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Stochastic Runge-Kutta methods with deterministic high order for ordinary differential equations

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Abstract

Our aim is to show that the embedding of deterministic Runge-Kutta methods with higher order than necessary order to achieve a weak order can enrich the properties of stochastic Runge-Kutta methods with respect to not only practical errors but also stability. This will be done through the comparisons between our new schemes and an efficient weak second order scheme with minimized error constant proposed by Debrabant and Rößler (2009).

1 Introduction

We are concerned with weak second order explicit stochastic Runge-Kutta (SRK) methods for non-commutative stochastic differential equations (SDEs). Among such methods, derivative-free methods are especially important because they can numerically solve SDEs with less computational effort, compared with other methods which need derivatives.

In fact, weak second order and derivative-free methods have been recently studied by many researchers. Kloeden and Platen [5, pp. 486–487] have proposed a derivative-free scheme of weak order two for non-commutative Itô SDEs. Tocino and Vigo-Aguiar [10] have also proposed it as an example in their SRK family. Komori [6] has proposed a different scheme which is for non-commutative Stratonovich SDEs and which has an advantage that it can reduce the random variables that need to be simulated. This scheme, however, still has a drawback that its computational costs for each diffusion coefficient linearly depend on the dimension of the Wiener process. Rößler [8] and Debrabant and Rößler [2] have proposed new schemes which overcome the drawback while keeping the advantage for Stratonovich and Itô SDEs, respectively.

Komori and Burrage [7] have also proposed an efficient SRK scheme which overcomes the drawback by improving the scheme in [6]. In addition, they have indicated that, even in a 10-dimensional Wiener process case, not only the scheme in [7] but also the other one in [6] can perform much better than an efficient scheme [8] in terms of computational costs. The classical Runge-Kutta (RK) method is embedded in both methods [6, 7]. This fact motivates us.

In the present paper we consider embedding deterministic high order RK methods into weak second order SRK methods proposed by Rößler [9] for non-commutative Itô SDEs. For these new SRK methods, we will study their stability properties and investigate their effectiveness in computation by numerical experiments, comparing them with the DRI1 scheme which is an efficient weak second order scheme with minimized error constant proposed by Debrabant and Rößler [2].

2 SRK methods for the weak approximation

Consider the autonomous d-dimensional Itô stochastic differential equation (SDE)

$$\mathrm{d}\boldsymbol{y}(t) = \boldsymbol{g}_0(\boldsymbol{y}(t))\mathrm{d}t + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}(t))\mathrm{d}W_j(t), \quad t > 0, \quad \boldsymbol{y}(0) = \boldsymbol{x}_0, \tag{2. 1}$$

where $W_j(t)$ is a scalar Wiener process and $\boldsymbol{x}_0$ is independent of $W_j(t) - W_j(0)$ for $t > 0$. We assume a global Lipschitz condition is satisfied such that the SDE has exactly one continuous global solution on the entire interval $[0, \infty)$ [1, p. 113]. For a given time T_{end}, let t_n be an equidistant grid point nh ($n = 0, 1, \ldots, M$) with step size $h \stackrel{\text{def}}{=} T_{end}/M < 1$ (M is a natural number) and $\boldsymbol{y}_n$ a discrete approximation to the solution $\boldsymbol{y}(t_n)$ of (2. 1). In addition, suppose that all moments of the initial value $\boldsymbol{x}_0$ exist and all components of $\boldsymbol{g}_j$ ($0 \leq j \leq m$) are sufficiently smooth, and define weak order in a usual way [5, p. 327].

On the base of the SRK framework proposed by Rößler [9], we consider the following

SRK method for (2. 1):

$$
\begin{aligned}
\boldsymbol{H}_i^{(0)} &= \boldsymbol{y}_n + \sum_{k=1}^{i-1} A_{ik}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) \quad (1 \le i \le s-2),\\
\boldsymbol{H}_i^{(0)} &= \boldsymbol{y}_n + \sum_{k=1}^{i-1} A_{ik}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) + \sum_{k=s-2}^{i-1}\sum_{l=1}^{m} B_{ik}^{(0)} \triangle\hat{W}_l \boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right) \quad (i = s-1, s),\\
\boldsymbol{H}_{s-2}^{(j)} &= \boldsymbol{y}_n + \sum_{k=1}^{s-2} A_{s-2,k}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) \quad (1 \le j \le m),\\
\boldsymbol{H}_i^{(j)} &= \boldsymbol{y}_n + \sum_{k=1}^{i} A_{ik}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) + \sum_{k=s-2}^{i-1}\sum_{l=1}^{m} B_{ik}^{(1)} \sqrt{h} \boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right)\\
&\qquad\qquad (i = s-1, s \text{ and } 1 \le j \le m),\\
\hat{\boldsymbol{H}}_i^{(j)} &= \boldsymbol{y}_n + \sum_{k=1}^{s} A_{ik}^{(2)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) + \sum_{k=s-2}^{s}\sum_{\substack{l=1\\ l\ne j}}^{m} B_{ik}^{(2)} \tilde{\eta}^{(j,l)} \boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right)\\
&\qquad\qquad (s-2 \le i \le s \text{ and } 1 \le j \le m),\\
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{i=1}^{s} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right) + \sum_{i=s-2}^{s}\sum_{j=1}^{m} \beta_i^{(1)} \triangle\hat{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_i^{(j)}\right)\\
&\quad + \sum_{i=s-2}^{s}\sum_{j=1}^{m} \beta_i^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\boldsymbol{H}_i^{(j)}\right)\\
&\quad + \sum_{i=s-2}^{s}\sum_{j=1}^{m} \beta_i^{(3)} \triangle\hat{W}_j \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_i^{(j)}\right) + \sum_{i=s-2}^{s}\sum_{j=1}^{m} \beta_i^{(4)} \sqrt{h} \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_i^{(j)}\right),
\end{aligned}
\tag{2. 2}
$$

where the α_i, $\beta_i^{(r_a)}$, $A_{ik}^{(r_b)}$, and $B_{ik}^{(r_b)}$ ($1 \le r_a \le 4$ and $0 \le r_b \le 2$) denote the parameters of the method and where $\tilde{\eta}_i^{(j,j)} \stackrel{\text{def}}{=} ((\triangle\hat{W}_j)^2 - h)/(2\sqrt{h})$,

$$
\tilde{\eta}_i^{(j,l)} \stackrel{\text{def}}{=} \begin{cases} (\triangle\hat{W}_j \triangle\hat{W}_l - \sqrt{h}\triangle\tilde{W}_j)/(2\sqrt{h}) & (j < l),\\ (\triangle\hat{W}_j \triangle\hat{W}_l + \sqrt{h}\triangle\tilde{W}_l)/(2\sqrt{h}) & (j > l),\end{cases}
$$

the $\triangle\tilde{W}_l$ ($1 \le l \le m-1$) are independent two-point distributed random variables with $P(\triangle\tilde{W}_j = \pm\sqrt{h}) = 1/2$ and the $\triangle\hat{W}_j$ ($1 \le j \le m$) are independent three-point distributed random variables with $P(\triangle\hat{W}_j = \pm\sqrt{3h}) = 1/6$ and $P(\triangle\hat{W}_j = 0) = 2/3$ [5, p. 225].

In addition to the SRK framework, Rößler [9] has given 59 order conditions for it to achieve weak order two. In order to satisfy the order conditions, we have to suppose $s \ge 3$ when we consider explicit SRK methods. In fact, Debrabant and Rößler [2] have supposed $s = 3$ and given the families of the solutions. Let us utilize some of their results because (2. 2) has the stochastic parts for $i = s-2, s-1, s$ only. That is, we assume

$$
\beta_{s-2}^{(1)} = \frac{-1 + 2\left(B_{s-1,s-2}^{(1)}\right)^2}{2\varepsilon_1\left(B_{s-1,s-2}^{(1)}\right)^2}, \quad \beta_{s-1}^{(1)} = \beta_s^{(1)} = \frac{1}{4\varepsilon_1\left(B_{s-1,s-2}^{(1)}\right)^2}, \quad \beta_{s-2}^{(2)} = 0,
$$

$$
\beta_{s-1}^{(2)} = -\beta_s^{(2)} = \frac{1}{2B_{s-1,s-2}^{(1)}}, \quad \beta_{s-2}^{(3)} = -\frac{1}{2\varepsilon_1 b_{s-1}^2}, \quad \beta_{s-1}^{(3)} = \beta_s^{(3)} = \frac{1}{4\varepsilon_1 b_{s-1}^2},
$$

$$
\beta_{s-2}^{(4)} = 0, \quad \beta_{s-1}^{(4)} = -\beta_s^{(4)} = \frac{1}{2b_{s-1}}, \quad B_{s,s-1}^{(0)} = 0, \quad B_{s,s-2}^{(1)} = -B_{s-1,s-2}^{(1)}, \tag{2. 3}
$$

$$B^{(1)}_{s,s-1} = 0, \quad B^{(2)}_{s-2,s-2} = B^{(2)}_{s-2,s-1} = B^{(2)}_{s-2,s} = 0, \quad B^{(2)}_{s-1,s} = B^{(2)}_{s-1,s-1},$$
$$B^{(2)}_{s,s-2} = -B^{(2)}_{s-1,s-2}, \quad B^{(2)}_{s,s-1} = B^{(2)}_{s,s} = -B^{(2)}_{s-1,s-1}$$

when $B^{(1)}_{s-1,s-2}$, $B^{(2)}_{s-1,s-2}$ and $B^{(2)}_{s-1,s-1}$ are given, where $\varepsilon_1 \stackrel{\mathrm{def}}{=} \pm 1$ and $b^2_{s-1} \stackrel{\mathrm{def}}{=} B^{(2)}_{s-1,s-2} + 2B^{(2)}_{s-1,s-1}$. Similarly, taking into their results into account as well as simplicity, we assume

$$A^{(1)}_{s-1,k} = A^{(1)}_{s,k} \quad (1 \le k \le s-2), \quad A^{(1)}_{s-1,s-1} = A^{(1)}_{s,s-1} = A^{(1)}_{s,s} = 0,$$
$$A^{(2)}_{s-2,k} = A^{(2)}_{s-1,k} = A^{(2)}_{s,k} \quad (1 \le k \le s).$$

After all, because we embed deterministic high order RK methods into our SRK methods, only the following three conditions remain to be solve:

$$1. \ \sum_{i=s-1}^{s} \alpha_i \left(B^{(0)}_{i,s-2} \right)^2 = \frac{1}{2}, \quad 2. \ \sum_{i=s-1}^{s} \alpha_i B^{(0)}_{i,s-2} = \frac{\varepsilon_1}{2}, \quad 3. \ \sum_{i=s-2}^{s} \beta^{(1)}_i \left(\sum_{k=1}^{s-2} A^{(1)}_{ik} \right) = \frac{\varepsilon_1}{2}.$$

Here, note that each of these corresponds to Conditions 11, 12 and 13 in [2], respectively.

From Conditions 1 and 2, we obtain

$$B^{(0)}_{s-1,s-2} = \frac{\alpha_{s-1}/\varepsilon_1 \pm \sqrt{\gamma_1}}{2\alpha_{s-1}(\alpha_{s-1}+\alpha_s)}, \qquad B^{(0)}_{s,s-2} = \frac{\alpha_s/\varepsilon_1 \mp \sqrt{\gamma_1}}{2\alpha_s(\alpha_{s-1}+\alpha_s)} \tag{2. 4}$$

(double sign in same order) if

$$\gamma_1 \stackrel{\mathrm{def}}{=} \alpha_{s-1}\alpha_s(-1 + 2(\alpha_{s-1}+\alpha_s)) \ge 0. \tag{2. 5}$$

Because of our assumption on $A^{(1)}_{ik}$, Condition 3 automatically holds if

$$\sum_{k=1}^{s-2} A^{(1)}_{s-2,k} = \sum_{k=1}^{s-2} A^{(1)}_{s-1,k} = \frac{1}{2}, \tag{2. 6}$$

or we have $B^{(1)}_{s-1,s-2} = \pm\sqrt{\gamma_2}$ from Condition 3 if

$$\gamma_2 \stackrel{\mathrm{def}}{=} \left(\sum_{k=1}^{s-2} A^{(1)}_{s-1,k} - \sum_{k=1}^{s-2} A^{(1)}_{s-2,k} \right) \Big/ \left(1 - 2\sum_{k=1}^{s-2} A^{(1)}_{s-2,k} \right) > 0. \tag{2. 7}$$

As an example satisfying (2. 6), we can choose the coefficients of the classical RK scheme for $A^{(0)}_{kj}$ and α_i, and can set

$$A^{(1)}_{s-2,k} = A^{(1)}_{s-1,k} = A^{(0)}_{s-2,k} \qquad (s = 4 \text{ and } 1 \le k \le s-2).$$

We will call it the SRKCL method. On the other hand, as an example satisfying (2. 7), we can choose the coefficients of the Fehlberg 4(5) scheme [3, p. 177] for $A^{(0)}_{kj}$ and α_i, and can set

$$A^{(1)}_{s-2,k} = A^{(0)}_{2,k}, \qquad A^{(1)}_{s-1,k} = A^{(0)}_{3,k} \qquad (s = 6 \text{ and } 1 \le k \le s-2).$$

We will call it the SRKF45 method. Of course, the SRKCL and SRKF45 methods are of order four and five for ODEs, respectively, and the both satisfy the critical restriction (2. 5).

3 Mean square stability

In order to study stability properties, let us deal with the scalar test SDE

$$\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + \sum_{j=1}^{m} \sigma_j y(t)\mathrm{d}W_j(t), \qquad t > 0, \qquad y(0) = x_0, \tag{3. 1}$$

where λ and σ_j $(1 \le j \le m)$ are real values and where $x_0 \neq 0$ with probability one (w. p. 1). By applying (2. 2) to (3. 1), we have

$$y_{n+1} = R\left(h, \lambda, \left\{\triangle\hat{W}_j\right\}_{j=1}^{m}, \left\{\triangle\tilde{W}_l\right\}_{l=1}^{m-1}, \{\sigma_j\}_{j=1}^{m}\right) y_n.$$

Here, by using $\beta_{s-1}^{(1)} = \beta_s^{(1)}$, $\beta_{s-2}^{(2)} = \sum_{i=s-2}^{s} \beta_i^{(3)} = \beta_{s-2}^{(4)} = B_{s,s-1}^{(0)} = B_{s,s-1}^{(1)} = B_{s-2,k}^{(2)} = 0$ $(s-2 \le k \le s)$, $\beta_{s-1}^{(2)} = -\beta_s^{(2)}$ and $\beta_{s-1}^{(4)} = -\beta_s^{(4)}$ from (2. 3), we have obtained

$$\begin{aligned} & R\left(h, \lambda, \left\{\triangle\hat{W}_j\right\}_{j=1}^{m}, \left\{\triangle\tilde{W}_l\right\}_{l=1}^{m-1}, \{\sigma_j\}_{j=1}^{m}\right) \\ = \; & 1 + \sum_{i=1}^{s} \alpha_i h\lambda Q_{i-1}(h\lambda) + \sum_{j=1}^{m} \triangle\hat{W}_j \sigma_j \left(\delta_1 + h\lambda\delta_2 + (h\lambda)^2\delta_3\right) \\ & + \sum_{j=1}^{m} \tilde{\eta}^{(j,j)}\sqrt{h}\sigma_j^2\delta_4 + \sum_{j=1}^{m}\sum_{\substack{l=1 \\ l\neq j}}^{m} \sqrt{h}\sigma_j\tilde{\eta}^{(j,l)}\sigma_l\delta_5, \end{aligned} \tag{3. 2}$$

where

$$\begin{aligned} & \delta_1 \stackrel{\text{def}}{=} \beta_{s-2}^{(1)}\hat{Q}_{s-2}(h\lambda) + 2\beta_{s-1}^{(1)}\hat{Q}_{s-1}(h\lambda), \quad \delta_2 \stackrel{\text{def}}{=} \left(\alpha_{s-1}B_{s-1,s-2}^{(0)} + \alpha_s B_{s,s-2}^{(0)}\right)\hat{Q}_{s-2}(h\lambda), \\ & \delta_3 \stackrel{\text{def}}{=} \alpha_s A_{s,s-1}^{(0)} B_{s-1,s-2}^{(0)}\hat{Q}_{s-2}(h\lambda), \quad \delta_4 \stackrel{\text{def}}{=} 2\beta_{s-1}^{(2)}B_{s-1,s-2}^{(1)}\hat{Q}_{s-2}(h\lambda), \\ & \delta_5 \stackrel{\text{def}}{=} 2\beta_{s-1}^{(4)}\left(B_{s-1,s-2}^{(2)}\hat{Q}_{s-2}(h\lambda) + 2B_{s-1,s-1}^{(2)}\hat{Q}_{s-1}(h\lambda)\right), \quad Q_0(z) \stackrel{\text{def}}{=} 1, \\ & Q_i(z) \stackrel{\text{def}}{=} 1 + z\sum_{k=1}^{i} A_{i+1,k}^{(0)} Q_{k-1}(z), \quad \hat{Q}_i(z) \stackrel{\text{def}}{=} 1 + z\sum_{k=1}^{i} A_{ik}^{(1)} Q_{k-1}(z) \quad (i \ge 1). \end{aligned}$$

Noting that

$$\begin{aligned} & E\left[\left(\tilde{\eta}^{(j,j)}\right)^2\right] = E\left[\left(\tilde{\eta}^{(j,l)}\right)^2\right] = \frac{h}{2} \quad (j \neq l), \\ & E\left[\tilde{\eta}^{(j,l_a)}\tilde{\eta}^{(j,l_b)}\right] = E\left[\tilde{\eta}^{(l_a,j)}\tilde{\eta}^{(l_b,j)}\right] = -E\left[\tilde{\eta}^{(j,l_a)}\tilde{\eta}^{(l_b,j)}\right] = \frac{h}{4} \quad (l_a \neq l_b \text{ and } l_a, l_b > j) \end{aligned}$$

and the expectation of the other terms concerning $\tilde{\eta}^{(j,l)}$ which appear when we square R vanishes, and by substituting (2. 3) and (2. 4) into $E[R^2]$, we obtain the stability function for (2. 2) as follows:

$$\begin{aligned} & \hat{R}(p, q_1, q_2, \ldots, q_m) \\ = \; & \left(1 + p\sum_{i=1}^{s} \alpha_i Q_{i-1}(p)\right)^2 + \sum_{j=1}^{m} q_j \left\{ \frac{1}{2(B_{s-1,s-2}^{(1)})^2}\left(\hat{Q}_{s-1}(p) - \hat{Q}_{s-2}(p)\right) \right. \end{aligned}$$

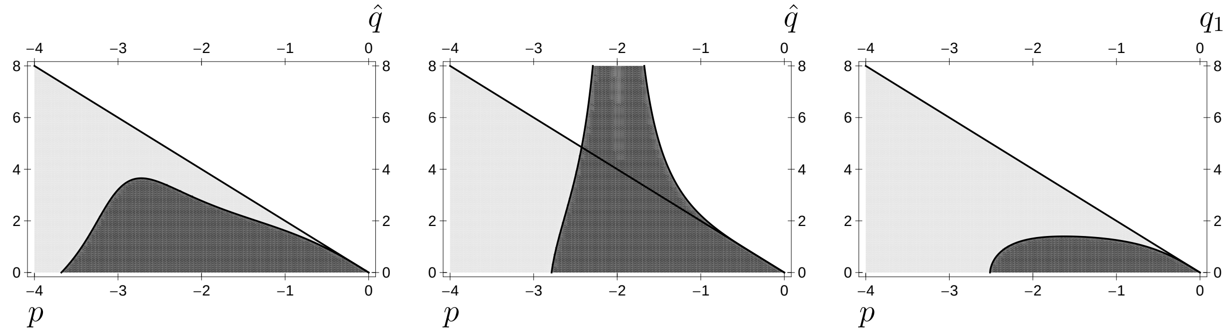

Figure 1: MS stability regions of SRK schemes

$$
+\left(1+\frac{1}{2}p+\frac{\alpha_s A^{(0)}_{s,s-1}(\alpha_{s-1}\pm\varepsilon_1\sqrt{\gamma_1})}{2\alpha_{s-1}(\alpha_{s-1}+\alpha_s)}p^2\right)\hat{Q}_{s-2}(p)\Bigg\}^2
$$

$$
+\frac{1}{2}\sum_{j=1}^{m} q_j^2\left(\hat{Q}_{s-2}(p)\right)^2
$$

$$
+\frac{1}{2}\sum_{j=1}^{m}\sum_{\substack{l=1\\ l\neq j}}^{m} q_j q_l\left(\frac{B^{(2)}_{s-1,s-2}\hat{Q}_{s-2}(p)+2B^{(2)}_{s-1,s-1}\hat{Q}_{s-1}(p)}{B^{(2)}_{s-1,s-2}+2B^{(2)}_{s-1,s-1}}\right)^2, \tag{3. 3}
$$

where $\hat{R}(p,q_1,q_2,\ldots,q_m)\stackrel{\text{def}}{=}E[R^2]$, $p\stackrel{\text{def}}{=}h\lambda$, and $q_j\stackrel{\text{def}}{=}h\sigma_j^2$. It is remarkable that the sum of the last two terms is equal to $\frac{1}{2}\left(\sum_{j=1}^{m}q_j\right)^2\left(\hat{Q}_{s-2}(p)\right)^2$ if $\hat{Q}_{s-2}(p)=\hat{Q}_{s-1}(p)$ or $B^{(2)}_{s-1,s-1}=0$, and then $\hat{R}$ simply becomes a function of p and $\hat{q}\stackrel{\text{def}}{=}\sum_{j=1}^{m}q_j$. The SRKCL method satisfies the former equality. For the SRKF45 method to satisfy the latter equality, let us set $B^{(2)}_{s-1,s-1}$ at 0 in the method. In addition, we set ε_1 at 1 and take the sign before $\sqrt{\gamma_1}$ plus.

The MS-stability regions of our methods, that is, $\{(p,\hat{q})|\hat{R}\leq 1\}$ [4], are given with dark-colored parts in the left-hand side and middle of Fig. 1. The parts enclosed by the two straight lines $\hat{q}=-2p$ and $\hat{q}=0$ indicate the region in which $\lim_{t\to\infty}E[|y(t)|^2]=0$ holds concerning (3. 1) [4]. Thus, light-colored parts indicate the region in which the test SDE is stable, but the SRK methods are not. On the other hand, because the DRI1 scheme neither satisfies $\hat{Q}_{s-2}(p)=\hat{Q}_{s-1}(p)$ nor $B^{(2)}_{s-1,s-1}=0$, its stability function cannot be expressed with p and $\hat{q}$. For this, under the assumption $m=1$ the MS-stability region of the scheme is given in the right-hand side in Fig. 1. We can see that the SRKF45 and SRKCL methods are better than the DRI1 scheme in terms of MS-stability. Because we have chosen parameter values such that $\hat{Q}_{s-1}(p)=\hat{Q}_{s-2}(p)=1+p/2$ in the SRKCL method, $\hat{R}$ does not depend on $\hat{q}$ when $p=-2$.

4 Numerical experiments and results

In order to investigate computational efficiency, we perform numerical experiments. Let us substitute $\varepsilon_1=B^{(2)}_{s-1,s-2}=1$, $B^{(2)}_{s-1,s-1}=0$ and $A^{(2)}_{s-2,k}=A^{(0)}_{2,k}$ $(1\leq k\leq s)$ into the both methods and $B^{(1)}_{s-1,s-2}=1$ into the SRKCL method. Then, we apply the numerical

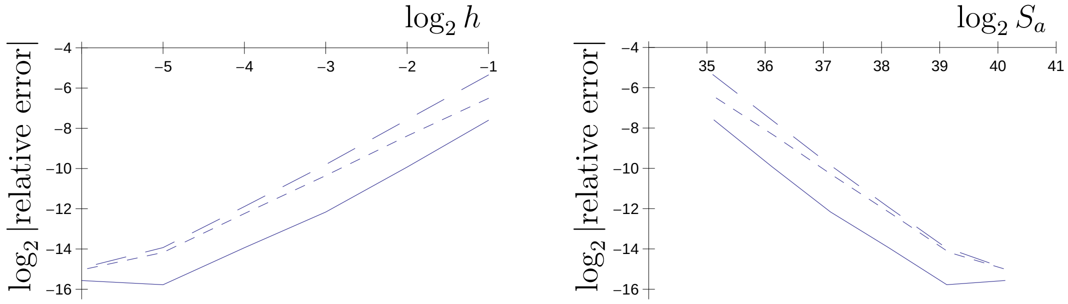

Figure 2: Relative errors about the fourth moment at $t = 1$.

schemes to the following SDE [2]:

$$\mathrm{d}y(t) = y(t)\mathrm{d}t + \sum_{j=1}^{10} \sigma_j \sqrt{y(t) + k_j}\mathrm{d}W_j(t), \qquad t > 0, \qquad y(0) = x_0,$$

where

$$\sigma_1 = \frac{1}{10}, \quad \sigma_2 = \sigma_8 = \frac{1}{15}, \quad \sigma_3 = \sigma_7 = \sigma_9 = \frac{1}{20}, \quad \sigma_4 = \sigma_6 = \sigma_{10} = \frac{1}{25}, \quad \sigma_5 = \frac{1}{40},$$
$$k_1 = k_6 = \frac{1}{2}, \quad k_2 = k_7 = \frac{1}{4}, \quad k_3 = k_8 = \frac{1}{5}, \quad k_4 = k_9 = \frac{1}{10}, \quad k_5 = k_{10} = \frac{1}{20},$$

and seek an approximation to the fourth moment of its solution when $x_0 = 1$ (w. p. 1) [7].

In the simulation, we simulate 256×10^6 independent trajectories for a given h. Here, remember that the DRI1 scheme is a scheme with minimized error constant and minimal stage number for weak order two. The results are indicated in Fig. 2. The solid, dotted or dash lines denote the SRKCL scheme, the SRKF45 scheme or the DRI1 scheme, respectively. In addition, S_a stands for the sum of the number of evaluations on the drift or diffusion coefficients and the number of generated pseudo random numbers. In this experiment we can see that the SRKCL scheme is better than the DRI1 scheme in terms of computational costs. We obtain similar results in numerical experiments concerning the other SDEs in [2].

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