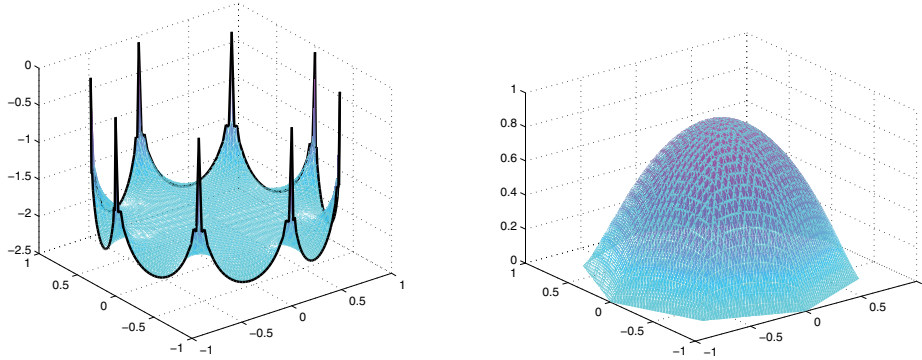
FIG. 4.2. *Left: Optimal state with boundary control. Right: Associated adjoint state.* $n = 8$.

Then we can write

$$\hat{u}_\rho^h = \partial_{\nu_h}^\rho \hat{\varphi}_\rho^h,$$

where $\partial_{\nu_h}^\rho \hat{\varphi}_\rho^h \in U_\rho^h$ is characterized by

$$\int_{\Gamma_h} \partial_{\nu_h}^\rho \hat{\varphi}_\rho^h \phi_\rho \, d\sigma_h = \int_{\Omega_h} \nabla \hat{\varphi}_\rho^h \cdot \nabla \phi_\rho dx - \int_{\Omega_h} 4\phi_\rho dx \;\; \forall \phi_\rho \in X_\rho^h.$$

With these settings, Theorem 7.1 in [3] states that there exists a constant $C_s > 0$ such that

$$(4.1) \qquad\qquad \|\bar{u}_h - \hat{u}_\rho^h\|_{L^2(\Gamma_h)} \le C_s \rho^s,$$

where $s < s^* = \frac{1}{2} + \frac{1}{n-2}$. Figure 4.2 illustrates an approximation of the optimal solution for $n = 8$.

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