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Scalar Sector of 331 Models Without Exotic Electric Charges

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“Órbitas Gauge para el Modelamiento del Potencial Escalar con Varios Tripletos
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1 Introduction

The Higgs boson or Higgs particle is an elementary particle initially theorised in 1964 [1], whose discovery was announced at CERN on 4 July 2012 [2]. The discovery has been called “monumental” because it appears to confirm the existence of the Higgs field, which is pivotal to the Standard Model (SM) [3, 13] and other theories within particle physics, like 3-3-1 models [8, 9, 10, 12, 14, 37]. It would explain why some fundamental particles have mass when the symmetries controlling their interactions should require them to be massless, and why the weak force has a much shorter range than the electromagnetic force. The discovery of a Higgs boson should allow physicists to finally validate the last untested area of the SM’s approach to fundamental particles and forces, guide other theories and discoveries in particle physics, and potentially lead to developments in “new” physics [4].

This unanswered question in fundamental physics is of such importance that it led to a search of more than 40 years for the Higgs boson and finally the construction of one of the world’s most expensive and complex experimental facilities to date, the Large Hadron Collider [5], able to create Higgs bosons and other particles for observation and study. On 4 July 2012, it was announced that a previously unknown particle with a mass between 125 and 127 GeV/ c^2 had been detected; physicists suspected at the time that it was the Higgs boson. By March 2013, the particle had been proven to behave, interact and decay in many of the ways predicted by the Standard Model, and was also tentatively confirmed to have positive parity and zero spin, two fundamental attributes of a Higgs boson. This appears to be the first elementary scalar particle discovered in nature. More data is needed to know if the discovered particle exactly matches the predictions of the Standard Model, or whether, as predicted by some theories, multiple Higgs bosons exist [6].

The Higgs boson arises as the direct physical manifestation of the origin of mass in the SM. Hence, the search for the Higgs boson continues to be of paramount importance to complete our understanding of the SM and, may soon completely change our understanding of the forces of nature. In the last few years the CDF and D0 collaborations at the Tevatron have ruled out a mass window for the SM Higgs boson in the range 158-173 GeV [7]. The study of scalar sector has become one of the booming subjects in particle physics. This study has been carried out within the SM framework as well as some extensions of the SM. One of these extensions is the $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ models [8, 12, 14, 15, 11].

In this work, we study the minimal set of Higgs scalars, for models based on the local gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ which do not contain particles with exotic electric charges. We show that only two Higgs $SU(3)_L$ triplets are needed in order to properly break the symmetry. The exact tree-level scalar mass matrices resulting from symmetry breaking are calculated at the minimum of the most general scalar potential, and the gauge bosons are obtained, together with their coupling to the physical scalar fields. We show how the scalar sector introduced is enough to produce masses for fermions in a particular model which is an E_6 subgroup. By using experimental results we constraint the scale of new physics to be above 1.3 TeV.

Also, a detailed study of the criteria for stability of the scalar potential and the proper electroweak symmetry breaking pattern in the economical 331 model, is presented. For the analysis we use, and improve, a method previously developed to study the scalar potential in the two-Higgs-doublet extension of the Standard Model (SM). A new theorem

related to the stability of the potential is stated. As a consequence of this study, the consistency of the economical 331 model emerges. Additionally, we concentrate in a scalar sector with three Higgs scalar triplets, with a potential that does not include the cubic term, due to the presence of a discrete symmetry. Our main result is to show the consistency of those 331 models without exotic electric charges.

Our study is organized as follows: in Sect. 2 we introduce the 331 models without exotic electric charges and review the different scalar sectors available in the literature for this type of models. In section 3 we study the (common) scalar sector of these models, including the analysis of its mass spectrum. We analyze the gauge boson structure common to all the models considered. We present the couplings between the neutral scalar fields in the model and the SM gauge bosons. One appendix A at the end shows how the Higgs scalars used to break the symmetry, can also be used to produce a consistent mass spectrum for the fermion fields, in the particular model which is an E_6 subgroup [9]. In Sect. 4 we briefly review the mathematical formalism in order to make the work self-contained, a new theorem that facilitates the stability criteria is proved; we apply the method to the scalar sector of the economical 331 model, which is followed in by the introduction of new parameterizations. We derive expressions for the masses of the scalar fields. Two appendixes with technical ones are presented at the end. In Appendix D two exceptional solutions for the global minimum of the potential are analyzed. And in Appendix C, it is verified that if only one scalar triplet acquires a nonzero Vacuum Expectation Value (VEV), the economical 331 model is inconsistent. In Sect. 5 we introduce three scalar triplets to study the scalar potential and analyze the consistency of the electroweak symmetry breaking pattern proposed; then we study the stability of the scalar potential, and we find its stationary points and its global minimum. Two appendixes with technical ones are presented at the end. Our conclusions are given in Sect. 6.

2 331 Models

The SM based on the local gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ [13] can be extended in several different ways: first by adding new fermion fields (adding a right-handed neutrino field constitute its simplest extension and has profound consequences, as for example the implementation of the see-saw mechanism, and the enlarging of the possible number of local gauge abelian symmetries that can be gauged simultaneously); second, by augmenting the scalar sector to more than one Higgs representation, and third by enlarging the local gauge group. In this last direction $SU(3)_L \otimes U(1)_X$ as a flavor group has been studied previously by many authors in the literature [8]-[14] who have explored possible fermion and Higgs-boson representation assignments. From now on, models based on the local gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ are going to be called 331 models.

There are in the literature several 331 models; the most popular one, the Pleitez-Frampton model [14], is far from being the simplest construction. Not only its scalar sector is quite complicated and messy (three triplets and one sextet [19]), but its physical spectrum is plagued with particles with exotic electric charges, namely: double charged gauge and Higgs bosons and exotic quarks with electric charges $5/3$ and $-4/3$. Other 331 models in the literature are just introduced or merely sketched in a few papers [9, 10, 11, 12], with a detailed phenomenological analysis of them still lacking. In particular, there is not published papers related to the study of the scalar sector for those other models.

All possible 331 models without exotic electric charges in their gauge boson sector and in their spin 1/2 fermion content are presented in Ref. [8], where it is shown that there are just a few anomaly free models for one or three families which share in common all of them the same gauge-boson content and, as we are going to show next, they may share a common scalar sector too. This scalar sector does not contain particles with exotic electric charges either.

2.1 Charge content of 331 models

In what follows we assume that the electroweak group is $SU(3)_L \otimes U(1)_X \supset SU(2)_L \otimes U(1)_Y$. We also assume that the left-handed quarks (color triplets) and left-handed leptons (color singlets) transform under the two fundamental representations of $SU(3)_L$ (the 3 and 3^*) and that $SU(3)_c$ is vectorlike as in the SM.

The most general electric charge operator in $SU(3)_L \otimes U(1)_X$ is a linear combination of the three diagonal generators of the gauge group

$$Q = aT_{3L} + \frac{2}{\sqrt{3}}bT_{8L} + XI_3, \quad (1)$$

where $T_{iL} = \lambda_{iL}/2$, being λ_{iL} the Gell-Mann matrices for $SU(3)_L$ normalized as $\mathbf{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}$, $I_3 = Dg(1, 1, 1)$ is the diagonal 3×3 unit matrix, and a and b are arbitrary parameters to be determined anon. The X values are fixed

by anomaly cancelation [8] and an eventual coefficient for XI_3 can be absorbed in the hypercharge definition.

If we assume that the usual isospin $SU(2)_L$ of the SM is such that $SU(2)_L \subset SU(3)_L$, then $a = 1$ and we have just a one parameter set of models, all of them characterized by the value of b . So, Eq.(1) allows for an infinite number of models in the context of the 331 gauge structure, each one associated to a particular value of the parameter b , with characteristic signatures that make each one quite different from each other.

There are a total of 17 gauge bosons in the gauge group under consideration, they are: one gauge field B^μ associated with $U(1)_X$, the 8 gluon fields associated with $SU(3)_c$ which remain massless after breaking the symmetry, and other 8 associated with $SU(3)_L$ and that we may write in the following way:

$$\frac{1}{2}\lambda_{\alpha L}A_\mu^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} D_{1\mu}^0 & W_\mu^+ & K_\mu^{(1/2+b)} \\ W_\mu^- & D_{2\mu}^0 & K_\mu^{-(1/2-b)} \\ K_\mu^{-(1/2+b)} & \bar{K}_\mu^{1/2-b} & D_{3\mu}^0 \end{pmatrix}$$

where $D_{1\mu}^0 = A_{3\mu}/\sqrt{2} + A_{8\mu}/\sqrt{6}$, $D_{2\mu}^0 = -A_{3\mu}/\sqrt{2} + A_{8\mu}/\sqrt{6}$, and $D_{3\mu}^0 = -2A_{8\mu}/\sqrt{6}$. The upper indices of the gauge bosons in the former expression stand for the electric charge of the corresponding particle, some of them functions of the b parameter as they should be [8]. Notice that the gauge bosons have integer electric charges only for $b = \pm 1/2, \pm 3/2, \pm 5/2, \dots, \pm(2n+1)/2$, $n = 0, 1, 2, 3, \dots$. A deeper analysis shows that each negative b value can be related to the positive one just by taking the complex conjugate in the covariant derivative, which in turn is equivalent to replace $3 \leftrightarrow 3^*$ in the fermion content of each particular model.

Our first conclusion is thus that if we want to avoid exotic electric charges in the gauge sector of our theory, then b must be equal to $1/2$, which is also the condition for excluding exotic electric charges in the fermion sector [8].

Now, contrary to the SM where only the abelian $U(1)_Y$ factor is anomalous, in the 331 theory both, $SU(3)_L$ and $U(1)_X$ are anomalous ($SU(3)_c$ is vectorlike as in the SM). So, special combination of multiplets must be used in each particular model in order to cancel the several possible anomalies, and end with physical acceptable models. The triangle anomalies we must take care of are: $[SU(3)_L]^3$, $[SU(3)_c]^2U(1)_X$, $[SU(3)_L]^2U(1)_X$, $[grav]^2U(1)_X$ (the gravitational anomaly), and $[U(1)_X]^3$.

In order to present specific examples let us see how the charge operator in Eq.(1) acts on the representations 3 and 3^* of $SU(3)_L$:

$$\begin{aligned} Q[3] &= Dg. \left(\frac{1}{2} + \frac{b}{3} + X, -\frac{1}{2} + \frac{b}{3} + X, -\frac{2b}{3} + X \right), \\ Q[3^*] &= Dg. \left(-\frac{1}{2} - \frac{b}{3} + X, \frac{1}{2} - \frac{b}{3} + X, \frac{2b}{3} + X \right). \end{aligned} \tag{2}$$

Notice from this expressions that, if we accommodate the known left-handed quark and lepton isodoublets in the two upper components of 3 and 3^* (or 3^* and 3), and forbid the presence of exotic electric charges in the possible models, then the electric charge of the third component in those representations must be equal either to the charge of the first or second component, which in turn implies $b = \pm 1/2$. Since the negative value is equivalent to the positive one, $b = 1/2$ is a necessary and sufficient condition in order to exclude exotic electric charges in the fermion sector too.

As an example of the former discussion let us take $b = 3/2$, then $Q[3] = Dg.(1 + X, X, X - 1)$ and $Q[3^*] = Dg.(X - 1, X, 1 + X)$. Then the following multiplets are associated with the respective $(SU(3)_c, SU(3)_L, U(1)_X)$ quantum numbers: $(e^-, \nu_e, e^+)_L^T \sim (1, 3^*, 0)$; $(u, d, j)_L^T \sim (3, 3, -1/3)$ and $(d, u, k)_L^T \sim (3, 3^*, 2/3)$, where j and k are isosinglet exotic quarks of electric charges $-4/3$ and $5/3$ respectively. This multiplet structure is the basis of the Pleitez-Frampton model [14] for which the anomaly-free arrangement for the three families is given by:

$$\begin{aligned} \psi_L^a &= (e^a, \nu^a, e^{ca})_L^T \sim (1, 3^*, 0), \\ q_L^i &= (u^i, d^i, j^i)_L^T \sim (3, 3, -1/3), \\ q_L^1 &= (d^1, u^1, k)_L^T \sim (3, 3^*, 2/3), \\ u_L^{ca} &\sim (3^*, 1, -2/3), \quad d_L^{ca} \sim (3^*, 1, 1/3), \\ k_L^a &\sim (3^*, 1, -5/3), \quad j_L^{ci} \sim (3^*, 1, -4/3), \end{aligned}$$

where the upper c symbol stands for charge conjugation, $a = 1, 2, 3$ is a family index and $i = 2, 3$ is related to two of the three families (in the 331 basis). As can be seen, there are six triplets of $SU(3)_L$ and six anti-triplets, which ensures cancelation of the $[SU(3)_L]^3$ anomaly. A power counting shows that the other four anomalies also vanish.

2.2 331 models without exotic electric charges

As discussed before, after fixing $a = 1$, the value $b = 1/2$ is a necessary condition in order to avoid particles with exotic electric charges in models based on the $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ gauge structure. For that particular value let us start first defining the following closed set of fermions (closed in the sense that they include the antiparticles of the charged particles):

- $S_1 = [(\nu_\alpha, \alpha^-, E_\alpha^-); \alpha^+; E_\alpha^+]$ with quantum numbers $[(1, 3, -2/3); (1, 1, 1); (1, 1, 1)]$.
- $S_2 = [(\alpha^-, \nu_\alpha, N_\alpha^0); \alpha^+]$ with quantum numbers $[(1, 3^*, -1/3); (1, 1, 1)]$.
- $S_3 = [(d, u, U); d^c; u^c; U^c]$ with quantum numbers $(3, 3^*, 1/3); (3^*, 1, 1/3); (3^*, 1, -2/3)$ and $(3^*, 1, -2/3)$, respectively.
- $S_4 = [(u, d, D); u^c; d^c; D^c]$ with quantum numbers $(3, 3, 0); (3^*, 1, -2/3); (3^*, 1, 1/3)$ and $(3^*, 1, 1/3)$, respectively.
- $S_5 = [(e^-, \nu_e, N_1^0); (E^-, N_2^0, N_3^0); (N_4^0, E^+, e^+)]$ with quantum numbers $(1, 3^*, -1/3); (1, 3^*, -1/3)$ and $(1, 3^*, 2/3)$, respectively.
- $S_6 = [(\nu_e, e^-, E_1^-); (E_2^+, N_1^0, N_2^0); (N_3^0, E_2^-, E_3^-); e^+; E_1^+; E_3^+]$ with quantum numbers $[(1, 3, -2/3); (1, 3, 1/3); (1, 3, -2/3); (1, 1, 1); (1, 1, 1); (1, 1, 1)]$.

Where the quantum numbers in parenthesis refer to $(SU(3)_c, SU(3)_L, U(1)_X)$ representations.

The several anomalies for the former six sets are presented in the following Table.

TABLE I. Anomalies for S_i .

Anomalies	S_1	S_2	S_3	S_4	S_5	S_6
$[SU(3)_c]^2 U(1)_X$	0	0	0	0	0	0
$[SU(3)_L]^2 U(1)_X$	-2/3	-1/3	1	0	0	-1
$[grav]^2 U(1)_X$	0	0	0	0	0	0
$[U(1)_X]^3$	10/9	8/9	-12/9	-6/9	6/9	12/9
$[SU(3)_L]^3$	1	-1	-3	3	-3	3

Table I allows us to build anomaly-free models without exotic electric charges, for one, two, three, four or more families. Let us extract out of the Table the possible models for one and three families:

2.2.1 One family models

There are just two anomaly-free one family structures that can be extracted from the Table. They are:

Model A: ($S_4 + S_5$). This models is associated with an E_6 subgroup and has been partially analyzed in Ref. [9]. (see also the appendix at the end of this paper).

Model B: ($S_3 + S_6$). This models is associated with an $SU(6)_L \otimes U(1)_X$ subgroup and has been partially analyzed in Ref. [10]

2.2.2 Three family models

Model C: ($3S_2 + S_3 + 2S_4$). This model deals with the following multiplets associated with the given quantum numbers: $(u, d, D)_L^T \sim (3, 3, 0)$, $(e^-, \nu_e, N^0)_L^T \sim (1, 3^*, -1/3)$ and $(d, u, U)_L^T \sim (3, 3^*, 1/3)$, where D and U are exotic quarks with electric charges $-1/3$ and $2/3$ respectively. With such a gauge structure the three family anomaly-free model is given by:

$$\begin{aligned}
\psi_L^{\prime a} &= (e^{-a}, \nu^a, N^{0a})_L^T \sim (1, 3^*, -1/3), \\
e_L^{+a} &\sim (1, 1, 1), \\
q_L^{\prime i} &= (u^i, d^i, D^i)_L^T \sim (3, 3, 0), \\
q_L^{\prime 1} &= (d^1, u^1, U)_L^T \sim (3, 3^*, 1/3), \\
u_L^{ca} &\sim (3^*, 1, -2/3), \quad d_L^{ca} \sim (3^*, 1, 1/3), \\
U_L^c &\sim (3^*, 1, -2/3), \quad D_L^{ci} \sim (3^*, 1, 1/3),
\end{aligned}$$

where $a = 1, 2, 3$ is a family index and $i = 1, 2$ is related to two of the three families. This models has been analyzed in the literature in Ref. [11, 12]. If needed, this model can be augmented with an undetermined number of neutral Weyl states $N_L^{0j} \sim (1, 1, 0)$, $j = 1, 2, \dots$ without violating the anomaly cancelation.

Model D: $(3S_1 + 2S_3 + S_4)$. It makes use of the same multiplets used in the previous model arranged in a different way, plus a new lepton multiplet $(\nu_e, e^-, E^-)_L^T \sim (1, 3, -2/3)$. The family structure of this new anomaly-free model is given by:

$$\begin{aligned}\psi_L''^a &= (\nu^a, e^a, E^a)_L^T \sim (1, 3, -2/3), \\ e_L^{ca} &\sim (1, 1, 1), \quad E_L^{ca} \sim (1, 1, 1), \\ q_L''^1 &= (u^1, d^1, D)_L^T \sim (3, 3, 0), \\ q_L''^i &= (d^i, u^i, U^i)_L^T \sim (3, 3^*, 1/3), \\ u_L^{ca} &\sim (3^*, 1, -2/3), \quad d_L^{ca} \sim (3^*, 1, 1/3), \\ D_L^c &\sim (3^*, 1, 1/3), \quad U_L^{ci} \sim (3^*, 1, 2/3).\end{aligned}$$

This model has been analyzed in the literature in Ref. [12].

Model E: $(S_1 + S_2 + S_3 + 2S_4 + S_5)$. **Model F:** $(S_1 + S_2 + 2S_3 + S_4 + S_6)$.

Besides the former four three family models, other four, carbon copy of the two one family models **A, B** can also be constructed. They are:

Model G: $3(S_4 + S_5)$. **Model H:** $3(S_3 + S_6)$. **Model I:** $2(S_4 + S_5) + (S_3 + S_6)$. **Model J:** $(S_4 + S_5) + 2(S_3 + S_6)$.

For a total of eight different three-family models, each one with a different fermion field content. Notice in particular that in models **E** and **F** each one of the three families is treated differently. As far as we know the last six models have not been studied in the literature so far.

If we wish we may construct also two, four, five, etc. family models (a two family model is given for example by $(S_1 + S_2 + S_3 + S_4)$), but we believe all those models are not realistic at all.

2.3 The scalar sector

As far as we know, for the 331 models without exotic electric charges, three different scalar sectors have been used in the literature, to deal with the spontaneous breaking of the gauge symmetry down to $U(1)_Q$ and, to produce at the same time, masses for the Fermion fields. Each set, as described anon, has its own advantages and disadvantages. They are:

2.3.1 The economical model

Introduced in the literature in Ref. [16] and further analyzed in Refs. [20, 43]. It makes use of only two scalar triplets, which together with their vacuum expectation values (VEV) are:

$$\Phi_1(1, 3^*, -1/3) = \begin{pmatrix} \phi_1^- \\ \phi_1^0 \\ \phi_1^0 \end{pmatrix}, \quad \text{with VEV: } \langle \Phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \\ V_1 \end{pmatrix}, \quad (3a)$$

$$\Phi_3(1, 3^*, 2/3) = \begin{pmatrix} \phi_3^0 \\ \phi_3^+ \\ \phi_3'^+ \end{pmatrix}, \quad \text{with VEV: } \langle \Phi_3 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_3 \\ 0 \\ 0 \end{pmatrix}. \quad (3b)$$

The former structure is the simplest one, able to break the 331 symmetry down to $U(1)_Q$ in a consistent way [16]. In spite of its simplicity, it has the disadvantage of being unable to produce a consistent Fermion mass spectrum at tree level. The claim in Ref. [43] is that the quantum fluctuations can generate non-zero mass terms for all the Fermion fields, but a systematic (tedious) numerical analysis reproducing the fermion mass spectrum has not been published yet, although probably, the most serious hurdle for the survival of this model, is the existence of flavor changing neutral currents (FCNC) at tree level, mediated by the Higgs scalar fields (only two sets of scalar fields producing masses for three Fermion families), neutral currents that severely constraint the parameters of the model.

For this economical model, the study of the scalar potential, using the method introduced in Refs. [23, 24] has been presented in full detail in Ref. [45].

2.3.2 The set with three scalar triplets

This set makes use of the two scalar Higgs fields of the economical model, plus the extra one

$$\Phi_2(1, 3^*, -1/3) = \begin{pmatrix} \phi_2^- \\ \phi_2^0 \\ \phi_2'^0 \\ \phi_2 \end{pmatrix}, \text{ with VEV: } \langle \Phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \\ V_2 \end{pmatrix}, \quad (4)$$

where we have assumed that all the five electrically neutral components do acquire non-zero VEV. This set of three scalar Higgs fields, with the vacuum aligned such that $V_1 = v_2 = 0$, was used for the first time in Refs. [12, 11]. The set has the advantage of being able to produce tree level masses for all the Fermion fields (which is true even for the particular alignment used in the original papers), but it can not completely avoid the presence of FCNC at tree level, coming from the scalar sector.

Notice that the VEV $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$ generate masses for the exotic quarks and the new heavy gauge bosons, while VEV $\langle \Phi_3 \rangle$ generates masses for ordinary fermions and for the SM gauge bosons. To keep the model consistent with low energy phenomenology, in this work we will use $\langle \Phi_3 \rangle \neq 0$ and the hierarchy

$$V_1, V_2 \gg v_1, v_2, v_3, \quad (5)$$

except for those cases when V_1 or V_2 are zero, when the hierarchy becomes

$$V_i \gg v_1, v_2, v_3; \quad i = 1, 2. \quad (6)$$

(Taking $\langle \Phi_3 \rangle = 0$ implies that several fermion fields remain massless.)

2.3.3 The extended scalar set

Introduced in the literature in Refs. [12], it consists of four scalar triplets: Φ_1 , Φ_2 and Φ_3 as above with the vacuum aligned such that $V_1 = v_2 = 0$ as in the original papers, plus a new scalar Higgs field

$$\Phi_4(1, 3^*, -1/3) = \begin{pmatrix} \phi_2^- \\ \phi_2^0 \\ \phi_2'^0 \\ \phi_2 \end{pmatrix}, \text{ with VEV: } \langle \Phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ v_4 \end{pmatrix}, \quad (7)$$

with the hierarchy $v_1 \sim v_3 \sim v_4 \ll V_2 \sim 1$ TeV. This set of four scalar fields combined with a convenient discrete symmetry [12], is able to generate several see-saw mechanisms, the basis of a consistent Fermion mass spectrum, and avoid, at the same time, tree level FCNC coming from the scalar sector.

In what follows we are going to use these scalar sectors to deal with the scalar potential built from them.

3 Minimal Scalar Sector of 331 Models Without Exotic Electric Charges

Extensions of the Standard Model based on the local gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ contain, in general, a scalar sector quite complicated to be analyzed in detail. For this type of models, three Higgs triplets, and in some cases one additional Higgs sextet are used, in order to break the symmetry and provide at the same time with masses to the fermion fields of each model [19].

Among the 331 models with the simplest scalar sector are the ones proposed for the first time in Ref. [16] and further analyzed in Refs. [20] (they make use of only two scalar Higgs field triplets). This class of models include eight different three-family models where the Higgs scalar fields, the gauge-boson sector and the fermion field representations are restricted to particles without exotic electric charges [8, 16]. Because of their minimal content of Higgs scalar fields they are named in the literature ‘‘economical 331 models’’.

3.1 The scalar sector

If we pretend to use the simplest $SU(3)_L$ representations in order to break the symmetry, at least two complex scalar triplets, equivalent to twelve real scalar fields, are required. For $b = 1/2$ there are just two Higgs scalars (together with their complex conjugates) which may develop nonzero Vacuum Expectation Values (VEV); they are $\phi_1(1, 3^*, -1/3)^T =$

$(\phi_1^-, \phi_1^{\prime 0}, \phi_1^0)$ with VEV $\langle \phi_1 \rangle^T = (0, v_1, V)$ and $\phi_2(1, 3^*, 2/3)^T = (\phi_2^0, \phi_2^+, \phi_2^{\prime +})$ with VEV $\langle \phi_2 \rangle^T = (v_2, 0, 0)$. As we will see ahead, to reach consistency with phenomenology we must have the hierarchy $V > v_1 \sim v_2$.

Our aim is to break the symmetry in one single step

$$SU(3)_c \otimes SU(3)_L \otimes U(1)_X \longrightarrow SU(3)_c \otimes U(1)_Q$$

which implies the existence of eight Goldstone bosons included in the scalar sector of the theory [17]. For the sake of simplicity we assume that the VEV are real. This means that the CP violation through the scalar exchange is not considered in this work. Now, for convenience in reading we rewrite the expansion of the scalar fields which acquire VEV as:

$$\phi_1^0 = V + \frac{H_{\phi_1}^0 + iA_{\phi_1}^0}{\sqrt{2}}, \quad \phi_1^{\prime 0} = v_1 + \frac{H_{\phi_1}^{\prime 0} + iA_{\phi_1}^{\prime 0}}{\sqrt{2}}, \quad \phi_2^0 = v_2 + \frac{H_{\phi_2}^0 + iA_{\phi_2}^0}{\sqrt{2}}. \quad (8)$$

In the literature, a real part H is called a CP-even scalar and an imaginary part A a CP-odd scalar or pseudoscalar field.

Now, the most general potential which includes ϕ_1 and ϕ_2 can then be written in the following form:

$$V(\phi_1, \phi_2) = \mu_1^2 \phi_1^\dagger \phi_1 + \mu_2^2 \phi_2^\dagger \phi_2 + \lambda_1 (\phi_1^\dagger \phi_1)^2 + \lambda_2 (\phi_2^\dagger \phi_2)^2 + \lambda_3 (\phi_1^\dagger \phi_1) (\phi_2^\dagger \phi_2) + \lambda_4 (\phi_1^\dagger \phi_2) (\phi_2^\dagger \phi_1). \quad (9)$$

Requiring that in the shifted potential $V(\phi_1, \phi_2)$, the linear terms in fields must be absent, we get in the tree-level approximation the following constraint equations:

$$\begin{aligned} \mu_1^2 + 2\lambda_1(v_1^2 + V^2) + \lambda_3 v_2^2 &= 0, \\ \mu_2^2 + \lambda_3(v_1^2 + V^2) + 2\lambda_2 v_2^2 &= 0. \end{aligned} \quad (10)$$

The analysis to the former equations shows that they are related to a minimum of the scalar potential with the value

$$V_{min} = -v_2^4 \lambda_2 - (v_1^2 + V^2)[(v_1^2 + V^2)\lambda_1 + v_2^2 \lambda_3] = V(v_1, v_2, V), \quad (11)$$

where $V(v_1 = 0, v_2, V) > V(v_1 \neq 0, v_2, V)$, implying that $v_1 \neq 0$ is preferred.

Substituting Eqs.(8) and (10) in Eq.(9) we get the following mass matrices:

3.1.1 Spectrum in the scalar neutral sector

In the $(H_{\phi_1}^0, H_{\phi_2}^0, H_{\phi_1}^{\prime 0})$ basis, the square mass matrix can be calculated using $M_{ij}^2 = 2 \frac{\partial^2 V(\phi_1, \phi_2)}{\partial H_{\phi_i}^0 \partial H_{\phi_j}^0}$. After imposing the constraints in Eq.(10) we get:

$$M_H^2 = 2 \begin{pmatrix} 2\lambda_1 V^2 & \lambda_3 v_2 V & 2\lambda_1 v_1 V \\ \lambda_3 v_2 V & 2\lambda_2 v_2^2 & \lambda_3 v_1 v_2 \\ 2\lambda_1 v_1 V & \lambda_3 v_1 v_2 & 2\lambda_1 v_1^2 \end{pmatrix}, \quad (12)$$

which has zero determinant, providing us with a Goldstone boson G_1 and two physical massive neutral scalar fields H_1 and H_2 with masses

$$M_{H_1, H_2}^2 = 2(v_1^2 + V^2)\lambda_1 + 2v_2^2 \lambda_2 \pm 2\sqrt{[(v_1^2 + V^2)\lambda_1 + v_2^2 \lambda_2]^2 + v_2^2(v_1^2 + V^2)(\lambda_3^2 - 4\lambda_1 \lambda_2)},$$

where real lambdas produce positive masses for the scalars only if $\lambda_1 > 0$ and $4\lambda_1 \lambda_2 > \lambda_3^2$ (which implies $\lambda_2 > 0$).

We may see from the former equations that in the limit $V > v_1 \sim v_2$, and for lambdas of order one, there is a neutral Higgs scalar with a mass of order V and other one with a mass of the order of $v_1 \sim v_2$, which may be identified with the SM scalar as we will see ahead.

The physical fields are related to the scalars in the weak basis by the linear transformation:

$$\begin{pmatrix} H_{\phi_1}^0 \\ H_{\phi_2}^0 \\ H_{\phi_1}^{\prime 0} \end{pmatrix} = \begin{pmatrix} \frac{v_2 V}{S_1} & \frac{v_2 V}{S_2} & -\frac{v_1}{\sqrt{v_1^2 + V^2}} \\ \frac{M_{H_1}^2 - 4(v_1^2 + V^2)\lambda_1}{2S_1 \lambda_3} & -\frac{(M_{H_1}^2 - 4v_2^2 \lambda_2)}{2S_2 \lambda_3} & 0 \\ \frac{v_1 v_2}{S_1} & \frac{v_1 v_2}{S_2} & \frac{V}{\sqrt{v_1^2 + V^2}} \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ G_1 \end{pmatrix},$$

where we have defined $S_1 = \sqrt{v_2^2(v_1^2 + V^2) + (M_{H_1}^2 - 4(v_1^2 + V^2)\lambda_1)^2/4\lambda_3^2}$ and $S_2 = \sqrt{v_2^2(v_1^2 + V^2) + (M_{H_1}^2 - 4v_2^2 \lambda_2)^2/4\lambda_3^2}$.

3.1.2 Spectrum in the pseudoscalar neutral sector

The analysis shows that $V(\phi_1, \phi_2)$ in Eq.(9), when expanded around the most general vacuum given by Eqs.(8) and using the constraints in Eq.(10), does not contain pseudoscalar fields $A_{\phi_i}^0$. This allows us to identify another three Goldstone bosons $G_2 = A_{\phi_1}^0$, $G_3 = A_{\phi_2}^0$ and $G_4 = A_{\phi_1}^{\prime 0}$.

3.1.3 Spectrum in the charged scalar sector

In the basis $(\phi_1^+, \phi_2^+, \phi_2^{\prime +})$ the square mass matrix is given by

$$M_+^2 = 2\lambda_4 \begin{pmatrix} v_2^2 & v_1 v_2 & v_2 V \\ v_1 v_2 & v_1^2 & v_1 V \\ v_2 V & v_1 V & V^2 \end{pmatrix}, \quad (13)$$

which has two eigenvalues equal to zero equivalent to four Goldstone bosons (G_5^\pm, G_6^\pm) and two physical charged Higgs scalars with large masses given by $\lambda_4(v_1^2 + v_2^2 + V^2)$, with the new constraint $\lambda_4 > 0$.

Our analysis shows that, after symmetry breaking, the original twelve degrees of freedom in the scalar sector have become eight Goldstone bosons (four electrically neutral and four charged), and four physical scalar fields, two neutrals (one of them the SM Higgs scalar) and two charged ones. The eight Goldstone bosons must be swallow up by eight gauge fields as we will see in the next section.

3.2 The Gauge boson sector

For $b = 1/2$, the nine gauge bosons in $SU(3)_L \otimes U(1)_X$ when acting on left-handed triplets can be arranged in the following convenient way:

$$\mathcal{A}_\mu = \frac{1}{2}g\lambda_{\alpha L}A_\mu^\alpha + g'XB_\mu I_3 = \frac{g}{\sqrt{2}} \begin{pmatrix} Y_{1\mu}^0 & W_\mu^+ & K_\mu^+ \\ W_\mu^- & Y_{2\mu}^0 & K_\mu^0 \\ K_\mu^- & \overline{K}_\mu^0 & Y_{3\mu}^0 \end{pmatrix},$$

where $Y_{1\mu}^0 = A_{3\mu}/\sqrt{2} + A_{8\mu}/\sqrt{6} + \sqrt{2}(g'/g)XB_\mu$, $Y_{2\mu}^0 = -A_{3\mu}/\sqrt{2} + A_{8\mu}/\sqrt{6} + \sqrt{2}(g'/g)XB_\mu$, and $Y_{3\mu}^0 = -2A_{8\mu}/\sqrt{6} + \sqrt{2}(g'/g)XB_\mu$; X being the hypercharge value of the given left-handed triplet (for example $-1/3$ and $2/3$ for ϕ_1 and ϕ_2 respectively).

After breaking the symmetry with $\langle \phi_i \rangle$, $i = 1, 2$, and using for the covariant derivative for triplets $D^\mu = \partial^\mu - i\mathcal{A}^\mu$, we get the following mass terms in the gauge boson sector:

3.2.1 Spectrum in the charged gauge boson sector

In the basis (K_μ^\pm, W_μ^\pm) the square mass matrix produced is

$$M_\pm^2 = \frac{g^2}{2} \begin{pmatrix} (V^2 + v_2^2) & v_1 V \\ v_1 V & (v_1^2 + v_2^2) \end{pmatrix}. \quad (14)$$

The former symmetric matrix give us the masses $M_{W'}^2 = g^2 v_2^2/2$ and $M_{K'}^2 = g^2(v_1^2 + v_2^2 + V^2)/2$, related to the physical fields $W'_\mu = \eta(v_1 K_\mu - V W_\mu)$, and $K'_\mu = \eta(V K_\mu + v_1 W_\mu)$ associated with the known charged weak current $W_\mu^{\prime \pm}$, and with a new one $K_\mu^{\prime \pm}$ predicted in the context of this model ($\eta^{-2} = v_1^2 + V^2$ is a normalization factor). From the experimental value $M_{W'} = 80.419 \pm 0.056$ GeV [18] we obtain $v_2 \simeq 174$ GeV as in the SM.

3.2.2 Spectrum in the neutral gauge boson sector

For the five electrically neutral gauge bosons we get first, that the imaginary part of $K_\mu^0 = (K_{\mu R}^0 + iK_{\mu I}^0)/\sqrt{2}$ decouples from the other four electrically neutral gauge bosons, acquiring a mass $M_{K^0}^2 = g^2(v_1^2 + V^2)/2$. Then, in the basis $(B^\mu, A_3^\mu, A_8^\mu, K_R^{0\mu})$, the following squared mass matrix is obtained:

$$M_0^2 = \begin{pmatrix} \frac{g'^2}{9}(v_1^2 + V^2 + 4v_2^2) & -\frac{gg'}{6}(v_1^2 + 2v_2^2) & -\frac{gg'}{3\sqrt{3}}(V^2 + v_2^2 - v_1^2/2) & gg'v_1V/3 \\ -\frac{gg'}{6}(v_1^2 + 2v_2^2) & g^2(v_1^2 + v_2^2)/4 & \frac{g^2}{4\sqrt{3}}(v_2^2 - v_1^2) & -g^2v_1V/4 \\ -\frac{gg'}{3\sqrt{3}}(V^2 + v_2^2 - v_1^2/2) & \frac{g^2}{4\sqrt{3}}(v_2^2 - v_1^2) & \frac{g^2}{12}(v_1^2 + v_2^2 + 4V^2) & -g^2v_1V/(4\sqrt{3}) \\ gg'v_1V/3 & -g^2v_1V/4 & -g^2v_1V/(4\sqrt{3}) & g^2(v_1^2 + V^2)/4 \end{pmatrix}$$

This matrix has determinant equal to zero which implies that there is a zero eigenvalue associated to the photon field with eigenvector

$$A^\mu = S_W A_3^\mu + C_W \left[\frac{T_W}{\sqrt{3}} A_8^\mu + (1 - T_W^2/3)^{1/2} B^\mu \right], \quad (15)$$

where $S_W = \sqrt{3}g'/\sqrt{3g^2 + 4g'^2}$ and C_W are the sine and cosine of the electroweak mixing angle ($T_W = S_W/C_W$). Orthogonal to the photon field A^μ we may define other two fields

$$\begin{aligned} Z^\mu &= C_W A_3^\mu - S_W \left[\frac{T_W}{\sqrt{3}} A_8^\mu + (1 - T_W^2/3)^{1/2} B^\mu \right] \\ Z'^\mu &= -(1 - T_W^2/3)^{1/2} A_8^\mu + \frac{T_W}{\sqrt{3}} B^\mu, \end{aligned} \quad (16)$$

where Z^μ corresponds to the neutral current of the SM and Z'^μ is a new weak neutral current predicted for these models.

We may also identify the gauge boson Y^μ associated with the SM hypercharge in $U(1)_Y$ as:

$$Y^\mu = \left[\frac{T_W}{\sqrt{3}} A_8^\mu + (1 - T_W^2/3)^{1/2} B^\mu \right].$$

In the basis $(Z'^\mu, Z^\mu, K_R^{0\mu})$ the mass matrix for the neutral sector reduces to:

$$\frac{g^2}{4C_W^2} \begin{pmatrix} \delta^2(v_1^2 C_{2W}^2 + v_2^2 + 4V^2 C_W^4) & \delta(v_1^2 C_{2W} - v_2^2) & \delta C_W v_1 V \\ \delta(v_1^2 C_{2W} - v_2^2) & v_1^2 + v_2^2 & -C_W v_1 V \\ \delta C_W v_1 V & -C_W v_1 V & C_W^2(v_1^2 + V^2) \end{pmatrix}, \quad (17)$$

where $C_{2W} = C_W^2 - S_W^2$ and $\delta = (4C_W^2 - 1)^{-1/2}$. The eigenvectors and eigenvalues of this matrix are the physical fields and their masses. In the approximation $v_1 = v_2 \equiv v \ll V$ and using $q \equiv v^2/V^2$ as an expansion parameter we get up, to first order in q , the following eigenvalues:

$$\begin{aligned} M_{Z_1}^2 &\approx \frac{1}{2} g^2 C_W^{-2} v^2 (1 - q T_W^4), \\ M_{Z_2}^2 &\approx \frac{g^2 V^2}{1 + 2C_{2W}} [1 + C_{2W} - q(S_{2W}^2 + C_{2W}^{-1})/2C_W^2], \\ M_{K_R^0}^2 &\approx g^2 V^2 [1 + q(1 + C_{2W}^{-1})]. \end{aligned}$$

So we have a neutral current associated to a gauge boson Z_1^0 , related to a mass scale $v \simeq 174$ GeV, which may be identified with the known experimental neutral current as we will see in what follows, and two new electrically neutral currents associated to a large mass scale $V \gg v$.

The former is the way how the eight would be Goldstone bosons are absorbed by the longitudinal components of the eight massive gauge bosons ($W'^{\pm}, K'^{\pm}, K_I^0, K_R^0, Z_1^0$ and Z_2^0) as expected.

From the expressions for $M_{W'}$ and M_{Z_1} we obtain $\rho_0 = M_{W'}^2/(M_{Z_1}^2 C_W^2) \approx 1 + T_W^4 q^2$, and the global fit for $\rho_0 = 1.0012_{-0.0014}^{+0.0023}$ [41] provides us with the lower limit $V \geq 1.3$ TeV (where we are using for $S_W^2 = 0.23113$ [18]). This result justifies the existence of the expansion parameter $q \leq 0.01$ which sets the scale of new physics, together with the hierarchy $V > v_1 \sim v_2$.

3.3 Higgs-SM gauge boson couplings

In order to identify the considered above Higgs bosons with the one in the SM, in this section we present the couplings of the two neutral scalar fields H_1 and H_2 from section 4 with the physical gauge bosons W'^{\pm} and Z_1^0 ; then we take the limit $V \gg v = v_1 = v_2$ which produces the couplings of the physical scalars H_1 and H_2 with the SM gauge bosons W^{\pm} and Z^0 .

When the algebra gets done we obtain the following trilinear couplings, provided $\lambda_3 < 0$:

$$\begin{aligned}
g(W'W'H_1) &= \frac{g^2 v_2 [M_{H_1}^2 - 4(v_1^2 + V^2)\lambda_1]}{2\sqrt{2}S_1\lambda_3} \xrightarrow{V \gg v} \frac{g^2 v_2^2 \lambda_3}{2\sqrt{2}\lambda_1 V} \\
g(W'W'H_2) &= \frac{g^2 v_2 (4v_2^2 \lambda_2 - M_{H_1}^2)}{2\sqrt{2}S_2\lambda_3} \xrightarrow{V \gg v} \frac{g^2 v_2}{\sqrt{2}} \\
g(Z_1^0 Z_1^0 H_1) &= \frac{g^2 v_1}{S_1} \left[\frac{M_{H_1}^2 - 4(v_1^2 + V^2)\lambda_1}{4\sqrt{2}C_W^2 \lambda_3} + q \frac{v_1^2 (\lambda_1 - \lambda_3 S_W^2)}{8\sqrt{2}C_W^2 \lambda_1} + \dots \right] \\
&\xrightarrow{V \gg v} \frac{g^2 v_1^2 \lambda_3}{4\sqrt{2}\lambda_1 C_W^2 V} \\
g(Z_1^0 Z_1^0 H_2) &= \frac{g^2 v_1 V^2}{S_2} \left[-\frac{\lambda_1}{\sqrt{2}\lambda_3 C_W^2} + q \frac{4\lambda_1 \lambda_2 - \lambda_3^2 + 2\lambda_1^2 (T_W^2 C_W^{-2} - 2)}{4\sqrt{2}C_W^2 \lambda_1 \lambda_3} + \dots \right] \\
&\xrightarrow{V \gg v} \frac{g^2 v_1}{2\sqrt{2}C_W^2},
\end{aligned}$$

where $g(W'W'H_i^0)$, $i = 1, 2$ are exact expressions and $g(Z_1^0 Z_1^0 H_i)$ are expansions in the parameter q up to first order.

The quartic couplings are determined to be:

$$\begin{aligned}
g(W'W'H_1 H_1) &= \frac{g^2 [M_{H_1}^2 - 4(v_1^2 + V^2)\lambda_1]^2}{16S_1^2 \lambda_3^2} \xrightarrow{V \gg v} \frac{g^2 v_2^2 \lambda_3^2}{16\lambda_1^2 V^2} \\
g(W'W'H_2 H_2) &= \frac{g^2 (M_{H_1}^2 - 4v_2^2 \lambda_2)^2}{16S_2^2 \lambda_3^2} \xrightarrow{V \gg v} \frac{g^2}{4} \\
g(Z_1^0 Z_1^0 H_1 H_1) &= \frac{g^2}{S_1^2} \left[\frac{[M_{H_1}^2 - 4(v_1^2 + V^2)\lambda_1]^2}{32C_W^2 \lambda_3^2} + q \frac{2v_1^4 \lambda_3^2 - S_W^2 [M_{H_1}^2 - 4(v_1^2 + V^2)\lambda_1]^2}{64C_W^6 \lambda_3^2} + \dots \right] \\
&\xrightarrow{V \gg v} \frac{g^2 v_1^2 \lambda_3^2}{32\lambda_1^2 C_W^2 V^2} \\
g(Z_1^0 Z_1^0 H_2 H_2) &= \frac{g^2 V^4}{S_2^2} \left[\frac{\lambda_1^2}{2\lambda_3^2 C_W^2} + q \frac{\lambda_3^2 - 4\lambda_1 \lambda_2 + \lambda_1^2 (4 - C_W^{-2} T_W^2)}{4C_W^2 \lambda_3^2} + \dots \right] \\
&\xrightarrow{V \gg v} \frac{g^2}{8C_W^2}
\end{aligned}$$

where as before $g(W'W'H_i^0 H_i^0)$, $i = 1, 2$ are exact expressions and $g(Z_1^0 Z_1^0 H_i^0 H_i^0)$ are expansions in the parameter q up to first order.

As can be seen, in the limit $V > v_1 \sim v_2$ the couplings $g(W'W'H_2)$, $g(Z_1^0 Z_1^0 H_2)$, $g(W'W'H_2 H_2)$ and $g(Z_1^0 Z_1^0 H_2 H_2)$ coincide with those in the SM as far as $\lambda_3 < 0$. This gives additional support to the hierarchy $V > v_1 \sim v_2$.

Summarizing, from the couplings of the SM gauge bosons with the physical Higgs scalars we can conclude, as anticipated before, that the scalar H_2 can be identified with the SM neutral Higgs particle, and that Z_1^0 can be associated with the known neutral current of the SM (more support to this last statement is presented in the Appendix A).

4 Stability of the Scalar Potential and Symmetry Breaking in the Economical 331 Model

A simple extension of the standard model (SM) consists of adding to the model a second Higgs scalar doublet [21], defining in this way the so-called two Higgs doublet model (THDM). The different ways how the two Higgs scalar doublets couple to the fermion sector, define the different versions of this extension[21, 22]. Many gauge group extensions of the SM have the THDM as an effective low energy theory (in this regard see the papers in [22] and references therein). In these extensions one intermediate step in the symmetry breaking chain leads to the $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ gauge theory with two Higgs doublets in one of its several versions.

A novel method for a detailed analysis of the scalar potential in the most general THDM was presented in Refs. [23, 24] where by using powerful algebraic techniques, the authors studied in detail the stationary points of the scalar potential. This allowed them to give, in a very concise way, clear criteria for the stability of the scalar potential and for the correct electroweak symmetry breaking pattern. By using different approaches, the authors in Refs. [42] reached also interesting conclusions for the scalar potential of the THDM, some of them related to the ones presented in Ref.[23]. In the present work we use this approach to analyse the scalar sector of the economical 331 model. No relevant new additional conditions are necessary to be imposed in order to implement the method in this last case.

One important advantage of the economical 331 model, compared with the THDM, concerns the Higgs potential. The 14 parameters required to describe the most general potential for the second case, should be compared with the six parameters required in the economical 331 model. For the THDM this is associated to the fact that the two Higgs doublets have the same $U(1)$ hypercharge [21, 22]. In the economical 331 model, by contrast, the two scalar triplets have different $U(1)_X$ hypercharges so that the most general Higgs potential shows itself in a very simple form.

In this work we deduce constraints on the parameters of the economical 3-3-1 scalar potential coming from the stability and from the electroweak symmetry breaking conditions. The stability of an scalar potential at the classical level, which is fulfilled when it is bounded from below, is a necessary condition in order to have a sound theory. The global minimum of the potential is found by determining its stationary points. Some of our results agree with those already presented in Refs. [16, 20]. Our study extends thus the method proposed in Refs. [23, 24] to the economical 331 model, where the results are very concise and should, in principle, be used as a guide in order to extend the method to other situations.

4.1 A review of the method

In this section, and following Refs. [23] and [24], we review a new algebraic approach used to determine the global minimum of the Higgs scalar potential, its stability, and the spontaneous symmetry breaking from $SU(2)_L \otimes U(1)_Y$ down to $U(1)_{em}$, in the extension of the SM known as the THDM, where φ_1 and φ_2 stand for two Higgs scalar field doublets with identical quantum numbers

Stability and the stationary points of the potential can be analyzed in terms of four real constants given by

$$K_0 = \sum_{i=1,2} \varphi_i^\dagger \varphi_i, \quad K_a = \sum_{i,j=1,2} (\varphi_i^\dagger \varphi_j) \sigma_{ij}^a, \quad (a = 1, 2, 3). \quad (18)$$

where $\sigma^a (a = 1, 2, 3)$ are the Pauli spin matrices. The four vector (K_0, \mathbf{K}) must lie on or inside the forward *light cone*, that is

$$K_0 \geq 0, \quad K_0^2 - \mathbf{K}^2 \geq 0. \quad (19)$$

Then the positive and hermitian 2×2 matrix

$$\underline{K} = \begin{pmatrix} \varphi_1^\dagger \varphi_1 & \varphi_2^\dagger \varphi_1 \\ \varphi_1^\dagger \varphi_2 & \varphi_2^\dagger \varphi_2 \end{pmatrix} \quad (20)$$

may be written as

$$\underline{K}_{ij} = \frac{1}{2}(K_0 \delta_{ij} + K_a \sigma_{ij}^a). \quad (21)$$

Inverting Eq. (18) it is obtained

$$\begin{aligned} \varphi_1^\dagger \varphi_1 &= (K_0 + K_3)/2, & \varphi_1^\dagger \varphi_2 &= (K_1 + iK_2)/2, \\ \varphi_2^\dagger \varphi_2 &= (K_0 - K_3)/2, & \varphi_2^\dagger \varphi_1 &= (K_1 - iK_2)/2. \end{aligned} \quad (22)$$

The most general $SU(2)_L \otimes U(1)_Y$ invariant Higgs scalar potential can thus be expressed as

$$V(\varphi_1, \varphi_2) = V_2 + V_4, \quad (23a)$$

$$V_2 = \xi_0 K_0 + \xi_a K_a, \quad (23b)$$

$$V_4 = \eta_{00} K_0^2 + 2K_0 \eta_a K_a + K_a \eta_{ab} K_b, \quad (23c)$$

where the 14 independent parameters ξ_0 , ξ_a , η_{00} , η_a and $\eta_{ab} = \eta_{ba}$ are real. Subsequently, it is defined $\mathbf{K} = (K_a)$, $\boldsymbol{\xi} = (\xi_a)$, $\boldsymbol{\eta} = (\eta_a)$ and $E = (\eta_{ab})$.

4.1.1 Stability

From (23), for $K_0 > 0$ and defining $\mathbf{k} = \mathbf{K}/K_0$, it is obtained

$$V_2 = K_0 J_2(\mathbf{k}), \quad J_2(\mathbf{k}) := \xi_0 + \boldsymbol{\xi}^T \mathbf{k}, \quad (24)$$

$$V_4 = K_0^2 J_4(\mathbf{k}), \quad J_4(\mathbf{k}) := \eta_{00} + 2\boldsymbol{\eta}^T \mathbf{k} + \mathbf{k}^T E \mathbf{k}, \quad (25)$$

where the functions $J_2(\mathbf{k})$ and $J_4(\mathbf{k})$ on the domain $|\mathbf{k}| \leq 1$ have been introduced. For the potential to be stable, it must be bounded from below. The stability is determined by the behavior of V in the limit $K_0 \rightarrow \infty$, and hence by the signs of $J_4(\mathbf{k})$ and $J_2(\mathbf{k})$ in (24) and (25). In this analysis only the *strong* criterion for stability is considered, that is, the stability is determined solely by the V quartic terms

$$J_4(\mathbf{k}) > 0 \quad \text{for all } |\mathbf{k}| \leq 1. \quad (26)$$

To assure that $J_4(\mathbf{k})$ is always positive, it is sufficient to consider its value for all its stationary points on the domain $|\mathbf{k}| < 1$, and for all the stationary points on the boundary $|\mathbf{k}| = 1$. This leads to bounds on η_{00} , η_a and η_{ab} , which parameterize the quartic term V_4 of the potential.

The regular solutions for the two cases $|\mathbf{k}| < 1$ and $|\mathbf{k}| = 1$ lead to

$$f(u) = u + \eta_{00} - \boldsymbol{\eta}^T (E - u)^{-1} \boldsymbol{\eta}, \quad (27)$$

$$f'(u) = 1 - \boldsymbol{\eta}^T (E - u)^{-2} \boldsymbol{\eta}, \quad (28)$$

so that for all ‘‘regular’’ stationary points \mathbf{k} of $J_4(\mathbf{k})$ both

$$f(u) = J_4(\mathbf{k})|_{stat}, \quad \text{and} \quad (29)$$

$$f'(u) = 1 - \mathbf{k}^2 \quad (30)$$

hold, where $u = 0$ must be set for the solution with $|\mathbf{k}| < 1$. There are stationary points of $J_4(\mathbf{k})$ with $|\mathbf{k}| < 1$ and $|\mathbf{k}| = 1$ exactly if $f'(0) > 0$ and $f'(u) = 0$, respectively, and the value of $J_4(\mathbf{k})$ is then given by $f(u)$.

In a basis where $E = \text{diag}(\mu_1, \mu_2, \mu_3)$ it is obtained

$$f(u) = u + \eta_{00} - \sum_{a=1}^3 \frac{\eta_a^2}{\mu_a - u}, \quad (31)$$

$$f'(u) = 1 - \sum_{a=1}^3 \frac{\eta_a^2}{(\mu_a - u)^2}. \quad (32)$$

The derivative $f'(u)$ has at most six zeros. Notice that there are no exceptional solutions if in this basis all three components of $\boldsymbol{\eta}$ are different from zero.

Consider now the functions $f(u)$ and $f'(u)$ and denote by I

$$I = \{u_1, \dots, u_n\} \quad (33)$$

the set of values u_j for which $f'(u_j) = 0$. Add $u_k = 0$ to I if $f'(0) > 0$. Consider then the eigenvalues μ_a ($a = 1, 2, 3$) of E . Add those μ_a to I where $f(\mu_a)$ is finite and $f'(\mu_a) \geq 0$. Then $n \leq 10$. The values of the function $J_4(\mathbf{k})$ at its stationary points are given by

$$J_4(\mathbf{k})|_{stat} = f(u_i) \quad (34)$$

with $u_i \in I$. In Appendix B we show that the stationary point in I having the smallest value, will produce the smallest value of $J_4(\mathbf{k})$ in the domain $|\mathbf{k}| \leq 1$. We now state the theorem.

Theorem 1. *The global minimum of the function $J_4(\mathbf{k})$, in the domain $|\mathbf{k}| \leq 1$, is given and guaranteed by the stationary point of the set I with the smallest value.*

This result guarantees strong stability if $f(u) > 0$, where u is the smallest value of I . The potential is unstable if we have $f(u) < 0$. If $f(u) = 0$ we have to consider in addition $J_2(\mathbf{k})$ in order to decide on the stability of the potential.

4.1.2 Location of stationary points and criteria for electroweak symmetry breaking

The next step after the stability analysis in the preceding section has been done is to determine the location of the stationary points of the potential, since among these points the local and global minima are found. To this end is defined

$$\tilde{\mathbf{K}} = \begin{pmatrix} K_0 \\ \mathbf{K} \end{pmatrix}, \quad \tilde{\boldsymbol{\xi}} = \begin{pmatrix} \xi_0 \\ \boldsymbol{\xi} \end{pmatrix}, \quad \tilde{E} = \begin{pmatrix} \eta_{00} & \boldsymbol{\eta}^T \\ \boldsymbol{\eta} & E \end{pmatrix}. \quad (35)$$

In this notation the potential (23) reads

$$V = \tilde{\mathbf{K}}^T \tilde{\boldsymbol{\xi}} + \tilde{\mathbf{K}}^T \tilde{E} \tilde{\mathbf{K}} \quad (36)$$

and is defined on the domain

$$\tilde{\mathbf{K}}^T \tilde{g} \tilde{\mathbf{K}} \geq 0, \quad K_0 \geq 0, \quad (37)$$

with

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \quad (38)$$

For the discussion of the stationary points of V , three different cases must be distinguished: $\tilde{\mathbf{K}} = 0$, $K_0 > |\mathbf{K}|$, which are the solutions inside the forward light cone, and $K_0 = |\mathbf{K}| > 0$, which are the solutions on the forward light cone.

The trivial configuration $\tilde{\mathbf{K}} = 0$ is a stationary point of the potential with $V = 0$, as a direct consequence of the definitions. The stationary points of V in the inner part of the domain, $K_0 > |\mathbf{K}|$, are given by

$$\tilde{E} \tilde{\mathbf{K}} = -\frac{1}{2} \tilde{\boldsymbol{\xi}}, \quad \text{with} \quad \tilde{\mathbf{K}}^T \tilde{g} \tilde{\mathbf{K}} > 0 \quad \text{and} \quad K_0 > 0. \quad (39)$$

The stationary points of V on the domain boundary $K_0 = |\mathbf{K}| > 0$ are stationary points of the function

$$\tilde{F}(\tilde{\mathbf{K}}, w) := V - w \tilde{\mathbf{K}}^T \tilde{g} \tilde{\mathbf{K}}, \quad (40)$$

where w is a Lagrange multiplier. The relevant stationary points of \tilde{F} are given by

$$(\tilde{E} - w \tilde{g}) \tilde{\mathbf{K}} = -\frac{1}{2} \tilde{\boldsymbol{\xi}}, \quad \text{with} \quad \tilde{\mathbf{K}}^T \tilde{g} \tilde{\mathbf{K}} = 0 \quad \text{and} \quad K_0 > 0. \quad (41)$$

For *any* stationary point the potential is given by

$$V|_{stat} = \frac{1}{2} \tilde{\mathbf{K}}^T \tilde{\boldsymbol{\xi}} = -\tilde{\mathbf{K}}^T \tilde{E} \tilde{\mathbf{K}}. \quad (42)$$

Similarly to the stability analysis in Sec. 4.1.1, a unified description for the regular stationary points of V with $K_0 > 0$ for both $|\mathbf{K}| < K_0$ and $|\mathbf{K}| = K_0$ can be used by defining the functions

$$\tilde{f}(w) = -\frac{1}{4} \tilde{\boldsymbol{\xi}}^T (\tilde{E} - w \tilde{g})^{-1} \tilde{\boldsymbol{\xi}}, \quad (43)$$

$$\tilde{f}'(w) = -\frac{1}{4} \tilde{\boldsymbol{\xi}}^T (\tilde{E} - w \tilde{g})^{-1} \tilde{g} (\tilde{E} - w \tilde{g})^{-1} \tilde{\boldsymbol{\xi}}. \quad (44)$$

Denoting the first component of $\tilde{\mathbf{K}}(w)$ as $K_0(w)$ the following theorem holds.

Theorem 2. *The stationary points of the potential are given by*

(Ia) $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}(0)$ if $\tilde{f}'(0) < 0$, $K_0(0) > 0$ and $\det \tilde{E} \neq 0$,

(Ib) solutions $\tilde{\mathbf{K}}$ of (39) if $\det \tilde{E} = 0$,

(IIa) $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}(w)$ for w with $\det(\tilde{E} - w \tilde{g}) \neq 0$, $\tilde{f}'(w) = 0$ and $K_0(w) > 0$,

(IIb) solutions $\tilde{\mathbf{K}}$ of (41) for w with $\det(\tilde{E} - w \tilde{g}) = 0$,

(III) $\tilde{\mathbf{K}} = 0$.

In what follows it is assumed that the potential is stable. For parameters fulfilling $\xi_0 \geq |\boldsymbol{\xi}|$, this immediately implies $J_2(\mathbf{k}) \geq 0$ and hence, from the strong condition (26), $V > 0$ for all $\tilde{\mathbf{K}} \neq 0$. Therefore for these parameters the global minimum is at $\tilde{\mathbf{K}} = 0$. This leads to the requirement

$$\xi_0 < |\boldsymbol{\xi}|. \quad (45)$$

Also, it is obtained

$$\left. \frac{\partial V}{\partial K_0} \right|_{\substack{\mathbf{k} \text{ fixed,} \\ K_0=0}} = \xi_0 + \boldsymbol{\xi}^T \mathbf{k} < 0 \quad (46)$$

for some \mathbf{k} , i.e. the global minimum of V lies at $\tilde{\mathbf{K}} \neq 0$ with

$$V|_{min} < 0. \quad (47)$$

Firstly, consider $p_0 = |\mathbf{p}|$. From (36) and (41) it follows that

$$\left. \frac{\partial V}{\partial K_0} \right|_{\substack{\mathbf{K} \text{ fixed,} \\ \tilde{\mathbf{K}}=\tilde{\mathbf{p}}}} = \xi_0 + 2(\tilde{E}\tilde{\mathbf{p}})_0 = 2w_p p_0. \quad (48)$$

If $w_p < 0$, there are points $\tilde{\mathbf{K}}$ with $K_0 > p_0$, $\mathbf{K} = \mathbf{p}$ and lower potential in the neighborhood of $\tilde{\mathbf{p}}$, which therefore cannot be a minimum. The conclusion is that in a theory with the required electroweak symmetry breaking (EWSB) the global minimum must have a Lagrange multiplier such that $w_0 \geq 0$, and for the THDM, the global minimum lies on the stationary points of the classes (IIa) and (IIb) of theorem 2, with the largest Lagrange multiplier [23] (contrary to what happens in the analysis that follows for the economical 331 model, where the global minimum must fall on the stationary points in classes (Ia) and (Ib)).

4.1.3 The scalar sector

If we pretend to use the simplest $SU(3)_L$ representations in order to break the symmetry, at least two complex scalar triplets, equivalent to twelve real scalar fields, are required (Eq. (3)). The two Higgs scalars (together with their complex conjugates) that may develop nonzero VEV, are

$$\phi_1(1, 3^*, -1/3) = \begin{pmatrix} \phi_1^- \\ \phi_1'^0 \\ \phi_1^0 \end{pmatrix}, \quad \phi_2(1, 3^*, 2/3) = \begin{pmatrix} \phi_2^0 \\ \phi_2^+ \\ \phi_2'^+ \end{pmatrix}. \quad (49)$$

Note that, unlike the THDM, these two scalar fields have different X hypercharge. For this reason, a change of basis of the Higgs fields in this model does not have any meaning.

The most general, renormalizable and 3-3-1 invariant scalar potential can thus be written as

$$V(\phi_1, \phi_2) = \mu_1^2 \phi_1^\dagger \phi_1 + \mu_2^2 \phi_2^\dagger \phi_2 + \lambda_1 (\phi_1^\dagger \phi_1)^2 + \lambda_2 (\phi_2^\dagger \phi_2)^2 + \lambda_3 (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + \lambda_4 (\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1). \quad (50)$$

The simplicity of this potential can be appreciated by noticing first the natural absence of a trilinear scalar coupling and by counting its number of free parameters: only six.

4.1.4 The orbital variables

Following the method presented in the previous section, the potential (50) can be expressed in terms of the orbital variables K_0 , K_1 , K_2 and K_3 which, for our case, are associated to the real parameters

$$\xi_0 = \frac{1}{2}(\mu_1^2 + \mu_2^2), \quad \boldsymbol{\xi} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}(\mu_1^2 - \mu_2^2) \end{pmatrix}, \quad (51)$$

$$\eta_{00} = \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3), \quad (52)$$

$$\boldsymbol{\eta} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4}(\lambda_1 - \lambda_2) \end{pmatrix}, \quad E = \begin{pmatrix} \frac{\lambda_4}{4} & 0 & 0 \\ 0 & \frac{\lambda_4}{4} & 0 \\ 0 & 0 & \frac{1}{4}(\lambda_1 + \lambda_2 - \lambda_3) \end{pmatrix}. \quad (53)$$

4.1.5 Stability

Note that E is a diagonal matrix. Then, we can calculate the functions $f(u)$ and $f'(u)$ directly from Eqs. (31) and (32). We obtain

$$f(u) = u + \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3) - \frac{(\lambda_1 - \lambda_2)^2}{4(\lambda_1 + \lambda_2 - \lambda_3) - 16u}, \quad (54)$$

$$f'(u) = 1 - \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2 - \lambda_3 - 4u)^2}. \quad (55)$$

For $\lambda_1 \neq \lambda_2$, the solutions of $f'(u) = 0$, which determine the stationary points of $J_4(\mathbf{k})$ on the boundary $|\mathbf{k}| = 1$, lead to the Lagrange multipliers

$$u_1 = \frac{1}{4}(2\lambda_1 - \lambda_3), \quad u_2 = \frac{1}{4}(2\lambda_2 - \lambda_3). \quad (56)$$

We must add the values

$$u_3 = 0, \quad u_4 = \frac{\lambda_4}{4}, \quad (57)$$

which correspond to the stationary point inside the sphere ($|\mathbf{k}| < 1$) and the exceptional solution, respectively. So, we have the set

$$I = \left\{ u_1 = \frac{1}{4}(2\lambda_1 - \lambda_3), u_2 = \frac{1}{4}(2\lambda_2 - \lambda_3), u_3 = 0, u_4 = \frac{\lambda_4}{4} \right\}, \quad (58)$$

which contains all the possible valid solutions. Among the solutions, the smallest value corresponds to the global minimum of $J_4(\mathbf{k})$ (See Appendix B for a demonstration). Let us now consider the different possibilities.

1. $u_1 < u_2, u_3, u_4$: i.e. the global minimum occurs at u_1 . In order to have a stable potential, in the strong sense, we impose the condition

$$f(u_1) > 0 \quad \Rightarrow \quad \lambda_1 > 0. \quad (59)$$

2. $u_2 < u_1, u_3, u_4$: in this case the strong stability leads to

$$f(u_2) > 0 \quad \Rightarrow \quad \lambda_2 > 0. \quad (60)$$

3. $u_3 < u_1, u_2, u_4$ (remember $u_3 = 0$): a valid solution requires a positive value for the function (55). Let us verify it:

$$f'(0) = \frac{16u_1u_2}{(\lambda_1 + \lambda_2 - \lambda_3)^2} = \frac{4u_1u_2}{(u_1 + u_2)^2} > 0. \quad (61)$$

Imposing the strong stability condition

$$\begin{aligned} f(0) &= \frac{\lambda_3^2 - 4\lambda_1\lambda_2}{4(\lambda_3 - \lambda_2 - \lambda_1)}, \\ &= \frac{4\lambda_1\lambda_2 - \lambda_3^2}{8(u_1 + u_2)} = \frac{4\lambda_1\lambda_2 - \lambda_3^2}{8(u_1 + u_2)} > 0, \end{aligned} \quad (62)$$

where $\lambda_3 - \lambda_2 - \lambda_1 = 2(u_1 + u_2) > 0$ we get

$$4\lambda_1\lambda_2 - \lambda_3^2 > 0 \quad \text{or} \quad 4\lambda_1\lambda_2 > \lambda_3^2. \quad (63)$$

4. $u_4 < u_3, u_1, u_2$ (again $u_3 = 0$): once more $f'(u_4)$ must be positive. Since each one of the factors $(u_1 - u_4)$, $(u_2 - u_4)$, $(u_1 + u_2 - 2u_4)$ are positive, we have

$$f'(u_4) = \frac{4(u_1 - u_4)(u_2 - u_4)}{(u_1 + u_2 - 2u_4)^2} > 0. \quad (64)$$

The strong stability condition produces

$$f(u_4) = \frac{4\lambda_1\lambda_2 - (\lambda_3 + \lambda_4)^2}{8(u_1 + u_2 - 2u_4)} > 0, \quad (65)$$

which means

$$4\lambda_1\lambda_2 > (\lambda_3 + \lambda_4)^2. \quad (66)$$

Summarizing, the following are sufficient conditions (but not necessary) to guarantee strong stability of the potential, for all the possible values of the parameters, including the special case $\lambda_1 = \lambda_2$:

$$\lambda_1 > 0, \quad (67a)$$

$$\lambda_2 > 0, \quad (67b)$$

$$4\lambda_1\lambda_2 > \lambda_3^2, \quad (67c)$$

$$4\lambda_1\lambda_2 > (\lambda_3 + \lambda_4)^2, \quad (67d)$$

where the first two inequalities are also necessary conditions.

4.1.6 Global minimum

According to the general notation introduced in (35), for the economical 331 model we have

$$\tilde{\xi} = \begin{pmatrix} \frac{1}{2}(\mu_1^2 + \mu_2^2) \\ 0 \\ 0 \\ \frac{1}{2}(\mu_1^2 - \mu_2^2) \end{pmatrix}, \quad (68)$$

$$\tilde{E} = \begin{pmatrix} \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3) & 0 & 0 & \frac{1}{4}(\lambda_1 - \lambda_2) \\ 0 & \frac{\lambda_4}{4} & 0 & 0 \\ 0 & 0 & \frac{\lambda_4}{4} & 0 \\ \frac{1}{4}(\lambda_1 - \lambda_2) & 0 & 0 & \frac{1}{4}(\lambda_1 + \lambda_2 - \lambda_3) \end{pmatrix}.$$

The condition (45), $\xi_0 < |\xi|$ thus implies that $\mu_1^2 + \mu_2^2 < |\mu_1^2 - \mu_2^2|$. This inequality is fulfilled if

$$\mu_1^2, \mu_2^2 < 0, \quad (69)$$

or when at least one of them is negative.

In order to determine the stationary points of the potential $V(\phi_1, \phi_2)$ in Eq. (50) we must solve Eq. (41):

$$\begin{aligned} (\tilde{E} - w\tilde{g})\tilde{\mathbf{K}} &= -\frac{1}{2}\tilde{\xi} \quad \text{with} \quad \tilde{\mathbf{K}}^T \tilde{g} \tilde{\mathbf{K}} = 0 \\ \left(\text{or } \tilde{\mathbf{K}}^T \tilde{g} \tilde{\mathbf{K}} > 0 \text{ when } w = 0 \right) &\quad \text{and} \quad K_0 > 0, \end{aligned} \quad (70)$$

where w is the Lagrange multiplier. As stated above, for regular values of w with $\det(\tilde{E} - w\tilde{g}) \neq 0$ we find solutions to the equation

$$\tilde{\xi}^T (\tilde{E} - w\tilde{g})^{-1} \tilde{g} (\tilde{E} - w\tilde{g})^{-1} \tilde{\xi} = 0,$$

which gives the following Lagrange multipliers

$$w_1 = \frac{1}{4} \left(\lambda_3 - \frac{2\lambda_1\mu_2^2}{\mu_1^2} \right), \quad w_2 = \frac{1}{4} \left(\lambda_3 - \frac{2\lambda_2\mu_1^2}{\mu_2^2} \right), \quad (71)$$

where we have assumed

$$\mu_1^2 \neq 0 \quad \text{and} \quad \mu_2^2 \neq 0, \quad (72)$$

($\mu_i = 0$ for $i = 1$ or 2 is not relevant as we will show at the end of this section).

The exceptional solutions are obtained from the equation $\det(\tilde{E} - w\tilde{g}) = 0$, which produces

$$w_3 = -\frac{\lambda_4}{4}, \quad w_4 = \frac{\lambda_3 - 2\sqrt{\lambda_1\lambda_2}}{4}, \quad w_5 = \frac{\lambda_3 + 2\sqrt{\lambda_1\lambda_2}}{4}. \quad (73)$$

Finally, for the case $\tilde{\mathbf{K}}^T \tilde{g} \tilde{\mathbf{K}} > 0$ we must add the possible solution

$$w_6 = 0. \quad (74)$$

Not all w obtained are solutions of Eq. (70). Let us denote by \tilde{I} the set of valid solutions which are related to the stationary points of the potential

$$\tilde{I} = \{ w \text{ values in expressions (71), (73) and that are solutions of Eq. (70)} \}. \quad (75)$$

The largest w in \tilde{I} corresponds to the global minimum of the Higgs potential.

4.1.7 Not allowed solutions.

The global minimum will be among the stationary points in \tilde{I} . By using the Schwarz inequality we can see that the regular and the exceptional solutions, corresponding to the possibility $K_0 = |\mathbf{K}|$, implies that the two scalar triplet vectors at VEV are linearly dependent, something which does not have any sense (the quantum numbers of the two triplets are different), situation which may be avoided in some cases if only one of the two triplets develops nonzero VEV along its neutral directions. Since at the same time, the global minimum must produce an adequate symmetry breaking pattern (see Appendix C) this kind of solutions are not allowed.

Theorem 3. *A global minimum with the correct EWSB pattern $SU(3)_L \otimes U(1)_X \rightarrow SU(2)_L \otimes U(1)_Y \rightarrow U(1)_{em}$, where the condition $\xi_0 < |\tilde{\xi}|$ is required, is given and guaranteed by the stationary points of the classes (Ia) or (IIa) of theorem 2 with $K_0 > |\vec{K}|$.*

Let us see this in more detail.

Regular solutions on the forward light cone: We start by considering the Lagrange multipliers w_1 and w_2 in Eq. (71). Let us define $\max\{\tilde{I}\}$ as the maximum value of the solutions in \tilde{I} . There are two possibilities:

1. $w_1 \in \tilde{I}$ and $w_1 = \max\{\tilde{I}\}$. That is, the point where the global minimum occurs is associated to w_1 . After solving (70), the global minimum is found at

$$\tilde{\mathbf{K}} = -\frac{1}{2}(\tilde{E} - w_1\tilde{g})^{-1}\tilde{\xi} = \left(-\frac{\mu_1^2}{2\lambda_1}, 0, 0, -\frac{\mu_1^2}{2\lambda_1}\right)^T, \quad (76)$$

under the condition $K_0 = -\mu_1^2/2\lambda_1 > 0$. Then we have the equivalence

$$w_1 \in \tilde{I} \iff \mu_1^2 < 0. \quad (77)$$

Substituting (76) into (21) we get

$$\frac{1}{2}(K_0\mathbf{1} + K_3\sigma^3) = \begin{pmatrix} -\frac{\mu_1^2}{2\lambda_1} & 0 \\ 0 & 0 \end{pmatrix}. \quad (78)$$

Comparing (20) and (78) we arrive at the conclusion that no VEV are found in the scalar elements of ϕ_2 , i.e. for this global minimum we have $\langle\phi_2\rangle = 0$, something that should not be accepted, as mentioned above.

2. $w_2 \in \tilde{I}$ and $w_2 = \max\{\tilde{I}\}$; in this case the global minimum is associated to w_2 , and it is found at

$$\tilde{\mathbf{K}} = \left(-\frac{\mu_2^2}{2\lambda_2}, 0, 0, \frac{\mu_2^2}{2\lambda_2}\right)^T, \text{ with } K_0 = -\frac{\mu_2^2}{2\lambda_2} > 0; \quad (79)$$

then, for this case we have

$$w_2 \in \tilde{I} \iff \mu_2^2 < 0, \quad (80)$$

and

$$\frac{1}{2}(K_0\mathbf{1} + K_3\sigma^3) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\mu_2^2}{2\lambda_2} \end{pmatrix}, \quad (81)$$

implying $\langle\phi_1\rangle = 0$ which should not be accepted either.

The two possibilities analyzed above must be discarded because they are unable to implement an adequate symmetry breaking pattern. This conclusion can be expressed in the following way:

$$\text{If } w_1 (w_2) \in \tilde{I} \Rightarrow w_1 (w_2) < \max\{\tilde{I}\}. \quad (82)$$

Exceptional solutions on the forward light cone The stability condition in (67c) implies $w_4 < 0$; so, according to the discussion following Eq. (48), w_4 cannot give a global minimum either.

In Appendix (D) we study and show in detail that the Lagrange multipliers (w_3 and w_5) do not satisfy the conditions to be global minima either.

We may conclude therefore that

$$\text{If } w_3 (w_4, w_5) \in \tilde{I} \Rightarrow w_3 (w_4, w_5) < \max\{\tilde{I}\}. \quad (83)$$

4.1.8 Allowed solution.

The only allowed solution to the global minimum lies inside the forward light cone and is associated to the value $w_6 = 0$, that is

$$\max\{\tilde{I}\} = w_6 = 0. \quad (84)$$

From Eq. (73), the value for w_3 allows us to say:

$$\text{If } w_3 \in \tilde{I} \Rightarrow w_3 < 0, \text{ that is } \lambda_4 > 0. \quad (85)$$

Also, from (67c) and the value for w_5 in Eq. (73) we have that $w_5 > 0$, implying $w_5 > w_6$, which means

$$w_5 \notin \tilde{I}. \quad (86)$$

This result, together with Eqs. (301) and (307) in Appendix (D), implies that

$$\sqrt{\lambda_1}\mu_2^2 + \sqrt{\lambda_2}\mu_1^2 \neq 0. \quad (87)$$

The conditions to have the global minimum at w_6 require that the solution must satisfy $-\frac{1}{4}\tilde{\xi}^T \tilde{E}^{-1} \tilde{g} \tilde{E}^{-1} \tilde{\xi} < 0$, which implies that

$$-\frac{64(w_1\mu_1^2)(w_2\mu_2^2)}{(4\lambda_1\lambda_2 - \lambda_3^2)^2} < 0, \quad (88)$$

reproducing the following stationary point:

$$\tilde{\mathbf{K}} = \begin{pmatrix} \frac{4\mu_1^2 w_1 + 4\mu_2^2 w_2}{4\lambda_1\lambda_2 - \lambda_3^2} \\ 0 \\ 0 \\ \frac{4\mu_2^2 w_2 - 4\mu_1^2 w_1}{4\lambda_1\lambda_2 - \lambda_3^2} \end{pmatrix}, \quad (89)$$

which is the global minimum as far as

$$K_0 > 0 \Rightarrow 4\mu_1^2 w_1 + 4\mu_2^2 w_2 > 0, \quad (90)$$

where the relation (67c) has been used.

Using equations (69), (77), (80) and (84), the inequalities in (88) and (90) are fulfilled in the following three different cases (this is going to be seen from another point of view in the following subsection):

$$\text{Case 1: } w_1, \mu_1^2 < 0 \text{ and } w_2, \mu_2^2 > 0, \quad (91)$$

$$\text{Case 2: } w_1, \mu_1^2 > 0 \text{ and } w_2, \mu_2^2 < 0, \quad (92)$$

$$\text{Case 3: } w_1, \mu_1^2 < 0 \text{ and } w_2, \mu_2^2 < 0. \quad (93)$$

A detailed analysis of the three cases shows that only the third one is realistic, and it is the only one consistent with a right implementation of the spontaneous symmetry breaking

Analysis of case 3. Let us consider the aforementioned Case 3 for which the condition (87) is immediately satisfied. The inequalities in Eq. (93) imply that $\lambda_4 > 0$ as we are going to see soon:

To prove it, let us assume that $\lambda_4 < 0$, that is $w_3 > 0$. Since w_1 and w_2 are negative we have $w_1 - w_3 < 0$ and $w_2 - w_3 < 0$. Then, Eq. (295) in Appendix D is satisfied, but Eq. (296) becomes $(K_0^2 - K_3^2) = \mu_1^2 \mu_2^2 (w_1 - w_3)(w_2 - w_3) > 0$, which allows for nonzero values in the directions K_1 and K_2 , which in turn implies $w_3 (> 0) \in \tilde{I}$, contrary to the conditions expressed in (84) and (85). In this development we have used the relation (67d) which in turn was used in Eq. (294). Then, we can claim that

$$\lambda_4 > 0. \quad (94)$$

This result ($\lambda_4 > 0$) makes redundant the inequality (67d), which may be replaced by the inequality (94).

Now, from (71) and (93) we have that

$$\lambda_3 < \frac{2\lambda_1\mu_2^2}{\mu_1^2}, \quad \text{and} \quad \lambda_3 < \frac{2\lambda_2\mu_1^2}{\mu_2^2}, \quad (95)$$

which does not rule out the possibility of a negative λ_3 value.

Using the fact that the global minimum occurs at the point given by Eq. (89), then from Eqs. (20) and (21), we may claim that

$$\langle \underline{K} \rangle = \begin{pmatrix} \frac{4\mu_2^2 w_2}{4\lambda_1\lambda_2 - \lambda_3^2} & 0 \\ 0 & \frac{4\mu_1^2 w_1}{4\lambda_1\lambda_2 - \lambda_3^2} \end{pmatrix}, \quad (96)$$

where the nonzero VEV must be in both scalar fields, ϕ_1 and ϕ_2 . Note also in (96) that the two off-diagonal entries are zero, which implies two things: first the orthogonality condition $\langle \phi_1 \rangle^T \cdot \langle \phi_2 \rangle = 0$, and second the electric charge conservation in the model. So, the VEV of the scalars can be written in the following form:

$$\langle \phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \\ V_1 \end{pmatrix}, \quad \langle \phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix}, \quad (97)$$

where the inclusion of complex phases does not affect the analysis of the global minimum, as can be seen from the structure of matrix (96). Note that ϕ_1 can get VEV at its two neutral directions due to the fact that the minimum state is best achieved in this way, as will be shown at the end of this section; but at this point, the possibility $v_1 = 0$ or $V_1 = 0$ is excluded by this analysis.

Now, using (20) we have

$$\frac{v_1^2 + V_1^2}{2} = \frac{4\mu_2^2 w_2}{4\lambda_1\lambda_2 - \lambda_3^2} = \frac{\lambda_3\mu_2^2 - 2\lambda_2\mu_1^2}{4\lambda_1\lambda_2 - \lambda_3^2}, \quad (98)$$

$$\frac{v_2^2}{2} = \frac{4\mu_1^2 w_1}{4\lambda_1\lambda_2 - \lambda_3^2} = \frac{\lambda_3\mu_1^2 - 2\lambda_1\mu_2^2}{4\lambda_1\lambda_2 - \lambda_3^2}. \quad (99)$$

These equations are equivalent to the tree level constraint equations

$$\mu_1^2 + \lambda_1(v_1^2 + V_1^2) + \lambda_3 \frac{v_2^2}{2} = 0, \quad (100)$$

$$\mu_2^2 + \lambda_3 \frac{(v_1^2 + V_1^2)}{2} + \lambda_2 v_2^2 = 0, \quad (101)$$

the same equations obtained in Refs. [16, 20] using a different approach.

At the global minimum the Higgs potential becomes

$$V|_{\min.} = \frac{1}{2} \tilde{\mathbf{K}}^T \tilde{\boldsymbol{\xi}} = \frac{2\mu_1^2\mu_2^2(w_1 + w_2)}{4\lambda_1\lambda_2 - \lambda_3} = \frac{2\mu_1^2\mu_2^2 w_2 + 2\mu_1^2\mu_2^2 w_1}{4\lambda_1\lambda_2 - \lambda_3^2}, \quad (102)$$

using (47), (98) and (99) we get

$$V|_{\min.} = \frac{\mu_1^2(v_1^2 + V_1^2)}{4} + \frac{\mu_2^2 v_2^2}{4} < 0. \quad (103)$$

Therefore, in order to have the deepest minimum value for the potential as stated by Nature, the following conditions are highly suggested:

$$\mu_1^2 < 0 \quad \text{and} \quad \mu_2^2 < 0, \quad (104)$$

$$v_1, V_1, v_2 \neq 0. \quad (105)$$

These last two expressions explain why Case 3 in (93) was chosen as the most viable solution. The expression (105) reveals, for the first time, that the elements of the Higgs scalar triplets develop VEV in all their neutral directions, although a hierarchy among the VEV cannot be concluded from the mathematical point of view.

Finally we must verify the remnant symmetry $U(1)_{\text{em}}$ left in the scalar potential after the spontaneous symmetry breakdown. For this purpose, we arrange the triplets in Eq. (49) using the following 2×3 matrix:

$$\Phi(x) = \begin{pmatrix} \phi_1^- & \phi_1^{\prime 0} & \phi_1^0 \\ \phi_2^0 & \phi_2^+ & \phi_2^{\prime +} \end{pmatrix}. \quad (106)$$

An $SU(3)_L \otimes U(1)_X$ gauge transformation $U_G(x)$ maps the scalar triplets as

$$\phi_i^\alpha \rightarrow \phi_i^{\prime \alpha} = [U_G(x)]_\beta^\alpha \phi_i^\beta, \quad i = 1, 2 \quad (107)$$

Then, the matrix $\Phi(x)$ transforms as

$$\Phi(x) \rightarrow \Phi'(x) = \Phi(x)U_G^T(x). \quad (108)$$

The scalar matrix $\Phi(x)$, in terms of the VEV of the scalar fields, acquires the form

$$\Phi_{vac} = \begin{pmatrix} 0 & v_1 & V_1 \\ v_2 & 0 & 0 \end{pmatrix}. \quad (109)$$

So, under the transformation (108) we have

$$\Phi'_{vac} = \Phi_{vac}U_G^T. \quad (110)$$

Note that the invariance of Φ_{vac} is always possible for $U_G \neq \mathbb{1}$, because in (110) we would have more variables than equations.

4.2 The scalar potential with explicit VEV content.

An alternative way of writing the scalar potential (50), showing explicitly its global minimum is

$$\begin{aligned} V(\phi_1, \phi_2) = & a \left[\phi_1^2 + \phi_2^2 - \frac{(v_2^2 + z^2)}{2} \right]^2 + b_1 \left(\phi_1^2 - \frac{z^2}{2} \right)^2 \\ & + b_2 \left(\phi_2^2 - \frac{v_2^2}{2} \right)^2 + \lambda(\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1), \end{aligned} \quad (111)$$

where $z^2 = v_1^2 + V_1^2$ and $\phi_i^2 = \phi_i^\dagger \phi_i$, $i = 1, 2$.

This way of writing the scalar potential and the analysis which follows parallels the study used in the first paper of Ref. [22] for the THDM; for this reason we may call this form of writing the scalar potential as the Gunion parameterization.

Notice first that $V(\phi_1, \phi_2)$ has six free parameters. A glance to Eq. (111) shows that a sufficient (but not necessary) condition to produce a global minimum at $\langle \phi_1 \rangle = (0, v_1/\sqrt{2}, V_1/\sqrt{2})$ and $\langle \phi_2 \rangle = (v_2/\sqrt{2}, 0, 0)$ is that

$$a, b_1, b_2, \lambda > 0 \quad (112)$$

(which by the way does not discard the possibility of negative values for some of them since the necessary conditions are $a + b_1 > 0$ and $a + b_2 > 0$).

At this point, the criteria for a local minimum becomes

$$\frac{\partial^2 V}{(\partial \phi_1^2)^2} > 0 \quad \Rightarrow \quad a + b_1 > 0, \quad (113)$$

$$\frac{\partial^2 V}{(\partial \phi_2^2)^2} > 0 \quad \Rightarrow \quad a + b_2 > 0. \quad (114)$$

and

$$\det \begin{pmatrix} \frac{\partial^2 V}{(\partial \phi_1^2)^2} & \frac{\partial^2 V}{\partial \phi_1^2 \partial \phi_2^2} \\ \frac{\partial^2 V}{\partial \phi_1^2 \partial \phi_2^2} & \frac{\partial^2 V}{(\partial \phi_2^2)^2} \end{pmatrix} > 0 \quad \Rightarrow \quad (a + b_1)(a + b_2) > a^2. \quad (115)$$

On the other hand, comparing (111) with (50), we see that the parameters in the two representations are related as follows:

$$\mu_1^2 = -(a + b_1)z^2 - av_2^2, \quad (116a)$$

$$\mu_2^2 = -az^2 - (a + b_2)v_2^2, \quad (116b)$$

$$\lambda_1 = a + b_1, \quad (116c)$$

$$\lambda_2 = a + b_2, \quad (116d)$$

$$\lambda_3 = 2a, \quad (116e)$$

$$\lambda_4 = \lambda, \quad (116f)$$

such that the relations (113), (114) and (115) correspond to the inequalities (67a), (67b) and (67c), respectively.

Examining now Eqs. (116a) and (116b) we have

$$\mu_1^2 = -(a + b_1)z^2 - av_2^2 = -\lambda_1 z^2 - \frac{\lambda_3}{2} v_2^2, \quad (117)$$

$$\mu_2^2 = -az^2 - (a + b_2)v_2^2 = -\frac{\lambda_3}{2} z^2 - \lambda_2 v_2^2, \quad (118)$$

which can be written as

$$\begin{pmatrix} \mu_1^2 \\ \mu_2^2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 & -\frac{\lambda_3}{2} \\ -\frac{\lambda_3}{2} & -\lambda_2 \end{pmatrix} \begin{pmatrix} z^2 \\ v_2^2 \end{pmatrix}. \quad (119)$$

Solving, we obtain

$$\frac{1}{2} \begin{pmatrix} z^2 \\ v_2^2 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_3 \mu_2^2 - 2\lambda_2 \mu_1^2}{4\lambda_1 \lambda_2 - \lambda_3^2} \\ \frac{\lambda_3 \mu_1^2 - 2\lambda_1 \mu_2^2}{4\lambda_1 \lambda_2 - \lambda_3^2} \end{pmatrix} = \begin{pmatrix} \frac{4\mu_2^2 w_2}{4\lambda_1 \lambda_2 - \lambda_3^2} \\ \frac{4\mu_1^2 w_1}{4\lambda_1 \lambda_2 - \lambda_3^2} \end{pmatrix}. \quad (120)$$

The fact that $z^2 > 0$ and $v_2^2 > 0$ implies that the following product must remain always positive:

$$\mu_2^2 w_2 > 0 \quad \text{and} \quad \mu_1^2 w_1 > 0, \quad (121)$$

which shows in a different way the validity of the classification introduced in (91)-(93) for the required symmetry breaking.

4.3 New parameterizations

The search and study of possible new parametrizations give us the possibility of checking some of the previously obtained results. New parameterizations for the invariant scalar products, different to the ones given in (18), can be constructed. We will partially study two cases and, for each one, we will verify the symmetry breaking $SU(3)_L \otimes U(1)_X \rightarrow U(1)_{\text{em}}$ following the analysis of Sect. 4.1.2.

A new parameterization for the scalar potential (50) is obtained by defining the variables

$$K_1 = \phi_1^\dagger \phi_1, \quad K_2 = \phi_2^\dagger \phi_2, \quad K_3 = \phi_1^\dagger \phi_2, \quad (122)$$

so that the potential is written as

$$\begin{aligned} V &= \mu_1^2 K_1 + \mu_2^2 K_2 + \lambda_1 K_1^2 + \lambda_2 K_2^2 + \lambda_3 K_1 K_2 + \lambda_4 K_3 K_3^*, \\ V &= \tilde{\mathbf{K}} \cdot \tilde{\boldsymbol{\xi}} + \tilde{\mathbf{K}} \cdot \tilde{\mathbf{E}} \cdot \tilde{\mathbf{K}}, \end{aligned} \quad (123)$$

with

$$\tilde{\mathbf{K}} = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_3^* \end{pmatrix}, \quad \tilde{\boldsymbol{\xi}} = \begin{pmatrix} \mu_1^2 \\ \mu_2^2 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{E}} = \begin{pmatrix} \lambda_1 & \frac{\lambda_3}{2} & 0 & 0 \\ \frac{\lambda_3}{2} & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda_4}{2} \\ 0 & 0 & \frac{\lambda_4}{2} & 0 \end{pmatrix}. \quad (124)$$

The new parameters satisfy the constraints

$$K_1 \geq 0 \quad (125)$$

$$K_2 \geq 0 \quad (126)$$

$$K_1 K_2 \geq K_3 K_3^*, \text{ or } \tilde{\mathbf{K}} \cdot \tilde{g} \cdot \tilde{\mathbf{K}} \geq 0, \quad (127)$$

with

$$\tilde{g} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & -1/2 & 0 \end{pmatrix}, \quad (128)$$

where (127) comes from the Schwarz inequality.

Now, for the case $K_1 K_2 > K_3 K_3^*$, we calculate the stationary point of the potential (123). To do this we solve the equation $\tilde{E} \cdot \tilde{\mathbf{K}} = -\frac{1}{2} \tilde{\xi}$, and we get

$$\tilde{\mathbf{K}} = \begin{pmatrix} \frac{\lambda_3 \mu_2^2 - 2\lambda_2 \mu_1^2}{4\lambda_1 \lambda_2 - \lambda_3^2} \\ \frac{\lambda_3 \mu_1^2 - 2\lambda_1 \mu_2^2}{4\lambda_1 \lambda_2 - \lambda_3^2} \\ 0 \\ 0 \end{pmatrix}, \quad (129)$$

which coincides with the results in (98) and (99).

To obtain another different parameterization, let us construct the following $SU(3)_L \otimes U(1)_X$ gauge invariant array

$$\underline{\mathbf{K}} = \begin{pmatrix} (\phi_1^\dagger \phi_1)^2 & (\phi_2^\dagger \phi_1)(\phi_1^\dagger \phi_2) \\ (\phi_2^\dagger \phi_1)(\phi_1^\dagger \phi_2) & (\phi_2^\dagger \phi_2)^2 \end{pmatrix}. \quad (130)$$

This matrix is real, symmetric and positive. We now write this matrix using the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

that is

$$\underline{\mathbf{K}} = K_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + K_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + K_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (131)$$

which, compared with (130) gives

$$(\phi_1^\dagger \phi_1)^2 = K_1 + K_2, \implies \phi_1^\dagger \phi_1 = \sqrt{K_1 + K_2}, \quad (132)$$

$$(\phi_2^\dagger \phi_2)^2 = K_1 - K_2, \implies \phi_2^\dagger \phi_2 = \sqrt{K_1 - K_2}, \quad (133)$$

$$(\phi_2^\dagger \phi_1)(\phi_1^\dagger \phi_2) = K_3. \quad (134)$$

Due to the positivity of (130) we have

$$K_1 \geq 0, \quad K_1^2 - K_2^2 - K_3^2 \geq 0. \quad (135)$$

Notice however that the scalar potential is not a polynomial function of the parameters K_1 , K_2 and K_3 [see the relations (132) and (133)].

4.4 The potential after the electroweak symmetry breaking

To analyze the form of the scalar potential after the electroweak symmetry has been broken, we may throw some insight into the physical problem, as we are now going to see. We start by assuming a stable potential which leads to the desired symmetry breaking pattern as discussed in the previous sections, and thus we see what the consequences are for the resulting physical fields. For this purpose we work in the unitary gauge and use a basis for the scalar fields such that the VEV in (97) hold. Furthermore, the relation

$$\text{Im } \phi_1^0 = 0 \quad (136)$$

immediately produces one Goldstone boson (G_{01}) which is eaten up by one of the CP-odd gauge bosons.

We use as usual the following shifted Higgs fields in the two triplets

$$\phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\phi_1^- \\ v_1 + H'_1 \\ V_1 + H_1 + iA_1 \end{pmatrix}, \phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} v_2 + H_2 + iA_2 \\ \sqrt{2}\phi_2^+ \\ \sqrt{2}\phi_2'^+ \end{pmatrix}. \quad (137)$$

We may now proceed to find the remaining Goldstone bosons and the physical Higgs fields (three CP-even and one CP-odd).

It is convenient to decompose $\tilde{\mathbf{K}}$ according to the power of the physical fields

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}_{\{0\}} + \tilde{\mathbf{K}}_{\{1\}} + \tilde{\mathbf{K}}_{\{2\}}, \quad (138)$$

with

$$\tilde{\mathbf{K}}_{\{0\}} = \begin{pmatrix} \frac{V_1^2}{2} + \frac{v_1^2}{2} + \frac{v_2^2}{2} \\ 0 \\ 0 \\ \frac{V_1^2}{2} + \frac{v_1^2}{2} - \frac{v_2^2}{2} \end{pmatrix}, \quad (139)$$

$$\tilde{\mathbf{K}}_{\{1\}} = \begin{pmatrix} V_1 H_1 + v_2 H_2 + v_1 H'_1 \\ \frac{V_1}{\sqrt{2}}\phi_2'^- + \frac{V_1}{\sqrt{2}}\phi_2'^+ + \frac{v_1}{\sqrt{2}}\phi_2^- + \frac{v_1}{\sqrt{2}}\phi_2^+ + \frac{v_2}{\sqrt{2}}\phi_1^- + \frac{v_2}{\sqrt{2}}\phi_1^+ \\ \frac{iV_1}{\sqrt{2}}\phi_2'^- - \frac{iV_1}{\sqrt{2}}\phi_2'^+ + \frac{i v_1}{\sqrt{2}}\phi_2^- - \frac{i v_1}{\sqrt{2}}\phi_2^+ + \frac{i v_2}{\sqrt{2}}\phi_1^- - \frac{i v_2}{\sqrt{2}}\phi_1^+ \\ V_1 H_1 + v_1 H'_1 - v_2 H_2 \end{pmatrix}, \quad (140)$$

$$\tilde{\mathbf{K}}_{\{2\}} = \frac{1}{2} \begin{pmatrix} H_2^2 + H_1^2 + A_2^2 + A_1^2 + H_1'^2 + 2\phi_2'^+ \phi_2'^- + 2\phi_2^+ \phi_2^- + 2\phi_1^+ \phi_1^- \\ \sqrt{2}\phi_2'^- H_1 + \sqrt{2}\phi_2'^+ H_1 + i\sqrt{2}\phi_2'^- A_1 - i\sqrt{2}\phi_2'^+ A_1 + \sqrt{2}\phi_2^- H_1' + \sqrt{2}\phi_2^+ H_1' \\ \sqrt{2}\phi_2^+ H_1' + \sqrt{2}\phi_1^- H_2 + \sqrt{2}\phi_1^+ H_2 - i\sqrt{2}\phi_1^- A_2 + i\sqrt{2}\phi_1^+ A_2 \\ i\sqrt{2}\phi_2'^- H_1 - i\sqrt{2}\phi_2'^+ H_1 - \sqrt{2}\phi_2'^- A_1 - \sqrt{2}\phi_2'^+ A_1 + i\sqrt{2}\phi_2^- H_1' - i\sqrt{2}\phi_2^+ H_1' + i\sqrt{2}\phi_1^- H_2 - i\sqrt{2}\phi_1^+ H_2 + \sqrt{2}\phi_1^- A_2 + \sqrt{2}\phi_1^+ A_2 \\ H_1^2 + A_1^2 + H_1'^2 - H_2^2 - A_2^2 - 2\phi_2'^+ \phi_2'^- - 2\phi_2^+ \phi_2^- - 2\phi_1^+ \phi_1^- \end{pmatrix}. \quad (141)$$

The global minimum of the potential occurs when $w = 0$ in Eq. (70). This leads to

$$\tilde{E} \tilde{\mathbf{K}}_{\{0\}} = -\frac{1}{2} \tilde{\boldsymbol{\xi}}. \quad (142)$$

Using equations (138) to (142), we get for the potential in Eq. (36)

$$V = V_{\{0\}} + V_{\{2\}} + V_{\{3\}} + V_{\{4\}}, \quad (143)$$

where $V_{\{k\}}$ are the terms of order k^{th} in the physical fields

$$V_{\{0\}} = \frac{1}{2} \tilde{\mathbf{K}}_{\{0\}} \cdot \tilde{\boldsymbol{\xi}}, \quad (144)$$

$$V_{\{2\}} = \tilde{\mathbf{K}}_{\{1\}} \cdot \tilde{E} \cdot \tilde{\mathbf{K}}_{\{1\}}, \quad (145)$$

$$V_{\{3\}} = 2\tilde{\mathbf{K}}_{\{1\}} \cdot \tilde{E} \cdot \tilde{\mathbf{K}}_{\{2\}}, \quad (146)$$

$$V_{\{4\}} = \tilde{\mathbf{K}}_{\{1\}} \cdot \tilde{E} \cdot \tilde{\mathbf{K}}_{\{2\}}. \quad (147)$$

The second order terms (145) determine the masses of the physical Higgs fields and the remaining Goldstone bosons

$$V_{\{2\}} = \frac{1}{2} \begin{pmatrix} H_1 & H_2 & H'_1 \end{pmatrix} \mathcal{M}_{\text{neutral}}^2 \begin{pmatrix} H_1 \\ H_2 \\ H'_1 \end{pmatrix} + \begin{pmatrix} \phi_1^+ & \phi_2^+ & \phi_2'^+ \end{pmatrix} \mathcal{M}_{\text{charged}}^2 \begin{pmatrix} \phi_1^- \\ \phi_2^- \\ \phi_2'^- \end{pmatrix}, \quad (148)$$

with

$$\mathcal{M}_{\text{neutral}}^2 = \begin{pmatrix} 2\lambda_1 V_1^2 & \lambda_3 v_2 V_1 & 2\lambda_1 v_1 V_1 \\ \lambda_3 v_2 V_1 & 2\lambda_2 v_2^2 & \lambda_3 v_1 v_2 \\ 2\lambda_1 v_1 V_1 & \lambda_3 v_1 v_2 & 2\lambda_1 v_1^2 \end{pmatrix}, \quad (149)$$

$$\mathcal{M}_{\text{charged}}^2 = \frac{\lambda_4}{2} \begin{pmatrix} v_2^2 & v_1 v_2 & v_2 V_1 \\ v_1 v_2 & v_1^2 & v_1 V_1 \\ v_2 V_1 & v_1 V_1 & V_1^2 \end{pmatrix}. \quad (150)$$

Clearly, the fields A_1 and A_2 are massless, providing two other CP-odd Goldstone bosons Go_2 and Go_3 . The neutral sector (149) provides a CP-even Goldstone boson Ge_4 and two CP-even massive scalars Hgg_1 and Hgg_2 with masses

$$M_{Hgg_1, Hgg_2}^2 = (v_1^2 + V_1^2)\lambda_1 + v_2^2\lambda_2 \pm \sqrt{[(v_1^2 + V_1^2)\lambda_1 + v_2^2\lambda_2]^2 + v_2^2(v_1^2 + V_1^2)(\lambda_3^2 - 4\lambda_1\lambda_2)}. \quad (151)$$

Now, the stability of the potential requires that $\lambda_1 > 0$, $\lambda_2 > 0$ and $4\lambda_1\lambda_2 > \lambda_3^2$ (see Eqs. (67a), (67b) and (67c)), which in turn implies a positive value for the former masses of the scalar fields predicted by the model.

For the charged sector (150) we get two zero eigenvalues corresponding to four Goldstone bosons G_5^\pm, G_6^\pm , two CP-even and two CP-odd, and two charged scalars, one CP-even and one CP-odd, with a degenerate mass $\frac{\lambda_4}{2}(v_1^2 + v_2^2 + V_1^2)$, which, according with Eq. (94), is positive.

The former analysis is in agreement with the results obtained in Refs. [16] and [20].

5 Scalar Potential Without Cubic Term in 331 Models Without Exotic Electric Charges

In the present work we make use of some of the new algebraic developments [23, 24] cited in the former section, to analyze the scalar sector of an extension to the SM based on the local gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$.

In general, the scalar sector for 331 models is quite complicated and difficult to analyze in detail. For example, for the minimal model (the Pisano-Pleitez-Frampton model [14]) three Higgs scalar triplets and one additional Higgs sextet must be used, in order to break the symmetry and provide, at the same time, masses to the fermion fields. For the 331 models without exotic electric charges [8, 12, 15, 11] the situation is simpler because it turns out that less Higgs scalar multiplets are needed [19]. For example, the so-called economical 331 model [16] makes use of only two scalar Higgs triplets which are able to break the symmetry in a consistent way, although they are not able to produce a consistent fermion mass spectrum at tree level. The alternative approach is to deal with three Higgs scalar triplets instead of two, as done for example in Refs. [12, 11].

In this work we pursue the study of the scalar sector of the 331 models without exotic electric charges started in Ref. [45], by considering this time a model with three Higgs scalar triplets. A discrete symmetry will be applied to the corresponding scalar potential [46, 47], which simplifies and facilitates its analysis, due that the cubic (or trilinear) term would be absent. In this analysis we will derive constraints on the parameters of the scalar potential coming from its stability and from the electroweak symmetry breaking conditions (the stability of an scalar potential at the classical level, which is fulfilled when it is bounded from below, is a necessary condition in order to have a consistent theory). The global minimum of the potential will also be found by determining its stationary points.

5.1 The scalar sector

In the following analysis we will concentrate only in the set with three scalar triplets as defined in Sects. (2.3.1) and (2.3.2), Eqs. (3) y (4).

$$\Phi_1(1, 3^*, -1/3) = \begin{pmatrix} \phi_1^- \\ \phi_1^0 \\ \phi_1'^0 \end{pmatrix}, \text{ with VEV: } \langle \Phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \\ V_1 \end{pmatrix}, \quad (152)$$

$$\Phi_3(1, 3^*, 2/3) = \begin{pmatrix} \phi_3^0 \\ \phi_3^+ \\ \phi_3'^+ \end{pmatrix}, \text{ with VEV: } \langle \Phi_3 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_3 \\ 0 \\ 0 \end{pmatrix}, \quad (153)$$

$$\Phi_2(1, 3^*, -1/3) = \begin{pmatrix} \phi_2^- \\ \phi_2^0 \\ \phi_2'^0 \end{pmatrix}, \text{ with VEV: } \langle \Phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \\ V_2 \end{pmatrix}, \quad (154)$$

with the most general VEV structure, but with the constraint derived in Appendix E.

5.2 The scalar Potential

The most general scalar potential which is 3-3-1 invariant, for the set of three scalar triplets Φ_1 , Φ_2 and Φ_3 is given by

$$\begin{aligned}
V'(\Phi_1, \Phi_2, \Phi_3) = & \mu_1^2 \Phi_1^\dagger \Phi_1 + \mu_2^2 \Phi_2^\dagger \Phi_2 + \mu_3^2 \Phi_3^\dagger \Phi_3 + \frac{1}{2}(\mu_4^2 \Phi_1^\dagger \Phi_2 + \mu_4^{2*} \Phi_2^\dagger \Phi_1) + \lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \lambda_2 (\Phi_2^\dagger \Phi_2)^2 \\
& + \lambda_3 (\Phi_3^\dagger \Phi_3)^2 + \lambda_4 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_5 (\Phi_1^\dagger \Phi_1)(\Phi_3^\dagger \Phi_3) + \lambda_6 (\Phi_2^\dagger \Phi_2)(\Phi_3^\dagger \Phi_3) + \lambda_7 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\
& + \lambda_8 (\Phi_1^\dagger \Phi_3)(\Phi_3^\dagger \Phi_1) + \lambda_9 (\Phi_2^\dagger \Phi_3)(\Phi_3^\dagger \Phi_2) + (f \Phi_1^\dagger \Phi_2 + f^* \Phi_2^\dagger \Phi_1)^2 + \frac{1}{2}(\lambda_{11} \Phi_1^\dagger \Phi_2 + \lambda_{11}^* \Phi_2^\dagger \Phi_1)(\Phi_1^\dagger \Phi_1) \\
& + \frac{1}{2}(\lambda_{12} \Phi_1^\dagger \Phi_2 + \lambda_{12}^* \Phi_2^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \frac{1}{2}(\lambda_{13} \Phi_1^\dagger \Phi_2 + \lambda_{13}^* \Phi_2^\dagger \Phi_1)(\Phi_3^\dagger \Phi_3) \\
& + \frac{1}{2} \left[\lambda_{14} (\Phi_1^\dagger \Phi_3)(\Phi_3^\dagger \Phi_2) + \lambda_{14}^* (\Phi_3^\dagger \Phi_1)(\Phi_2^\dagger \Phi_3) \right] + (g \epsilon_{ijk} \Phi_1^i \Phi_2^j \Phi_3^k + h.c.).
\end{aligned} \tag{155}$$

Since μ_4^2 , f , λ_{11} , λ_{12} , λ_{13} , λ_{14} and the trilinear coupling constant g can be complex numbers, there are 26 free parameters in $V'(\Phi_1, \Phi_2, \Phi_3)$ and 5 VEV, in principle all of them different from zero. The last element of V' correspond to the so-called cubic term of the potential, which is closely related to a determinant function of Higgs fields due to the Levi-Civita component ϵ_{ijk} .

For the sake of simplicity we are going to assume real VEV throughout this paper, which means that spontaneous CP violation is not going to be considered in our analysis. Notice also that the most general scalar potential $V'(\Phi_1, \Phi_2, \Phi_3)$ in (155) is invariant under the local Gauge group $SU(3)_L \otimes U(1)_X$, invariance that is spontaneously broken by the VEV in $\langle \Phi_i \rangle$, $i = 1, 2, 3$ down to $U(1)_Q$, where Q is the electric charge generator in equation (2). So, after the breaking of the symmetry, a consistent model will emerge only if eight massless Goldstone bosons show up, coming from the transformed potential obtained from (155); zero mass bosons that should be eaten up by the Gauge bosons associated with the $SU(3)_L \otimes U(1)_X$ broken symmetry.

The scalar potential in Eq. (155) is quite complicated and very difficult to study in a systematic way, and as far as we know, it has not been studied in full detail in the literature yet (and we do not intend to do it here either). A partial analysis of this general potential, for the particular vacuum alignment $V_1 = v_2 = 0$, has been done in Ref. [19]. However, as mentioned in Refs. [46, 47], by introducing discrete symmetries, the form of the potential simplifies largely and can be analyzed in detail, as we are going to do next.

5.2.1 Discrete symmetry in the scalar potential

Under assumption of the discrete symmetry $\Phi_1 \rightarrow -\Phi_1$, the most general potential obtained from (155), is presented in Appendix F, where it is demonstrated that f can be taken as a single parameter. As a consequence of this, the reduced potential

$$\begin{aligned}
V(\Phi_1, \Phi_2, \Phi_3) = & \mu_1^2 \Phi_1^\dagger \Phi_1 + \mu_2^2 \Phi_2^\dagger \Phi_2 + \mu_3^2 \Phi_3^\dagger \Phi_3 + \lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_3^\dagger \Phi_3)^2 \\
& + \lambda_4 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_5 (\Phi_1^\dagger \Phi_1)(\Phi_3^\dagger \Phi_3) + \lambda_6 (\Phi_2^\dagger \Phi_2)(\Phi_3^\dagger \Phi_3) + \lambda_7 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\
& + \lambda_8 (\Phi_1^\dagger \Phi_3)(\Phi_3^\dagger \Phi_1) + \lambda_9 (\Phi_2^\dagger \Phi_3)(\Phi_3^\dagger \Phi_2) + \frac{\lambda_{10}}{2} (\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1)^2,
\end{aligned} \tag{156}$$

contains only 13 free parameters (instead of 26) and does not include the cubic term $\epsilon_{ijk} \Phi_1^i \Phi_2^j \Phi_3^k$. The rest of the paper will be dedicated to study this potential (156).

A careful analysis shows now that, due to the absence of the cubic term, the potential $V(\Phi_1, \Phi_2, \Phi_3)$ turns out to be $U(3) \otimes U(1)_X$ invariant [instead of $SU(3) \otimes U(1)_X$] with the consequence that the most general VEV breaks this symmetry down to $U(1)_Q$ as before, producing now nine Goldstone bosons, instead of the eight that can be Gauged away, due to the fact that the generator $I_3 = Dg(1, 1, 1)$ gets also broken. This leaves an (unphysical?) extra zero mass scalar after the implementation of the Higgs mechanism. The simplest way to avoid this situation is by restoring the cubic term in the scalar potential (a dynamical breaking of the $U(3)$ symmetry), something does not allowed by the discrete symmetry imposed. But as shown in appendix E, for the case when $\langle \Phi_3 \rangle \neq 0$ the problem can be solved by demanding that $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$ became linearly dependent (LD); this avoids an spontaneous breaking of the $U(3) \otimes U(1)_X$ symmetry down to $SU(3) \otimes U(1)_X$, with the consequence that the VEV must satisfy the constraint

$$v_2 V_1 = v_1 V_2, \tag{157}$$

which can be used to express at least one VEV in terms of the rest.

Notice that if $\langle \Phi_3 \rangle = 0$, the $U(3)$ generator $Diag.(1, 0, 0)$ remains unbroken by $\langle \Phi_1 \rangle \oplus \langle \Phi_2 \rangle \oplus \langle \Phi_3 \rangle$, restoring in this way the eight Goldstone bosons required. But we are not going to consider this unphysical situation as previously mentioned.

Before continuing, let us emphasize that constraint (157) is a consequence of demanding a consistent implementation of the Higgs mechanism for the breaking of the original $SU(3) \otimes U(1)_X$ local Gauge symmetry, respecting the electromagnetic $U(1)_Q$ invariance, and it is not coming from the minimization of the scalar potential. On the contrary, this constraint is taken into account when we study the stability and minimization of the potential.

Notice that first two papers in Ref. [11, 12] and all papers in Ref. [46], the reduced potential $V(\Phi_1, \Phi_2, \Phi_3)$ was studied using the particular vacuum alignment $V_1 = v_2 = 0$, with $V_2 \gg v_1 \neq 0$, in clear contradiction with equation (157). As an immediate consequence, in those papers an extra zero mass Goldstone boson which cannot be Gauged away appears, making the analysis and some of the conclusions in all those papers dubious. To add in proof, notice that the four papers in Ref. [47] make use of that extra Goldstone boson to implement the Peccei-Quinn symmetry [44] in the context of the 331 model with right handed neutrinos, with the inconvenience of having in their analysis an unrealistic axion that is hidden by the introduction of an extra scalar field.

In what follows we are going to study the consistency of the scalar potential $V(\Phi_1, \Phi_2, \Phi_3)$ in Eq. (156), under the linear dependent (LD) constraint equation: $v_2 V_1 = v_1 V_2$.

To start with, let us define as usual the scalar fields in the way:

$$\phi_1^0 = \frac{v_1 + H_1 + iA_1}{\sqrt{2}}, \quad \phi_1'^0 = \frac{V_1 + H_1' + iA_1'}{\sqrt{2}}, \quad (158a)$$

$$\phi_2^0 = \frac{v_2 + H_2 + iA_2}{\sqrt{2}}, \quad \phi_2'^0 = \frac{V_2 + H_2' + iA_2'}{\sqrt{2}}, \quad (158b)$$

$$\phi_3^0 = \frac{v_3 + H_3 + iA_3}{\sqrt{2}}, \quad (158c)$$

where a real part H is called in the literature a CP-even scalar and an imaginary part A a CP-odd scalar or pseudoscalar field.

5.3 Independent vacuum structures

Assuming for the VEV the hierarchy in (5) or in (6), and using the LD constraint relation (157), we classify in Table 1, all the possible 3-3-1 vacuum structures of Φ_1 and Φ_2 , the two scalar triplets with identical quantum numbers, where at least one VEV is different from zero.

A careful analysis shows that not all the nine structures are independent. As a matter of fact, by performing an $SU(3)_L$ transformation on $\langle \Phi_1 \rangle$ and on $\langle \Phi_2 \rangle$ in structure 1 of Table 1, we can obtain either the structure configuration 2 or the structure configuration 5. But it is not possible to make an $SU(3)_L \otimes U(1)_X$ transformation followed by a change of basis of the Higgs fields $\Phi_i \rightarrow \Phi_i'$ of the form

$$\begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} = U \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad (159)$$

where U is a 2×2 unitary matrix, such that the configuration 3 can be obtained; this is because the transformation (159) violates the discrete symmetry $\Phi_1 \rightarrow -\Phi_1$ previously imposed on the scalar potential. It is also possible to show that structures 3 and 4 are equivalent to each other due to the symmetry of the potential under the exchange $\Phi_1 \leftrightarrow \Phi_2$, with some parameters renamed appropriately. In conclusion, the analysis shows that only structures 1 and 3 in Table 1 are independent and are the only cases we are going to consider in our analysis.

5.3.1 Vacuum structure with $v_1, V_1, v_2, V_2 \neq 0$

It will be shown further below that, by minimization methods applied on potential, the scalars acquiring non-zero VEVs along their electrically neutral entries, is highly suggested (269).

Structure	VEV	Vacuum alignments
1	$v_1, V_1, v_2, V_2 \neq 0$ and $v_2 V_1 = v_1 V_2$	$\langle \Phi_1 \rangle \propto (0 \ v_1 \ V_1)^T$, $\langle \Phi_2 \rangle \propto (0 \ v_2 \ V_2)^T$
2	$v_1, v_2 = 0; V_1, V_2 \neq 0$	$\langle \Phi_1 \rangle \propto (0 \ 0 \ V_1)^T$, $\langle \Phi_2 \rangle \propto (0 \ 0 \ V_2)^T$
3	$v_2, V_2 = 0; v_1, V_1 \neq 0$	$\langle \Phi_1 \rangle \propto (0 \ v_1 \ V_1)^T$, $\langle \Phi_2 \rangle = (0 \ 0 \ 0)^T$
4	$v_1, V_1 = 0; v_2, V_2 \neq 0$	$\langle \Phi_1 \rangle = (0 \ 0 \ 0)^T$, $\langle \Phi_2 \rangle \propto (0 \ v_2 \ V_2)^T$
5	$V_1, V_2 = 0; v_1, v_2 \neq 0$	$\langle \Phi_1 \rangle \propto (0 \ v_1 \ 0)^T$, $\langle \Phi_2 \rangle \propto (0 \ v_2 \ 0)^T$
6	$v_2, V_2, V_1 = 0; v_1 \neq 0$	$\langle \Phi_1 \rangle \propto (0 \ v_1 \ 0)^T$, $\langle \Phi_2 \rangle = (0 \ 0 \ 0)^T$
7	$v_2, V_2, v_1 = 0; V_1 \neq 0$	$\langle \Phi_1 \rangle \propto (0 \ 0 \ V_1)^T$, $\langle \Phi_2 \rangle = (0 \ 0 \ 0)^T$
8	$v_1, V_1, V_2 = 0; v_2 \neq 0$	$\langle \Phi_1 \rangle = (0 \ 0 \ 0)^T$, $\langle \Phi_2 \rangle \propto (0 \ v_2 \ 0)^T$
9	$v_1, V_1, v_2 = 0; V_2 \neq 0$	$\langle \Phi_1 \rangle = (0 \ 0 \ 0)^T$, $\langle \Phi_2 \rangle \propto (0 \ 0 \ V_2)^T$

Table 1: Different VEV structures for scalars Φ_1 and Φ_2 .

$$\langle \Phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \\ V_1 \end{pmatrix}, \quad \langle \Phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \\ V_2 \end{pmatrix}, \quad \langle \Phi_3 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_3 \\ 0 \\ 0 \end{pmatrix}, \quad (160)$$

with $v_2 V_1 = v_1 V_2$. And requiring that in the shifted potential obtained from $V(\Phi_1, \Phi_2, \Phi_3)$, the linear terms in the fields must be absent, we get in the tree level approximation the following constraint equations:

$$2\mu_1^2 + (\lambda_7 + \lambda_4 + 2\lambda_{10}) (V_2^2 + v_2^2) + 2\lambda_1 (V_1^2 + v_1^2) + \lambda_5 v_3^2 = 0, \quad (161a)$$

$$2\mu_2^2 + 2\lambda_2 (V_2^2 + v_2^2) + (\lambda_7 + \lambda_4 + 2\lambda_{10}) (V_1^2 + v_1^2) + \lambda_6 v_3^2 = 0, \quad (161b)$$

$$2\mu_3^2 + \lambda_6 (V_2^2 + v_2^2) + \lambda_5 (V_1^2 + v_1^2) + 2\lambda_3 v_3^2 = 0. \quad (161c)$$

where the VEVs must satisfy the constraint (157). In section 5.5.2, we will express $\{v_1^2 + V_1^2, v_2^2 + V_2^2, v_3^2\}$ in terms of the parameters of the potential by using the orbital variables method.

Spectrum in the scalar neutral sector In the $H_1, H_2, H'_1, H'_2, H_3$ basis, the square mass matrix can be calculated by using $M_{ij}^2 = [\partial V(\Phi_1, \Phi_2, \Phi_3)/\partial H_i \partial H_j]_{fields=0}$. After imposing constraints (161), we get

$$M_H^2 = \begin{pmatrix} 2(u_5 V_2^2 + \lambda_1 v_1^2) & (\lambda_4 - 4u_5) v_1 v_2 - 2u_5 V_1 V_2 & 2(\lambda_1 v_1 V_1 - u_5 v_2 V_2) & (\lambda_4 - 2u_5) v_2 V_1 & \lambda_5 v_3 v_1 \\ (\lambda_4 - 4u_5) v_1 v_2 - 2u_5 V_1 V_2 & 2(u_5 V_1^2 + \lambda_2 v_2^2) & (\lambda_4 - 2u_5) v_2 V_1 & 2(\lambda_2 v_2 V_2 - u_5 v_1 V_1) & \lambda_6 v_3 v_2 \\ 2(\lambda_1 v_1 V_1 - u_5 v_2 V_2) & (\lambda_4 - 2u_5) v_2 V_1 & 2(\lambda_1 V_1^2 + u_5 v_2^2) & (\lambda_4 - 4u_5) V_1 V_2 - 2u_5 v_1 v_2 & \lambda_5 v_3 V_1 \\ (\lambda_4 - 2u_5) v_2 V_1 & 2(\lambda_2 v_2 V_2 - u_5 v_1 V_1) & (\lambda_4 - 4u_5) V_1 V_2 - 2u_5 v_1 v_2 & 2(\lambda_2 V_2^2 + u_5 v_1^2) & \lambda_6 u V_2 \\ \lambda_5 v_3 v_1 & \lambda_6 v_3 v_2 & \lambda_5 v_3 V_1 & \lambda_6 u V_2 & 2\lambda_3 v_3^2 \end{pmatrix},$$

where $u_5 = -(\lambda_7 + 2\lambda_{10})/4$ has been used. This mass matrix has zero determinant providing us with a Goldstone boson G_1 and four massive scalar fields. The analytic mass values are not easy to find, but in the approximation

$V_1 \sim V_2 \gg v_1 \sim v_2 \sim v_3$ they are

$$M_{he_1}^2 \approx \lambda_2 V_2^2 + \lambda_1 V_1^2 + \sqrt{(\lambda_1 V_1^2 - \lambda_2 V_2^2)^2 + (\lambda_4 + \lambda_7 + 2\lambda_{10})^2 V_1^2 V_2^2}, \quad (162)$$

$$M_{he_2}^2 \approx \lambda_2 V_2^2 + \lambda_1 V_1^2 - \sqrt{(\lambda_1 V_1^2 - \lambda_2 V_2^2)^2 + (\lambda_4 + \lambda_7 + 2\lambda_{10})^2 V_1^2 V_2^2}, \quad (163)$$

$$M_{he_3}^2 \approx -\frac{(\lambda_7 + 2\lambda_{10})(V_2^2 + V_1^2)}{2}, \quad (164)$$

$$M_{he_4}^2 \approx 2\lambda_3 v_3^2 \quad \text{with} \quad he_4 \approx H_3, \quad (165)$$

where the scalar he_4 is light and can be identified as the SM Higgs boson scalar. In order to have positive masses for all the former scalars, the following constraints must be satisfied

$$\lambda_1, \lambda_2, \lambda_3 > 0, \quad (166a)$$

$$4\lambda_1 \lambda_2 > (\lambda_4 + \lambda_7 + 2\lambda_{10})^2, \quad (166b)$$

$$(\lambda_7 + 2\lambda_{10}) < 0. \quad (166c)$$

Spectrum in the pseudoscalar neutral sector In the $A_1, A_2, A'_1, A'_2, A_3$ basis the square mass matrix is given by

$$M_A^2 = \begin{pmatrix} 2u_5 V_2^2 - \lambda_{10} v_2^2 & \lambda_{10} v_1 v_2 - 2u_5 V_1 V_2 & \frac{\lambda_7 v_2 V_2}{2} & -\frac{\lambda_7 v_2 V_1}{2} & 0 \\ \lambda_{10} v_1 v_2 - 2u_5 V_1 V_2 & 2u_5 V_1^2 - \lambda_{10} v_1^2 & -\frac{\lambda_7 v_2 V_1}{2} & \frac{\lambda_7 v_1 V_1}{2} & 0 \\ \frac{\lambda_7 v_2 V_2}{2} & -\frac{\lambda_7 v_2 V_1}{2} & 2u_5 v_2^2 - \lambda_{10} V_2^2 & \lambda_{10} V_1 V_2 - 2u_5 v_1 v_2 & 0 \\ -\frac{\lambda_7 v_2 V_1}{2} & \frac{\lambda_7 v_1 V_1}{2} & \lambda_{10} V_1 V_2 - 2u_5 v_1 v_2 & 2u_5 v_1^2 - \lambda_{10} V_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (167)$$

which is a rank-2 matrix, giving three Goldstone bosons and two heavy pseudoscalar particles with masses given by

$$M_{ho_1}^2 \approx -\frac{(\lambda_7 + 2\lambda_{10})(V_1^2 + V_2^2)}{2}, \quad (168)$$

$$M_{ho_2}^2 \approx -\lambda_{10}(V_1^2 + V_2^2), \quad (169)$$

where $M_{ho_1}^2 > 0$ due to the constraint (166c), and the condition $M_{ho_2}^2 > 0$ implies the new constraint

$$\lambda_{10} < 0. \quad (170)$$

Spectrum in the charged scalar sector In the $\phi_1^\pm, \phi_2^\pm, \phi_3^\pm, \phi_3^{\prime\pm}$ basis, the square mass matrix is given by

$$M_\phi^2 = \frac{1}{2} \begin{pmatrix} 4u_5 (V_2^2 + v_2^2) + \lambda_8 v_3^2 & -4u_5 (V_1 V_2 + v_1 v_2) & \lambda_8 v_3 v_1 & \lambda_8 v_3 V_1 \\ -4u_5 (V_1 V_2 + v_1 v_2) & 4u_5 (V_1^2 + v_1^2) + \lambda_9 v_3^2 & \lambda_9 v_3 v_2 & \lambda_9 v_3 V_2 \\ \lambda_8 v_3 v_1 & \lambda_9 v_3 v_2 & \lambda_9 v_2^2 + \lambda_8 v_1^2 & \lambda_9 v_2 V_2 + \lambda_8 v_1 V_1 \\ \lambda_8 v_3 V_1 & \lambda_9 v_3 V_2 & \lambda_9 v_2 V_2 + \lambda_8 v_1 V_1 & \lambda_9 V_2^2 + \lambda_8 V_1^2 \end{pmatrix} \quad (171)$$

which is a rank-2 matrix, implying the existence of four Goldstone bosons and four massive charged scalars, with masses given by

$$M_{h_1^\pm}^2 \approx \frac{\lambda_8 V_1^2 + \lambda_9 V_2^2}{2} > 0, \quad (172)$$

$$M_{h_2^\pm}^2 \approx -\frac{(\lambda_7 + 2\lambda_{10})(V_1^2 + V_2^2)}{2}, \quad (173)$$

where again we have $M_{h_2^\pm}^2 > 0$ due to the constraint (166c). Counting Goldstone bosons we have a total of eight: an scalar and three pseudoscalars which are used to provide with masses to four electrically neutral gauge bosons (Z^0, Z'^0, K^0 and \bar{K}^0), and four charged ones which are used to provide with masses to W^\pm and to K^\pm . This shows the consistency of our analysis.

5.3.2 Vacuum structure with $v_2 = V_2 = 0, v_1, V_1 \neq 0$

In this section we are going to study the other independent structure given in Table 1, where the LD between $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$ must be respected. The VEV configuration structure is

$$\langle \Phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \\ V_1 \end{pmatrix}, \quad \langle \Phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \Phi_3 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_3 \\ 0 \\ 0 \end{pmatrix}. \quad (174)$$

In the tree level approximation, the constraint equations are now

$$\mu_1^2 + \lambda_1 (V_1^2 + v_1^2) + \frac{\lambda_5}{2} v_3^2 = 0, \quad (175a)$$

$$\mu_3^2 + \frac{\lambda_5}{2} (V_1^2 + v_1^2) + \lambda_3 v_3^2 = 0, \quad (175b)$$

where there is no a constraint relation for μ_2^2 , which becomes now a free parameter of the model.

Spectrum in the scalar neutral sector The first result obtained is that the fields H_1, H'_1 and H_3 do not mix with H_2 and H'_2

In the H_1, H'_1, H_3 basis, the square mass matrix is

$$M_{e2}^2 = \begin{pmatrix} 2 \lambda_1 v_1^2 & 2 \lambda_1 v_1 V_1 & \lambda_5 v_1 v_3 \\ 2 \lambda_1 v_1 V_1 & 2 \lambda_1 V_1^2 & \lambda_5 v_3 V_1 \\ \lambda_5 v_1 v_3 & \lambda_5 v_3 V_1 & 2 \lambda_3 v_3^2 \end{pmatrix}, \quad (176)$$

which is a rank-2 matrix, implying the existence of one Goldstone boson.

Now, in the H_2, H'_2 basis, the rank-2 mass matrix is.

$$M_{e3}^2 = \begin{pmatrix} \frac{\lambda_4 V_1^2 + S v_1^2 + \lambda_6 v_3^2}{2} + \mu_2^2 & \frac{(\lambda_7 + 2\lambda_{10}) v_1 V_1}{2} \\ \frac{(\lambda_7 + 2\lambda_{10}) v_1 V_1}{2} & \frac{S V_1^2 + \lambda_4 v_1^2 + \lambda_6 v_3^2}{2} + \mu_2^2 \end{pmatrix}, \quad (177)$$

where $S = \lambda_4 + \lambda_7 + 2\lambda_{10}$. For this vacuum structure the analytic scalar mass values can be calculated exactly; they are

$$M_{he'_1}^2 = \lambda_1 (v_1^2 + V_1^2) + \lambda_3 v_3^2 + \sqrt{[\lambda_1 (V_1^2 + v_1^2) - \lambda_3 v_3^2]^2 + \lambda_5^2 v_3^2 (V_1^2 + v_1^2)}, \quad (178)$$

$$M_{he'_2}^2 = \lambda_1 (v_1^2 + V_1^2) + \lambda_3 v_3^2 - \sqrt{[\lambda_1 (V_1^2 + v_1^2) - \lambda_3 v_3^2]^2 + \lambda_5^2 v_3^2 (V_1^2 + v_1^2)}, \quad (179)$$

$$M_{he'_3}^2 = \frac{(\lambda_7 + \lambda_4 + 2\lambda_{10}) (V_1^2 + v_1^2) + \lambda_6 v_3^2}{2} + \mu_2^2 > 0, \quad (180)$$

$$M_{he'_4}^2 = \frac{\lambda_4 (V_1^2 + v_1^2) + \lambda_6 v_3^2}{2} + \mu_2^2 > 0. \quad (181)$$

In order to have positive masses for the first two scalars, the following constraint equations must be satisfied:

$$\lambda_1, \lambda_3 > 0 \quad \text{and} \quad 4 \lambda_1 \lambda_3 > \lambda_5^2. \quad (182)$$

Notice by the way that the masses for the scalar fields (178) and (179) correspond to the masses of the CP-even physical fields in the economical model [16, 43, 45], and thus he'_2 can be identified as the SM Higgs boson scalar.

Spectrum in the pseudoscalar neutral sector. In this sector the fields A_1, A'_1, A_3 do not get mass entries, becoming automatically 3 odd Goldstone bosons. Now, in the basis A_2, A'_2 the rank-2 square mass matrix M_0^2 is

$$\begin{pmatrix} \frac{\lambda_4 V_1^2 + (\lambda_7 + \lambda_4) v_1^2 + \lambda_6 v_3^2}{2} + \mu_2^2 & \frac{\lambda_7 v_1 V_1}{2} \\ \frac{\lambda_7 v_1 V_1}{2} & \frac{(\lambda_7 + \lambda_4) V_1^2 + \lambda_4 v_1^2 + \lambda_6 v_3^2}{2} + \mu_2^2 \end{pmatrix},$$

with eigenvalues for the physical fields given by

$$M_{ho'_1}^2 = \frac{(\lambda_7 + \lambda_4)(V_1^2 + v_1^2) + \lambda_6 v_3^2}{2} + \mu_2^2 > 0, \quad (183)$$

$$M_{ho'_2}^2 = \frac{\lambda_4 (V_1^2 + v_1^2) + \lambda_6 v_3^2}{2} + \mu_2^2 > 0. \quad (184)$$

Spectrum in the charged scalar sector In the $\phi_1^\pm, \phi_2^\pm, \phi_3^\pm, \phi_3^{\prime\pm}$ basis the 4×4 square mass matrix M_c^2 is

$$\begin{pmatrix} \frac{\lambda_8 v_3^2}{2} & 0 & \frac{(\lambda_8 v_1 v_3)}{2} & \frac{\lambda_8 v_3 V_1}{2} \\ 0 & \frac{\lambda_4 V_1^2 + \lambda_4 v_1^2 + (\lambda_9 + \lambda_6) v_3^2 + 2\mu_2^2}{2} & 0 & 0 \\ \frac{\lambda_8 v_1 v_3}{2} & 0 & \frac{\lambda_8 v_1^2}{2} & \frac{\lambda_8 v_1 V_1}{2} \\ \frac{\lambda_8 v_3 V_1}{2} & 0 & \frac{\lambda_8 v_1 V_1}{2} & \frac{\lambda_8 V_1^2}{2} \end{pmatrix},$$

which is a rank-2 mass matrix producing in this way four Goldstone bosons. The remaining physical fields have square masses:

$$M_{h_1^\pm}^2 = \frac{\lambda_8(V_1^2 + v_1^2 + v_3^2)}{2}, \quad (185)$$

$$M_{h_2^\pm}^2 = \frac{\lambda_4(V_1^2 + v_1^2) + (\lambda_9 + \lambda_6)v_3^2}{2} + \mu_2^2 > 0, \quad (186)$$

where $h_2^\pm = \phi_2^\pm$. Now, for $M_{h_1^\pm}^2 > 0$ it must hold

$$\lambda_8 > 0. \quad (187)$$

Notice again that the masses of the two physical charged scalars coincide with that masses in the economical 331 model.

Counting Goldstone bosons we get again a consistent spectrum.

In the following two sections we are going to derive bounds on the parameters of the scalar potential (156) that result from the following conditions:

- The potential $V(\Phi_1, \Phi_2, \Phi_3)$ must be stable,
- The potential must be able to break the symmetry $SU(3)_L \otimes U(1)_X$ down to $U(1)_Q$, in a consistent way.

5.4 Stability of the scalar Potential

The scalar potential $V(\Phi_1, \Phi_2, \Phi_3)$ in (156) is stable if it is bounded from below; this guarantees the existence of a global minimum in the potential. The stability of the scalar potential turns out to be independent of the values taken by the VEV: v_1, v_2, v_3, V_1 and V_2 , as it is going to be shown in the following analysis. In other words, the results obtained below are valid, independent of the vacuum structure chosen.

5.4.1 The orbital variables

The most general gauge invariant and renormalizable scalar potential $V(\Phi_1, \Phi_2, \Phi_3)$ in (156), that does not contain the cubic term, for the three Higgs scalar triplets Φ_1, Φ_2 , and Φ_3 , is an Hermitian linear combination of terms of the form

$$\Phi_i^\dagger \Phi_j, \quad (\Phi_i^\dagger \Phi_j)(\Phi_k^\dagger \Phi_l), \quad (188)$$

where $i, j, k, l \in 1, 2, 3$.

Following the method presented in section 4.1, it is convenient to discuss the properties of the scalar potential, such as its stability and its spontaneous symmetry breaking, in terms of gauge invariant expressions. For this purpose we arrange the $SU(3)_L$ invariant scalar products into the the following three 2×2 hermitian matrices:

$$\begin{aligned} \underline{K} &= \begin{pmatrix} \Phi_1^\dagger \Phi_1 & \Phi_2^\dagger \Phi_1 \\ \Phi_1^\dagger \Phi_2 & \Phi_2^\dagger \Phi_2 \end{pmatrix}, & \underline{L} &= \begin{pmatrix} \Phi_1^\dagger \Phi_1 & \Phi_3^\dagger \Phi_1 \\ \Phi_1^\dagger \Phi_3 & \Phi_3^\dagger \Phi_3 \end{pmatrix}, \\ \underline{M} &= \begin{pmatrix} \Phi_2^\dagger \Phi_2 & \Phi_3^\dagger \Phi_2 \\ \Phi_2^\dagger \Phi_3 & \Phi_3^\dagger \Phi_3 \end{pmatrix}, \end{aligned} \quad (189)$$

where according with Eq. (21), each matrix is related to the following four real parameters

$$\underline{K} : \begin{cases} \Phi_1^\dagger \Phi_1 = (K_0 + K_3)/2, & \Phi_2^\dagger \Phi_2 = (K_0 - K_3)/2, \\ \Phi_1^\dagger \Phi_2 = (K_1 + i K_2)/2, & \Phi_2^\dagger \Phi_1 = (K_1 - i K_2)/2, \end{cases} \quad (190a)$$

$$\underline{L} : \begin{cases} \Phi_1^\dagger \Phi_1 = (L_0 + L_3)/2, & \Phi_3^\dagger \Phi_3 = (L_0 - L_3)/2, \\ \Phi_1^\dagger \Phi_3 = (L_1 + i L_2)/2, & \Phi_3^\dagger \Phi_1 = (L_1 - i L_2)/2, \end{cases} \quad (190b)$$

$$\underline{M} : \begin{cases} \Phi_2^\dagger \Phi_2 = (M_0 + M_3)/2, & \Phi_3^\dagger \Phi_3 = (M_0 - M_3)/2, \\ \Phi_2^\dagger \Phi_3 = (M_1 + i M_2)/2, & \Phi_3^\dagger \Phi_2 = (M_1 - i M_2)/2, \end{cases} \quad (190c)$$

with the constraints

$$K_0 \geq 0, \quad K_0^2 - K_1^2 - K_2^2 - K_3^2 = K_0^2 - \mathbf{K}^2 \geq 0, \quad (191a)$$

$$L_0 \geq 0, \quad L_0^2 - L_1^2 - L_2^2 - L_3^2 = L_0^2 - \mathbf{L}^2 \geq 0, \quad (191b)$$

$$M_0 \geq 0, \quad M_0^2 - M_1^2 - M_2^2 - M_3^2 = M_0^2 - \mathbf{M}^2 \geq 0. \quad (191c)$$

The scalar products $\Phi_1^\dagger \Phi_1$, $\Phi_2^\dagger \Phi_2$ and $\Phi_3^\dagger \Phi_3$, present in the expressions (190a), (190b) and (190c), allow us to eliminate three of the 12 variables due to the fact that

$$K_3 = L_0 - M_0, \quad (192a)$$

$$L_3 = K_0 - M_0, \quad (192b)$$

$$M_3 = K_0 - L_0, \quad (192c)$$

ending up with only the following nine real orbital variables, used to describe the full scalar potential

$$K_0, L_0, M_0, K_1, K_2, L_1, L_2, M_1, M_2. \quad (193)$$

With the help of the former variables, the scalar potential (156) may be written as

$$V(\Phi_1, \Phi_2, \Phi_3) = (V_{2K} + V_{4K}) + (V_{2L} + V_{4L}) + (V_{2M} + V_{4M}), \quad (194)$$

where as can be seen, the general space splits as the direct sum of three subspaces, due to the particular simple form of the scalar potential in (156) and to the fact that $\langle \Phi_3 \rangle$ is orthogonal to $\langle \Phi_1 \rangle$ and to $\langle \Phi_2 \rangle$, something which guarantees the validity of the generalized Schwartz's Inequality

$$\langle \Phi_1 | \Phi_1 \rangle \langle \Phi_2 | \Phi_2 \rangle \langle \Phi_3 | \Phi_3 \rangle \geq \langle \Phi_1 | \Phi_2 \rangle \langle \Phi_1 | \Phi_3 \rangle \langle \Phi_2 | \Phi_3 \rangle.$$

With the use of the real parameters $\xi_{k(l,m)0}$, $\xi_{k(l,m)a}$, $\eta_{k(l,m)00}$, $\eta_{k(l,m)a}$ and $\eta_{k(l,m)ab} = \eta_{k(l,m)ba}$, the following functions defined in the domain $|\mathbf{k}|, |\mathbf{l}|, |\mathbf{m}| \leq 1$.

$$J_{k2}(\mathbf{k}) = \xi_{k0} + \boldsymbol{\xi}_k \cdot \mathbf{k}, \quad (195a)$$

$$J_{k4}(\mathbf{k}) = \eta_{k00} + 2\boldsymbol{\eta}_k \cdot \mathbf{k} + \mathbf{k} \cdot \mathbf{E}_k \cdot \mathbf{k}, \quad (195b)$$

$$J_{l2}(\mathbf{l}) = \xi_{l0} + \boldsymbol{\xi}_l \cdot \mathbf{l}, \quad (195c)$$

$$J_{l4}(\mathbf{l}) = \eta_{l00} + 2\boldsymbol{\eta}_l \cdot \mathbf{l} + \mathbf{l} \cdot \mathbf{E}_l \cdot \mathbf{l}, \quad (195d)$$

$$J_{m2}(\mathbf{m}) = \xi_{m0} + \boldsymbol{\xi}_m \cdot \mathbf{m}, \quad (195e)$$

$$J_{m4}(\mathbf{m}) = \eta_{m00} + 2\boldsymbol{\eta}_m \cdot \mathbf{m} + \mathbf{m} \cdot \mathbf{E}_m \cdot \mathbf{m}, \quad (195f)$$

where in according to (191)

$$\mathbf{k} = \mathbf{K}/K_0, \quad (|\mathbf{k}| \leq 1); \quad (196a)$$

$$\mathbf{l} = \mathbf{L}/L_0, \quad (|\mathbf{l}| \leq 1); \quad (196b)$$

$$\mathbf{m} = \mathbf{M}/M_0, \quad (|\mathbf{m}| \leq 1), \quad (196c)$$

for $K_0, L_0, M_0 > 0$, allows us to write the terms of potential (194).

$$V_{2K} = \xi_{k0}K_0 + \xi_{ka}K_a = K_0J_{k2}(\mathbf{k}), \quad (197a)$$

$$\begin{aligned} V_{4K} &= \eta_{k00}K_0^2 + 2K_0\eta_{ka}K_a + K_a\eta_{kab}K_b \\ &= K_0^2J_{k4}(\mathbf{k}), \end{aligned} \quad (197b)$$

$$V_{2L} = \xi_{l0}L_0 + \xi_{la}L_a = L_0J_{l2}(\mathbf{l}), \quad (197c)$$

$$\begin{aligned} V_{4L} &= \eta_{l00}L_0^2 + 2L_0\eta_{la}L_a + L_a\eta_{lab}L_b \\ &= L_0^2J_{l4}(\mathbf{l}), \end{aligned} \quad (197d)$$

$$V_{2M} = \xi_{m0}M_0 + \xi_{ma}M_a = M_0J_{m2}(\mathbf{m}), \quad (197e)$$

$$\begin{aligned} V_{4M} &= \eta_{m00}M_0^2 + 2M_0\eta_{ma}M_a + M_a\eta_{mab}M_b \\ &= M_0^2J_{m4}(\mathbf{m}), \end{aligned} \quad (197f)$$

where sum over the indices a and b from 1 to 3 must be understood. In the former expressions, the following notation has been used: $E_k = \eta_{kab}$, $E_l = \eta_{lab}$, $E_m = \eta_{mab}$. The parametrization employed in (189)-(193) should not invalidate the stability conditions (in the strong sense) as far as sufficient conditions are concerned (necessary and sufficient conditions should be affected).

On the other hand, the parameters given in (193) does not imply that the matrix arrangements (189) can be established. In that way, the parameters (193) may help in the procedure to find the stationary points in the scalar potential, but it is necessary to verify at the end, if all matrices (189) are consistent with the stationary points found. That is the analysis given below.

5.4.2 Stability conditions

For the potential to be stable, it must be bounded from below. The stability is determined by the behavior of V in the limit $K_0 \rightarrow \infty$, $L_0 \rightarrow \infty$ and/or $M_0 \rightarrow \infty$; hence, by the signs of $J_{k(l,m)2}(\mathbf{k}, \mathbf{l}, \mathbf{m})$ and $J_{k(l,m)4}(\mathbf{k}, \mathbf{l}, \mathbf{m})$ in (197), (approach which conduces only to sufficiency conditions but not to necessary conditions).

In the strong sense, the stability of the potential is guaranteed when $V \rightarrow \infty$ for \mathbf{k} , \mathbf{l} and \mathbf{m} taking any value, which means that

$$J_{k4}(\mathbf{k}), J_{l4}(\mathbf{l}), J_{m4}(\mathbf{m}) > 0 \text{ for all } |\mathbf{k}|, |\mathbf{l}|, |\mathbf{m}| \leq 1. \quad (198)$$

To assure the existence of a positive (semi-)definite value for $J_{k(l,m)4}(\mathbf{k}, \mathbf{l}, \mathbf{m})$, it is sufficient to consider its value for all the stationary points of $J_{k(l,m)4}(\mathbf{k}, \mathbf{l}, \mathbf{m})$ in the domain $|\mathbf{k}|, |\mathbf{l}|, |\mathbf{m}| < 1$, and for all stationary points on the boundary $|\mathbf{k}|, |\mathbf{l}|, |\mathbf{m}| = 1$. This holds, because the global minimum of the continuous function $J_{k(l,m)4}(\mathbf{k}, \mathbf{l}, \mathbf{m})$ is reached on the compact domain $|\mathbf{k}|, |\mathbf{l}|, |\mathbf{m}| \leq 1$, and it is located among those stationary points. This leads to bounds on $\eta_{k(l,m)00}$, $\eta_{k(l,m)a}$ and $\eta_{k(l,m)ab}$, which parametrise the quartic term $V_{4K(L,M)}$ of the potential. A detailed analysis of the stability criteria for a scalar potential can be found in Refs [23, 45].

With the help of Eqs. (190) and (192), the parameters defined in (197), for the scalar potential in (156) are:

$$\begin{aligned} \xi_{k0} &= (\mu_1^2 + \mu_2^2 - \mu_3^2)/2, \quad \eta_{k00} = (\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4)/4, \\ \boldsymbol{\xi}_k &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\eta}_k = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad E_k = \begin{pmatrix} (\lambda_7 + 2\lambda_{10})/4 & 0 & 0 \\ 0 & \lambda_7/4 & 0 \\ 0 & 0 & (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)/4 \end{pmatrix}, \\ \xi_{l0} &= (\mu_1^2 - \mu_2^2 + \mu_3^2)/2, \quad \eta_{l00} = (\lambda_1 - \lambda_2 + \lambda_3 + \lambda_5)/4, \\ \boldsymbol{\xi}_l &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\eta}_l = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad E_l = \begin{pmatrix} \lambda_8/4 & 0 & 0 \\ 0 & \lambda_8/4 & 0 \\ 0 & 0 & (\lambda_1 - \lambda_2 + \lambda_3 - \lambda_5)/4 \end{pmatrix}, \\ \xi_{m0} &= (-\mu_1^2 + \mu_2^2 + \mu_3^2)/2, \quad \eta_{m00} = (-\lambda_1 + \lambda_2 + \lambda_3 + \lambda_6)/4, \\ \boldsymbol{\xi}_m &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\eta}_m = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad E_m = \begin{pmatrix} \lambda_9/4 & 0 & 0 \\ 0 & \lambda_9/4 & 0 \\ 0 & 0 & (-\lambda_1 + \lambda_2 + \lambda_3 - \lambda_6)/4 \end{pmatrix}. \end{aligned} \quad (199)$$

The stability criteria established in the previous section allow us to bound the parameters of the potential in the following way:

For K :

$$\lambda_1 + \lambda_2 - \lambda_3 > 0, \quad (200a)$$

$$\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 > 0, \quad (200b)$$

$$\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 + \lambda_7 > 0, \quad (200c)$$

$$\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 + \lambda_7 + 2\lambda_{10} > 0, \quad (200d)$$

For L :

$$\lambda_1 - \lambda_2 + \lambda_3 > 0, \quad (201a)$$

$$\lambda_1 - \lambda_2 + \lambda_3 + \lambda_5 > 0, \quad (201b)$$

$$\lambda_1 - \lambda_2 + \lambda_3 + \lambda_5 + \lambda_8 > 0, \quad (201c)$$

For M :

$$-\lambda_1 + \lambda_2 + \lambda_3 > 0, \quad (202a)$$

$$-\lambda_1 + \lambda_2 + \lambda_3 + \lambda_6 > 0, \quad (202b)$$

$$-\lambda_1 + \lambda_2 + \lambda_3 + \lambda_6 + \lambda_9 > 0. \quad (202c)$$

In this way, when the constraints (200),(201) and (202) are satisfied, the potential is stable in the strong sense. The former constraints can be combined: summing (200a)+(201a), (200a)+(202a), and (201a)+(202a), we have respectively

$$\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_3 > 0. \quad (203)$$

From the sums (200a)+(201b), (200b)+(201a), (200a)+(202b), (200b)+(202a), (201a)+(202b), (201b)+(202a), we have

$$\begin{aligned} 2\lambda_1 &> -\lambda_5, & 2\lambda_1 &> -\lambda_4, & 2\lambda_2 &> -\lambda_6, \\ 2\lambda_2 &> -\lambda_4, & 2\lambda_3 &> -\lambda_6, & 2\lambda_3 &> -\lambda_5. \end{aligned} \quad (204)$$

The following operations (200b)+(201b), (200d)+(201a), (200a)+(201c), (200b)+(202b), (200d)+(202a), (200a)+(202c), (200b)+(202b), (201c)+(202a), (201a)+(202c) give, respectively

$$\begin{aligned} 2\lambda_1 &> -(\lambda_4 + \lambda_5), & 2\lambda_1 &> -(\lambda_4 + \lambda_7), \\ 2\lambda_1 &> -(\lambda_5 + \lambda_8), & 2\lambda_2 &> -(\lambda_4 + \lambda_6), \\ 2\lambda_2 &> -(\lambda_4 + \lambda_7), & 2\lambda_2 &> -(\lambda_6 + \lambda_9), \\ 2\lambda_3 &> -(\lambda_5 + \lambda_6), & 2\lambda_3 &> -(\lambda_5 + \lambda_8), \\ 2\lambda_3 &> -(\lambda_6 + \lambda_9). \end{aligned} \quad (205)$$

Other two interesting conditions are (200d)+(201a), (200d)+(202a)

$$\begin{aligned} 2\lambda_1 &> -(\lambda_4 + \lambda_7 + 2\lambda_{10}), \\ 2\lambda_2 &> -(\lambda_4 + \lambda_7 + 2\lambda_{10}). \end{aligned} \quad (206)$$

And else inequalities that have not yet derived here can be found.

Notice that conditions (203), (204) and (206) derived by stability conditions, are compatible with some constraints (166a), (166b) and (182) derived by positive masses (positive concavity) conditions.

When k_3, l_3 and m_3 take fixed values, we can make use of Eqs. (192) and (196) in order to write the orbital variables as

$$K_0 = K_0, \quad L_0 = K_0 \left(\frac{1+k_3}{1+l_3} \right) \quad \text{and} \quad M_0 = K_0 \left(\frac{1-l_3 k_3}{1+l_3} \right), \quad (207)$$

where K_0 is an independent free parameter, such that the following conditions

$$\xi_{k0} < |\xi_k|, \quad \text{and} \quad \xi_{l0} < |\xi_l|, \quad \text{and} \quad \xi_{m0} < |\xi_m|, \quad (208)$$

imply

$$\left. \frac{\partial V}{\partial K_0} \right|_{\substack{\mathbf{k} \text{ fixed,} \\ K_0=0}} = \xi_{k0} + \boldsymbol{\xi}_k \cdot \mathbf{k} < 0, \text{ and} \quad (209)$$

$$\left. \frac{\partial V}{\partial L_0} \right|_{\substack{\mathbf{l} \text{ fixed,} \\ L_0=0}} = \xi_{l0} + \boldsymbol{\xi}_l \cdot \mathbf{l} < 0, \text{ and} \quad (210)$$

$$\left. \frac{\partial V}{\partial M_0} \right|_{\substack{\mathbf{m} \text{ fixed,} \\ M_0=0}} = \xi_{m0} + \boldsymbol{\xi}_m \cdot \mathbf{m} < 0. \quad (211)$$

for some fixed values \mathbf{k} , \mathbf{l} , and \mathbf{m} while varying K_0 , L_0 , and M_0 in the form given by Eqs. (207). This guarantees that the global minimum of V lies at $\Phi_i \neq 0$. For our case, from (199) and (208) we have that

$$(\mu_1^2 + \mu_2^2 - \mu_3^2), (\mu_1^2 - \mu_2^2 + \mu_3^2), (-\mu_1^2 + \mu_2^2 + \mu_3^2) < 0,$$

which implies

$$\mu_1^2 < 0, \quad \text{and} \quad \mu_2^2 < 0, \quad \text{and} \quad \mu_3^2 < 0. \quad (212)$$

5.5 Stationary Points

Now let us find the stationary points of the scalar potential, since among those points the local and global minima are located. To start with, let us define the following nine component vector

$$\tilde{\mathbf{P}} = (K_0 \quad L_0 \quad M_0 \quad K_1 \quad K_2 \quad L_1 \quad L_2 \quad M_1 \quad M_2)^T \quad (213)$$

and let's also define the following nine vectors, each one with eight components

$$\tilde{\mathbf{P}}_{\{i\}}, \quad \text{for } i = 1, 2, \dots, 9, \quad (214)$$

where $\tilde{\mathbf{P}}_{\{i\}}$ is the vector $\tilde{\mathbf{P}}$ with the i^{th} entry suppressed [for example $\tilde{\mathbf{P}}_{\{1\}} = (L_0 \quad M_0 \quad K_1 \quad K_2 \quad L_1 \quad L_2 \quad M_1 \quad M_2)^T$, etc.]

With the help of this notation the potential (156) reads

$$V = \tilde{\mathbf{P}} \cdot \tilde{\boldsymbol{\xi}} + \tilde{\mathbf{P}} \cdot \tilde{\mathbf{E}} \cdot \tilde{\mathbf{P}} \quad (215)$$

where

$$\tilde{\boldsymbol{\xi}} = \begin{pmatrix} \frac{\mu_1^2 + \mu_2^2 - \mu_3^2}{2} \\ \frac{\mu_1^2 - \mu_2^2 + \mu_3^2}{2} \\ \frac{-\mu_1^2 + \mu_2^2 + \mu_3^2}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{E}} = \begin{pmatrix} \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_6 - \lambda_5}{4} & \frac{\lambda_1 + \lambda_6 - \lambda_3 - \lambda_2}{4} & \frac{\lambda_5 - \lambda_3 + \lambda_2 - \lambda_1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda_1 + \lambda_6 - \lambda_3 - \lambda_2}{4} & \frac{\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6}{4} & \frac{\lambda_4 + \lambda_3 - \lambda_2 - \lambda_1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda_5 - \lambda_3 + \lambda_2 - \lambda_1}{4} & \frac{\lambda_4 + \lambda_3 - \lambda_2 - \lambda_1}{4} & \frac{\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 + \lambda_6}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda_7 + 2\lambda_{10}}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\lambda_7}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\lambda_8}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\lambda_8}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\lambda_9}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\lambda_9}{4} \end{pmatrix}. \quad (216)$$

The domain of orbital variables, Eqs. (191), can be written in the following form.

$$\tilde{\mathbf{P}} \cdot \tilde{\mathbf{g}}_1 \cdot \tilde{\mathbf{P}} \geq 0, \quad \tilde{\mathbf{P}} \cdot \tilde{\mathbf{g}}_2 \cdot \tilde{\mathbf{P}} \geq 0, \quad \tilde{\mathbf{P}} \cdot \tilde{\mathbf{g}}_3 \cdot \tilde{\mathbf{P}} \geq 0, \quad K_0 \geq 0, \quad L_0 \geq 0, \quad M_0 \geq 0, \quad \text{with} \quad (217)$$

$$\tilde{\mathbf{g}}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{g}}_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{g}}_3 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (218)$$

The trivial configuration $\tilde{\mathbf{P}} = 0$ is a stationary point of the potential with $V = 0$, as a direct consequence of the definitions.

For the discussion of the stationary points of V , we must distinguish among the following cases

$$\tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} > 0, \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} > 0, \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}} > 0; \quad (219a)$$

$$\tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} = 0, \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} > 0, \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}} > 0; \quad (219b)$$

$$\tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} > 0, \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} = 0, \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}} > 0; \quad (219c)$$

$$\tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} > 0, \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} > 0, \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}} = 0; \quad (219d)$$

$$\tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} = 0, \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} = 0, \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}} > 0; \quad (219e)$$

$$\tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} = 0, \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} > 0, \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}} = 0; \quad (219f)$$

$$\tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} > 0, \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} = 0, \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}} = 0; \quad (219g)$$

$$\tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} = 0, \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} = 0, \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}} = 0. \quad (219h)$$

The stationary points of V in the inner part of the domain, cases (219a), (219c), (219d) and (219g), imply linear independence between $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$, something which is not allowed. The global minimum for the case (219h) implies LD among $\langle \Phi_1 \rangle$, $\langle \Phi_2 \rangle$ and $\langle \Phi_3 \rangle$, not allowed either. Case (219e) implies $\langle \Phi_1 \rangle = 0$. So, the only two cases of concern for us here are (219b) corresponding to the general vacuum structure with $v_1, v_2, v_3, V_1, V_2 \neq 0$ studied in Sec. (5.3.1), and case (219f) which corresponds to the vacuum structure $\langle \Phi_2 \rangle = 0$ studied in Sec. (5.3.2).

In general, the stationary points of the scalar potential in (215), for any domain in (219), are stationary points of the function

$$\tilde{F}(\tilde{\mathbf{P}}, u, v, w) = V - u \tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} - v \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} - w \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}}, \quad (220)$$

where u, v and w are Lagrange multipliers. The relevant stationary points of \tilde{F} are thus given as solutions to the equation

$$(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3)\tilde{\mathbf{P}} = -\frac{1}{2}\tilde{\xi}; \quad \text{with} \quad (221a)$$

$$\tilde{\mathbf{P}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}} \geq 0, \tilde{\mathbf{P}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}} \geq 0, \tilde{\mathbf{P}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}} \geq 0, \quad (221b)$$

and

$$K_0, L_0, M_0 > 0, \quad (222)$$

with the inequality (> 0) in Eq. (221b) taking place, for the case when the Lagrange multipliers are excluded. For regular values of u, v and w , with the determinant

$$\det(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3) \neq 0$$

we have

$$\tilde{\mathbf{P}} = -\frac{1}{2}(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3)^{-1}\tilde{\xi}. \quad (223)$$

The Lagrange multipliers are thus obtained by inserting (223) in the constraint Eqs. (221b):

$$\begin{aligned} \tilde{\xi}(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3)^{-1}\tilde{g}_1(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3)^{-1}\tilde{\xi} &= 0, \\ \tilde{\xi}(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3)^{-1}\tilde{g}_2(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3)^{-1}\tilde{\xi} &= 0, \\ \tilde{\xi}(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3)^{-1}\tilde{g}_3(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3)^{-1}\tilde{\xi} &= 0, \end{aligned} \quad (224)$$

and $K_0, L_0, M_0 > 0$.

Additionally, there may be up to 9 values $u = \tilde{\mu}_a$ (and also 9 values for $v = \tilde{\mu}_a$, and for $w = \tilde{\mu}_a$) with $a = 1, \dots, 9$, for which $\det(\tilde{E} - u\tilde{g}_1 - v\tilde{g}_2 - w\tilde{g}_3) = 0$. Depending on the form of the potential; some, or all of them, may lead to exceptional solutions of (221a).

For *any* stationary point of the potential we have

$$V|_{stat} = \frac{1}{2}\tilde{\mathbf{P}} \cdot \tilde{\xi} = -\tilde{\mathbf{P}} \cdot \tilde{E} \cdot \tilde{\mathbf{P}}. \quad (225)$$

Suppose now that the strong stability condition (198) holds. Then (225) gives for non-trivial stationary points where $\tilde{\mathbf{P}} \neq 0$:

$$V|_{stat} < 0. \quad (226)$$

Firstly, in the context that only one Lagrange multiplier is non-zero, for instance $u_p \neq 0, v = w = 0$ in (221a), let us consider $\tilde{\mathbf{p}} = (p_{k_0}, p_{l_0}, p_{m_0}, p_{k_1}, p_{k_2}, p_{l_1}, p_{l_2}, p_{m_1}, p_{m_2})^T$ be an stationary point for this case. Then, from (215) and (221a) we have

$$\left. \frac{\partial V}{\partial K_0} \right|_{\substack{\tilde{\mathbf{P}}_{\{1\}} \text{ fixed,} \\ \tilde{\mathbf{P}} = \tilde{\mathbf{p}}}} = 2 u_p p_{k_0}, \quad (227a)$$

$$\left. \frac{\partial V}{\partial L_0} \right|_{\substack{\tilde{\mathbf{P}}_{\{2\}} \text{ fixed,} \\ \tilde{\mathbf{P}} = \tilde{\mathbf{p}}}} = 2 u_p p_{l_0}, \quad (227b)$$

$$\left. \frac{\partial V}{\partial M_0} \right|_{\substack{\tilde{\mathbf{P}}_{\{3\}} \text{ fixed,} \\ \tilde{\mathbf{P}} = \tilde{\mathbf{p}}}} = 2 u_p p_{m_0}, \quad (227c)$$

where the notation established in (214) has been used. For the analysis which follows, only the most convenient partial derivative from (227) is chosen, in such a way that only two, out of the three values in (192) lower down, and also that the inequalities (191) hold for the new points; (for example, let us take M_0 . From (192) we have $(K_3, L_3) \rightarrow (K'_3 = K_3 - \Delta M_0), (L'_3 = L_3 - \Delta M_0)$. Here $|K'_3| < |K_3|$ and $|L'_3| < |L_3|$, if $K_3, L_3 > 0$). If $u_p < 0$, there are points $\tilde{\mathbf{P}}$ with $K_0 > p_{k_0}$ (or $L_0 > p_{l_0}, M_0 > p_{m_0}$), $\tilde{\mathbf{P}}_{\{1\}} = \tilde{\mathbf{p}}_{\{1\}}$ (or $\tilde{\mathbf{P}}_{\{2\}} = \tilde{\mathbf{p}}_{\{2\}}, \tilde{\mathbf{P}}_{\{3\}} = \tilde{\mathbf{p}}_{\{3\}}$) for which the potential decreases in its neighborhood, and as a consequence cannot be a minimum. We conclude that in a theory with the required EWSB, a stationary point coming from an unique no null Lagrange multiplier (for example $u \neq 0, v = w = 0$), to be a global minimum candidate, it must hold

$$u_0 > 0. \quad (228)$$

Secondly, for the stationary points $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$, we have from (221a) and (225), the following relation

$$\begin{aligned} V(\tilde{\mathbf{p}}) - V(\tilde{\mathbf{q}}) &= \frac{1}{2} \tilde{\mathbf{p}} \cdot \tilde{\boldsymbol{\xi}} - \frac{1}{2} \tilde{\mathbf{q}} \cdot \tilde{\boldsymbol{\xi}} \\ &= \tilde{\mathbf{p}} \cdot (u_q \tilde{g}_1 + v_q \tilde{g}_2 + w_q \tilde{g}_3 - \tilde{E}) \cdot \tilde{\mathbf{q}} - \tilde{\mathbf{q}} \cdot (u_p \tilde{g}_1 + v_p \tilde{g}_2 + w_p \tilde{g}_3 - \tilde{E}) \cdot \tilde{\mathbf{p}} \\ &= (u_q - u_p) \tilde{\mathbf{p}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{q}} + (v_q - v_p) \tilde{\mathbf{p}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{q}} + (w_q - w_p) \tilde{\mathbf{p}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{q}}, \end{aligned} \quad (229)$$

where $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ are vectors on the forward light cone, and $\tilde{\mathbf{p}} \cdot \tilde{g}_1 \cdot \tilde{\mathbf{q}}, \tilde{\mathbf{p}} \cdot \tilde{g}_2 \cdot \tilde{\mathbf{q}}, \tilde{\mathbf{p}} \cdot \tilde{g}_3 \cdot \tilde{\mathbf{q}}$ are always non-negative.

In Table 2, an exhaustive of all the possible stationary points of the potential are presented (even if it includes unphysical VEVs). Lagrange multipliers coming from solutions of (221a) belonging to regulars and exceptional values for the cases are stated. Those cases of Lagrange multipliers giving the same stationary point, only one of them were included in Table 2. Solutions which imply specific relations among the parameters $\mu_1^2, \mu_2^2, \mu_3^2$ were excluded too (see for example [45]). Incidentally, it is important to say that all stationary points written in Table 2 give physical VEVs, i.e, they are consistent in relation with matrices (189).

To end our analysis, let us apply our findings to the two independent vacuum structures given by the constraints (219b) and (219f), which were studied in detail in Sects. (5.3.1) and (5.3.2) respectively.

5.5.1 Case $v_2 = V_2 = 0$

We want a global minimum with the configuration (174), which implies solutions satisfying (219f). For this purpose we use the results in Table (2), the stability conditions stated in Sec. (5.4.2) and taking into account expression (229). In the case of an unique no null Lagrange multiplier, the restriction in (228) can be used. In that way, the conditions found below are sufficient (but not necessary) to have the minimum of the scalar potential (156) at $\tilde{\mathbf{P}}_1$.

From the Table (2), we want the global minimum be associated to Lagrange multiplier u_1 (where w_2 gives the same stationary point) which does not coincide with solutions inside the forward light cone (219a).

Domains	Lagrange Multipliers	Stationary Points
$\tilde{P}_1 \cdot \tilde{g}_1 \cdot \tilde{P}_1 = 0,$ $\tilde{P}_1 \cdot \tilde{g}_2 \cdot \tilde{P}_1 = 4 K_{01} M_{01},$ $\tilde{P}_1 \cdot \tilde{g}_3 \cdot \tilde{P}_1 = 0.$	$u_1 = \frac{\mu_2^2 + \lambda_4 K_{01} + \lambda_6 M_{01}}{4 K_{01}}, \quad w_1 = 0$	$\underline{\tilde{P}}_1 = \begin{pmatrix} K_{01} = \frac{\lambda_5 \mu_3^2 - 2 \lambda_3 \mu_1^2}{4 \lambda_1 \lambda_3 - \lambda_5^2} \\ L_{01} = K_{01} + M_{01} \\ M_{01} = \frac{\lambda_5 \mu_1^2 - 2 \lambda_1 \mu_3^2}{4 \lambda_1 \lambda_3 - \lambda_5^2} \end{pmatrix}$
	$u_2 = 0, \quad w_2 = \frac{\mu_2^2 + \lambda_4 K_{01} + \lambda_6 M_{01}}{4 M_{01}}$	$\underline{\tilde{P}}_2 = \underline{\tilde{P}}_1$
$\tilde{P}_3 \cdot \tilde{g}_1 \cdot \tilde{P}_3 = 0,$ $\tilde{P}_3 \cdot \tilde{g}_2 \cdot \tilde{P}_3 = 0,$ $\tilde{P}_3 \cdot \tilde{g}_3 \cdot \tilde{P}_3 = 4 K_{03} L_{03}.$	$u_3 = \frac{\mu_1^2 + \lambda_4 K_{03} + \lambda_5 L_{03}}{4 K_{03}}, \quad v_3 = 0$	$\underline{\tilde{P}}_3 = \begin{pmatrix} K_{03} = \frac{\lambda_6 \mu_3^2 - 2 \lambda_3 \mu_2^2}{4 \lambda_2 \lambda_3 - \lambda_6^2} \\ L_{03} = \frac{\lambda_6 \mu_2^2 - 2 \lambda_2 \mu_3^2}{4 \lambda_2 \lambda_3 - \lambda_6^2} \\ K_{03} + L_{03} \end{pmatrix}$
$\tilde{P}_4 \cdot \tilde{g}_1 \cdot \tilde{P}_4 = 4 L_4 M_4,$ $\tilde{P}_4 \cdot \tilde{g}_2 \cdot \tilde{P}_4 = 0,$ $\tilde{P}_4 \cdot \tilde{g}_3 \cdot \tilde{P}_4 = 0.$	$v_4 = 0, \quad w_4 = \frac{\mu_3^2 + \lambda_5 L_4 + \lambda_6 M_4}{4 M_4}$	$\underline{\tilde{P}}_4 = \begin{pmatrix} L_4 + M_4 \\ L_4 = \frac{\lambda_4 \mu_3^2 - 2 \lambda_2 \mu_1^2}{4 \lambda_1 \lambda_2 - \lambda_4^2} \\ M_4 = \frac{\lambda_4 \mu_1^2 - 2 \lambda_1 \mu_3^2}{4 \lambda_1 \lambda_2 - \lambda_4^2} \end{pmatrix}$
$\tilde{P}_{\{5,6\}} \cdot \tilde{g}_1 \cdot \tilde{P}_{\{5,6\}} = 0,$ $\tilde{P}_{\{5,6\}} \cdot \tilde{g}_2 \cdot \tilde{P}_{\{5,6\}} \neq 0,$ $\tilde{P}_{\{5,6\}} \cdot \tilde{g}_3 \cdot \tilde{P}_{\{5,6\}} \neq 0.$	Exceptional solution: $u_5 = -\frac{\lambda_7 + 2 \lambda_{10}}{4}$	$\underline{\tilde{P}}_5$
	Exceptional solution: $u_6 = -\frac{\lambda_7}{4}$	$\underline{\tilde{P}}_6$
$\tilde{P}_7 \cdot \tilde{g}_1 \cdot \tilde{P}_7 \neq 0,$ $\tilde{P}_7 \cdot \tilde{g}_2 \cdot \tilde{P}_7 = 0,$ $\tilde{P}_7 \cdot \tilde{g}_3 \cdot \tilde{P}_7 \neq 0.$	Exceptional solution: $v_7 = -\frac{\lambda_8}{4}$	$\underline{\tilde{P}}_7$
$\tilde{P}_8 \cdot \tilde{g}_1 \cdot \tilde{P}_8 \neq 0,$ $\tilde{P}_8 \cdot \tilde{g}_2 \cdot \tilde{P}_8 \neq 0,$ $\tilde{P}_8 \cdot \tilde{g}_3 \cdot \tilde{P}_8 = 0.$	Exceptional solution: $w_8 = -\frac{\lambda_9}{4}$	$\underline{\tilde{P}}_8$
$\tilde{P}_9 \cdot \tilde{g}_1 \cdot \tilde{P}_9 = 0,$ $\tilde{P}_9 \cdot \tilde{g}_2 \cdot \tilde{P}_9 = 0,$ $\tilde{P}_9 \cdot \tilde{g}_3 \cdot \tilde{P}_9 = 0.$	Exceptional solution: $u_9 = -\frac{\lambda_7 + 2 \lambda_{10}}{4},$ $v_9 = \frac{\mu_3^2 + \lambda_5 L_9 + \lambda_6 M_9}{4 L_9}, \quad w_9 = 0$	$\underline{\tilde{P}}_9 = \begin{pmatrix} L_9 + M_9 \\ L_9 = \frac{(\lambda_4 + \lambda_7 + 2 \lambda_{10}) \mu_2^2 - 2 \lambda_2 \mu_1^2}{4 \lambda_1 \lambda_2 - (\lambda_4 + \lambda_7 + 2 \lambda_{10})^2} \\ M_9 = \frac{(\lambda_4 + \lambda_7 + 2 \lambda_{10}) \mu_1^2 - 2 \lambda_1 \mu_2^2}{4 \lambda_1 \lambda_2 - (\lambda_4 + \lambda_7 + 2 \lambda_{10})^2} \\ \pm 2 \sqrt{L_9 M_9} \end{pmatrix}$
$\tilde{P}_{10} \cdot \tilde{g}_1 \cdot \tilde{P}_{10} = 0,$ $\tilde{P}_{10} \cdot \tilde{g}_2 \cdot \tilde{P}_{10} = 0,$ $\tilde{P}_{10} \cdot \tilde{g}_3 \cdot \tilde{P}_{10} = 0.$	Exceptional solution: $u_{10} = -\frac{\lambda_7}{4},$ $v_{10} = \frac{\mu_3^2 + \lambda_5 L_{10} + \lambda_6 M_{10}}{4 L_{10}}, \quad w_{10} = 0$	$\underline{\tilde{P}}_{10} = \begin{pmatrix} L_{10} + M_{10} \\ L_{10} = \frac{(\lambda_4 + \lambda_7) \mu_2^2 - 2 \lambda_2 \mu_1^2}{4 \lambda_1 \lambda_2 - (\lambda_4 + \lambda_7)^2} \\ M_{10} = \frac{(\lambda_4 + \lambda_7) \mu_1^2 - 2 \lambda_1 \mu_2^2}{4 \lambda_1 \lambda_2 - (\lambda_4 + \lambda_7)^2} \\ \pm 2 \sqrt{L_{10} M_{10}} \end{pmatrix}$
$\tilde{P}_{11} \cdot \tilde{g}_1 \cdot \tilde{P}_{11} = 0,$ $\tilde{P}_{11} \cdot \tilde{g}_2 \cdot \tilde{P}_{11} = 0,$ $\tilde{P}_{11} \cdot \tilde{g}_3 \cdot \tilde{P}_{11} = 0.$	Exceptional solution: $u_{11} = \frac{\mu_3^2 + \lambda_4 K_{11} + \lambda_6 M_{11}}{4 K_{11}},$ $v_{11} = -\frac{\lambda_8}{4}, \quad w_{11} = 0$	$\underline{\tilde{P}}_{11} = \begin{pmatrix} K_{11} = \frac{(\lambda_5 + \lambda_8) \mu_3^2 - 2 \lambda_3 \mu_1^2}{4 \lambda_1 \lambda_3 - (\lambda_5 + \lambda_8)^2} \\ K_{11} + M_{11} \\ M_{11} = \frac{(\lambda_5 + \lambda_8) \mu_1^2 - 2 \lambda_1 \mu_3^2}{4 \lambda_1 \lambda_3 - (\lambda_5 + \lambda_8)^2} \\ \pm 2 \sqrt{K_{11} M_{11}} \end{pmatrix}$
$\tilde{P}_{12} \cdot \tilde{g}_1 \cdot \tilde{P}_{12} = 0,$ $\tilde{P}_{12} \cdot \tilde{g}_2 \cdot \tilde{P}_{12} = 0,$ $\tilde{P}_{12} \cdot \tilde{g}_3 \cdot \tilde{P}_{12} = 0.$	Exceptional solution: $u_{12} = \frac{\mu_1^2 + \lambda_4 K_{12} + \lambda_5 L_{12}}{4 K_{12}},$ $v_{12} = -\frac{\lambda_9}{4}, \quad w_{12} = 0$	$\underline{\tilde{P}}_{12} = \begin{pmatrix} K_{12} = \frac{(\lambda_6 + \lambda_9) \mu_3^2 - 2 \lambda_3 \mu_2^2}{4 \lambda_2 \lambda_3 - (\lambda_6 + \lambda_9)^2} \\ L_{12} = \frac{(\lambda_6 + \lambda_9) \mu_2^2 - 2 \lambda_2 \mu_3^2}{4 \lambda_2 \lambda_3 - (\lambda_6 + \lambda_9)^2} \\ K_{12} + L_{12} \\ \pm 2 \sqrt{K_{12} L_{12}} \end{pmatrix}$
$\tilde{P} \cdot \tilde{g}_1 \cdot \tilde{P} = 0,$ $\tilde{P} \cdot \tilde{g}_2 \cdot \tilde{P} = 0,$ $\tilde{P} \cdot \tilde{g}_3 \cdot \tilde{P} = 0.$	$u_{13} = \frac{\lambda_4 \mu_1^2 - 2 \lambda_1 \mu_2^2}{4 \mu_1^2}, \quad v_{13} = \frac{\lambda_5 \mu_1^2 - 2 \lambda_1 \mu_3^2}{4 \mu_1^2}, \quad w_{13} = 0$	$\underline{\tilde{P}}_{13} = \begin{pmatrix} -\frac{\mu_1^2}{2 \lambda_1} \\ -\frac{\mu_1^2}{2 \lambda_1} \\ 0 \end{pmatrix}$
	$u_{14} = \frac{\lambda_4 \mu_2^2 - 2 \lambda_2 \mu_1^2}{4 \mu_2^2}, \quad v_{14} = 0, \quad w_{14} = \frac{\lambda_6 \mu_2^2 - 2 \lambda_2 \mu_3^2}{4 \mu_2^2}$	$\underline{\tilde{P}}_{14} = \begin{pmatrix} -\frac{\mu_2^2}{2 \lambda_2} \\ 0 \\ -\frac{\mu_2^2}{2 \lambda_2} \end{pmatrix}$
	$u_{15} = 0, \quad v_{15} = \frac{\lambda_5 \mu_3^2 - 2 \lambda_3 \mu_1^2}{4 \mu_3^2}, \quad w_{15} = \frac{\lambda_6 \mu_3^2 - 2 \lambda_3 \mu_2^2}{4 \mu_3^2}$	$\underline{\tilde{P}}_{15} = \begin{pmatrix} 0 \\ -\frac{\mu_3^2}{2 \lambda_3} \\ -\frac{\mu_3^2}{2 \lambda_3} \end{pmatrix}$

Table 2: Lagrange Multipliers. The stationary points ($\underline{\tilde{P}}$) were underlined indicating that only the upper non-zero entries of column vector are written, with the remaining ones entries filled by zeros.

Let's assume the following conditions

$$\lambda_4 > 0, \quad \lambda_5 < 0 \quad \text{and} \quad \lambda_6 > 0, \quad (230)$$

which in combination with the inequalities (203), (204) and (212), we have that $\lambda_5\mu_3^2 - 2\lambda_3\mu_1^2 > 0$, $\lambda_5\mu_1^2 - 2\lambda_1\mu_3^2 > 0$, and $4\lambda_1\lambda_3 - \lambda_5^2 > 0$, hence $K_{01} > 0$, $M_{01} > 0$ and $L_{01} > 0$. Additionally, with same arguments, we can verify that the Lagrange multipliers $v_{13}, v_{15} < 0$.

As you can see, there is not inconvenient to impose the following condition

$$u_1 > 0, \quad (231)$$

as is required by the global minimum condition (228). Therefore, for the moment, the point $\tilde{\mathbf{P}}_1$ satisfy all requirements to be a stationary point. The other aspect to take into account, it is to show that this point is the global minimum of potential. For that, let's see the other Lagrange multipliers and their points, and to establish conditions over them such that the global are not found there.

For example, if we assume

$$\lambda_7 + 2\lambda_{10} > 0, \quad \lambda_7 > 0, \quad \lambda_8 > 0 \quad \lambda_9 > 0, \quad (232)$$

it immediately discards out the points $\tilde{\mathbf{P}}_5, \tilde{\mathbf{P}}_6, \tilde{\mathbf{P}}_7$ and $\tilde{\mathbf{P}}_8$ as minimal global points of potential, because the corresponding Lagrange multipliers u_5, u_6, v_7 and w_8 are negative numbers.

If we assume

$$\lambda_6\mu_2^2 - 2\lambda_2\mu_3^2 > 0, \quad (233)$$

$$\lambda_6\mu_3^2 - 2\lambda_3\mu_2^2 < 0, \quad (234)$$

it gives either the condition $K_{03} > 0$ or $L_{03} > 0$, but not both conditions satisfied simultaneously. And, in similar way, considering the case

$$\lambda_4\mu_2^2 - 2\lambda_2\mu_1^2 > 0, \quad (235)$$

$$\lambda_4\mu_1^2 - 2\lambda_1\mu_2^2 < 0 \quad (236)$$

we conclude that L_4, M_4 are not positive numbers simultaneously. Then, $\tilde{\mathbf{P}}_3$ and $\tilde{\mathbf{P}}_4$ are not stationary points of potential.

From (212), (235) and (233) we see that the Lagrange multipliers $u_{14}, w_{14} < 0$. Thus, the minimum of potential is not present at $\tilde{\mathbf{P}}_{14}$.

We derived above that $v_{13} < 0$ and $v_{15} < 0$, that together with conditions (212) and (230), it is easy to verify that $u_1 - u_{13} = [v_{13}(\lambda_6\mu_1^2 - \lambda_5\mu_2^2)] / (4\mu_3^2v_{15}) > 0$, which implies that $u_1 > u_{13}$. In the same way, $w_2 - w_{15} = [v_{15}(\lambda_4\mu_3^2 - \lambda_5\mu_2^2)] / (4\mu_1^2v_{13}) > 0$, that is, $w_2 > w_{15}$. Therefore, in the points $\tilde{\mathbf{P}}_{13}$ and $\tilde{\mathbf{P}}_{15}$ the global minimum are not found.

Remain to see the points $\tilde{\mathbf{P}}_9, \tilde{\mathbf{P}}_{10}, \tilde{\mathbf{P}}_{11}$ and $\tilde{\mathbf{P}}_{12}$. In order to discard these points as global minima, we can proceed in the same way as we did with the points $\tilde{\mathbf{P}}_3$ and $\tilde{\mathbf{P}}_4$. Let us consider the numerator of $L_9, L_{10}, K_{11}, L_{12}$ as positive and the numerator of $M_9, M_{10}, M_{11}, K_{12}$ as negative. After that, we obtain the following conditions,

$$\frac{2\lambda_1\mu_2^2}{\mu_1^2} < \lambda_4 < \frac{2\lambda_2\mu_1^2}{\mu_2^2} - \max\{\lambda_7, (\lambda_7 + 2\lambda_{10})\}, \quad (237)$$

$$\frac{2\lambda_3\mu_2^2}{\mu_3^2} < \lambda_6 < \frac{2\lambda_2\mu_3^2}{\mu_2^2} - \lambda_9, \quad (238)$$

$$\frac{2\lambda_1\mu_3^2}{\mu_1^2} < (\lambda_5 + \lambda_8) < \frac{2\lambda_3\mu_1^2}{\mu_3^2}, \quad (239)$$

where the inequalities (233) to (236) are derived from these new ones. And where the function max takes the largest value from a set.

Finally, we conclude, under above conditions the global minimum of the potential lies on point $\tilde{\mathbf{P}}_1$, where

$$\tilde{\mathbf{P}}_1 \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}}_1 = 4 K_{01} M_{01} > 0. \quad (240)$$

Also

$$\langle \underline{K} \rangle = \begin{pmatrix} \langle \Phi_1^\dagger \Phi_1 \rangle & \langle \Phi_2^\dagger \Phi_1 \rangle \\ \langle \Phi_1^\dagger \Phi_2 \rangle & \langle \Phi_2^\dagger \Phi_2 \rangle \end{pmatrix} = \begin{pmatrix} \frac{\lambda_5 \mu_3^2 - 2\lambda_3 \mu_1^2}{4\lambda_1 \lambda_3 - \lambda_5^2} & 0 \\ 0 & 0 \end{pmatrix}, \quad (241)$$

$$\langle \underline{L} \rangle = \begin{pmatrix} \langle \Phi_1^\dagger \Phi_1 \rangle & \langle \Phi_3^\dagger \Phi_1 \rangle \\ \langle \Phi_1^\dagger \Phi_3 \rangle & \langle \Phi_3^\dagger \Phi_3 \rangle \end{pmatrix} = \begin{pmatrix} \frac{\lambda_5 \mu_3^2 - 2\lambda_3 \mu_1^2}{4\lambda_1 \lambda_3 - \lambda_5^2} & 0 \\ 0 & \frac{\lambda_5 \mu_1^2 - 2\lambda_1 \mu_3^2}{4\lambda_1 \lambda_3 - \lambda_5^2} \end{pmatrix}, \quad (242)$$

$$\langle \underline{M} \rangle = \begin{pmatrix} \langle \Phi_2^\dagger \Phi_2 \rangle & \langle \Phi_3^\dagger \Phi_2 \rangle \\ \langle \Phi_2^\dagger \Phi_3 \rangle & \langle \Phi_3^\dagger \Phi_3 \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\lambda_5 \mu_1^2 - 2\lambda_1 \mu_3^2}{4\lambda_1 \lambda_3 - \lambda_5^2} \end{pmatrix}, \quad (243)$$

which implies the configuration of VEVs vectors given by (174). Let us mention here that the former results does not change if we consider complex phases in the VEV. Then we have

$$\frac{v_1^2 + V_1^2}{2} = \frac{\lambda_5 \mu_3^2 - 2\lambda_3 \mu_1^2}{4\lambda_1 \lambda_3 - \lambda_5^2}, \quad (244)$$

$$\frac{v_3^2}{2} = \frac{\lambda_5 \mu_1^2 - 2\lambda_1 \mu_3^2}{4\lambda_1 \lambda_3 - \lambda_5^2}, \quad (245)$$

solutions that agree with the constraint equations given in (175). Using the former relations we can write the Lagrange multiplier u_1 in the following way

$$u_1 = \frac{1}{4} \frac{(4\lambda_1 \lambda_3 - \lambda_5^2)}{(\lambda_5 \mu_3^2 - 2\lambda_3 \mu_1^2)} \left(\frac{\lambda_4 (v_1^2 + V_1^2) + \lambda_6 v_3^2 + \mu_2^2}{2} \right) > 0. \quad (246)$$

Since the value in (244) is positive, the large parenthesis in (246) is also positive and thus, the square mass in (181) is also positive. Using conditions (232), we can conclude that the square masses in (180), (183), (184) and (186) are also positive quantities, with the hierarchy $(M_{he_3}^2, M_{ho_1}^2, M_{h_\pm}^2) > M_{he_4}^2 = M_{ho_2}^2$.

At the global minimum, the Higgs potential now becomes

$$V_{\min.} = \frac{1}{4} \mu_1^2 (v_1^2 + V_1^2) + \frac{1}{4} \mu_3^2 v_3^2 < 0. \quad (247)$$

Therefore, in order to have the deepest minimum value for the potential for this particular vacuum structure, the following conditions are highly suggested

$$v_1, V_1, v_3 \neq 0. \quad (248)$$

5.5.2 The general case $v_1, V_1, v_2, V_2 \neq 0$

In this case, we want the global minimum of potential be located at $\tilde{\mathbf{P}}_5$. Looking at the Table (2), we choose this point as the global minimum, because for it $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$ are LD and the VEV configuration presents in Eqs. (3) and (4) are reproduced. For that, it is necessary that

$$\lambda_7 + 2\lambda_{10} < 0. \quad (249)$$

At the same time let us eliminate the possibility that the exceptional values v_7 and w_8 became global minima, which is reached by making them negatives, that is

$$\lambda_8, \lambda_9 > 0, \quad (250)$$

in such a way that the square mass in (172) becomes positive.

To exclude $\tilde{\mathbf{P}}_6$ as the global minimum it is sufficient to assume that $u_5 > u_6$ which is achieved as far as

$$\lambda_{10} < 0. \quad (251)$$

If we assume that

$$\lambda_4 < 0, \quad \lambda_5 < 0, \quad \text{and} \quad \lambda_6 < 0, \quad (252)$$

and taking into account Eqs. (203), (204), (206), (212) and (249), it implies that $u_1 < 0, w_2 < 0, u_3 < 0, w_4 < 0, v_9 < 0, u_{13} < 0, v_{13} < 0, u_{14} < 0, w_{14} < 0, v_{15} < 0$ and $w_{15} < 0$. Finally, the remaining Lagrange multipliers

$u_{10} < 0, v_{10} < 0, u_{11} < 0, v_{11} < 0, u_{12} < 0$ and $u_{12} < 0$ are negative, when the corresponding points, are stationary points respectively. From conditions (203), (206), (249), (250), (251) and (252), the inequalities (166), (170) and (172) are immediately satisfied.

As you can observe, being exhaustive in our reasoning, the global minimum remains at $\tilde{\mathbf{P}}_5$, and it is given by

$$\begin{pmatrix} K_5 = \frac{8\mu_3^2 [w_{15}(u_5-u_3) + v_{15}(u_5-u_1)]}{d} \\ L_5 = \frac{8\mu_3^2 [w_{15}(u_5-u_3) + u_5^2] - 8\mu_1^2 u_{13} w_4 - 2(2\lambda_4 \mu_3^2 - \lambda_5 \mu_2^2 - \lambda_6 \mu_1^2) u_5}{d} \\ M_5 = \frac{8\mu_3^2 [v_{15}(u_5-u_1) + u_5^2] - 8\mu_1^2 u_{13} w_4 - 2(2\lambda_4 \mu_3^2 - \lambda_5 \mu_2^2 - \lambda_6 \mu_1^2) u_5}{d} \\ \frac{16|\mu_3^2| \sqrt{(u_5-u_1)(u_5-u_3)} v_{15} w_{15}}{|d|} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (253)$$

with $d = \lambda_1 (4\lambda_2 \lambda_3 - \lambda_6^2) + \lambda_2 (4\lambda_1 \lambda_3 - \lambda_5^2) - \lambda_3 [4\lambda_1 \lambda_2 + (\lambda_4 - 4u_5)^2] + \lambda_5 \lambda_6 (\lambda_4 - 4u_5)$, where the first two terms of d are positive and the last ones negative. In light of the above results, we can choose λ_1, λ_2 and λ_3 as larger as necessary such that

$$d > 0, \quad (254)$$

as can be observed from $\lim_{\lambda_{1,2,3} \rightarrow +\infty} d \approx 4\lambda_1 \lambda_2 \lambda_3 = \mathcal{O}(\lambda^3) > 0$ and similarly $\lim_{\lambda_{1,2,3} \rightarrow +\infty} L_5, M_5 = \mathcal{O}(1/\lambda) > 0$, such that it is possible to find cases for which

$$L_5 > 0, \quad \text{and} \quad M_5 > 0. \quad (255)$$

The fourth entry in (253) is taken positive by assuming positive VEV. Since we are in the domain given by (219b), we must have

$$\tilde{\mathbf{P}}_5 \cdot \tilde{g}_1 \cdot \tilde{\mathbf{P}}_5 = 0, \quad (256)$$

$$\tilde{\mathbf{P}}_5 \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}}_5 = -\frac{64\mu_3^2 w_{15} (u_5 - u_3) [4\mu_1^2 u_{13} w_4 - 4\mu_3^2 u_5^2 - (\lambda_6 \mu_1^2 + \lambda_5 \mu_2^2 - 2\lambda_4 \mu_3^2) u_5]}{d^2} > 0, \quad (257)$$

$$\tilde{\mathbf{P}}_5 \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}}_5 = -\frac{64\mu_3^2 v_{15} (u_5 - u_1) [4\mu_1^2 u_{13} w_4 - 4\mu_3^2 u_5^2 - (\lambda_6 \mu_1^2 + \lambda_5 \mu_2^2 - 2\lambda_4 \mu_3^2) u_5]}{d^2} > 0. \quad (258)$$

where you can observe too that $\lim_{\lambda_{1,2,3} \rightarrow +\infty} (\tilde{\mathbf{P}}_5 \cdot \tilde{g}_2 \cdot \tilde{\mathbf{P}}_5, \tilde{\mathbf{P}}_5 \cdot \tilde{g}_3 \cdot \tilde{\mathbf{P}}_5) = \mathcal{O}(1/\lambda^2) > 0$.

The expectation values satisfy also

$$\langle \underline{K} \rangle = \begin{pmatrix} \langle \Phi_1^\dagger \Phi_1 \rangle & \langle \Phi_2^\dagger \Phi_1 \rangle \\ \langle \Phi_1^\dagger \Phi_2 \rangle & \langle \Phi_2^\dagger \Phi_2 \rangle \end{pmatrix} = \begin{pmatrix} \frac{8\mu_3^2 w_{15} (u_5 - u_3)}{d} & \frac{8|\mu_3^2| \sqrt{(u_5-u_1)(u_5-u_3)} v_{15} w_{15}}{|d|} \\ \frac{8|\mu_3^2| \sqrt{(u_5-u_1)(u_5-u_3)} v_{15} w_{15}}{|d|} & \frac{8\mu_3^2 v_{15} (u_5 - u_1)}{d} \end{pmatrix}, \quad (259)$$

$$\langle \underline{L} \rangle = \begin{pmatrix} \langle \Phi_1^\dagger \Phi_1 \rangle & \langle \Phi_3^\dagger \Phi_1 \rangle \\ \langle \Phi_1^\dagger \Phi_3 \rangle & \langle \Phi_3^\dagger \Phi_3 \rangle \end{pmatrix} = \begin{pmatrix} \frac{8\mu_3^2 w_{15} (u_5 - u_3)}{d} & 0 \\ 0 & -\frac{2[4\mu_1^2 u_{13} w_4 - 4\mu_3^2 u_5^2 - (\lambda_6 \mu_1^2 + \lambda_5 \mu_2^2 - 2\lambda_4 \mu_3^2) u_5]}{d} \end{pmatrix}, \quad (260)$$

$$\langle \underline{M} \rangle = \begin{pmatrix} \langle \Phi_2^\dagger \Phi_2 \rangle & \langle \Phi_3^\dagger \Phi_2 \rangle \\ \langle \Phi_2^\dagger \Phi_3 \rangle & \langle \Phi_3^\dagger \Phi_3 \rangle \end{pmatrix} = \begin{pmatrix} \frac{8\mu_3^2 v_{15} (u_5 - u_1)}{d} & 0 \\ 0 & -\frac{2[4\mu_1^2 u_{13} w_4 - 4\mu_3^2 u_5^2 - (\lambda_6 \mu_1^2 + \lambda_5 \mu_2^2 - 2\lambda_4 \mu_3^2) u_5]}{d} \end{pmatrix}. \quad (261)$$

This shows that $\langle \Phi_3 \rangle$ is orthogonal to $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$, and at the same time $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$ are LD due to the relation (256). It also shows that generality is not lost by taking positive VEV in (3) and (4). We then have

$$\frac{v_1^2 + V_1^2}{2} = \frac{8\mu_3^2 w_{15} (u_5 - u_3)}{d}, \quad (262)$$

$$\frac{v_2^2 + V_2^2}{2} = \frac{8\mu_3^2 v_{15} (u_5 - u_1)}{d}, \quad (263)$$

$$\begin{aligned} \frac{v_3^2}{2} &= -\frac{8\mu_1^2 u_{13} w_4 - 8\mu_3^2 u_5^2}{d} \\ &+ \frac{2(\lambda_6 \mu_1^2 + \lambda_5 \mu_2^2 - 2\lambda_4 \mu_3^2) u_5}{d}, \end{aligned} \quad (264)$$

$$\frac{v_1 v_2 + V_1 V_2}{2} = \frac{8|\mu_3^2| \sqrt{(u_5 - u_1)(u_5 - u_3)} v_{15} w_{15}}{|d|}, \quad (265)$$

The solution to (262), (263) and (264) coincide with the expressions given in (161). Also, from the relations (157), (262), (263) and (265) we can see that the VEV satisfy the following relations

$$\frac{v_1^2}{v_2^2} = \frac{V_1^2}{V_2^2} = \frac{w_{15}(u_5 - u_3)}{v_{15}(u_5 - u_1)} = \alpha^2, \quad (266)$$

where

$$\alpha = \sqrt{\frac{w_{15}(u_5 - u_3)}{v_{15}(u_5 - u_1)}} \quad (267)$$

is the proportionality factor between $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$ as stated in (310), which allow us to connect the value of α to the Lagrange multipliers.

At the global minimum the Higgs potential becomes

$$V_{\min.} = \frac{1}{4}\mu_1^2(v_1^2 + V_1^2) + \frac{1}{4}\mu_2^2(v_2^2 + V_2^2) + \frac{1}{4}\mu_3^2 v_3^2 < 0, \quad (268)$$

which reproduces Eq. (247) in the limit $v_2 = V_2 = 0$. Therefore, in order to have the deepest minimum value for the potential for this particular vacuum structure, the following conditions are highly suggested

$$v_1, V_1, v_2, V_2, v_3 \neq 0. \quad (269)$$

6 Conclusions

In this work we have studied in detail the minimal scalar sector of some models based on the local gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$. By restricting the field representations to particles without exotic electric charges we end up with ten different models, two one family models and eight models for three families. The two one family models are studied in the papers in Refs. [9, 10], but enough attention was not paid to the scalar sector in the analysis done. As far as we know, most of the three family models are new in the literature, but models **C** and **D**, which has been partially analyzed in Refs. [11] and [12] respectively.

We have also considered the mass spectrum eigenstates of the most general scalar potential specialized for the 331 models without exotic electric charges, with two Higgs triplets with the most general VEV possible. It is shown that in the considered models there is just one light neutral Higgs scalar which can be identified with the SM Higgs scalar; there are besides three more heavy scalars, one charged and its charge conjugate and one extra neutral one.

The two triplets of $SU(3)_L$ scalars with the most general VEV possible produces a consistent fermion mass spectrum at least for one of the models in the literature and the scale of the new physics predicted by the class of models analyzed in this work lies above 1.3 TeV as shown in the main text. This scale is consistent with the analysis done in other papers [9, 10] using a different phenomenological analysis.

Finally notice that our analysis allows us to constraint all the parameters in the scalar potential; that is, our model is a consistent one as far as $\lambda_1 > 0$, $\lambda_2 > 0$, $4\lambda_1\lambda_2 > \lambda_3^2$, $\lambda_3 < 0$ and $\lambda_4 > 0$.

A detailed study of the scalar potential for the economical 331 model has been carried through. In order to have an acceptable theory, this potential should be stable; that is, it should be bounded from below and lead to the correct EWSB pattern observed in Nature.

For the scalar potential as presented in Eq. (50), the following are the conditions which guarantee strong stability:

1. Necessary and sufficient conditions:

$$\lambda_1 > 0 \quad \text{and} \quad \lambda_2 > 0.$$

2. Sufficient (but not necessary) conditions

$$4\lambda_1\lambda_2 > \lambda_3^2 \quad \text{and} \quad 4\lambda_1\lambda_2 > (\lambda_3 + \lambda_4)^2.$$

Now, at the global minimum of the potential, $\lambda_4 > 0$ is required; a condition which makes redundant the last inequality. And the inequality $4\lambda_1\lambda_2 > \lambda_3^2$ is a necessary condition in order that the square mass for the physical Higgs be positive.

Additional constraints coming from our analysis are:

- The criteria used to find the minimum state leads us to assure that the second order coefficients μ_1^2 and μ_2^2 in the scalar potential must be negative.
- The required EWSB allows us to conclude that both scalar triplets must develop nonzero VEV. Additionally, the VEV are found to be necessary along the three electrically neutral directions of the scalar fields.
- In the main text, specific new relations among several parameters of the scalar potential were derived, as for example that $\sqrt{\lambda_1}\mu_2^2 + \sqrt{\lambda_2}\mu_1^2 \neq 0$. This condition is related to the existence of a *critical point* on the scalar potential.
- In Refs. [16, 20] λ_3 was declared as a negative value parameter. Here we have shown that under special circumstances it can take positive values, constrained by

$$\lambda_3 < \min\{2\lambda_1|\mu_2^2/\mu_1^2|, 2\lambda_2|\mu_1^2/\mu_2^2|\}$$

- Unfortunately, from the mathematical point of view we could not establish a hierarchy among V_1 , v_1 and v_2 , unless a fine tuning is introduced (from the physical point of view we know that $V_1 \gg v_2 \gg v_1$ [20]).

But the most important conclusion of our study is that the conditions for strong stability of the scalar potential, guarantee positive masses for the scalar fields predicted by the model. This outstanding result shows the consistency of the economical 331 model, something that should not be taken for granted due to the scarce number of parameters to deal with.

Notice that the inclusion of imaginary VEV do not alter the minimum of the scalar potential, due to the fact that $\langle\phi_1\rangle^T \cdot \langle\phi_2\rangle = 0$ in Eq. (96).

Notice also that in order to implement the mathematical method in this particular model, the criteria for stability were straightened, with a new theorem proved in Appendix B.

The mathematical analysis presented here may be extended to other 331 models with three or more Higgs scalar triplets (work in progress). For these other models the Gunion parameterization may not be implemented easily.

The parameterization given in Sect. 4.1 for the scalars, using orbital variables, is not unique. Other acceptable parameterizations can be found in Sect. 4.3. These new schemes seem to work well and deserve more attention, in particular the last parameterization used has the additional property that the scalar product terms are $SU(3)_L \otimes U(1)_X$ gauge invariant.

Finally we want to mention that some results presented here, either coincide or are compatible with partial results already published in Refs. [16, 20].

In this work we have presented original results related to the scalar sector of some 331 models without exotic electric charges. An exhaustive study of the scalar potential with 3 scalar triplets Φ_1 , Φ_2 and Φ_3 and VEV as introduced in Sects. (2.3.1) and (2.3.2); potential which does not include the possible cubic term according to the discrete symmetry $\Phi_1 \rightarrow -\Phi_1$ imposed, has been carried through. This problem partially analyzed in the literature [11, 12, 46, 47] had not been studied in a systematic way.

In concrete we have:

- Looked for a consistent implementation of the Higgs mechanism.
- Implemented a consistent electroweak symmetry breaking pattern.
- Established the strong stability conditions for the scalar potential.
- Found the stationary points of the scalar potential, except the ones coming from specific relations among the parameters μ_1^2 , μ_2^2 and μ_3^2 that we assume are not satisfied, in general.

One outstanding new result is that $\langle\Phi_1\rangle$ and $\langle\Phi_2\rangle$, the VEV of the two Higgs scalars with identical quantum numbers, must be proportional to each other, a necessary condition in order to properly implement the Higgs mechanism, and achieve a consistent electroweak symmetry breaking; besides, the proportionality constant is connected with the Lagrange multipliers (which in turn are connected to the other parameters of the potential) via Eq. (267).

Other important result is that, from the nine possible vacuum structures compatible with the stated LD constraint, only two are independent. Our analysis has been done for both structures.

But probably, the most important conclusion of our study is the existence (as a sufficient condition), in the scalar potential, of a global minimum stationary point for each one of the two cases of VEV considered, being it, at the same time, compatible with the stability conditions imposed in the strong sense. Stability conditions and the global minimum point were found via the orbit gauge method, implemented in this case for three scalar triplets. Although the method can give unphysical VEVs.

A Masses for Fermion Fields in the Minimal Scalar Sector

In this appendix we show how the fermion fields of a particular model acquire masses with the Higgs scalars and VEV introduced in the main text. The analysis is model dependent, so let us use the one family model **A**, for which the fermion multiplets are [9] $\chi_L^T = (u, d, D)_L \sim (3, 3, 0)$; $u_L^c \sim (3^*, 1, -2/3)$; $d_L^c \sim (3^*, 1, 1/3)$, $D_L^c \sim (3^*, 1, 1/3)$; $\psi_{1L}^T = (e^-, \nu_e, N_1^0)_L \sim (1, 3^*, -1/3)$, $\psi_{2L}^T = (E^-, N_2^0, N_3^0)_L \sim (1, 3^*, -1/3)$, and $\psi_{3L}^T = (N_4^0, E^+, e^+)_L \sim (1, 3^*, 2/3)$. As shown in Ref. [9], this structure corresponds to an E_6 subgroup.

A.1 Bare Masses for fermion fields

The most general Yukawa Lagrangian that the Higgs scalars in Section 4 produce for the fermion fields in this model, can be written as $\mathcal{L}_Y = \mathcal{L}_Y^Q + \mathcal{L}_Y^l$, with

$$\begin{aligned}\mathcal{L}_Y^Q &= \chi_L^T C (h_u \phi_2 u_L^c + h_D \phi_1 D_L^c + h_d \phi_1 d_L^c) + h.c., \\ \mathcal{L}_Y^l &= \epsilon_{abc} [\psi_{1L}^a C (h_1 \psi_{2L}^b \phi_2^c + h_2 \psi_{3L}^b \phi_1^c) + \psi_{2L}^a C h_3 \psi_{3L}^b \phi_1^c] + h.c.,\end{aligned}$$

where h_η , $\eta = u, d, D, 1, 2, 3$ are Yukawa couplings of order one; a, b, c are $SU(3)_L$ tensor indices and C is the charge conjugation operator.

Using for $\langle \phi_i \rangle$, $i = 1, 2$ the VEV in section 4 we get $m_u = h_u v_2$ for the mass of the up type-quark and for the down sector in the basis (d, D) we get the mass matrix

$$M_d = \begin{pmatrix} h_d v_1 & h_d V \\ h_D v_1 & h_D V \end{pmatrix}; \quad (270)$$

now, looking for the eigenvalues of $M_d M_d^\dagger$, we get $\sqrt{(h_d^2 + h_D^2)(v_1^2 + V^2)}$ and zero. Notice that for $h_u = 1$ and assuming for example that we are referring to the third family, we obtain the correct mass for the top quark (remember from Section 5 that $v_2 \simeq 174$ GeV), the bottom quark remains massless at zero level, and there is an exotic Bottom quark with a very large mass. Since there is no way to distinguish between d_L^c and D_L^c in the Yukawa Lagrangian it is just natural to impose the discrete symmetry $h_d = h_D \equiv h$.

For the charged lepton sector the mass eigenvalues are 0 and $\sqrt{(h_2^2 + h_3^2)(v_1^2 + V^2)}$, with similar consequences as in the down quark sector, where again it is natural to impose the symmetry $h_2 = h_3 \equiv h'$.

The analysis of the neutral lepton sector is more elaborated; at zero level and in the basis $(\nu, N_1, N_2, N_3, N_4)$ we get the mass matrix:

$$M_N = \begin{pmatrix} 0 & 0 & 0 & h_1 v_2 & -h_2 V \\ 0 & 0 & -h_1 v_2 & 0 & h_2 v_1 \\ 0 & -h_1 v_2 & 0 & 0 & -h_3 V \\ h_1 v_2 & 0 & 0 & 0 & h_3 v_1 \\ -h_2 V & h_2 v_1 & -h_3 V & h_3 v_1 & 0 \end{pmatrix},$$

with eigenvalues 0, $\pm h_1 v_2$ and $\pm \sqrt{h_1^2 v_2^2 + (h_2^2 + h_3^2)(V^2 + v_1^2)}$, which implies a Majorana neutrino of zero mass and two Dirac neutral particles with masses one of them at the electroweak mass scale and the other one at the TeV scale.

So, at zero level the charged exotic particles get large masses of order $V > 1.3$ TeV, the top quark and a Dirac neutral particle get masses of order $v_2 \sim 174$ GeV, there is a Dirac neutral particle with a mass of order V , and the bottom quark, charged lepton and a Majorana neutrino remain massless. In what follows we will see that they pick up a radiative mass in the context of the model studied here.

A.2 Currents

The interactions among the charged gauge fields in Section 5 with the fermions of **Model A** are [9]:

$$\begin{aligned}
H^{CC} &= \frac{g}{\sqrt{2}} [W_\mu^+ (\bar{u}_L \gamma^\mu d_L - \bar{\nu}_{eL} \gamma^\mu e_L - \bar{N}_{2L}^0 \gamma^\mu E_L^- - \bar{E}_L^+ \gamma^\mu N_{4L}^0) \\
&+ K_\mu^+ (\bar{u}_L \gamma^\mu D_L - \bar{N}_{1L}^0 \gamma^\mu e_L - \bar{N}_{3L}^0 \gamma^\mu E_L^- - \bar{e}_L^+ \gamma^\mu N_{4L}^0) \\
&+ K_\mu^0 (\bar{d}_L \gamma^\mu D_L - \bar{N}_{1L}^0 \gamma^\mu \nu_{eL} - \bar{N}_{3L}^0 \gamma^\mu N_{2L}^0 - \bar{e}_L^+ \gamma^\mu E_L^+)] + h.c.,
\end{aligned}$$

where the first two terms constitute the charged weak current of the SM, and K^\pm , K^0 and \bar{K}^0 are related to new charged currents which violate weak isospin.

The algebra also shows that the neutral currents $J_\mu(EM)$, $J_\mu(Z)$ and $J_\mu(Z')$, associated with the Hamiltonian $H^0 = eA^\mu J_\mu(EM) + \frac{g}{C_W} Z^\mu J_\mu(Z) + \frac{g'}{\sqrt{3}} Z'^\mu J_\mu(Z')$ (where A_μ is the photon field in Eq.(15) and Z_μ and Z'_μ are the neutral gauge bosons introduced in Eq.(16)) are:

$$J_\mu(EM) = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} (\bar{d} \gamma_\mu d + \bar{D} \gamma_\mu D) - \bar{e}^- \gamma_\mu e^- - \bar{E}^- \gamma_\mu E^-, \quad (271)$$

$$J_\mu(Z) = J_{\mu,L}(Z) - S_W^2 J_\mu(EM), \quad (272)$$

$$J_\mu(Z') = T_W J_\mu(EM) - J_{\mu,L}(Z'), \quad (273)$$

where $e = gS_W = g' C_W \sqrt{(1 - T_W^2/3)} > 0$ is the electric charge, $J_\mu(EM)$ is the (vectorlike) electromagnetic current, and the two neutral left-handed currents are given by:

$$\begin{aligned}
J_{\mu,L}(Z) &= \bar{u}_L \gamma_\mu u_L - \bar{d}_L \gamma_\mu d_L + \bar{\nu}_{eL} \gamma_\mu \nu_{eL} - \bar{e}_L^- \gamma_\mu e_L^- + \bar{N}_2^0 \gamma_\mu N_2^0 - \bar{E}^- \gamma_\mu E^- \\
&= \sum_f T_{3f} \bar{f}_L \gamma_\mu f_L,
\end{aligned} \quad (274)$$

$$\begin{aligned}
J_{\mu,L}(Z') &= S_{2W}^{-1} (\bar{u}_L \gamma_\mu u_L - \bar{e}_L^- \gamma_\mu e_L^- - \bar{E}_L^- \gamma_\mu E_L^- - \bar{N}_{4L}^0 \gamma_\mu N_{4L}^0) \\
&+ T_{2W}^{-1} (\bar{d}_L \gamma_\mu d_L - \bar{E}_L^+ \gamma_\mu E_L^+ - \bar{\nu}_{eL} \gamma_\mu \nu_{eL} - \bar{N}_{2L}^0 \gamma_\mu N_{2L}^0) \\
&- T_W^{-1} (\bar{D}_L \gamma_\mu D_L - \bar{e}_L^+ \gamma_\mu e_L^+ - \bar{N}_{1L}^0 \gamma_\mu N_{1L}^0 - \bar{N}_{3L}^0 \gamma_\mu N_{3L}^0),
\end{aligned} \quad (275)$$

where $S_{2W} = 2S_W C_W$, $T_{2W} = S_{2W}/C_{2W}$, $\bar{N}_2^0 \gamma_\mu N_2^0 = \bar{N}_{2L}^0 \gamma_\mu N_{2L}^0 + \bar{N}_{2R}^0 \gamma_\mu N_{2R}^0 = \bar{N}_{2L}^0 \gamma_\mu N_{2L}^0 - \bar{N}_{2L}^{0c} \gamma_\mu N_{2L}^{0c} = \bar{N}_{2L}^0 \gamma_\mu N_{2L}^0 - \bar{N}_{4L}^0 \gamma_\mu N_{4L}^0$, similarly $\bar{E}^- \gamma_\mu E^- = \bar{E}_L^- \gamma_\mu E_L^- - \bar{E}_L^+ \gamma_\mu E_L^+$ and $T_{3f} = Dg(1/2, -1/2, 0)$ is the third component of the weak isospin acting on the representation 3 of $SU(3)_L$ (the negative when acting on $\bar{3}$). Notice that $J_\mu(EM)$ and $J_\mu(Z)$ are just the generalization of the electromagnetic and neutral weak currents of the SM, as they should be, implying that Z_μ can be identified as the neutral gauge boson of the SM.

A.3 Radiative masses for fermion fields

Using the currents in the previous section and the off diagonal entries in matrix in Eq.(17), we may draw the four diagrams in Fig. 1 which allow for non diagonal entries in the mass matrix for the down quark sector of the form $(\Delta_D D_L d_R + h.c.)$ and $(\Delta_d d_L D_R + h.c.)$ respectively, which in turn produce a radiative mass for the ordinary down quark. Notice that due to the presence of $K_R^{0\mu}$ in the graphs, mass entries of the form $d_L d_R$ and $D_L D_R$ are not present. The equations in this work imply for the diagrams in Fig. 1 that: $\alpha_\mu = g\gamma_\mu/2$, $\beta_\mu = g\gamma_\mu S_W T_W/3$, $\beta'_\mu = -g'\gamma_\mu T_W/\sqrt{27}$, $\epsilon = -C_W v_1 V$ and $\epsilon' = C_W v_1 V/\sqrt{4C_W^2 - 1}$.

In a similar way we achieve radiative masses for the charged lepton and for the Majorana neutrino. The detailed analysis for these leptons will be presented elsewhere.

B The smallest Lagrange multiplier as the global minimum of the function $J_4(\mathbf{k})$

Let \mathbf{p} and \mathbf{q} be two stationary points with Lagrange multipliers u_p and u_q respectively, with $|\mathbf{p}| = |\mathbf{q}| = 1$ (we will consider later the case $u_p = 0$, $|\mathbf{p}| < 1$). Both \mathbf{p} and \mathbf{q} must satisfy

$$(E - u_p)\mathbf{p} = -\boldsymbol{\eta} \quad \text{and} \quad (E - u_q)\mathbf{q} = -\boldsymbol{\eta}. \quad (276)$$

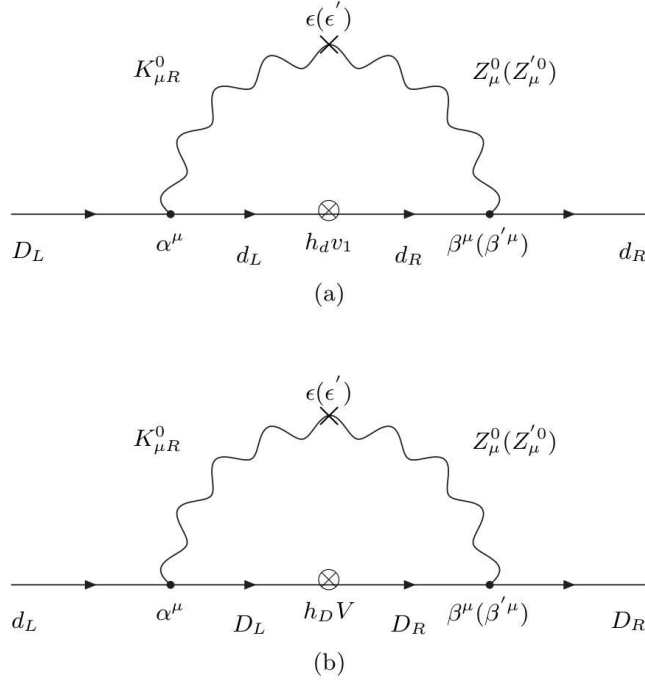


Figure 1: Four one-loop diagrams contributing to the radiative generation of the ordinary down quark mass. For the meaning of α , β and ϵ see the main text.

At these two stationary points, $J_4(\mathbf{k})$ takes the values

$$J_4(\mathbf{p}) = \eta_{00} + u_p + \boldsymbol{\eta}^T \cdot \mathbf{p}, \quad (277)$$

$$J_4(\mathbf{q}) = \eta_{00} + u_q + \boldsymbol{\eta}^T \cdot \mathbf{q}, \quad (278)$$

where we have used Eqs. (27), (29) and (276). Subtracting we obtain

$$J_4(\mathbf{p}) - J_4(\mathbf{q}) = u_p - u_q + \boldsymbol{\eta}^T \cdot (\mathbf{p} - \mathbf{q}). \quad (279)$$

Now, recalling that $(E - u_p)^T = E - u_p$, we transpose Eqs. (276)

$$\mathbf{p}^T (E - u_p) = -\boldsymbol{\eta}^T, \quad \mathbf{q}^T (E - u_q) = -\boldsymbol{\eta}^T. \quad (280)$$

Multiplying by \mathbf{q} and \mathbf{p} , we have

$$\mathbf{p}^T \cdot (E - u_p) \mathbf{q} = -\boldsymbol{\eta}^T \cdot \mathbf{q}, \quad (281)$$

$$\mathbf{q}^T \cdot (E - u_q) \mathbf{p} = -\boldsymbol{\eta}^T \cdot \mathbf{p}. \quad (282)$$

Subtracting Eqs. (281) and (282) it is obtained that

$$(u_q - u_p) \mathbf{p}^T \cdot \mathbf{q} = \boldsymbol{\eta}^T \cdot (\mathbf{p} - \mathbf{q}), \quad (283)$$

which we place into (279) to finally obtain

$$\begin{aligned} J_4(\mathbf{p}) - J_4(\mathbf{q}) &= u_p - u_q + (u_q - u_p) \mathbf{p}^T \cdot \mathbf{q}, \\ &= (u_p - u_q) (1 - \mathbf{p}^T \cdot \mathbf{q}), \end{aligned} \quad (284)$$

where $\mathbf{p}^T \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}| \cos \theta = \cos \theta < 1$ ¹. Notice that $\cos \theta$ cannot be equal to 1, because \mathbf{p} and \mathbf{q} cannot be parallel: if we assume that they are parallel to each other, Eq. (276) leads to

$$(u_p - u_q) \mathbf{p} = 0, \quad \text{and then} \quad u_p = u_q, \quad (285)$$

¹If $|\mathbf{p}| < 1$, then $\mathbf{p}^T \mathbf{q} = |\mathbf{p}||\mathbf{q}| \cos \theta < 1$

but we have assumed $u_p \neq u_q$. So, in all the cases we would have

$$(1 - \mathbf{p}^T \cdot \mathbf{q}) > 0. \quad (286)$$

From Eq. (284), we finally conclude that

$$\text{if } u_p < u_q \Leftrightarrow J_4(\mathbf{p}) < J_4(\mathbf{q}). \quad \square \quad (287)$$

C EWSB in the case $w_0 > 0$

It still remains to see if the economical 331 model is consistent, when the global minimum is found at $K_0 = |\mathbf{K}|$, i.e. if it is related to the Lagrange multiplier $w_0 > 0$ (this situation was addressed in section 4.1.7). In this case the vacuum expectation vectors $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ become linearly dependent, which implies that either $V_1 = v_1 = 0$ or $v_2 = 0$ (cases where the electric charge generator is broken are not considered).

Following a similar approach to the one presented in Sect. 4.4, we analyze the second order term of the scalar potential, the one responsible to provide with masses to the physical Higgs fields. This term takes the form

$$V_{\{2\}} = \tilde{\mathbf{K}}_{\{1\}}^T \tilde{E} \tilde{\mathbf{K}}_{\{1\}} + 2w_0 \tilde{\mathbf{K}}_{\{0\}}^T \tilde{g} \tilde{\mathbf{K}}_{\{2\}}. \quad (288)$$

Let us examine the two possible cases:

- $V_1 = v_1 = 0$: in this case all particles are decoupled. There are a total of six massive scalar particles with masses given by

$$\begin{aligned} M_{H_1}^2 &= 2w_0 v_2^2, \quad M_{H_2}^2 = 2\lambda_2 v_2^2, \quad M_{H_1'}^2 = 2w_0 v_2^2, \\ M_{A_1}^2 &= 2w_0 v_2^2, \quad M_{\phi_1^\pm(\phi_1^-)}^2 = \lambda_4 v_2^2/2, \end{aligned} \quad (289)$$

leaving the model with only six Goldstone bosons, which are not enough to provide with masses to the eight gauge bosons associated to the same number of broken generators present in 331 models.

- $v_2 = 0$: for the notation established in (148) we have

$$\mathcal{M}_{neutral}^2 = \begin{pmatrix} 2\lambda_1 V_1^2 & 0 & 2\lambda_1 v_1 V_1 \\ 0 & 2w_0(v_1^2 + V_1^2) & 0 \\ 2\lambda_1 v_1 V_1 & 0 & 2\lambda_1 v_1^2 \end{pmatrix}, \quad (290)$$

where $m_{H_2}^2 = 2w_0(v_1^2 + V_1^2)$. The remaining submatrix has null determinant. In this way a total of two massive CP-even particles show up. For the CP-odd sector a massive particle $M_{A_2}^2 = 2w_0(v_1^2 + V_1^2)$ is found.

In the charged sector we have

$$\mathcal{M}_{charged}^2 = \begin{pmatrix} 0 & & 0 \\ 0 & 2w_0(v_1^2 + V_1^2) + \lambda_4 v_1^2/4 & \lambda_4 v_1 V_1/2 \\ 0 & \lambda_4 v_1 V_1/2 & 2w_0(v_1^2 + V_1^2) + \lambda_4 V_1^2/4 \end{pmatrix}, \quad (291)$$

where at least two additional massive charged particles are present, for a total of five massive particles; there remaining in this way seven Goldstone bosons, which is not enough to implement the Higgs mechanism in a consistent way.

D The exceptional solutions w_3 and w_5

In what follows we are going to find the conditions which avoid that the Lagrange multipliers $w_3 = -\frac{\lambda_4}{4}$ and $w_5 = \frac{\lambda_3 + 2\sqrt{\lambda_1 \lambda_2}}{4}$ be global minima.

D.1 The exceptional solution w_3 :

Let us assume that w_3 is the largest value among the acceptable solutions in \tilde{I} , that is $w_3 = \max\{\tilde{I}\}$. For w_3 , let us solve the equation $(\tilde{E} - w_3\tilde{g})\tilde{\mathbf{K}} = -\frac{1}{2}\tilde{\boldsymbol{\xi}}$, where

$$\tilde{E} - w_3\tilde{g} = \begin{pmatrix} \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4} & 0 & 0 & \frac{\lambda_1-\lambda_2}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\lambda_1-\lambda_2}{4} & 0 & 0 & \frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{4} \end{pmatrix}. \quad (292)$$

By looking the parameters in Eq. (68), we see that the orbital variables K_1 and K_2 would be arbitrary. But by the use of Eq. (20) the cases $K_1 \neq 0$ or $K_2 \neq 0$ imply that $\phi_1^\dagger\phi_2 \neq 0$, i.e. we would have electric charge breaking.

If $K_1 = 0$ and $K_2 = 0$, we focus on the variables K_0 and K_3 :

$$\begin{pmatrix} \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4} & \frac{\lambda_1-\lambda_2}{4} \\ \frac{\lambda_1-\lambda_2}{4} & \frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{4} \end{pmatrix} \begin{pmatrix} K_0 \\ K_3 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} \mu_1^2 + \mu_2^2 \\ \mu_1^2 - \mu_2^2 \end{pmatrix}, \quad (293)$$

then

$$\begin{pmatrix} K_0 \\ K_3 \end{pmatrix} = a \begin{pmatrix} (-2\lambda_1 + \lambda_3 + \lambda_4)\mu_2^2 + (-2\lambda_2 + \lambda_3 + \lambda_4)\mu_1^2 \\ -(2\lambda_2 + \lambda_3 + \lambda_4)\mu_1^2 + (2\lambda_1 + \lambda_3 + \lambda_4)\mu_2^2 \end{pmatrix}, \quad (294)$$

with $a = 1/(4\lambda_1\lambda_2 - (\lambda_3 + \lambda_4)^2)$. The global minimum requires that

$$\begin{aligned} K_0 > 0 &\Rightarrow (-2\lambda_1 + \lambda_3 + \lambda_4)\mu_2^2 + (-2\lambda_2 + \lambda_3 + \lambda_4)\mu_1^2 = \\ &(-2\lambda_1\mu_2^2 + \lambda_4\mu_1^2 + \lambda_3\mu_2^2) + (-2\lambda_2\mu_1^2 + \lambda_4\mu_2^2 + \lambda_3\mu_2^2) = \\ &4\mu_1^2(w_1 - w_3) + 4\mu_2^2(w_2 - w_3) > 0, \end{aligned} \quad (295)$$

and

$$\begin{aligned} K_0^2 - K_3^2 &= 0 \Rightarrow \\ (2\lambda_1\mu_2^2 - \lambda_3\mu_1^2 - \lambda_4\mu_1^2)(2\lambda_2\mu_1^2 - \lambda_3\mu_2^2 - \lambda_4\mu_2^2) &= \\ \mu_1^2\mu_2^2(w_1 - w_3)(w_2 - w_3) &= 0, \end{aligned} \quad (296)$$

which implies either $w_3 = w_1$ or $w_3 = w_2$. These solutions were already studied in Sect. 4.1.7.

D.2 The exceptional solution w_5 :

In this case we solve the equation $(\tilde{E} - w_5\tilde{g})\tilde{\mathbf{K}} = -\frac{1}{2}\tilde{\boldsymbol{\xi}}$, where the matrix $\tilde{E} - w_5\tilde{g}$ is equal to

$$\begin{pmatrix} \frac{\lambda_1+\lambda_2-2\sqrt{\lambda_1\lambda_2}}{4} & 0 & 0 & \frac{\lambda_1-\lambda_2}{4} \\ 0 & \frac{\lambda_3+\lambda_4+2\sqrt{\lambda_1\lambda_2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\lambda_3+\lambda_4+2\sqrt{\lambda_1\lambda_2}}{4} & 0 \\ \frac{\lambda_1-\lambda_2}{4} & 0 & 0 & \frac{\lambda_1+\lambda_2+2\sqrt{\lambda_1\lambda_2}}{4} \end{pmatrix}. \quad (297)$$

From (67d) we have $\frac{\lambda_3+\lambda_4+2\sqrt{\lambda_1\lambda_2}}{4} > 0$, then $K_1 = K_2 = 0$. The equation relating K_0 and K_3 is

$$\frac{1}{4} \begin{pmatrix} (\sqrt{\lambda_1} - \sqrt{\lambda_2})^2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & (\sqrt{\lambda_1} + \sqrt{\lambda_2})^2 \end{pmatrix} \begin{pmatrix} K_0 \\ K_3 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} \mu_1^2 + \mu_2^2 \\ \mu_1^2 - \mu_2^2 \end{pmatrix}. \quad (298)$$

Notice that the 2×2 matrix in the left hand side of (298) is not invertible. Its entries are therefore linearly dependent

$$(\sqrt{\lambda_1} - \sqrt{\lambda_2})^2 K_0 + (\lambda_1 - \lambda_2)K_3 = -(\mu_1^2 + \mu_2^2), \quad (299)$$

$$(\lambda_1 - \lambda_2)K_0 + (\sqrt{\lambda_1} + \sqrt{\lambda_2})^2 K_3 = -(\mu_1^2 - \mu_2^2). \quad (300)$$

We will solve these equations in the following two cases:

i) $\lambda_1 = \lambda_2$: then, from (299), we have

$$\mu_1^2 + \mu_2^2 = 0, \quad (301)$$

which together with (300), gives

$$K_3 = -\frac{\mu_1^2}{2\lambda_1}. \quad (302)$$

Additionally

$$K_0^2 - K_3^2 = 0; \quad (303)$$

then

$$K_0 = \pm K_3. \quad (304)$$

In both cases

$$\underline{K} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{K_0+K_3}{2} \end{pmatrix} \quad \text{or} \quad \underline{K} = \begin{pmatrix} \frac{K_0+K_3}{2} & 0 \\ 0 & 0 \end{pmatrix}. \quad (305)$$

ii) $\lambda_1 \neq \lambda_2$: in this case, taking into account Eqs. (299) and (300), the entries in the right hand side of (298) must be such that

$$(\mu_1^2 + \mu_2^2) = \alpha(\mu_1^2 - \mu_2^2), \quad (306)$$

with $\alpha = \frac{(\sqrt{\lambda_1} - \sqrt{\lambda_2})^2}{\lambda_1 - \lambda_2} = \frac{(\sqrt{\lambda_1} - \sqrt{\lambda_2})}{\sqrt{\lambda_1} + \sqrt{\lambda_2}}$, and $|\alpha| < 1$. The former implies

$$\sqrt{\lambda_1}\mu_2^2 + \sqrt{\lambda_2}\mu_1^2 = 0. \quad (307)$$

Using (299) and (300) together with the condition (303), we have two solutions. The first one is

$$K_0 = \frac{\mu_1^2(\sqrt{\lambda_1} + \sqrt{\lambda_2})}{2(\lambda_1\sqrt{\lambda_2} - \lambda_2\sqrt{\lambda_1})} = -K_3, \quad (308)$$

where $K_0 > 0$ if $\mu_1^2 > 0$, $\lambda_1 > \lambda_2$, or $\mu_1^2 < 0$, $\lambda_1 < \lambda_2$.

The second solution is

$$K_0 = \frac{\mu_1^2(\sqrt{\lambda_1} + \sqrt{\lambda_2})^2}{2(\lambda_1\lambda_2 - \lambda_1^2)} = K_3, \quad (309)$$

where $K_0 > 0$ if $\mu_1^2 > 0$, $\lambda_2 > \lambda_1$, or $\mu_1^2 < 0$, $\lambda_2 < \lambda_1$.

E Linear dependence between $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$

In this appendix we study the consequences of a linear dependence between $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$.

As in the main text we use the VEV

$$\begin{aligned} \langle \Phi_1 \rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \\ V_1 \end{pmatrix}, & \langle \Phi_2 \rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \\ V_2 \end{pmatrix}, \\ \langle \Phi_3 \rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} v_3 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The LD between $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$ can be written as

$$\langle \Phi_1 \rangle = \alpha \langle \Phi_2 \rangle, \quad (310)$$

where α is a constant. Eq. (310) implies that $v_1 = \alpha v_2$ and $V_1 = \alpha V_2$, which combine to produce the constraint $v_2 V_1 = v_1 V_2$.

Now, the nine $U(3)$ generators are

$$\begin{aligned}
I_3 &= \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\end{aligned} \tag{311}$$

where $\lambda_i (i = 1, \dots, 8)$ are the eight Gell-Mann unitary matrices for $SU(3)$.

Let us now show that the LD in Eq. (310) with the additional constraint $\langle \Phi_3 \rangle \neq 0$, implies that either I_3 , or a linear combination of the generators in (311) which includes I_3 , remains unbroken, with the consequence that the appearance of an extra zero mass Goldstone bosons is avoided.

The algebra shows that the most general new unbroken generator is given by the following linear combination:

$$G = aI_3 + b\lambda_3 + c\lambda_8 + d\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & V_1^2 & -v_1 V_1 \\ 0 & -v_1 V_1 & v_1^2 \end{pmatrix}, \tag{312}$$

where

$$\begin{aligned}
a &= \frac{V_1^2 + v_1^2}{2}, & b &= -\frac{V_1^2}{2}, \\
c &= \frac{V_1^2 - 2v_1^2}{2\sqrt{3}}, & d &= -v_1 V_1.
\end{aligned}$$

That G remains unbroken can be seen by the fact that $G\langle \Phi_1 \rangle = 0$ by direct calculation, $G\langle \Phi_2 \rangle = 0$ is a consequence of the relation (157), and $G\langle \Phi_3 \rangle = 0$ is trivial.

Since $Tr.G = v_1^2 + V_1^2 \neq 0$, the new unbroken generator in Eq. (312) is such that $G \in U(3)$ but $G \notin SU(3)$.

F Discrete symmetry in the scalar potential

Under assumption of the discrete symmetry $\Phi_1 \rightarrow -\Phi_1$, the most general potential obtained from (155), can then be written in the following form:

$$\begin{aligned}
V(\Phi_1, \Phi_2, \Phi_3) &= \mu_1^2 \Phi_1^\dagger \Phi_1 + \mu_2^2 \Phi_2^\dagger \Phi_2 + \mu_3^2 \Phi_3^\dagger \Phi_3 + \lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_3^\dagger \Phi_3)^2 \\
&+ \lambda_4 (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + \lambda_5 (\Phi_1^\dagger \Phi_1) (\Phi_3^\dagger \Phi_3) + \lambda_6 (\Phi_2^\dagger \Phi_2) (\Phi_3^\dagger \Phi_3) + \lambda_7 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) \\
&+ \lambda_8 (\Phi_1^\dagger \Phi_3) (\Phi_3^\dagger \Phi_1) + \lambda_9 (\Phi_2^\dagger \Phi_3) (\Phi_3^\dagger \Phi_2) + (f \Phi_1^\dagger \Phi_2 + f^* \Phi_2^\dagger \Phi_1)^2.
\end{aligned} \tag{313}$$

where the complex value f is going to be used as $f = f_1 + if_2$, with f_j , $j = 1, 2$ are two real parameters. With the new definitions of the scalar fields introduced in (158), and by demanding that the VEV in (3) and (4) became stationary

points of the potential, the following nine constraints must be satisfied:

$$\left. \frac{\partial V}{\partial H_1} \right|_{\text{fields}=0} = \frac{2\mu_1^2 v_1 + \lambda_4 v_1 V_2^2 + (\lambda_7 + 4f_1^2) v_2 V_1 V_2 + 2\lambda_1 v_1 V_1^2 + (\lambda_7 + \lambda_4 + 4f_1^2) v_1 v_2^2 + 2\lambda_1 v_1^3 + \lambda_5 v_3^2 v_1}{2} = 0, \quad (314)$$

$$\left. \frac{\partial V}{\partial H'_1} \right|_{\text{fields}=0} = \frac{2\mu_1^2 V_1 + (\lambda_7 + \lambda_4 + 4f_1^2) V_1 V_2^2 + (\lambda_7 + 4f_1^2) v_1 v_2 V_2 + 2\lambda_1 V_1^3 + (\lambda_4 v_2^2 + 2\lambda_1 v_1^2 + \lambda_5 v_3^2) V_1}{2} = 0, \quad (315)$$

$$\left. \frac{\partial V}{\partial H_2} \right|_{\text{fields}=0} = \frac{2\mu_2^2 v_2 + 2\lambda_2 v_2 V_2^2 + (\lambda_7 + 4f_1^2) v_1 V_1 V_2 + \lambda_4 v_2 V_1^2 + 2\lambda_2 v_2^3 + [(\lambda_7 + \lambda_4 + 4f_1^2) v_1^2 + \lambda_6 v_3^2] v_2}{2} = 0, \quad (316)$$

$$\left. \frac{\partial V}{\partial H'_2} \right|_{\text{fields}=0} = \frac{2\mu_2^2 V_2 + 2\lambda_2 V_2^3 + [(\lambda_7 + \lambda_4 + 4f_1^2) V_1^2 + 2\lambda_2 v_2^2 + \lambda_4 v_1^2 + \lambda_6 v_3^2] V_2 + (\lambda_7 + 4f_1^2) v_1 v_2 V_1}{2} = 0, \quad (317)$$

$$\left. \frac{\partial V}{\partial H_3} \right|_{\text{fields}=0} = \frac{v_3 (2\mu_3^2 + \lambda_6 V_2^2 + \lambda_5 V_1^2 + \lambda_6 v_2^2 + \lambda_5 v_1^2 + 2\lambda_3 v_3^2)}{2} = 0, \quad (318)$$

$$\left. \frac{\partial V}{\partial A_1} \right|_{\text{fields}=0} = 2f_1 f_2 v_2 (V_1 V_2 + v_1 v_2) = 0, \quad (319)$$

$$\left. \frac{\partial V}{\partial A'_1} \right|_{\text{fields}=0} = 2f_1 f_2 V_2 (V_1 V_2 + v_1 v_2) = 0, \quad (320)$$

$$\left. \frac{\partial V}{\partial A_2} \right|_{\text{fields}=0} = -2f_1 f_2 v_1 (V_1 V_2 + v_1 v_2) = 0, \quad (321)$$

$$\left. \frac{\partial V}{\partial A'_2} \right|_{\text{fields}=0} = -2f_1 f_2 V_1 (V_1 V_2 + v_1 v_2) = 0. \quad (322)$$

A simple algebra shows that both operations $[v_1 \times (315) - V_1 \times (314)]$ and $[V_2 \times (316) - v_2 \times (317)]$ produce the same relation

$$(\lambda_7 + 4f_1^2)(v_1 V_2 - v_2 V_1) (V_1 V_2 + v_1 v_2) = 0, \quad (323)$$

which must be satisfied in order to have a consistent set of equations (314)–(317).

The two possible solutions to (323) are $(v_1 V_2 - v_2 V_1) = 0$ and/or $(V_1 V_2 + v_1 v_2) = 0$. Obviously, (323) is satisfied if $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$ are LD. (For the unphysical case $\langle \Phi_3 \rangle = 0$ with $\langle \Phi_1 \rangle$ and $\langle \Phi_2 \rangle$ being linearly independent, the mathematical solution $V_1 V_2 = -v_1 v_2$ is still available.)

But at the same time, the relations (319)–(322) must be satisfied, the alternative which remains for the physical case is that either the real or the imaginary part of f become zero, that is

$$f_1 = 0 \quad \text{or} \quad f_2 = 0, \quad (324)$$

meaning that f represents only one parameter, something which allow us to introduce the usual notation $|f|^2 = \frac{\lambda_{10}}{2}$, with λ_{10} either positive or negative.

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