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High order explicit exponential Runge-Kutta methods for the weak approximation of solutions of stochastic differential equations

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Abstract

We are concerned with numerical methods which give weak approximations for stiff Itô stochastic differential equations (SDEs). It is well known that the numerical solution of stiff SDEs leads to a stepsize reduction when explicit methods are used. However, there are some classes of explicit methods that are well suited to solving some types of stiff SDEs. One such class is the class of stochastic orthogonal Runge-Kutta Chebyshev (SROCK) methods. SROCK methods reduce to Runge-Kutta Chebyshev methods when applied to ordinary differential equations (ODEs). Another promising class of methods is the class of explicit methods that reduce to explicit exponential Runge-Kutta (RK) methods when applied to semilinear ODEs. In this paper, we will propose new exponential RK methods which achieve weak order one or two for multi-dimensional, non-commutative SDEs with a semilinear drift term, whereas they are of order one, two or three for semilinear ODEs. We will analytically investigate their stability properties in mean square, and will check their performance in numerical examples.

1 Introduction

For stiff ordinary differential equations (ODEs), there are some classes of explicit methods that are well suited to solving them. One such class is the class of Runge-Kutta Chebyshev (RKC) methods. They are useful for stiff problems whose eigenvalues lie near the negative real axis. van der Houwen and Sommeijer [26] have constructed a family of first order RKC methods. Abdulle and Medovikov [3] have modified it and have proposed a family of the second order RKC methods. Another suitable class of methods is the class of explicit exponential Runge-Kutta (RK) methods for semilinear problems [9, 11, 12, 13, 19, 22]. Although these methods were proposed many years ago, until recently they have not been regarded as practical because of the cost of calculations for matrix exponentials, especially for large problems. In order to overcome this problem, new methods have been proposed [11, 12, 13].

Similarly, for stochastic differential equations (SDEs) explicit RK methods who have excellent stability properties have been developed. Abdulle and Cirilli [1] have proposed a family of explicit stochastic orthogonal Runge-Kutta Chebyshev (SROCK) methods with extended mean square (MS) stability regions. Their methods have strong order one half and weak order one for non-commutative Stratonovich SDEs, whereas they reduce to the first order RKC methods when applied to ODEs. Abdulle and Li [2] have proposed SROCK methods of the same order for non-commutative Itô SDEs. Komori and Burrage [17] have developed these ideas and have proposed weak second order SROCK methods for non-commutative Stratonovich SDEs. If the methods are applied to ODEs, they reduce to the second order RKC methods of Abdulle and Medovikov [3]. Komori and Burrage [18] have also proposed strong first order SROCK methods for non-commutative Itô and Stratonovich SDEs, which reduce to the first or second order RKC methods for ODEs. The weak second order SROCK methods given by Komori and Burrage [17] have an advantage that the stability region is large along the negative real axis, but they still have a drawback, that is, their stability region is not so wide. In order to overcome the drawback, Abdulle, Vilmart and Zygalakis [4] have proposed a new family of weak second order SROCK methods for non-commutative Itô SDEs, in which another family of second order RKC methods is embedded.

On the other hand, Shi, Xiao and Zhang [25] have proposed an exponential Euler scheme for the strong approximation of solutions of SDEs with multiplicative noise driven by a scalar Wiener process. Adamu [5] has proposed exponential integrators for stochastic partial differential equations with a semilinear drift term and multiplicative noise. Komori and Burrage [16] have proposed another explicit exponential Euler scheme for non-commutative Itô SDEs with a semilinear drift term, which is of strong order one half and A-stable in MS.

In the present paper, we devote ourselves to deriving stochastic exponential Runge-Kutta (SERK) methods for the weak approximation of solutions of non-commutative Itô SDEs with a semilinear drift term. We will achieve it on the basis of a stochastic Runge-Kutta (SRK) family proposed by Rößler [24] and explicit exponential RK methods for ODEs proposed by Hochbruck and Ostermann [12]. In Section 2 we will briefly introduce explicit exponential RK methods for ODEs. In Section 3 we will derive our SERK methods, and in Section 4 we will give their stability analysis. Section 5 will present numerical results and Section 6 our conclusions.

2 Explicit exponential RK methods for ODEs

We consider autonomous semilinear ODEs given by

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + \boldsymbol{f}(\boldsymbol{y}(t)), \qquad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{2. 1}$$

where $\boldsymbol{y}$ is an $\mathbb{R}^d$-valued function on $[0,\infty)$, A is a $d \times d$ matrix and $\boldsymbol{f}$ is an $\mathbb{R}^d$-valued nonlinear function on $\mathbb{R}^d$ or a constant vector. By the variation-of-constants formula, we have

$$\boldsymbol{y}(t_{n+1}) = \mathrm{e}^{Ah}\boldsymbol{y}_n + \int_{t_n}^{t_{n+1}} \mathrm{e}^{A(t_{n+1}-s)}\boldsymbol{f}(\boldsymbol{y}(s))\mathrm{d}s \tag{2. 2}$$

if $\boldsymbol{y}(t_n) = \boldsymbol{y}_n$. Here, $\boldsymbol{y}_n$ denotes a discrete approximation to the solution $\boldsymbol{y}(t_n)$ of (2. 1) for an equidistant grid point $t_n \stackrel{\text{def}}{=} nh$ $(n = 1, 2, \ldots, M)$ with step size h (M is a natural number). By interpolating $\boldsymbol{f}(\boldsymbol{y}(s))$ at $\boldsymbol{f}(\boldsymbol{y}_n)$ only, we obtain the simplest exponential scheme for (2. 1) [13]:

$$\boldsymbol{y}_{n+1} = \mathrm{e}^{Ah}\boldsymbol{y}_n + h\varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n), \tag{2. 3}$$

where $\varphi_1(Z) \stackrel{\text{def}}{=} Z^{-1}(e^Z - I)$ and I stands for the $d \times d$ identity matrix. This is called the explicit exponential Euler scheme.

In addition, higher order exponential RK methods have been proposed in [12, 13]. The following is a second order exponential RK method [13]:

$$\begin{aligned} \boldsymbol{Y}_1 &= \mathrm{e}^{c_2hA}\boldsymbol{y}_n + c_2h\varphi_1(c_2hA)\boldsymbol{f}(\boldsymbol{y}_n), \\ \boldsymbol{y}_{n+1} &= \mathrm{e}^{hA}\boldsymbol{y}_n + h\left\{\varphi_1(hA) - \frac{1}{c_2}\varphi_2(hA)\right\}\boldsymbol{f}(\boldsymbol{y}_n) + \frac{1}{c_2}h\varphi_2(hA)\boldsymbol{f}(\boldsymbol{Y}_1), \end{aligned} \tag{2. 4}$$

where c_2 is a parameter and $\varphi_2(Z) \stackrel{\text{def}}{=} Z^{-2}(e^Z - I - Z)$. The following is a third order exponential RK method [12]:

$$\begin{aligned} \boldsymbol{Y}_1 &= \mathrm{e}^{c_2hA}\boldsymbol{y}_n + c_2h\varphi_1(c_2hA)\boldsymbol{f}(\boldsymbol{y}_n), \\ \boldsymbol{Y}_2 &= \mathrm{e}^{c_3hA}\boldsymbol{y}_n + h\{c_3\varphi_1(c_3hA) - \xi(hA)\}\boldsymbol{f}(\boldsymbol{y}_n) + h\xi(hA)\boldsymbol{f}(\boldsymbol{Y}_1), \\ \boldsymbol{y}_{n+1} &= \mathrm{e}^{hA}\boldsymbol{y}_n + h\left\{\varphi_1(hA) - \frac{\gamma+1}{\gamma c_2 + c_3}\varphi_2(hA)\right\}\boldsymbol{f}(\boldsymbol{y}_n) \\ &\quad + \frac{1}{\gamma c_2 + c_3}h\varphi_2(hA)\{\gamma\boldsymbol{f}(\boldsymbol{Y}_1) + \boldsymbol{f}(\boldsymbol{Y}_2)\}, \end{aligned} \tag{2. 5}$$

where c_2, c_3 and γ are parameters satisfying

$$2(\gamma c_2 + c_3) = 3\left(\gamma c_2^2 + c_3^2\right)$$

and $\xi(Z) \stackrel{\text{def}}{=} \frac{c_2}{\gamma}\varphi_2(c_2Z) + \frac{c_3^2}{c_2}\varphi_2(c_3Z)$ (It should be noted that there is a typographical error in (5.9) of [12]).

3 Weak order SERK methods

We derive SERK methods of weak order one or two by utilizing some results in SRK methods. For this, we give a brief introduction to SRK methods in the first subsection. After it, we will derive and show SERK methods in the second and third subsections.

3.1 SRK methods

Similarly to the case of ODEs, we are concerned with autonomous SDEs with the semi-linear drift term given by

$$\mathrm{d}\boldsymbol{y}(t) = (A\boldsymbol{y}(t) + \boldsymbol{f}(\boldsymbol{y}(t)))\mathrm{d}t + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}(t))\mathrm{d}W_j(t), \qquad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \quad (3.\ 1)$$

where $\boldsymbol{g}_j$, $j = 1, 2, \ldots, m$ are $\mathbb{R}^d$-valued functions on $\mathbb{R}^d$, the $W_j(t)$, $j = 1, 2, \ldots, m$ are independent Wiener processes and $\boldsymbol{y}_0$ is independent of $W_j(t) - W_j(0)$ for $t > 0$.

In order to deal with weak approximations for (3. 1), let $\boldsymbol{g}_0(\boldsymbol{y})$ be $A\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{y})$ and let us consider the following SRK method with the stage number s and $r \le s$ [15], which is based on the SRK framework proposed by Rößler [24]:

$$\begin{aligned}\boldsymbol{y}_{n+1} = \boldsymbol{y}_n &+ \sum_{i=1}^{s} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right) + \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(1)} \triangle\hat{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_i^{(j)}\right)\\ &+ \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\boldsymbol{H}_i^{(j)}\right)\\ &+ \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(3)} \triangle\hat{W}_j \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_i^{(j)}\right) + \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(4)} \sqrt{h} \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_i^{(j)}\right),\end{aligned} \quad (3.\ 2)$$

where

$$\begin{aligned}\boldsymbol{H}_i^{(0)} &= \boldsymbol{y}_n + \sum_{k=1}^{i-1} A_{ik}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) \quad (1 \le i \le r),\\ \boldsymbol{H}_i^{(0)} &= \boldsymbol{y}_n + \sum_{k=1}^{i-1} A_{ik}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) + \sum_{k=r}^{i-1}\sum_{l=1}^{m} B_{ik}^{(0)} \triangle\hat{W}_l \boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right) \quad (r < i \le s),\\ \boldsymbol{H}_r^{(j)} &= \boldsymbol{y}_n + \sum_{k=1}^{r} A_{rk}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right),\\ \boldsymbol{H}_i^{(j)} &= \boldsymbol{y}_n + \sum_{k=1}^{i} A_{ik}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) + \sum_{k=r}^{i-1} B_{ik}^{(1)} \sqrt{h} \boldsymbol{g}_j\left(\boldsymbol{H}_k^{(j)}\right) \quad (r < i \le s),\\ \hat{\boldsymbol{H}}_i^{(j)} &= \boldsymbol{y}_n + \sum_{k=1}^{s} A_{ik}^{(2)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) + \sum_{k=r}^{s}\sum_{\substack{l=1\\ l\ne j}}^{m} B_{ik}^{(2)} \tilde{\eta}^{(j,l)} \boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right) \quad (r \le i \le s)\end{aligned}$$

for $j = 1, 2, \ldots, m$ and where the α_i, $\beta_i^{(r_a)}$, $A_{ik}^{(r_b)}$, and $B_{ik}^{(r_b)}$ ($1 \le r_a \le 4$ and $0 \le r_b \le 2$) denote the parameters of the method. The random variables involved in the method are given by $\tilde{\eta}^{(j,j)} \stackrel{\text{def}}{=} ((\triangle\hat{W}_j)^2 - h)/(2\sqrt{h})$,

$$\tilde{\eta}^{(j,l)} \stackrel{\text{def}}{=} \begin{cases} (\triangle\hat{W}_j\triangle\hat{W}_l - \sqrt{h}\triangle\tilde{W}_j)/(2\sqrt{h}) & (j < l),\\ (\triangle\hat{W}_j\triangle\hat{W}_l + \sqrt{h}\triangle\tilde{W}_l)/(2\sqrt{h}) & (j > l),\end{cases}$$

the $\triangle\tilde{W}_l$ ($1 \le l \le m-1$) are independent two-point distributed random variables with $P(\triangle\tilde{W}_j = \pm\sqrt{h}) = 1/2$ and the $\triangle\hat{W}_j$ ($1 \le j \le m$) are independent three-point distributed random variables with $P(\triangle\hat{W}_j = \pm\sqrt{3h}) = 1/6$ and $P(\triangle\hat{W}_j = 0) = 2/3$ [14, p.

Table 1: Butcher tableau for (3. 2) with $r = s - 2$

$A_{21}^{(0)}$										
$\vdots$	$\ddots$									
$A_{s-1,1}^{(0)}$	$\cdots$	$A_{s-1,s-2}^{(0)}$			$B_{s-1,s-2}^{(0)}$					
$A_{s,1}^{(0)}$	$\cdots$	$A_{s,s-2}^{(0)}$	$A_{s,s-1}^{(0)}$		$B_{s,s-2}^{(0)}$	$B_{s,s-1}^{(0)}$				
$A_{s-2,1}^{(1)}$	$\cdots$	$A_{s-2,s-2}^{(1)}$								
$A_{s-1,1}^{(1)}$	$\cdots$	$A_{s-1,s-2}^{(1)}$	$A_{s-1,s-1}^{(1)}$		$B_{s-1,s-2}^{(1)}$					
$A_{s,1}^{(1)}$	$\cdots$	$A_{s,s-2}^{(1)}$	$A_{s,s-1}^{(1)}$	$A_{s,s}^{(1)}$	$B_{s,s-2}^{(1)}$	$B_{s,s-1}^{(1)}$				
$A_{s-2,1}^{(2)}$	$\cdots$	$A_{s-2,s-2}^{(2)}$	$A_{s-2,s-1}^{(2)}$	$A_{s-2,s}^{(2)}$	$B_{s-2,s-2}^{(2)}$	$B_{s-2,s-1}^{(2)}$	$B_{s-2,s}^{(2)}$			
$A_{s-1,1}^{(2)}$	$\cdots$	$A_{s-1,s-2}^{(2)}$	$A_{s-1,s-1}^{(2)}$	$A_{s-1,s}^{(2)}$	$B_{s-1,s-2}^{(2)}$	$B_{s-1,s-1}^{(2)}$	$B_{s-1,s}^{(2)}$			
$A_{s,1}^{(2)}$	$\cdots$	$A_{s,s-2}^{(2)}$	$A_{s,s-1}^{(2)}$	$A_{s,s}^{(2)}$	$B_{s,s-2}^{(2)}$	$B_{s,s-1}^{(2)}$	$B_{s,s}^{(2)}$			
α_1	$\cdots$	α_{s-2}	α_{s-1}	α_s	$\beta_{s-2}^{(1)}$	$\beta_{s-1}^{(1)}$	$\beta_s^{(1)}$	$\beta_{s-2}^{(2)}$	$\beta_{s-1}^{(2)}$	$\beta_s^{(2)}$
					$\beta_{s-2}^{(3)}$	$\beta_{s-1}^{(3)}$	$\beta_s^{(3)}$	$\beta_{s-2}^{(4)}$	$\beta_{s-1}^{(4)}$	$\beta_s^{(4)}$

225]. If we assume $r = s - 2$, for example, (3. 2) is characterized by the Butcher tableau in Table 1.

Let $C_P^L(\mathbb{R}^d, \mathbb{R})$ be the family of L times continuously differentiable real-valued functions on $\mathbb{R}^d$, whose partial derivatives of order less than or equal to L have polynomial growth. Whenever we deal with weak convergence of order q, we will assume the following on SDEs [14, p. 474] (also see [6, p. 113]):

Assumption 3.1 *All moments of the initial value* $\boldsymbol{y}_0$ *exist and* $\boldsymbol{g}_j$ $(j = 0, 1, \ldots, m)$ *are Lipschitz continuous with all their components belonging to* $C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$.

Then, we can give the definition of weak convergence of order q [14, p. 327]:

Definition 3.1 *When discrete approximations* $\boldsymbol{y}_n$ *are given by a numerical scheme, we say that the scheme is of weak (global) order* q *if for all* $G \in C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$, *constants* $C > 0$ (*independent of* h) *and* $\delta_0 > 0$ *exist, such that*

$$|E[G(\boldsymbol{y}(t_M)] - E[G(\boldsymbol{y}_M)]| \le Ch^q, \quad h \in (0, \delta_0).$$

In order to consider numerical schemes of weak order q, Rößler [23] has made use of the following theorem due to Milstein [21], which is stated with an appropriate notation.

Theorem 3.1 *In addition to Assumption 3.1, suppose that the following conditions hold:*

(1) for sufficiently large r, *the moments* $E[\|\boldsymbol{y}_n\|^{2r}]$ *exist and are uniformly bounded with respect to* M *and* $n = 0, 1, \ldots, M$;

(2) for all $G \in C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$, *the local error estimation*

$$\left|E[f(\boldsymbol{y}(t_{n+1}))] - E[f(\boldsymbol{y}_{n+1})]\right| \le K(\boldsymbol{y}_n)h^{q+1}$$

holds if $\boldsymbol{y}(t_n) = \boldsymbol{y}_n$, *where* $K \in C_P^0(\mathbb{R}^d, \mathbb{R})$.

Then, the scheme that gives $\boldsymbol{y}_n$ $(n = 0, 1, \ldots, M)$ is of weak (global) order q.

The second condition concerning the local error in the theorem provides us order conditions for SRK methods of weak order q. For further information, see [23].

If we want to derive a scheme of weak order one from (3. 2), for example, we need to find a set of parameter values satisfying the following nine order conditions [24]:

$$
\begin{gathered}
1.\ \sum_{i=1}^{s} \alpha_i = 1, \quad 2.\ \sum_{i=r}^{s} \beta_i^{(4)} = 0, \quad 3.\ \sum_{i=r}^{s} \beta_i^{(3)} = 0, \quad 4.\ \left(\sum_{i=r}^{s} \beta_i^{(1)}\right)^2 = 1, \\
5.\ \sum_{i=r}^{s} \beta_i^{(2)} = 0, \quad 6.\ \sum_{i=r+1}^{s} \beta_i^{(1)} \left(\sum_{k=r}^{i-1} B_{ik}^{(1)}\right) = 0, \quad 7.\ \sum_{i=r}^{s} \beta_i^{(4)} \left(\sum_{k=1}^{s} A_{ik}^{(2)}\right) = 0, \\
8.\ \sum_{i=r}^{s} \beta_i^{(3)} \left(\sum_{k=r}^{s} B_{ik}^{(2)}\right) = 0, \quad 9.\ \sum_{i=r}^{s} \beta_i^{(4)} \left(\sum_{k=r}^{s} B_{ik}^{(2)}\right)^2 = 0.
\end{gathered}
$$

We will refer to these in the next subsection.

In the case of weak order two we have 59 order conditions including the above nine order conditions, and we need three stages at least to satisfy them [24]. Let us suppose $s = 3$. In order to solve the order conditions in a simple way, we can assume

$$
\begin{aligned}
&\beta_1^{(1)} = \frac{-1 + 2\left(B_{21}^{(1)}\right)^2}{2\varepsilon_1 \left(B_{21}^{(1)}\right)^2}, \quad \beta_2^{(1)} = \beta_3^{(1)} = \frac{1}{4\varepsilon_1 \left(B_{21}^{(1)}\right)^2}, \quad \beta_1^{(2)} = 0, \\
&\beta_2^{(2)} = -\beta_3^{(2)} = \frac{1}{2B_{21}^{(1)}}, \quad \beta_1^{(3)} = -\frac{1}{2\varepsilon_1 b_2^2}, \quad \beta_2^{(3)} = \beta_3^{(3)} = \frac{1}{4\varepsilon_1 b_2^2}, \quad \beta_1^{(4)} = 0, \\
&\beta_2^{(4)} = -\beta_3^{(4)} = \frac{1}{2b_2}, \quad B_{32}^{(0)} = 0, \quad B_{31}^{(1)} = -B_{21}^{(1)}, \quad B_{32}^{(1)} = 0, \\
&B_{11}^{(2)} = B_{12}^{(2)} = B_{13}^{(2)} = 0, \quad B_{23}^{(2)} = B_{22}^{(2)}, \quad B_{31}^{(2)} = -B_{21}^{(2)}, \quad B_{32}^{(2)} = B_{33}^{(2)} = -B_{22}^{(2)}, \\
&A_{21}^{(1)} = A_{31}^{(1)}, \quad A_{22}^{(1)} = A_{32}^{(1)} = A_{33}^{(1)} = 0, \quad A_{1,k}^{(2)} = A_{2,k}^{(2)} = A_{3,k}^{(2)} \quad (1 \le k \le 3)
\end{aligned}
\tag{3. 3}
$$

when $B_{21}^{(1)}$, $B_{21}^{(2)}$ and $B_{22}^{(2)}$ are given [15]. Here, $\varepsilon_1 \stackrel{\text{def}}{=} \pm 1$ and $b_2 \stackrel{\text{def}}{=} B_{21}^{(2)} + 2B_{22}^{(2)}$. Then, only the following three order conditions remain to be solved [15]:

$$
10.\ \sum_{i=2}^{3} \alpha_i \left(B_{i,1}^{(0)}\right)^2 = \frac{1}{2}, \quad 11.\ \sum_{i=2}^{3} \alpha_i B_{i,1}^{(0)} = \frac{\varepsilon_1}{2}, \quad 12.\ \sum_{i=1}^{3} \beta_i^{(1)} A_{i,1}^{(1)} = \frac{\varepsilon_1}{2}.
$$

3.2 SERK methods

As preparations, we start with a simple case. Let us assume $s = r = 1$ in (3. 2) and consider

$$
\begin{aligned}
&\boldsymbol{H}_1^{(0)} = \boldsymbol{y}_n, \qquad \boldsymbol{H}_1^{(j)} = \boldsymbol{y}_n + h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) \quad (1 \le j \le m), \\
&\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + \sum_{j=1}^{m} \triangle\tilde{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_1^{(j)}\right),
\end{aligned}
\tag{3. 4}
$$

which means

$$
A_{11}^{(1)} = \alpha_1 = \beta_1^{(1)} = 1, \quad \beta_1^{(2)} = \beta_1^{(3)} = \beta_1^{(4)} = 0.
$$

Table 2: Butcher tableau of (3. 4)

0	$\bar{0}$	
1	0	
0	0	
1	1	0
	0	0

Because Conditions 1 to 9 are satisfied, (3. 4) is of weak order one. Here, note that $\triangle\tilde{W}_j$ is available for weak order one instead of $\triangle\hat{W}_j$. The Butcher tableaux of (3. 4) is given in Table 2.

Incidentally, since the Euler scheme and (2. 3) are of order one for (2. 1),

$$\left\| \mathrm{e}^{Ah}\boldsymbol{y}_n + \varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n)h - \left(\boldsymbol{y}_n + h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right)\right) \right\| = O(h^2)$$

as $h \to 0$. For this, the replacement of $\boldsymbol{y}_n + h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right)$ with $\mathrm{e}^{Ah}\boldsymbol{y}_n + \varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n)h$ in (3. 4) does not violate the weak order of convergence. Thus, we can obtain the following SERK scheme of weak order one:

$$\begin{aligned} \boldsymbol{H}_1^{(j)} &= \mathrm{e}^{Ah}\boldsymbol{y}_n + h\varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n) \quad (1 \le j \le m), \\ \boldsymbol{y}_{n+1} &= \mathrm{e}^{Ah}\boldsymbol{y}_n + h\varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n) + \sum_{j=1}^{m} \triangle\tilde{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_1^{(j)}\right). \end{aligned} \tag{3. 5}$$

It is remarkable that (3. 5) reduces to (3. 4) if A goes to the zero matrix, whereas they have the same weak order. Taking this into account, now let us consider a way of finding SERK methods who achieve weak order q $(= 1, 2)$ when (3. 2) is of the same weak order q. The following lemma will be helpful for us to do this.

Lemma 3.1 *Assume that $\boldsymbol{y}_{n+1}$ is given by (3. 2) and another approximation $\hat{\boldsymbol{y}}_{n+1}$ is given by*

$$\begin{aligned} \hat{\boldsymbol{y}}_{n+1} = \tilde{\boldsymbol{y}}_{n+1} &+ \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(1)} \triangle\hat{W}_j \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_i^{(j)}\right) + \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_i^{(j)}\right) \\ &+ \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(3)} \triangle\hat{W}_j \boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_i^{(j)}\right) + \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(4)} \sqrt{h}\boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_i^{(j)}\right), \end{aligned} \tag{3. 6}$$

where $\tilde{\boldsymbol{H}}_i^{(j)}, \bar{\boldsymbol{H}}_i^{(j)}$ ($i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, m$) *and $\tilde{\boldsymbol{y}}_{n+1}$ satisfy the deterministic conditions*

$$\begin{aligned} &\left\| \tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)} \right\| = O(h^q), \quad \left\| \bar{\boldsymbol{H}}_i^{(j)} - \hat{\boldsymbol{H}}_i^{(j)} \right\| = O(h^{q+1/2}), \\ &\left\| \tilde{\boldsymbol{y}}_{n+1} - \left\{ \boldsymbol{y}_n + \sum_{i=1}^{s} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right) \right\} \right\| = O(h^{q+1/2}), \end{aligned} \tag{3. 7}$$

the expectation condition

$$\left\| E\left[\tilde{\boldsymbol{y}}_{n+1} - \left\{ \boldsymbol{y}_n + \sum_{i=1}^{s} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right) \right\} \right] \right\| = O(h^{q+1}) \tag{3. 8}$$

and the covariance conditions

$$\left\| E\left[\triangle \hat{W}_j \left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\| = O(h^{q+1}), \qquad (3.\ 9)$$
$$\left\| E\left[\tilde{\eta}^{(j,j)} \left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\| = O(h^{q+1})$$

as $h \to 0$ *for a given* $q = 1$ *or* 2 *under the condition that* $\boldsymbol{y}_n$ *is given. Then, for all* $G \in C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$

$$|E[G(\hat{\boldsymbol{y}}_{n+1}) - G(\boldsymbol{y}_{n+1})]| = O(h^{q+1})$$

as $h \to 0$ *under the condition that* $\boldsymbol{y}_n$ *is given.*

Proof. From (3. 2), (3. 6), (3. 7) and (3. 8), we have

$$\left\|E\left[\hat{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}\right]\right\| \le \left\| E\left[\sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(1)} \triangle \hat{W}_j \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\|$$
$$+ \left\| E\left[\sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(2)} \tilde{\eta}^{(j,j)} \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\| + O(h^{q+1}).$$

Here,

$$\left\| E\left[\triangle \hat{W}_j \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\|$$
$$= \left\| E\left[\triangle \hat{W}_j \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{y}_n\right)\left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\| + O(h^{q+1})$$

because of (3. 2) and (3. 7). This and (3. 9) lead to

$$\left\| E\left[\triangle \hat{W}_j \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\| = O(h^{q+1})$$

under the condition that $\boldsymbol{y}_n$ is given. Similarly,

$$\left\| E\left[\tilde{\eta}^{(j,j)} \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\| = O(h^{q+1}).$$

Hence, we have

$$\left\|E\left[\hat{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}\right]\right\| = O(h^{q+1}) \qquad (3.\ 10)$$

under the condition that $\boldsymbol{y}_n$ is given.

On the other hand,

$$\left\|\hat{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}\right\| = O(h^{q+1/2})$$

because of (3. 2), (3. 6) and (3. 7). For all $G \in C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$, thus,

$$G\left(\hat{\boldsymbol{y}}_{n+1}\right) - G\left(\boldsymbol{y}_{n+1}\right) = \frac{\partial G}{\partial \boldsymbol{y}}\left(\boldsymbol{y}_{n+1}\right)\left(\hat{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}\right) + O(h^{2q+1})$$
$$= \frac{\partial G}{\partial \boldsymbol{y}}\left(\boldsymbol{y}_{n}\right)\left(\hat{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}\right) + O(h^{q+1}).$$

Consequently, because of (3. 10) we obtain

$$E\left[G\left(\hat{\boldsymbol{y}}_{n+1}\right) - G\left(\boldsymbol{y}_{n+1}\right)\right] = O(h^{q+1})$$

as $h \to 0$ under the condition that $\boldsymbol{y}_n$ is given. □

This lemma and Theorem 3.1 give us a way of finding SERK methods. That is, if $\boldsymbol{y}_{n+1}$ given by (3. 2) is of weak order q and $\hat{\boldsymbol{y}}_{n+1}$ given by an SERK method satisfies the assumption in the lemma, then $\hat{\boldsymbol{y}}_{n+1}$ is also of weak order q.

3.3 Examples of SERK methods

In Subsection 3.2 we have derived (3. 5), which is of weak order one. In the present subsection we will derive other SERK methods by utilizing the results in the previous subsection.

3.3.1 Another method of weak order one

When we set $s = r = 2$ and $\beta_2^{(2)} = \beta_2^{(3)} = \beta_2^{(4)} = 0$ in (3. 2), we have

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \alpha_1 h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + \alpha_2 h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right) + \sum_{j=1}^{m} \beta_2^{(1)} \triangle \tilde{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_2^{(j)}\right), \tag{3. 11}$$

where

$$\boldsymbol{H}_1^{(0)} = \boldsymbol{y}_n, \qquad \boldsymbol{H}_2^{(0)} = \boldsymbol{y}_n + A_{21}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right),$$
$$\boldsymbol{H}_2^{(j)} = \boldsymbol{y}_n + A_{21}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + A_{22}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right)$$

for $j = 1, 2, \ldots, m$. When $\alpha_1 + \alpha_2 = \beta_2^{(1)} = 1$, this method is of weak order one because Conditions 1 to 9 are satisfied. Here, note that $\triangle \tilde{W}_j$ is available for weak order one instead of $\triangle \hat{W}_j$.

Taking this and (2. 4) into account, let us consider the following SERK method

$$\boldsymbol{y}_{n+1} = \tilde{\boldsymbol{y}}_{n+1} + \sum_{j=1}^{m} \beta_2^{(1)} \triangle \tilde{W}_j \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_2^{(j)}\right), \tag{3. 12}$$

where

$$\tilde{\boldsymbol{H}}_1^{(0)} = \boldsymbol{y}_n, \qquad \tilde{\boldsymbol{H}}_2^{(0)} = \mathrm{e}^{A_{21}^{(0)} hA} \boldsymbol{y}_n + A_{21}^{(0)} h \varphi_1\left(A_{21}^{(0)} hA\right) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right),$$
$$\tilde{\boldsymbol{H}}_2^{(j)} = \mathrm{e}^{hA} \boldsymbol{y}_n + h\left\{\varphi_1(hA) - \frac{1}{A_{21}^{(0)}} \varphi_2(hA)\right\} \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right)$$
$$+ \frac{1}{A_{21}^{(0)}} h \varphi_2(hA) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right), \qquad \tilde{\boldsymbol{y}}_{n+1} = \tilde{\boldsymbol{H}}_2^{(j)}$$

for $j = 1, 2, \ldots, m$. When $A_{21}^{(1)} = \alpha_1$ and $A_{22}^{(1)} = \alpha_2$ as well as

$$\alpha_1 = 1 - \frac{1}{2A_{21}^{(0)}}, \qquad \alpha_2 = \frac{1}{2A_{21}^{(0)}}, \tag{3. 13}$$

we have

$$\left\|\tilde{\boldsymbol{H}}_2^{(j)} - \boldsymbol{H}_2^{(j)}\right\| = O(h^3), \qquad \left\|\tilde{\boldsymbol{y}}_{n+1} - \left\{\boldsymbol{y}_n + \sum_{i=1}^{2} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right)\right\}\right\| = O(h^3).$$

Moreover, if $\beta_2^{(1)} = 1$, (3. 12) is of weak order one because (3. 11) is of weak order one, whereas (3. 12) is of order two for (2. 1). After all, the Butcher tableaux of (3. 11) is given in Table 3 when both of (3. 11) and (3. 12) are of weak order one.

Table 3: Butcher tableau of (3. 11) when it and (3. 12) are of weak order one

$A_{21}^{(0)}$		0	
$1-\frac{1}{2A_{21}^{(0)}}$	$\frac{1}{2A_{21}^{(0)}}$	0	
0	0	0	
$1-\frac{1}{2A_{21}^{(0)}}$	$\frac{1}{2A_{21}^{(0)}}$	1	0
		0	0

3.3.2 A method of weak order two

Let us suppose $s=3$, $r=1$ and $\alpha_3 = A_{11}^{(2)} = A_{13}^{(2)} = 0$ in (3. 2) as well as (3. 3), and consider

$$\begin{aligned}
\boldsymbol{y}_{n+1} = \boldsymbol{y}_n &+ \alpha_1 h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + \alpha_2 h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right) + \sum_{i=1}^{3}\sum_{j=1}^{m} \beta_i^{(1)} \triangle \hat{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_i^{(j)}\right) \\
&+ \sum_{j=1}^{m} \beta_2^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\boldsymbol{H}_2^{(j)}\right) + \sum_{j=1}^{m} \beta_3^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\boldsymbol{H}_3^{(j)}\right) \\
&+ \sum_{i=1}^{3}\sum_{j=1}^{m} \beta_i^{(3)} \triangle \hat{W}_j \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_i^{(j)}\right) \\
&+ \sum_{j=1}^{m} \beta_2^{(4)} \sqrt{h} \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_2^{(j)}\right) + \sum_{j=1}^{m} \beta_3^{(4)} \sqrt{h} \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_3^{(j)}\right),
\end{aligned} \tag{3. 14}$$

where

$$\begin{aligned}
\boldsymbol{H}_1^{(0)} &= \boldsymbol{y}_n, \qquad \boldsymbol{H}_1^{(j)} = \boldsymbol{y}_n + A_{11}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right), \\
\boldsymbol{H}_2^{(0)} &= \boldsymbol{y}_n + A_{21}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + \sum_{l=1}^{m} B_{21}^{(0)} \triangle \hat{W}_l \boldsymbol{g}_l\left(\boldsymbol{H}_1^{(l)}\right), \\
\boldsymbol{H}_i^{(j)} &= \boldsymbol{y}_n + A_{i,1}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + B_{i,1}^{(1)} \sqrt{h} \boldsymbol{g}_j\left(\boldsymbol{H}_1^{(j)}\right), \\
\hat{\boldsymbol{H}}_1^{(j)} &= \boldsymbol{y}_n + A_{12}^{(2)} h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right), \\
\hat{\boldsymbol{H}}_i^{(j)} &= \boldsymbol{y}_n + A_{i,2}^{(2)} h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right) + \sum_{k=1}^{3}\sum_{\substack{l=1 \\ l\neq j}}^{m} B_{ik}^{(2)} \tilde{\eta}^{(j,l)} \boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right)
\end{aligned}$$

for $i=2,3$ and $j=1,2,\ldots m$.

Corresponding to this and (2. 4), let us suppose the following SERK method

$$
\begin{aligned}
\boldsymbol{y}_{n+1} = \tilde{\boldsymbol{y}}_{n+1} &+ \sum_{i=1}^{3}\sum_{j=1}^{m} \beta_i^{(1)} \triangle \tilde{W}_j \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_i^{(j)}\right) \\
&+ \sum_{j=1}^{m} \beta_2^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_2^{(j)}\right) + \sum_{j=1}^{m} \beta_3^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_3^{(j)}\right) \\
&+ \sum_{i=1}^{3}\sum_{j=1}^{m} \beta_i^{(3)} \triangle \tilde{W}_j \boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_i^{(j)}\right) \\
&+ \sum_{j=1}^{m} \beta_2^{(4)} \sqrt{h}\boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_2^{(j)}\right) + \sum_{j=1}^{m} \beta_3^{(4)} \sqrt{h}\boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_3^{(j)}\right),
\end{aligned}
\tag{3. 15}
$$

where

$$
\begin{aligned}
\tilde{\boldsymbol{H}}_1^{(0)} &= \boldsymbol{y}_n, \qquad \tilde{\boldsymbol{H}}_1^{(j)} = \mathrm{e}^{A_{11}^{(1)}hA}\boldsymbol{y}_n + A_{11}^{(1)} h\varphi_1\left(A_{11}^{(1)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right), \\
\tilde{\boldsymbol{H}}_2^{(0)} &= \mathrm{e}^{A_{21}^{(0)}hA}\boldsymbol{y}_n + A_{21}^{(0)} h\varphi_1\left(A_{21}^{(0)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) + \sum_{l=1}^{m} B_{21}^{(0)} \triangle\tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right), \\
\tilde{\boldsymbol{H}}_i^{(j)} &= \mathrm{e}^{A_{i,1}^{(1)}hA}\boldsymbol{y}_n + A_{i,1}^{(1)} h\varphi_1\left(A_{i,1}^{(1)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) + B_{i,1}^{(1)}\sqrt{h}\boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_1^{(j)}\right), \\
\bar{\boldsymbol{H}}_1^{(j)} &= \mathrm{e}^{A_{12}^{(2)}hA}\boldsymbol{y}_n + A_{12}^{(2)} hA\varphi_1\left(A_{12}^{(2)}hA\right)\sum_{l=1}^{m} B_{21}^{(0)} \triangle\tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right) \\
&\quad + A_{12}^{(2)} h\varphi_1\left(A_{12}^{(2)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right), \\
\bar{\boldsymbol{H}}_i^{(j)} &= \mathrm{e}^{A_{i,2}^{(2)}hA}\boldsymbol{y}_n + A_{i,2}^{(2)} hA\varphi_1\left(A_{i,2}^{(2)}hA\right)\sum_{l=1}^{m} B_{21}^{(0)} \triangle\tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right) \\
&\quad + A_{i,2}^{(2)} h\varphi_1\left(A_{i,2}^{(2)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right) + \sum_{k=1}^{3}\sum_{\substack{l=1 \\ l\neq j}}^{m} B_{ik}^{(2)} \tilde{\eta}^{(j,l)} \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_k^{(l)}\right) \\
\tilde{\boldsymbol{y}}_{n+1} &= \mathrm{e}^{hA}\boldsymbol{y}_n + \frac{1}{A_{21}^{(0)}} hA\varphi_2(hA)\sum_{l=1}^{m} B_{21}^{(0)} \triangle\tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right) \\
&\quad + h\left\{\varphi_1(hA) - \frac{1}{A_{21}^{(0)}}\varphi_2(hA)\right\}\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) + \frac{1}{A_{21}^{(0)}} h\varphi_2(hA)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right)
\end{aligned}
$$

for $i = 2, 3$ and $j = 1, 2, \ldots m$.

From these,

$$
\begin{aligned}
\left\|\tilde{\boldsymbol{H}}_1^{(j)} - \left\{\boldsymbol{H}_1^{(j)} + \frac{1}{2}\left(A_{11}^{(1)}h\right)^2 A\left(A\boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)\right)\right\}\right\| &= O(h^3), \\
\left\|\tilde{\boldsymbol{H}}_i^{(j)} - \left\{\boldsymbol{H}_i^{(j)} + \frac{1}{2}\left(A_{i,1}^{(1)}h\right)^2 A\left(A\boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)\right)\right\}\right\| &= O(h^{5/2})
\end{aligned}
$$

for $i=2,3$ and $j=1,2,\ldots m$. Again, let us assume (3. 13). Then, we have

$$\left\| \tilde{\boldsymbol{y}}_{n+1} - \left\{ \boldsymbol{y}_n + \sum_{i=1}^{2} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right) \right.\right. \\ \left.\left. + \frac{1}{6A_{21}^{(0)}} h^2 A \left(A + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{y}_n)\right) \sum_{l=1}^{m} B_{21}^{(0)} \triangle \tilde{W}_l \boldsymbol{g}_l \left(\tilde{\boldsymbol{H}}_1^{(l)}\right) \right\} \right\| = O(h^3).$$

In addition, because

$$\left\| \bar{\boldsymbol{H}}_i^{(j)} - \hat{\boldsymbol{H}}_i^{(j)} - \left(\frac{1}{2}A_{i,2}^{(2)} - A_{21}^{(0)}\right) A_{i,2}^{(2)} h^2 A \left(A\boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)\right) \right\| = O(h^{5/2})$$

for $i=1,2,3$, let us assume

$$A_{12}^{(2)} = A_{22}^{(2)} = A_{32}^{(2)} = 2A_{21}^{(0)}. \tag{3. 16}$$

Thus, if (3. 14) is of weak order two when (3. 13) and (3. 16) hold, (3. 15) is also weak order two.

We can find a solution for (3. 14) to achieve weak order two as follows. The substitution of $\alpha_3 = 0$ into Conditions 10 and 11 yields $B_{21}^{(0)} = \varepsilon_1$ and $\alpha_2 = \frac{1}{2}$, which means $A_{21}^{(0)} = 1$ due to (3. 13). Taking into account that $B_{21}^{(0)}$, $\beta_i^{(1)}$ and $\beta_i^{(3)}$ ($i=1,2,3$) are multiplied by $\triangle \hat{W}_j$ ($1 \le j \le m$) in (3. 14), we can suppose $\varepsilon_1 = 1$ without loss of generality. Because of (3. 3), we have $B_{21}^{(1)} = \pm\sqrt{\gamma_0}$ from Condition 12 if $\gamma_0 \stackrel{\text{def}}{=} (A_{21}^{(1)} - A_{11}^{(1)})/(1-2A_{11}^{(1)}) > 0$. The Butcher tableaux of this method will be given in the next section.

3.3.3 Another method of weak order two

Let us suppose $s=3$, $r=1$ and $A_{11}^{(2)} = A_{13}^{(2)} = 0$ in (3. 2) as well as (3. 3), and consider

$$\begin{aligned} \boldsymbol{y}_{n+1} = \boldsymbol{y}_n &+ \sum_{i=1}^{3} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right) + \sum_{i=1}^{3}\sum_{j=1}^{m} \beta_i^{(1)} \triangle \hat{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_i^{(j)}\right) \\ &+ \sum_{j=1}^{m} \beta_2^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\boldsymbol{H}_2^{(j)}\right) + \sum_{j=1}^{m} \beta_3^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\boldsymbol{H}_3^{(j)}\right) \\ &+ \sum_{i=1}^{3}\sum_{j=1}^{m} \beta_i^{(3)} \triangle \hat{W}_j \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_i^{(j)}\right) \\ &+ \sum_{j=1}^{m} \beta_2^{(4)} \sqrt{h} \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_2^{(j)}\right) + \sum_{j=1}^{m} \beta_3^{(4)} \sqrt{h} \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_3^{(j)}\right), \end{aligned} \tag{3. 17}$$

where

$$\begin{aligned} \boldsymbol{H}_1^{(0)} &= \boldsymbol{y}_n, \qquad \boldsymbol{H}_1^{(j)} = \boldsymbol{y}_n + A_{11}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right), \\ \boldsymbol{H}_2^{(0)} &= \boldsymbol{y}_n + A_{21}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + \sum_{l=1}^{m} B_{21}^{(0)} \triangle \hat{W}_l \boldsymbol{g}_l\left(\boldsymbol{H}_1^{(l)}\right), \\ \boldsymbol{H}_2^{(j)} &= \boldsymbol{y}_n + A_{21}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + B_{21}^{(1)} \sqrt{h} \boldsymbol{g}_j\left(\boldsymbol{H}_1^{(j)}\right), \\ \boldsymbol{H}_3^{(0)} &= \boldsymbol{y}_n + A_{31}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + A_{32}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right) + \sum_{l=1}^{m} B_{31}^{(0)} \triangle \hat{W}_l \boldsymbol{g}_l\left(\boldsymbol{H}_1^{(l)}\right), \\ \boldsymbol{H}_3^{(j)} &= \boldsymbol{y}_n + A_{31}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + B_{31}^{(1)} \sqrt{h} \boldsymbol{g}_j\left(\boldsymbol{H}_1^{(j)}\right), \end{aligned}$$

$$\hat{\boldsymbol{H}}_1^{(j)} = \boldsymbol{y}_n + A_{12}^{(2)} h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right),$$

$$\hat{\boldsymbol{H}}_i^{(j)} = \boldsymbol{y}_n + A_{i,2}^{(2)} h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right) + \sum_{k=1}^{3}\sum_{\substack{l=1\\ l\neq j}}^{m} B_{ik}^{(2)} \tilde{\eta}^{(j,l)} \boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right)$$

for $i = 2, 3$ and $j = 1, 2, \ldots m$.

Corresponding to this and (2. 5), let us suppose the following SERK method

$$\begin{aligned}
\boldsymbol{y}_{n+1} = \tilde{\boldsymbol{y}}_{n+1} &+ \sum_{i=1}^{3}\sum_{j=1}^{m} \beta_i^{(1)} \triangle \tilde{W}_j \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_i^{(j)}\right) \\
&+ \sum_{j=1}^{m} \beta_2^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_2^{(j)}\right) + \sum_{j=1}^{m} \beta_3^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_3^{(j)}\right) \\
&+ \sum_{i=1}^{3}\sum_{j=1}^{m} \beta_i^{(3)} \triangle \tilde{W}_j \boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_i^{(j)}\right) \\
&+ \sum_{j=1}^{m} \beta_2^{(4)} \sqrt{h} \boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_2^{(j)}\right) + \sum_{j=1}^{m} \beta_3^{(4)} \sqrt{h} \boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_3^{(j)}\right),
\end{aligned} \tag{3. 18}$$

where

$$\tilde{\boldsymbol{H}}_1^{(0)} = \boldsymbol{y}_n, \qquad \tilde{\boldsymbol{H}}_1^{(j)} = \mathrm{e}^{A_{11}^{(1)} hA} \boldsymbol{y}_n + A_{11}^{(1)} h \varphi_1\left(A_{11}^{(1)} hA\right) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right),$$

$$\tilde{\boldsymbol{H}}_2^{(0)} = \mathrm{e}^{A_{21}^{(0)} hA} \boldsymbol{y}_n + A_{21}^{(0)} h \varphi_1\left(A_{21}^{(0)} hA\right) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) + \sum_{l=1}^{m} B_{21}^{(0)} \triangle \tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right),$$

$$\tilde{\boldsymbol{H}}_2^{(j)} = \mathrm{e}^{A_{21}^{(1)} hA} \boldsymbol{y}_n + A_{21}^{(1)} h \varphi_1\left(A_{21}^{(1)} hA\right) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) + B_{21}^{(1)} \sqrt{h} \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_1^{(j)}\right),$$

$$\begin{aligned}
\tilde{\boldsymbol{H}}_3^{(0)} = \mathrm{e}^{c_3^{(0)} hA} \boldsymbol{y}_n &+ A_{32}^{(0)} hA \varphi_1(A_{32}^{(0)} hA) \sum_{l=1}^{m} B_{21}^{(0)} \triangle \tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right) \\
&+ h\left\{c_3^{(0)} \varphi_1(c_3^{(0)} hA) - \xi^{(0)}(hA)\right\} \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) \\
&+ h \xi^{(0)}(hA) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right) + \sum_{l=1}^{m} B_{31}^{(0)} \triangle \tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right),
\end{aligned}$$

$$\tilde{\boldsymbol{H}}_3^{(j)} = \mathrm{e}^{A_{31}^{(1)} hA} \boldsymbol{y}_n + A_{31}^{(1)} h \varphi_1\left(A_{31}^{(1)} hA\right) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) + B_{31}^{(1)} \sqrt{h} \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_1^{(j)}\right),$$

$$\begin{aligned}
\bar{\boldsymbol{H}}_1^{(j)} = \mathrm{e}^{A_{12}^{(2)} hA} \boldsymbol{y}_n &+ A_{12}^{(2)} hA \varphi_1\left(A_{12}^{(2)} hA\right) \sum_{l=1}^{m} B_{21}^{(0)} \triangle \tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right) \\
&+ A_{12}^{(2)} h \varphi_1\left(A_{12}^{(2)} hA\right) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right),
\end{aligned}$$

$$\begin{aligned}
\bar{\boldsymbol{H}}_i^{(j)} = \mathrm{e}^{A_{i,2}^{(2)} hA} \boldsymbol{y}_n &+ A_{i,2}^{(2)} hA \varphi_1\left(A_{i,2}^{(2)} hA\right) \sum_{l=1}^{m} B_{21}^{(0)} \triangle \tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right) \\
&+ A_{i,2}^{(2)} h \varphi_1\left(A_{i,2}^{(2)} hA\right) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right) + \sum_{k=1}^{3}\sum_{\substack{l=1\\ l\neq j}}^{m} B_{ik}^{(2)} \tilde{\eta}^{(j,l)} \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_k^{(l)}\right),
\end{aligned}$$

$$
\begin{aligned}
\tilde{\boldsymbol{y}}_{n+1} = {} & \mathrm{e}^{hA}\boldsymbol{y}_n + \frac{\gamma B_{21}^{(0)} + B_{31}^{(0)}}{\gamma A_{21}^{(0)} + c_3^{(0)}} hA\varphi_2(hA) \sum_{l=1}^{m} \triangle\tilde{W}_l \boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right) \\
& + h\left\{\varphi_1(hA) - \frac{\gamma+1}{\gamma A_{21}^{(0)} + c_3^{(0)}}\varphi_2(hA)\right\}\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) \\
& + h\frac{1}{\gamma A_{21}^{(0)} + c_3^{(0)}}\varphi_2(hA)\left\{\gamma\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right) + \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_3^{(0)}\right)\right\}
\end{aligned}
$$

for $i = 2, 3$ and $j = 1, 2, \ldots m$ as well as

$$
c_3^{(0)} \stackrel{\mathrm{def}}{=} A_{31}^{(0)} + A_{32}^{(0)}, \qquad \xi^{(0)}(Z) \stackrel{\mathrm{def}}{=} \frac{A_{21}^{(0)}}{\gamma}\varphi_2\left(A_{21}^{(0)}Z\right) + \frac{(c_3^{(0)})^2}{A_{21}^{(0)}}\varphi_2\left(c_3^{(0)}Z\right).
$$

Because this $\tilde{\boldsymbol{y}}_{n+1}$ comes from (2. 5), we assume

$$
2\left(\gamma A_{21}^{(0)} + c_3^{(0)}\right) = 3\left\{\gamma(A_{21}^{(0)})^2 + (c_3^{(0)})^2\right\}. \tag{3. 19}
$$

From these,

$$
\begin{aligned}
\left\|\tilde{\boldsymbol{H}}_1^{(j)} - \left\{\boldsymbol{H}_1^{(j)} + \frac{1}{2}\left(A_{11}^{(1)}h\right)^2 A\left(A\boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)\right)\right\}\right\| &= O(h^3), \\
\left\|\tilde{\boldsymbol{H}}_i^{(j)} - \left\{\boldsymbol{H}_i^{(j)} + \frac{1}{2}\left(A_{i,1}^{(1)}h\right)^2 A\left(A\boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)\right)\right\}\right\| &= O(h^{5/2})
\end{aligned}
$$

for $i = 2, 3$ and $j = 1, 2, \ldots m$. Because if

$$
\frac{1}{2}\left(\frac{A_{21}^{(0)}}{\gamma} + \frac{(c_3^{(0)})^2}{A_{21}^{(0)}}\right) = A_{32}^{(0)} \tag{3. 20}
$$

then $\|\tilde{\boldsymbol{H}}_3^{(0)} - \boldsymbol{H}_3^{(0)}\| = O(h^2)$ holds, let us assume (3. 20). In addition, assume

$$
\alpha_1 = 1 - \frac{\gamma+1}{2(\gamma A_{21}^0 + c_3^{(0)})}, \quad \alpha_2 = \frac{\gamma}{2(\gamma A_{21}^0 + c_3^{(0)})}, \quad \alpha_3 = \frac{1}{2(\gamma A_{21}^0 + c_3^{(0)})}. \tag{3. 21}
$$

Then, we have

$$
\begin{aligned}
& \Bigg\|\tilde{\boldsymbol{y}}_{n+1} - \Bigg\{\boldsymbol{y}_n + \sum_{i=1}^{3}\alpha_i h\boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right) \\
& \qquad + \frac{\gamma B_{21}^{(0)} + B_{31}^{(0)} - 3A_{32}^{(0)}B_{21}^{(0)}}{6\left(\gamma A_{21}^{(0)} + c_3^{(0)}\right)} h^2 A\left(A + \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{y}}(\boldsymbol{y}_n)\right)\sum_{l=1}^{m}\triangle\tilde{W}_l\boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right)\Bigg\}\Bigg\| \\
& = O(h^3).
\end{aligned}
$$

Further again, let us assume (3. 16) because this leads to

$$
\left\|\bar{\boldsymbol{H}}_i^{(j)} - \hat{\boldsymbol{H}}_i^{(j)}\right\| = O(h^{5/2})
$$

for $i = 1, 2, 3$. From Lemma 3.1, thus, if (3. 17) is of weak order two when (3. 19), (3. 20) and (3. 21) hold, (3. 18) is also of weak order two.

We can find a solution for (3. 17) to achieve weak order two as follows. For simplicity, let us set γ at 1. The substitution of $\gamma = 1$ into (3. 19) and (3. 20) and simplification yield

$$A_{21}^{(0)} = \frac{2A_{32}^{(0)}}{1 + (3A_{32}^{(0)} - 1)^2}, \qquad A_{31}^{(0)} = \frac{-A_{32}^{(0)}(3A_{32}^{(0)} - 2)^2}{1 + (3A_{32}^{(0)} - 1)^2}.$$

Because $\alpha_2 = \alpha_3$ from $\gamma = 1$ and (3. 21), Conditions 10 and 11 give us

$$B_{21}^{(0)} = \frac{\varepsilon_1 \pm \sqrt{-1 + 4\alpha_2}}{4\alpha_2}, \qquad B_{31}^{(0)} = \frac{\varepsilon_1 \mp \sqrt{-1 + 4\alpha_2}}{4\alpha_2}$$

(double sign in same order) if $-1 + 4\alpha_2 \geq 0$. Taking into account that $B_{21}^{(0)}$, $B_{31}^{(0)}$, $\beta_i^{(1)}$ and $\beta_i^{(3)}$ $(i = 1, 2, 3)$ are multiplied by $\triangle\hat{W}_j$ $(1 \leq j \leq m)$ in (3. 17), we can suppose $\varepsilon_1 = 1$ without loss of generality. Because of (3. 3), we have $B_{21}^{(1)} = \pm\sqrt{\gamma_0}$ from Condition 12 if $\gamma_0 > 0$. The Butcher tableaux of this method will be given in the next section.

4 MS stability analysis for SERK methods

Let us investigate stability properties for our SERK methods. We consider the following test scalar SDE [10]:

$$\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + \sum_{j=1}^{m} \sigma_j y(t)\mathrm{d}W_j(t), \qquad t > 0, \qquad y(0) = y_0, \tag{4. 1}$$

where $y_0 \neq 0$ with probability one (w. p. 1) and where λ and σ_j $(1 \leq j \leq m)$ are complex values and they satisfy

$$2\Re(\lambda) + \sum_{j=1}^{m} \left|\sigma_j\right|^2 < 0. \tag{4. 2}$$

Because of (4. 2), the solution of (4. 1) is MS stable ($\lim_{t\to\infty} E[|y(t)|^2] = 0$) [10].

If we apply (3. 5) to (4. 2), then, we have

$$y_{n+1} = \left(1 + \sum_{j=1}^{m} \sigma_j \triangle\tilde{W}_j\right) \mathrm{e}^{h\lambda} y_n.$$

From this, the MS stability function $\hat{R}$ of (3. 5) is given by

$$\hat{R}(p_r, q) \stackrel{\text{def}}{=} E\left[\left|\left(1 + \sum_{j=1}^{m} \sigma_j \triangle\tilde{W}_j\right) \mathrm{e}^{h\lambda}\right|^2\right] = (1 + q)\mathrm{e}^{2p_r},$$

where $p_r \stackrel{\text{def}}{=} \Re(\lambda)h$ and $q \stackrel{\text{def}}{=} \sum_{j=1}^{m} |\sigma_j|^2 h$. Because we can rewrite (4. 2) by

$$2p_r + q < 0, \tag{4. 3}$$

we have

$$\hat{R}(p_r, q) < (1 - 2p_r)\mathrm{e}^{2p_r}.$$

The function in the right-hand side is less than 1 for any $p_r < 0$. Thus,

$$\hat{R}(p_r, q) < 1, \quad \forall p_r < 0$$

under the condition (4. 3). Consequently, (3. 5) is A-stable in MS [10]. In addition, if we apply (3. 12) to (4. 2), we have the same one as the above $\hat{R}$. Thus, (3. 12) is also A-stable in MS.

If we apply (3. 15) to (4. 2) and utilize (3. 3), then, we have

$$y_{n+1} = R\left(h\lambda, \left\{\triangle\hat{W}_j\right\}_{j=1}^{m}, \left\{\triangle\tilde{W}_l\right\}_{l=1}^{m-1}, \{\sigma_j\}_{j=1}^{m}\right) y_n$$

for which

$$\begin{aligned}
&R\left(h\lambda, \left\{\triangle\hat{W}_j\right\}_{j=1}^{m}, \left\{\triangle\tilde{W}_l\right\}_{l=1}^{m-1}, \{\sigma_j\}_{j=1}^{m}\right) \\
&\quad \stackrel{\text{def}}{=} \mathrm{e}^{h\lambda} + \sum_{j=1}^{m} \triangle\hat{W}_j d_j + \sum_{j=1}^{m} \tilde{\eta}^{(j,j)}\sqrt{h}v_{jj} + \sum_{j=1}^{m}\sum_{\substack{l=1\\l\neq j}}^{m} \tilde{\eta}^{(j,l)}\sqrt{h}v_{jl}, \\
&d_j \stackrel{\text{def}}{=} \sigma_j \left\{\frac{B_{21}^{(0)}(\mathrm{e}^{h\lambda}-1-h\lambda)}{A_{21}^{(0)}h\lambda}\mathrm{e}^{A_{11}^{(1)}h\lambda} + \beta_1^{(1)}\mathrm{e}^{A_{11}^{(1)}h\lambda} + 2\beta_2^{(1)}\mathrm{e}^{A_{21}^{(1)}h\lambda}\right\}, \\
&v_{jl} \stackrel{\text{def}}{=} \begin{cases} \sigma_j^2 \mathrm{e}^{A_{11}^{(1)}h\lambda} & (j = l), \\ 2\beta_2^{(4)}\sigma_j\sigma_l\left(B_{21}^{(2)}\mathrm{e}^{A_{11}^{(1)}h\lambda} + 2B_{22}^{(2)}\mathrm{e}^{A_{21}^{(1)}h\lambda}\right) & (j \neq l). \end{cases}
\end{aligned}$$

Similarly to (3.5) in [15], we have

$$E\left[|R|^2\right] = |\mathrm{e}^{h\lambda}|^2 + \sum_{j=1}^{m} h|d_j|^2 + \frac{1}{2}\sum_{j=1}^{m} h^2|v_{jj}|^2 + \frac{1}{2}\sum_{j=1}^{m}\sum_{\substack{l=1\\l\neq j}}^{m} h^2|v_{jl}|^2. \tag{4. 4}$$

If $\Im(\lambda) = 0$, thus,

$$\begin{aligned}
E[|R|^2] &= \mathrm{e}^{2p_r} + q\left(\frac{(\mathrm{e}^{p_r}-1-p_r)}{p_r}\mathrm{e}^{A_{11}^{(1)}p_r} + \frac{-1+2\gamma_0}{2\gamma_0}\mathrm{e}^{A_{11}^{(1)}p_r} + \frac{1}{2\gamma_0}\mathrm{e}^{A_{21}^{(1)}p_r}\right)^2 \\
&\quad + \frac{1}{2}\sum_{j=1}^{m} q_j^2 \mathrm{e}^{2A_{11}^{(1)}p_r} \\
&\quad + \frac{1}{2}\sum_{j=1}^{m}\sum_{\substack{l=1\\l\neq j}}^{m} q_j q_l \left(\frac{B_{21}^{(2)}\mathrm{e}^{A_{11}^{(1)}p_r} + 2B_{22}^{(2)}\mathrm{e}^{A_{21}^{(1)}p_r}}{B_{21}^{(2)} + 2B_{22}^{(2)}}\right)^2,
\end{aligned} \tag{4. 5}$$

where $q_j \stackrel{\text{def}}{=} h|\sigma_j|^2$, due to (3. 3) and the last paragraph in Subsection 3.3.2. In addition, let us assume $B_{22}^{(2)} = 0$. Then, we obtain the following stability function for (3. 15):

$$\begin{aligned}
\hat{R}(p_r, q) &\stackrel{\text{def}}{=} E[|R|^2] \\
&= \mathrm{e}^{2p_r} + q\left(\frac{(\mathrm{e}^{p_r}-1-p_r)}{p_r}\mathrm{e}^{A_{11}^{(1)}p_r} + \frac{-1+2\gamma_0}{2\gamma_0}\mathrm{e}^{A_{11}^{(1)}p_r} + \frac{1}{2\gamma_0}\mathrm{e}^{A_{21}^{(1)}p_r}\right)^2 \\
&\quad + \frac{1}{2}q^2\mathrm{e}^{2A_{11}^{(1)}p_r}.
\end{aligned}$$

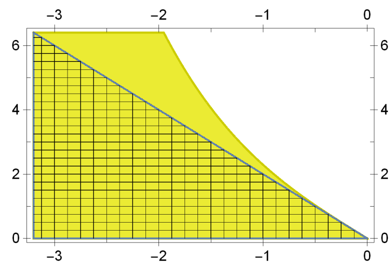

Figure 1: MS stability region for (3. 15)

For simplicity and stability, let us assume $\gamma_0 = 1/2$. Because this leads to $A_{21}^{(1)} = 1/2 \neq A_{11}^{(1)}$, thus,

$$\hat{R}(p_r, q) = \mathrm{e}^{2p_r} + q\left(\frac{1}{p_r}(\mathrm{e}^{p_r} - 1 - p_r)\mathrm{e}^{A_{11}^{(1)}p_r} + \mathrm{e}^{p_r/2}\right)^2 + \frac{1}{2}q^2\mathrm{e}^{2A_{11}^{(1)}p_r}.$$

The application of (4. 3) to this and simplification yield

$$\hat{R}(p_r, q) < \mathrm{e}^{p_r}\psi_1(p_r),$$

where

$$\psi_1(p_r) \stackrel{\mathrm{def}}{=} \mathrm{e}^{p_r} - 2p_r\left(\psi_2(p_r)/p_r + 1\right)^2 + 2p_r^2\mathrm{e}^{(2A_{11}^{(1)}-1)p_r},$$
$$\psi_2(p_r) \stackrel{\mathrm{def}}{=} (\mathrm{e}^{p_r} - 1 - p_r)\mathrm{e}^{(2A_{11}^{(1)}-1)p_r/2}.$$

If we set $A_{11}^{(1)} = 1$, then, by plotting the graph of $\psi_2(p_r)$ we can clearly see $\psi_2(p_r) < \frac{1}{2}$ $(\forall p_r < 0)$. This fact and $A_{11}^{(1)} = 1$ lead to

$$\begin{aligned}&\psi_1(p_r) + \psi_1'(p_r)\\ &= 2\left(\psi_2(p_r)/p_r + 1\right)\left\{\psi_2(p_r)/p_r - p_r - 4\psi_2(p_r) - 2p_r\mathrm{e}^{p_r/2} - 1\right\}\\ &\quad + 2\mathrm{e}^{p_r} + 4p_r(p_r+1)\mathrm{e}^{p_r}\\ &\geq 2\left(\psi_2(p_r)/p_r + 1\right)\left\{1/(2p_r) - p_r - 3\right\}\end{aligned}$$

if $p_r < -1$. From these, we have

$$\{\mathrm{e}^{p_r}\psi_1(p_r)\}' = \mathrm{e}^{p_r}\left(\psi_1(p_r) + \psi_1'(p_r)\right) > 0 \ (\forall p_r \leq -3.2).$$

For any $p_r \leq -3.2$, thus,

$$\hat{R}(p_r, q) < \mathrm{e}^{-3.2}\psi_1(-3.2) \approx 0.228 < 1$$

holds under the condition (4. 3). For this, we plot the MS stability region only in the interval $(-3.2, 0)$ of p_r. The MS stability region is indicated by the colored part in Figure 1. Here, the other part enclosed by the mesh indicates the region in which the solution of the test SDE is MS stable. From these results, we can see that (3. 15) is conditionally A-stable in MS, that is, if $\Im(\lambda) = 0$, it is A-stable. After all, the Butcher tableaux of (3. 14) is given in Table 4 when both of (3. 14) and (3. 15) are of weak order two.

Table 4: Butcher tableau of (3. 14) when it and (3. 15) are of weak order two

1			1					
0	0		0	0				
1								
$\frac{1}{2}$	0		$\frac{1}{\sqrt{2}}$					
$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{2}}$	0				
0	2	0	0	0	0			
0	2	0	$B_{21}^{(2)}$	0	0			
0	2	0	$-B_{21}^{(2)}$	0	0			
$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
			$-\frac{1}{2\left(B_{21}^{(2)}\right)^2}$	$\frac{1}{4\left(B_{21}^{(2)}\right)^2}$	$\frac{1}{4\left(B_{21}^{(2)}\right)^2}$	0	$\frac{1}{2B_{21}^{(2)}}$	$-\frac{1}{2B_{21}^{(2)}}$

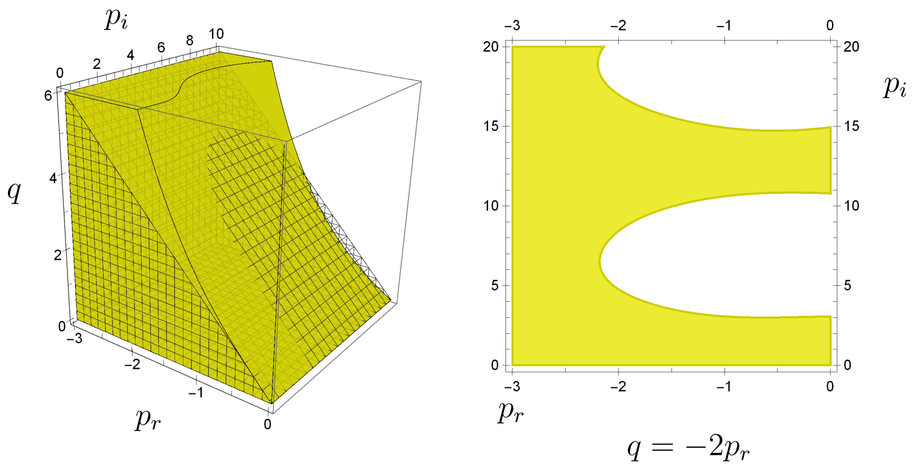

Figure 2: MS stability domain (left) and its profile (right) for (3. 14)

On the other hand, if $\Im(\lambda) \neq 0$, by employing (4. 4) we obtain the following stability function for (3. 15) with $A_{11}^{(1)} = 1$:

$$\begin{aligned}
&\tilde{R}(p_r, p_i, q) \\
&\stackrel{\text{def}}{=} E[|R|^2] \\
&= \mathrm{e}^{2p_r} + \frac{q\mathrm{e}^{p_r}}{p_r^2 + p_i^2} \left\{(1 + \mathrm{e}^{2p_r} - 2\mathrm{e}^{p_r} \cos p_i + 2p_r)\mathrm{e}^{p_r}\right. \\
&\qquad - 2p_r \left[\mathrm{e}^{2p_r} \cos p_i + \mathrm{e}^{p_r/2} \cos(p_i/2) - \mathrm{e}^{3p_r/2} \cos(3p_i/2)\right] \\
&\qquad \left. -2p_i \left[\mathrm{e}^{2p_r} \sin p_i + \mathrm{e}^{p_r/2} \sin(p_i/2) - \mathrm{e}^{3p_r/2} \sin(3p_i/2)\right]\right\} \\
&\quad + q\mathrm{e}^{p_r} \left\{1 + \mathrm{e}^{p_r} - 2\mathrm{e}^{p_r/2} \cos(p_i/2)\right\} + \frac{1}{2} q^2 \mathrm{e}^{2p_r},
\end{aligned}$$

where $p_i \stackrel{\text{def}}{=} \Im(\lambda)h$. Now, we can plot the MS stability domain for (3. 14). The MS stability domain and its profile are given in Figure 2. The MS stability domain is indicated

Table 5: Butcher tableau of (3. 17) when it and (3. 18) are of weak order two

$\frac{2}{3}$			$\frac{2+\sqrt{2}}{3}$					
0	$\frac{2}{3}$		$\frac{2-\sqrt{2}}{3}$	0				
1								
$\frac{1}{2}$	0		$\frac{1}{\sqrt{2}}$					
$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{2}}$	0				
0	$\frac{4}{3}$	0	0	0	0			
0	$\frac{4}{3}$	0	$B_{21}^{(2)}$	0	0			
0	$\frac{4}{3}$	0	$-B_{21}^{(2)}$	0	0			
$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
			$-\frac{1}{2\left(B_{21}^{(2)}\right)^2}$	$\frac{1}{4\left(B_{21}^{(2)}\right)^2}$	$\frac{1}{4\left(B_{21}^{(2)}\right)^2}$	0	$\frac{1}{2B_{21}^{(2)}}$	$-\frac{1}{2B_{21}^{(2)}}$

by the colored part in the left of the figure. Here, the other part enclosed by the mesh indicates the domain in which the solution of the test SDE is MS stable. In the right of the figure, the colored area indicates the profile of the MS stability domain when $q = -2p_r$, which is the boundary of the stability region of the test SDE. From these results, we can see that (3. 14) is not A-stable any more in MS if $\Im(\lambda) \neq 0$.

When we apply (3. 18) to (4. 2), similarly to the case of (3. 15) we have (4. 5) if $\Im(\lambda) = 0$, due to (3. 3) and the last paragraph in Subsection 3.3.3. Also for (3. 18), thus, we set $B_{22}^{(2)} = 0$, $A_{11}^{(1)} = 1$ and $A_{21}^{(1)} = 1/2$. Then, we have the same $\tilde{R}(p_r, p_i, q)$ as that of (3. 15) if $\Im(\lambda) \neq 0$. When we set $A_{32}^{(0)}$ at 2/3 for simplicity, the Butcher tableaux of (3. 17) is given in Table 5. Under this parameter setting, both of (3. 17) and (3. 18) are of weak order two.

5 Numerical Experiments

In the previous sections, we have derived four SERK schemes or methods. For example, (3. 5) is of weak order one and deterministic order one. In the sequel, thus, let us call it the SERKW1D1 scheme. Next, (3. 12) with $\beta_2^{(1)} = 1$ is of weak order one and deterministic order two, and it has the free parameter $A_{21}^{(0)}$. By setting $A_{21}^{(0)}$ at 1 simply, we call it the SERKW1D2 scheme. If (3. 15) has the same parameter values in Table 4, it is of weak order two and deterministic order two, and it has the free parameter $B_{21}^{(2)}$. By setting $B_{21}^{(2)}$ at 1 simply, we call it the SERKW2D2 scheme. Similarly, if (3. 18) has the same parameter values in Table 5 and $B_{21}^{(2)} = 1$, we call it the SERKW2D3 scheme.

In order to confirm the performance of the schemes, we investigate some statistics in numerical experiments. We also investigate computational costs. In simulation results, we will indicate $S_a \stackrel{\text{def}}{=} n_e d + n_r$, where n_e and n_r stand for the number of evaluations on the drift or diffusion coefficients and the number of generated pseudo random numbers, respectively.

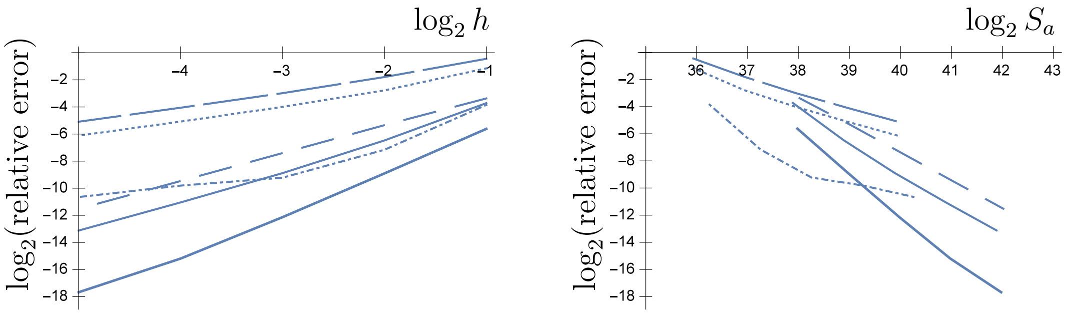

Figure 3: Log-log plots of the relative error of $E[\boldsymbol{y}(1)]$ versus h or S_a (Thick solid: SERKW2D3, Solid: SERKW2D2, dash-dotted: SERKW1D2, dotted: SERKW1D1, long dash: EM, dash: SROCK2)

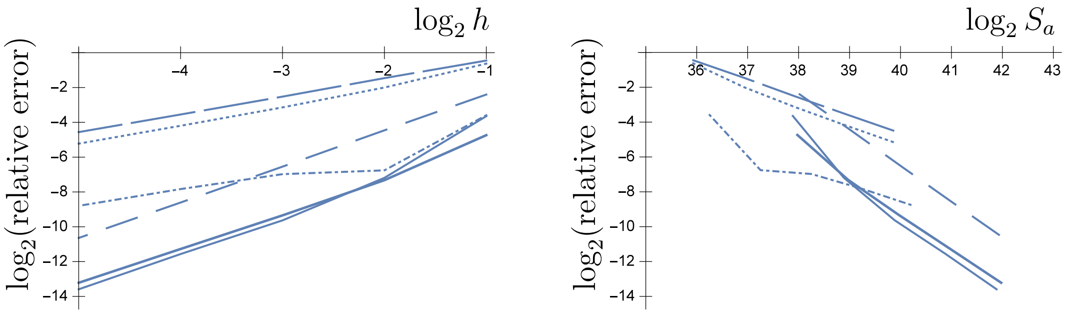

Figure 4: Log-log plots of the relative error of the second moment versus h or S_a (Thick solid: SERKW2D3, Solid: SERKW2D2, dash-dotted: SERKW1D2, dotted: SERKW1D1, long dash: EM, dash: SROCK2)

As a first example, let us consider the following non-commutative SDE, which is obtained by adding a non-linear term to (36) in [8]:

$$
\begin{aligned}
\mathrm{d}\boldsymbol{y}(t) &= \left\{ \begin{bmatrix} -\frac{273}{512} & 0 \\ -\frac{1}{160} & -\frac{785}{512} + \frac{\sqrt{2}}{8} \end{bmatrix} \boldsymbol{y}(t) - \begin{bmatrix} (y_1(t))^3 \\ (y_2(t))^3 \end{bmatrix} \right\} \mathrm{d}t \\
&\quad + \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1-2\sqrt{2}}{4} \end{bmatrix} \boldsymbol{y}(t) \mathrm{d}W_1(t) + \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{10} & \frac{1}{16} \end{bmatrix} \boldsymbol{y}(t) \mathrm{d}W_2(t), \quad t > 0, \quad (5.\ 1) \\
\boldsymbol{y}(0) &= [1\ 1]^\top \ (\text{w.p.1}).
\end{aligned}
$$

We seek an approximation to the expectation of $\boldsymbol{y}(1)$ or to the second moment of each element of $\boldsymbol{y}(1)$, that is, $[E[(y_1(1))^2]\ \ E[(y_2(1))^2]]^\top$. As we do not know the exact solution of the SDE, we seek numerical approximations by the SRKCL scheme [15] with $h = 2^{-7}$ and use them instead of the exact expectation and second moment.

In this example, using the Mersenne twister [20] we simulate 4096×10^6 independent trajectories for a given h. The results are indicated in Figures 3 and 4. As the solution is a vector, the Euclidean norm is used. The thick solid, solid, dash-dotted, dotted, long dash or dash lines denote the SERKW2D3 scheme, the SERKW2D2 scheme, the SERKW1D2 scheme, the SERKW1D1 scheme, the Euler-Maruyama (EM) scheme or the SROCK2 scheme with the stage number 3 [4], respectively. The SERKW2D3 scheme shows high accuracy both in approximations to the expectation and to the second moment.

Table 6: CPU time to solve (5. 1) for 4×10^6 trajectories (the unit is seconds)

$\log_2 h$	-1	-2	-3	-4	-5
SERKW2D3	3	4	9	18	36
SERKW2D2	3	4	8	15	31
SERKW1D2	1	2	4	7	13
SERKW1D1	1	1	3	6	12
EM	1	1	3	6	12
SROCK2	2	2	6	11	22

In Table 6, the schemes are compared in terms of CPU time to solve the same SDE. For 4×10^6 trajectories, it has been measured by Intel C++ Compiler on Windows 7, Intel Core i7 CPU, 2.80 GHz. From these results, we can see that CPU time depends on the weak order rather than the deterministic order.

The second example comes from a stochastic Burgers equation with white noise in time only. Da Prato and Gatarek [7] have proved the existence and uniqueness of the global solution of a scalar Burgers equation with multiplicative noise driven by a scalar Wiener process. Now, we consider an extended version of their equation, that is, the following stochastic Burgers equation:

$$
\begin{aligned}
\mathrm{d}u(t,x) &= \left(\frac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\frac{\partial u}{\partial x}(t,x)\right)\mathrm{d}t + u(t,x)\mathrm{d}W_1(t) \\
&\qquad + \sqrt{1+\big(u(t,x)\big)^2}\,\mathrm{d}W_2(t), \qquad t>0, \quad x\in[0,1], \qquad (5.\ 2)\\
u(t,0) &= u(t,1) = 0 \ (\text{w.p.1}), \qquad t>0,\\
u(0,x) &= 2\sin(\pi x)\ (\text{w.p.1}), \qquad x\in[0,1].
\end{aligned}
$$

If we discretize the space interval by $N+2$ equidistant points x_i $(0 \le i \le N+1)$ and define a vector-valued function by $\boldsymbol{y}(t) \stackrel{\text{def}}{=} [u(t,x_1)\ u(t,x_2)\ \cdots\ u(t,x_N)]^\top$, then we obtain the following non-commutative SDE

$$
\begin{aligned}
\mathrm{d}\boldsymbol{y}(t) &= \big(A\boldsymbol{y}(t) + \boldsymbol{f}(\boldsymbol{y}(t))\big)\mathrm{d}t + \boldsymbol{y}(t)\mathrm{d}W_1(t) + \boldsymbol{b}\big(\boldsymbol{y}(t)\big)\mathrm{d}W_2(t), \quad t>0, \qquad (5.\ 3)\\
\boldsymbol{y}(0) &= [2\sin(\pi x_1)\ 2\sin(\pi x_2)\ \cdots\ 2\sin(\pi x_N)]^\top \ (\text{w. p. 1})
\end{aligned}
$$

by applying the central difference scheme to (5. 2), where

$$
A \stackrel{\text{def}}{=} (N+1)^2 \begin{bmatrix} -2 & 1 & & & \mathbf{0} \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ \mathbf{0} & & & 1 & -2 \end{bmatrix},
$$

$$
\boldsymbol{f}(\boldsymbol{y}) \stackrel{\text{def}}{=} \frac{N+1}{2}\begin{bmatrix} y_1 y_2 \\ y_2(y_3 - y_1) \\ \vdots \\ y_{N-1}(y_N - y_{N-2}) \\ y_N(-y_{N-1}) \end{bmatrix}, \qquad \boldsymbol{b}(\boldsymbol{y}) \stackrel{\text{def}}{=} \begin{bmatrix} \sqrt{1+y_1^2} \\ \sqrt{1+y_2^2} \\ \vdots \\ \sqrt{1+y_N^2} \end{bmatrix}.
$$

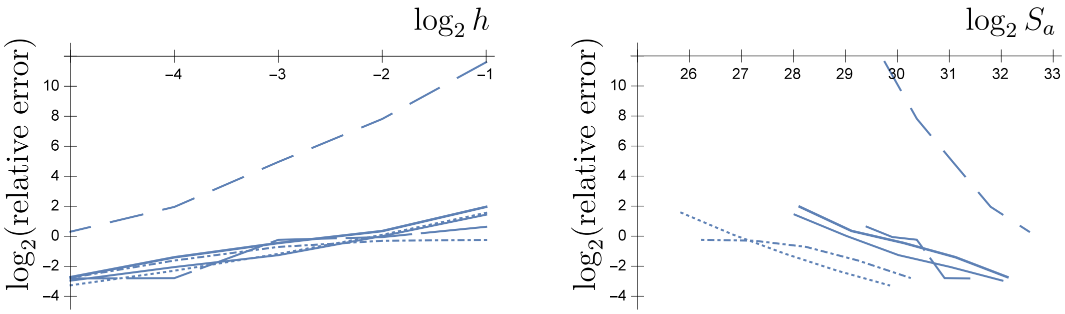

Figure 5: Log-log plots of the relative error of $E[\boldsymbol{y}(1)]$ versus h or S_a (Thick solid: SERKW2D3, solid: SERKW2D2, dash-dotted: SERKW1D2, dotted: SERKW1D1, long dash: SROCK, dash: SROCK2)

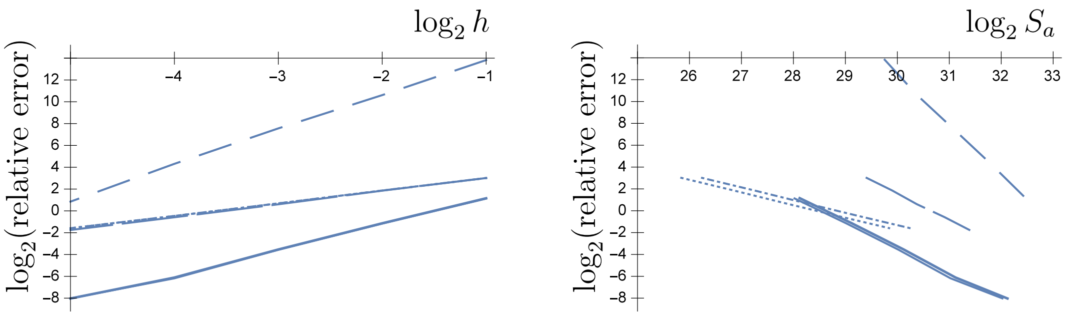

Figure 6: Log-log plots of the relative error of the variance versus h or S_a (Thick solid: SERKW2D3, solid: SERKW2D2, dash-dotted: SERKW1D2, dotted: SERKW1D1, long dash: SROCK, dash: SROCK2)

For $N = 15$ we seek an approximation to the expectation of $\boldsymbol{y}(t)$ or to the variance of each element of $\boldsymbol{y}(t)$. As we do not know the exact solution of the SDE, we seek numerical approximations by the SRKCL scheme with $h = 2^{-9}$ and use them instead of the exact expectation and variance. Here, note that we cannot choose a larger step size 2^{-i} $(1 \leq i \leq 8)$ for the SRKCL scheme to solve the SDE numerically stably.

In this example, we simulate 64×10^4 independent trajectories for a given h. Because the example is a stiff problem, we use the SROCK scheme [2] instead of the EM scheme. The scheme is a kind of stabilized EM scheme. In order to solve the SDE numerically stably with reasonable cost by the SROCK2 scheme, we set the stage number of the scheme at 35, 24, 17, 12 or 8 corresponding to the step size $2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}$ or 2^{-5}, respectively. Similarly, we set the stage number of the SROCK scheme at 35, 24, 16, 11 or 7 corresponding to the step size $2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}$ or 2^{-5}. The results are indicated in Figures 5 and 6. The thick solid, solid, dash-dotted, dotted, long dash or dash lines denote the SERKW2D3 scheme, the SERKW2D2 scheme, the SERKW1D2 scheme, the SERKW1D1 scheme, the SROCK scheme or the SROCK2 scheme, respectively.

In Figure 5, we cannot see big differences among the schemes except the SROCK2 concerning the amount of error, but the SERKW1D1 and SERKW1D2 schemes show low computational cost to achieve the precision. Figure 6 indicates that the SERKW1D1, SERKW1D2 and SROCK schemes have almost the same amount of error, whereas The SERKW2D2 and SERKW2D3 schemes have the almost the same amount of error. The

Table 7: CPU time to solve (5. 3) for 64×10^4 trajectories (the unit is seconds)

$\log_2 h$	-1	-2	-3	-4	-5
SERKW2D3	2	5	9	17	36
SERKW2D2	2	3	7	13	28
SERKW1D2	0	2	2	5	9
SERKW1D1	0	1	2	3	5
SROCK	1	2	2	3	5
SROCK2	1	3	4	6	12

SERKW2D2 and SERKW2D3 schemes, however, show lower computational cost than the SROCK scheme. It is remarkable that the SROCK2 is much worse than the other schemes in both figures.

In Table 7, the schemes are compared in terms of CPU time to solve the same SDE. Similarly to the previous example, the CPU time strongly depends on not only the weak order but also the deterministic order. There is no big difference between the SERKW1D1 scheme and the SROCK scheme in the CPU time, although they are different in computational cost.

6 Conclusions

By utilizing Lemma 3.1 we have constructed SERK methods to achieve weak order one or two for non-commutative Itô SDEs with a semilinear drift term, and simultaneously to achieve order one, two or three for ODEs. Using a scalar test SDE with complex coefficients, we have investigated stability properties for the methods. As a result, we have derived unconditionally A-stable SERK schemes, that is, they are A-stable in MS for a test SDE whose drift and diffusion terms have complex coefficients. They are weak first order schemes, which are of order one or two when applied to ODEs. In addition, we have also derived conditionally A-stable SERK schemes, that is, they are A-stable in MS for a test SDE whose drift term has a real coefficient. They are weak second order schemes, which are of order two or three when applied to ODEs.

In order to check numerical performance of the schemes as well as their stability properties, we have performed two numerical experiments. In the first experiment, our SERK schemes have been compared with the EM scheme and the SROCK2 scheme. The experiment has shown the advantage of our weak second order SERK schemes in computational accuracy. The second experiment is a stiff problem. In the experiment our SERK schemes have been compared with the SROCK scheme and the SROCK2 scheme. All schemes have confirmed their good stability properties, but whereas our weak second order SERK schemes have shown high accuracy, the SROCK2 scheme has shown poor accuracy.

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