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著者	Suzuki Tomonari
journal or publication title	Journal of Mathematical Analysis and Applications
volume	340
number	1
page range	752-755
year	2008-04-01
URL	http://hdl.handle.net/10228/1237

doi: 10.1016/j.jmaa.2007.08.022

AUTHOR FINAL VERSION

Journal of Mathematical Analysis and Applications, 340 (2008), 752–755.

MIZOGUCHI-TAKAHASHI'S FIXED POINT THEOREM IS A REAL GENERALIZATION OF NADLER'S

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Abstract. We give an example which says that Mizoguchi-Takahashi's fixed point theorem for set-valued mappings is a real generalization of Nadler's. The example is a counterexample to a recent result in Eldred, Anuradha and Veeramani [J. Math. Anal. Appl. (2007), doi:10.1016/j.jmaa.2007.01.087]. We also give a very simple proof of Mizoguchi-Takahashi's theorem.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Let (X, d) be a metric space. We denote by $\mathrm{CB}(X)$ the class of all nonempty bounded closed subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) = \max\left\{ \sup_{u \in A} d(u, B),\ \sup_{v \in B} d(v, A) \right\}$$

for every $A, B \in \mathrm{CB}(X)$, where $d(u, B) = \inf\{d(u, y) : y \in B\}$. In 1969, Nadler [8] extended the Banach contraction principle [2] to set-valued mappings.

Theorem 1 (Nadler [8]). *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that*

$$H(Tx, Ty) \le r\, d(x, y) \tag{1}$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Mizoguchi and Takahashi [6] proved a generalization (Theorem 2 below) of Theorem 1; see Theorem 2 in Alesina, Massa and Roux [1]. Theorem 2 is a partial answer of Problem 9 in Reich [9]. See also [4, 7, 10].

Theorem 2 (Mizoguchi and Takahashi [6]). *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume*

$$H(Tx, Ty) \le \alpha\big(d(x, y)\big)\, d(x, y) \tag{2}$$

for all $x, y \in X$, where α is a function from $[0, \infty)$ into $[0, 1)$ satisfying $\limsup_{s \to t+0} \alpha(s) < 1$ for all $t \in [0, \infty)$. Then there exists $z \in X$ such that $z \in Tz$.

Remark. The domain of α is $(0, \infty)$ in the original statement. However the both are equivalent because $d(x, y) = 0$ implies $H(Tx, Ty) = 0$.

2000 *Mathematics Subject Classification.* Primary 54H25, Secondary 54C60, 54E50.

Key words and phrases. Nadler's fixed point theorem.

The author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

Very recently, Eldred et al [5] claimed that Theorem 2 is equivalent to Theorem 1 in the following sense: If a mapping T from X into $\mathrm{CB}(X)$ satisfies (2), then there exists a nonempty complete subset M of X satisfying the following:

- M is T-invariant, that is, $Tx \subset M$ for all $x \in M$.
- T satisfies (1) for all $x, y \in M$.

In this paper, we give a counterexample to the claim. We also give a very simple proof of Theorem 2.

2. Results

We first give a counterexample to the claim due to Eldred et al [5].

Example 1. Let ℓ^∞ be the Banach space consisting of all bounded real sequences with supremum norm and let $\{e_n\}$ be the canonical basis of ℓ^∞. Let $\{\tau_n\}$ be a bounded, strictly increasing sequence in $(0, \infty)$. Put $x_n = \tau_n \, e_n$ and $X_n = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$ for $n \in \mathbb{N}$. Define a bounded, complete subset X of ℓ^∞ by $X = X_1$ and a mapping T from X into $\mathrm{CB}(X)$ by

$$Tx_n = \begin{cases} X & \text{if } n = 1, \\ X_{n-1} & \text{if } n > 1 \end{cases}$$

for $n \in \mathbb{N}$. Define a function α from $[0, \infty)$ into $[0, 1)$ by

$$\alpha(t) = \begin{cases} \tau_{n-1}/\tau_n & \text{if } t = \tau_n \text{ for some } n \in \mathbb{N} \text{ with } n > 2, \\ 0 & \text{otherwise} \end{cases}$$

Then the following hold:

(i) There is no T-invariant subset M such that $M \neq \varnothing$ and (1) holds for all $x, y \in M$.
(ii) T satisfies (2) for all $x, y \in X$.
(iii) $\limsup_{s \to t+0} \alpha(s) < 1$ for all $t \in [0, \infty)$.

Proof. We first note that X is a unique T-invariant subset of X because $x_{n-1} \in Tx_n$ for $n \in \mathbb{N}$ with $n > 1$ and $Tx_1 = X$. The following are easily proved:

- If $m > n$, then $d(x_m, x_n) = \tau_m$.
- If $m > n$ and $m > 2$, then $H(Tx_m, Tx_n) = \tau_{m-1}$.
- $H(Tx_2, Tx_1) = 0$.

Thus, we obtain

$$\lim_{n \to \infty} \frac{H(Tx_n, Tx_{n+1})}{d(x_n, x_{n+1})} = \lim_{n \to \infty} \frac{\tau_n}{\tau_{n+1}} = \frac{\tau_\infty}{\tau_\infty} = 1,$$

where $\tau_\infty := \lim_n \tau_n < \infty$. Therefore we have shown (i). Fix $m, n \in \mathbb{N}$ with $m > n$. In the case where $m > 2$, we have

$$\alpha\big(d(x_m, x_n)\big)\, d(x_m, x_n) = \alpha(\tau_m)\, \tau_m = \tau_{m-1} = H(Tx_m, Tx_n).$$

In the other case, where $m = 2$, noting $n = 1$, we have

$$\alpha\big(d(x_2, x_1)\big)\, d(x_2, x_1) = \alpha(\tau_2)\, \tau_2 = 0 = H(Tx_2, Tx_1).$$

Thus, we obtain (ii). Since $\{\tau_n\}$ is strictly increasing, $\limsup_{s \to t+0} \alpha(s) = 0 < 1$ for all $t \in [0, \infty)$. Therefore we obtain (iii). $\square$

We next give an alternative proof of Theorem 2 because the proof in [6] is not simple. Another proof in [3] is not yet simple. We remark that we do not use reductio ad absurdum in our proof.

Proof of Theorem 2. Define a function β from $[0,\infty)$ into $[0,1)$ by $\beta(t) = \big(\alpha(t)+1\big)/2$ for $t \in [0,\infty)$. Then the following hold:

- $\limsup_{s\to t+0} \beta(s) < 1$ for all $t \in [0,\infty)$.
- For $x, y \in X$ and $u \in Tx$, there exists an element v of Ty such that $d(u,v) \le \beta\big(d(x,y)\big)\, d(x,y)$.

Putting $u = y$, we obtain the following:

- For $x \in X$ and $y \in Tx$, there exists an element v of Ty such that $d(y,v) \le \beta\big(d(x,y)\big)\, d(x,y)$.

Thus, we can define a sequence $\{x_n\}$ in X satisfying

$$x_{n+1} \in Tx_n \quad \text{and} \quad d(x_{n+1}, x_{n+2}) \le \beta\big(d(x_n, x_{n+1})\big)\, d(x_n, x_{n+1})$$

for $n \in \mathbb{N}$. Since $\beta(t) < 1$ for all $t \in [0,\infty)$, $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence in $\mathbb{R}$. Hence $\{d(x_n, x_{n+1})\}$ converges to some nonnegative real number τ. Since $\limsup_{s\to\tau+0} \beta(s) < 1$ and $\beta(\tau) < 1$, there exist $r \in [0,1)$ and $\varepsilon > 0$ such that $\beta(s) \le r$ for all $s \in [\tau, \tau+\varepsilon]$. We can take $\nu \in \mathbb{N}$ such that $\tau \le d(x_n, x_{n+1}) \le \tau + \varepsilon$ for all $n \in \mathbb{N}$ with $n \ge \nu$. Then since

$$d(x_{n+1}, x_{n+2}) \le \beta\big(d(x_n, x_{n+1})\big)\, d(x_n, x_{n+1}) \le r\, d(x_n, x_{n+1})$$

for $n \in \mathbb{N}$ with $n \ge \nu$. we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \le \sum_{n=1}^{\nu} d(x_n, x_{n+1}) + \sum_{n=1}^{\infty} r^n\, d(x_\nu, x_{\nu+1}) < \infty$$

and hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some point $z \in X$. Since

$$\begin{aligned} d(z, Tz) &= \lim_{n\to\infty} d(x_{n+1}, Tz) \le \lim_{n\to\infty} H(Tx_n, Tz) \\ &\le \lim_{n\to\infty} \beta\big(d(x_n, z)\big)\, d(x_n, z) \le \lim_{n\to\infty} d(x_n, z) = 0 \end{aligned}$$

and Tz is closed, we obtain $z \in Tz$. □

Theorems 1 and 2 are not equivalent, however, from the above proof, we can think that the both are very close. The essence of the both is the following theorem, which we can prove as in the above proof.

Theorem 3. *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume there exists $\varepsilon > 0$ satisfying the following:*

- *There exists $r \in [0,1)$ such that (1) holds for all $x, y \in X$ with $d(x,y) < \varepsilon$.*
- *There exists $x \in X$ such that $d(x, Tx) < \varepsilon$.*

Then there exists $z \in X$ such that $z \in Tz$.

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