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SINGULAR SOLUTIONS OF TRAVELING WAVES IN A CHEMOTACTIC MODEL

By

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1. Introduction

Some biological phenomena exhibit a waveform which travels without change of shape and has a constant speed. Keller and Segel [4] have put forward a mathematical model for chemotaxis in a bacteria-substrate mixture to describe the phenomena of traveling bands observed by Adler [1]. The simplified model is described by two partial differential equations

$$\frac{\partial b}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial b}{\partial x} - \alpha \frac{b}{s}\frac{\partial s}{\partial x}\right), \tag{1.1}$$

$$\frac{\partial s}{\partial t} = \varepsilon \frac{\partial^2 s}{\partial x^2} - b, \tag{1.2}$$

where $\alpha > 0$ and $\varepsilon \geq 0$. In the model, $b(x, t)$ is the density of the bacteria at position x and time t and $s(x, t)$ denotes the concentration of the critical substrate.

A traveling wave of (1.1), (1.2) is a solution having the form $(b(x, t), s(x, t)) = (B(x - ct), S(x - ct))$, where the constant c is refered to as the wave speed. By introducing the traveling coordinate $z = x - ct$, the partial differential equations (1.1) and (1.2) are reduced to the system of two ordinary differential equations on $\mathbf{R}$

$$0 = \left(B' - \alpha \frac{B}{S} S' + cB\right)', \tag{1.3}$$

$$0 = \varepsilon S'' + cS' - B, \tag{1.4}$$

where $' = d/dz$. As mentioned in [8], this system is considered under the conditions

$$B \geq 0, \int_{-\infty}^{\infty} B(z)\,dz = N, \tag{1.5}$$

$$S \geq 0, \ S(-\infty) = 0, \ S(\infty) = s_\infty, \tag{1.6}$$

where N and s_∞ are given positive numbers.

When $\varepsilon = 0$, for $\alpha > 1$ Keller and Segel [4] have given explicit traveling waves, which have compared with experimental results. For $\alpha \leq 1$ Odell and Keller [9] obtained an explicit solution $(B(z), S(z))$, which has an interesting feature such that

$$B(z) = S(z) = 0 \, (z \leq z_0) \quad \text{and} \quad B(z) > 0, \; S(z) > 0 \, (z > z_0)$$

for some z_0 if $\alpha < 1$. Such a solution is called a *singular* solution. More general systems have been studied by [12] numerically, and by [2, 5, 10, 11] analytically. When $\varepsilon > 0$ and $\alpha > 1$, the band propagation has been studied by [3] both numerically and experimentally. Nagai and Ikeda [8] have proved the existence of traveling waves and studied the linearized instability of traveling waves for some classes of perturbations decaying at infinity. We refer to [7, 13] for mathematical models of biological waves.

In this paper we deal with the case

$$\varepsilon > 0 \quad \text{and} \quad \alpha \leq 1.$$

Under the condition the existence of traveling waves will be shown, and for the case $\alpha < 1$ the traveling waves are shown to be singular. We also discuss the dependency of traveling waves with respect to α and ε.

2. Main results

Solutions of (1.3)–(1.6) are defined in the following way because of singularity in (1.3). $\mathbf{T}(\mathbf{R})$ consists of all real valued functions $(B, S) \in C(\mathbf{R}) \times C^1(\mathbf{R})$ such that $B \in C^1(P(S))$, where

$$P(S) = \{z \in \mathbf{R} \mid S(z) > 0\}.$$

For $(B, S) \in \mathbf{T}(\mathbf{R})$ we put

$$J(z) = \begin{cases} B' - \alpha \dfrac{B}{S} S' + cB & \text{for } z \in P(S), \\ 0 & \text{for } z \notin P(S). \end{cases}$$

Our problem is to find $(B, S) \in \mathbf{T}(\mathbf{R})$ and $c \in \mathbf{R}$ satisfying

$$J \in C^1(\mathbf{R}) \quad \text{and} \quad J' = 0 \text{ in } \mathbf{R}, \tag{2.1}$$

$$S \in C^2(\mathbf{R}) \quad \text{and} \quad \varepsilon S'' + cS' - B = 0 \text{ in } \mathbf{R} \tag{2.2}$$

under the conditions (1.5) and (1.6).

As will be shown in Lemma 3.1 the traveling wave solution (B, S) satisfies the relation

$$B = C \times F(z, S) \qquad \text{for some positive constant } C,$$

where $F(z, S)$ is given by

(2.3) $$F(z, S) = e^{-cz} S^{\alpha}.$$

Using a shift in z-coordinate system, we can assume $B = F(z, S)$. Such a solution (B, S) is called the *normalized* traveling wave, for which the problem is reduced to a problem for a single equation with respect to S to obtain the following theorem.

THEOREM 2.1. (i) *For given N, $s_\infty > 0$ there exists a traveling wave solution (B, S) which is unique up to shift in z-coordinate system, and the wave speed c is uniquely determined by $c = N/s_\infty$.*

(ii) *When $\alpha = 1$, $B(z) > 0$ and $S(z) > 0$ for $z \in \mathbf{R}$.*

(iii) *When $\alpha < 1$, $P(S) = (z_0, \infty)$ for some $z_0 \in \mathbf{R}$, that is, (B, S) is a singular solution.*

For $\varepsilon \geq 0$ we denote the normalized traveling wave solution of (1.3)–(1.6) by $(B_\varepsilon, S_\varepsilon)$. When $\varepsilon = 0$, such a solution is given explicitly in [9] as follows: In the case $\alpha = 1$

$$S_0(z) = s_\infty \exp\left(-\frac{1}{c^2}\exp(-cz)\right)$$

and in the case $\alpha < 1$

$$S_0(z) = \left\{\left(s_\infty^{1-\alpha} + \frac{\alpha - 1}{c^2} e^{-cz}\right)^+\right\}^{\frac{1}{1-\alpha}},$$

where $(a)^+ = \max\{a, 0\}$. We then have the following theorem.

THEOREM 2.2. *$\| S_\varepsilon - S_0 \|_{L^\infty} = O(\varepsilon)$ and $\| B_\varepsilon - B_0 \|_{L^\infty} = O(\varepsilon^\beta)$ as $\varepsilon \to 0$, where $\beta = \alpha$ if $0 < \alpha < 1$ and $0 < \beta < 1$ if $\alpha = 1$.*

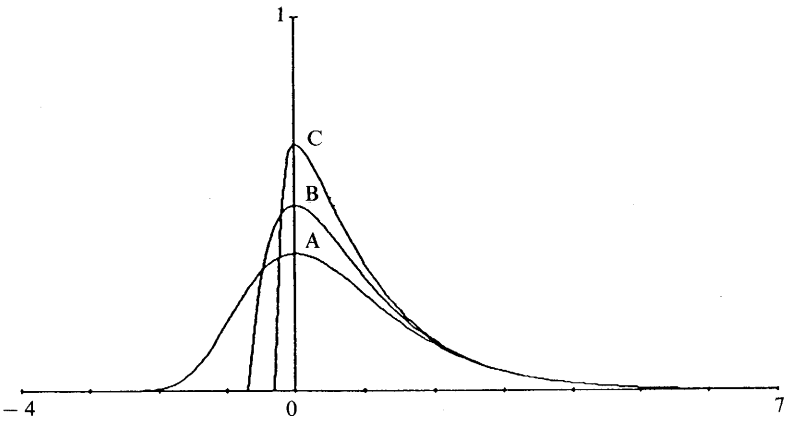

Figure 1. Profiles of the exact solutions ($\varepsilon = 0.0$).

By Theorem 2.2, (B_0, S_0) is approximated by $(B_\varepsilon, S_\varepsilon)$ when ε is sufficiently small. We are then concerned with the dependency of the band shape (the shape of B_ε) with respect to α and ε. Figures 1–3 show the profiles of bacterial bands. The curve A is for the case $\alpha = 1.0$, the curve B for the case $\alpha = 0.5$ and the curve C for the case $\alpha = 0.2$. When $\varepsilon = 0$, the profiles of the exact solutions are shown in Figure 1. The bacterial bands become steeper in the rear and narrower as α decreases. When $\varepsilon > 0$, the profiles of numerical solutions say that the bacterial bands have the same properties as in the case $\varepsilon = 0$ (see Figures 2 and 3).

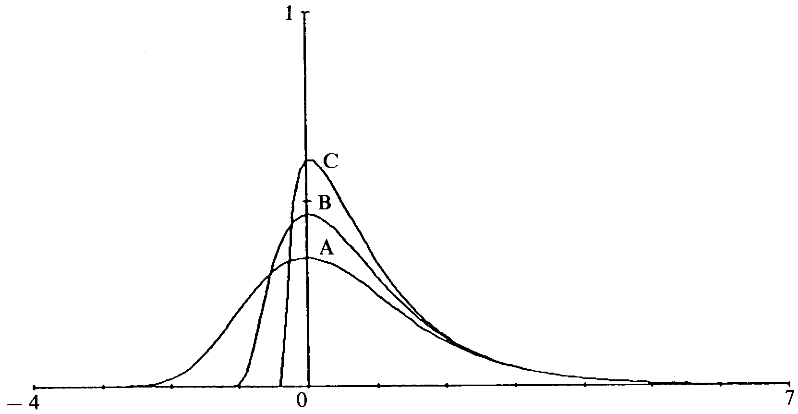

Figure 2. Profiles of numerical solutions ($\varepsilon = 0.1$).

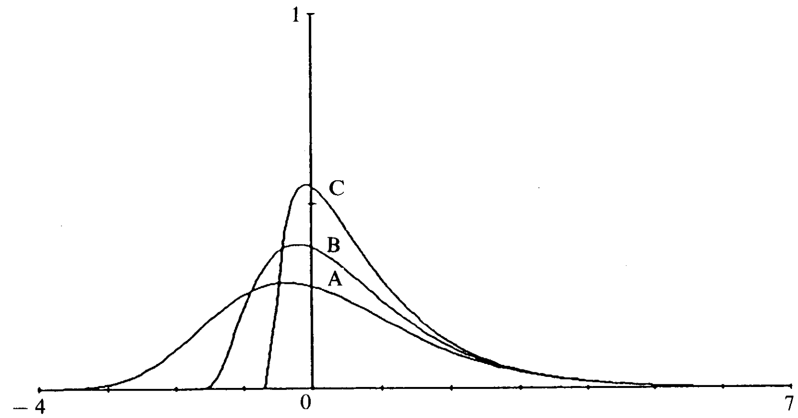

Figure 3. Profiles of numerical solutions ($\varepsilon = 1.0$).

3. A single equation related to (1.3), (1.4)

For given N, $s_\infty > 0$ let (B, S) be the traveling wave solution with the wave speed c. In order to reduce (1.3), (1.4) to a single equation, we need the following lemma.

LEMMA 3.1. (i) *$S'(\pm\infty)=0$ and $S'\geq 0$.*

(ii) *$B(z)=Ce^{-cz}S^{\alpha}(z)$ for a constant $C>0$.*

(iii) $c=N/s_{\infty}$.

Proof. By using the same way as in the proof of Lemma 2.1 [8], (i) and (iii) are proved. To prove (ii) we put

$$z_0=\inf\{z\in\mathbf{R}\,|\,S(z)>0\}.$$

When $z_0=-\infty$, $S(z)>0$ for $z\in\mathbf{R}$. Hence, (ii) is proved in the same way as in Lemma 2.1 [8]. We will prove (ii) in the case when $-\infty<z_0<\infty$.

By $S'\geq 0$ and the definition of z_0, we have $P(S)=(z_0,\infty)$. Hence, by (2.1) we obtain

$$B'-\alpha\frac{S'}{S}B+cB=0 \qquad \text{for } z>z_0,$$

which implies that $B(z)>0$ for $z>z_0$ and

$$B(z)=Ce^{-cz}S^{\alpha}(z) \qquad \text{for } z\geq z_0, \tag{3.1}$$

where C is a positive constant. For $z<z_0$, it follows from (2.2) that $B(z)=0$. Therefore (ii) holds for $z\in\mathbf{R}$. Thus the proof is complete.

Using a shift of (B, S) in z-coordinate sysytem, we can take C in (3.1) to be 1, which means that (B, S) can be a normalized traveling wave. $S(z)$ in this case is a solution of the problem

$$\varepsilon S''+cS'-F(z,S)=0 \quad \text{in } \mathbf{R}, \tag{3.2}$$

$$S\geq 0,\ S(-\infty)=0,\ S(\infty)=s_{\infty}, \tag{3.3}$$

$$F(z,S)\in L^1(\mathbf{R}), \tag{3.4}$$

where $c=N/s_{\infty}$ and $F(z, S)$ is the same one as in (2.3).

The following lemma is used to prove the uniqueness of solutions of (3.2)–(3.4) and that a solution of (3.2)–(3.4) is singular when $\alpha<1$.

LEMMA 3.2. *Let $-\infty\leq a<b\leq\infty$. Suppose that $S_1, S_2\in C^2(a,b)$ and*

$$\varepsilon S_1''+cS_1'-F(z,S_1)\leq 0,\ \varepsilon S_2''+cS_2'-F(z,S_2)\geq 0 \qquad \text{in } (a,b). \tag{3.5}$$

If $S_2(a)\leq S_1(a)$ and $S_2(b)\leq S_1(b)$, then $S_2(z)\leq S_1(z)$ on (a, b).

Proof. For $W=S_2-S_1$, it follows from (3.5) that

$$\varepsilon W''+cW'\geq F(z,S_2)-F(z,S_1). \tag{3.6}$$

Let us assume that $\sup\{W(z)|a<z<b\}>0$. Since $W(a)\leq 0$ and $W(b)\leq 0$, there exists $z_0\in(a, b)$ such that $W(z_0)=\max\{W(z)|a<z<b\}>0$. At the point z_0, $W''\leq 0$ and $W'=0$. Using these in (3.6), we have

$$0\geq\varepsilon W''(z_0)\geq F(z_0, S_2(z_0))-F(z_0, S_1(z_0))>0,$$

which gives a contradiction. Hence, our assertion has been proved.

Let S be a solution of (3.2)–(3.4) and put $B=F(z, S)$. It is easily seen that $S'\geq 0$ and $S'(\pm\infty)=0$, from which $P(S)$ is an interval, $J\equiv 0$ and $\int_{-\infty}^{\infty}B(z)\,dz=N$. Hence, (B, S) is a normalized traveling wave, which means that the problem for (1.3)–(1.6) is reduced to the problem for (3.2)–(3.4). We also note that the uniqueness of solutions holds for (3.2)–(3.4) by Lemma 3.2.

LEMMA 3.3. (i) *A normalized traveling wave of* (1.3)–(1.6) *is a solution of* (3.2)–(3.4).

(ii) *For a solution S of* (3.2)–(3.4), (B, S) *with* $B=F(z, S)$ *is a normalized traveling wave of* (1.3)–(1.6) *with wave speed* $c=N/s_\infty$.

(iii) *The uniqueness holds for the problem* (3.2)–(3.4).

LEMMA 3.4. *For a solution S of* (3.2)–(3.4) *the following assertions hold.*

(i) *When* $\alpha=1$, $S(z)>0$ *for* $z\in\mathbf{R}$.

(ii) *When* $\alpha<1$, *for* $\varepsilon_0>0$ *there exists* z_1 *such that for* $0<\varepsilon<\varepsilon_0$

$$S(z)\equiv 0 \qquad (z\leq z_1).$$

Proof. Since the assertion (i) is easily shown, we prove (ii) by using Lemma 3.2. We choose β and z_0 so that

$$\beta>\frac{2}{1-\alpha}, \qquad \varepsilon_0\beta(\beta-1)+c\beta-e^{-cz_0}s_\infty^{\alpha-1}\leq 0,$$

and define the function W on $(-\infty, z_0)$ by

$$W(z)=\begin{cases} s_\infty(z-z_0+1)^\beta & \text{for } z_0-1<z\leq z_0, \\ 0 & \text{for } z\leq z_0-1. \end{cases}$$

Then it is clear that

$$S(-\infty)=W(-\infty), \qquad S(z_0)\leq W(z_0).$$

For $z_0-1<z<z_0$,

$$\begin{aligned} &\varepsilon W''+cW'-F(z, W)\\ &=s_\infty(z-z_0+1)^{\beta-2}\{\varepsilon\beta(\beta-1)+c\beta(z-z_0+1)-e^{-cz}s_\infty^{\alpha-1}(z-z_0+1)^{\alpha\beta-\beta+2}\}\\ &\leq s_\infty(z-z_0+1)^{\beta-2}\{\varepsilon\beta(\beta-1)+c\beta-e^{-cz_0}s_\infty^{\alpha-1}\}\\ &\leq 0, \end{aligned}$$

from which we have

$$\varepsilon W'' + cW' - F(z, W) \leq 0 \qquad \text{in } (-\infty, z_0).$$

Hence, by Lemma 3.2 we obtain

$$S(z) \leq W(z) \qquad (z \leq z_0),$$

which implies (ii). Thus the proof is complete.

4. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. By Lemmas 3.1 and 3.3, we have $c = N/s_\infty$ and the uniqueness of solutions up to shift in z-coordinate system. The assertion (iii) follows from Lemma 3.4. For the proof of existence it sufficies to show the existence of solutions of (3.2)–(3.4).

Let us consider the problem

$$w''(y) = py^q w^\alpha(y) \qquad \text{in } (0, \infty), \tag{4.1}$$

$$w(+0) = s_\infty \quad \text{and} \quad w(y) \geq 0,\ w'(y) \leq 0 \qquad (0 < y < \infty). \tag{4.2}$$

Here p and q are given by

$$p = \frac{1}{\varepsilon}\left(\frac{c}{\varepsilon}\right)^{\varepsilon - 2} \quad \text{and} \quad q = \varepsilon - 2.$$

The existence of solutions of (4.1), (4.2) is guaranteed by [6]. With a solution $w(y)$ of (4.1), (4.2), we define $S(z)$ by

$$S(z) = w(y), \qquad y = \frac{\varepsilon}{c}\exp\left(-\frac{c}{\varepsilon}z\right).$$

We see that $S(z)$ satisfies (3.2), $S \geq 0$ and $S(\infty) = s_\infty$. In order to prove that S is a solution of (3.2)–(3.4), we have to show that (3.4) and $S(-\infty) = 0$.

Let us prove (3.4). It follows from (4.1) that

$$(yw')' = py^{q+1}w^\alpha + w'.$$

By integrating this relation on $(y_1, y_2)(y_1 > 0)$, we have

$$p\int_{y_1}^{y_2} y^{q+1} w^\alpha(y)\, dy \leq -y_1 w'(y_1) + w(y_1), \tag{4.3}$$

from which we get

$$\int_{y_1}^{\infty} y^{q+1} w^\alpha(y)\, dy < +\infty. \tag{4.4}$$

Since $w \in C[0, \infty)$, (4.4) implies that $y^{q+1}w^{\alpha}(y)$ is integrable on $(0, \infty)$. Then (3.4) follows from

$$\int_{-\infty}^{\infty} e^{-cz} S^{\alpha}(z)\, dz = \left(\frac{c}{\varepsilon}\right)^{\varepsilon - 1} \int_{0}^{\infty} y^{q+1} w^{\alpha}(y)\, dy.$$

We next show $S(-\infty) = 0$ for the case $\alpha = 1$. By (4.2) there exists $w(+\infty)(\geq 0)$. Assume $w(+\infty) > 0$. This assumption together with (4.3) implies

$$\int^{\infty} y^{q+1} dy < \infty,$$

which contradicts $\int^{\infty} y^{q+1} dy = \infty$. Hence we have $w(+\infty) = 0$, which implies that $S(-\infty) = w(+\infty) = 0$. Thus the proof of Theorem 2.1 is complete.

Proof of Theorem 2.2. For $U = S_{\varepsilon} - S_0$ we begin by showing the relation

$$\varepsilon(U^2(z))' + cU^2(z) \leq 2\varepsilon \int_{-\infty}^{\infty} |S_0'|\, dz\, \| U \|_{L^{\infty}} \qquad \text{for } z \in \mathbf{R}. \tag{4.5}$$

When $\alpha > 1/2$, $S_0 \in C^2(\mathbf{R})$ and $S_0'' \in L^1(\mathbf{R}) \cap L^{\infty}(\mathbf{R})$. Hence, we have (4.5) in the same way as in the proof of Theorem 1.2 in [8]. Let us consider the case $\alpha \leq 1/2$. For z_0 such that $S_0(z) = 0 (z \leq z_0)$ and $S_0 > 0 (z > z_0)$, S_0'' is discontinuous at $z = z_0$. But we see $S_0'' \in L^1(\mathbf{R})$. Hence, a slight modification of the proof in the case $\alpha > 1/2$ gives (4.5).

By using the same way as in Theorem 1.2 [8], it follows from (4.5) that

$$\| U \|_{L^{\infty}} \leq \frac{2\varepsilon}{c} \| S_0'' \|_{L^1},$$

which implies

$$\| S_{\varepsilon} - S_0 \|_{L^{\infty}} = O(\varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0. \tag{4.6}$$

For the case $\alpha < 1$, by Lemma 3.4 we have

$$S_{\varepsilon}(z) \equiv 0 \qquad (z \leq z_0)$$

for $0 \leq \varepsilon < 1$, where z_0 is independent of ε. Hence,

$$\| B_{\varepsilon} - B_0 \|_{L^{\infty}} \leq e^{-cz_0} \| S_{\varepsilon} - S_0 \|_{L^{\infty}}^{\alpha},$$

from which we have $\| B_{\varepsilon} - B_0 \|_{L^{\infty}} = O(\varepsilon^{\alpha})$ as $\varepsilon \to 0$.

Let us consider the case $\alpha = 1$. Choose p satisfying $p > c$, and take z_0 and a such that

$$e^{-cz_0} = 2p^2, \qquad a = s_{\infty} e^{-pz_0}.$$

Define the function $W(z)$ on $(-\infty, z_0]$ by

$$W(z) = ae^{pz}.$$

For $0 < \varepsilon \leq 1$ and $z < z_0$, we have

$$\begin{aligned} \varepsilon w'' + cW' - e^{-cz} W &= (\varepsilon p^2 + cp - e^{-cz}) W \\ &< (2p^2 - e^{-cz}) W \leq 0. \end{aligned}$$

Since $S_\varepsilon(-\infty) = W(-\infty)$ and $S_\varepsilon(z_0) \leq s_\infty = W(z_0)$, by Lemma 3.2 we obtain

(4.7) $$S_\varepsilon(z) \leq ae^{pz} \qquad \text{for } z \leq z_0 \text{ and } 0 < \varepsilon \leq 1.$$

For β with $0 < \beta < 1$, we choose p in (4.7) so that $(1-\beta)p > c$, and put $q = (1-\beta)p$. For $z \leq z_0$, it follows from (4.6) and (4.7) that

$$\begin{aligned} |B_\varepsilon(z) - B_0(z)| &= e^{(q-c)z} \{e^{-pz} |S_\varepsilon(z) - S_0(z)|\}^{1-\beta} |S_\varepsilon(z) - S_0(z)|^\beta \\ &\leq \text{Const.} |S_\varepsilon(z) - S_0(z)|^\beta \\ &\leq \text{Const.}\, \varepsilon^\beta. \end{aligned}$$

For $z \geq z_0$, by (4.6) we have

$$|B_\varepsilon(z) - B_0(z)| \leq e^{-cz_0} |S_\varepsilon(z) - S_0(z)| \leq \text{Const.}\, \varepsilon^\beta.$$

Hence, we obtain $\| B_\varepsilon - B_0 \|_{L^\infty} = O(\varepsilon^\beta)$ as $\varepsilon \to 0$. Thus Theorem 2.2 has been proved.

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