CSSE-42 July 27, 2015

On the Nonlinear Stabilization Problem via Quadratic Immersion

Part I: Sigma/Pi-systems and biased solutions of driver-type systems.

Francesco Carravetta *

Istituto di Analisi dei Sistemi ed Informatica
Consiglio Nazionale delle Ricerche

Via dei Taurini, 19
00185 Roma, Italy
francesco.carravetta@iasi.cnr.it

Abstract

Based on our recently issued paper on Quadratic Immersion (QI), (also said 'exact quadratization') for nonlinear control systems, the present work aims to define the main features of a possible new approach in nonlinear control, able to exploit the structural properties of a quadratized system in order to design global, state-feedback-regulators, with an exponential, and tunable, performance, for a meaningful class of nonlinear systems. The paper is divided into two parts. In the Part I we go through the properties of QI and explore the possibility of writing explicitly the solution for the class of the, so called, Sigma/Pi-systems. The main result consists in showing that, under certain conditions, Sigma/Pi-systems are always forward complete, and the solution can be calculated at the steady-state.

* At the time of submission of this manuscript, Francesco Carravetta was visiting the Department of Systems Design and Informatics, Kyushu Institute of Technology, under a program of the JSPS Invitation Fellowship for Research in Japan.

 ISSN 1344-8803

1. Introduction. [1] The aim of this paper is to show that a kind of *systems immersion*, namely the *quadratic immersion* (QI), that we have introduced in the paper [1], can be fruitfully used for the design of nonlinear feedback regulators for certain classes of nonlinear control systems[2] described by ordinary differential equations. A *systems immersion*, as was defined in [2], [3], is a smooth map, say Φ , between the state *spaces* (in general, *smooth manifolds*) of two given systems, that allows to represent one system (the lower dimensional one, said *original system*) as a *subsystem* (said: immersed system) of the other (the larger dimensional one), in the following way: first, the image of the immersion (which is a sub-manifold of the recipient space, or manifold) is the state-manifold of the sub-system, and second, every state trajectory, say the one passing throughout x, of the original system can be recovered – by means of some smooth map – from the trajectory of the immersed system state passing throughout $\Phi(x)$. A quadratic immersion – as we have defined it in [1] – is an immersion into a quadratic system, with the particular feature that it is not required to be defined in *all* of the original system domain, but just on a dense subset of it, leaving aside a zero measure set or, in purely topologic terms, a subset of the original system manifold having a nonempty interior. The main result of [1] is that a QI exists for every analytic integral-closed-form (ICF) nonlinear system, which is a wide class (within the linear in control class) including all systems whose systems functions can be written as any finite composition of the most common transcendent functions with any composition of the elementary algebraic operations, with even any composition of integrals of the above functions, provided that the final function is analytic[3]. Among the analytic ICF systems, a sub class is to be distinguished, namely the class of *$\sigma\pi$-systems*, for which the QI applies in a very direct way, in that the immersed system, i.e. the system satisfying a quadratic differential equation, can be build up directly from the parameters of the original $\sigma\pi$-system. These parameters, that can be time-varying in general, are of two kinds: *exponents* and *coefficients*. As a matter of fact a $\sigma\pi$-system is nothing else that a *formal polynomial* in $\mathbb{R}^n$, where the exponent are allowed to be real numbers in general. A result of [1] is that for any analytic ICF system, there exist an immersion (a complete immersion in this case, not only a dense immersion) into a $\sigma\pi$-system. For this reason $\sigma\pi$-system occupies a central role in a QI-based vision of nonlinearity (as it is our vision), indeed a sort of *nonlinear paradigm*, in that an *huge* class of nonlinear systems[4] can be reduced to a $\sigma\pi$-system by systems immersion.

With the motivation above discussed, in this paper we focus on the class of $\sigma\pi$-system, with the aim of giving a first answer to the following question: how the property of representability through a quadratic system can be exploited for the main problem of interest in control, that is the design of nonlinear regulators, and in general of nonlinear controllers? The answer we propose is the final result of the paper where we present a systematic way, QI-based, for building up state-feedback regulators for $\sigma\pi$-systems. A sufficient condition will be presented for such regulators to be global

[1]The author wishes to thank Professor Hiroshi Ito for the many helpful discussions had during his stay at the Kyushu Institute of Technology, Japan, where the present technical report has been completed. Many of the suggestions arisen during these discussions have surely contributed in improving the technical report in many of his aspects.

[2]Besides further sub categorizations, all systems here considered are linear in the control

[3]Note that, if the composition does not include integrals, the composed function is surely analytic in every point where it is smooth.

[4]Said in a very rough way, ICF systems are all systems that one can write down through *a formula*, since a 'formula' is in practice just a finite composition of 'simple' operations, or in other words a finite composition of the more commonly used functions.

and having an *exponential*, and *tunable* performance, meaning this that the regulation task can be achieved at an exponential speed, that can be even fixed in advance by the designer. A nonlinear regulator designed with the method that we present here has thus a performance similar to the performance of regulators for linear systems, and as a matter of fact, as we will see, the design method will be by itself formally similar to a classical linear regulator design, in that all the matter will be finally reduced to an *eigenvalue assignment problem* for a few suitable matrices, that will be precisely defined in the following, related to the original $\sigma\pi$-system at issue.

We believe that the method we are going to present in this paper constitutes a new approach in nonlinear control, and has few concern with the existing methods classically used in control literature, as methods based on finding Lyapunov functions, or methods based on the exact linearization. Peraphs, the more closely related approach, is the exact linearization one [4]–[6], in that QI-based methods share with exact linearization a common basic guideline consisting in searching a solution to a nonlinear problem by reducing it to a simpler and known problem. The QI reduces indeed a wide class of nonlinear system to a *simpler* one, but not to a kind of nonlinear system for which solutions to the regulation problem are well known. The present paper aims to reduce this gap, by showing that the quadratic representation obtained through QI has, as a matter of fact, some nice features that can be indeed exploited for control purposes.

The paper is sub divided into two Parts. The above described QI-based regulator design method is developed in the Part II. In the present Part I, a few topics will be considered with the aim of building up the main tools used next in Part II. First of all, we perform a review of the main concepts related to the QI, and also go further by focusing on certain additional results that are consequent to the main results of [1], but weren't included therein. We also introduce a particular notation, as well as a classification of $\sigma\pi$-systems, that will be widely used in the sequel to manipulate this kind of systems. The main result of Part I is Theorem 5.3, where it is shown that, under certain conditions, a particular kind of $\sigma\pi$-system, so called *self-driver*: 1) admits a solution defined (and, hence, continuous) on a right-unbounded interval (kind of $[\bar{t}, +\infty) \subset \mathbb{R}$ for some $\bar{t} \in \mathbb{R}$), namely a *steady-state solution*, and: 2) the steady state solution can be calculated as a ratio of free-evolution *modes* of a certain linear system associated to the self-driver, namely the *bilinear frame*[5]

Part I of the paper is organized into 5 sections. In §2 a few notations are introduced, and in particular the *assembling notation*, that is widely used throughout the paper in both Parts I and II. §3 is focused on $\sigma\pi$-systems. A classification is introduced for this kind of systems, as for the various equivalent forms in which they could be given (S-form, C-form, assembled, ordered-form etc.). Also, the main features of a $\sigma\pi$-system, like *size* and *order*, are defined. In §4 the homogeneous quadratic part of the immersed subsystem (cf. supra), namely the *driver*, associated to a $\sigma\pi$-system is focused. Since the driver constitutes the central component of a QI, the review of the main definitions and concepts related to QI are given in this section. Also, as above mentioned, $\sigma\pi$-systems of the type *self-driver* – which, as the name suggests, are $\sigma\pi$-systems whose associated driver is the system itself – are taken under consideration. §6 includes the main result, given in Theorem 5.3. Some top-up issues, as

[5] It will be clear later why we use the term 'bilinear' for an actually *linear* system.

the existence of an *inverse* counterpart of a driver (namely the *inverse driver*), and related *inverse results* are explored as well.

2. Some remarks on the notation. In this paper we adopt a few conventions concerning indices, below explained. Intermediate lowercase letters, such as i, j, k, l, will be preferably used to indicate indices taking values in some *countable* set. For the purposes of the present section let us denote by $\mathcal{I}_i$ the set of values of an index i. The indices mostly used in the present paper are finite subsets of the natural numbers: $\mathcal{I}_i = \{1, \ldots, \nu_i\}$, having cardinality $\nu_i \in \mathbb{N}$ (depending in general by i) and, in general, *n-tuples* from the above set: $j \in \mathcal{I}_j$, $\mathcal{I}_j = \mathcal{I}_{j_1} \times \ldots \times \mathcal{I}_{j_{\nu_j}}$, where $\nu_j = (\nu_{j_1}, \ldots, \nu_{j_{\nu_j}})$, and $\mathcal{I}_j$ is canonically ordered as a cartesian product, that is to say, it forms the sequence: $(1, \ldots, 1), (1, \ldots, 2), \ldots, (1, \ldots, \nu_{j_1}), (1, \ldots, 2, 1), (1, \ldots, 2, 2), \ldots, (\nu_{j_1}, \ldots, \nu_{j_{\nu_j}})$.

The following *assembling convention* will be used: if at a point of the paper a *real* scalar quantity has been named with a subscripted and/or superscripted symbol of the type $\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_m}$ thereinafter the omission of the l-th superscript $l = 1, \ldots, n$ (resp. subscript $l = 1, \ldots, m$), which is marked by a bar:

$$\xi^{i_1,\ldots,i_{l-1},-,i_{l+1}\ldots,i_n}_{j_1,\ldots,j_m}, \tag{2.1}$$

(resp. $\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_{l-1},-,j_{l+1},\ldots j_m}$) shall indicate the corresponding aggregated vector. Thus:

$$\xi^{i_1,\ldots,i_{l-1},-,i_{l+1}\ldots,i_n}_{j_1,\ldots,j_m} = [\xi^{i_1,\ldots,i_{l-1},1,i_{l+1}\ldots,i_n}_{j_1,\ldots,j_m}, \ldots, \xi^{i_1,\ldots,i_{l-1},\nu_l,i_{l+1}\ldots,i_n}_{j_1,\ldots,j_m}]^T, \tag{2.2}$$

$$\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_{l-1},-,j_{l+1},\ldots j_m} = [\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_{l-1},1,j_{l+1},\ldots j_m}, \ldots, \xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_{l-1},\nu_l,j_{l+1},\ldots j_m}]^T. \tag{2.3}$$

The omission of a further index in the above vectors, for instance j_s in (2.1), shall symbolize the stacked vector:

$$\xi^{i_1,\ldots,i_{l-1},-,i_{l+1}\ldots,i_n}_{j_1,\ldots,j_{s-1},-,j_{s+1},\ldots,j_m} = [\xi^{i_1,\ldots,i_{l-1},-,i_{l+1}\ldots,i_n}_{j_1,\ldots,j_{s-1},1,j_{s+1},\ldots,j_m}{}^T, \ldots, \xi^{i_1,\ldots,i_{l-1},-,i_{l+1}\ldots,i_n}_{j_1,\ldots,j_{s-1},\nu_s,j_{s+1},\ldots,j_m}{}^T]^T, \tag{2.4}$$

and so on. In this way ξ shall represent the composed stack of the above vectors, in a certain order, given by the sequence of indices $i_l, j_m, \ldots$, indicating in which sequence the stacks have been performed. If the latter sequence is undefined, i.e. at a certain point of the paper we write ξ, just after the definition of $\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_m}$, then it shall be understood that the sequence used for the recursive stack is as follows: the superscripts are removed first *in the order right to left* and then the subscripts *in the order right to left as well.* Thus, by successive stacks one passes from scalar quantity $\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_m}$, to the ν_{i_n}-dimensional vector $\xi^{i_1,\ldots,i_{n-1}}_{j_1,\ldots,j_m}$, then to the $\nu_{i_{n-1}} \cdot \nu_{i_n}$-vector $\xi^{i_1,\ldots,i_{n-2}}_{j_1,\ldots,j_m}$, and so on, up to ξ, which is $\nu_{i_1} \cdots \nu_{i_n} \nu_{j_1} \cdots \nu_{j_m}$-dimensional.

If $\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_m}$ is a p-vector, then the apposition of a further *subscript*, say j_{m+1} shall denote its j_{m+1}-th component, with $j_{m+1} = 1, \ldots, p$. In this way, $\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_{m+1}}$ has received the definition as a scalar quantity, and thus thereinafter the assembling convention applies. Note that a subscripted/superscripted symbol as $\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_m}$ is not necessarily a vector, or a scalar. In case, for instance, it is defined as a *matrix*, say a $p \times q$-matrix, the assembling convention shall not be applied to that symbol, but the following *matrix convention* is applied: the apposition of *two subscripts*, say j_{m+1}, j_{m+2}: $\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_{m+2}}$, indicates the (j_{m+1}, j_{m+2})-th element of the matrix, with $j_{m+1} = 1, \ldots, p$ (row index) and $j_{m+2} = 1, \ldots, q$ (column index). Also, any index in the indexed quantity $\xi^{i_1,\ldots,i_n}_{j_1,\ldots,j_m}$ is not necessarily a *single* index (a natural number), but can be a *multi-index* as well (cf. *supra*). Thus, for instance, if i_1 (or any other index) is equal to $(l_1,\dots,l_s)$, where the $l.$'s are scalar indices, then replacing for i_1 is allowed, and $\xi^{l_1,\dots,l_s,i_2,\dots,i_n}_{j_1,\dots,j_m}$ shall represent the same quantity.

In case a symbol is defined as anything else than expected – i.e. the definition is inconsistent with the above rules – than the convention is suspended for that symbol, which shall be considered a new symbol with no more a predefined meaning.

Finally, in this paper we widely use summation with respect to multi-indices. The basic convention we use is to indicate the summation bounds when a single index is involved, thus (with r being i or j):

$$\sum_{r_k=1}^{\nu_{r_k}} \xi^{i_1,\dots,i_n}_{j_1,\dots,j_m} \quad , \tag{2.5}$$

otherwise, when a multi-index, say $(r_{k_1},\dots,r_{k_s})$, is involved, we simplify the notation as follows

$$\sum_{r_{k_1},\dots,r_{k_s}}^{\nu_{r_{k_1}},\dots,\nu_{r_{k_s}}} \xi^{i_1,\dots,i_n}_{j_1,\dots,j_m} \quad . \tag{2.6}$$

Sometimes, when the lower bounds are not clear from the context, we write $r_{k_1},\dots,r_{k_s} = (b_1,\dots,b_s)$ below the summation, thus explicitly indicating lower bounds for $r_{k_1},\dots,r_{k_s}$ equal to $b_1,\dots,b_s$ in the order.

3. $\sigma\pi$-systems. In the paper [1] we considered equations having the following form (written componentwise for $i=1,\dots,n$):

$$\dot{x}_i(t) = \sum_{i'=1}^{\nu_i} f_i^{i'}(x(t))v_{i,i'}(t), \qquad x(t_0)=\bar{x}, \tag{3.1}$$

where $f_i^{(i')}$ are C^∞ scalar functions all defined on some open set $\mathcal{X}$ included in $\mathbb{R}^n$. The initial value $\bar{x}$ belongs to $\mathcal{X}$, and the functions $v_{i,i'}$'s are Lebesgue measurable into some real interval including zero and represent perturbations, time varying parameters, or controls. Note the *controls* $v_{i,i'}$ could be dependent of each other, nonetheless we assume that there is always an underlying set of *independent* scalar controls $\{u_1,\dots,u_q\}$ from which the controls $v_{i,i'}$ can be determined, and the map $u\mapsto v_{i,i'}$ is linear[6], which means: for any control $v_{i,i'}(u)$ there exist q time-varying coefficients $b^{i'}_{i,0},\dots,b^{i'}_{i,q}$ such that

$$v_{i,i'}(u) = \sum_{s=1}^{q} b^{i'}_{i,s}u_s + b^{i'}_{i,0}. \tag{3.3}$$

We assume here that system (3.1) is $\sigma\pi$-*algebraic*, that is: either $f_i^{i'}\equiv 0$, or

$$f_i^{i'}(x) = X_{i,i'} = \prod_{j=1}^{n} x_j^{p^{i'}_{i,j}}, \tag{3.4}$$

[6]This is a direct implication for any nonlinear system of the type

$$\dot{x} = f(x,t) + g(x,t)u, \tag{3.2}$$

i.e. *linear in the control.* The class of system (3.1) that we are considering, is just another way of writing the more classical (3.2), where the explicit dependence on t of the functions f, g comes just from another linear binding with time-varying parameters of the type $v_{i,i'}$

where $p_{i,j}^{i'}$ are real numbers. Any $\sigma\pi$-algebraic system ($\sigma\pi$-*system* for short) has a maximal C^∞-domain, $\mathcal{D}$, which is the maximal open set on which all the functions in (3.4) are well defined and C^∞, and can be directly derived as soon as all the exponents $p_{i,j}^{l}$ are given. In particular we have that $\mathcal{D}$ is not in general a connected set, but it is has always a finite number of connected components, each one of which is an open convex set[7]. Also note that any $\sigma\pi$-system is analytic on $\mathcal{D}$.

As shown in [1] if a system of the type (3.1) is analytic and ICF[8], then it is proto-$\sigma\pi$-algebraic, and then undergoes a quadratization. Thus, the $\sigma\pi$-algebraic case considered in this paper includes actually the more general analytic ICF case, as the latter case can be always reduced to the former by analytic systems immersion.

Note that a $\sigma\pi$-system has two basic constituents: the *monomials* $X_{i,i'}$ and the coefficients $v_{i,i'}$. The monomials are always non-zero quantities (non zero functions of x) whereas the coefficients could be zero (the zero function in general). The form (3.1), by replacing (3.4) rewrites as follows

$$\dot{x}_i = \sum_{i'=1}^{\nu_i} v_{i,i'} X_{i,i'}. \tag{3.5}$$

3.1. S-form. We call (3.5) the *standard form* (S-form) of a $\sigma\pi$-systems, if the control-dependent coefficients satisfy (3.3) with $b_{i,0}^{l} = 0$. The S-form of a $\sigma\pi$-system is the one used in [1]: it is the form corresponding to the usual way in which a system of scalar equations is written, where only the terms that are not *a priori* zero are actually written, and labeled following some order (being unessential which order is chosen) with any distinction between control dependent coefficients and 'pure' parameters. If the monomials $X_{i,i'}$'s are labeled so to obey to the restriction: $X_{i,l} \neq X_{i,m}$ if $l \neq m$, we say that (3.5) is in an *assembled* S-form. However, note that, in an assembled S-form, identity (3.3) holds in general with $b_{i,0}^{l} \neq 0$.

3.2. C-form. Besides the S-form, in this paper we extensively use another form obtained from the S-form by *distinguishing* the $v_{i,l}$'s depending on whether they are control-dependent. For any i define $\mathcal{I}_i = \{1, \ldots, \nu_i\}$, and let $\mathcal{I}_i^{\mathbf{c}} \subset \mathcal{I}_i$ such that $v_{i,i'}$ depends of the control u – i.e. satisfies (3.3) with $b_{i,0}^{i'} = 0$ – for any $i' \in \mathcal{I}_i^{\mathbf{p}}$. Also, denote $\mathcal{I}_i^{\mathbf{p}} = \mathcal{I}_i \backslash \mathcal{I}^{\mathbf{c}}$. Let $\nu_i^{\mathbf{c}}$ (resp. $\nu_i^{\mathbf{p}}$) be the cardinality of $\mathcal{I}_i^{\mathbf{c}}$ (resp. of $\mathcal{I}_i^{\mathbf{p}}$), and $i_*(i')$, (resp. $i^*(i')$) an invertible change of indices defined as: $i' \in \mathcal{I}_i^{\mathbf{c}} \mapsto i_* \in \{1, \ldots, \nu_i^{\mathbf{c}}\}$ (resp, $i' \in \mathcal{I}_i^{\mathbf{p}} \mapsto i^* \in \{1, \ldots, \nu_i^{\mathbf{p}}\}$). We settle $i'(i_*(i')) = i'(i^*(i')) = i'$ so that $i'(i_*)$ (resp. $i'(i^*)$) shall denote the inverse map $i_* \in \{1, \ldots, \nu_i^{(\mathbf{c})}\} \mapsto i' \in \mathcal{I}_i^{\mathbf{c}}$ (resp. $i^* \in \{1, \ldots, \nu_i^{(\mathbf{p})}\} \mapsto i' \in \mathcal{I}_i^{\mathbf{p}}$). Then we define

$$v_{i,i_*}^{\mathbf{c}} = v_{i,i'(i_*)}, \quad X_{i,i_*}^{\mathbf{c}} = X_{i,i'(i_*)}; \tag{3.6}$$

$$v_{i,i^*}^{\mathbf{p}} = v_{i,i'(i^*)}, \quad X_{i,i^*}^{\mathbf{p}} = X_{i,i'(i^*)}. \tag{3.7}$$

We call $v_{i,i_*}^{\mathbf{c}}$ and $X_{i,i_*}^{\mathbf{c}}$ (resp. $v_{i,i^*}^{\mathbf{p}}$ and $X_{i,i^*}^{\mathbf{p}}$) the *control coefficients* and the *control monomials* (resp. the *parameters* and the *parametric monomials*). That said, system (3.5), can be rewritten

$$\dot{x}_i = \sum_{i^*=1}^{\nu_i^{\mathbf{p}}} v_{i,i^*}^{\mathbf{p}} X_{i,i^*}^{\mathbf{p}} + \sum_{i_*=1}^{\nu_i^{\mathbf{c}}} v_{i,i_*}^{\mathbf{c}} X_{i,i_*}^{\mathbf{c}}. \tag{3.8}$$

[7]For a characterization of $\mathcal{D}$ see [1], Proposition 2.2

[8]ICF: Integral Closed Form. For the definition of ICF system see [1] §3.).

We call (3.8) a $\sigma\pi$-system in *control form* (C-form). For the parametric and control monomials we have:

$$X^{\mathbf{p}}_{i,i^*} = \prod_{j=1}^{n} x_j^{p^{\mathbf{p},i^*}_{i,j}}, \quad \text{with} \quad p^{\mathbf{p},i^*}_{i,j} = p^{i'(i^*)}_{i,j} \tag{3.9}$$

$$X^{\mathbf{c}}_{i,i_*} = \prod_{j=1}^{n} x_j^{p^{\mathbf{c},i_*}_{i,j}}, \quad \text{with} \quad p^{\mathbf{c},i_*}_{i,j} = p^{i'(i_*)}_{i,j}. \tag{3.10}$$

From (3.3) [9] we have

$$v^{\mathbf{c}}_{i,i_*} = \sum_{s=1}^{q} b^{\mathbf{c},i_*}_{i,s} u_s, \tag{3.11}$$

where $b^{\mathbf{c},i_*}_{i,s} = b^{i'(i_*)}_{i,s}$, and u denotes a q-vector of scalar independent controls. By substituting (3.11) into (3.8), we obtain another type of C-form (we call it Cu-form) of the system:

$$\dot{x}_i = \sum_{i^*=1}^{\nu^{\mathbf{p}}_i} v^{\mathbf{p}}_{i,i^*} X^{\mathbf{p}}_{i,i^*} + \sum_{i_*,s}^{\nu^{\mathbf{c}}_i,q} b^{\mathbf{c},i_*}_{i,s} X^{\mathbf{c}}_{i,i_*} u_s, \tag{3.12}$$

3.3. Size. For a $\sigma\pi$-system, the triple (i', i^*, i_*) associated to a C-form will be said *size triple*, and in particular i': *size index*; i^*: *parametric index*; i_*: *control index.* A size triple depends of what is named i in (3.5) e.g. : the *equation index* of the system. As a matter of fact, in the following we generally use the convention of naming the size, the parametric, and the control indices, the same as the equation index with a prime, a superscripted star and a subscripted star respectively. Given a $\sigma\pi$-system, hereinafter we say that it is i*-labeled*, with this understanding that the equation index is named i, the triple (i', i^*, i_*) labels the C-form and the Cu-form as in (3.8), (3.12), and the S-form is labeled by i' as in in (3.5). The number ν_i in (3.5) or $\nu_i = \nu^{\mathbf{p}}_i + \nu^{\mathbf{c}}_i$ in (3.8) will be said the i-th *size* of the system. The numbers

$$\nu = \max_i \nu_i; \qquad \left(\text{resp.} \quad d = \sum_{i=1}^{n} \nu_i,\right) \tag{3.13}$$

are named the size (resp. the total size) of the system. Moreover, the numbers

$$\nu^{\mathbf{q}} = \max_i \nu^{\mathbf{q}}_i; \qquad \left(\text{resp.} \quad d^{\mathbf{q}} = \sum_{i=1}^{n} \nu^{\mathbf{q}}_i\right), \tag{3.14}$$

with $\mathbf{q} \in \{\mathbf{p}, \mathbf{c}\}$, are said (resp. total) parametric (if $\mathbf{q} = \mathbf{p}$) or control (if $\mathbf{q} = \mathbf{c}$) size. If $\nu > n$ (resp. $\nu^{(\mathbf{q})} > n$) the system is said to be *oversized* (resp. *parametrically oversized* for $\mathbf{q} = \mathbf{p}$ and *oversized in control* for $\mathbf{q} = \mathbf{c}$).

It should be noted that the *same* $\sigma\pi$-system may have different sizes, i.e. its size can be always increased by adding terms of the type $v_{i,i'} X_{i,i'}$ with $v_{i,i'} = 0$ and $X_{i,i'}$

[9] Hereinafter, when an S-form is handled, we understand that it is $b^l_{i,0} = 0$ in (3.3).

being *any* monomial. Moreover, an oversized system can be always reduced in size by defining the additional variables $\{x_{n+1}, \dots, x_\nu\}$, by adding the new equations $\dot{x}_i = 0$, for $i = n+1, \dots, \nu$, pinned at the initial condition $x_i(t_0) = 1$, and redefining the monomials $X_{i,i'} := X_{i,i'} x_{n+1}^0 \cdots x_\nu^0$.

That said, we can always consider a $\sigma\pi$-system rewritten in *constant* (resp. *parametric* and/or *control*) *size*, that is to say with $\nu_i = \nu \ \forall i$ (resp. $\nu_i^{\mathbf{p}} = \nu^{\mathbf{p}}$, and/or $\nu_i^{\mathbf{c}} = \nu^{\mathbf{c}}$, $\forall i$), or in *square* (resp. parametric and/or control) *size*, which means that it is in constant (resp. parametric and/or control) size with $\nu = n$ (resp. $\nu^{\mathbf{p}} = n$ and/or $\nu^{\mathbf{c}} = n$). Moreover, we can always write a $\sigma\pi$-system in *aligned* (resp. parametric and/or control) size, which means that $X_{i,i'} = X_{j,i'}$, (resp. $X^{\mathbf{p}}_{i,i^*} = X^{\mathbf{p}}_{j,i^*}$ and $X^{\mathbf{c}}_{i,i_*} = X^{\mathbf{c}}_{j,i_*}$) for any pair of equation indices i, j such that the monomials are defined.

3.4. Double-indexed $\sigma\pi$-systems. As well as i-indexed $\sigma\pi$-systems, in this paper we have concern with *double* indexed $\sigma\pi$-systems of the type:

$$\dot{x}_{i,i'} = \sum_{j,j'}^{n,\nu_j} v_{i,i'}^{j,j'} X_{i,i'}^{j,j'}, \tag{3.15}$$

where i, j are two equation indices – of some underlying i-indexed $\sigma\pi$-system – spanning $\{1, \dots, n\}$, whereas i', j' are the corresponding size indices. System (3.15) is then another $\sigma\pi$-system having the couple (i, i') as equation index, and the couple (j, j') as size index. v and X represent as usual the coefficients and the monomials of this $\sigma\pi$-system, though they have now each four indices: the lower pair the new composite equation index, and the upper pair the new composite size index. We suppose that the control-coefficients and control-monomials (resp: the parameters and the parametric-monomials) in (3.15) are just those indexed by (j, j_*) (resp. by j, j^*), and thus we can set:

$$v_{i,i'}^{\mathbf{c},j,j_*} = v_{i,i'}^{j,j'(j_*)}, \quad X_{i,i'}^{\mathbf{c},j,j_*} = X_{i,i'}^{j,j'(j_*)}; \tag{3.16}$$

$$v_{i,i'}^{\mathbf{p},j,j^*} = v_{i,i'}^{j,j'(j^*)}, \quad X_{i,i'}^{\mathbf{p},j,j^*} = X_{i,i'}^{j,j'(j^*)}. \tag{3.17}$$

We call (3.15), featuring (3.16), (3.17), the S-form of a (i, j)-*indexed* $\sigma\pi$-system. With the symbolic association: $i \leftrightarrow (i, i')$, and $i' \leftrightarrow (j, j')$, (on right sides the equation and size indices of the underlying system, and on the left sides their counterparts in the double indexed system) we can derive the C-form of (3.15) in the same way as in the mono-index case, though now the equation index (i, i'), and the size index (j, j'), both range in the set [10] :

$$\{(1,1), \dots, (1,\nu_1), (2,1), \dots, (2,\nu_2), \dots, (n,1), \dots, (n,\nu_n)\}. \tag{3.18}$$

With the above guidelines, the C-form of (3.15) is easily derived by, first, splitting the summation in (3.15) according to (3.16), (3.17), which gives

$$\dot{x}_{i,i'} = \sum_{j,j^*}^{n,\nu_j^{\mathbf{p}}} v_{i,i'}^{\mathbf{p},j,j^*} X_{i,i'}^{\mathbf{p},j,j^*} + \sum_{j,j_*}^{n,\nu_j^{\mathbf{c}}} v_{i,i'}^{(\mathbf{c},j,j_*)} X_{i,i'}^{\mathbf{c},j,j_*} \quad ,$$

[10] As a matter of fact, (3.15) is always a square form, even though the underlying system were not. There is no problem in considering more general expressions that (3.15), but this is the more general case we are concerned with in this paper.

and, second, by gathering the above set of equations into two distinct subsets, distinguishing between those labeled by (i, i^*) and those labeled by (i, i_*):

$$\dot{x}^{\mathbf{p}}_{i,i^*} = \sum_{j,j^*}^{n,\nu^{\mathbf{p}}_j} v^{\mathbf{pp},j,j^*}_{i,i^*} X^{\mathbf{pp},j,j^*}_{i,i^*} + \sum_{j,j_*}^{n,\nu^{\mathbf{c}}_j} v^{\mathbf{cp},j,j_*}_{i,i^*} X^{\mathbf{cp},j,j_*}_{i,i^*} \quad , \tag{3.19}$$

$$\dot{x}^{\mathbf{c}}_{i,i_*} = \sum_{j,j^*}^{n,\nu^{\mathbf{p}}_j} v^{\mathbf{pc},j,j^*}_{i,i_*} X^{\mathbf{pc},j,j^*}_{i,i_*} + \sum_{j,j_*}^{n,\nu^{\mathbf{c}}_j} v^{\mathbf{cc},j,j^*}_{i,i_*} X^{\mathbf{cc},j,j_*}_{i,i_*} \quad , \tag{3.20}$$

where – for χ equal to any of the two symbols v, X:

$$x^{\mathbf{p}}_{i,i^*} = x_{i,i'(i^*)}; \quad x^{\mathbf{c}}_{i,i_*} = x_{i,i'(i_*)}, \tag{3.21}$$

$$\chi^{\mathbf{pp},j,j^*}_{i,i^*} = \chi^{\mathbf{p},j,j^*}_{i,i'(i^*)}; \quad \chi^{\mathbf{cp},j,j^*}_{i,i_*} = \chi^{(\mathbf{p},j,j^*)}_{i,i'(i_*)}; \tag{3.22}$$

$$\chi^{\mathbf{pc},j,j^*}_{i,i_*} = \chi^{\mathbf{p},j,j^*}_{i,i'(i_*)}; \quad \chi^{\mathbf{cc},j,j_*}_{i,i^*} = \chi^{\mathbf{c},j,j_*}_{i,i'(i_*)}; \tag{3.23}$$

We call (3.19), (3.20), the C-form of a (i, j)-*indexed* $\sigma\pi$-system whose S-form is (3.15) and whose coefficients and monomials satisfy (3.16), (3.17).

3.5. Ordering. For $\sigma\pi$-systems we sometimes use quantities that depend of the particular *ordering*, i.e. the map $i' \mapsto X_{i,i'}$, used for writing the i-th, i-labeled, system equation. Let us define the set of the *single monomials* in the variable x_i:

$$\mathcal{M}_i = \{m_i = x_i^p; \quad p \in \mathbb{R}\}$$

we define a *canonic* total order relation '$\prec$' (CTOR) on $\mathcal{M}_i$ by setting $m_i \prec m'_i \Leftrightarrow p_i \geq p'_i$. The CTOR on the set $\mathcal{M}^i$ of all monomials of the type $m^i = m_i m_{i+1} \cdots m_n$, with $m_j \in \mathcal{M}_j$ $j = i, \ldots, n$ is then directly obtained recursively as follows: since $\mathcal{M}^n = \mathcal{M}_n$ the two sets have the same CTOR, the CTOR on $\mathcal{M}_i$ is defined by $m_i m^{i+1} = m^i \prec \bar{m}^i = \bar{m}_i \bar{m}^{i+1}$ if and only if either $m_i \prec \bar{m}_i$ and $m_i \neq \bar{m}_i$, or $m_i = \bar{m}_i$ and $m^{i+1} \prec \bar{m}^{i+1}$. We say that a $\sigma\pi$-system is *canonically ordered* if the maps $i' \mapsto X_{i,i'}$ are, for each $i = 1, \ldots, n$, accordingly defined by the CTOR on $\mathcal{M}^1$ (i.e. they are monotone increasing).

In conclusion, different forms can be considered for a given $\sigma\pi$-system. All of such forms are equivalent (i.e. are different expressions of the same differential equation): every $\sigma\pi$-system in S-form can be put in a C-form and vice-versa, as well as enlarged in size and/or (canonically) ordered. All these different forms will be said different *versions* of the same $\sigma\pi$-system.

3.6. Dynamic matrix, control matrix and generator of a $\sigma\pi$-system. To any $\sigma\pi$-system of order n, and with q independent controls, we can associate a couple of matrices (A, B), $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, defined as (for an i-labeled system):

$$A_{i,j} = \sum_{i'=1}^{\nu_i} v_{i,i'} \frac{\partial X_{i,i'}}{\partial x_j} = \sum_{i^*=1}^{\nu^{\mathbf{p}}_i} v^{\mathbf{p}}_{i,i^*} \frac{\partial X^{\mathbf{p}}_{i,i^*}}{\partial x_j} + \sum_{i_*=1}^{\nu^{\mathbf{c}}_i} v^{\mathbf{c}}_{i,i_*} \frac{\partial X^{\mathbf{c}}_{i,i_*}}{\partial x_j}, \tag{3.24}$$

$$B_{i,s} = \sum_{i_*=1}^{\nu^{\mathbf{c}}_i} b^{\mathbf{c},i_*}_{i,s} X^{\mathbf{c}}_{i,i_*}, \qquad s = 1, \ldots, q. \tag{3.25}$$

We call A the *dynamic matrix*, and B the *control matrix* of the system. Note that $A = A(x, u)$ and $B = B(x)$, i.e. the control matrix depends (in general) of x, and the dynamic matrix depends in general of the control (through the control coefficients $v^{\mathbf{c}}$). Moreover, notice that the definition of A and B does not depend of the particular *version* to which (3.24) and (3.25) are applied. The reason why we use the well known terms 'dynamic' and 'control' for the matrix defined in (3.24) and (3.25) will be clear in the Part II of the present paper.

Along with the dynamic and control matrix, we associate to a $\sigma\pi$-system a third matrix, that we name *the generator*, namely $V \in \mathbb{R}^{n\times n}$, of the system, defined as follows: $V_{i,j} = v_{i,j}$ where $v_{i,j}$ are the coefficients of the *assembled*, *square sized* and *canonically ordered* S-form of the system. The generator of a $\sigma\pi$-system may depend of u, but never depends of x. It is in general a time-varying matrix.

4. Drivers, self-drivers, and driver-type $\sigma\pi$-systems.

4.1. Quadratic Immersion. We briefly summarize the concept of quadratic immersion, and refer the reader to our former paper [1] for all the details. We also go deeper into certain issues that were only mentioned in [1].

A *systems immersion* (resp. a *dense immersion*) (cf. [1]-[2]-[3]) from a system, $\mathcal{S}_1$ in $\mathbb{R}^n$, into another system $\mathcal{S}_2$ living in $\mathbb{R}^m$ (with $m \geq n$) is a smooth map from the domain[11] (resp. the domain with possibly the exception of a zero-measure set) of $\mathcal{S}_1$ onto a smooth manifold, say $\mathcal{M}$, included in the domain of $\mathcal{S}_2$, such that any trajectory of $\mathcal{S}_2$ starting from $\mathcal{M}$ includes a trajectory of $\mathcal{S}_1$, and all the trajectories of $\mathcal{S}_1$ can be generated by trajectories of $\mathcal{S}_2$ starting from $\mathcal{M}$. A quadratic immersion is a dense immersion into a quadratic system.

The basic results proved in [1] is the following: if (3.5) is $\sigma\pi$-algebraic, with domain $\mathcal{D} \subset \mathbb{R}^n$, then the (scalar) variables

$$Z_{i,i'} = \frac{X_{i,i'}}{x_i}, \tag{4.1}$$

satisfy the following system of quadratic ordinary differential equations (QODEs):

$$\dot{Z}_{i,i'} = \Big(\sum_{j=1}^{n} p_{i,j}^{(i')} Z_j^T v_j - Z_i^T v_i\Big) Z_{i,i'}, \tag{4.2}$$

every time they are well defined. The QODE (4.2) constitutes a new quadratic system (written component wise) said *the driver* associated to the $\sigma\pi$-system (3.5). Note that (4.2) is constituted by d equations, where d is the total size (cf. §3.3) of the system, and thus $Z \in \mathbb{R}^d$. Looking at (3.5), (3.4), and (4.1) we have

$$\dot{x}_i = \Big(\sum_{i'=1}^{\nu_i} Z_{i,i'} v_{i,i'}\Big) x_i = (Z_i^T v_i) x_i \quad , \tag{4.3}$$

[11] Hereinafter by 'system domain' we mean the maximal open set which includes all system trajectories. In system-theoretic terms it is the 'state space', but it is not supposed to be a (vector) space. In fact, in our framework, where the system function is always given through finite compositions of algebraic and/or transcendent well known maps (cf. [1]), the system-domain is always the maximal open set, in some analytic sub-manifold of $\mathbb{R}^n$, where the system function is well defined and analytic.

whose solution can be written as

$$x_i(t) = e^{\int_{t_o}^{t} Z_i^T v_i} x_i(t_0) \quad , \tag{4.4}$$

provided that $Z_i(\tau)$ is defined $\forall \tau \in [t_0, t]$. Another result ([1] Theorem 2.5) is that even the monomials $X_{i,i'}$ are differentially related to the driver components. Indeed the $X_{i,i'}$'s satisfy[12] the following set of differential equations:

$$\dot{X}_{i,i'} = \left(\sum_{j=1}^{n} p_{i,j}^{(i')} Z_j^T v_j \right) X_{i,i'} \quad , \tag{4.5}$$

and thus, similarly to (4.4), we can write each monomial $X_{i,i'}$ as well, as a function of the driver:

$$X_{i,i'}(t) = e^{\sum_{j=1}^{n} \int_{t_o}^{t} p_{i,j}^{(i')} Z_j^T v_j} X_{i,i'}(t_0) = \prod_{j=1}^{n} e^{\int_{t_o}^{t} p_{i,j}^{(i')} Z_j^T v_j} X_{i,i'}(t_0) \quad , \tag{4.6}$$

which holds under the same conditions we have already seen as for x_i.

The general solution of a $\sigma\pi$-system can be written as in (4.4), i.e. as a function of the solution of the associated driver, every time both the solutions are defined on a time interval (T_1, T_2), with $T_1, T_2 \in [-\infty, +\infty]$, and $t_0, t \in (T_1, T_2)$. The quadratic system described by eqs. (4.2) can be thought of as 'driving' the 'final stage', i.e. another system given by eqs. (4.3), which is said *the final* system, and giving back the original state components. An *alternative* 'cascade decomposition' (cf [1]) can be obtained by involving (4.5). As a matter of fact, equations (4.5) can be viewed as a *bilinear* system, which in [1] is named *the medial system*, whose input is Z. Assuming this representation, we can interprete the original system equations (3.5) a final system (different than (4.4)) given by an integral action on the *medial state* (i.e. the collection of all $\sigma\pi$-system monomials), and thus the $\sigma\pi$-system can be viewed as the *cascade* of three systems: the driver (feeding the medial) and the medial feeding an integral action (cf [1] Fig. 1).

The map $x \mapsto Z$ has the same domain as the original $\sigma\pi$-system, except all the *coordinate hyperplanes*[13] in $\mathbb{R}^n$, namely $\mathcal{X}_i$:

$$\mathcal{X}_i = \{x \in \mathbb{R}^n : x_i = 0\} \quad . \tag{4.7}$$

By denoting $\mathcal{D}'$ such a domain, we have that $\mathcal{D} \setminus \mathcal{D}'$ has zero measure[14] in $\mathbb{R}^n$. Moreover, the driver (4.2), if starts from a $\bar{Z} \in \mathbb{R}^d$ satisfying (4.1) for $i = 1, \dots n$, gives the original state x through eq. (4.3). Thus the map $x \mapsto (x, Z)$, namely $\Phi : \mathbb{R}^n \supset \mathcal{D}' \to \mathbb{R}^{n \times d}$:

$$(x, Z) = \Phi(x) \quad , \tag{4.8}$$

is a dense immersion into the quadratic system constituted by (4.2), (4.3), that is: a quadratic immersion.

---

[12] In [1] the monomials of a $\sigma\pi$-system are indicated with the symbol $Z_{i,0}^{(l)}$, which in the present notation would became $X_{i,l}$.

[13] Note that it could be lacking coordinate hyperplanes the original domain itself.

[14] Or, in pure topologic terms: has non empty interior. The *only* case in which $\mathcal{D} \setminus \mathcal{D}'$ is empty (and not only has an empty interior) is when the original domain lacks *all* coordinate hyperplanes.

We conclude this eye-bird view of the main results of [1], by pointing out a fact that in [1] has not been highlighted: as a matter of fact (4.5) holds in general for *any* monomial of the type (3.4), even if it is not included in the system expression. This can be directly entailed by a few new concepts we have introduced in previous sections, and in particular by the fact that we can always enlarge the *size* of a $\sigma\pi$-system, by adding a new monomial, multiplied by zero, to *any* of the system equations. Also, since the driver components, are $\sigma\pi$-functions of x – i.e. $Z_{i,i'} = Z_{i,i'}(x)$ where the map $x \mapsto Z_{i,i'}$ is a $\sigma\pi$-function – it follows that any monomial can be included into a new $\sigma\pi$-system, equivalent to the original *on the common domain*, and with order increased by one: i.e. by adjoining equation (4.5), considered as a $\sigma\pi$-equation of x,[15] as a further equation to (3.5).

Example. Consider the $\sigma\pi$-system:

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \quad , \\ \dot{x}_2 &= x_1^2 \quad . \end{aligned}$$

We want to write a differential equation for the term $x_1^5 x_2^7$. Let us enlarge the system size by introducing the fictitious term $0 \cdot X_{2,1}$ in the second equation, with $X_{2,1} = x_1^5 x_2^7$, where the other monomials are $X_{1,1} = x_1 x_2$, and $X_{2,2} = x_1^2$. It is easily seen that the enlarged-size-system has three driver components: $Z_{1,1} = x_2$, $Z_{2,1} = x_1^5 x_2^6$, and $Z_{2,2} = x_1^2/x_2$, thus by formula (4.5) we have

$$\dot{X}_{2,1} = \left(p_{2,1}^{(1)}(Z_1^T v_1) + p_{2,2}(Z_2^T v_2) \right) X_{2,1} = (5Z_{1,1} + 7Z_{2,1}) X_{2,1} \quad . \tag{4.9}$$

Thus, if we want to adjoin $X_{2,1}$ as a new state variable of the original system, we adjoin the above equation with the drivers components replaced by their expressions in x:

$$\dot{X}_{2,1} = 5x_2 X_{2,1} + 7\frac{x_1^2}{x_2} X_{2,1} = 5x_1^5 x_2^9 + 7x_1^7 x_2^6 \quad , \tag{4.10}$$

where we can use the first or the second expression above, as the right hand side of the new equation. In both cases the new equation has to be initialized to $x_1^5(t_0) x_2^6(t_0)$.

4.2. Some more insight into Quadratic Immersion. In this paper we go deeper into the structure of the quadratic representation (4.2), (4.3), thus, in order to avoid ambiguities, we introduce some new terminology.

In the following we call 'subsystem' of a system described by a set of differential equations of the type (3.5) defined on some analytic sub manifold of $\mathbb{R}^n$, say $\mathcal{M}$, the system described by a subset of these equations, say those labeled $\{i_1, \ldots, i_L\}$, with domain given by the trace of $\mathcal{M}$ on $\text{span}\{x_{i_1}, \ldots, x_{i_L}\}$. The dependent variables of the system that appears without derivative in the subsystem are re-interpreted as parameters of the subsystem, solutions of the *complement* sub-system, given by the equations labeled $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_L\}$, with domain given by the trace of $\mathcal{M}$ on $\text{span}^{\perp}\{x_{i_1}, \ldots, x_{i_L}\}$.

[15] As shown in [1] products and sums of $\sigma\pi$-functions are $\sigma\pi$-functions. However, note that the domain of the new system might change

We call the system described by (4.2), (4.3), whose domain is all of $\mathbb{R}^{n\times d}$, *the abode system*, whereas the same system *restricted* to the manifold[16]:

$$\mathcal{M}_\Phi = \{(x,Z) \in \mathbb{R}^{n\times d}: \quad (x,Z) = \Phi(x)\} \quad , \tag{4.11}$$

i.e. restricted to the image of the immersion Φ defined in (4.8), is said *the image system*[17]. The system undergoing the quadratization, here system (3.5), is said the *original system* (or *the object system*), whereas the *subsystem* of the image system constituted by the final equations (4.4), is said *the immersed system* (and its complement the *immersed driver*). It should be stressed that the domain $\mathcal{D}$ of the original system, and the domain $\mathcal{M}_\Phi$ of the image system, need not to be connected. As a matter of fact, as we will see in a moment, it is very common the situation where $\mathcal{M}_\Phi$ is the union of disconnected components.

Following the terminology of Fliess-Kupka [3] we can rephrase the quadratic immersion theorem of [1] as follows: any $\sigma\pi$-system can be represented as a subsystem – namely the *immersed system* (4.3) – of a quadratic system, namely the image system: which is a quadratic system evolving on the manifold (4.11). However, there are some differences with respect to [2], as we are here concerned with *dense* immersions, and thus even though all the trajectories of the immersed system are trajectories of the original system, the converse is true only for the *pieces* of the original trajectory that lies in the domain of the immersed system.

In order to get more insight from this point, first of all note that the domain of the immersed system agrees with the domain $\mathcal{D}'$ of the immersion Φ, and thus is included in $\mathcal{D}$. Now, let $x(t)$, with $t \in (a,b) \subset \mathbb{R}$, the solution of the original $\sigma\pi$-system (3.5) passing through a point $\bar{x} \in \mathcal{D}'$ at some time $t_0 \in (a,b)$. The function $x(\cdot)$ describes a smooth curve in $\mathcal{D}$, thus it may well happen that some piece of this curve lies in $\mathcal{D} \setminus \mathcal{D}'$ and not in $\mathcal{D}'$. In this case the set:

$$\mathcal{C}_{\bar{x},t_0} = \{(x,t) \in \mathcal{D}' \times (a,b): \qquad x = x(t), \quad t \in (a,b)\}, \tag{4.12}$$

is just a union of *disconnected* curves, and what the quadratic immersion theorem tells us is not that the original and the immersed system have *the same* trajectories (which is not exact) but that: any *connected component* of $\mathcal{C}_{\bar{x},t_0}$ is a sub-curve[18] of the curve $x(t)$. In the forthcoming lemma we see that, unless trivial cases, $(x(t),t) \notin \mathcal{C}_{\bar{x},t_0}$ holds at most at isolated time points. Denote by $\mathbf{T}_{\bar{x},t_0} \subset (a,b)$ the set

$$\mathbf{T}_{\bar{x},t_0} = \{t \in (a,b): \quad x(t) \in \mathcal{D} \setminus \mathcal{D}'\}, \tag{4.13}$$

and consider a $\sigma\pi$-system which does not contain *trivial* equations, that is zero equations of the type $\dot{x}_0 = 0$ with initial condition $x_i(t_0) = 0$.

Lemma 4.1. *For any $\sigma\pi$-system without trivial equations, either the set $\mathbf{T}$ defined in (4.13) is empty, or is a union of isolated points in (a,b), corresponding to the times in which $x(t)$ crosses some coordinate hyperplane.*

---

[16]Indeed a smooth sub-manifold of $\mathbb{R}^{n+d}$

[17]The quadratic immersion theorem, stated in other words, guarantees that the image system is well defined, or equivalently that $\mathcal{M}_\Phi$ is an invariant set of the abode system.

[18]Literally: it is a *sub-graph* of the graph of the curve $x(t)$, but we identify the function with its graph.

Proof. Suppose that there exists an open interval $(\alpha,\beta) \subset \mathbf{T}$. Then we have $x(\tau) \in \mathcal{D} \setminus \mathcal{D}'$ $\forall \tau \in (\alpha,\beta)$. But we have seen that $\mathcal{D} \setminus \mathcal{D}'$ either is empty or there exists an i, $1 \leq i \leq n$, such that $\mathcal{X}_i \subset \mathcal{D} \setminus \mathcal{D}'$, where $\mathcal{X}_i$ is given by (4.7). In the latter case it is $x_i(\tau) = 0$ $\forall \tau \in (\alpha,\beta)$, and hence the original system includes the trivial equation $\dot{x}_i = 0$ with $x(t_0) = 0$. If $\mathbf{T}$ cannot include an open interval the thesis follows. □

Thus, the solution of the original system may be defined on some coordinate hyperplane, while the immersed system may be not defined therein. In this case, the original system – provided has not trivial equations – has a solution that can *cross* the hyperplane, but cannot *lie* therein, moving in the hyperplane for a non zero length time interval, and then leave it. The time t such that $x_i(t) = 0$ shall be always a single time. Obviously, trivial equations are useless, and can be always skipped while applying the immersion (4.1). On the contrary, zero equations of the type $\dot{x}_i = 0$ with non-zero initial condition, which is the case when we hide *fictitious monomials* in the zero equation, plays an important role in the regulator design method we are going to describe in the Part II of the present paper, but in this case as well, as claimed by Lemma 4.1, the system trajectory $x(t)$ can only cross the hyperplane $\mathcal{X}_i$ and not lie within for a non-zero length time interval.

We highlight that (4.4) is the solution of (4.3), i.e.: is the solution of the *final* system, and *not* of the original system. As we have already seen, the latter is a smooth connected curve that just *includes* $\mathcal{C}_{\bar{x},t_0}$, which is a union of *distinct* solutions of (4.3) passing through distinct points of $\mathcal{D}'$. Thus, formula (4.4) describes only the connected component of $\mathcal{C}_{\bar{x},t_0}$ which $(x(t_0), t_0)$ belongs to, and not all the trajectory of the original system passing through $x(t_0)$, which might be larger. It should be noted that what above described is not a sort of 'pathologic' event, and in fact is a common situation occurring while applying the quadratic immersion, even in very simple cases, as shown in the following example.

Example. Let a, b be two fixed positive real numbers, $t_0 \in \mathbb{R}$, and let us consider the linear scalar system:

$$\dot{x} = -ax - b, \qquad x(t_0) = \bar{x}, \qquad \bar{x} \in \mathbb{R}, \tag{4.14}$$

Whose domain is $\mathcal{D} = \mathbb{R}$. As a $\sigma\pi$-system, (4.14) agrees with (3.5) with $n = 1$, $\nu_i \equiv \nu_1 = 2$, $v_{1,1} = -a$, $v_{1,2} = -b$, $X_{1,1} = x$, $X_{1,2} = 1$, $p^1_{1,1} = 1$, $p^2_{1,1} = 0$. The driver components are given by (4.1), in this case: $Z_{1,1} = 1$, $Z_{1,2} = x^{-1}$, and applying formula (4.2) we obtain the driver equations:

$$\dot{Z}_{1,1} = 0, \tag{4.15}$$

$$\dot{Z}_{1,2} = aZ_{1,1}Z_{1,2} + bZ^2_{1,2}. \tag{4.16}$$

and by (4.3) the final equation:

$$\dot{x} = (-aZ_{1,1} - bZ_{1,2})x. \tag{4.17}$$

The (analytic) immersion is $\Phi = (1, x^{-1})$ on the domain $\mathcal{D}' = \mathbb{R}^+ \cup \mathbb{R}^-$. The abode system is given by the three equations (4.15)-(4.17) with domain $\mathbb{R}^3$ (which means: with initial conditions $(Z_{1,1}(t_0), Z_{1,2}(t_0), x(t_0)) \in \mathbb{R}^3$). The image system is the abode system restricted to the manifold (4.11), i.e. $\mathcal{M}_\Phi = \Phi(\mathcal{D}') \subset \mathbb{R}^3$, or, which is the same: is given by eqs. (4.15)-(4.17) with $(Z_{1,1}(t_0), Z_{1,2}(t_0), x(t_0)) \in \mathcal{M}_\Phi$, that is:

$$Z_{1,1}(t_0) = 1; \quad Z_{1,2}(t_0) = \bar{x}^{-1}; \quad x(t_0) = \bar{x}, \qquad \bar{x} \in \mathcal{D}' = \mathbb{R}^+ \cup \mathbb{R}^-. \tag{4.18}$$

The immersed system is then the subsystem of the image system constituted by equation (4.17) on the domain $\mathcal{D}'$:

$$\dot{x} = (-aZ_{1,1} - bZ_{1,2})x, \qquad x(t_0) = \bar{x} \neq 0. \tag{4.19}$$

with $Z_{1,1}, Z_{1,2}$ solutions of the immersed driver:

$$\dot{Z}_{1,1} = 0, \qquad Z_{1,1}(t_0) = 1, \tag{4.20}$$

$$\dot{Z}_{1,2} = aZ_{11}Z_{1,2} + bZ_{1,2}^2, \qquad Z_{1,2}(t_0) = \bar{x}^{-1}, \qquad (\bar{x} \neq 0). \tag{4.21}$$

Let $x(t)$ the solution of (4.14) with $x(t_0) = \bar{x} \in \mathcal{D}$ and define

$$\mathbf{T}_{\bar{x},t_0} = \{t \in \mathbb{R} : x(t) \in \mathcal{D} \setminus \mathcal{D}'\} \qquad (= \{t \in \mathbb{R} : x(t) = 0\}).$$

$\mathbf{T}_{t_0,\bar{x}}$ is a countable set, by Lemma 4.1, and we can calculate it as follows. The solution of (4.14) is

$$x(t) = \bar{x}e^{-a(t-t_0)} - \frac{b}{a}(1 - e^{-a(t-t_0)}), \tag{4.22}$$

thus $x(t) \to -(b/a)$ for $t \to +\infty$. There are three cases that shall be considered:

(i) $\bar{x} \geq 0$. In this case $\mathbf{T}_{t_0,\bar{x}} = \{\bar{t}\}$, with

$$\bar{t} = t_0 + \frac{c}{a}; \qquad c = -\ln\left(\frac{b}{a\bar{x} + b}\right) > 0. \tag{4.23}$$

and $t_0 \leq \bar{t}$.

(ii) $-(b/a) < \bar{x} < 0$. In this case $\mathbf{T}_{t_0,\bar{x}} = \{\bar{t}\}$, with $\bar{t}$ as in (4.23), but $t_0 > \bar{t}$.

(iii) $\bar{x} \leq -(b/a)$. In this case $\mathbf{T}_{0,\bar{x}} = \emptyset$.

Consider first the case where $\mathbf{T}_{0,\bar{x}} \neq \emptyset$, That is $\bar{x} > -(-b/a)$. In this case the set (4.12) is $\mathcal{C}_{\bar{x},t_0} = \mathcal{C}_1 \cup \mathcal{C}_2$, where, denoted $\mathcal{T}_1 = (-\infty, \bar{t})$ and $\mathcal{T}_2 = (\bar{t}, +\infty)$:

$$\mathcal{C}_i = \{(x,t) : \quad t \in \mathcal{T}_i, \quad x = x(t)\}, \qquad i = 1, 2. \tag{4.24}$$

By naming $\mathcal{G}$ the graph of $x(t)$, $\mathcal{C}_1, \mathcal{C}_2$ are two pieces of $\mathcal{G}$ separated by the point $(0, \bar{t})$, i.e. $\mathcal{G} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{(0, \bar{t})\}$.

The quadratic immersion theorem states that if $(\bar{x}, t_0) \in \mathcal{C}_i$ $(i = 1, 2)$, then, denoted $x'(t)$ the solution of the immersed system (4.19), we have

$$\{(x'(t), t) : t \in \mathcal{T}_i\} = \mathcal{C}_i, \qquad i = 1, 2. \tag{4.25}$$

We can use formula (4.4) in order to express only a single piece of the solution $x(t)$: the piece which $(x(t_0), t_0)$ belongs to, with $x(t_0) \neq 0$. In the example, for $\bar{x} \neq 0$ we can write

$$x(t) = e^{\int_{t_0}^t Z_1^T v_1} \bar{x} = e^{-\int_{t_0}^t (aZ_{1,1} + bZ_{1,2})} \bar{x}, \tag{4.26}$$

where for $\bar{x} > 0$ we are in the case *(i)*, thus $t_0 < \bar{t}$, $(x(t_0), t_0) \in \mathcal{C}_1$, and (4.26), with $Z_{1,1}, Z_{1,2}$ given by (4.20), (4.21) with $\bar{x} > 0$, expresses the solution $x(t)$ for $t \in (-\infty, \bar{t})$. For $-(b/a) < \bar{x} < 0$, we are in the case *(ii)*, thus $t_0 > \bar{t}$, $(x(t_0), t_0) \in \mathcal{C}_2$, and (4.26) with $Z_{1,1}, Z_{1,2}$ given by (4.20), (4.21) with $-(b/a) < \bar{x} < 0$, expresses the solution $x(t)$ for $t \in (\bar{t}, +\infty)$.

We can perform a qualitative verification of (4.26) through the explicit calculation of the the two components $Z_{1,1}, Z_{1,2}$ of the immersed driver. By (4.20) we immediately realize that $Z_{1,1} \equiv 1$ for any t. The component $Z_{2,2}$ is given by the ODE (4.16), which is a *Bernoulli* quadratic differential equation, and thus the general solution can be calculated by a well known method: define $W_{1,2} = Z_{1,2}^{-1}$, then (as it is easy to verify) $W_{1,2}$ satisfies the *linear* equation:

$$\dot{W}_{1,2} = -aZ_{1,1}W_{1,2} - b. \tag{4.27}$$

As $Z_{1,1} = 1$, notice that (4.27) is the same equation as the original system (4.14). Moreover $W_{1,2}(t_0) = Z_{1,2}^{-1}(t_0) = \bar{x}$. The solution is then equal to (4.22), and by $Z_{1,2} = W_{1,2}^{-1}$, we get the general expression for $Z_{1,2}$:

$$Z_{1,2}(t) = \frac{a}{(a\bar{x} + b)e^{-a(t-t_0)} - b}. \tag{4.28}$$

The above function is (as expected) not defined for $t = \bar{t}$.

For $\bar{x} > 0$ we already saw that $t \in (-\infty, \bar{t})$ and $t_0 < \bar{t}$, and thus the piece of function (4.28) for $t < \bar{t}$ is to be used in (4.26). By calculating the limits, we have: for $t \to -\infty$, $Z_{1,2} \to 0^+$, and thus $\phi = aZ_{1,1} + bZ_{1,2} \to a^+$ and $-\int_{t_0}^{t} \phi \to +\infty \Rightarrow x(t) \to +\infty$; for $t \to \bar{t}^-$, $Z_{1,2} \to +\infty$, $\phi \to +\infty$, and thus $-\int_{t_0}^{t} \phi \to -\infty \Rightarrow x(t) \to 0^+$.

For $0 < \bar{x} < -(b/a)$ we have $t \in (\bar{t}, +\infty)$ and $t_0 > \bar{t}$, and thus the piece of (4.28) for $t > \bar{t}$ shall be used in (4.26): we have in this case $a\bar{x} + b > 0$ again, therefore for $t \to \bar{t}^+$, $Z_{1,2} \to -\infty$, $\phi \to -\infty$, and recalling that $t_0 > \bar{t}$ we have $-\int_{t_0}^{t} \phi \to -\infty$, which entails $x(t) \to 0^-$. Finally, for $t \to +\infty$, $Z_{1,2} \to -(a/b)^-$, $\phi \to 0^-$ and $-\int_{t_0}^{t} \phi \to k$ for some negative real number k. Therefore, by recalling that $\bar{x} < 0$, we have $x(t) \to e^k \bar{x} < 0$, which is enough for our qualitative verification, since is consistent with $-(b/a) = e^k \bar{x}$, that one would expect ($c = -(b/a\bar{x}) > 0$ and apparently $k = \ln c$).

The case $\mathbf{T}_{0,\bar{x}} = \emptyset$, amounts to *(iii)*: $\bar{x} \leq -b/a$. In this case the original system solution is equal to the immersed system solution for any $t \in \mathbb{R}$. The immersed driver, i.e. (4.21) with $\bar{x} \leq -b/a$ has a solution defined for all $t \in \mathbb{R}$, since the denominator in the expression of the general solution (4.28), never vanishes. We have $x(t) \to -\infty$, $Z_{1,2} \to 0-$ for $t \to -\infty$, and $x(t) \to (-b/a)^-$, $Z_{1,2} \to (-a/b)^+$ for $t \to +\infty$, accordingly with $Z_{1,2} = 1/x$. In the case $\bar{x} = 0$, the original solution $x(t)$ still has two separate branches (separated by $(0, t_0)$), but any of them can be expressed by (4.26). However, notice that this is just a formal issue, which is readily overtaken by re-defining t_0 in order that $x(t_0) \neq 0$ (we have excluded the trivial case $\dot{x} = 0$ thus $x(t_0 + \epsilon)$ shall be non zero for some $\epsilon > 0$).

4.3. Drivers in C-form. Let us introduce the symbol $\pi_{i,j}^{i'}$ defined as follows:

$$\pi_{i,j}^{i'} = \begin{cases} p_{i,j}^{i'} & \text{for} \quad i \neq j; \\ p_{i,i}^{i'} - 1 & \text{otherwise.} \end{cases} \tag{4.29}$$

Then (4.2) can be rewritten as

$$\dot{Z}_{i,i'} = \sum_{j=1}^{n} \pi^{i'}_{i,j}(Z_j^T v_j) Z_{i,i'} = \sum_{j,j'}^{n,\nu_j} \pi^{i'}_{i,j} v_{j,j'} Z_{j,j'} Z_{i,i'}, \tag{4.30}$$

where j' is another size-index related to j [19]. Given an i-indexed $\sigma\pi$-system of order n, and total size d, we can directly write the associated driver, which is an homogeneous quadratic system, and hence in particular another $\sigma\pi$-system having *always* order equal to d. Note that the expression (4.30) describes a double-indexed $\sigma\pi$-system of the type (3.15), where

$$v^{j,j'}_{i,i'} = \pi^{i'}_{i,j} v_{j,j'}; \qquad X^{j,j'}_{i,i'} = Z_{j,j'} Z_{i,i'}. \tag{4.31}$$

In the following we always assume the square form expression in (4.30) as the S-form of an (i,j)-indexed driver.

The C-form of a driver is then derived from (3.19), (3.20), using the setting (4.31). The result is:

$$\dot{Z}^{\mathbf{p}}_{i,i^*} = \sum_{j,j^*}^{n,\nu^{\mathbf{p}}_j} \pi^{\mathbf{p},i^*}_{i,j} v^{\mathbf{p}}_{j,j^*} Z^{\mathbf{p}}_{i,i^*} Z^{\mathbf{p}}_{j,j^*} + \sum_{j,j_*}^{n,\nu^{\mathbf{c}}_j} \pi^{\mathbf{p},i^*}_{i,j} v^{\mathbf{c}}_{j,j_*} Z^{\mathbf{p}}_{i,i^*} Z^{\mathbf{c}}_{j,j_*} \tag{4.32}$$

$$\dot{Z}^{\mathbf{c}}_{i,i_*} = \sum_{j,j^*}^{n,\nu^{\mathbf{p}}_j} \pi^{\mathbf{c},i_*}_{i,j} v^{\mathbf{p}}_{j,j^*} Z^{\mathbf{c}}_{i,i_*} Z^{\mathbf{p}}_{j,j^*} + \sum_{j,j_*}^{n,\nu^{\mathbf{c}}_j} \pi^{\mathbf{c},i_*}_{i,j} v^{\mathbf{c}}_{j,j_*} Z^{\mathbf{c}}_{i,i_*} Z^{\mathbf{c}}_{j,j_*} \quad , \tag{4.33}$$

where the $v^{(\mathbf{q})}$'s $\mathbf{q} \in \{\mathbf{p}, \mathbf{c}\}$ has been defined in (3.6), (3.7), and, with l a generic equation-index, we have

$$Z^{(\mathbf{p})}_{l,l^*} = Z_{l,l'(l^*)}; \qquad Z^{(\mathbf{c})}_{l,l_*} = Z_{l,l'(l_*)}. \tag{4.34}$$

$$\pi^{(\mathbf{p},i^*)}_{i,j} = \pi^{i'(i^*)}_{i,j}; \qquad \pi^{(\mathbf{c},i_*)}_{i,j} = \pi^{i'(i^*)}_{i,j}. \tag{4.35}$$

We will refer to (4.32), (4.33), as an (i,j)-indexed driver in C-form, or as the C-form of the driver of an i-indexed $\sigma\pi$-system as (3.5) (or: as (3.8), if we want to relate it to the C-form of the underlying system).

4.4. Inverse driver. The driver equations (4.2) are quadratic differential equation of the Bernoulli type (Riccati type with no zero degree term). The basic properties of this kind of equations are well known and can be find in any textbook of ordinary differential equations. Let us define

$$W_{i,i'} = Z^{-1}_{i,i'} \quad , \tag{4.36}$$

where $Z_{i,i'}$ is the (i,i')-th component of the state of the driver (4.2), then we have that $W_{i,i'}$ satisfies the equation:

$$\begin{aligned} \dot{W}_{i,i'} &= \left(Z_i^T v_i - \sum_{j=1}^{n} p^{i'}_{i,j} Z_j^T v_j \right) W_{i,i'} = -\left(\sum_{j=1}^{n} \pi^{i'}_{i,j} Z_j^T v_j \right) W_{i,i'} \\ &= -\sum_{j,j'}^{n,\nu_j} \pi^{i'}_{i,j} v_{j,j'} W^{-1}_{j,j'} W_{i,i'}. \end{aligned} \tag{4.37}$$

[19] As a matter of fact the index j used in (4.2) spans the system equations, and thus it is a second equation-index. The index j' spans the monomials on the j-th equation, and thus it is a size-index.

Thus, the aggregate W is the state of another system, no more quadratic (but still $\sigma\pi$-algebraic), that we call *the inverse driver* (associated to the system (3.5)). We see that the inverse driver, as well as its *direct* counterpart, entails the calculation of the original state, provided it feeds an *inverse final* system:

$$\dot{x}_i = \Big(\sum_{i'=1}^{\nu_i} W_{i,i'}^{-1} v_{i,i'}\Big) x_i, \qquad x_i = e^{\sum_{i'=1}^{\nu_i}\int_o^t W_{i,i'}^{-1} v_{i,i'}} x_i(0). \tag{4.38}$$

4.5. Self-drivers and canonic forms of bilinear and linear systems. There are three particular cases of $\sigma\pi$-algebraic system that deserves a separate attention. We define them as follows.

DEFINITION 4.2. *An i-labeled $\sigma\pi$-system, is said to be*

(i) a canonic linear (CL) system, if it is in constant size $\nu_i = n+1$ and

$$X_{i,i'} = \begin{cases} x_{i'} & \text{for} \quad i' = 1,\dots,n; \\ 1 & \text{for} \quad i' = n+1. \end{cases} \tag{4.39}$$

where the $v_{i,n+1}$ only can be control-dependent.

(ii) a canonic bilinear (CB) system, if it is in constant size $\nu_i = 2n$ and

$$X_{i,i'} = \begin{cases} x_{i'} & \text{for} \quad i' = 1,\dots,n; \\ x_{i'-n} & \text{for} \quad i' = n+1,\dots,2n. \end{cases} \tag{4.40}$$

where the $v_{i,i'}$, for $i' > n$, only can be control-dependent.

(iii) a self-driver, if the S-form is in square size and $X_{i,i'} = x_i x_{i'}$, $\forall i, i' = 1,\dots n$.

Note that any CB or CL system, always have a canonic version, provided it is suitably i-labeled. A few features of *i)-ii)-iii)* will be used later in the paper.

4.5.1. CL systems. By Definition 4.2 *(i)*

$$\dot{x}_i = \sum_{i'=1}^{n+1} v_{i,i'} X_{i,i'} = \sum_{i'=1}^{n} v_{i,i'} x_{i'} + v_{i,n+1},$$

and thus $\nu_i^{\mathbf{p}} = n$, $\nu_i^{\mathbf{c}} = 1$, $\mathcal{I}^{\mathbf{p}} = \{1,\dots,n\}$, $\mathcal{I}^{\mathbf{c}} = \{1\}$, $i_*(i')$ is the identity map, $i_*(i')$ is unique. Therefore, the C-form is constant in both parametric and control-size: $\nu_i^p = n$, $\nu_i^{\mathbf{c}} = 1$, and we have

$$\dot{x}_i = \sum_{i^*=1}^{n} v_{i,i^*}^{\mathbf{p}} x_{i^*} + v_{i,1}^{\mathbf{c}} = \sum_{i^*=1}^{n} v_{i,i^*}^{\mathbf{p}} x_{i^*} + \sum_{s=1}^{q} b_{i,s}^{\mathbf{c},1} u_s, \tag{4.41}$$

where in the left side we have made explicit the control, by (3.11). For the dynamic and control matrix, by (3.24), (3.25) we have

$$A_{i,j} = \sum_{i^*=1}^{n} v_{i,i^*}^{\mathbf{p}} \frac{\partial x_{i^*}}{\partial x_j} + v_{i,1}^{\mathbf{c}} \frac{\partial 1}{\partial x_1} = v_{i,j}^{\mathbf{p}}, \tag{4.42}$$

$$B_{i,s} = b_{i,s}^{\mathbf{c},1}, \qquad s = 1,\dots,q. \tag{4.43}$$

and thus (A, B) agree with the pair dynamic/control matrix in the standard sense for linear systems. We can use the couple (A, B) to rewrite system (4.41) in vector form. The result is the familiar *Kalman form* of a linear control system:

$$\dot{x} = Ax + Bu. \tag{4.44}$$

The reader can readily verify that, for a *non-canonic* linear system, e.g. such that $X^{\mathbf{p}}_{i,i^*} = x_l$, where in general $l \neq i^*$, formulas (3.24), (3.25) give the correct couple A, B of the corresponding vector form as well. Also, any non-canonical linear system can be turned into a CL system by simply writing the monomials in a way canonically ordered.

The driver of a CL system can be readily derived from the general formula (4.30), taking into account that Definition 4.2 *(i)* is equivalent to assume that the exponents of the $\sigma\pi$-system satisfy: $p^{i'}_{i,j} = 0$, for $j \neq i'$, $p^{j}_{i,j} = 1$, and thus (after some arguing):

$$\pi^{(i')}_{i,j} = \begin{cases} 1 & \text{for} \quad i \neq j = i', \\ -1 & \text{for} \quad i = j \neq i' \\ 0 & \text{otherwise.} \end{cases} \tag{4.45}$$

The result is

$$\dot{Z}_{i,m} = \sum_{j'=1}^{n+1} v_{m,j'} Z_{m,j'} Z_{i,m} - \sum_{j'=1}^{n+1} v_{i,j'} Z_{i,j'} Z_{i,m}, \qquad i \neq m \tag{4.46}$$

$$\dot{Z}_{i,i} = 0, \tag{4.47}$$

$$\dot{Z}_{i,n+1} = -\sum_{j'=1}^{n+1} v_{i,j'} Z_{i,j'} Z_{i,n+1}. \tag{4.48}$$

with the initial conditions

$$Z_{i,i'}(t_0) = \frac{X_{i,i'}(t_0)}{x_i(0)} = \frac{x_{i'}(t_0)}{x_i(t_0)}, \quad \text{for} \quad i' = 1, \dots, n, \tag{4.49}$$

$$Z_{i,n+1}(t_0) = \frac{1}{x_i(t_0)}. \tag{4.50}$$

Note that the initial conditions for the n equations $\dot{Z}_{i,i} = 0$, $i = 1, \dots, n$, are $Z_{i,i}(t_0) = 1$, that is $Z_{i,i} \equiv 1$, as expected by the general immersion formula (4.1). Thus, the driver of a CL system, which has order equal to the total size: $n(n+1) = n^2 + n$, has n redundant equations, those labeled (i, i). By removing the redundant equations, we obtain a *reduced* driver of order n^2 given by $(i = 1, \dots, n)$:

$$\dot{Z}_{i,i'} = \sum_{j'=1}^{n+1} v_{i',j'} Z_{i',j'} Z_{i,i'}, \quad i \neq i' = 1, \dots, n+1. \tag{4.51}$$

4.5.2. CB systems. By Definition 4.2 *(ii)*

$$\dot{x}_i = \sum_{i'=1}^{2n} v_{i,i'} X_{i,i'} = \sum_{i'=1}^{n} v_{i,i'} x_{i'} + \sum_{i'=n+1}^{2n} v_{i,i'} x_{i'-n},$$

and thus $\nu_i^{\mathbf{p}} = \nu_i^{\mathbf{c}} = n$, $\mathcal{I}^{\mathbf{p}} = \{1, \ldots, n\}$, $\mathcal{I}^{\mathbf{c}} = \{2, \ldots, 2n\}$, $i_*(i')$ is the identity map, $i_*(i')$ is the map $i' \mapsto i' - n$, and the C-form is in both parametric and control square size:

$$\dot{x}_i = \sum_{i^*=1}^{n} v_{i,i^*}^{\mathbf{p}} x_{i^*} + \sum_{i_*=1}^{n} v_{i,i_*}^{\mathbf{c}} x_{i_*}. \tag{4.52}$$

System (4.52) can be rewritten

$$\dot{x}_i = \sum_{j=1}^{n} w_{i,j} x_j, \quad \text{with } w_{i,j} = v_{i,j}^{\mathbf{p}} + v_{i,j}^{\mathbf{c}}, \tag{4.53}$$

which is an *assembled* S-form, and thus, the generator V is defined as $V_{i,j} = w_{i,j}$, and we have the vector form

$$\dot{x} = Vx. \tag{4.54}$$

For the dynamic and control matrix, by (3.24), (3.25) we have

$$A_{i,j} = \sum_{i^*=1}^{n} v_{i,i^*}^{(\mathbf{p})} \frac{\partial x_{i^*}}{\partial x_j} + \sum_{i_*=1}^{n} v_{i,i_*}^{\mathbf{c}} \frac{\partial x_{i_*}}{\partial x_j} = v_{i,j}^{\mathbf{p}} + v_{i,j}^{\mathbf{c}} = w_{i,j}, \tag{4.55}$$

$$B_{i,s} = \sum_{i_*=1}^{n} b_{i,s}^{\mathbf{c},i_*} x_{i_*}, \qquad s = 1, \ldots, q, \tag{4.56}$$

and thus, for a CB system one has the important property $V = A$, as stated in the following Proposition.

PROPOSITION 4.3. *For a CB system the generator is equal to the dynamic matrix.*

Since the dynamic matrix of a $\sigma\pi$-algebraic system does not depend of the version, we can also rephrase Proposition 4.3 as follows: *the dynamic matrix of a bilinear system is equal to the generator of the system canonic version.*

Also, note that, differently than in the linear case, if we write a non-canonical bilinear system in a canonically ordered form, we do not obtain the S-form, but the assembled S-form (4.53)

4.5.3. Self-drivers. By definition, a self-driver is a $\sigma\pi$-system whose S-form is an homogeneous quadratic system of the type

$$\dot{x}_i = \sum_{i'=1}^{n} v_{i,i'} x_i x_{i'}. \tag{4.57}$$

Since (4.57) is quadratic, is $\sigma\pi$-algebraic as well, and thus undergoes a quadratization, and has in turn an associated driver. The name 'self-driver' comes from the fact that system (4.57) has a driver which generates the components of system (4.57) itself, and only these. Indeed, from (4.1), it is $Z_{i,l} = x_l$, and thus, although the driver has n^2 entries, only n of these are distinct, and these are just the n original state components. From (4.2) the equations of the driver of (4.57) are:

$$\dot{Z}_{l,i} = (Z_i^T v_i + Z_l^T v_l - Z_l^T v_l) Z_{l,i} = \sum_{i'=1}^{n} v_{i,i'} Z_{i,i'} Z_{l,i}, \quad Z_{l,i}(t_0) = x_i(t_0) \tag{4.58}$$

thus, as expected, since $Z_{l,i} = x_i$ and $Z_{i,i'} = x_{i'}$, (4.58) is equal to the equation (4.57), and all the driver equations, for $(l,i) = (1,1),\dots,(n,n)$, are simply n copies of the system of equations (4.57).

From (4.57) we see that the assembled S-form of a self-driver is the S-form itself, and is square sized and canonically ordered. Thus, the generator of a self-driver is directly $V \in \mathbb{R}^{n\times n}$, with $V_{i,j} = v_{i,j}$, by definition as usual.

4.6. Bilinear frame of a self-driver. Here we give what is a basic notion of this paper: the *bilinear frame* of a self-driver type system.

DEFINITION 4.4. *We define the bilinear frame of a self-driver type system with generator V, as the canonical bilinear system whose generator is V.*

We generally will use the symbol z for the state *vector* of the bilinear frame of some self-driver type system having generator $V \in \mathbb{R}^{n\times n}$:

$$\dot z = Vz. \qquad \Leftrightarrow \qquad \dot z_i = \sum_{l=1}^{n} v_{i,l} z_l, \tag{4.59}$$

where in the right side we have indicated the corresponding scalar equation.

5. Self-drivers and biased-solutions. Let V be the generator of the self-driver (4.57), and let $\phi^V(z)$ the flow of the linear vector field Vz passing through z. We also denote by $\phi_i^V(z)$ the i-th component of the flow. Let us choose $i, i' \in \{1,\dots,n\}$ and define the time function:

$$\Psi_{i,i'}(z) = -\frac{\phi_{i'}^V(z)}{(\phi_i^V(z))^2}\dot\phi_i^V(z), \tag{5.1}$$

which we call *the bias* at z. We give the following Definition.

DEFINITION 5.1. *Let ζ_i the general solution of the i-th subsystem of the following n^2 scalar differential equations:*

$$\dot\zeta_{i,i'} = \sum_{j=1}^{n} v_{i',j}\zeta_{i',j}\zeta_{i,i'} + \Psi_{i,i'}(z). \tag{5.2}$$

where $\Psi_{i,i'}(z)$ is the bias at z, defined in (5.1), and the $v_{i',j}$'s are the coefficients of a self-driver, as (4.57). Then ζ_i is said the be the i-th biased-solution at z of the self-driver (4.57). We also call the the i-th subsystem of the system of equations (5.2) the i-th biased-driver of (4.57).

One nice property, for a system of self-driver type, is that all of its n biased-solutions can be explicitly calculated as stated in the following theorem.

THEOREM 5.2. *The i-th biased-solution at z of the self-driver (4.57) is given by:*

$$\zeta_{i,i'} = \frac{\phi_{i'}^V(z)}{\phi_i^V(z)}. \tag{5.3}$$

Proof. Let us consider the identity (5.1), defining the bias at z, and simplify the notation of the flows $\phi^V_{i'}(z)$ as $\phi_{i'}$. By adding the quantity $\dot\phi_{i'}/\phi_i$ on both sides we have:

$$\frac{\dot\phi_{i'}}{\phi_i} = \frac{\phi_{i'}}{\phi_i^2}\dot\phi_i + \frac{\dot\phi_{i'}}{\phi_i} + \Psi_{i,i'}. \tag{5.4}$$

Now $\dot\phi = V\phi$, and thus

$$\dot\phi_{i'} = \sum_{j=1}^{n} v_{i',j}\phi_j. \tag{5.5}$$

Let us multiply and divide by ϕ_i the left hand side of (5.4), and use (5.5) in the right hand side, we have

$$\frac{\dot\phi_{i'}}{\phi_i^2}\phi_i - \frac{\phi_{i'}}{\phi_i^2}\dot\phi_i = \frac{1}{\phi_i}\sum_{j=1}^{n} a_{i',j}\phi_j + \Psi_{i,i'} = \sum_{j=1}^{n} a_{i',j}\frac{\phi_j}{\phi_i} + \Psi_{i,i'}, \tag{5.6}$$

and thus

$$\sum_{j=1}^{n} a_{i',j}\frac{\phi_j}{\phi_{i'}}\frac{\phi_{i'}}{\phi_i} + \Psi_{i,i'} = \frac{\dot\phi_{i'}}{\phi_i^2}\phi_i - \frac{\phi_{i'}}{\phi_i^2}\dot\phi_i = \frac{\dot\phi_{i'}\phi_i - \phi_{i'}\dot\phi_i}{\phi_i^2} = \frac{d}{dt}\frac{\phi_{i'}}{\phi_i}, \tag{5.7}$$

which proves the Theorem. □

If $z(t)$ is the solution of the bilinear frame (4.59) passing at $t = 0$ through z, it is $z(t) = \phi^V(z)(t)$, and thus (5.3) can be written as

$$\zeta_{i,i'}(t) = \frac{z_{i'}(t)}{z_i(t)}. \tag{5.8}$$

It is useful to express the above formula in words: the i'-th component, with $i' \neq i$, of the i-th biased solution (of a given self-driver) at z is equal to the ratio – the i'-th over the i-th – between components of the solution, starting from z, of the driver bilinear frame. Similarly, the i'-th component of the bias at z of the i-th biased driver (at z, and with $i' \neq i$) is given – in terms of the solution $z(t)$ of the bilinear frame starting from z – by the formula:

$$\Psi_{i,i'}(z)(t) = -\frac{z_{i'}(t)}{z_i^2(t)}\dot z_i(t). \tag{5.9}$$

The biased-solutions of a self-driver type system have the following important property: they all converge – except in one component – to the true solution of the system, provided a basic condition is verified, roughly speaking: that the bias at a suitably defined *pivoted point* goes to zero for $t \to +\infty$

THEOREM 5.3. *Let $x(t)$ be the solution at time t of the self-driver type system (4.57) passing through x at $t = 0$. Let us choose an i, said pivot-index, and let $z(t)$ the solution of the bilinear frame (4.59), associated to (4.57), passing at $t = 0$ through the point $x^{(i)} \in \mathbb{R}^n$, said the pivoted (on i) initial state, defined as*

$$x^i_{i'} = \begin{cases} \alpha x_{i'} & \text{for } i' \neq i; \\ \alpha & \text{otherwise;} \end{cases} \tag{5.10}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$. *Suppose that*

$$\lim_{t\to+\infty} \Psi_{i,i'}(x^i) = 0, \qquad \forall i, i' \in 1,\dots,n, \tag{5.11}$$

then, if ζ_i *is the i-th biased-solution at* x^i *of (4.57) we have*

$$\lim_{t\to+\infty} (\zeta_{i,i'}(t) - x_{i'}(t)) = 0, \qquad \forall i' \neq i. \tag{5.12}$$

Proof. By (5.8) it is, for $i' \neq i$

$$\zeta_{i,i'}(t_0) = \frac{z_{i'}(t_0)}{z_i(t_0)} = \frac{x^i_{i'}}{x^i_i} = x_{i'}. \tag{5.13}$$

From (5.2), the biased solution $\zeta_{i,i'}(t)$ at x^i satisfies

$$\dot{\zeta}_{i,i'} = \sum_{j=1}^{n} v_{i',j}\zeta_{i',j}\zeta_{i,i'} + \Psi_{i,i'}(x^i), \tag{5.14}$$

with initial condition given by (5.13). Thus, on account of hypothesis (5.11), it follows that $\zeta_{i,i'} \to Z_{i,i'}$, $\forall i' \neq i$, where $Z_{i,i'}$ is the solution of (4.58), with initial condition $Z_{i,i'}(t_0) = x_{i'}$. As the system is a self-driver the thesis follows. □

5.1. Remark: pivot components. If a self-driver has an associated bilinear frame giving rise to (5.11) then all of its entries can be calculated by setting a component as a *pivot* component, say the i-th, and then for any $i' \neq i$, by applying Theorem 5.3 which gives $x_{i'}$ as a ratio of components of the bilinear-frame solution starting from a suitable initial point x^i *pivoted on* i. The remaining i-th component can be calculated by applying again the same Theorem with another pivot component.

5.2. When the bias goes to zero?. There are some important cases in which the condition (5.11) is verified. For instance, if the self-driver (4.57) is *stationary*, i.e. the associated matrix V is a constant matrix, and V has at least one zero eigenvalue. In this case, by the expression of $\Psi_{i,l}$ given in (5.1), it's easy to see that, for any x, the time derivative of the flow: $\dot{\phi}^V(x)$ either goes to zero (for a single pole in the origin) and $\phi^V(x) \to const$, or it goes to a limit (finite or infinite, which happens for multiple poles in the origin) slower than the flow, and thus condition (5.11) is verified even for any starting point. In the same case, note also that the convergence of the biased-solution to the system solution occurs at an exponential rate.
In general, the linear vector field Vx is a time-varying linear vector field, and the matrix V depends of the control u, i.e. $V \equiv V(t,u)$. Thus, we can calculate the (unbiased) steady-state solution of a self-driver for all controls u that bring about all pivoted flows to converge to a non zero constant.

5.3. Inverse driver of a self-driver. Let us derive the inverse driver (cf. §4.4) of the self-driver (4.57). By (4.37) we readily get

$$\dot{W}_{i,i'} = -\sum_{j=1}^{n} v_{i',j} Z_{i',j} W_{i,i'}, \qquad W_{i,l}(0) = \frac{1}{\bar{x}_l}. \tag{5.15}$$

The inverse driver of a self driver gives the inverses, and only these, of the components of the self-driver itself. All it has been shown before for the driver has a inverse counterpart, in terms of inverse driver as show below.

We define the *starred bias at* z as follows

$$\Psi^*_{i,i'}(z) = \frac{\dot{\phi}^V_i(z)}{\phi^V_{i'}(z)}, \tag{5.16}$$

and give the following Definition.

DEFINITION 5.4. *Let* ω_i *the general solution of the* i*-th subsystem of the following* n^2 *scalar differential equations:*

$$\dot{\omega}_{i,i'} = -\sum_{j=1}^{n} v_{i',j}\zeta_{i',j}\omega_{i,i'} + \Psi^*_{i,i'}(z). \tag{5.17}$$

where $\Psi^*_{i,l}(x)$ *is the time function (5.16), the* $v_{i',j}$*'s are the coefficients of the self-driver (4.57). and* ζ_i *is the* i*-th biased-solution at* z *of the self-driver (4.57). We call* ω_i *the* i*-th inverse biased-solution of (4.57). Moreover, we call the* i*-th subsystem of the system of equations (5.17) the* i*-th inverse biased-driver of (4.57).*

THEOREM 5.5. *The* i*-th inverse biased-solution at* x *of the self-driver (4.57) is given by:*

$$\omega_{i,l} = \frac{\phi^V_i(z)}{\phi^V_{i'}(z)}. \tag{5.18}$$

Proof. Let us simplify the notation of the flows, as in the proof of Theorem 5.2, and let us add and subtract the quantity $\dot{\phi}_l\phi_i/\phi^2_{i'}$ on the left hand side of (5.16), we have:

$$\frac{\dot{\phi}_i\phi_{i'}}{\phi^2_{i'}} = \frac{\dot{\phi}_i}{\phi_{i'}} = \frac{\phi_i}{\phi^2_{i'}}\dot{\phi}_{i'} - \frac{\phi_i}{\phi^2_{i'}}\dot{\phi}_{i'} + \Psi^*_{i,i'} = \frac{\phi_i}{\phi^2_{i'}}\dot{\phi}_{i'} - \frac{\phi_i}{\phi^2_{i'}}\sum_{j=1}^{n} a_{i',j}\phi_j + \Psi^*_{i,i'} \tag{5.19}$$

from which we have

$$\frac{d}{dt}\frac{\phi_i}{\phi_{i'}} = \frac{\dot{\phi}_i\phi_{i'} - \phi_i\dot{\phi}_{i'}}{\phi^2_{i'}} = -\sum_{j=1}^{n} a_{i',j}\frac{\phi_j}{\phi_{i'}}\frac{\phi_i}{\phi_{i'}} + \Psi^*_{i,i'}, \tag{5.20}$$

which proves the Theorem. □

The inverse biased-solutions of a self-driver type system have the following property: each converges – in all of the components except one – to the inverse of the true solution of the system, provided the starred bias converges to zero for $t \to +\infty$.

THEOREM 5.6. *Let* $x(t)$ *be the solution of the self-driver type system (4.57) passing through* x *at* $t = 0$*. For any pivot index* i *consider the flow of the vector field* Vx *passing at* $t = 0$ *through the pivoted point* $x^{(i)} \in \mathbb{R}^n$ *defined as in Theorem 5.3, and suppose that*

$$\lim_{t\to+\infty} \Psi^*_{i,i'}(x^{(i)}) = 0, \qquad \forall i, i' \in 1, \ldots, n, \tag{5.21}$$

Then, if ω_i is the i-th inverse biased-solution at $x^{(i)}$ of (4.57) we have

$$\lim_{t\to+\infty}\left(\omega_{i,i'}(t)-\frac{1}{x_{i'}(t)}\right)=0,\qquad \forall l\neq i. \tag{5.22}$$

Proof. By (5.3) it is, for $i'\neq i$

$$\zeta_{i,i'}(t_0)=\frac{\phi_i^V(x^{(i)})(t_0)}{\phi_{i'}^V(x^{(i)})(t_0)}=\frac{x_i^{(i)}}{x_{i'}^{(i)}}=\frac{1}{x_{i'}}. \tag{5.23}$$

Then, a similar argument as in Theorem 5.3 concludes the proof. □

6. Conclusion and final remarks. Here's a brief summary of the most important points of the Part I of the paper. First of all we have given some further theoretical insight for the concept of QI first issued in [1], that is important in order to well understand the sequel of the paper, and can be summarized as follows. Identity (4.1) is the definition of 'driver state'. The right hand side of (4.1) gives the *formula* of the dense immersion $\Phi(x)$, defining a QI. The QODE (4.2) is the driver equation, which, with (4.3), constitutes the *abode system*, into which a given $\sigma\pi$-system is to be immersed. The QI consists in the property of the trajectories of (4.2) passing through a point of the manifold $\Phi(\mathcal{D})$, with $\mathcal{D}$ the domain of the original system, of remaining confined in the manifold $\Phi(\mathcal{D})$ itself. Thus the basic feature of the QI can be rephrased with the following two simple statements: 1) for any $\sigma\pi$-system having domain $\mathcal{D}$ there exists a map Φ, analytic almost everywhere on $\mathcal{D}$, and a quadratic system, the driver, such that the flow of the driver is $\Phi(\mathcal{D})$ invariant, 2) any trajectory of the original system is given as a solution of the bilinear differential equation (4.3) driven by some driver trajectory lying in $\Phi(\mathcal{D})$. As argued in §4.2, there might be many *pieces* of disconnected drivers trajectories for *one* original state trajectory, but, with the possible exception of isolated points, any $x(t)$ can be expressed through formula (4.4) – which is is the *integral form* of the bilinear equation (4.3) – by means of some of the above pieces of driver trajectory.

As for the specific contribution of Part I, the main result is Theorem 5.2, where, for a particular kind of $\sigma\pi$-system, namely the *self-drivers*, defined in §4.5.3, under the hypothesis that the *bias* – given by (5.1) – converges to zero for $t\to+\infty$, a *steady state solution* exists and can be calculated as the limit of a *biased solution* (Definition 5.1) of the self-driver. In particular, a new system has been defined, said *bilinear frame* (Definition 4.4), that can be always associated to any self-driver type system, and is the *linear*, and in general time-varying, autonomous system whose dynamic matrix is the *generator* (see §4.4) of the self-driver itself. Such a system can be viewed as a time-varying linear system if we consider the control u as a kind of system parameter, whereas if we in fact distinguish controls from other kind of parameters, it is indeed a *bilinear* system. The i-th biased solution at z of a self-driver type system is then given by formula (5.3), which shows that it is the ratio of two 'free evolution' modes of the bilinear frame, starting from suitable *pivoted points* defined in (5.10).

REFERENCES

[1] F. Carravetta *Global Exact Quadratization for Continuous-Time Nonlinear Control Systems* SIAM J. Control Optim., 53 (2015), pp. 235 - 261.

[2] P. Jouan *Immersion of Nonlinear Systems into Linear Systems Modulo Output Injection*. SIAM J. Control Optim., 41 (2003), pp. 1756 - 1778.

[3] M. Fliess, I. Kupka *A finiteness Criterion for Nonlinear Input-Output Differential Systems*. SIAM J. Control Optim., 21 (1983), pp. 721 - 728.

[4] A. J. Krener, *On the Equivalence of Control Systems and the Linearization of Nonlinear Systems*, SIAM J. Control, 11 (1973), pp. 670 - 676.

[5] A. Isidori, *Nonlinear Control Systems*, Communication and Control Engineering Series, First Ed., Springer Verlag, London, UK 1985.

[6] B. Jakubczyk and W. Respondek, *On linearization of control systems*. Bull. Acad. Polon. Sci. Śer. Sci. Math, 28 (1980), 517522, pp. 1980.