Numerical Instability of Magnetic Damping Problem of Elastic Plate

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***Abstract*—Numerical instability occurs in an analysis of a vibration with magnetic damping, or an electromagnetic and structural coupled problem. In this paper, the numerical instability of the coupledanalysis is examined by the finite element in time. It is confirmed that the simultaneous method is unconditionally stable even if the magnetic field and the time increment are large. For the staggered method, we obtain the conditions where the numerical instability occurs.**

***Index Terms*—Finite element methods, fusion reactors, magnetomechanical effects, numerical stability.**

I. Introduction

IN FUSION reactors and magnetically levitated vehicles, large Lorentz force that is produced by eddy current and magnetic field is applied to conductive thin structure. When the structure is deformed, the electromotive force induced by deformation velocity and magnetic field reduces the eddy current. Therefore, electromagnetic and structural coupled analysis is needed for the design of these components. In recent years, various coupled analysis methods have been proposed for this problem [1], [2].

In the coupled analysis, the solution diverges as a result of numerical instability under specific conditions. A similar numerical instability is encountered in the eddy current analysis of moving conductor in magnetic field. To avoid this numerical instability, upwind finite element schemes for electromagnetic field problems in moving media have been proposed [3], [4]. Influences of the moving and the fixed coordinate systems on the numerical instability have also been investigated [5]. The stability of the numerical analyzes for this electromagnetic field problem can be discussed by Peclet number.

The numerical instabilities of the eddy current analysis for moving conductor have been studied; however, little attention has been given to the numerical instability of the analysis for the vibration of elastic plate with magnetic damping. In this paper, the numerical instability of magnetic damping analysis with elastic deformation is discussed for some coupled analysis methods. In order to obtain the conditions where the numerical instability occurs, the finite element in time [6] is applied to the equations of one-degree-of-freedom coupled problem [7].

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II. Methods of the Analysis

A. Matrix Equation of the Coupled Problem

In the coupled problem of a thin shell structure, the matrix equation of the eddy current [8] including the electromotive force is expressed using the normal component T of the current vector potential and the displacement u as

$$[U]\{\dot{T}\} + [C_e]\{\dot{u}\} + [R]\{T\} = \{\dot{B}^{ex}\} \tag{1}$$

where matrices $[U]$, $[R]$, $[C_e]$ and $\{\dot{B}^{ex}\}$ are the inductance matrix, the resistance matrix, the coupling sub-matrix by the electromotive force and the change of the external magnetic field, respectively. The matrix equation of the structure [8] is expressed as

$$[M]\{\ddot{u}\} + [K]\{u\} + [C_s]\{T\} = \{F^{ex}\} \tag{2}$$

where $[M]$, $[K]$, $[C_s]$ and $\{F^{ex}\}$ are the mass matrix, the stiffness matrix, the coupling sub-matrix by the electromagnetic force and the external mechanical force, respectively.

B. Staggered Method

In the conventional coupled analysis methods, both matrix equations (1) and (2) are solved one by one in each time step. To solve (1) at time step $n+1$, $\{\dot{u}\}$ at time step n is used to evaluate $[C_e]\{\dot{u}\}$. The solution $\{T\}$ of (1) at time step $n+1$ is substituted into $[C_s]\{T\}$ to solve (2) at time step $n+1$. In this study, the Crank–Nicolson method and Newmark's β method are applied to (1) and (2) as the time integration.

C. Simultaneous Method

Combining (1) and (2), we obtain the matrix equation

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{T} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ C_e & U \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{T} \end{Bmatrix} + \begin{bmatrix} K & C_s \\ 0 & R \end{bmatrix} \begin{Bmatrix} u \\ T \end{Bmatrix} = \begin{Bmatrix} F^{ex} \\ \dot{B}^{ex} \end{Bmatrix} \tag{3}$$

for the eddy current and the structural coupled system [8]. The solution $\{u\}$ and $\{T\}$ can be obtained simultaneously from (3). Newmark's β method is applied to (3) for the time integration.

III. Numerical Instability of the Coupled Problem

The coupled problem of bending vibration is analyzed. A copper rectangular plate rigidly clamped at one end is placed in a transient magnetic field B_z and a steady magnetic field B_x as shown in Fig. 1. The interaction between the eddy current, which is induced by the transient magnetic field B_z, and the steady magnetic field B_x causes bending deformation of the plate. While the plate is vibrating, the electromotive force by the

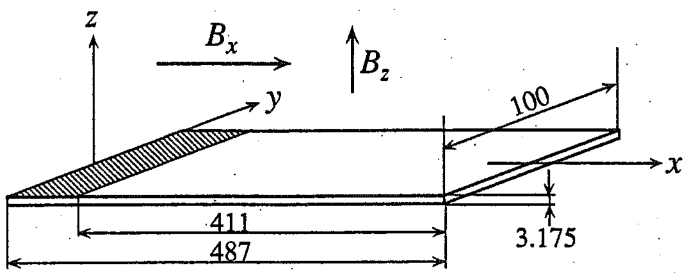

Fig. 1. Schematic diagram of a plate in electromagnetic field.

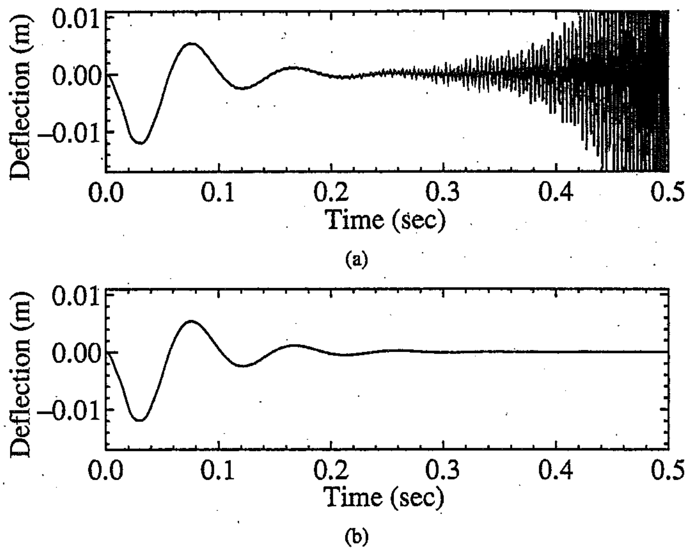

Fig. 2. Deflection of the bending plate (a) Staggered analysis (b) Simultaneous analysis.

deflection velocity and the steady magnetic field B_x influences the eddy current and causes magnetic damping. The conditions of this problem are the same as TEAM (Testing Electromagnetic Analysis Methods) problem 12 [1].

Fig. 2 shows the time history of the deflection at the free end of the plate by the coupled analyzes. The conditions of these analyzes are as follows: time increment Δt is 9.3×10^{-4} sec, magnetic field B_x is 0.5 T, the parameter β of Newmark's β method is 0.25 and the parameter α of the Crank–Nicolson method is 0.5. In general, these time integration schemes are unconditionally stable when parameters β and α are set to values described above. Although the result obtained by the simultaneous method is stable, the numerical instability occurs for the staggered method under the same analysis conditions.

IV. Instability Analysis of the Coupled Methods

The numerical instability of the coupled method is discussed using the finite element in time [6]. Since this method is a general form of the time integration methods, the numerical instability of various time integration methods can be examined.

A. Finite Element Analysis in Time

The equation of motion for damped vibration is generally expressed by

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{4}$$

for one-degree-of-freedom problems. Assuming u_{n-1}, u_n and u_{n+1} are solutions of (4) at time t_{n-1}, t_n and t_{n+1}, and applying the finite element in time to (4), we obtain the residual equation with weight like

$$\int_{-1}^{1} W(\xi)\left(m \sum_{i=n-1}^{n+1} \ddot{N}_i(\xi)u_i + c \sum_{i=n-1}^{n+1} \dot{N}_i(\xi)u_i + k \sum_{i=n-1}^{n+1} N_i(\xi)u_i\right) d\xi = 0, \tag{5}$$

where $W(\xi)$, $N_i(\xi)$ and ξ are the weight function, interpolation function of second order and the local coordinate of finite element in time, respectively. If we rewrite u_n and u_{n+1} using the amplification factor λ as

$$u_n = \lambda u_{n-1}, \quad u_{n+1} = \lambda u_n = \lambda^2 u_{n-1}, \tag{6}$$

(5) becomes

$$\begin{aligned}\lambda^2\{m + \gamma\,\Delta tc + \beta\,\Delta t^2 k\} & \\ + \lambda\left\{-2m(1-2\gamma)\,\Delta tc + \left(\tfrac{1}{2} - 2\beta + \gamma\right)\,\Delta t^2 k\right\} & \\ + \left\{m - (1-\gamma)\,\Delta tc + \left(\tfrac{1}{2} + \beta - \gamma\right)\,\Delta t^2 k\right\} &= 0,\end{aligned} \tag{7}$$

where γ and β are function of $W(\xi)$.

From (6), the solution u is not converged when the absolute value of λ is lager than 1.0, the numerical instability occurs. Therefore, the condition where the numerical instability occurs can be obtained by considering the absolute value of λ in (7).

B. Numerical Instability of the Simultaneous Method

In order to examine the numerical instability of the coupled methods, we consider the coupled problem without the change of the external magnetic field and the external mechanical force. Equations (1) and (2) reduce to

$$U\dot{T} + C_e\dot{u} + RT = 0 \tag{8}$$

and

$$m\ddot{u} + ku + C_s T = 0 \tag{9}$$

for the one-degree-of-freedom coupled problem. The numerical instability of the coupled problem is discussed based on these equations.

In the simultaneous method, (8) and (9) can be written as

$$\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{u} \\ \ddot{T} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ C_e & U \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{T} \end{Bmatrix} + \begin{bmatrix} k & C_s \\ 0 & R \end{bmatrix}\begin{Bmatrix} u \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10}$$

for the one-degree-of-freedom coupled problem. If we rewrite the solution of (10) using λ as

$$\begin{aligned}\begin{Bmatrix} u \\ T \end{Bmatrix}_n &= \lambda \begin{Bmatrix} u \\ T \end{Bmatrix}_{n-1}, \\ \begin{Bmatrix} u \\ T \end{Bmatrix}_{n+1} &= \lambda \begin{Bmatrix} u \\ T \end{Bmatrix}_n = \lambda^2 \begin{Bmatrix} u \\ T \end{Bmatrix}_{n-1},\end{aligned} \tag{11}$$

and applying the finite element in time to (10), this equation becomes

$$\left[\begin{array}{c|c} \begin{array}{l} \lambda^2(m+\Delta t^2 k/4) \\ +\lambda(-2m+\Delta t^2 k/2) \\ +(m+\Delta t^2 k/4) \end{array} & \begin{array}{l} \lambda^2\,\Delta t C_s/4 \\ +\gamma\,\Delta t^2 C_s/2 \\ +\Delta t^2 C_s/4 \end{array} \\ \hline \begin{array}{l} \gamma^2\,\Delta C_e/2 \\ -\Delta t C_e/2 \end{array} & \begin{array}{l} \lambda^2(\Delta t U/2+\Delta t^2 R/4) \\ +\lambda\,\Delta t^2 R/2 \\ -\Delta t U/2+\Delta t^2 R/4 \end{array} \end{array}\right] \cdot \left\{\begin{array}{c} u \\ \hline T \end{array}\right\}_{n-1} = \left\{\begin{array}{c} 0 \\ \hline 0 \end{array}\right\} \tag{12}$$

for $\beta = 1/4$ and $\gamma = 1/2$. These values are obtained using a specific weight function W for the finite element in time. In this case, the time integration method derived from the finite element in time is equivalent to Newmark's β method. In order to obtain nontrivial solutions of (12), it is necessary to satisfy the equation

$$\begin{aligned}(1+\lambda)\{&\lambda^3(2p+4q-2r+pq+8) \\ &+\lambda^2(2p-4q-2r+3pq-24) \\ &+\lambda(-2p-4q+2r+3pq+24) \\ &+(-2p+4q+2r+pq-8)\} = 0,\end{aligned} \tag{13}$$

where

$$p = \Delta t^2\frac{k}{m}, \quad q = \Delta t\frac{R}{U}, \quad r = \Delta t^2\frac{C_eC_s}{mU}.$$

The numerical instability of the coupled analysis for simultaneous method can be discussed using the solution λ of (13).

C. Numerical Instability of the Staggered Method

Applying the Crank–Nicolson method to (8), this equation becomes to

$$T_{i+1} = \left(\frac{U}{\Delta t}T_i - \frac{R}{2}T_i - C_e\dot{u}_i\right)\left(\frac{U}{\Delta t}+\frac{R}{2}\right)^{-1} \tag{14}$$

for $\alpha = 1/2$. Using (14) for all time steps $i = 0, 1, 2, \cdots, n+1$, we obtain

$$\begin{aligned}T_{n+1} = &\left(\frac{\frac{U}{\Delta t}-\frac{R}{2}}{\frac{U}{\Delta t}+\frac{R}{2}}\right)^{n+1} T_0 \\ &- C_e\sum_{i=1}^{n+1}\dot{u}_{i-1}\frac{\left(\frac{U}{\Delta t}-\frac{R}{2}\right)^{n+1-i}}{\left(\frac{U}{\Delta t}+\frac{R}{2}\right)^{n+2-i}}.\end{aligned} \tag{15}$$

Assuming that T_0 is zero, and substituting (15) into (9), the structural equation including the coupled effect becomes

$$m\ddot{u}_{n+1} + ku_{n+1} - C_eC_s\sum_{i=1}^{n+1}\dot{u}_{i-1}\frac{\left(\frac{U}{\Delta t}-\frac{R}{2}\right)^{n+1-i}}{\left(\frac{U}{\Delta t}+\frac{R}{2}\right)^{n+2-i}} = 0. \tag{16}$$

Combining (16) for time step n and that for $n+1$, we obtain

$$m\ddot{u}_{n+1} + ku_{n+1} - \frac{\left(\frac{U}{\Delta t}-\frac{R}{2}\right)}{\left(\frac{U}{\Delta t}+\frac{R}{2}\right)}(m\ddot{u}_n + ku_n)$$

$$-\frac{C_eC_s}{\left(\frac{U}{\Delta t}\frac{R}{2}\right)}\dot{u}_n = 0. \tag{17}$$

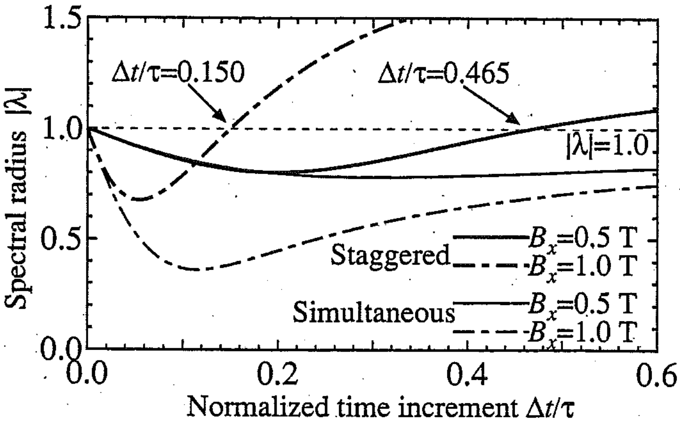

Fig. 3. Change of the spectral radius with time increment.

Applying the finite element in time ($\beta = 1/4, \gamma = 1/2$) to (17), the residual equation is expressed by

$$\begin{aligned}&\lambda^3(2p+4q+pq+8) \\ &+\lambda^2(2p-4q-4r+3pq-24) \\ &+\lambda(-2p-4q+3pq+24) \\ &+(-2p+4q+4r+pq-8) = 0.\end{aligned} \tag{18}$$

The numerical instability of the coupled analysis for the staggered method can be discussed using the solution λ of (18).

V. Results and Discussion

To verify that the numerical instability of the coupled methods can be described by the absolute values of λ in (13) and (18), numerical analyzes of the coupled problem of Fig. 1 were performed. For the problem of the rectangular plate, values of k/m, R/U and C_eC_s/mU can be determined for each mode using the finite element analysis [7]. Substituting these values into (13) and (18), we obtain the value of λ for each method. Fig. 3 shows the dependence of the absolute value of λ on time increment, which is normalized by the natural period τ of the plate, for the first structural mode. For the simultaneous method, the absolute value of λ is always smaller than 1.0 for this condition. According to (13), the absolute value of λ approaches to 1.0 when the magnetic field or time increment becomes infinity. Therefore, the simultaneous method is unconditionally stable for the coupled analysis.

The absolute value of λ of the first structural mode, however, is larger than 1.0 for the staggered method as shown in Fig. 3. The numerical instability occurs when $\Delta t/\tau > 0.465$ for $B_x = 0.5\ T$ and $\Delta t/\tau > 0.150$ for $B_x = 1.0\ T$. To confirm these conditions of the numerical instability, numerical calculations for $B_x = 0.5\ T$ were performed. The deflection of the plate is shown in Fig. 4, which was obtained by the time integration of (2) for the first structural mode and the time integration of (1) for all eddy current modes. The numerical solution for $\Delta t/\tau = 0.408$ ($\Delta t = 0.0382$ sec), which was predicted to be in the stable region, is stable. The numerical instability occurs for the case of $\Delta t/\tau = 0.533$ (0.0501 sec), which was predicted to be in unstable region. Therefore, it is confirmed that the conditions

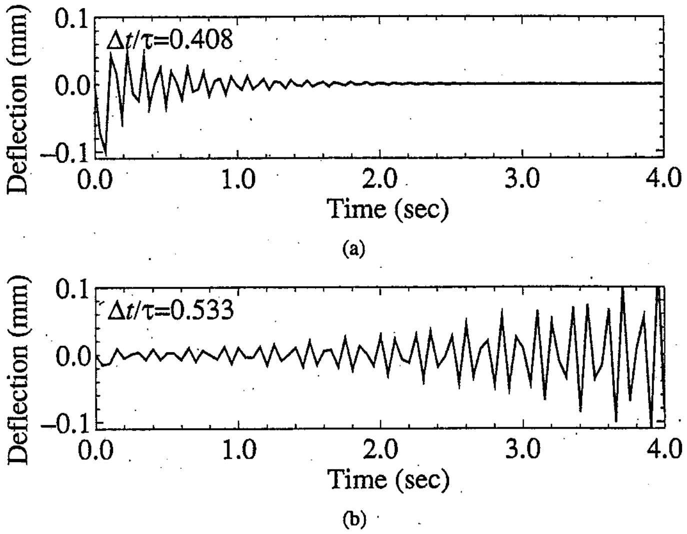

Fig. 4. Deflection of the plate for different time increment by staggered analysis.

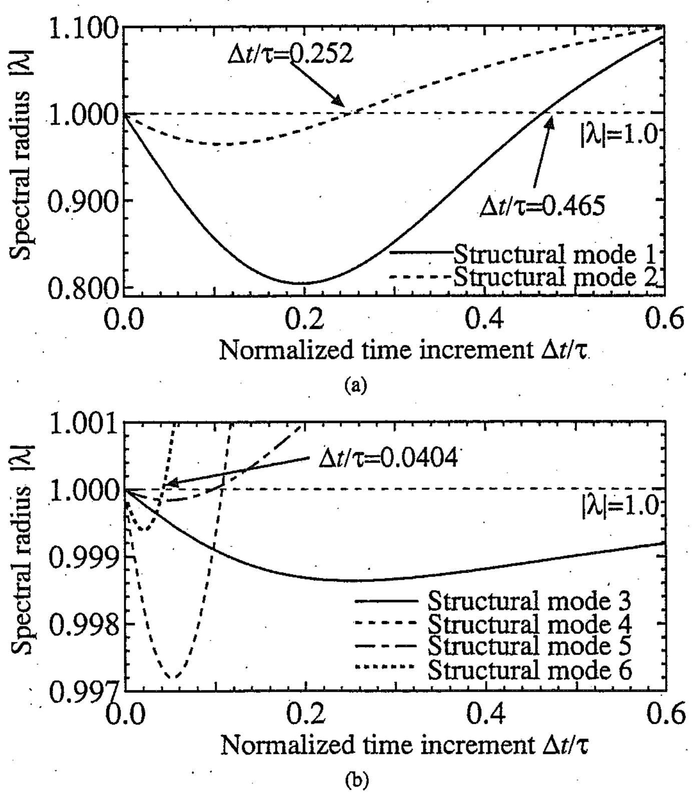

Fig. 5. Change of the spectral radius for six structural mode ($B_x = 0.5\ T$) (a) Structural mode 1, 2 (b) Structural mode 3–6.

to predict the occurrence of numerical instability are obtained from (18).

Since several vibration modes are needed in general in the mode superposition method to obtain dynamic response for the vibration problem, numerical instability analysis for the problem with more than one mode should be examined. Fig. 5 shows $|\lambda|$ for six structural modes in the staggered method for $B_x = 0.5\ T$. Since all $|\lambda|$ for six structural modes are smaller than 1.0 when $\Delta t/\tau$ is smaller than 0.0404 ($\Delta t < 0.110 \times 10^{-3}$ sec), all vibration modes are predicted to be stable under this condition for this problem. To verify this condition, finite

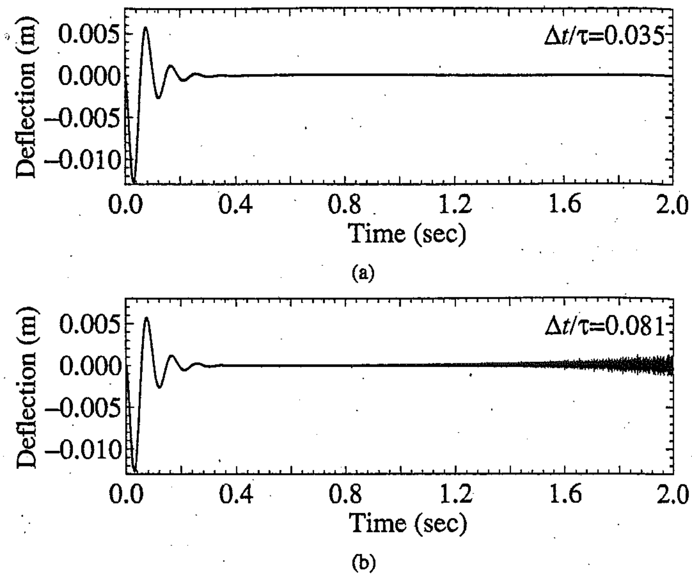

Fig. 6. Deflection of the plate by the staggered method using six structural modes ($B_x = 0.5\ T$) (a) $\Delta t/\tau = 0.035$ ($\Delta t = 0.095 \times 10^{-3}$ sec) (b) $\Delta t/\tau = 0.081$ ($\Delta t = 0.219 \times 10^{-3}$ sec).

element analyzes with six structural modes and all eddy current modes were performed for the cases with $\Delta t/\tau$ of 0.035 (stable region) and 0.081 (unstable region). The deflection of the plate is shown in Fig. 6. The numerical instability occurs in the case of $\Delta t/\tau = 0.081$ because of the instability of solution of mode 6. Therefore, it is required that $|\lambda|$ of all modes should be smaller than 1.0 to obtain the stable solution of the vibration problem with magnetic damping.

VI. Conclusions

The numerical instability analysis of time integration methods using finite element in time can be applied to the vibration problem with magnetic damping. The simultaneous method is unconditionally stable even if the magnetic field or the time increment is large. As for the staggered method, the conditions in which the numerical instability occurs can be obtained by this instability analysis.

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