On the Nonlinear Stabilization Problem via Quadratic Immersion

Part II: A QI-based approach for the design of nonlinear regulators having exponential-tunable performance

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Abstract

The results of Part I of the paper are here exploited in order to define a regulator design method for Sigma/Pi-systems, even non stationary, which makes use of a static state-feedback working in the same domain as the open-loop system. Sufficient conditions are given in order that a fixed performance, in terms of convergence rate, be achievable. Provided these conditions are satisfied the regulator steers the system state to zero, at an exponential, tunable, rate and the domain of 'attraction' of the zero will include all points from which the orbits of the closed-loop system are bounded. For stationary systems additional results are proven as far as the accessibility property.

1. Introduction. [1] $\sigma\pi$-systems, the basic class of nonlinear system we have introduced in [1] are still focused in the Part II of the present paper, where we come to the point: how to exploit the quadratic immersion (QI) in order to build up regulators for nonlinear systems in the class $\sigma\pi$. A few conditions are defined that, whether satisfied, allows to build up a state-feedback regulator able to steer the state of the system to zero at an exponential speed that can be fixed in advance by the designer by suitably tuning a set of *gain*-like parameters. As for the global/local nature of the regulator, it will be shown that, for a regulator satisfying all requirements, the behavior of the closed-loop system is such that, from any given initial point: either the solution blows up *in finite time*, or it goes to zero for $t \to +\infty$. Thus the domain of attraction of the zero can be simply calculated as the set of all points of the original system domain from which the closed-loop system admits a bounded response. The basic condition, of the above sketched QI-based regulation method, is the so-called $\sigma\pi$-*controllability*, which can be easily tested on the original $\sigma\pi$-system. Whether such a condition is satisfied, the next steps of the design procedure can be carried out and some further condition is to be tested as well. This give place to a QI-based design method composed by a series of a few steps. If the design method is brought off successfully, then the exponential performance of the controlled system is assured. The regulator is build up as a static feedback of the state (we assume that all the components of the original $\sigma\pi$-system are directly measurable) and the feedback function is a $\sigma\pi$-function, thus leading to a $\sigma\pi$ closed-loop system. Moreover, it will be shown that it is always possible to determine the monomials of the feedback in such a way that the closed loop system has the same domain as the original system. A simple example is presented at the end of the paper, consisting in a $\sigma\pi$ controllable system, which is *accessible* at every point of its domain, but is not *controllable* to zero from all points (we refer readers to [2]–[4] as for the mathematical definitions of nonlinear accessibility and controllability). Nonetheless, for this system the set of initial points that can *in principle* (i.e. using some input, no bounds being imposed on the input function) be steered to zero is known. It will be shown that the QI-based method at issue, can be brought off successfully for this case, and, as a matter of fact, the global regulator steers to zero, exponentially, all the states that can in principle be steered to zero, and by reason of that can be reasonably be called a *global* regulator.

Part II is organized into six chapters. §2 includes a top-up issue, that can be even skipped at a first reading as it is not directly involved with the regulation method that constitutes the main topic of the paper. This is the topic of the *accessibility* of a $\sigma\pi$ system. We show that certain matrices that we have defined in the Part I of this paper, that is the *dynamic* and *control* matrix, and that can be associated to any $\sigma\pi$-system[2], can be used, in the stationary case, for defining an accessibility test very easy to carry on, which allows, for $\sigma\pi$-systems, to calculate the domain of accessibility by simple rank-tests performed just in a few points of the domain itself. In §3, §4, and §5, some further notational tools are introduced in order to better manipulate the closed-loop $\sigma\pi$-systems that are used in the sequel. The main sections are the 4-th and the 5-th. In §5 is given the main result, above sketched, which is Theorem

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[2]In a way consistent with the similar well known matrices characterizing linear systems, as soon as the latter are regarded as $\sigma\pi$-systems.

5.2. §6 includes the example, and in §7 we take the conclusions, and summarize the main result of the paper by pointing out the most important technical details of it.

2. Accessibility of stationary $\sigma\pi$-systems. Let us consider a stationary $\sigma\pi$-system in Cu-form (cf. Part I, eq. (3.12)) :

$$(2.1) \qquad \dot{x}_i = \sum_{i^*=1}^{\nu_i^{\mathbf{P}}} v_{i,i^*}^{\mathbf{P}} X_{i,i^*}^{\mathbf{P}} + \beta_i^T u, \qquad \left(\beta_{i,i_*} = \sum_{i_*=1}^{\nu_i^c} b_{i,s}^{\mathbf{c},i_*} X_{i,m}^{\mathbf{c}} \right)$$

and let $\mathcal{D}$ its C^∞-domain (and thus, also, its domain of analiticity). Moreover, let us define $\alpha_{i,j}$:

$$(2.2) \qquad \alpha_{i,j} \ (= \alpha_{i,j}(x,u)) \ = \sum_{i^*=1}^{\nu_i^{\mathbf{P}}} v_{i,i^*}^{\mathbf{P}} \frac{\partial X_{i,l}^{\mathbf{P}}}{\partial x_j} + \frac{\partial \beta_i^T}{\partial x_j} u.$$

Note that $\alpha_{i,j}$ is analytic on $\mathcal{D} \times \mathbb{R}^q$ – as it is a linear function of monomials' derivatives (cf. [1], §2.1). Let us define the matrices (functions of x, u) $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, as $A = \{\alpha_{i,j}\}$, $B = \{\beta_{i,s}\}$, which, by §3.6 of Part I, are the *Dynamic* and *Control* matrices, respectively, associate to the $\sigma\pi$-system (2.1). Let us define the *controllability matrix $\mathcal{C}$ ($= \mathcal{C}(x,u)$), of the $\sigma\pi$-system (2.1)* as follows:

$$(2.3) \qquad \mathcal{C} = \left[\begin{array}{ccccc} B & AB & A^2B & \dots & A^nB \end{array} \right].$$

By §4.5.1 of Part I we see that the above definition is consistent with the usual definition of controllability matrix associated to a linear system, provided that the linear system, as a particular kind of $\sigma\pi$-system, is expressed in CL (canonic linear) form.

In the following we denote by $\mathrm{Ker}(M)$ (resp. $\mathcal{R}(M)$) the kernel (resp. the range) spaces of a matrix M. Also, since the superscript $'\mathbf{p}'$ become redundant, we'll skip it throughout the present section. Before giving the main result of the section (Theorem 2.4), we shall to prove a few preliminary results, which are given in the forthcoming Lemmas 2.1, 2.2, and 2.3. In the following of this section some concepts will be used taken from the algebraic theory of nonlinear systems, for which we refer readers to [2].

LEMMA 2.1. *The following two claims are equivalent:*

(i) System (2.1) is strongly accessible in $x \in \mathcal{D}$.

(ii) The following conditions

$$(2.4) \qquad \sum_{i=1}^n f_i^{(k)} \beta_{i,s} = 0, \qquad \forall s, \forall u \in \mathbb{R}^q, \forall f_i^{(1)}, \dots, f_i^{(n)} \in \mathbf{M}(\mathcal{D}), k = 1, \dots n,$$

where for any integer $k \geq 2$, the $f^{(k)}$'s are the (meromorphic) functions

$$(2.5) \qquad f_i^{(k)} = \dot{f}_i^{(k-1)} + \sum_{j=1}^n f_j^{(k-1)} \alpha_{j,i},$$

with $\alpha_{i,j}$ defined in (2.2), imply that $f_1^{(1)}, \ldots, f_n^{(1)}$ are identically zero.

Proof. Let us build up a sequence $\{\mathcal{H}_k\}$ – of spaces of one-forms on the differential field [2], say **K**, associated to system (2.1) – recursively defined as $\mathcal{H}_k = \{\omega \in \mathcal{H}_{k-1}; \quad \dot{\omega} \in \mathcal{H}_{k-1}\}$ with $\mathcal{H}_1 = \mathcal{X}$, where $\mathcal{X} = \mathrm{span}_{\mathbf{K}}\{dx_1, \ldots, dx_n\}$. We have

$$(2.6) \qquad \mathcal{H}_1 = \mathcal{X} = \left\{ \omega = \sum_{i=1}^{n} f_i^{(1)} dx_i; \quad \text{for some } f_1^{(1)}, \ldots, f_n^{(1)} \in \mathbf{M}(\mathcal{D}) \right\}.$$

Then, let us calculate $\mathcal{H}_2 = \{\omega = \sum_{i=1}^{n} f_i^{(1)} dx_i; \quad \dot{\omega} \in \mathcal{H}_1\}$. By using the system equation, after some easy calculation, we obtain:

$$(2.7) \qquad \dot{\omega} = \sum_{i=1}^{n} f_i^{(2)} dx_i + \sum_{s,i}^{q,n} f_i^{(1)} \beta_{i,s} du_s,$$

where

$$f_i^{(2)} = \phi_i(f_i^{(1)}) = \dot{f}_i^{(1)} + \sum_{j=1}^{n} f_j^{(1)} \left(\sum_{l=1}^{\nu_j} v_{j,l} \frac{\partial X_{j,l}}{\partial x_i} + \frac{\partial \beta_j^T}{\partial x_i} u \right)$$

$$(2.8) \qquad\qquad = \dot{f}_i^{(1)} + \sum_{j=1}^{n} f_j^{(1)} \alpha_{j,i}.$$

Now, $\alpha_{i,j}$ is analytic on $\mathcal{D} \times \mathbb{R}^q$, and thus $f_i^{(2)} \in \mathbf{M}(\mathcal{D})$. Looking at (2.5) we realize that, if $f_i^{(k-1)} \in \mathbf{M}(\mathcal{D})$ then $f_i^{(k)} \in \mathbf{M}(\mathcal{D})$, and thus by induction all the $f_i^{(k)}$'s are meromorphic functions of $\mathbf{M}(\mathcal{D})$. That said, it is $\dot{\omega} \in \mathcal{H}_1$ if and only if

$$(2.9) \qquad 0 = \sum_{s,i}^{q,n} f_i^{(1)} \beta_{i,s} du_s = \sum_{s=1}^{q} \left(\sum_{i}^{n} f_i^{(1)} \beta_{i,s} \right) du_s,$$

which implies that $\mathcal{H}_2$ can be equivalently written as follows

$$\mathcal{H}_2 = \left\{ \omega = \sum_{i=1}^{n} f_i^{(1)} dx_i : \qquad \sum_{i}^{n} f_i^{(1)} \beta_{i,s} = 0 \quad \forall s \right\}.$$

Now, suppose that $\mathcal{H}_k$ can be written as

$$\mathcal{H}_k = \left\{ \omega = \sum_{i=1}^{n} f_i^{(1)} dx_i : \quad \sum_{i}^{n} f_i^{(l)} \beta_{i,s} = 0 \quad \forall s, \forall l = 1, \ldots, k-1 \right\},$$

where $f_i^{(l)} = \phi_i^l(f_i^{(1)})$, with $\phi_i^l = \phi_i \circ \phi_i^{l-1}$, and ϕ_i is the map defined by (2.8). Let us calculate $\mathcal{H}_{k+1} = \{\omega \in \mathcal{H}_k; \quad \dot{\omega} \in \mathcal{H}_k\}$. From (2.7), since $\omega \in \mathcal{H}_k$, it is

$$(2.10) \qquad \dot{\omega} = \sum_{i=1}^{n} f_i^{(2)} dx_i,$$

and thus, in order to condition $\dot{\omega} \in \mathcal{H}_k$ be verified, there shall be verified:

$$\sum_{i}^{n} \phi_i^l(f_i^{(2)}) \beta_{i,s} = 0 \quad \forall s, \forall l = 1, \ldots, k-1.$$

Since $\phi_i^l(f_i^{(2)}) = f_i^{(l+1)}$, we have

$$\mathcal{H}_{k+1} = \left\{ \omega = \sum_{i=1}^n f_i^{(1)} dx_i : \quad \sum_{i=1}^n f_i^{(l)} \beta_{i,s} = 0 \quad \forall s, \forall l = 1, \ldots, k \right\}$$

$$(2.11) \qquad = \left\{ \omega \in \mathcal{H}_k : \quad \sum_{i=1}^n f_i^{(k)} \beta_{i,s} = 0 \quad \forall s \right\},$$

Thus, by induction, the result is that (2.11) is the general expression for any space in the sequence $\{\mathcal{H}_k\}$. Now, it is well known (see for instance [2]) that a system of the type of (2.1) is strongly accessible if and only if there exists an integer k^* such that

$$(2.12) \qquad \mathcal{H}_1 \supset \mathcal{H}_2 \supset \ldots \supset \mathcal{H}_{k^*} \neq \emptyset, \qquad \mathcal{H}_{k^*+1} = \emptyset$$

where the inclusions are all strict-sense, and thus it is $k^* \leq n$. By reason of (2.11), such a condition is equivalent to the statement of the Lemma. $\square$

By using the matrices A, B we can rewrite (2.5) and the condition (2.4) in vector form:

$$(2.13) \qquad f^{(k)} = \dot{f}^{(k-1)} + A^T f^{(k-1)},$$

$$(2.14) \qquad B^T f^{(k)} = 0, \qquad k = 1, \ldots n.$$

LEMMA 2.2. *System (2.1) is strongly accessible $\forall x \in U$, $U \subset \mathcal{D}$ an open set, if and only if for almost all (x, u), $x \in U$, $u \in \mathbb{R}^q$:*

$$(2.15) \qquad \operatorname{rank}\{\mathcal{C}\} = n.$$

Proof. By successive substitutions of (2.14) in (2.13) the following formula is readily obtained

$$(2.16) \qquad B^T f^{(k)} = A^{k-1^T} B^T f^{(1)} + \sum_{r=2}^k A^{k-r^T} B^T \dot{f}^{(r-1)},$$

and thus condition (2.4) is equivalent to

$$(2.17) \qquad \begin{bmatrix} B^T & 0 & \cdots & \\ B^T A^T & B^T & 0 & \\ \vdots & & \ddots & \\ B^T A^{n-1^T} & B^T A^{n-2^T} & \cdots & B^T \end{bmatrix} \begin{bmatrix} f^{(1)} \\ \dot{f}^{(1)} \\ \vdots \\ \dot{f}^{(n-1)} \end{bmatrix} = 0, \quad \forall f^{(1)} \in \mathbf{M}(\mathcal{D}).$$

Let us denote by S the matrix in (2.17), and consider the following condition:

Condition 1. (2.17) being verified $\forall (x, u) \in \mathcal{U}$ implies $f^{(1)} \equiv 0$ at each point of its domain[3].

Moreover, let us consider the following strong (resp. weak) claim:

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[3]Recall that $f^{(1)}$ is a generic meromorphic function of a finite number of indeterminates chosen in the (infinite) set of indeterminates $\{x \in U, u \in \mathbb{R}^q, u^{(1)} \in \mathbb{R}^q, u^{(2)} \in \mathbb{R}^q, \ldots\}$.

Strong claim (resp. *Weak claim*). System (2.1) is strongly accessible in U if and only if $\mathcal{N}(S) = \{0\}$ $\forall (x, y) \in \mathcal{U} = U \times \mathbb{R}^q$ (resp. almost everywhere in $\mathcal{U}$)

By Lemma 2.1 system (2.1) is strongly accessible in U if and only if *Condition 1* is verified. By using (2.13) it is easy to verify that $f^{(1)} \equiv 0$ on an open set entails $\dot{f}^{(k)} \equiv 0$ on the same open set, for any k, and thus *Strong claim* holds. By reason of the continuity of the $\dot{f}^{(k)}$'s, *Weak claim* holds as well. Now, it is $(\mathcal{N}(S))^{\perp} = \mathcal{R}(S^T)$, and thus $\mathcal{N}(S) = \{0\}$ implies the surjecivity of S^T. Finally S^T being surjective implies that the controllability matrix $\mathcal{C}$, defined in (2.3), is surjective as well, since it is the first block row of S^T. $\square$

LEMMA 2.3. *System (2.1) is strongly accessible $\forall x \in \bar{\mathcal{D}}$, where $\bar{\mathcal{D}}$ is a connected component of the domain $\mathcal{D}$, if and only if, denoted by $\mathcal{M}_{\mathcal{C}}$ the set of all main minors of $\mathcal{C}$, there exists an $M \in \mathcal{M}_{\mathcal{C}}$ and a pair $(x, u) \in \bar{\mathcal{D}} \times \mathbb{R}^q$ such that*

$$(2.18) \qquad \qquad \mathrm{Det}\{M(x, u)\} \neq 0.$$

Proof. Let us define $\mathcal{F} \subset \bar{\mathcal{D}} \times \mathbb{R}^q$ as follows:

$$(2.19) \qquad \mathcal{F} = \{(x, u) : \mathrm{Det}\{M(x, u)\} = 0, \quad \forall M \in \mathcal{M}_{\mathcal{C}}\}.$$

Then – as $\bar{\mathcal{D}}$ is an open set – by Lemma 2.2, system (2.1) is strongly accessible $\forall x \in \bar{\mathcal{D}}$ if and only if $\mathcal{F}$ is a set of zero measure (or, which is the same, is a set with a void interior). Now, all entries of every matrix M are functions of x, u, analytic $\forall x \in \bar{\mathcal{D}}$, and the determinant is an analytic function of these entries. It follows that – as the set $\mathcal{M}_{\mathcal{C}}$ is finite, and $\bar{\mathcal{D}}$ is an open and convex set – we have either $\mathcal{F} = \bar{\mathcal{D}} \times \mathbb{R}^q$ or $\mathrm{Int}\{\mathcal{F}\} = \emptyset$. Thus, the condition $\mathrm{Int}\{\mathcal{F}\} = \emptyset$ is true if and only if $\mathcal{F} = \bar{\mathcal{D}} \times \mathbb{R}^q$ is false. The proof is completed by noticing that $\mathcal{F} = \bar{\mathcal{D}} \times \mathbb{R}^q$ is falsified as soon as (2.18) is verified for just a minor $M \in \mathcal{M}_{\mathcal{C}}$ and a pair $(x, u) \in \bar{\mathcal{D}} \times \mathbb{R}^q$. $\square$

Now we can state the main theorem of the section as a simple rephrasing of Lemma 2.3.

THEOREM 2.4. *System (2.1) is strongly accessible at every point of a connected component, $\bar{\mathcal{D}}$, of the domain $\mathcal{D}$, if and only if there exists an a pair $(x, u) \in \bar{\mathcal{D}} \times \mathbb{R}^q$ such that*

$$(2.20) \qquad \qquad \mathrm{rank}\{\mathcal{C}\} = n.$$

2.1. Example. Consider the system

$$\dot{x}_1 = x_2 + x_3 u,$$
$$\dot{x}_2 = -x_1,$$
$$\dot{x}_3 = -x_1 u,$$

which is bilinear, and thus is a $\sigma\pi$-system. The system domain is all $\mathbb{R}^3$. This system has been studied in [3] (example 4.2, p. 102) where it is shown that it is free to evolve only on the sphere centered at the origin passing through the initial state. Such a

restriction implies that the system is *not* accessible at any point of $\mathbb{R}^3$, since the set of accessibility of any point is included in a two dimensional manifold (and thus cannot include an open set of $\mathbb{R}^3$). By Corollary 4 the controllability matrix $\mathcal{C}$ is expected to have a rank lower than three. In order to verify this let us build up the dynamical matrix A and the control matrix B according to the formulas of $\alpha_{i,j}, \beta_{i,s}$ given in (2.2) and in (2.1). The result is

$$(2.21) \qquad A = \begin{bmatrix} 0 & 1 & u \\ -1 & 0 & 0 \\ -u & 0 & 0 \end{bmatrix}; \qquad B = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix};$$

from which we derive $\mathcal{C}$:

$$(2.22) \qquad \mathcal{C} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ u & 0 & 0 \end{bmatrix},$$

which has always rank 2 for any x, u.

3. $\sigma\pi$-systems in closed loop. Let us consider an i-indexed $\sigma\pi$-system in C-form (cf. Part I):

$$(3.1) \qquad \dot{x}_i = \sum_{i^*=1}^{\nu_i^{(\mathbf{p})}} v_{i,i^*}^{(\mathbf{p})} X_{i,i^*}^{(\mathbf{p})} + \sum_{i_*=1}^{\nu_i^{(\mathbf{c})}} v_{i,i_*}^{(\mathbf{c})} X_{i,i_*}^{(\mathbf{c})},$$

$$(3.2) \qquad v_{i,i_*}^{(\mathbf{c})} = \sum_{s=1}^{q} b_{i,s}^{(\mathbf{c},i_*)} u_s = b_i^{\mathbf{c},i_*}{}^{T} u,$$

and the corresponding (i, j)-indexed driver, whose S-form is:

$$(3.3) \qquad \dot{Z}_{i,i'} = \sum_{j,j'}^{n,\nu_j} \pi_{i,j}^{(i')} v_{j,j'} Z_{j,j'} Z_{i,i'},$$

where

$$(3.4) \qquad \pi_{i,j}^{(i')} = \begin{cases} p_{i,j}^{(i')} & \text{for} \quad i \neq j; \\ p_{i,i}^{(i')} - 1 & \text{otherwise,} \end{cases}$$

while the C-form is

$$(3.5) \qquad \dot{Z}_{i,i^*}^{(\mathbf{p})} = \sum_{j,j^*}^{n,\nu_j^{(\mathbf{p})}} \pi_{i,j}^{(\mathbf{p},i^*)} v_{j,j^*}^{(\mathbf{p})} Z_{i,i^*}^{(\mathbf{p})} Z_{j,j^*}^{(\mathbf{p})} + \sum_{j,j_*}^{n,\nu_j^{(\mathbf{c})}} \pi_{i,j}^{(\mathbf{p},i^*)} v_{j,j_*}^{(\mathbf{c})} Z_{i,i^*}^{(\mathbf{p})} Z_{j,j_*}^{(\mathbf{c})}$$

$$(3.6) \qquad \dot{Z}_{i,i_*}^{(\mathbf{c})} = \sum_{j,j^*}^{n,\nu_j^{(\mathbf{p})}} \pi_{i,j}^{(\mathbf{c},i_*)} v_{j,j^*}^{(\mathbf{p})} Z_{i,i_*}^{(\mathbf{c})} Z_{j,j^*}^{(\mathbf{p})} + \sum_{j,j_*}^{n,\nu_j^{(\mathbf{c})}} \pi_{i,j}^{(\mathbf{c},i_*)} v_{j,j_*}^{(\mathbf{c})} Z_{i,i_*}^{(\mathbf{c})} Z_{j,j_*}^{(\mathbf{c})} \quad,$$

where the $v^{(\mathbf{q})}$'s and $Z^{(\mathbf{q})}$'s, $\mathbf{q} \in \{\mathbf{p}, \mathbf{c}\}$, are given by

$$(3.7) \qquad v_{i,i_*}^{(\mathbf{c})} = v_{i,i'(i_*)}, \qquad Z_{l,l^*}^{(\mathbf{p})} = Z_{l,l'(l^*)}; \qquad \pi_{i,j}^{(\mathbf{p},i^*)} = \pi_{i,j}^{i'(i^*)};$$

$$(3.8) \qquad v_{i,i^*}^{(\mathbf{p})} = v_{i,i'(i^*)}, \qquad Z_{l,l_*}^{(\mathbf{c})} = Z_{l,l'(l_*)}, \qquad \pi_{i,j}^{(\mathbf{c},i_*)} = \pi_{i,j}^{i'(i_*)};.$$

We consider the following class of feedbacks ($s = 1, \ldots, q;\ m = 1, \ldots, \mu$):

$$(3.9) \qquad u_s = \sum_{m=1}^{\mu} k_{s,m} X_{s,m}^{(\mathbf{F})}; \qquad X_{s,m}^{(\mathbf{F})} = \prod_{l=1}^{n} x_l^{p_l^{(\mathbf{F},s,m)}},$$

where μ is some positive integer, and $k_{s,m}, p_l^{(\mathbf{F},s,m)}$ are real numbers. By replacing (3.9) in (3.1) we obtain the *closed-loop system*:

$$(3.10) \qquad \dot{x}_i = \sum_{i^*=1}^{\nu_i^{(\mathbf{P})}} v_{i,i^*}^{(\mathbf{P})} X_{i,i^*}^{(\mathbf{P})} + \sum_{i_{**}=(1,1,1)}^{\nu_i^{(\mathbf{L})}} v_{i,i_{**}}^{(\mathbf{L})} X_{i,i_{**}}^{(\mathbf{L})},$$

where $\nu_i^{(\mathbf{L})} = (\nu_i^{(\mathbf{c})}, q, \mu)$, $i_{**} = (i_*, s, m)$ is a triple-index ranging the set

$$(3.11)\ \{(1,1,1), \ldots, (1,1,\mu), (1,2,1), \ldots, (1,2,\mu), \ldots, (\nu_i^{(\mathbf{c})}, q, 1), \ldots, (\nu_i^{(\mathbf{c})}, q, \mu)\},$$

and

$$(3.12) \qquad v_{i,i_{**}}^{(\mathbf{L})} = b_{i,s}^{(\mathbf{c},i_*)} k_{s,m}, \qquad X_{i,i_{**}}^{(\mathbf{L})} = X_{s,m}^{(\mathbf{F})} X_{i,i_*}^{(\mathbf{c})} = \prod_{l=1}^{n} x_l^{p_{i,l}^{(\mathbf{L},i_{**})}},$$

$$(3.13) \qquad p_{i,l}^{(\mathbf{L},i_{**})} = p_{i,l}^{(\mathbf{c},i_*)} + p_l^{(\mathbf{F},s,m)},$$

and the symbol $\mathbf{L}$ stands for (closed)'loop'. Note that (3.10) has a larger size, namely $\mu_i = \nu_i^{(\mathbf{P})} + q \cdot \mu \cdot \nu_i^{(\mathbf{c})}$, than the open-loop system (3.1), and in particular what changes is the control size $\nu_i^{(\mathbf{c})}$ which increases up to $q\mu\nu_i^{(\mathbf{c})}$.

3.1. Closed-loop driver. The driver associated to (3.1), i.e. the *open-loop driver* (3.5), (3.6), can be written from (3.1) through certain well defined transformations of symbols. Notice that (3.1) and (3.10) are formally the same provided we identify the symbols $\mathbf{c}$, 1, and the subscript $\cdot_*$ (of i_*) in (3.1) with the symbols $\mathbf{L}$, $(1,1,1)$, and the subscript $\cdot_{**}$ (of i_{**}) in (3.10). Therefore, the driver of the closed-loop system (3.10) (the *closed-loop driver*) can be directly written by applying on the open-loop driver the same substitutions of symbols. The result is

$$(3.14) \quad \dot{Z}_{i,i^*}^{(\mathbf{P})} = \sum_{j,j^*}^{n,\nu_j^{(\mathbf{P})}} \pi_{i,j}^{(\mathbf{P},i^*)} v_{j,j^*}^{(\mathbf{P})} Z_{i,i^*}^{(\mathbf{P})} Z_{j,j^*}^{(\mathbf{P})} + \sum_{j,j_{**}}^{n,\nu_j^{(\mathbf{L})}} \pi_{i,j}^{(\mathbf{P},i^*)} v_{j,j_{**}}^{(\mathbf{L})} Z_{i,i^*}^{(\mathbf{P})} Z_{j,j_{**}}^{(\mathbf{L})}$$

$$(3.15) \quad \dot{Z}_{i,i_{**}}^{(\mathbf{L})} = \sum_{j,j^*}^{n,\nu_j^{(\mathbf{P})}} \pi_{i,j}^{(\mathbf{L},i_{**})} v_{j,j^*}^{(\mathbf{P})} Z_{i,i_{**}}^{(\mathbf{L})} Z_{j,j^*}^{(\mathbf{P})} + \sum_{j,j_{**}}^{n,\nu_j^{(\mathbf{L})}} \pi_{i,j}^{(\mathbf{L},i_{**})} v_{j,j_*}^{(\mathbf{L})} Z_{i,i_{**}}^{(\mathbf{L})} Z_{j,j_{**}}^{(\mathbf{L})} \quad,$$

where there has been created the new symbols: $Z_{l,l_{**}}^{(\mathbf{L})}$, (for $l = i$ or $l = j$) – which has the obvious interpretation $Z_{l,l_{**}}^{(\mathbf{L})} = Z_{l,l'(l_{**})}$ – and $\pi_{i,j}^{(\mathbf{L},l_{**})}$, which is easily recognized as

$$(3.16) \quad \pi_{i,j}^{(\mathbf{L},i_{**})} = \begin{cases} p_{i,j}^{(\mathbf{L},i_{**})} & \text{for} \quad i \neq j; \\ p_{i,i}^{(\mathbf{L},i_{**})} - 1 & \text{otherwise}, \end{cases} \qquad \left(= \pi_{i,l}^{(\mathbf{c},i_*)} + p_l^{(\mathbf{F},s,m)} \right).$$

3.2. Generator of the open-loop driver. A driver is obviously a self-driver (cf. Part I), and as such it has a generator, namely G. In order to write G, we are faced with the fact that (3.3), differently than the definition of self-driver we gave in Part I, is double indexed. Let us rewrite (3.3) as

$$(3.17) \qquad \dot{Z}_{i,i'} = \sum_{j,j'}^{n,\nu_j} v_{i,i'}^{(j,j')} Z_{j,j'} Z_{i,i'},$$

$$(3.18) \qquad v_{i,i'}^{(j,j')} = \pi_{i,j}^{(i')} v_{j,j'},$$

then, the generator G is the matrix collecting the coefficients $v_{i,i'}^{(j,j')}$ by using the pair (i, i') as row (double) index, and the pair (j, j') as (double) column index. Both indices span the set:

$$(3.19) \qquad \{(1,1), \ldots, (1,\nu_1), (2,1), \ldots, (2,\nu_2), \ldots, (n,1), \ldots (n,\nu_n)\},$$

having cardinality equal to d, the total size of the underlying $\sigma\pi$-system, thus $G \in \mathbb{R}^{d \times d}$. We use a particular *block form* for the generator G of a driver, obtained as follows. We state beforehand the following notation: if $a_{i_1,\ldots,i_\alpha}$ is a real quantity depending of a number, α, of indices (which could be subscripted as well as superscripted), then the symbol

$$(3.20) \qquad [a_{i_1,\ldots,i_n}]_{i_l}^{i_m},$$

shall denote the matrix obtained by using i_l (resp. i_m) as row (resp. column) index, and pinning the other indices. That said, we have that the generator G of the driver (3.17) can be written in the following *block-form*:

$$(3.21)\; G = \begin{bmatrix} G_{1,1} & G_{1,2} & \ldots & G_{1,n} \\ G_{2,1} & G_{2,2} & & G_{1,n} \\ \vdots & & \ddots & \vdots \\ G_{n,1} & G_{n,2} & \ldots & G_{n,n} \end{bmatrix} \in \mathbb{R}^{d \times d}; \qquad G_{i,j} = [v_{i,i'}^{(j,j')}]_{i'}^{j'} \in \mathbb{R}^{\nu_i \times \nu_j}.$$

In the following we identify G with its i, j-th block, $G_{i,j}$, so there will be $G_{i,j}$ referred to as 'the generator' of the driver (3.3). By recalling that i', j' are two size indices of the $\sigma\pi$-system (3.1) underlying the driver (3.3), we can write the following four-block matrix, $G_{i,j}^*$, which is equal to $G_{i,j}$ unless a permutation of rows and columns:

$$G_{i,j}^* = \begin{bmatrix} [v_{i,i'}^{(j,j')}]_{i' \in \mathcal{I}^\mathbf{P}}^{j' \in \mathcal{I}^\mathbf{P}} & [v_{i,i'}^{(j,j')}]_{i' \in \mathcal{I}^\mathbf{P}}^{j' \in \mathcal{I}^\mathbf{c}} \\ [v_{i,i'}^{(j,j')}]_{i' \in \mathcal{I}^\mathbf{c}}^{j' \in \mathcal{I}^\mathbf{P}} & [v_{i,i'}^{(j,j')}]_{i' \in \mathcal{I}^\mathbf{c}}^{j' \in \mathcal{I}^\mathbf{c}} \end{bmatrix} = \begin{bmatrix} [\pi_{i,j}^{(\mathbf{P},i^*)} v_{j,j^*}^{(\mathbf{P})}]_{i^*}^{j^*} & [\pi_{i,j}^{(\mathbf{P},i^*)} v_{j,j_*}^{(\mathbf{c})}]_{i^*}^{j_*} \\ [\pi_{i,j}^{(\mathbf{c},i_*)} v_{j,j^*}^{(\mathbf{P})}]_{i_*}^{j^*} & [\pi_{i,j}^{(\mathbf{c},i_*)} v_{j,j_*}^{(\mathbf{c})}]_{i_*}^{j_*} \end{bmatrix};$$

(3.22)

where in the last matrix on right hand side we have replaced (3.18) and then (3.7) (3.8). Notice that the matrix $G^* = [G_{i,j}^*]_i^j$ is the generator of the driver in C-form (3.5), (3.6). We can now derive the generator of the closed-loop driver (3.14), (3.15), through the usual symbolic substitution $\mathbf{c} \to \mathbf{L}$, $\cdot_* \to \cdot_{**}$, thus, by using the same symbol G^*, the generator of the closed-loop driver is

$$(3.23) \qquad G_{i,j}^* = \begin{bmatrix} [\pi_{i,j}^{(\mathbf{P},i^*)} v_{j,j^*}^{(\mathbf{P})}]_{i^*}^{j^*} & [\pi_{i,j}^{(\mathbf{P},i^*)} v_{j,j_{**}}^{(\mathbf{L})}]_{i^*}^{j_{**}} \\ [\pi_{i,j}^{(\mathbf{L},i_{**})} v_{j,j^*}^{(\mathbf{P})}]_{i_{**}}^{j^*} & [\pi_{i,j}^{(\mathbf{L},i_{**})} v_{j,j_{**}}^{(\mathbf{L})}]_{i_{**}}^{j_{**}} \end{bmatrix}.$$

Now, for any vector $\mathbf{a} \in \mathbb{R}^{n_2}$, and $\mathbf{b} \in \mathbb{R}^{n_3}$ we have

$$(3.24) \qquad [\mathbf{a}_i \mathbf{b}_j]_i^j = \mathbf{a}\mathbf{b}^T.$$

Moreover let us denote

$$(3.25) \qquad [/\mathbf{b_j}/]_j = \operatorname{diag}\{\mathbf{b}_1, \ldots, \mathbf{b}_{n_2}\}.$$

By using (3.24), $G_{i,j}$ and $G_{i,j}^*$ can be rewritten

$$(3.26) \qquad G_{i,j} = \left[\begin{array}{cc} \pi_{i,j}^{\mathbf{P}} v_j^{\mathbf{P}T} & \pi_{i,j}^{\mathbf{P}} v_j^{\mathbf{c}T} \\ \pi_{i,j}^{\mathbf{c}} v_j^{\mathbf{P}T} & \pi_{i,j}^{\mathbf{c}} v_j^{\mathbf{c}T} \end{array}\right], \qquad G_{i,j}^* = \left[\begin{array}{cc} \pi_{i,j}^{\mathbf{P}} v_j^{\mathbf{P}T} & \pi_{i,j}^{\mathbf{P}} v_j^{\mathbf{L}T} \\ \pi_{i,j}^{\mathbf{L}} v_j^{\mathbf{P}T} & \pi_{i,j}^{\mathbf{L}} v_j^{\mathbf{L}T} \end{array}\right].$$

By definition, given in (3.12), it is

$$(3.27) \qquad v_j^{\mathbf{L}T} = [\alpha_1, \ldots \alpha_{\nu_j^{\mathbf{c}}}],$$

where, denoting $K = [k_{s,m}]_s^m$, it is

$$(3.28) \qquad \alpha_{j_*} = [b_{j,1}^{\mathbf{c},j_*} k_1^T, \ldots, b_{j,q}^{\mathbf{c},j_*} k_q^T] = \mathbf{st}^T\{K^T[/b_j^{\mathbf{c},j_*}/]\}.$$

The vector $\pi_{i,j}^{\mathbf{L}}$ can be rewritten as well, showing the dependence of $\pi^{\mathbf{c}}$ and $p^{\mathbf{F}}$:

$$(3.29) \qquad \pi_{i,j}^{\mathbf{L}} = [\pi_{i,j}^{(\mathbf{c},i_*)}]_{i_{**}} + [p_j^{(\mathbf{F},s,m)}]_{i_{**}} = \pi_{i,j}^{(\mathbf{c})} \otimes \mathbf{1}_{q\mu} + \mathbf{1}_{\nu_i^{(\mathbf{c})}} \otimes p_j^{(\mathbf{F})}$$

where $\mathbf{1}_l \in \mathbb{R}^n$ denotes the column vector of l ones : $[1, 1, \ldots, 1]^T$.

4. A class of closed-loop systems and its properties. Let us consider, without loss of generality, a $\sigma\pi$-system in constant and aligned parametric/control size (cf. Part I), and thus

$$(4.1) \qquad X_{i,i^*}^{\mathbf{P}} = X_{j,i^*}^{\mathbf{P}}, \qquad X_{i,i_*}^{\mathbf{c}} = X_{j,i_*}^{\mathbf{c}},$$

for any i, j such that the monomials are well defined. For such a system we have, for $\mathbf{q} \in \{\mathbf{p}, \mathbf{c}\}$, $\nu_i^{\mathbf{q}} = \nu^{\mathbf{q}}$, and $\mathcal{I}^{\mathbf{q}} = \{1, \ldots, \nu^{\mathbf{q}}\}$, and the monomials in (4.1) are defined $\forall i^* \in \mathcal{I}^{\mathbf{P}}$ and $\forall i_* \in \mathcal{I}^{\mathbf{c}}$. Now, consider the corresponding closed-loop system (3.10), with some feedback given by (3.9), and suppose that the following property holds:

P1) there exists a surjective map $\iota : \mathcal{I}^{\mathbf{c}} \times \{1, \ldots, \mu\} \ni (i_*, m) \mapsto \iota(i_*, m) \in \mathcal{I}^{\mathbf{P}}$ such that:

$$(4.2) \qquad X_{i,i_{**}}^{\mathbf{L}} \quad \left(= X_{i,i_*,s,m}^{\mathbf{L}} = X_{s,m}^{\mathbf{F}} X_{i,i_*}^{\mathbf{c}}\right) \quad = X_{i,\iota(i_*,m)}^{\mathbf{P}},$$

and thus, in particular, $X_{i,i_{**}}^{\mathbf{L}}$ – as well as $X_{s,m}^{\mathbf{F}}$ – is constant with respect to s.

Then, we can define the following quantities

$$(4.3) \qquad \delta_{i,m}^{i_*} = \sum_{s=1}^q b_{i,s}^{\mathbf{c},i_*} k_{s,m}.$$

$$(4.4) \qquad d_{i,i^*} = \sum_{(i_*,m)\in\iota^{-1}(i^*)} \delta_{i,m}^{i_*}, \qquad \iota^{-1}(i^*) = \{(i_*, m) : \quad i^* = \iota(i_*, m)\}.$$

By using (4.2), (4.3), the closed loop system (3.10) becomes

$$\dot{x}_i = \sum_{i^*=1}^{\nu^{\mathbf{P}}} v_{i,i^*}^{\mathbf{P}} X_{i,i^*}^{\mathbf{P}} + \sum_{i_*,s,m}^{\nu^{\mathbf{c}},q,\mu} b_{i,s}^{\mathbf{c}.i^*} k_{s,m} X_{i,i_{**}}^{\mathbf{L}}$$

$$(4.5) \qquad = \sum_{i^*=1}^{\nu^{\mathbf{P}}} v_{i,i^*}^{\mathbf{P}} X_{i,i^*}^{\mathbf{P}} + \sum_{i_*,m}^{\nu^{\mathbf{c}},\mu} \delta_{i,m}^{i_*} X_{i,\iota(i_*,m)}^{\mathbf{P}},$$

thus, by using (4.4) in (4.5), and since the map ι is surjective, we have

$$\dot{x}_i = \sum_{i^*=1}^{\nu^{\mathbf{P}}} v_{i,i^*}^{\mathbf{P}} X_{i,i^*}^{\mathbf{P}} + \sum_{i^*=1}^{\nu^{\mathbf{P}}} \left(\sum_{(i_*,m)\in\iota^{-1}(i^*)} \delta_{i,m}^{i_*} \right) X_{i,i^*}^{\mathbf{P}}$$

$$(4.6) \qquad = \sum_{i^*=1}^{\nu^{\mathbf{P}}} v_{i,i^*}^{\mathbf{P}} X_{i,i^*}^{\mathbf{P}} + \sum_{i^*=1}^{\nu^{\mathbf{P}}} d_{i,i^*} X_{i,i^*}^{\mathbf{P}} = \sum_{i^*=1}^{\nu^{\mathbf{P}}} (v_{i,i^*}^{\mathbf{P}} + d_{i,i^*}) X_{i,i^*}^{\mathbf{P}}.$$

Now, we can prove the following theorem.

THEOREM 4.1. *For any $\sigma\pi$-system as (3.2), in aligned and constant parametric/control size, there exists a feedback of the form (3.9) such that the corresponding closed-loop system satisfies Property* **P1**. *In particular, there exists a feedback with*

$$(4.7) \qquad \mu = \nu^{\mathbf{P}} \cdot \nu^{\mathbf{c}},$$

whose monomials $X_{s,m}^{\mathbf{F}}$ are given by [4]

$$(4.8) \qquad X_{s,m}^{\mathbf{F}} = X_{i,i^*}^{\mathbf{P}} X_{i,i_*}^{\mathbf{c}}{}^{-1}, \qquad i^* = \frac{1}{\nu^{\mathbf{c}}}(m - i_*) + 1. \qquad \forall s = 1, \ldots, q.$$

Proof. Consider a system version with aligned and constant parametric/control size. Then, build up the closed-loop system by using a feedback in the form (3.9) where μ is given by (4.7), and the set of monomials $\{X_{s,m}^F : (s,m) : (1,1), \ldots, (q, \nu^{\mathbf{P}}\nu^{\mathbf{c}})\}$ are defined by the following algorithm:

Step 1. For $i^* = 1, \ldots, \nu^{\mathbf{P}}$ perform Step 2.

Step 2. For $i_* = 1, \ldots, \nu^{\mathbf{c}}$: set $m = i_* + (i^* - 1)\nu^{\mathbf{c}}$, define $\iota(i_*, m) = i^*$, and set $X_{s,m}^{\mathbf{F}} = X_{i,i^*}^{\mathbf{P}} X_{i,i_*}^{\mathbf{c}}{}^{-1}$, which is (4.8). (end of Step 2, and of the algorithm).

Clearly, the monomials $X_{s,m}^{\mathbf{F}}$ and the map ι – whose range is $\mathcal{I}^{\mathbf{P}}$, and thus it is surjective – satisfy (4.2). $\square$

4.0.1. Remark. Without loss of generality we can assume that, for any i_*, we have either $X_{i,i_*}^{\mathbf{c}} = X_{i,i^*}^{\mathbf{P}}$, for some i^*, or $X_{i,i_*}^{\mathbf{c}} = 1$. Indeed, if $X_{i,i_*}^{\mathbf{c}} \neq X_{i,i^*}^{\mathbf{P}}$, we can always add the term $0 \cdot X_{i,i^*}^{\mathbf{P}}$ to all equations, and relabel $X_{i,i^*}^{\mathbf{c}}$ as a parametric monomial. Note that, in this way, the feedback monomials defined in (4.8), yield a

---

[4]Note that, since the system is aligned, (4.8) can be calculated at any $i = 1, \ldots, n$.

closed-loop system having always the same domain as the open-loop system.

Let us apply the feedback (4.8), and notice that

$$(4.9)\ \iota^{-1}(i^*) = \{(i_*, m) : i_* = 1, \ldots, \nu^{\mathbf{c}}; \quad m = (i^* - 1)\nu^{\mathbf{c}} + 1, \ldots, (i^* - 1)\nu^{\mathbf{c}} + \nu^{\mathbf{c}}\}.$$

Thus, from (4.3), (4.4), we have

$$d_{i,i^*} = \sum_{(i_*,m)\in\iota^{-1}(i^*)} \delta_{i,m}^{i^*} = \sum_{i_*=1}^{\nu^{\mathbf{c}}} \sum_{m=(i^*-1)\nu^{\mathbf{c}}+1}^{(i^*-1)\nu^{\mathbf{c}}+\nu^{\mathbf{c}}} \sum_{s=1}^{q} b_{i,s}^{\mathbf{c},i_*} k_{s,m}$$

$$(4.10) \qquad = \sum_{s=1}^{q} \left(\sum_{i_*=1}^{\nu^{\mathbf{c}}} b_{i,s}^{\mathbf{c},i_*} \right) \cdot \left(\sum_{m=(i^*-1)\nu^{\mathbf{c}}+1}^{(i^*-1)\nu^{\mathbf{c}}+\nu^{\mathbf{c}}} k_{s,m} \right) = \sum_{s=1}^{q} B_{i,s} K_{s,i^*}$$

where $B \in \mathbb{R}^{n \times q}$ is the matrix[5]:

$$(4.11) \qquad\qquad B = \sum_{i_*=1}^{\nu^{\mathbf{c}}} B^{i_*}; \qquad B^{i_*} = [b_{i,s}^{\mathbf{c},i_*}]_i^s,$$

and $K \in \mathbb{R}^{q \times \nu^p}$ is the matrix:

$$(4.12)\ K_{s,i^*} = \sum_{m=(i^*-1)\nu^{\mathbf{c}}+1}^{(i^*-1)\nu^{\mathbf{c}}+\nu^{\mathbf{c}}} k_{s,m} = \bar{K}[/\mathbf{1}_{\nu^{\mathbf{c}}}/]_{i^*}; \qquad \text{with}\ \ \mathbb{R}^{q \times \nu^{\mathbf{p}} \cdot \nu^{\mathbf{c}}} \ni \bar{K} = [k_{s,m}]_s^m.$$

By defining

$$(4.13) \qquad\qquad \mathbb{R}^{n \times \nu^{\mathbf{p}}} \ni A = [v_{i,i^*}^{\mathbf{p}}]_i^{i^*}; \qquad \mathbb{R}^{n \times \nu^{\mathbf{p}}} \ni D = [d_{i,i^*}]_i^{i^*};$$

$$(4.14) \qquad\qquad \mathbb{R}^{\nu^{\mathbf{p}}} \ni X^{\mathbf{p}} = X_i^{\mathbf{p}},$$

where in (4.14) $X_i^{\mathbf{p}} = [X_{i,i^*}^{\mathbf{p}}]_{i^*}$ following the usual convention, we can rewrite (4.6) in the *almost-familiar*, for automatic control scholars, vector form:

$$(4.15) \qquad\qquad\qquad \dot{x} = FX^{\mathbf{p}},$$

$$(4.16) \qquad\qquad\qquad F = A + BK.$$

Let us denote $f_{i,i^*} = F_{i,i^*}$ – thus, by (4.6), $f_{i,i^*} = v_{i,i^*}^{\mathbf{p}} + d_{i,i^*}$ – and let $G \in \mathbb{R}^{n^2 \times n^2}$ the generator of the driver associated to (4.15), which has the block structure depicted in (3.21) with (notice that the size indices i', j', in the closed loop case, are equal to parametric-size indices i^*, j^*):

$$(4.17) \qquad\qquad G_{i,j} = [f_{i,i^*}^{j,j^*}]_{i^*}^{j^*} \in \mathbb{R}^{n \times n}; \qquad f_{i,i^*}^{j,j^*} = \pi_{i,j}^{(i^*)} f_{j,j^*},$$

and thus

$$(4.18) \qquad\qquad\qquad G_{i,j} = \pi_{i,j} f_j^T.$$

Moreover, since $\pi_{i,i} = p_{i,i} - \mathbf{1}_n$, and $\pi_{i,j} = p_{i,j}$ for $i \neq j$, by defining

$$(4.19) \qquad\qquad\qquad G'_{i,j} = p_{i,j} f_j^T,$$

we have

$$(4.20) \qquad\qquad G_{i,j} = \begin{cases} G'_{i,j} & \text{for} \quad i \neq j; \\ G'_{i,i} - \mathbf{1}_n f_i^T & \text{otherwise,} \end{cases}$$

---

[5]Recall the notation introduced in (3.20), and (3.25).

4.0.2. Remark. Note that, since the $\sigma\pi$-system at issue has aligned size, i.e. identities (4.1) hold, $p_{i,j}$ and, hence, $G'_{i,j}$ does not depend actually of i. Nevertheless, we maintain the subscript i for the sake of notational consistency [6].

The generator of the driver associated to the closed loop system (4.15) is given by (3.21) with the $G_{i,j}$'s given by (4.18), and we have:

$$(4.21) \qquad G = \begin{bmatrix} \pi_{-,1} f_1^T & \cdots & \pi_{-,n} f_n^T \end{bmatrix}$$

Now, let us suppose that the couple (A, B) is controllable (in order to fix ideas also suppose that $n \geq q$, as well as B is full-rank) . Than, there exists an invertible matrix T, and q non-negative integers (controllability indices) $m_1, \ldots, m_q$ ($m_1 + \ldots + m_q = n$) such that $A^{\mathbf{c}} = TAT^{-1}$, $B^{\mathbf{c}} = TB$, where $A^{\mathbf{c}}$, $B^{\mathbf{c}}$, have the following structure:

$$(4.22) \qquad A^{\mathbf{c}} = \begin{bmatrix} \mathbf{e}_2^{nT} \\ \vdots \\ \mathbf{e}_{\mu_1}^{n\ T} \\ \alpha_{\mu_1}^T \\ \mathbf{e}_{\mu_1+2}^{n\ T} \\ \vdots \\ \alpha_{\mu_2}^T \\ \vdots \\ \alpha_n^T \end{bmatrix}, \qquad B^{\mathbf{c}} = \begin{bmatrix} 0_{m_1-1\times q} \\ \beta_{\mu_1}^T \\ 0_{\mu_2-1\times q} \\ \beta_{\mu_2}^T \\ \vdots \\ \beta_n^T \end{bmatrix},$$

where $\mu_s = m_1 + \ldots + m_s$ (and, thus, $\mu_1 = m_1$, and $\mu_q = n$), $\mathbf{e}_i^n \in \mathbb{R}^n$ denotes the i-th vector of the canonical base, $0_{d\times l} \in \mathbb{R}^{d\times l}$ is a zero matrix, $\alpha_{l,j} \in \mathbb{R}$, $\beta_{l,s} \in \mathbb{R}$ for $l \in \{\mu_1, \ldots, \mu_q\}$. By defining

$$(4.23) \qquad F^{\mathbf{c}} = A^{\mathbf{c}} + B^{\mathbf{c}} K^{\mathbf{c}}, \qquad K^{\mathbf{c}} = KT^{-1},$$

by the structure of $A^{\mathbf{c}}$ and $B^{\mathbf{c}}$, shown in (4.22) we have

$$(4.24) \qquad F^{\mathbf{c}} = \begin{bmatrix} \mathbf{e}_2^{nT} \\ \vdots \\ \mathbf{e}_{\mu_1}^{n\ T} \\ \alpha_{\mu_1}^T + \beta_{\mu_1}^T K^{\mathbf{c}} \\ \mathbf{e}_{\mu_1+2}^{n\ T} \\ \vdots \\ \alpha_{\mu_2}^T + \beta_{\mu_2}^T K^{\mathbf{c}} \\ \vdots \\ \alpha_n^T + \beta_n^T K^{\mathbf{c}} \end{bmatrix}.$$

Let us define $\bar{A}^{\mathbf{c}} = [\alpha_{\mu_s,j}]_s^j \in \mathbb{R}^{q\times n}$, $\bar{B}^{\mathbf{c}} = [\beta_{\mu_s,s}]_s^s \in \mathbb{R}^{q\times q}$. Since B is full (column) rank, and so does $B^{\mathbf{c}}$ defined in (4.22), $\bar{B}^{\mathbf{c}}$ is invertible, and for any matrix $M \in \mathbb{R}^{q\times n}$, by setting

$$(4.25) \qquad K^{\mathbf{c}} = \bar{B}^{\mathbf{c}-1}(M - \bar{A}^{\mathbf{c}}),$$

---

[6]Indeed, dropping an index, in our notational system has the meaning of aggregated vector.

we have $\bar{A}^{\mathbf{c}} + \bar{B}^{\mathbf{c}} K^{\mathbf{c}} = M$. Thus, all the rows having indices $\mu_1, \ldots, \mu_q$ in (4.24) can be set to any value, so let us set for $s = 1, \ldots, q-1$:

$$(4.26) \qquad M_{\mu_s} = \mathbf{e}^n_{\mu_s+1},$$

and

$$(4.27) \qquad M_n^T = \begin{bmatrix} -c_0 & -c_1 & \cdots & -c_{n-1} \end{bmatrix},$$

where $c_0, \ldots, c_{n-1}$ are the real coefficients of a polynomial $p(\lambda)$, whose roots are $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then the matrix $F^{\mathbf{c}}$, with $K^{\mathbf{c}}$ as in (4.25), is the Frobenius companion matrix of $p(\lambda)$, and the complex numbers $\lambda_1, \ldots, \lambda_n$, whatever chosen in conjugate pairs, are the eigenvalues of $F^{\mathbf{c}}$. Since $F = T^{-1} F^{\mathbf{c}} T$, $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of F as well. In conclusion, with $K^{\mathbf{c}}$ given by (4.25), that is, taking into account of the second equation in (4.23), with

$$(4.28) \qquad K = (\bar{B}^{\mathbf{c}^{-1}}(M - \bar{A}^{\mathbf{c}}))T,$$

we can assign eigenvalues to $F = A + BK$ to be equal to any n-tuple of complex numbers in conjugate couples.

Without loss of generality, we can assume that the matrices A, B in (4.16) have an even number of rows, say $2n$, $n \in \mathbb{N}$, and the following structure:

$$(4.29) \qquad A = \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}; \qquad B = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix} \in \mathbb{R}^{2n \times q}$$

where all the blocks in A are $n \times n$-dimensioned, and $\bar{B}$ is $n \times q$. Indeed, we can otherwise enlarge the original system size by adding fictitious parametric monomials like $0 \cdot X^{\mathbf{P}}_{i,i*}$, and/or increase up to $2n$ the upperbound of the equation index i, by introducing new independent variables x_i with the corresponding null equation: $\dot{x}_i = 0$, and initial condition $x_i(t_0) = 1$. Note that even though a monomial $X^{\mathbf{P}}_{i,i*}$ could be fictitious, such a monomial is actually multiplied for some non zero coefficient $k_{s,i*}$ in the closed-loop system (4.15). For the matrix K we recognize the following structure:

$$(4.30) \qquad K = [K^1, K^2] \in \mathbb{R}^{q \times 2n}; \qquad K^1 \in \mathbb{R}^{q \times n}, \quad K^2 \in \mathbb{R}^{q \times n},$$

and thus, from (4.29), we have

$$(4.31) \qquad A + BK = \begin{bmatrix} \bar{A} + \bar{B}K^1 & \bar{B}K^2 \\ 0 & 0 \end{bmatrix} = A^* + B^* K^*,$$

where

$$(4.32) \qquad A^* = \begin{bmatrix} \bar{A} + \bar{B}K^1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad B^* = \begin{bmatrix} 0 & \bar{B} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2q},$$

$$(4.33) \qquad K^* = \begin{bmatrix} K^2 & 0 \\ 0 & K^2 \end{bmatrix} \in \mathbb{R}^{2q \times 2n}, \qquad K^2 \in \mathbb{R}^{q \times n}.$$

Let us denote $a^*_{i,i*} = A^*_{i,i*}$, $b^*_{i,s} = B^*_{i,s}$. Then, apparently we have $f_j^T = a_j^{*T} + b_j^{*T} K$ – which is a $2n$-dimensional vector $\forall j = 1, \ldots, 2n$ – and from (4.21), since

$a_j^* = 0$, $b_j^* = 0$ for $n + 1 \leq j \leq 2n$, we can rewrite G as follows

$$(4.34) \quad G = \mathcal{A}^* + \mathcal{B}^*\mathcal{K}^* = \left[\begin{array}{cccc} \pi_{-,1}f_1^T & \cdots & \pi_{-,n}f_n^T, 0, \ldots, 0 \end{array} \right] \in \mathbb{R}^{4n^2 \times 4n^2},$$

$$(4.35) \quad \mathcal{A}^* = \left[\begin{array}{cccc} \pi_{-,1}a_1^{*T} & \cdots & \pi_{-,n}a_n^{*T}, 0, \ldots, 0 \end{array} \right] \in \mathbb{R}^{4n^2 \times 4n^2},$$

$$(4.36) \quad \mathcal{B}^* = \left[\begin{array}{cccc} \pi_{-,1}b_1^{*T} & \cdots & \pi_{-,n}b_n^{*T}, 0, \ldots, 0 \end{array} \right] \in \mathbb{R}^{4n^2 \times 4nq},$$

$$(4.37) \quad \mathcal{K}^* = [/K^*/]_i \in \mathbb{R}^{4nq \times 4n^2}.$$

Let us denote by $\mathcal{C}$ the controllability matrix associated to $(\mathcal{A}^*, \mathcal{B}^*)$.

4.0.3. Remark. Note that, since the *feedback* $u = u(x)$ is a functions of monomials, $X^{\mathbf{P}}$, that, in principle, are freely chosen by the designer, the vectors $\pi_{-,j}$, appearing in (4.34)–(4.36), and collecting *powers* of the monomials $X_{i,j}^{\mathbf{P}}$ [7], can be always chosen in order to satisfy the following property:

P2) *the vectors* $\pi_{-,1}, \ldots, \pi_{-,n}$ *are linearly independent, and the matrix* $T_{2,1} \in \mathbb{R}^{n \times n}$:

$$(4.38) \quad T_{2,1} = \left[\begin{array}{ccc} \pi_{1,1}^{n+1} & \cdots & \pi_{1,n}^{n+1} \\ \vdots & \cdots & \vdots \\ \pi_{1,1}^{2n} & \cdots & \pi_{1,n}^{2n} \end{array} \right],$$

is invertible.

Now, let us suppose that property **P2** holds. We can prove the following lemma.

LEMMA 4.2. *Assume that the matrices* A, B *of the closed-loop system have a structure as in (4.29), and suppose that* $(\bar{A}, \bar{B})$ *is controllable. Then, for any choice of* K^1 *such that* $\bar{A} + \bar{B}K^1$ *is invertible, we have:*

(i) The driver generator, $G = G(K^2)$ *as a matrix linear function of* K^2*, for any given* $K^2 \in \mathbb{R}^{q \times n}$*, has at least* $4n^2 - n$ *zero eigenvalues. Moreover,* $\text{rank}(G(K^2)) = n$ *for any* $K^2 \in \mathbb{R}^{q \times n}$*.*

(ii) $\text{rank}(\mathcal{C}) \leq n$*;*

(iii) if $\text{rank}(\mathcal{C}) = n$*, then, for any given* n*-tuple of complex numbers in conjugate pairs,* $\lambda_1, \ldots, \lambda_n$*, there exists a* K^2 *such that* $G(K^2)$ *has* n *eigenvalues equal to* $\lambda_1, \ldots, \lambda_n$*.*

Proof. Looking at (4.34), since the blocks $\pi_{-,j}f_j^T$ have each at most rank one, $\mathcal{R}(G) \leq n$, therefore the kernel of G has a dimension at least equal to $4n^2 - n$, which proves *(i)*. From (4.34), and since $\bar{A} + \bar{B}K^1$ is invertible, we see that the first n row of A^*: $a_1^{*T}, \ldots, a_n^{*T}$ (resp: the first n rows of $A^* + B^*K^*$: $f_1^{*T}, \ldots, f_n^{*T}$) are linearly independent, and thus they have each at least one non zero entry: let $a_{1,i_1}^*, \ldots, a_{n,i_n}^*$ (resp. $f_{1,i_1}^*, \ldots, f_{n,i_n}^*$) be a choice of such non zero elements. Then, since the vectors $\pi_{-,1}, \ldots, \pi_{-,n}$ are linearly independent, by (4.35), (resp. by (4.34))

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[7]exactly, the powers of x_j in all monomials for $i = 1, \ldots, n$, with $i \neq j$, and the same powers minus one, for $i = j$

the set of n vectors, $\pi_{-1}a^*_{1,i_1}, \ldots \pi_{-,n}a^*_{n,i_n}$, (resp. $\pi_{-1}f^*_{1,i_1}, \ldots \pi_{-,n}f^*_{n,i_n}$) and, hence, the set $\pi_{-1}, \ldots \pi_{-,n}$ itself, is a basis for $\mathcal{R}(\mathcal{A}^*)$ (resp. for $\mathcal{R}(G)$). Thus, $\text{rank}(\mathcal{A}^*) = \text{rank}(G) = n$. Moreover, as well known, $\mathcal{R}(\mathcal{C}) \subset \mathcal{R}(\mathcal{B}^*) \cup \mathcal{R}(\mathcal{A}^*)$; but, from (4.36), we see that $\mathcal{R}(\mathcal{B}^*) \subset \text{span}\{\pi_{-,1}, \ldots, \pi_{-,n}\}$, therefore $\mathcal{R}(\mathcal{B}^*) \subset \mathcal{R}(\mathcal{A}^*)$ and thus $\mathcal{R}(\mathcal{C}) \subset \mathcal{R}(\mathcal{A}^*)$, which entails *(ii)*, and, under the hypothesis of *(iii)* implies $\mathcal{R}(\mathcal{C}) = \mathcal{R}(\mathcal{A}^*)$ as well. Therefore, the set $\pi_{-,1}, \ldots, \pi_{-,n}$ is a basis in $\mathcal{R}(\mathcal{C})$ as well. In order to complete the proof of *(iii)*, we follow an argument which uses sometimes standard steps: we here retain only those steps pertaining to the present proof. Since $\mathcal{R}(\mathcal{C})$ is invariant with respect to $\mathcal{A}^*$, and $\mathcal{R}(\mathcal{B}^*) \subset \mathcal{R}(\mathcal{C})$, we can write

(4.39) $$\mathcal{A}^*[\pi_{-,1}, \ldots, \pi_{-,n}] = [\pi_{-,1}, \ldots, \pi_{-,n}]\mathcal{A}',$$

(4.40) $$\mathcal{B}^* = [\pi_{-,1}, \ldots, \pi_{-,n}]\begin{bmatrix} \mathcal{B}^{\mathbf{c}}_{\mathbf{11}} & 0 \end{bmatrix}$$

for some matrix $\mathcal{A}' \in \mathbb{R}^{n \times 4n^2}$, and $\mathcal{B}^{\mathbf{c}}_{\mathbf{11}} \in \mathbb{R}^{n \times 2nq}$. Note in particular that, by (4.36), the matrix $\mathcal{B}^{\mathbf{c}}_{\mathbf{11}}$ is given by

(4.41) $$\mathcal{B}^{\mathbf{c}}_{\mathbf{11}} = \begin{bmatrix} b^{*T}_1 & 0 & \ldots & 0 \\ 0 & b^{*T}_2 & & \\ \vdots & & \ddots & \\ 0 & \ldots & & b^{*T}_n \end{bmatrix}.$$

Now, define

(4.42) $$T = [\pi_{-,1}, \ldots, \pi_{-,n}, \phi_{n+1}, \ldots, \phi_{4n^2}],$$

where the vectors ϕ_j, $j = n+1, \ldots, 4n^2$ are chosen such that T is invertible. Let us partition the matrix T in $4n \times 4n$ blocks, each $n \times n$ dimensioned:

(4.43) $$T = \begin{bmatrix} T_{1,1} & \ldots & T_{1,4n} \\ \vdots & \ddots & \vdots \\ T_{4n,1} & \ldots & T_{4n,4n} \end{bmatrix},$$

where by feedback property **P2** (see Remark 4.0.3) the block $T_{2,1}$ is invertible. By using T we see that identities (4.39), (4.40) implies

(4.44) $$\mathcal{A}^*T = T\mathcal{A}^{\mathbf{c}},$$

(4.45) $$\mathcal{B}^* = T\mathcal{B}^{\mathbf{c}}$$

where

(4.46) $$\mathcal{A}^c = \begin{bmatrix} \mathcal{A}^{\mathbf{c}}_{\mathbf{11}} & \mathcal{A}^{\mathbf{c}}_{\mathbf{12}} \\ 0 & \mathcal{A}^{\mathbf{c}}_{\mathbf{22}} \end{bmatrix}; \qquad \mathcal{B}^c = \begin{bmatrix} \mathcal{B}^{\mathbf{c}}_{\mathbf{11}} & 0 \\ 0 & 0 \end{bmatrix};$$

(4.47) $$\mathcal{A}^{\mathbf{c}}_{\mathbf{11}} \in \mathbb{R}^{n \times n}; \qquad \mathcal{B}^{\mathbf{c}}_{\mathbf{11}} \in \mathbb{R}^{n \times 2nq}.$$

Notice that $\text{rank}\{\mathcal{B}^{\mathbf{c}}_{\mathbf{11}}\} = q$, and thus there exists a matrix, $M \in \mathbb{R}^{2nq \times 2nq}$, such that

(4.48) $$\mathcal{B}^{\mathbf{c}}_{\mathbf{11}}M = \begin{bmatrix} \bar{B} & 0 \end{bmatrix}.$$

It is easy to check that a possible M is the following:

(4.49) $$M = \left.\begin{bmatrix} \bar{M} \\ \vdots \\ \bar{M} \end{bmatrix}\right\} \; n \text{ times}, \qquad \bar{M} = \begin{bmatrix} 0_q & I_q & 0_q & 0_q & \ldots & 0_q \\ I_q & 0_q & 0_q & 0_q & \ldots & 0_q \end{bmatrix},$$

where I_q (resp: 0_q) is the identity (resp: the zero matrix) in $\mathbb{R}^q$. As well known the couple $(\mathcal{A}_{11}^{\mathbf{c}}, \mathcal{B}_{11}^{\mathbf{c}})$ is controllable, which implies that $(\mathcal{A}_{11}^{\mathbf{c}}, \bar{B})$ is controllable as well [8], then there exists a matrix, say $\bar{K}_{11}^{\mathbf{c}} \in \mathbb{R}^{q \times n}$ such that $\mathcal{A}_{11}^{\mathbf{c}} + \bar{B}\bar{K}_{11}^{\mathbf{c}}$ has the n eigenvalues $\lambda_1, \ldots, \lambda_n$. It is easy to see that the matrix

$$(4.50) \qquad \mathbb{R}^{2nq \times n} \ni \mathcal{K}_{11}^{\mathbf{c}} = M \left[\begin{array}{c} \bar{K}_{11}^{\mathbf{c}} \\ K_{12}^{\mathbf{c}} \end{array} \right],$$

for any matrix $K_{12}^{\mathbf{c}} \in \mathbb{R}^{(2n-1)q \times n}$ assigns the same eigenvalues $\lambda_1, \ldots, \lambda_n$ to the matrix $\mathcal{A}_{11}^{\mathbf{c}} + \mathcal{B}_{11}^{\mathbf{c}}\mathcal{K}_{11}^{\mathbf{c}}$. By property $\mathbf{P}$ we can define

$$(4.51) \qquad K_{11}^{\mathbf{c}} = \bar{K}_{11}^{\mathbf{c}} T_{2,1}^{-1}.$$

Moreover let us choose the block $K_{12}^{\mathbf{c}}$ in (4.50) as follows

$$(4.52) \qquad K_{12}^{\mathbf{c}} = \left[\begin{array}{c} K_{11}^{\mathbf{c}} T_{1,1} \\ K_{11}^{\mathbf{c}} T_{4,1} \\ K_{11}^{\mathbf{c}} T_{3,1} \\ \vdots \\ K_{11}^{\mathbf{c}} T_{2n,1} \\ K_{11}^{\mathbf{c}} T_{2n-1,1} \end{array} \right]$$

and define the matrix

$$(4.53) \qquad \mathcal{K}^{\mathbf{c}} = \left[\begin{array}{cc} \mathcal{K}_{11}^{\mathbf{c}} & \mathcal{K}_{12}^{\mathbf{c}} \\ \mathcal{K}_{21}^{\mathbf{c}} & \mathcal{K}_{22}^{\mathbf{c}} \end{array} \right] \in \mathbb{R}^{4nq \times 4n^2}.$$

We have

$$(4.54) \qquad \mathcal{A}^{\mathbf{c}} + \mathcal{B}^{\mathbf{c}}\mathcal{K}^{\mathbf{c}} = \left[\begin{array}{cc} A_{11}^{\mathbf{c}} + \mathcal{B}_{11}^{\mathbf{c}}\mathcal{K}_{11}^{\mathbf{c}} & A_{12}^{\mathbf{c}} + \mathcal{B}_{11}^{\mathbf{c}}\mathcal{K}_{12}^{\mathbf{c}} \\ 0 & \mathcal{A}_{22}^{\mathbf{c}} \end{array} \right]$$

and thus, for any $\mathcal{K}_{12}^{\mathbf{c}}, \mathcal{K}_{21}^{\mathbf{c}}, \mathcal{K}_{22}^{\mathbf{c}}$ the matrix $\mathcal{A}^{\mathbf{c}} + \mathcal{B}^{\mathbf{c}}\mathcal{K}^{\mathbf{c}}$ has $\lambda_1, \ldots, \lambda_n$ among its $4n^2$ eigenvalues. Now, by (4.37), (4.33), and (4.43) we have:

$$(4.55) \qquad \mathcal{K}^* T = \left[\begin{array}{ccc} K^2 T_{1,1} & \ldots & K^2 T_{1,4n} \\ \vdots & \ddots & \vdots \\ K^2 T_{4n,1} & \ldots & K^2 T_{4n,4n} \end{array} \right].$$

By (4.53) and (4.55), since the matrices $\mathcal{K}_{12}^{\mathbf{c}}, \mathcal{K}_{21}^{\mathbf{c}}, \mathcal{K}_{22}^{\mathbf{c}}$ are arbitrary, taking into account of (4.50)–(4.52), and of the structure of M, given in (4.49), we have:

$$\mathcal{K}^{\mathbf{c}} = \mathcal{K}^* T \quad \Leftrightarrow \quad \mathcal{K}_{11}^{\mathbf{c}} = \left[\begin{array}{c} K^2 T_{1,1} \\ \vdots \\ K^2 T_{2n,1} \end{array} \right]$$

$$\Leftrightarrow \quad \left[\begin{array}{c} K^2 T_{1,1} \\ K^2 T_{2,1} \\ \vdots \\ K^2 T_{2n-1,1} \\ K^2 T_{2n,1} \end{array} \right] = M \left[\begin{array}{c} K_{11}^{\mathbf{c}} T_{2,1} \\ K_{11}^{\mathbf{c}} T_{1,1} \\ \vdots \\ K_{11}^{\mathbf{c}} T_{2n,1} \\ K_{11}^{\mathbf{c}} T_{2n-1,1} \end{array} \right] = \left[\begin{array}{c} K_{11}^{\mathbf{c}} T_{1,1} \\ K_{11}^{\mathbf{c}} T_{1,2} \\ \vdots \\ K_{11}^{\mathbf{c}} T_{2n-1,1} \\ K_{11}^{\mathbf{c}} T_{2n,1} \end{array} \right]$$

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[8]This readily comes from the fact that the columns of the controllability matrix of the latter couple are linear combination of the controllability matrix of the former.

and thus, setting $\mathcal{K}^{\mathbf{c}} = \mathcal{K}^* T$ and

$$(4.56) \qquad\qquad K^2 = K^{\mathbf{c}}_{\mathbf{11}},$$

the matrix (4.54) shall have the prescribed eigenvalues. Since

$$(4.57) \qquad\qquad G = \mathcal{A}^* + \mathcal{B}^* \mathcal{K}^* = T(\mathcal{A}^{\mathbf{c}} + \mathcal{B}^c \mathcal{K}^{\mathbf{c}}) T^{-1},$$

the matrix (4.54) is similar to the driver generator G, which concludes the proof. $\square$

5. Synthesis of global exponential regulators with tunable rate. In this section we describe a design procedure for a feedback controller that can be always applied to a certain class of original $\sigma\pi$-systems, the so called class of $\sigma\pi$-*controllable* systems that we are going to define, and a few sufficient conditions that guarantee that the feedback controller asymptotically stabilize the original system with an exponential, and tunable, convergence rate, in all the region $\mathcal{S} \subset \mathbb{R}^n$ of the state space from which the closed loop system is *forward complete* that is to say: the set of all $x \in \mathbb{R}^n$ such that the closed-loop system trajectory $x(t)$ such that $x(t_0) = x$, is defined on all the interval $[t_0, +\infty)$[9]

Let us consider an n-dimensional $\sigma\pi$-system characterized by a couple of matrices (A, B) as in (4.16). This means that a feedback in the class (3.9) has been applied to an object system as in (2.1), according to Theorem 4.1. We call this feedback, characterized by the $q \times n$ matrix K appearing in (4.16), the *internal loop*. Next, enlarge the system size and order up to $2n$, and modify the feedback in such a way the new system is characterized by a couple A, B given by (4.29) and a feedback matrix K given by (4.30), where the old A (resp B) has been renamed $\bar{A}$ (resp. $\bar{B}$). The new K, $q \times 2n$-dimensional, has the structure described in (4.30), where K^1 is the old K, and K^2 defines a new feedback that we name *the outer feedback*. As we have already seen in the previous section, the monomials of the new system can be always chosen in such a way that property $\mathbf{P}$ is verified: hereinafter we do understand that that property $\mathbf{P}$ is satisfied. As yet, the term 'object system' and 'original system' has been used as synonyms, hereinafter we differentiate the meanings and call 'object system' the new system with enlarged size and order $2n$, whereas the term 'original' will be reserved to the n-dimensional system before the enlargement, characterized by the couple $\bar{A}, \bar{B}$.

The original and the object systems, considered in open-loop, i.e. identified with maps $\Sigma_{\bar{x}} : u \mapsto x$, $\bar{x}$ belonging to a set of allowable 'initial states', are equivalent, in the sense that the object system has n state variables following the same trajectories as the n state variables of the original system, whereas the remaining n state variables are identically equal to one. However, in closed-loop they are different, as the object system has an *additional* feedback, given by the outer feedback defined above. Hereinafter, as we talk about original or object systems we understand the underlying feedback, and thus the object system is the real closed-loop system, whereas the original system has to be though of as the open-loop system to which only the internal feedback has been applied, and is characterized by the couple of matrices $\bar{A}, \bar{B}$, *as if it were an open-loop linear system*, except that the matrix $\bar{B}$ is actually defined only *after* the insertion of the internal feedback.

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[9]In other words: the controller gives place to a closed-loop system whose trajectories converge exponentially to zero, provided just they are defined for any $t_0 \leq t < +\infty$.

5.1. Statement of the regulation problem. Paying attention to the terminology above introduced, let be given an open-loop $\sigma\pi$-system and an original system characterized by the couple $\bar{A}, \bar{B}$. In following definition we give the main assumption under which the regulator design method we are going to describe holds.

DEFINITION 5.1. *We say that an original system (associated to some open-loop $\sigma\pi$-system through some internal feedback) is $\sigma\pi$-controllable, if the couple $\bar{A}, \bar{B}$ is controllable, i.e. the controllability matrix $[\bar{B}, \bar{A}\bar{B}, \ldots \bar{A}^n\bar{B}]$ has rank equal to n.*

The *object* system is the system we want to *regulate* in a certain set, $\mathcal{S} \subset \mathbb{R}^n$ of the *original* state space. This means: we want to find a feedback $K = [K^1, K^2]$ – i.e. an internal feedback K^1, and an outer feedback K^2 – such that the closed-loop system has the first n entries[10] converging asymptotically to zero, for any initial state having the first n-entries[11] in the set $\mathcal{S}$. By the way, we recall that for $\sigma\pi$-systems the zero, even thought could not belong to the original system domain, is always an adherence point of this domain, and thus convergence to zero is always a well defined notion in our framework, no matter which the *physical meaning* is for such a notion [12]. Also, we point out that a linear change of coordinates in the state space does not turn, in general, a $\sigma\pi$-system into a another $\sigma\pi$-system, but into an *algebraic* system (see [1]). Thus, the convergence-to-zero criterion is not a *general* way, for $\sigma\pi$-systems, for setting the *target point* of a regulator. The general case of a non-zero target point for a $\sigma\pi$-regulator is in general a quite different problem that will be considered in future works.

5.2. Main result. We give the main result of this paper in the theorem below. Before stating the theorem, we need to recall some facts and give some further definition. First of all, recall (cf. Part I of this paper) that, as it is in general for any $\sigma\pi$-system, any monomial $X_{i,i'}$ of the object system is a continuous time function such that, for any $x \in \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^{2n}$ is the maximal open domain where $Z = Z(x)$ and $X = X(x)$, are well defined C^∞ functions of x, there exist time points $t_0, T \in \mathbb{R}$, $t_0 < T \le +\infty$, such that if $X_{i,i'}(t_0) = x$ then

$$(5.1) \qquad X_{i,i'}(t) = e^{\int_{t_0}^t \gamma_{i,i'}(Z(\tau))d\tau} X_{i,i'}(t_0)$$

for any $t \in [t_0, T)$, where $\gamma_{i,i'}(\cdot)$ is the following (scalar) function of the driver state Z:

$$(5.2) \qquad \gamma_{i,i'}(Z) = \sum_{j=1}^{2n} p_{i,j}^{i'} Z_j^T v_j$$

Also notice that, since the object system is characterized by the matrix $A + BK$ given in (4.31), the monomials $X_{i,i'}$'s appear in fictitious terms of the type $0 \cdot X_{i,i'}$ in the last n equations of the object system, and since $x_i \equiv 1$ for any $i = n+1, \ldots, 2n$, we have for the corresponding drivers components $Z_{i,i'}$:

$$(5.3) \qquad Z_{i,i'} = \frac{X_{i,i'}}{x_i} = X_{i,i'} \qquad \forall i = n+1, \ldots, 2n: \quad \forall i' = 1, \ldots, 2n$$

---

[10]Recall that the last n entries of the object system are identically equal to one.

[11]Therefore, the last n entries of the initial state are always one.

[12]For at least all $\sigma\pi$-systems having nonnegative exponents, and no monomials with all zero exponents (identically one), the physical meaning is the ordinary notion of equilibrium point.

That is: *all monomials of the object system are equal to the last n drivers components.*

To the object system we associate the subset $\mathcal{S} \subset \mathbb{R}^n$ defined as follows: let $x(t) \in \mathbb{R}^{2n}$ the solution of the object system such that $x(t_0) = x$, where $x \in \mathbb{R}^{2n}$ is such that $x_j = 1$, for $j = n+1, \ldots, 2n$, and $T(x) \in \mathbb{R} \cup \{+\infty\}$ such that $x(t)$ is defined $\forall t \in [t_0, T(x))$, then

$$(5.4) \qquad\qquad \mathcal{S} = \{x' \in \mathbb{R}^n : \quad T(x) = +\infty\},$$
$$(5.5) \qquad\qquad x' = [x_1, \ldots, x_n]^T,$$

Note that the set $\mathcal{S}$ above defined depends of the feedback K.

THEOREM 5.2. *Suppose that the following hypotheses are satisfied:*

(i) the original system is $\sigma\pi$-controllable[13];

(ii) the controllability matrix associated to the matrices $\mathcal{A}^*, \mathcal{B}^*$ defined in (4.35) (4.36) has rank n;

then, there exists a K such that the (immersed[14]) driver of the object system has all trajectories defined on some interval $[\bar{t}, +\infty)$, and

$$(5.6) \qquad \lim_{t \to +\infty} Z(t) = Z^\infty \in \mathbb{R}^{4n^2},$$
$$(5.7) \qquad \lim_{t \to +\infty} \gamma_{i,i'}(Z(t)) = \gamma_{i,i'}^\infty = \gamma_{i,i'}(Z^\infty) \in \mathbb{R}, \qquad \forall i, i' = 1, \ldots, 2n,$$

where $\gamma_{i,i'}$ are defined in (5.2), the constant vectors $Z^\infty = Z^\infty(x)$, $\bar{\gamma} = \gamma(x)$, depend in general of $x = x(t_0)$, and the convergence occurs at a whatever fixed exponential rate.

If (5.6) holds then under the additional hypoteses:

(iii) $Z_{i,i'}^\infty \neq 0$, for $i = 1, \ldots, n$;

(iv) $\gamma_{i,i'}^\infty \neq 0$; for $i = n+1, \ldots, 2n$;

the object system is exponentially asymptotically stable on $\mathcal{S}$ in the first n components x', that is: if $x'(t)$ is the vector collecting the first n components of the object system solution with initial point $x'(t_0) = \tilde{x}$, then $x'(t)$ is defined $\forall \tilde{x} \in \mathcal{S}$, $\forall t \in [t_0, +\infty)$ and

$$(5.8) \qquad\qquad \lim_{t \to +\infty} x'(t) = 0, \qquad \forall \tilde{x} \in \mathcal{S},$$

at the same exponential rate as the driver, whereas, if $x'(t_0) \notin \mathcal{S}$, $x'(t)$ blows up in a finite time.

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[13]For simplicity we assume that $\bar{B}$ has full (column) rank, and $q < n$. The extension to the general case entails just a more complex calculation.

[14]We remind that the immersed driver is the subsystem of the driver equations that includes only the trajectories that lie in the image of the immersion (cf. Part I).

Proof. Let G be the generator of the driver of the object system. Obviously (cf. Part I) any driver, of some given $\sigma\pi$-system, is a self-driver, and thus, from Theorem 5.3 of Part I, in order to guarantee the existence of the immersed driver solution $Z(t)$ on some right unbounded interval $[\bar{t}, +\infty)$ we have to look at the bias at a pivoted point of the biased driver. In the present case, denoted z the solution of the bilinear frame[15] i.e.: $\dot{z} = Gz$, z is a vector in $\mathbb{R}^{4n^2}$ whose components can be organized as the driver components $Z_{i,i'}$. Thus, we address a component of z by a double index $(i,j) = (1,1), \ldots, (4n^2, 4n^2)$. The bias at z (see eq. (5.1) of Part I) is a function of two entries of the flow $z(t)$, of vector field Gz, passing through z, and thus in the present case is a function of a double double-index, say (i,j) and (i',j') and is given by:

$$(5.9) \qquad \Psi_{i,j}^{i',j'}(z)(t) = -\frac{z_{i',j'}(t)}{z_{i,j}^2(t)}\dot{z}_{i,j}(t).$$

By Lemma 4.2, G has a zero eigenvalue with algebraic multiplicity at least equal to $4n^2 - n$, and a K can be found such that the remaining n eigenvalues are n whatever chosen, real numbers, namely $\lambda_1^*, \ldots, \lambda_n^*$[16]. Now, let $K = [K^1, K^2]$ such that $\text{rank}(G) = n$ – for which, by Lemma 4.2, it is sufficient that K^1 makes invertible $\bar{A} + \bar{B}K^1$ – and the λ_k^*'s are all negative, which is achieved of course for some K^2. Then the zero eigenvalue has algebraic multiplicity equal to $4n^2 - n$, and since $\mathcal{R}(G)$ has dimension n, the geometric multiplicity [17] of the zero eigenvalue is equal to $4n^2 - n$ as well. Therefore, we can write the flow $z(t)$ as

$$(5.10) \qquad z(t) = \sum_{k=1}^{n} e^{\lambda_k^*(t-t_o)}\mathbf{u}_k\mathbf{v}_k^T z + \sum_{k'=1}^{4n^2-n} \mathbf{u}'_{k'}\mathbf{v}'^T_{k'}z,$$

where $\mathbf{u}_k$, (resp. $\mathbf{v}_k$) $k = 1, \ldots, n$, $k \neq k^*$ are $n-1$ eigenvectors (resp. are $n-1$ left-eigenvectors) associated to λ_k^*, $\mathbf{u}'_{k'}$, (resp. $\mathbf{v}_k$), $k' = 1, \ldots, 4n^2 - n$ form a base for $\text{Ker}(G)$ (resp. form a base for $\text{Ker}(G^T)$). As well known, the set of all eigenvectors: $\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}'_1, \ldots, \mathbf{u}'_{4n^2-n}\}$ forms a base in $\mathbb{R}^{4n^2}$ and

$$(5.11) \qquad [\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}'_1, \ldots, \mathbf{v}'_{4n^2-n}]^T = [\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}'_1, \ldots, \mathbf{u}'_{4n^2-n}]^{-1},$$

which used in (5.10) yields $z(t_0) = z$, as expected, and the scalar terms $\mathbf{v}_k^T z$, $\mathbf{v}'^T_{k'}z$ are the components of z in the base of eigenvectors. The flow $z(t)$ written component wise is

$$(5.12) \qquad z_{i',j'}(t) = \sum_{k=1}^{n} e^{\lambda_k^*(t-t_o)}(\mathbf{v}_k^T z)\mathbf{u}_{k,i',j'} + \chi_{i',j'}$$

$$(5.13) \qquad \chi_{i',j'}(z) = \sum_{k'=1}^{4n^2-n}(\mathbf{v}'^T_{k'}z)\mathbf{u}'_{k',i',j'}.$$

---

[15] Notice that the bilinear frame, defined in Part I of the paper, in the case of closed-loop systems, as it is the present case, is indeed a *linear* frame.

[16] Actually many of such K's can be found: for any K^1 such that $\bar{A} + \bar{B}K^1$ is invertible, we can find a K^2, such that $\lambda_1^*, \ldots, \lambda_n^*$ are n eigenvalues of G.

[17] In this paper, for 'geometric multiplicity' of an eigenvalue, say λ_i, of a matrix G, we mean the dimension of kernel of $G - \lambda_i I$, thus following the definition more usual in the literature. Another definition that can be found in literature for 'geometric multiplicity' is: the multiplicity of λ_i in the minimal polynomial annihilating G (see for instance [5]). It should be stressed that the two definitions are *not* equivalent.

Note that, if $z \neq 0$, as the right-eigenvectors matrix in (5.11) is invertible, the scalar terms $\mathbf{v}_k^T z^{i,j}, \mathbf{v'}_{k'}^T z^{i,j}$ cannot be zero $\forall k, k'$. Also note that $\chi(z)$ stands for the projection[18] of z onto the kernel of G. Following Theorem 5.6 of Part I, let (i, j) be a pivot index, and let $z^{i,j} \in \mathrm{I\!R}^{4n^2}$ be the pivot point at (i, j), defined as

$$(5.14) \qquad z_{i',j'}^{i,j} = \alpha \bar{z}_{i',j'}^{i,j} \ ,$$

$$(5.15) \qquad \bar{z}_{i',j'}^{i,j} = \begin{cases} Z_{i',j'}(t_0) & \text{for } (i', j') \neq (i, j); \\ 1 & \text{otherwise;} \end{cases} ,$$

where α is any nonzero real constant, Z is the state of the immersed driver – hence $Z_{i',j'}(t_0) \neq 0, \forall i', j'$ – and thus by choosing $z = z^{i,j}$ in (5.12), $z_{i',j'}(t)$ are not identically zero for all i', j', and there exists a t^* such that $z_{i',j'}(t) \neq 0 \ \forall t \in [t^*, +\infty)$. Keeping to follow Theorem 5.3 of Part I, the (i, j)-biased solution, namely $\zeta_{i,j}^{i',j'}$, is defined on $[t^*, +\infty)$, and is given by

$$(5.16) \qquad \zeta_{i,j}^{i',j'} = \frac{z_{i',j'}(t)}{z_{i,j}(t)}.$$

Now, let us prove the following Claims *C1*, and *C2*:

Claim C1: *it is always possible to set the pivot index to an (i, j) such that the projection of the pivoted point $z^{i,j}$ onto $\mathrm{Ker}(G)$, i.e. $\chi(z^{i,j})$, has a non zero (i, j)-th component, i.e. for any $z \neq 0$, $\exists (i, j)$ such that*

$$(5.17) \qquad \chi_{i,j}(z^{i,j}) \neq 0.$$

Claim C2: *there are at least two of such pivot indices.*

In order to show Claim *C1*, first of all note that, for any $z \neq 0$, the corresponding pivot $z^{i,j}$ has always a non zero component along (i, j) – i.e. α – whatever the pivot index (i, j) has been chosen. Let $\mathbf{e}_{i,j}$ stand for the element of the canonical base along the (i, j) axis, then the vector component along (i, j) of $z^{i,j}$ is $\alpha \mathbf{e}_{i,j}$. The projection of $z^{i,j}$ on $\mathrm{Ker}(G)$ is an orthogonal projection, and thus it is a summation of terms including $\alpha \mathbf{e}_{i,j}^*$, with $\mathbf{e}_{i,j}^*$ denoting the projection $\mathbf{e}_{i,j}$ onto $\mathrm{Ker}(G)$. Now, $\mathbf{e}_{i,j}^*$ can be zero if and only if

$$(5.18) \qquad \mathrm{Ker}(G) \subset \mathrm{span}\{\mathbf{e}_{i',j'} : (i', j') \neq (i, j)\} = \mathcal{H}_0.$$

If $\mathrm{Ker}(G)$ satisfies (5.18) for some (i, j) then let us choose the pivot into the set $\{(i', j') : (i', j') \neq (i, j)\}$, and let it be denoted by (i_1, j_1). Then $\mathbf{e}_{i_1,j_1}^*$ can be zero if and only if

$$(5.19) \qquad \mathrm{Ker}(G) \subset \mathrm{span}\{\mathbf{e}_{i',j'} : (i', j') \neq (i_1, j_1), (i, j)\} = \mathcal{H}_1.$$

In this way we build up a finite sequence $\mathbf{e}_{i,j}^*, \mathbf{e}_{i_1,j_1}^*, \ldots, \mathbf{e}_{i_l,j_l}^*$, that can be a sequence of zero vectors if and only if

$$(5.20) \qquad \mathrm{Ker}(G) \subset \mathrm{span}\{\mathbf{e}_{i',j'} : (i', j') \neq (i, j), (i_1, j_1), \ldots, (i_l, j_l)\} = \mathcal{H}_l.$$

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[18]Which is indeed the *orthogonal* projection, as we have $z = \chi(z) + z^*$ in a unique way, where z^* is the vector component along $\mathrm{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = \mathcal{R}(G)$.

Therefore, Claim *C1* is proven as soon as one notice that $\dim\{\mathcal{H}_l\} = 4n^2 - l - 1$, whereas $\text{Ker}\{G\}$ has dimension $4n^2 - n$, and thus (5.20) is falsified at most for $l = n$. Moreover, once a pivot (i, j) satisfying (5.17) has been found, since $\text{Ker}\{G\}$ is at least three-dimensional, we can skip it and apply the above procedure until another pivot satisfying (5.17) is found. This proves Claim *C2* as well.

Since $\lambda_k^* \leq 0$, $k = 1, \ldots, n$, by (5.12) we have

$$(5.21) \qquad \lim_{t \to +\infty} z_{i',j'}(t) = \chi_{i',j'}(z^{i,j}).$$

Moreover:

$$(5.22) \qquad \dot{z}_{i,j}(t) = \sum_{k=1}^{n} \lambda_k^* e^{\lambda_k^*(t - t_o)}(\mathbf{v}_k^T z^{i,j})\mathbf{u}_{k,i,j} \to 0, \quad \text{for} \quad t \to +\infty.$$

From (5.12), (5.17) we have $z_{i,j}^2(t) \to \chi_{i,j}^2(z^{i,j}) \neq 0$, and thus, taking into account of (5.22) we have from (5.9) that the bias $\Psi_{i,j}^{i',j'}(t)$ is defined $\forall t \in [t^*, +\infty)$, and

$$(5.23) \qquad \lim_{t \to +\infty} \Psi_{i,j}^{i',j'}(z)(t) = 0.$$

In conclusion we have $\zeta_{i,j}^{i',j'} - Z_{i',j'} \to 0$ for $t \to +\infty$, and thus there exists a left-bounded interval $[\bar{t}, +\infty)$ with $\bar{t} \geq t^*$ in which $Z_{i',j'}(t)$ is well defined. By using (5.16), and (5.21), on account of (5.17) we have, for $(i', j') \neq (i, j)$:

$$(5.24) \qquad \lim_{t \to +\infty} Z_{i',j'} = \lim_{t \to +\infty} \zeta_{i,j}^{i',j'} = \frac{\chi_{i',j'}(z^{i,j})}{\chi_{i,j}(z^{i,j})} = Z_{i',j'}^{\infty} < \infty.$$

Calling for another pivot satisfying (5.17), whose existence is guaranteed by Claim *C2*, we obtain the steady state value for $Z_{i,j}$ as well.

In order to prove (5.8), first of all notice that if *(iv)* holds, then $\gamma_{i,i'}^{\infty} < 0$. Indeed, if $\gamma_{i,i'}^{\infty} > 0$, by (5.1) we would have $X_{i,i'} \to +\infty$, for $i = n+1, \ldots, 2n$, which contradicts (5.6) and (5.3). Then, recalling that the object system is aligned, that is $X_{i,i'} = X_{j,i'}$, $\forall j$, we have $Z_{i,i'} = X_{\bar{i},i'}/x_i$, for $i = 1, \ldots, n$, where $\bar{i}$ is any index $n + 1 < \bar{i} \leq 2n$. Finally, by hypothesis *(iii)*, since $X_{\bar{i},i'} \to 0$, then necessarily the orbit $x'(t)$ of the (first n components of the) object system, is defined (and, hence, continuoys) for $t \in [\bar{t}, +\infty)$ and $x'(t) \to 0$ at the same rate of $X_{\bar{i},i'}$ and, hence, at the same rate of Z. From the well know properties of the solution of ODEs, for such a continuous component of the orbit there are only two cases: either there is a component, say $1 \leq i \leq n$, that comes from infinity, e.g. $x_i'(t) \to \pm\infty$ for $t \to \underline{t}^+$ (from the right), for some $\underline{t}$ such that $\mathbb{R} \cup \{-\infty\} \ni \underline{t} \leq \bar{t}$, or there is an $x \in \mathbb{R}^n$ such that $x'(t_0) = x$, and $x'(t)$ is defined (and continuous) $\forall t \in [t_0, +\infty)$[19]. The latter case amounts to (5.8), and thus, if $x'(t_0) \notin \mathcal{S}$, the solution blows up in finite time. □

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[19]Defined and continuous, obviously, in the object system domain. In the *abode* system domain (cf Part I §2) it may well happen that $x'(t)$ is undefined on some countable set $\mathbf{T} \subset (\mathbf{t_0}, +\infty)$ corresponding to the times in which the orbit *crosses* the *non-common* border between the abode system domain and the object system domain. Such common border, as we have seen in Part I, is constituted by (possibly a subset of) the union of the coordinate hyperplanes in the lifted state-space.

5.2.1. Practical implementation. As for the *gain* $K = [K^1, K^2] \in \mathbb{R}^{q \times 2n}$, of which Theorem 5.2 gives the guidelines for the setting, some features should be kept in mind. We summarize these features as follows: first of all, $K^1 \in \mathbb{R}^{q \times n}$, as required by Lemma (4.2), has to be fixed to a value such that the matrix $\bar{A} + \bar{B}K^1$ is invertible, which is always possible under the hypothesis of Theorem 5.2 (controllability of $(\bar{A}, \bar{B})$), since we can fix all the n eigenvalues of the matrix[20]. Then, following the steps of the proof of Lemma 4.2 (it is indeed a constructive proof) we fix $K^2 \in \mathbb{R}^{q \times n}$ in order to fix to whatever chosen n values the eigenvalues of the matrix $\mathcal{A}_{\mathbf{11}}^{\mathbf{c}} + \bar{B}\bar{K}_{\mathbf{11}}^{\mathbf{c}}$, where $\mathcal{A}_{\mathbf{11}}^{\mathbf{c}} \in \mathbb{R}^{n \times n}$ is read out from (4.46), and $\bar{K}_{\mathbf{11}}^{\mathbf{c}}$ is a certain gain-matrix related to K^2, as described in the proof of Lemma 4.2. Such an eigenvalues assignment is always possible provided the hypotheses of the Lemma are satisfied. Moreover, provided we assign zero to one eigenvalue, and choose negative (and different from each other, for simplicity) the other eigenvalues, we steer all drivers components – and, hence, all the object system monomials – to constant values at steady state. Finally, provided that the additional hypotheses *iii)* and *iv)* of Theorem 5.2 are satisfied, the regulator actually works, at an exponential rate, steering the system state to zero *from any initial state such that the corresponding system trajectory does not blow up in finite time.*

A practical way for implementing the design method above described can be the following. First of all notice that the procedure entails that $K^2 = K^2(K^1)$, i.e. K^2 is a function of K^1. In other words, the feedback regulator can be found by leaving the entries of K^1 'free' variables, within the set of values:

$$(5.25) \qquad \mathcal{K}^1 = \{K^1 \in \mathbb{R}^{q \times n}: \quad \mathrm{Ker}\{\bar{A} + \bar{B}K^1\} \neq \{0\}\},$$

which is the set of K^1-values such that $\bar{A} + \bar{B}K^1$ has not all zero eigenvalues. Thus, at first, the designer shall have to set $K^1(K^2)$ in order to guarantee the required speed of response (i.e. the overall exponential rate of convergence). Next the regulator design has to be fine tuned by moving – the simpler way is by tentatives – K^1 in the set $\mathcal{K}^1$ defined in (5.25) in order to assure that the conditions *iii)* and *iv)* of Theorem 5.2 are satisfied[21]. In order to complete the design, the designer shall have finally to perform an analysis on the closed-loop system equations – on the final object system with K fixed – in order to determine which is the set of initial points for which the system has a bounded response, i.e. the set $\mathcal{S}$ defined in (5.4), which represent the 'domain of attraction' of the zero of the stabilized system.

6. An example. The regulator design method described in the previous sections has in general a low computational burden. However, without suitable dedicated software tools for symbolic calculation, it might give rise to cumbersome calculations, if one has to carry them out by hand, as well as a uselessly lengthy exposition. The example we propose below, has the quality of being not too heavy, as for the calculation, and at the same time it give rise to most of the critic features related to the method. So, there isn't any theoretical or computational problem in testing a more general system, but an increase in the length of exposition.

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[20]In order that $\bar{A} + \bar{B}K^1$ be invertible, it is sufficient that there isn't a zero eigenvalue. The eigenvalues could be chosen even with positive real part, so making *unstable* the original system.

[21]Up to now we are not able to say whether such conditions can be always satisfied. Thus, the method described still belongs to a category of 'methods by tentatives'. Nevertheless such a method by tentatives is not unrelated with control-system history, and indeed has a similarity with root-locus based design method for linear regulators.

Let us consider the following system in $\mathbb{R}^2$:

$$\dot{x}_1 = x_2^2 \tag{6.1}$$

$$\dot{x}_2 = u, \tag{6.2}$$

which is a well know one in the nonlinear systems literature, as it is *intuitively* controllable to zero from any initial condition in the left open orthant: $\mathcal{O}_2 = \{(x_1, x_2) : x_2 < 0\}$, but for sure is *not* controllable to zero from points in the right orthant. Indeed, the first equation tells us that, whatever be u, x_1 has a non negative derivative, and thus it can only increase. Moreover, it is easy to check – by using classical methods, or our accessibility test presented in §1 – that this system is accessible *at any point* of $\mathbb{R}^2$. We now show that, by building up a feedback following the guidelines described in §4, the closed-loop system satisfies all requirements of Theorem 5.2 and thus we can find a gain that exponentially stabilizes the equilibrium zero. What we have to expect is that the set $\mathcal{S}$ defined in (5.4) will be *included* in the left open orthant $\mathcal{O}_2$. As a matter of fact we will see that even $\mathcal{S} = \mathcal{O}_2$ holds, and thus we can say that the feedback *globally* stabilizes the zero, in the sense that the zero becomes an attractor of all points that *in principle* are not ruled out as candidates to be steered to zero, in other words: the largest performance that can be theoretically achieved for system (6.1), (6.2), is actually achieved, on account that the convergence rate is even tunable, and can be arbitrarily fixed.

Following the construction described in §4, we associate an object system to (6.1), (6.2), characterized by a pair A, B having the structure (4.29), in the following way. First, let us enlarge the system size and order up to 4, by introducing the new variables x_3, x_4 satisfying zero equations starting from the unity, and new monomials $x_1, x_1^{\gamma_1}, x_2^{\gamma_2}$, where $\gamma_1, \gamma_2 \in \mathbb{R}$ are to be determined later. For the sake of presentation we show all fictitious monomial used:

$$\dot{x}_1 = 0 \cdot x_1 + x_2^2 + 0 \cdot x_1^{\gamma_1} + 0 \cdot x_2^{\gamma_2} \tag{6.3}$$

$$\dot{x}_2 = u, \tag{6.4}$$

$$\dot{x}_1 = 0 \cdot x_1 + 0 \cdot x_2^2 + 0 \cdot x_1^{\gamma_1} + 0 \cdot x_2^{\gamma_2} \qquad x_3(0) = 1 \tag{6.5}$$

$$\dot{x}_2 = 0 \cdot x_1 + 0 \cdot x_2^2 + 0 \cdot x_1^{\gamma_1} + 0 \cdot x_2^{\gamma_2} \qquad x_4(0) = 1 \tag{6.6}$$

Then, let us consider the feedback:

$$u = k_1 x_1 + k_2 x_2^2 + k_3 x_1^{\gamma_1} + k_4 x_2^{\gamma_2} \tag{6.7}$$

Keeping on §4, after (4.29), by substituting (6.7) into (6.4), we obtain the object system, characterized by:

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad K = \begin{bmatrix} K^{1T} & K^{2T} \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \tag{6.8}$$

As for the matrices A^*, B^*, K^*, related to A, B, K through (4.31)–(4.33), in the present case they are:

$$A^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_1 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad K^* = \begin{bmatrix} k_3 & k_4 & 0 & 0 \\ 0 & 0 & k_3 & k_4 \end{bmatrix} \tag{6.9}$$

Here's a few sentences useful as a mnemonic for the exponents $\pi_{i,j}^{i'}$, and $p_{i,j}^{i'}$:

a) $p_{i,j}^{i'}$ is the exponent of the variable x_j in the i'-th monomial of the i-th equation.

b) $\pi_{i,j}^{i'} = p_{i,j}^{i'}$, except for $i = j$, which yields $\pi_{i,i}^{i'} = p_{i,i}^{i'} - 1$.

c) $\pi_{-,j}$ is the stack of $\pi_{i,j}$, $i = 1, \ldots, 2n$. $\pi_{i,j}$ is the column vector aggregating $\pi_{i,j}^{i'}$, for $i' = 1, \ldots, 2n$.

By keeping in mind the above mnemonics, it is easy to read out[22] from (6.3)–(6.6), with (6.7) replacing u, the vectors $\pi_{-,1}, \pi_{-,2} \in \mathbb{R}^{16}$ pertaining (4.34) (4.36):

(6.10) $\qquad \pi_{-,1} = [0, -1, \gamma_1 - 1, -1, 1, 0, \gamma_1, 0, 1, 0, \gamma_1, 0, 1, 0, \gamma_1, 0]^T$,

(6.11) $\qquad \pi_{-,2} = [0, 2, 0, \gamma_2, -1, 1, -1, \gamma_2 - 1, 0, 2, 0, \gamma_2, 0, 2, 0, \gamma_2]^T$.

Let us choose γ_1, γ_2 in such a way that the matrix $T_{2,1}$, given in (4.38) is invertible. In the present example, the matrix $T_{2,1} \in \mathbb{R}^{2 \times 2}$ is

$$(6.12) \qquad T_{2,1} = \begin{bmatrix} \gamma_1 - 1 & 0 \\ -1 & \gamma_2 \end{bmatrix},$$

The choice $\gamma_2 = 1$, $\gamma_1 = 2$, satisfies the requirement, and thus property **P2** of Remark 4.0.3 is satisfied, and we have

$$(6.13) \qquad T_{2,1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \qquad T_{2,1}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

By using the chosen values of γ_1, γ_2 in (6.10), (6.11), and on account of (4.34), we readily calculate $\mathcal{A}^*$:

$$(6.14) \; \mathcal{A}^* = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 2k_1 & 2k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & k_1 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -k_1 & -k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_1 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & -k_1 & -k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2k_1 & 2k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_1 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2k_1 & 2k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_1 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

as well as $\mathcal{B}^* \in \mathbb{R}^{16 \times 8}$, which we write shortly as

$$(6.15) \qquad \mathcal{B}^* = \begin{bmatrix} 0 & 0 & 0 & \pi_{-,2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

---

[22]It's worth noting that, if for some j, x_j do not appear in a monomial, say the i'-th monomial of the i-th equation, then $p_{i,j}^{i'} = 0$.

where $'0'$ stands for the zero vector in $\mathbb{R}^{16}$, and $\mathcal{K}^* \in \mathbb{R}^{8 \times 16}$:

$$(6.16) \qquad \mathcal{K}^* = \begin{bmatrix} K^* & 0 & 0 & 0 \\ 0 & K^* & 0 & 0 \\ 0 & 0 & K^* & 0 \\ 0 & 0 & 0 & K^* \end{bmatrix} \qquad K^* = \begin{bmatrix} k_3 & k_4 & 0 & 0 \\ 0 & 0 & k_3 & k_4 \end{bmatrix}$$

It is easy to check that the controllability matrix $\mathcal{C}$, associated to $\mathcal{A}^*$, $\mathcal{B}^*$, has rank n ($= 2$), [23] for any value of k_1 and k_2. The matrix $\mathcal{A}^{\mathbf{c}}_{\mathbf{11}} \in \mathbb{R}^{2 \times 2}$, namely:

$$(6.17) \qquad \mathcal{A}^{\mathbf{c}}_{\mathbf{11}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

which is the top right block of $\mathcal{A}^{\mathbf{c}}$ in (4.46), and satisfying identity (4.44), can be calculated by using (4.45):

$(6.18) \quad -\pi_{-,1} + k_1 \pi_{-,2} = a_{11}\pi_{-,1} + a_{21}\pi_{-,2} \quad \Rightarrow a_{11} = -1, \quad a_{21} = k_1,$

$(6.19) \quad 2\pi_{-,1} - k_1\pi_{-,2} + k_2\pi_{-,2} = a_{12}\pi_{-,1} + a_{22}\pi_{-,2} \quad \Rightarrow a_{12} = -2, \quad a_{22} = k_2 - k_1,$

Similarly $\mathcal{B}^{\mathbf{c}}_{11}$ is calculated by (4.45). Finally we have:

$$(6.20) \qquad \mathcal{A}^{\mathbf{c}}_{\mathbf{11}} = \begin{bmatrix} -1 & 2 \\ k_1 & k_2 - k_1 \end{bmatrix} \qquad \mathcal{B}^{\mathbf{c}}_{\mathbf{11}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By (4.48), (4.49) we have:

$$(6.21) \qquad \mathcal{B}^{\mathbf{c}}_{\mathbf{11}}M = \begin{bmatrix} \bar{B} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We see that, $\forall k_1, k_2$ the pair $(\mathcal{A}^{\mathbf{c}}_{\mathbf{11}}, \bar{B})$ is controllable, thus defining $\bar{K}^{\mathbf{c}}_{\mathbf{11}} = [\bar{k}_1, \bar{k}_2]^T$, for λ_1^*, λ_2^* whatever fixed real, there exists real values of $\bar{k}_1, \bar{k}_2$ such that the matrix

$$(6.22) \qquad \mathcal{A}^{\mathbf{c}}_{\mathbf{11}} + \bar{B}\bar{K}^{\mathbf{c}}_{\mathbf{11}} = \begin{bmatrix} -1 & 2 \\ k_1 + \bar{k}_1 & k_2 - k_1 + \bar{k}_2 \end{bmatrix}$$

has eigenvalues equal to λ_1^*, λ_2^*. By Lemma 4.2, by setting $K^2 = \bar{K}^{\mathbf{c}}_{\mathbf{11}}T_{2,1}^{-1}$, we set as well equal to λ_1^*, λ_2^* two eigenvalues of the driver generator $G = \mathcal{A}^* + \mathcal{B}^*\mathcal{K}^*$, (all the other eigenvalues are necessarily zero). Thus, on account of (6.13) the result is:

$$(6.23) \qquad K^2 = [k_3, k_4] = [\bar{k}_1 + \bar{k}_2, \bar{k}_2],$$

which yields the feedback:

$$(6.24) \qquad \begin{aligned} u &= k_1 x_1 + k_2 x_2^2 + (\bar{k}_1 + \bar{k}_2)x_1^2 + \bar{k}_2 x_2 \\ &= (k_1 + \bar{k}_2)x_2 + (k_2 + \bar{k}_1 + \bar{k}_2)x_2^2. \end{aligned}$$

Therefore, by Lemma 4.2, given two (real numbers) λ_1^*, λ_2^*, for any k_1, k_2 such that:

$$(6.25) \qquad \bar{A} + \bar{B}K^1 = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}$$

---

[23]The range of $\mathcal{C}$, whose dimension we know cannot be greater than n (i.e 2 in the present case) includes the span of the columns of $\mathcal{A}^*$ and $\mathcal{B}^*$. Thus, it is enough notice that the unique non zero column of $\mathcal{B}^*$, and the unique column of $\mathcal{A}^*$ independent of k_1, k_2, are linearly independent.

has non zero eigenvalues, there exist functions of k_1, k_2, i.e. $\bar{k}_1 = \bar{k}_1(k_1, k_2)$, $\bar{k}_2 = \bar{k}_2(k_1, k_2)$, such that the matrix (6.22) has eigenvalues equal to λ_1^*, λ_2^*.

Let z denote a point in $\mathbb{R}^{16}$, which, in our framework stands for the 'initial point' of the linear frame. Moreover, let z^o be the orthogonal projection of z on Ker(G). By (6.14), and by the condition $Gz^o = 0$, we have

$$(6.26) \qquad z_{12}^o G_{-,2} + z_{21}^o G_{-,5} + z_{22}^o G_{-,6} + z_{23}^o G_{-,7} + z_{24}^o G_{-,8} = 0$$

where of course $G_{-,h}$ stands for the h-th column of G according with our assembling convention. By noticing that: $G_{-,6} = (k_2/k_1)G_{-,5}$, $G_{-,7} = (k_3/k_1)G_{-,5}$, and $G_{-,8} = (k_4/k_1)G_{-,5}$, we have that (6.26) turns into

$$(6.27) \qquad z_{12}^o G_{-,2} + \left(z_{21}^o + z_{22}^o \frac{k_2}{k_1} + z_{23}^o \frac{k_3}{k_1} + z_{24}^o \frac{k_4}{k_1} \right) G_{-,5} = 0.$$

$G_{-,2}$, and $G_{-,5}$ are linearly independent[24] and thus (6.27) implies:

$$(6.28) \qquad\qquad\qquad z_{12}^o = 0,$$

$$(6.29) \qquad\qquad\qquad z_{21}^o = -z_{22}^o \frac{k_2}{k_1} - z_{23}^o \frac{k_3}{k_1} - z_{24}^o \frac{k_4}{k_1}.$$

The relationships (6.28), (6.29), characterize all elements of Ker(G), and in particular the orthogonal projection on Ker(G) of the pivot $z^{i,j}$, for any $(i,j) = (1,1), \ldots, (4,4)$. Let us choose the pivot index (i,j) as one of the last eight entries[25] (between $(3,1)$ and $(4,4)$), and denote the projection of $z^{i,j}$ – which is $\chi(z^{i,j})$ in the proof of Theorem 5.2 – with the same symbol, z^o, used above for the generic projection. By (5.24) all the relevant components, which in our case are: $Z_{12}^\infty, \ldots, Z_{24}^\infty$, are directly read out from (6.28), (6.29), as $z_{12}^o, \ldots, z_{21}^o$ respectively. In particular we have:

$$(6.30) \qquad\qquad\qquad Z_{12}^\infty = 0,$$

$$(6.31) \qquad\qquad\qquad Z_{21}^\infty = -Z_{22}^\infty \frac{k_2}{k_1} - Z_{23}^\infty \frac{k_3}{k_1} - Z_{24}^\infty \frac{k_4}{k_1}.$$

Let us choose k_1, k_2 such that (6.25) is invertible, and z_{21}^o, i.e. the right hand side of (6.31) is equal to some real constant $c \neq 0$. This is possible in many ways: for instance let $k_2 = 0$. Then for any $k_1 \neq 0$ (6.25) is invertible, and thus we can set k_1 as the unique solution[26] of

$$(6.32) \qquad\qquad\qquad Z_{21}^\infty = c.$$

By (6.3), (6.6) the algebraic values[27] of Z_{12}^∞, and Z_{21}^∞ are:

$$(6.33) \qquad\qquad Z_{12}^\infty = \frac{x_2^2}{x_1}, \qquad Z_{21}^\infty = \frac{x_1}{x_2},$$

whereas

$$(6.34) \qquad\qquad Z_{22}^\infty = x_2, \qquad Z_{23}^\infty = \frac{x_1^2}{x_2}, \qquad Z_{24}^\infty = 1.$$

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[24]They are indeed a base for $\mathcal{R}(G)$

[25]With this choice the pivot index labels an element of a base for Ker(G), as one immediately realizes looking at (6.14), and thus falsifies (5.18), which is equivalent to verify (5.17).

[26]Recall that k_3, k_4 has been set earlier as well defined functions of k_1, k_2

[27]i.e. the values of the variable Z in the image of the immersion $Z = \Phi(x)$.

By (6.30), and the first of (6.33), we have $x_2^2/x_1 \to 0$, which, with the second in (6.33), and (6.32), implies $x_2 \to Z_{12}^\infty Z_{21}^\infty = cZ_{12}^\infty = 0$. $x_2 \to 0$ and $x_1/x_2 \to c$ entails $x_1 \to 0$, at the same rate as x_2, which, by the second identity in (6.34), implies $Z_{23}^\infty = 0$. Also, $x_2 \to 0$ implies, by the first of (6.34), that $Z_{22}^\infty = 0$. By using in (6.31) the values just above calculated for $Z_{22}^\infty, Z_{23}^\infty$, and using the last identity in (6.34) we calculate the value $k_1 = k_4/c$. As c is an arbitrary non zero constant we conclude that, for any $k_1 \neq 0$, (and with $k_2 = 0$) the pair k_3, k_4 (which is a function of k_1, k_2) assigning – as we have seen above – the two negative eigenvalues λ_1^*, λ_2^*, performs the zero-regulation task with an exponential performance, at a rate equal to $\max_{i \in \{1,2\}} |\lambda_i^*|$ (thus, a *tunable* rate), *from some initial condition*. Indeed, as we have already seen in Part I, the 'initial point' is undetermined, as Z^∞ is the limit of the *last continuous segment* of the curve $\{Z(t), t_0 \leq t < +\infty\}$, which, as we have seen in the proof of Theorem 5.2 (see for instance (5.24)) is defined and continuous in general only on a subinterval $[\bar{t}, +\infty)$. On this sub interval we can write the solution x_1 in integral form:

$$(6.35) \qquad x_1(t) = x_1(\bar{t})e^{\int_{\bar{t}}^t Z_{12}(\tau)d\tau},$$

We point out in passing that in the case at issue, the conditions *(iii), (iv)* of Theorem 5.2 are not all verified, as $Z_{12}^\infty = 0$ and thus (6.35) is not effective for predicting the steady-state behavior of x_1. Nonetheless, $x_1 \to 0$ is predictable in this case in a different way, through an analysis of the algebraic values of Z^∞, as we have shown a short earlier. In fact, formula (6.35) is effective for predicting the initial values from which x_1 is steered to zero. Indeed, from the first of (6.33) we see that $Z_{12}(t)$ has the same sign of x_1, thus if $x_1(\bar{t}) > 0$ then $Z_{12}(t) > 0 \; \forall t > \bar{t}$, and $x_1 \to 0$ is impossible. Thus we conclude that, for any $x_1(t_0) < 0$ we have $x_1 \to 0$, whereas for $x_1(t_0) > 0$, $x_1 \to +\infty$ *in a finite time*, say $t^* - t_0$, so that, for any $\bar{t} > t^*$, $x_1(\bar{t}) < 0$, $x_1(\bar{t}) \to -\infty$ for $\bar{t} \to t^{*+}$, and again $x_1 \to 0$, for $t \to +\infty$ on the segment $[\bar{t}, +\infty)$. In conclusion, the domain of attraction of the zero for the closed loop system is the half-plane $x_1 \leq 0$. Outside of this domain, we knew in advance that there are no points with $x_1 > 0$ that can be steered to zero. As a matter of fact, what comes out is that all trajectories goes to zero, but not all of them are continuous. A trajectory starting from a point ouside the domain of attraction shall be discontinuous, and in particular it will be composed by two countinuous disconnected branches: the first one, blowing up in a finite time, and the second one following then a trajectory included in the half-plane $x_1 < 0$ and converging to zero.

7. Conclusion and final remarks. The basic structure of the regulator built up through the QI-based design method described in the paper, is the following: first of all we have an open-loop system, not to be confused with the original system, which is the system to be controlled, an initial $\sigma\pi$-system, of order n, depending of u, the q-dimensional control, or written as a function of generical 'control coefficients' $v^{\mathbf{c}}$, i.e. (3.1) (3.2). To such a system we can associate a pair of dynamic and control matrices (as generally described in Part I) which, in the stationary case, allows the calculation of the accessibility system domain, as described in §2. Dynamic and control matrices are not to be confused with the couple (A, B), or $(\bar{A}, \bar{B})$, associated to the *closed-loop* system. These matrices depend of the choice of a feedback in the class (3.9), characterized by a $q \times 2n$ dimensioned matrix K. Such a feedback is in turn the composition of two lower dimensioned feedbacks: the *internal* and the *external* feedbacks. The *internal feedback*, which is characterized by a $q \times n$ matrix K^1, is the first feedback to be chosen in the class (3.9), and makes – by the procedure

described in §4, leading to (4.16) – the matrices $(\bar{A}, \bar{B})$ to appear in the system structure. The $\sigma\pi$-controllability (Definition 5.1) depends of this couple $(\bar{A}, \bar{B})$, as it is just the controllability (in the usual sense of linear systems) of such a couple. Thus, the $\sigma\pi$-controllability is a property of the system closed with the internal feedback K^1, and it depends in general of the choice of the feedback monomials, as well as of the *order* used for assembling the monomials in the vector $X^{\mathbf{p}}$. It makes in general no sense to say that an open loop system, as (3.1) (3.2) is '$\sigma\pi$-controllable', as there aren't matrices $\bar{A}, \bar{B}$ defined at this stage, nevertheless, by Remark 4.0.1 there are some 'natural' and standard ways for choosing the feedback monomials. A standard way is to choose nothing else that the *same* monomials as the open-loop system, and when the number of monomials is lower that n, to fill the n-dimensional vector $X^{\mathbf{p}}$ with linear monomials, kind of x_i. As we have made in the example, indeterminate exponents can be used as well in the feedback monomials, to be adjusted later in order to satisfy the further requirements of the method. Thus, with some abuse of terminology, we can use the expression 'the system is $\sigma\pi$-controllable' with reference to the open-loop system, meaning this that there is a 'standard' choice of the monomials and an order in the vector $X^{\mathbf{p}}$ that give place to a controllable pair $(\bar{A}, \bar{B})$. The system in internal closed-loop is called the *original system*, but is not yet the system to which the quadratization is applied. Another system, that we call the *object system* has to be build up by adding an *external feedback* characterized by another $q \times n$ matrix K^2. Unless the original, the object system is build up systematically without further choices: simply the original system is doubled in size and order (by adding fictitious monomials and new fictitious state components), and the external feedback is formed as $K^2 X^{\mathbf{p}}$ by using the *same* monomials set (with the same indeterminate exponents to be adjusted later), then it is simply *added* to the internal feedback $K^1 X^{\mathbf{p}}$. The object system has then dimension $2n$, and is characterized by the matrices (A, B) given by (4.29), that are directly derived from the original couple $(\bar{A}, \bar{B})$. The QI is then applied to the object system, which means that we are going to consider the driver associated to object system, characterized by its *generator* G, which can be directly calculated from the triple A, B, K, through (4.34). The indeterminate exponents of the monomials are then chosen in such a way to satisfy (which is always possible) property P2 of Remark 4.0.3. Once these preliminaries has been performed, the regulator design method is completed by setting (if possible) suitable values for the *gains* K^1, K^2, so that the conditions of the main theorem, i.e. Theorem (5.2) are satisfied. Condition *(i)* of the Theorem has yet been satisfied. K^1 has to be set with the only purpose to make the matrix (6.25) *invertible*. There are of course many way for achieve the invertibility of (6.25), so K^1 is actually a quite *free* matrix parameter. In general it can be set in order to satisfy condition *(ii)* as well, which also leaves in general K^1 with a residual degree of freedom. Next, K^2 can be chosen, by means of (4.56), in such a way to fix the n *rates* of exponential convergence of the n components of the original system. Once K^2 has been fixed, if *(iii), (iv)* of the main theorem are satisfied, the convergence to zero is assured, but it should be stressed that nothing assures that conditions *(iii), (iv)* are satisfied. Nonetheless, even though not satisfied, the design procedure can be brought off successfully as well if allowed by the particular nature of the problem at issue. This is the case, for instance, of the example presented in §6, where all conditions are satisfied except *(iii), (iv)*, and nonetheless all points are steered by a feedback designed following the previous described QI-based method. The example shows that the trajectories that are send to zero are not necessarily *continuous* trajectories. This is expected, since

we knew for this case that the zero cannot be reached from the points in the right half plane. The trajectories starting from there will be then *discontinuous* in that: they blows up in a finite time and appears again (at infinity) in the left right plane, where they are steered continuously to zero. What was not predicted is that: every of such unbounded-discontinuous trajectories originates *exactly* from the set of points of the system domain that are *accessible* but *not controllable.* This gives rise to the interesting – and probably very important from both theoretical and practical points of view – issue, that we leave for another occasion, about whether the points that are accessible but not controllable to zero are *characterizable* through the fact of giving rise to unbounded-discontinuous trajectories of the type above described.

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