

A GEOMETRIC DERIVATION OF NEW CONSERVATION LAWS

Fumitake MIMURA, Toshiharu IKEDA and Fumiyo FUJIWARA

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1. Introduction

Noether's theorem [7] has been extensively initiated for the derivation of conservation laws based on the symmetries in the Lagrangian or the Hamiltonian structures. However, without using the structures (which may fail to exist), Caviglia [1, 3] determined the new operative procedure for the laws via the application of a suitable version of Noether's theorem to the composite variational principle. And the procedure was analyzed by Mimura and Nôno [5] (also [6]) with various viewpoints for the derivation of the laws of a given second-order (partial) differential system.

In this paper, the local version of [5] is reformulated with a geometric notion of the calculus on manifolds (refer, e.g., to Sarlet and Cantrijn [9]). In 2, on a setting of a bundle $J = TM \times \mathbb{R}$ for a configuration manifold M , a set of 1-forms \mathfrak{X}_F^* (symmetry 1-forms of Γ) and a set of vector fields \mathfrak{X}_F on J or its subset \mathfrak{X}_K^ω ($\omega \in \mathfrak{X}_F^*$, $K \in \mathfrak{F}$: differentiable functions on J) are introduced associating with the equation field Γ of a given differential system in the kinematical form. In 3, conserved quantities of Γ , i.e., quantities C on J satisfying $\Gamma(C) = 0$, can be constructed from elements of \mathfrak{X}_F^* and \mathfrak{X}_K^ω , or particularly $\mathfrak{X}_K^{d\Omega}$ ($d\Omega \in \mathfrak{X}_F^*$, where Ω is a conserved quantity). In 4, a further derivation of constructing conserved quantities is given under a correspondence between an element of \mathfrak{X}_F and an element of \mathfrak{X}_F^* defined by a given regular Lagrangian, or particularly an element $d\Omega \in \mathfrak{X}_F^*$ given by a conserved quantity Ω . In 5, such a correspondence is explicitly realized by virtue of a closed 2-form defined by the exterior derivative of Poincaré-Cartan form. The realization contributes to make the ring of all conserved quantities of Γ into an infinite dimensional Lie algebra. In 6, we re-examine the equations of two-dimensional harmonic oscillator, of single particle motion under a central force and that arises in gas dynamics [8], which are illustratively used in [5], [4, 5] and [11], respectively.

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

2. Geometric characters associated with second-order system

Adding the time-axis \mathbb{R} to the tangent bundle TM of m -dimensional configuration manifold M , let $J = TM \times \mathbb{R}$ and U be its local chart with coordinate functions $(\dot{q}, q, t) = (\dot{q}^\alpha(t), q^\alpha(t), t)$ ($\alpha = 1, \dots, m$). On the setting, we consider a given second-order

system of m differential equations in the kinematical form:

$$(1) \quad \ddot{q}^\alpha = f^\alpha(\dot{q}, q, t),$$

together with its equation field

$$\Gamma = f^\alpha \frac{\partial}{\partial \dot{q}^\alpha} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \frac{\partial}{\partial t}.$$

To a contact system $\{\theta^\alpha\}$, where θ^α are given in coordinates as $\theta^\alpha = dq^\alpha - \dot{q}^\alpha dt$ on U , there corresponds a contact distribution, i.e., a subset Δ of vector fields \mathfrak{X} on J (characteristic vector fields of θ^α):

$$\Delta = \{X \in \mathfrak{X} \mid i_X \theta^\alpha = 0; \alpha = 1, \dots, m\},$$

where i_X denotes the interior product (contraction) by X . Associating with the equation field Γ , the distribution is used to define

$$\mathfrak{X}_\Gamma = \{X \in \mathfrak{X} \mid [\Gamma, X] \in \Delta\}.$$

Then, in view of the relations

$$\begin{aligned} \left[\frac{\partial}{\partial \dot{q}^\alpha}, \Gamma \right] &= \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \frac{\partial}{\partial \dot{q}^\beta} + \frac{\partial}{\partial q^\alpha}, \\ \left[\frac{\partial}{\partial q^\alpha}, \Gamma \right] &= \frac{\partial f^\beta}{\partial q^\alpha} \frac{\partial}{\partial \dot{q}^\beta}, \end{aligned}$$

a vector field $X \in \mathfrak{X}$, when expressed with respect to the basis $\{\partial/\partial \dot{q}^\alpha, \partial/\partial q^\alpha, \Gamma\}$:

$$X = \eta^\alpha \frac{\partial}{\partial \dot{q}^\alpha} + \xi^\alpha \frac{\partial}{\partial q^\alpha} + \psi \Gamma,$$

has the following commutator with Γ

$$(2) \quad \begin{aligned} [\Gamma, X] &= \Gamma(\eta^\alpha) \frac{\partial}{\partial \dot{q}^\alpha} + \Gamma(\xi^\alpha) \frac{\partial}{\partial q^\alpha} + \Gamma(\psi) \Gamma - \eta^\alpha \left[\frac{\partial}{\partial \dot{q}^\alpha}, \Gamma \right] - \xi^\alpha \left[\frac{\partial}{\partial q^\alpha}, \Gamma \right] \\ &= \left(\Gamma(\eta^\alpha) - \eta^\beta \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} \right) \frac{\partial}{\partial \dot{q}^\alpha} + (\Gamma(\xi^\alpha) - \eta^\alpha) \frac{\partial}{\partial q^\alpha} + \Gamma(\psi) \Gamma. \end{aligned}$$

So that $i_{[\Gamma, X]} \theta^\alpha = 0$ imply $\eta^\alpha = \Gamma(\xi^\alpha)$. Therefore we have a local expression of $X \in \mathfrak{X}_\Gamma$ (cf. [9], the dynamical symmetry of Γ in Lemma 3.1; Eq. (17) and (18) in [1]):

THEOREM 1. *A vector field $X \in \mathfrak{X}_\Gamma$ is written as follows with respect to the basis $\{\partial/\partial \dot{q}^\alpha, \partial/\partial q^\alpha, \Gamma\}$:*

$$(3) \quad X = \Gamma(\xi^\alpha) \frac{\partial}{\partial \dot{q}^\alpha} + \xi^\alpha \frac{\partial}{\partial q^\alpha} + \psi \Gamma.$$

Particularly a vector field X satisfying $[\Gamma, X] = 0$ is an element of \mathfrak{X}_Γ . In this case, (3) is provided with the conditions $\Gamma(\psi) = 0$ and

$$(4) \quad \Gamma^2(\xi^\alpha) - \Gamma(\xi^\beta) \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} = 0.$$

Alternatively, in terms of the Lie derivative \mathfrak{L}_Γ , a subset of 1-forms \mathfrak{X}^* on J (symmetry 1-forms of Γ) is defined:

$$\mathfrak{X}_\Gamma^* = \{\omega \in \mathfrak{X}^* \mid \mathfrak{L}_\Gamma \omega = 0\}.$$

Here introduce 1-forms $\phi^\alpha = d\dot{q}^\alpha - f^\alpha dt$ for the basis $\{\phi^\alpha, \theta^\alpha, dt\}$ to write $\omega \in \mathfrak{X}^*$ on U as

$$\omega = \mu_\alpha \phi^\alpha + \nu_\alpha \theta^\alpha + \tau dt.$$

Then, in view of

$$\begin{aligned} \mathfrak{L}_\Gamma \phi^\alpha &= \frac{\partial f^\alpha}{\partial \dot{q}^\beta} \phi^\beta + \frac{\partial f^\alpha}{\partial q^\beta} \theta^\beta, \\ \mathfrak{L}_\Gamma \theta^\alpha &= \phi^\alpha, \end{aligned}$$

it follows that

$$\mathfrak{L}_\Gamma \omega = \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} + \nu_\alpha \right) \phi^\alpha + \left(\Gamma(\nu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial q^\alpha} \right) \theta^\alpha + \Gamma(\tau) dt.$$

Therefore, the condition $\mathfrak{L}_\Gamma \omega = 0$ for $\omega \in \mathfrak{X}_\Gamma^*$ leads to $\Gamma(\tau) = 0$ and

$$\begin{aligned} \Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} + \nu_\alpha &= 0, \\ \Gamma(\nu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial q^\alpha} &= 0, \end{aligned}$$

from which we have a local expression of $\omega \in \mathfrak{X}_\Gamma^*$ (cf. the adjoint symmetry in [10]; Eq. (14) in [1]):

THEOREM 2. *A 1-form $\omega \in \mathfrak{X}_\Gamma^*$ is written as follows with respect to the basis $\{\phi^\alpha, \theta^\alpha, dt\}$:*

$$(5) \quad \omega = \mu_\alpha \phi^\alpha - \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \right) \theta^\alpha + \tau dt,$$

which is provided with the conditions $\Gamma(\tau) = 0$ and

$$(6) \quad \Gamma \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \right) - \mu_\beta \frac{\partial f^\beta}{\partial q^\alpha} = 0.$$

Particularly a multiple $\psi\Gamma$ ($\psi \in \mathfrak{F}$: differentiable functions on J) of Γ is an element of \mathfrak{X}_Γ (see (3) with ξ^α). So we regard that X_1 and X_2 in \mathfrak{X}_Γ are equivalent if $X_1 - X_2$ is equal to such a multiple; and in each equivalent class of \mathfrak{X}_Γ , we can take particularly an element

$$(3) \quad X = \Gamma(\xi^\alpha) \frac{\partial}{\partial \dot{q}^\alpha} + \xi^\alpha \frac{\partial}{\partial q^\alpha}.$$

Then, for $\omega \in \mathfrak{X}_\Gamma^*$ of the form (5), through (2) with $\psi = 0$ and $\eta^\alpha = \Gamma(\xi^\alpha)$, we have

$$(7) \quad i_{[\Gamma, X]} \omega = \mu_\alpha \left(\Gamma^2(\xi^\alpha) - \Gamma(\xi^\beta) \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} \right).$$

For an arbitrary 1-form $\omega \in \mathfrak{X}_\Gamma^*$, associating with an element $K \in \mathfrak{F}$, we now define a subset \mathfrak{X}_K^ω of \mathfrak{X}_Γ :

$$\mathfrak{X}_K^\omega = \{X \in \mathfrak{X}_\Gamma \mid i_{[\Gamma, X]} \omega = \Gamma(K)\}.$$

Then it follows that (cf. Eq. (3.8b) in [3])

THEOREM 3. *A vector field X in each equivalent class of \mathfrak{X}_Γ can be put as (3) with respect to the basis $\{\partial/\partial \dot{q}^\alpha, \partial/\partial q^\alpha, \Gamma\}$. And then, for $K \in \mathfrak{F}$ and $\omega \in \mathfrak{X}_\Gamma^*$ of the form (5), $X \in \mathfrak{X}_K^\omega$ if and only if*

$$(8) \quad \mu_\alpha \left(\Gamma^2(\xi^\alpha) - \Gamma(\xi^\beta) \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} \right) = \Gamma(K).$$

REMARK 1. For $X \in \mathfrak{X}_\Gamma$ of the form (3), since in view of $\Gamma(\tau) = 0$ and $i_\Gamma \phi^\alpha = i_\Gamma \theta^\alpha = 0$:

$$i_{[\Gamma, \psi\Gamma]} \omega = \Gamma(\psi) i_\Gamma \omega = \Gamma(\psi) \tau = \Gamma(\psi\tau),$$

the right hand side of (7) is modified by adding a term $\Gamma(\psi\tau)$. So the condition $i_{[\Gamma, X]} \omega = \Gamma(K)$ for \mathfrak{X}_K^ω differs from (8) essentially nothing but in that K must be replaced with $K - \psi\tau$.

3. Conserved quantities associated with \mathfrak{X}_Γ^* and \mathfrak{X}_K^ω

Elements of \mathfrak{X}_Γ^* and \mathfrak{X}_K^ω can be used effectively to construct conserved quantities of Γ (first integrals of the system (1)), i.e., quantities $C(\dot{q}, q, t)$ on J satisfying $\Gamma(C) = 0$. An element $\omega \in \mathfrak{X}_\Gamma^*$ vanishes by \mathfrak{L}_Γ , i.e., $\mathfrak{L}_\Gamma \omega = 0$, so that

$$i_{[\Gamma, X]} \omega = \Gamma(i_X \omega) - i_X(\mathfrak{L}_\Gamma \omega) = \Gamma(i_X \omega);$$

which implies $\Gamma(i_X \omega - K) = 0$ if $X \in \mathfrak{X}_K^\omega$. Therefore it is deduced (cf. Eq. (22) in [1]):

THEOREM 4. *Elements $\omega \in \mathfrak{X}_\Gamma^*$ and $X \in \mathfrak{X}_K^\omega$ give rise to a conserved quantity C :*

$$(9) \quad C = i_X \omega - K.$$

The respective appearance (5) and (3)' of $\omega \in \mathfrak{X}_F^*$ and $X \in \mathfrak{X}_K^\omega$ give a local version of the theorem 4. In fact the solutions μ_α of (6) and ξ^α of (8) yield the conserved quantity ([5], Remark 4; cf. [3], Theorem)

$$(10) \quad i_X \omega - K = \mu_\alpha \Gamma(\xi^\alpha) - \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \right) \xi^\alpha - K.$$

Particularly, together with the solutions μ_α of (6), an element $X \in \mathfrak{X}_0^\omega$ ($K = 0$) satisfying $[\Gamma, X] = 0$, i.e., ξ^α of (3)' satisfying (4) yield the conserved quantity $i_X \omega$, i.e., the quantity (10) with $K = 0$ ([5], Theorem 2).

Whenever f^α do not depend explicitly on the time t , i.e., $f^\alpha = f^\alpha(\dot{q}, q)$, the equation field Γ supplies $[\Gamma, \Gamma - \partial/\partial t] = 0$, so that $\Gamma_0 \equiv \Gamma - \partial/\partial t \in \mathfrak{X}_0^\omega$ for arbitrary $\omega \in \mathfrak{X}_F^*$. Therefore solutions μ_α of (6) yield the conserved quantity ([5], Theorem 5; cf. Eq. (21) in [1])

$$(11) \quad i_{\Gamma_0} \omega = \mu_\alpha f^\alpha - \dot{q}^\alpha \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \right).$$

REMARK 2. Similarly as in the remark 1, if $X \in \mathfrak{X}_K^\omega$ is not of the form (3)' but (3), K is replaced with $K - \psi\tau$ in the resulting conserved quantity (10).

A conserved quantity Ω satisfies $\Omega_\Gamma(d\Omega) = d\Gamma(\Omega) = 0$, i.e., $\omega \equiv d\Omega \in \mathfrak{X}_F^*$. In view of $i_X d\Omega = X(\Omega)$, it reduces the theorem 4 to

THEOREM 5. Together with a vector field $X \in \mathfrak{X}_K^{d\Omega}$, a conserved quantity Ω gives rise to a new one C :

$$(12) \quad C = X(\Omega) - K.$$

In the coordinates, since $d\Omega$ of a conserved quantity Ω is expressed as

$$(13) \quad d\Omega = \frac{\partial \Omega}{\partial \dot{q}^\alpha} \phi^\alpha + \frac{\partial \Omega}{\partial q^\alpha} \theta^\alpha + \Gamma(\Omega) dt = \frac{\partial \Omega}{\partial \dot{q}^\alpha} \phi^\alpha + \frac{\partial \Omega}{\partial q^\alpha} \theta^\alpha,$$

the theorem 3 implies that X of the form (3)' is an element of $\mathfrak{X}_K^{d\Omega}$ if and only if

$$\frac{\partial \Omega}{\partial \dot{q}^\alpha} \left(\Gamma^2(\xi^\alpha) - \Gamma(\xi^\beta) \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} \right) = \Gamma(K),$$

under which the resulting conserved quantity (12) is written as

$$(14) \quad X(\Omega) - K = \Gamma(\xi^\alpha) \frac{\partial \Omega}{\partial \dot{q}^\alpha} + \xi^\alpha \frac{\partial \Omega}{\partial q^\alpha} - K.$$

Particularly an element $X \in \mathfrak{X}_0^{d\Omega}$ ($K = 0$) satisfying $[F, X] = 0$, i.e., ξ^α of (3)' satisfying (4), yields the conserved quantity (14) with $K = 0$ ([5], Theorem 4).

4. Euler-Lagrange systems

Let $L(\dot{q}, q, t)$ be a regular Lagrangian, i.e., $\det(W_{\alpha\beta}) \neq 0$ where $W_{\alpha\beta} = \partial^2 L / \partial \dot{q}^\alpha \partial \dot{q}^\beta$, and the system (1) have resulted from the Euler-Lagrange equations

$$(15) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} = \ddot{q}^\gamma W_{\alpha\gamma} + \dot{q}^\gamma \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\gamma} + \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial t} - \frac{\partial L}{\partial q^\alpha} = 0.$$

Then, after replacing \ddot{q}^γ with f^γ in (15), the equations are differentiated with respect to \dot{q}^β and q^β to obtain respectively

$$(16) \quad W_{\alpha\gamma} \frac{\partial f^\gamma}{\partial \dot{q}^\beta} = -\Gamma(W_{\alpha\beta}) - \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} + \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha},$$

$$(17) \quad W_{\alpha\gamma} \frac{\partial f^\gamma}{\partial q^\beta} = -\Gamma \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} \right) + \frac{\partial^2 L}{\partial q^\alpha \partial q^\beta};$$

where the skew-symmetric parts of (17) for the indices α and β lead to

$$(18) \quad W_{\beta\gamma} \frac{\partial f^\gamma}{\partial q^\alpha} - W_{\alpha\gamma} \frac{\partial f^\gamma}{\partial q^\beta} = \Gamma \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right).$$

Now $\mu_\alpha = W_{\alpha\beta} \xi^\beta$ are substituted for the coefficient of θ^α in (5), and then (16) is used to see

$$(19) \quad \begin{aligned} v_\alpha &\equiv \Gamma(\mu_\alpha) + \mu_\gamma \frac{\partial f^\gamma}{\partial \dot{q}^\alpha} \\ &= W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\Gamma(W_{\alpha\beta}) + W_{\gamma\beta} \frac{\partial f^\gamma}{\partial \dot{q}^\alpha} \right) \xi^\beta \\ &= W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi^\beta; \end{aligned}$$

and also for (6), together with the above v_α , to see

$$\begin{aligned} &\Gamma \left(W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi^\beta \right) - W_{\gamma\beta} \xi^\beta \frac{\partial f^\gamma}{\partial q^\alpha} \\ &= W_{\alpha\beta} \Gamma^2(\xi^\beta) + \left(\Gamma(W_{\alpha\beta}) + \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \Gamma(\xi^\beta) \\ &\quad + \left(\Gamma \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) - W_{\beta\gamma} \frac{\partial f^\gamma}{\partial q^\alpha} \right) \xi^\beta = 0, \end{aligned}$$

in which, by (18) and (17), the terms in the last parenthesis are moreover rewritten as

$$\begin{aligned} \Gamma\left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha}\right) - W_{\beta\gamma} \frac{\partial f^\gamma}{\partial q^\alpha} &= -W_{\alpha\gamma} \frac{\partial f^\gamma}{\partial q^\beta} \\ &= \Gamma\left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta}\right) - \frac{\partial^2 L}{\partial q^\alpha \partial q^\beta}. \end{aligned}$$

Therefore from the theorem 2 it follows (cf. Eq. (3.7) in [2]; Eq. (15) in [5]):

THEOREM 6. *A 1-form ω_L associated with the regular Lagrangian L :*

$$(20) \quad \omega_L = W_{\alpha\beta} \xi^\beta \phi^\alpha - \left(W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi^\beta \right) \theta^\alpha + \tau dt$$

is an element of \mathfrak{X}_F^* if and only if $\Gamma(\tau) = 0$ and ξ^α satisfy

$$(21) \quad W_{\alpha\beta} \Gamma^2(\xi^\beta) + \left(\Gamma(W_{\alpha\beta}) + \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \Gamma(\xi^\beta) + \left(\Gamma\left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta}\right) - \frac{\partial^2 L}{\partial q^\alpha \partial q^\beta} \right) \xi^\beta = 0.$$

The left hand side of (21), for which (16) and (17) are substituted, is just the same as

$$W_{\alpha\gamma} \left(\Gamma^2(\xi^\gamma) - \Gamma(\xi^\beta) \frac{\partial f^\gamma}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\gamma}{\partial q^\beta} \right).$$

So that, since $\det(W_{\alpha\beta}) \neq 0$, the equations (4) and (21) are to be equivalent. Therefore from the theorem 1 it follows:

THEOREM 7. *To a 1-form $\omega_L \in \mathfrak{X}_F^*$ of the form (20) associated with the regular Lagrangian L , there corresponds a vector field $X \in \mathfrak{X}_F$ of the form (3) uniquely up to a multiple of Γ satisfying $[\Gamma, X] \equiv 0 \pmod{\Gamma}$, and vice versa.*

The correspondence in the theorem 7 can be applied to the theorem 4 for a derivation of conserved quantities of the Euler-Lagrange equations (15).

At first, in a pair of elements ω_L^1 and ω_L^2 of \mathfrak{X}_F^* , i.e., in those of solutions ξ_i^α ($i = 1, 2$) of (21), ξ_1^α define an element $X \in \mathfrak{X}_F$ of the form (3) satisfying $[\Gamma, X] = 0$, while $\xi^\alpha = \xi_2^\alpha$ are left in the appearance (20) of ω_L^2 . Then, by the theorem 4, such elements $\omega_L^2 \in \mathfrak{X}_F^*$ and $X \in \mathfrak{X}_F$ (of course $X \in \mathfrak{X}_0$ for arbitrary $\omega \in \mathfrak{X}_F^*$) yield a conserved quantity (see (10) with $K = 0$)

$$-i_X \omega_L^2 = v_\alpha \xi_1^\alpha - W_{\alpha\beta} \xi_2^\beta \Gamma(\xi_1^\alpha),$$

where v_α are given as (19) in which $\xi^\alpha = \xi_2^\alpha$ and $\mu_\alpha = W_{\alpha\beta} \xi_2^\beta$. So that ([5], Theorem 6)

$$-i_X \omega_L^2 = W_{\alpha\beta} (\xi_1^\alpha \Gamma(\xi_2^\beta) - \xi_2^\beta \Gamma(\xi_1^\alpha)) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi_1^\alpha \xi_2^\beta.$$

Whenever $L = L(\dot{q}, q)$, i.e., $f^\alpha = f^\alpha(\dot{q}, q)$, by $\omega_L \in \mathfrak{X}_F^*$ of the form (20), i.e., by ξ^α satisfying (21), the resulting conserved quantity (11) is written as

$$-i_{\Gamma_0}\omega_L = \dot{q}^\alpha v_\alpha - f^\alpha W_{\alpha\beta} \xi^\beta,$$

for which v_α of the last expression of (19) are substituted and then (15) is used to lead ([5], Theorem 7)

$$(22) \quad \begin{aligned} -i_{\Gamma_0}\omega_L &= \dot{q}^\alpha W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\dot{q}^\alpha \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \dot{q}^\alpha \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} - f^\alpha W_{\alpha\beta} \right) \xi^\beta \\ &= \dot{q}^\alpha W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\dot{q}^\alpha \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial L}{\partial q^\beta} \right) \xi^\beta. \end{aligned}$$

REMARK 3. In the case of $L = L(\dot{q}, q)$ the Hamiltonian

$$H = \dot{q}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha} - L$$

is a conserved quantity. It is interesting to note that, together with $X \in \mathfrak{X}_\Gamma$ of the form (3)' satisfying $[\Gamma, X] = 0$, i.e., solutions ξ^α of (4) or equivalently of (21), the Hamiltonian gives rise to a new one (see (12) with $K = 0$):

$$i_X dH = X(H) = \Gamma(\xi^\alpha) \frac{\partial H}{\partial \dot{q}^\alpha} + \xi^\alpha \frac{\partial H}{\partial q^\alpha},$$

which is just the same as (22).

A further application of the theorem 7 begins with a conserved quantity Ω . Since $d\Omega \in \mathfrak{X}_F^*$ as seen before, it can be written as (5). In fact, in view of $\Gamma(\Omega) = 0$ in

$$(23) \quad \begin{aligned} \left[\frac{\partial}{\partial \dot{q}^\alpha}, \Gamma \right] (\Omega) &= \left(\frac{\partial f^\beta}{\partial \dot{q}^\alpha} \frac{\partial}{\partial \dot{q}^\beta} + \frac{\partial}{\partial q^\alpha} \right) (\Omega), \\ \text{i.e., } -\Gamma \left(\frac{\partial \Omega}{\partial \dot{q}^\alpha} \right) &= \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \frac{\partial \Omega}{\partial \dot{q}^\beta} + \frac{\partial \Omega}{\partial q^\alpha}, \end{aligned}$$

$d\Omega$ of (13) leads to

$$(24) \quad d\Omega = \frac{\partial \Omega}{\partial \dot{q}^\alpha} \phi^\alpha - \left(\Gamma \left(\frac{\partial \Omega}{\partial \dot{q}^\alpha} \right) + \frac{\partial \Omega}{\partial \dot{q}^\beta} \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \right) \theta^\alpha.$$

This appearance follows also from (20) by putting $\xi^\alpha = W^{\alpha\beta} \partial \Omega / \partial \dot{q}^\beta$ where $(W^{\alpha\beta}) = (W_{\alpha\beta})^{-1}$, i.e., $\mu_\alpha = W_{\alpha\beta} \xi^\beta = \partial \Omega / \partial \dot{q}^\alpha$; which are substituted for the first expression of v_α in (19) to confirm that the coefficients of θ^α in (20) lead to those in the above $d\Omega$. Therefore in the theorem 7, the corresponding vector field $X_\Omega \in \mathfrak{X}_\Gamma$ to $d\Omega \in \mathfrak{X}_F^*$ is

$$(25) \quad X_{\Omega} = \Gamma \left(W^{\alpha\beta} \frac{\partial \Omega}{\partial \dot{q}^{\beta}} \right) \frac{\partial}{\partial \dot{q}^{\alpha}} + W^{\alpha\beta} \frac{\partial \Omega}{\partial \dot{q}^{\beta}} \frac{\partial}{\partial q^{\alpha}},$$

which is provided with $[\Gamma, X_{\Omega}] = 0$. So that, together with an element $\omega \in \mathfrak{X}_{\mathcal{F}}^*$ of the form (5), i.e., μ_{α} satisfying (6), the conserved quantity Ω , i.e., the corresponding vector field X_{Ω} , gives rise to a new one (see (9) with $K = 0$)

$$-i_{X_{\Omega}}\omega = -\mu_{\alpha}\Gamma\left(W^{\alpha\beta}\frac{\partial\Omega}{\partial\dot{q}^{\beta}}\right) + \left(\Gamma(\mu_{\alpha}) + \mu_{\gamma}\frac{\partial f^{\gamma}}{\partial\dot{q}^{\alpha}}\right)W^{\alpha\beta}\frac{\partial\Omega}{\partial\dot{q}^{\beta}},$$

in which the terms are rewritten by (23) as

$$\Gamma\left(W^{\alpha\beta}\frac{\partial\Omega}{\partial\dot{q}^{\beta}}\right) = \Gamma(W^{\alpha\beta})\frac{\partial\Omega}{\partial\dot{q}^{\beta}} - W^{\alpha\beta}\left(\frac{\partial f^{\gamma}}{\partial\dot{q}^{\beta}}\frac{\partial\Omega}{\partial\dot{q}^{\gamma}} + \frac{\partial\Omega}{\partial q^{\beta}}\right);$$

so that

$$-i_{X_{\Omega}}\omega = \mu_{\alpha}\left(W^{\alpha\gamma}\frac{\partial f^{\beta}}{\partial\dot{q}^{\gamma}} + W^{\beta\gamma}\frac{\partial f^{\alpha}}{\partial\dot{q}^{\gamma}} - \Gamma(W^{\alpha\beta})\right)\frac{\partial\Omega}{\partial\dot{q}^{\beta}} + W^{\alpha\beta}\left(\Gamma(\mu_{\alpha})\frac{\partial\Omega}{\partial\dot{q}^{\beta}} + \mu_{\alpha}\frac{\partial\Omega}{\partial q^{\beta}}\right).$$

Moreover the symmetric parts of (16) for the indices α and β :

$$\Gamma(W_{\alpha\beta}) = -\frac{1}{2}\left(W_{\alpha\gamma}\frac{\partial f^{\gamma}}{\partial\dot{q}^{\beta}} + W_{\beta\gamma}\frac{\partial f^{\gamma}}{\partial\dot{q}^{\alpha}}\right)$$

are substituted for the differentiation of $W_{\alpha\beta}W^{\beta\gamma} = \delta_{\alpha}^{\gamma}$ by Γ :

$$(26) \quad \begin{aligned} \Gamma(W_{\alpha\beta})W^{\beta\gamma} + W_{\alpha\beta}\Gamma(W^{\beta\gamma}) &= 0, \text{ i.e.,} \\ \Gamma(W^{\alpha\beta}) &= -W^{\alpha\gamma}W^{\beta\sigma}\Gamma(W_{\gamma\sigma}), \end{aligned}$$

to obtain

$$\Gamma(W^{\alpha\beta}) = \frac{1}{2}\left(W^{\alpha\gamma}\frac{\partial f^{\beta}}{\partial\dot{q}^{\gamma}} + W^{\beta\gamma}\frac{\partial f^{\alpha}}{\partial\dot{q}^{\gamma}}\right);$$

which are used to have the final appearance ([5], Theorem 9)

$$(27) \quad -i_{X_{\Omega}}\omega = \frac{1}{2}\mu_{\alpha}\left(W^{\alpha\gamma}\frac{\partial f^{\beta}}{\partial\dot{q}^{\gamma}} + W^{\beta\gamma}\frac{\partial f^{\alpha}}{\partial\dot{q}^{\gamma}}\right)\frac{\partial\Omega}{\partial\dot{q}^{\beta}} + W^{\alpha\beta}\left(\Gamma(\mu_{\alpha})\frac{\partial\Omega}{\partial\dot{q}^{\beta}} + \mu_{\alpha}\frac{\partial\Omega}{\partial q^{\beta}}\right), \text{ or}$$

$$(27)' \quad -i_{X_{\Omega}}\omega = \Gamma(\mu_{\alpha}W^{\alpha\beta})\frac{\partial\Omega}{\partial\dot{q}^{\beta}} + \mu_{\alpha}W^{\alpha\beta}\frac{\partial\Omega}{\partial q^{\beta}}.$$

Particularly for conserved quantities Ω_1 and Ω_2 , ω in (27) is replaced with $d\Omega_1$, i.e., $\mu_{\alpha} = \partial\Omega_1/\partial\dot{q}^{\alpha}$, while $\Omega = \Omega_2$, to derive

$$-i_{X_{\Omega_2}} d\Omega_1 = \frac{1}{2} \left(W^{\alpha\gamma} \frac{\partial f^\beta}{\partial \dot{q}^\gamma} + W^{\beta\gamma} \frac{\partial f^\alpha}{\partial \dot{q}^\gamma} \right) \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \left(\Gamma \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \right) \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} \right),$$

which, in view of (23), can be written as ([5], Theorem 10)

(28)

$$-i_{X_{\Omega_2}} d\Omega_1 = \frac{1}{2} W^{\alpha\gamma} \frac{\partial f^\beta}{\partial \dot{q}^\gamma} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} - \frac{\partial \Omega_1}{\partial \dot{q}^\beta} \frac{\partial \Omega_2}{\partial \dot{q}^\alpha} \right) + W^{\alpha\beta} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} - \frac{\partial \Omega_1}{\partial \dot{q}^\beta} \frac{\partial \Omega_2}{\partial \dot{q}^\alpha} \right);$$

while (27) leads to

$$\begin{aligned} -i_{X_{\Omega_2}} d\Omega_1 &= \Gamma \left(W^{\alpha\beta} \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \right) \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} \\ &= \Gamma(W^{\alpha\beta}) \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \left(\Gamma \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \right) \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} \right) \\ &= \left(\Gamma(W^{\alpha\beta}) - W^{\beta\gamma} \frac{\partial f^\alpha}{\partial \dot{q}^\gamma} \right) \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} - \frac{\partial \Omega_1}{\partial \dot{q}^\beta} \frac{\partial \Omega_2}{\partial \dot{q}^\alpha} \right). \end{aligned}$$

Since by (16) and (26):

$$\begin{aligned} W^{\alpha\beta} W^{\beta\sigma} \left(\frac{\partial^2 L}{\partial \dot{q}^\gamma \partial q^\sigma} - \frac{\partial^2 L}{\partial \dot{q}^\sigma \partial q^\gamma} \right) &= -W^{\alpha\gamma} W^{\beta\sigma} \left(\Gamma(W_{\gamma\sigma}) + W_{\gamma\tau} \frac{\partial f^\tau}{\partial \dot{q}^\sigma} \right) \\ &= \Gamma(W^{\alpha\beta}) - W^{\beta\gamma} \frac{\partial f^\alpha}{\partial \dot{q}^\gamma}, \end{aligned}$$

it follows, beside (28), the other expression

(28)'

$$\begin{aligned} -i_{X_{\Omega_2}} d\Omega_1 &= W^{\alpha\gamma} W^{\beta\sigma} \left(\frac{\partial^2 L}{\partial \dot{q}^\gamma \partial q^\sigma} - \frac{\partial^2 L}{\partial \dot{q}^\sigma \partial q^\gamma} \right) \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} - \frac{\partial \Omega_1}{\partial \dot{q}^\beta} \frac{\partial \Omega_2}{\partial \dot{q}^\alpha} \right) \\ &= W^{\alpha\gamma} W^{\beta\sigma} \frac{\partial^2 L}{\partial \dot{q}^\gamma \partial q^\sigma} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} - \frac{\partial \Omega_1}{\partial \dot{q}^\beta} \frac{\partial \Omega_2}{\partial \dot{q}^\alpha} \right) + W^{\alpha\beta} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} - \frac{\partial \Omega_1}{\partial \dot{q}^\beta} \frac{\partial \Omega_2}{\partial \dot{q}^\alpha} \right). \end{aligned}$$

5. Lie algebra structure on conserved quantities

Similarly as in 4, let the system (1) have resulted from the Euler-Lagrange equations (15) with the regular Lagrangian L . Then, within the context of the calculus on differential forms, a procedure of constructing an infinite dimensional Lie algebra structure on the ring \mathfrak{R} of all conserved quantities of Γ , will begin with a closed 2-form \mathcal{E} which is given by the exterior derivative of Poincaré-Cartan form \mathcal{O} associated with the regular Lagrangian L :

$$\Theta = \frac{\partial L}{\partial \dot{q}^\alpha} \theta^\alpha + L dt.$$

In the derivative, $d\dot{q}^\alpha$ and dq^α are replaced respectively with $\phi^\alpha + f^\alpha dt$ and $\theta^\alpha + \dot{q}^\alpha dt$ to have the appearance of the form $\Xi = d\Theta$:

$$\Xi = W_{\alpha\beta} \phi^\alpha \wedge \theta^\beta - \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} \theta^\alpha \wedge \theta^\beta - \left(f^\beta W_{\alpha\beta} + \dot{q}^\beta \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} + \frac{\partial^2 L}{\partial q^\alpha \partial t} - \frac{\partial L}{\partial q^\alpha} \right) \theta^\alpha \wedge dt,$$

which is, by the Euler-Lagrange equations (15) and its equivalent form (1), reduced to

$$\Xi = W_{\alpha\beta} \phi^\alpha \wedge \theta^\beta - \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} \theta^\alpha \wedge \theta^\beta.$$

For the vector field $X \in \mathfrak{X}_T$ expressed as (3), in view of (16) with an alternation of the indices α and β :

$$\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} = \Gamma(W_{\alpha\beta}) + W_{\beta\gamma} \frac{\partial f^\gamma}{\partial \dot{q}^\alpha},$$

it follows that (note: $i_T \phi^\alpha = i_T \theta^\alpha = 0$)

$$\begin{aligned} (29) \quad -i_X \Xi &= W_{\alpha\beta} \xi^\beta \phi^\alpha - \left(W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi^\beta \right) \theta^\alpha \\ &= W_{\alpha\beta} \xi^\beta \phi^\alpha - \left(\Gamma(W_{\alpha\beta} \xi^\beta) + W_{\beta\gamma} \xi^\beta \frac{\partial f^\gamma}{\partial \dot{q}^\alpha} \right) \theta^\alpha. \end{aligned}$$

Therefore, by the regularity condition of L : $\det(W_{\alpha\beta}) \neq 0$, $i_X \Xi = 0$ implies $W_{\alpha\beta} \xi^\beta = 0$, i.e., $\xi^\beta = 0$, consequently $X \equiv 0 \pmod{\Gamma}$. So that Ξ is non-degenerate on the set of equivalent classes of \mathfrak{X}_T .

Now, for an arbitrary element $X \in \mathfrak{X}_T$, define a 1-form ω_X by

$$(30) \quad \omega_X = -i_X \Xi.$$

Then, since $d\Xi = d^2\Theta = 0$ and since $i_T \Xi = 0$ by $i_T \phi^\alpha = i_T \theta^\alpha = 0$, it follows that

$$\mathfrak{L}_T \Xi = i_T d\Xi + di_T \Xi = 0,$$

and so, moreover that

$$\begin{aligned} i_{[T, X]} \Xi &= \mathfrak{L}_T i_X \Xi - i_X \mathfrak{L}_T \Xi \\ &= \mathfrak{L}_T i_X \Xi = -\mathfrak{L}_T \omega_X. \end{aligned}$$

Therefore the non-degeneracy of Ξ implies that $\omega_X \in \mathfrak{X}_T^*$, i.e., $\mathfrak{L}_T \omega_X = 0$, if and only if $[T, X] \equiv 0 \pmod{\Gamma}$. Thus the correspondence in the theorem 7 can be realized by Ξ .

THEOREM 8. *Under the relation (30), the form Ξ defines a bijection between the set \mathfrak{X}_F^* and the equivalent classes of a subset of \mathfrak{X}_F :*

$$\mathfrak{X}_F^0 = \{X \in \mathfrak{X}_F \mid [F, X] \equiv 0 \pmod{F}\}.$$

Particularly for an element $d\Omega$ of a subset \mathfrak{X}_0^* of \mathfrak{X}_F^* :

$$\mathfrak{X}_0^* = \{d\Omega \mid \Omega \in \mathfrak{R}, \text{ i.e., } \Gamma(\Omega) = 0\},$$

there exists an element $X_\Omega \in \mathfrak{X}_F^0$, uniquely up to a multiple of F , satisfying

$$d\Omega = -i_{X_\Omega}\Xi;$$

whose appearance in coordinates follows from (29) by putting X as X_Ω of (25), i.e., ξ^α as $W^{\alpha\beta}\partial\Omega/\partial\dot{q}^\beta$, and consequently (25) leads to $d\Omega$ of (24). So that, since $\Gamma(\Omega) = 0$, it is well-defined a product $\{\Omega_1, \Omega_2\} \in \mathfrak{R}$ of elements $\Omega_i \in \mathfrak{R}$ ($i = 1, 2$):

$$(31) \quad \{\Omega_1, \Omega_2\} = -i_{X_{\Omega_2}}d\Omega_1 = -X_{\Omega_2}(\Omega_1),$$

which is written in coordinates as (28) or equivalently as (28)'. Then, in a familiar calculations on differential forms, we can see the anti-commutativity for $\Omega_i \in \mathfrak{R}$ ($i = 1, 2$):

$$\begin{aligned} \{\Omega_1, \Omega_2\} &= i_{X_{\Omega_2}}i_{X_{\Omega_1}}\Xi = -i_{X_{\Omega_1}}i_{X_{\Omega_2}}\Xi \\ &= -\{\Omega_2, \Omega_1\}, \end{aligned}$$

and the Leibniz identity for $\Omega_i \in \mathfrak{R}$ ($i = 1, 2, 3$):

$$\begin{aligned} \{\Omega_1\Omega_2, \Omega_3\} &= -X_{\Omega_3}(\Omega_1\Omega_2) = -\Omega_1X_{\Omega_3}(\Omega_2) - \Omega_2X_{\Omega_3}(\Omega_1) \\ &= \Omega_1\{\Omega_2, \Omega_3\} + \Omega_2\{\Omega_1, \Omega_3\}. \end{aligned}$$

Moreover, in view of

$$\begin{aligned} \mathfrak{L}_{X_{\Omega_1}}\Xi &= (di_{X_{\Omega_1}} + i_{X_{\Omega_1}}d)\Xi \\ &= -d^2\Omega_1 + i_{X_{\Omega_1}}d^2\Theta = 0, \end{aligned}$$

it follows that

$$\begin{aligned} i_{[X_{\Omega_1}, X_{\Omega_2}]} \Xi &= (\mathfrak{L}_{X_{\Omega_1}}i_{X_{\Omega_2}} - i_{X_{\Omega_2}}\mathfrak{L}_{X_{\Omega_1}})\Xi \\ &= -\mathfrak{L}_{X_{\Omega_1}}d\Omega_2 = -dX_{\Omega_1}(\Omega_2) \\ &= d\{\Omega_2, \Omega_1\} = -d\{\Omega_1, \Omega_2\}; \end{aligned}$$

so, by the theorem 8, that

$$X_{\{\Omega_1, \Omega_2\}} \equiv [X_{\Omega_1}, X_{\Omega_2}] \pmod{F}.$$

Therefore, since $\Gamma(\Omega_3) = 0$ for $\Omega_3 \in \mathfrak{R}$, we can see

$$\begin{aligned}
\{\Omega_3, \{\Omega_1, \Omega_2\}\} &= -X_{\{\Omega_1, \Omega_2\}}(\Omega_3) = -[X_{\Omega_1}, X_{\Omega_2}](\Omega_3) \\
&= -X_{\Omega_1}(X_{\Omega_2}(\Omega_3)) + X_{\Omega_2}(X_{\Omega_1}(\Omega_3)) \\
&= X_{\Omega_1}\{\Omega_3, \Omega_2\} - X_{\Omega_2}\{\Omega_3, \Omega_1\} \\
&= -\{\{\Omega_3, \Omega_2\}, \Omega_1\} + \{\{\Omega_3, \Omega_1\}, \Omega_2\},
\end{aligned}$$

which is rearranged to conclude the Jacobi identity

$$\{\{\Omega_1, \Omega_2\}, \Omega_3\} + \{\{\Omega_2, \Omega_3\}, \Omega_1\} + \{\{\Omega_3, \Omega_1\}, \Omega_2\} = 0.$$

Thus the product (31) gives the Poisson algebra structure on \mathfrak{R} .

THEOREM 9. *The ring \mathfrak{R} of all conserved quantities of Γ forms an infinite dimensional Lie algebra under the Poisson product defined by (31).*

6. Illustrative examples

In illustration of the geometric derivation of conservation laws, we first re-examine the second-order equations of two-dimensional harmonic oscillator

$$\ddot{q}^\alpha + q^\alpha = 0 \quad (\alpha = 1, 2),$$

with the Lagrangian

$$L = \frac{1}{2}(\dot{q}^\alpha \dot{q}^\alpha - q^\alpha q^\alpha).$$

As in [5], since $f^\alpha = -q^\alpha$, the equations (4) and (6) are written as the similar forms:

$$\Gamma^2(\xi^\alpha) + \xi^\alpha = 0, \quad \Gamma^2(\mu_\alpha) + \mu_\alpha = 0,$$

which have solutions ξ_i^α ($i = 1, 2, 3, 4$):

$$\begin{aligned}
\xi_1^\alpha &= \mu_\alpha^1 = q^\alpha, \quad \xi_2^\alpha = \mu_\alpha^2 = a_\alpha \cos t, \\
\xi_3^\alpha &= \mu_\alpha^3 = a_\alpha \sin t, \quad \xi_4^\alpha = \mu_\alpha^4 = \dot{q}^\alpha,
\end{aligned}$$

where a_α are some constants. For the respective solutions ξ_i^α , elements $X_i \in \mathfrak{X}_\Gamma$ satisfying $[L, X_i] = 0$ are determined (Theorem 1):

$$\begin{aligned}
X_1 &= \dot{q}^\alpha \frac{\partial}{\partial \dot{q}^\alpha} + q^\alpha \frac{\partial}{\partial q^\alpha}, \quad X_2 = -a_\alpha \sin t \frac{\partial}{\partial q^\alpha} + a_\alpha \cos t \frac{\partial}{\partial q^\alpha}, \\
X_3 &= a_\alpha \cos t \frac{\partial}{\partial \dot{q}^\alpha} + a_\alpha \sin t \frac{\partial}{\partial q^\alpha}, \quad X_4 = -q^\alpha \frac{\partial}{\partial \dot{q}^\alpha} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha};
\end{aligned}$$

and for the respective solutions μ_α^i , elements $\omega_i \in \mathfrak{X}_\Gamma^*$ are determined also up to a multiple of dt (Theorem 2):

$$\begin{aligned}\omega_1 &= q^\alpha \phi^\alpha - \dot{q}^\alpha \theta^\alpha, \quad \omega_2 = (a_\alpha \cos t) \phi^\alpha + (a_\alpha \sin t) \theta^\alpha, \\ \omega_3 &= (a_\alpha \sin t) \phi^\alpha - (a_\alpha \cos t) \theta^\alpha, \quad \omega_4 = \dot{q}^\alpha \phi^\alpha + q^\alpha \theta^\alpha.\end{aligned}$$

The couples of elements $X_1 \in \mathfrak{X}_T$ and $\omega_i \in \mathfrak{X}_T^*$ ($i = 2, 3, 4$) are carried into (9). Then, through

$$\begin{aligned}i_{X_1} \omega_2 &= a_\alpha (\dot{q}^\alpha \cos t + q^\alpha \sin t), \\ i_{X_1} \omega_3 &= a_\alpha (\dot{q}^\alpha \sin t - q^\alpha \cos t), \\ i_{X_1} \omega_4 &= \dot{q}^\alpha \dot{q}^\alpha + q^\alpha q^\alpha,\end{aligned}$$

the following conserved quantities are obtained:

$$\begin{aligned}\Omega_1^\alpha &= \dot{q}^\alpha \cos t + q^\alpha \sin t, \\ \Omega_2^\alpha &= \dot{q}^\alpha \sin t - q^\alpha \cos t, \\ \Omega_3 &= \frac{1}{2}(\dot{q}^\alpha \dot{q}^\alpha + q^\alpha q^\alpha),\end{aligned}$$

where Ω_1^α and Ω_2^α are independent, while $\Omega_3 = \frac{1}{2}(\Omega_1^\alpha \Omega_1^\alpha + \Omega_2^\alpha \Omega_2^\alpha)$. These quantities can be derived also from (11) with $\Gamma_0 = X_4$ and ω_i ($i = 1, 2, 3$):

$$i_{\Gamma_0} \omega_1 = -\Omega_3, \quad i_{\Gamma_0} \omega_2 = a_\alpha \Omega_2^\alpha, \quad i_{\Gamma_0} \omega_3 = -a_\alpha \Omega_1^\alpha.$$

By (12), one of the above Ω 's yields the other for suitable $X_i \in \mathfrak{X}_T$, e.g.,

$$X_2(\Omega_3) = -a_\alpha \Omega_2^\alpha, \quad X_4(\Omega_2^\alpha) = -\Omega_1^\alpha.$$

Since $W_{\alpha\beta} = \partial^2 L / \partial \dot{q}^\alpha \partial \dot{q}^\beta = \delta_{\alpha\beta}$ accordingly $W^{\alpha\beta} = \delta^{\alpha\beta}$, the corresponding vector fields of (25) to the conserved quantities are

$$\begin{aligned}X_{\Omega_1^\alpha} &= -\sin t \frac{\partial}{\partial \dot{q}^\alpha} + \cos t \frac{\partial}{\partial q^\alpha}, \\ X_{\Omega_2^\alpha} &= \cos t \frac{\partial}{\partial \dot{q}^\alpha} + \sin t \frac{\partial}{\partial q^\alpha}, \\ X_{\Omega_3} &= -q^\alpha \frac{\partial}{\partial \dot{q}^\alpha} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha};\end{aligned}$$

which are used in the last term of (31) to obtain the Poisson product

$$\{\Omega_1^\alpha, \Omega_2^\beta\} = -\delta^{\alpha\beta}, \quad \{\Omega_2^\alpha, \Omega_3\} = \Omega_1^\alpha, \quad \{\Omega_3, \Omega_1^\alpha\} = \Omega_2^\alpha.$$

Next example is a single particle moving under a central force, whose equations of the motion are (refer to [4])

$$\begin{aligned}\ddot{q}^1 + 2\mu\dot{q}^1 + \omega^2 q^1 - q^1(\dot{q}^2)^2 &= 0, \\ q^1 \ddot{q}^2 + 2\mu q^1 \dot{q}^2 + 2\dot{q}^1 \dot{q}^2 &= 0,\end{aligned}$$

with the Lagrangian given by

$$L = \frac{1}{2} e^{2\mu} ((\dot{q}^1)^2 + (q^1 \dot{q}^2)^2 - (\omega q^1)^2),$$

for the generalized coordinates $q^1 = r$ and $q^2 = \varphi$, where μ and ω are some constants. In this case, since

$$\begin{aligned} f^1 &= -2\mu \dot{q}^1 + q^1 (\dot{q}^2)^2 - \omega^2 q^1, \\ f^2 &= -\frac{2\dot{q}^1 \dot{q}^2}{q^1} - 2\mu \dot{q}^2, \end{aligned}$$

the equations of (4) are reduced to

$$\begin{aligned} \Gamma^2(\xi^1) + 2\mu \Gamma(\xi^1) - 2q^1 \dot{q}^2 \Gamma(\xi^2) - ((\dot{q}^2)^2 - \omega^2) \xi^1 &= 0, \\ \Gamma^2(\xi^2) + 2\left(\frac{\dot{q}^1}{q^1} + \mu\right) \Gamma(\xi^2) + \frac{2\dot{q}^2}{q^1} \Gamma(\xi^1) - \frac{2\dot{q}^1 \dot{q}^2}{(q^1)^2} \xi^1 &= 0. \end{aligned}$$

According to the respective solutions ξ_i^a ($i = 1, 2$) obtained in [5]:

$$\xi_1^1 = a\dot{q}^1, \quad \xi_1^2 = a\dot{q}^2; \quad \xi_2^1 = bq^1, \quad \xi_2^2 = k,$$

where a, b and k are some constants, elements $X_i \in \mathfrak{X}_F$ satisfying $[I, X_i] = 0$ are written as (Theorem 1)

$$\begin{aligned} X_1 &= af^1 \frac{\partial}{\partial \dot{q}^1} + af^2 \frac{\partial}{\partial \dot{q}^2} + a\dot{q}^1 \frac{\partial}{\partial q^1} + a\dot{q}^2 \frac{\partial}{\partial q^2}, \\ X_2 &= b\dot{q}^1 \frac{\partial}{\partial \dot{q}^1} + bq^1 \frac{\partial}{\partial q^1} + k \frac{\partial}{\partial q^2}. \end{aligned}$$

Corresponding to the elements $X_i \in \mathfrak{X}_F$, we can find elements $\omega_L^i \in \mathfrak{X}_F^*$ (Theorem 7). In fact, in view of

$$\begin{aligned} (W_{\alpha\beta}) &= e^{2\mu} \begin{pmatrix} 1 & 0 \\ 0 & (q^1)^2 \end{pmatrix}, \\ \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) &= 2e^{2\mu} \begin{pmatrix} 0 & -q^1 \dot{q}^2 \\ q^1 \dot{q}^2 & 0 \end{pmatrix}, \end{aligned}$$

they are determined up to a multiple of dt (Theorem 6):

$$\begin{aligned} \omega_L^i &= e^{2\mu} (\phi^1 \phi^2) \begin{pmatrix} 1 & 0 \\ 0 & (q^1)^2 \end{pmatrix} \begin{pmatrix} \xi_i^1 \\ \xi_i^2 \end{pmatrix} \\ &\quad - e^{2\mu} (\theta^1 \theta^2) \left(\begin{pmatrix} 1 & 0 \\ 0 & (q^1)^2 \end{pmatrix} \begin{pmatrix} \Gamma(\xi_i^1) \\ \Gamma(\xi_i^2) \end{pmatrix} + 2 \begin{pmatrix} 0 & -q^1 \dot{q}^2 \\ q^1 \dot{q}^2 & 0 \end{pmatrix} \begin{pmatrix} \xi_i^1 \\ \xi_i^2 \end{pmatrix} \right) \end{aligned}$$

$$= e^{2\mu}(\phi^1 \phi^2) \begin{pmatrix} \xi_i^1 \\ (q^1)^2 \xi_i^2 \end{pmatrix} - e^{2\mu}(\theta^1 \theta^2) \begin{pmatrix} \Gamma(\xi_i^1) - 2q^1 \dot{q}^2 \xi_i^2 \\ (q^1)^2 \Gamma(\xi_i^2) + 2q^1 \dot{q}^2 \xi_i^1 \end{pmatrix}.$$

Therefore ω_L^1 and ω_L^2 lead respectively to

$$\begin{aligned} \omega_L^1 &= ae^{2\mu}(\phi^1 \phi^2) \begin{pmatrix} \dot{q}^1 \\ (q^1)^2 \dot{q}^2 \end{pmatrix} - ae^{2\mu}(\theta^1 \theta^2) \begin{pmatrix} f^1 - 2q^1 (\dot{q}^2)^2 \\ (q^1)^2 f^2 + 2q^1 \dot{q}^1 \dot{q}^2 \end{pmatrix} \\ &= ae^{2\mu}(\dot{q}^1 \phi^1 + (q^1)^2 \dot{q}^2 \phi^2) - ae^{2\mu}((f^1 - 2q^1 (\dot{q}^2)^2)\theta^1 + ((q^1)^2 f^2 + 2q^1 \dot{q}^1 \dot{q}^2)\theta^2), \\ \omega_L^2 &= e^{2\mu}(\phi^1 \phi^2) \begin{pmatrix} bq^1 \\ k(q^1)^2 \end{pmatrix} - e^{2\mu}(\theta^1 \theta^2) \begin{pmatrix} b\dot{q}^1 - 2kq^1 \dot{q}^2 \\ 2b(q^1)^2 \dot{q}^2 \end{pmatrix} \\ &= e^{2\mu}(bq^1 \phi^1 + k(q^1)^2 \phi^2) - e^{2\mu}((b\dot{q}^1 - 2kq^1 \dot{q}^2)\theta^1 + 2b(q^1)^2 \dot{q}^2 \theta^2). \end{aligned}$$

Now (9) is calculated with the elements $X_i \in \mathfrak{X}_F$ and $\omega_L^i \in \mathfrak{X}_F^*$ ($i = 1, 2$), e.g.,

$$\begin{aligned} i_{X_2} \omega_L^1 &= abe^{2\mu}((\dot{q}^1)^2 - q^1 f^1 + 2(q^1 \dot{q}^2)^2) - ake^{2\mu}((q^1)^2 f^2 + 2q^1 \dot{q}^1 \dot{q}^2) \\ &= abe^{2\mu}((\dot{q}^1)^2 + (q^1 \dot{q}^2)^2 + (\omega q^1)^2 + 2\mu q^1 \dot{q}^1) + 2ak\mu e^{2\mu}(q^1)^2 \dot{q}^2, \end{aligned}$$

in which the following independent conserved quantities are observed:

$$\begin{aligned} \Omega_1 &= e^{2\mu}(\frac{1}{2}(\dot{q}^1)^2 + \frac{1}{2}(q^1 \dot{q}^2)^2 + \frac{1}{2}(\omega q^1)^2 + \mu q^1 \dot{q}^1), \\ \Omega_2 &= e^{2\mu}(q^1)^2 \dot{q}^2, \end{aligned}$$

while Ω_1 was obtained by Djukic [4] in the illustration of the gauge-variant Lagrangians. Here note that (11) with $\Gamma_0 = X_1$ and $\omega = \omega_L^2$ can be written also as

$$i_{\Gamma_0} \omega_L^2 = -2ab\Omega_1 - 2ak\mu\Omega_2.$$

The corresponding vector field $X_{\Omega_2} = \partial/\partial q^2$ to Ω_2 follows from (25) with

$$(W^{\alpha\beta}) \begin{pmatrix} \partial \Omega \\ \partial \dot{q}^\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (q^1)^{-2} \end{pmatrix} \begin{pmatrix} 0 \\ (q^1)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So, by (31), the Poisson product of Ω_1 and Ω_2 leads to

$$\{\Omega_1, \Omega_2\} = -X_{\Omega_2}(\Omega_1) = 0.$$

Final example is the equation arises in gas dynamics (refer to [8], also [11])

$$\ddot{q} - aq^{-1/2} = 0 \quad (a: \text{const.}),$$

with the Lagrangian

$$L = \frac{1}{2} \dot{q}^2 + 2aq^{1/2}.$$

Since $f = aq^{-1/2}$, the equations (4) and (6) are written as the similar forms:

$$\Gamma^2(\xi) + \frac{1}{2}aq^{-3/2}\xi = 0, \quad \Gamma^2(\mu) + \frac{1}{2}aq^{-3/2}\mu = 0.$$

Here, assuming that ξ (also μ) is of the form:

$$\xi = k\dot{q}^2 + g(q, t)\dot{q} + h(q, t) \quad (k: \text{const.}),$$

the equation of ξ (also of μ) can be reduced to

$$\begin{aligned} \frac{\partial^2 g}{\partial q^2} \dot{q}^3 + \left(2 \frac{\partial^2 g}{\partial q \partial t} + \frac{\partial^2 h}{\partial q^2} - \frac{1}{2}akq^{-3/2} \right) \dot{q}^2 + \left(\frac{\partial^2 g}{\partial t^2} + 3aq^{-1/2} \frac{\partial g}{\partial t} + 2 \frac{\partial^2 h}{\partial q \partial t} \right) \dot{q} \\ + 2aq^{1/2} \frac{\partial g}{\partial t} + \frac{\partial^2 h}{\partial t^2} + aq^{-1/2} \frac{\partial h}{\partial q} + \frac{1}{2}aq^{-3/2}h + 2a^2kq^{-1} = 0, \end{aligned}$$

which is satisfied for arbitrary \dot{q} if

$$(32a) \quad \frac{\partial^2 g}{\partial q^2} = 0,$$

$$(32b) \quad 2 \frac{\partial^2 g}{\partial q \partial t} + \frac{\partial^2 h}{\partial q^2} - \frac{1}{2}akq^{-3/2} = 0,$$

$$(32c) \quad \frac{\partial^2 g}{\partial t^2} + 3aq^{-1/2} \frac{\partial g}{\partial t} + 2 \frac{\partial^2 h}{\partial q \partial t} = 0,$$

$$(32d) \quad 2aq^{-1/2} \frac{\partial g}{\partial t} + \frac{\partial^2 h}{\partial t^2} + aq^{-1/2} \frac{\partial h}{\partial q} + \frac{1}{2}aq^{-3/2}h + 2a^2kq^{-1} = 0.$$

In view of (32a), by putting

$$g = \varphi(t)q + \psi(t),$$

(32b) leads to

$$\frac{\partial^2 h}{\partial q^2} = -2\varphi' + \frac{1}{2}akq^{-3/2},$$

which is integrated as

$$h = -\varphi'q^2 - 2akq^{1/2} + \gamma(t)q + \tau(t).$$

So, by substituting these g and h for (32c), it follows that

$$3\varphi''q - 3a\varphi q^{-1/2} - \psi'' - 2\gamma' = 0,$$

accordingly $\varphi = 0$ and

$$(33) \quad \psi'' + 2\gamma' = 0;$$

and also for (32d), it follows that

$$2\gamma''q + a(4\psi' + 3\gamma)q^{-1/2} + a\tau q^{-3/2} + 2\tau'' = 0,$$

accordingly $\gamma'' = 0$, $\tau = 0$ and

$$(34) \quad 4\psi' + 3\gamma = 0.$$

By (33) and the differentiation of (34), ψ is determined as $\psi = 3mt + n$ (m, n : const.), and then $\gamma = -4m$. So that $g = 3mt + n$ and $h = -2akq^{1/2} - 4mq$, consequently

$$\xi = k(\dot{q}^2 - 2aq^{1/2}) + m(3t\dot{q} - 4q) + n\dot{q}.$$

Now from the respective solutions:

$$\xi_1 = \mu_1 = \dot{q}^2 - 2aq^{1/2}, \quad \xi_2 = \mu_2 = 3t\dot{q} - 4q, \quad \xi_3 = \mu_3 = \dot{q},$$

it follows the elements $X_i \in \mathfrak{X}_T$ satisfying $[\Gamma, X_i] = 0$ (Theorem 1):

$$X_1 = a\dot{q}q^{-1/2} \frac{\partial}{\partial \dot{q}} + (\dot{q}^2 - 2aq^{1/2}) \frac{\partial}{\partial q},$$

$$X_2 = -(\dot{q} - 3atq^{-1/2}) \frac{\partial}{\partial \dot{q}} + (3t\dot{q} - 4q) \frac{\partial}{\partial q},$$

$$X_3 = aq^{-1/2} \frac{\partial}{\partial \dot{q}} + \dot{q} \frac{\partial}{\partial q};$$

and also the elements $\omega_i \in \mathfrak{X}_T^*$ up to a multiple of dt (Theorem 2):

$$\omega_1 = (\dot{q}^2 - 2aq^{1/2})\phi - a\dot{q}q^{-1/2}\theta,$$

$$\omega_2 = (3t\dot{q} - 4q)\phi + (\dot{q} - 3atq^{-1/2})\theta,$$

$$\omega_3 = \dot{q}\phi - aq^{-1/2}\theta.$$

Therefore conserved quantities of (9), e.g., $\Omega_1 = -i_{X_2}\omega_3$ and $\Omega_2 = i_{X_1}\omega_2$ are written respectively as

$$\Omega_1 = \dot{q}^2 - 4aq^{1/2},$$

$$\Omega_2 = \dot{q}^3 - 6a\dot{q}q^{1/2} + 6a^2t,$$

while (11) yields $-i_{\Gamma_0}\omega_2 = \Omega_1$ ($\Gamma_0 = X_3$) also. By putting $t = x$, $q = V + \varepsilon^2$ and $a = 2^{3/2}(e/m)^{-1/2}\pi I$, the conserved quantities Ω_1 and Ω_2 lead respectively to those (Eq. (5) and (6) in [8]) obtained by Parsons. For Ω_2 , since $\partial^2 L / \partial \dot{q}^2 = 1$, (25) is written as

$$X_{\Omega_2} = 3a\dot{q}q^{-1/2} \frac{\partial}{\partial \dot{q}} + 3(\dot{q}^2 - 2aq^{1/2}) \frac{\partial}{\partial q},$$

which is used in (31) to obtain the Poisson product

$$\{\Omega_1, \Omega_2\} = -X_{\Omega_2}(\Omega_1) = -12a^2.$$

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*Department of Mathematics
Kyushu Institute of Technology
Tobata, Kitakyushu, 804, Japan*