UNIVERSITÀ DEGLI STUDI DI SALERNO

DIPARTIMENTO DI SCIENZE ECONOMICHE E STATISTICHE

CORSO DI DOTTORATO IN
INGEGNERIA ED ECONOMIA DELL'INNOVAZIONE
XIII CICLO

Theoretical problems in Cause – Specific Mortality forecasting and diagnosis rates. Solutions and actuarial applications.

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ANNO ACCADEMICO 2013 – 2014

Introduction

The study of cause-specific mortality in the actuarial field is one of the main sources of information for public health monitoring. Among the number of challenges that need to be addressed, two of them are the main aims of this work. First, the World Health Organization manages a cause-specific mortality database, based on the International Classification of Diseases (ICD). The ICD changed three times between 1950 and 2010 in order to account for progress in science and technology and to achieve more refined cause descriptions. Thus, the ICD revision often causes major discontinuities in trends in mortality and morbidity statistics, requiring then an appropriate correction for any time series analyses or forecasts. Second, models for trends in mortality rates for different ages and sexes as well as for different countries are often based on the assumption of independence between the causes of death. Actually, in literature we cannot find models taking into account both the questions. Our aim is to suggest a new method developed considering simultaneously the ICD changes (discontinuities in the data) and the dependence among several causes of death. To this end, basing on an extension of the Lee – Carter Model (Lee R.D. and Carter L., 1992) we mitigate the structural breaks in mortality rates and contextually the VECM (Vector Error Correction Model) is used in order to project the cause-specific time component of the Lee-Carter model. This methodology allows to include long-term stationary relations between the different causes of death, that is cause-of death dependence in the mortality forecasting model. Results are compared to the more

traditional forecasting approach based on ARIMA processes.

In particular we show that the proposed method produces more precise projections in order to better understand the cause – specific mortality. This is crucial in different topics for example in social security, health, socio-economic strategies, having implications in different decision choices.

The application in pricing assessment of the methodology here discussed is developed in the insurance and banking filed, in order to design tailored and more individual contracts. In particular several insured loans built within the critical illness policy model are proposed and priced. The new products insure the loan, covering the risk to suffer several dread disease and/or the event in case of death for a specific cause. The inclusion of the benefit in case of a specific cause of death does not involve additional cost to the life office beyond the critical illness benefit. On the contrary the new designs ensure less expensive conditions in comparison with the standard policy and are very appealing from the market point of view, looking for more and more personalized and cheap clauses.

The layout of the thesis is as follows. In Chapter 1, the problems related to the evaluation and the prediction for cause – specific deaths are exposed from a critical viewpoint. Chapter 2 illustrates the new method here proposed, aimed to mitigate the structural breaks and capture the dependencies among all causes of death. Based on a recent work of Haberman S. and Villegas A. (2014), we adjust the mortality time series mitigating the break points. Then, after having shown that the cause –

specific deaths are competitive risks, the Vector Error Correction Model is studied in its application to the adjusted probabilities to the aim of forecasting their future trend. Chapter 3 is dedicated to the application of the proposed methodology in the actuarial pricing assessment. Specifically, we price different new proposals for insured loan, in which the loan is saved in case of specific events as the cause – specific deaths. To this aim we use the death and survivor probabilities correctly calculated taking into account contextually structural breaks and dependences among causes of death, accordingly with the topic of the work.

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THEORETICAL PROBLEMS IN CAUSE – SPECIFIC MORTALITY FORECASTING.
Solutions and actuarial applications.

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Chapter I

MODELING FOR CAUSE – MORTALITY

1.1 Introduction

Models for trends in mortality rates for different ages and sexes as well as for different countries are often based on the assumption that past trends in historical data will continue in the future. Mortality trends and related fluctuations determine changes in the causes of deaths. These causes have different age patterns and have shown different trends over recent years. At the same time, systematic changes in causes of death have been common across the industrialized economies.

Recent literature has addressed the issue of cause-specific mortality analysis. In particular, Wilmoth J.R. (1995) shows how taking into account causes of death can influence projected trends and effectively highlights how cause of death influence is hidden in aggregated data. Tuljapurkar S. et al. (2000) show how mortality declines have had common trends in the G7 countries, although there is evidence of variability in those trends. Booth H. et al. (2006) also demonstrate the difficulties related to the projections obtained by the decomposition of the population according to causes of death. Maccheroni C. et al. (2007) examine how the standard Lee-Carter model is not suitable for the analysis by causes of death. Sherris M. et. al. (2010) discuss the factors driving mortality changes based on causes of death.

The modelling of cause – specific mortality is a very delicate focus and it gives origin to a lively debate in the scientific community. In particular, we want to highlight that two different types of problems come out from the decomposition of the mortality time series in different causes of death.

The first one is the presences of some structural breaks affecting the estimation of the historical and the future trend of cause – specific mortality. In particular, the WHO (World Health Organization, 2009) has revised the international classification of diseases (ICD) approximately every 20 years since 1900. The purpose of revision is to stay abreast of advances in medical sciences, changes in medical terminology and to ensure the international comparability of health statistics. However, the ICD revision often causes major discontinuities in trends of mortality and morbidity statistics because of changes in classification rules for selecting underlying causes of death. The ranking of leading causes of death is also affected by this revision.

The second problem consists in the dependences among all causes of death. All the mortality models consider the (quite unrealistic) hypothesis of independences between them.

These discontinuities lead not only to a misinterpretation of trends in mortality, but also to misinformation about the changes in life expectancy (Kochanek K.D. et al., 1994). Furthermore, without properly correcting these discontinuities, trends in age-specific death-rates may become biased; this distortion may lead to unreliable forecasts of life expectancy. In the following sections we analyze the most popular models to estimate the future

trend of mortality, focusing on the criticalities they present in forecasting cause - specific mortality.

1.2 The Lee – Carter Model

The Lee-Carter model (1992) and its extensions have been used by actuaries for multiple purposes. Essentially, the model assumes that the dynamic of mortality trends over the time is only ruled by a single parameter called mortality index. The mortality forecast is based on the index extrapolation obtained through the selection of an appropriate time series model. Box-Jenkins models, also known as autoregressive moving average process ARIMA (Box G.E.P. and Jenkins G.M., 1976), are usually used on forecasting.
The model's basic premise is that there is a linear relationship among the logarithm of age-specific death rates $m_{x,t}$ and two explanatory factors: the age x, and time t. Information is distributed in age intervals, so the interval starting in age x will be called " x age interval". The equation describing the model is the following:

$$m_{x,t} = exp(\alpha_x + \beta_x k_t + \varepsilon_{x,t}) \quad (1)$$

$$\ln (m_{x,t}) = \alpha_x + \beta_x k_t + \varepsilon_{x,t} \quad (2)$$

where:

- $m_{x,t}$ is the age-specific death rate for the x interval and the year t;
- α_x is the average age-specific mortality;

- k_t is the mortality index that describes the variation in the level of mortality to t;
- β_x is a deviation in mortality due to changes in the index;
- $\varepsilon_{x,t}$ is the random error.

To evaluate the parameter α_x we impose:

$$\sum_t k_t = 0, \tag{3}$$

the following equation holds:

$$\sum_{t=t_1}^{t_n} \ln(m_{x,t}) = n\alpha_x + \beta_x \sum_{t=t_1}^{t_n} k_t + \sum_{t=t_1}^{t_n} \varepsilon_{x,t} \tag{4}$$

Posing $\varepsilon_{x,t} = 0$ and being $m_{x,t}$ observable from life tables, we can calculate $\hat{\alpha}_x$ as follows:

$$\frac{\sum_{t=t_1}^{t_n} \ln(m_{x,t})}{n} = \ln\left[\left(\prod_{t=t_1}^{t_n} m_{x,t}\right)^{\frac{1}{n}}\right] = \hat{\alpha}_x \tag{5}$$

Posing:

$$\sum_x \beta_x = 1 \tag{6}$$

it is possible to evaluate the parameter k_t by means of the following equation:

$$\sum_{x=0}^{\omega} \ln(m_{x,t}) = \sum_{x=0}^{\omega} \alpha_x + k_t \sum_{x=0}^{\omega} \beta_x + \sum_{x=0}^{\omega} \varepsilon_{x,t} \tag{7}$$

Being $\varepsilon_{x,t} = 0$, we can write:

$$\hat{k}_t = \sum_{x=0}^{\omega} \ln(m_{x,t}) - \sum_{x=0}^{\omega} \alpha_x \tag{8}$$

Finally, we can calculate the parameter β_x with a simple regression in formulas (1) and (2).
The model assumes the constancy of α_x and β_x in respect of t and k_t is the only parameter to project. The k_t description will be made by means of an appropriate ARIMA process, determined using a procedure Box-Jenkins. According to the model, the mortality rate follows a linear trend on the basis of an ARIMA (0,1,0), which fits well the representation of the index evolution over time. Following Box G.E.P. and Jenkins G.M. (1976), the model is the following:

$$k_t = k_{t-1} - c - \varepsilon_t \qquad (9)$$

where:

- k_t is the index of time t;
- c is a drift parameter;
- ε_t is the term error at time t.

Basing on the characteristic equation of the model (2) we can determine the mortality tables:

$$q_x = \frac{2m_{x,t}}{2+m_{x,t}} \quad \text{and} \quad p_x = 1 - q_x \qquad (10)$$

where q_x and p_x indicate respectively death and survival probability rates.

The Lee – Carter model is fitted to a matrix of age - specific observed forces of mortality using singular value

decomposition (SVD). Specifically, α_x, β_x and k_t have to minimize:

$$\sum_{x,t}\left(\ln\left(m_{x,t}\right) - \alpha_x - \beta_x k_t\right)^2 \quad (11)$$

It is worth mentioning the characteristic equation of the model (2) is not a simple regression model, since there are no observed covariates in the right-hand side. The minimization consists in taking for $\hat{\alpha}_x$ the row average of the $\ln\left(m_{x,t}\right)$ and to get $\hat{\beta}_x$ and $\hat{k}_t$ from the first term of an SVD of the matrix $\ln\left(m_{x,t}\right) - \alpha_x$. This yields a single time-varying index of mortality $\hat{k}_t$ (Alho J.M., 2000).
Before proceeding to modeling the parameter k_t as a time series process, the k_t's are adjusted (taking $\hat{\alpha}_x$ and $\hat{\beta}_x$ estimates as given), to reproduce the observed number of deaths $\sum_x D_{x,t}$; $\hat{k}_t$ is the solution of the following equation (Lee R.D. – Carter L., 1992):

$$\sum_x D_{x,t} = \sum_x E_{x,t}\, exp(\alpha_x + \beta_x k_t + \varepsilon_{x,t}) \quad (12)$$

At this point, k_t is estimated again, so that the obtained death rates (with the previously estimated $\hat{\alpha}_x$ and $\hat{\beta}_x$), applied to the actual risk exposure, produce the total number of deaths observed in the data for the years under consideration. There are several advantages to make this second-stage estimation of the parameter k_t. In particular, it avoids sizable discrepancies between predicted and actual deaths (occurring because the first step is based on logarithms of death rates). The original Lee – Carter method was used to aggregate (sexes combined) US data. Carter and Lee (1992) implemented their model for males

and females separately, showing that the two decreasing series are best treated independently. Wilmoth J.R. (1996) applied Lee – Carter methods to forecast Japanese mortality and also experimented some variants of this model. Lee R.D. and Nault F. (1993) applied Lee – Carter methods to model Canadian mortality.

It should be noted that the Lee – Carter method does not attempt to incorporate assumptions about advances in medical science or specific environmental changes; no information other than previous history is taken into account. This means that this approach could be unable to forecast sudden improvements in mortality due to the discovery of new medical treatments or revolutionary cures including antibiotics. Similarly, future deteriorations caused by epidemics, the apparition of new diseases or the aggravation of pollution cannot enter the model.

The Lee–Carter methodology is a mere extrapolation of past trends. All purely extrapolative forecasts assume that the future will be in some sense like the past. Some authors (see, e.g. Gutterman S. and Vanderhoof I.T. (2000)) severely criticized this approach because it seems to ignore underlying mechanisms. As pointed out by Wilmoth J.R. (2000), such a critique is valid only if such mechanisms are understood with sufficient precision to offer a legitimate alternative method of prediction. The understanding of the complex interactions of social and biological factors, which determine mortality levels is still imprecise. This means that the extrapolative approach is particularly compelling in the case of human mortality.

1.3 The Poisson log-bilinear Lee Carter model.

According to Alho J.M. (2000), the basic Lee – Carter model (1992) is not a well suited construction of projected life tables. The main drawback of the OLS (ordinary last square) estimation via SVD is that the errors are assumed homoskedastic.
This is due to the assumption that the errors are normally distributed.
The logarithm of the observed force of mortality is much more variable at older ages than at younger ones because of the much smaller absolute number of deaths at older ages. Because the number of deaths is a counting random variable, according to Brillinger D.R. (1986), the Poisson assumption appears to be plausible. In order to circumvent the problems associated with the OLS method (see Brouhns N., Denuit M. and Vermunt J., 2002), we now consider:

$$D_{x,t} \sim Poisson\big(E_{x,t}(m_{x,t})\big)$$

with:

$$m_{x,t} = exp(\alpha_x + \beta_x k_t) \qquad (13)$$

where the parameters are still subjected to the constraints $\sum_t k_t = 0$, $\sum_x \beta_x = 1$. The force of mortality is thus assumed to have the log-bilinear form $\ln(m_{x,t}) = \alpha_x + \beta_x k_t$. The meaning of the α_x, β_x, and k_t parameters is essentially the same as in the classical Lee - Carter model.

Instead of resorting to SVD for estimating α_x, β_x, and k_t, we now determine these parameters by maximizing the log-likelihood based on the Lee – Carter Poisson log-bilinear model, which is given by:

$$L(\alpha, \beta, k) = \sum_{x,t}\left[D_{x,t}(\alpha_x + \beta_x k_t) - E_{x,t} exp(\alpha_x + \right.$$
$$\left. + \beta_x k_t)\right] + constant \quad (14)$$

Because of the presence of the bilinear term $\beta_x k_t$, it is not possible to estimate the proposed model with statistical packages implementing Poisson regression.
Goodman (1979) was the first who proposed an iterative method for estimating log-linear models with bilinear terms. In iteration step $v + 1$, a single set of parameters is updated fixing the other parameters at their current estimates using the following updating scheme.

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\partial L^{(v)}/\partial\theta}{\partial\theta^2 L(v)/\partial\theta^2}, \quad (15)$$

where $L^{(v)} = L^{(v)}\left(\hat{\theta}^{(v)}\right)$.

In our application, there are three sets of parameters, α_x, β_x, and k_t. The updating scheme is the following (Brouhns N. et al. 2002), starting with $\hat{\alpha}_x^{(0)} = 0$, $\hat{\beta}_x^{(0)} = 1$, and $\hat{k}_t^{(0)} = 0$ (random values can also be used),

$$\hat{\alpha}_x^{(v+1)} = \hat{\alpha}_x^{(v)} - \frac{\sum_t (D_{x,t} - \hat{D}_{x,t}^{(v)})}{-\sum_t \hat{D}_{x,t}^{(v)}}, \quad \hat{\beta}_x^{(v+1)} = \hat{\beta}_x^{(v)}, \quad \hat{k}_t^{(v+1)} =$$
$$= \hat{k}_t^{(v)} \quad (16)$$

$$\hat{k}_t^{(v+2)} = \hat{k}_t^{(v+1)} - \frac{\sum_t (D_{x,t} - \hat{D}_{x,t}^{(v+1)})\hat{\beta}_x^{(v+1)}}{-\sum_t \hat{D}_{x,t}^{(v)} (\hat{\beta}_x^{(v+1)})^2}, \ \hat{\alpha}_x^{(v+2)} = \hat{\alpha}_x^{(v+1)},$$
$$\hat{\beta}_x^{(v+2)} = \hat{\beta}_x^{(v+1)} \quad (17)$$

$$\hat{\beta}_x^{(v+3)} = \hat{\beta}_x^{(v+2)} - \frac{\sum_t (D_{x,t} - \hat{D}_{x,t}^{(v+2)})\hat{k}_x^{(v+2)}}{-\sum_t \hat{D}_{x,t}^{(v+2)} (\hat{k}_t^{(v+2)})^2}, \hat{\alpha}_x^{(v+3)} = \hat{\alpha}_x^{(v+2)},$$
$$\hat{k}_t^{(v+3)} = \hat{k}_t^{(v+2)} \quad (18)$$

where $\hat{D}_{x,t}^{(v)} = E_{x,t} exp(\alpha_x + \beta_x k_t)$, are the estimated number of deaths after iteration step v. The procedure stops when the log-likelihood function has a sufficiently small increment. After updating the k_t parameters, we have to impose two constraints, $\sum_t k_t = 0, \sum_x \beta_x = 1$, which are the same constraints as in the Lee–Carter parameterization. The evaluations of α_x and β_x are used with the forecasted k_t to generate the life table functions.

1.4 Criticalities in these background for modeling cause-specific mortality.

The models presented in the preceding sections are quite unfit to describe mortality by causes of death. Firstly, these models assume the independence between different causes of death. This unrealistic assumption produces a systematic overestimation or underestimation of the mortality phenomenon.
Moreover, the classification of the diseases has been adapted over the years through the “bridge coding” (Istat, 2011). The ranking of leading causes of death is also affected by this revision that produces structural breaks in

the mortality series. These discontinuities lead not only to a misinterpretation of trends in mortality, but also to misinformation about the changes in life expectancy because the mortality time series is not stationary.

Chapter II

SOLUTIONS TO THEORETICAL PROBLEMS IN MODELING MORTALITY BY CAUSE OF DEATH

2.1 Introduction

As in Villegas A. et al (2014), in any country, mortality rates and indices such as life expectancy usually vary among subpopulations. Subpopulations can differ each other for gender, geographic area, cause – specific deaths, socioeconomic variables (e.g., occupation, level of education, or income) and so on. These differentials, and in particular those related to cause – specific mortality, pose important challenges to design public policies for tackling social inequalities, as well as pension systems and to manage longevity risk in pension funds and annuity portfolios.

The models that we discussed in the previous chapter are quite unfit for describing the cause – specific mortality. First, as mentioned, they do not take into account the discontinuity caused by the ICD. In addition, they assume the independence between the different causes of death. These two problems as well as causing a wrong interpretation of mortality also do not produce an adequate mortality trend forecasting.

Here we consider contextually the two questions, always focused separately in literature. To this aim, using well-known criteria and models, we suggest a methodology

representing an extension of the procedure proposed by Arnold S. and Sherris M. (2013).
Following the guide lines of this paper, we suggest the method here presented can be synthesized as follows. As a first step we will adjust the data about mortality with respect to the structural breaks; as a second step we use the procedure VECM (Vector Error Correction Model) in order to project the cause-specific time component of the Lee-Carter model: in this way we can include long-term stationary relations between the different causes of death and thus cause of death dependence in the mortality forecasting model.
The new application of VECM to cause–specific death we propose takes into account contextually the discontinuities and the dependencies between causes.
The procedure is illustrated and analyzed in several numerical evidences (Arnold S., Passannante V., 2014).
Data concerning mortality, disaggregated for causes of death, are available at the Mortality Database administered by the World Health Organization [2009] (WHO), containing several demographic information as the number of deaths for many countries over the last 50 years for five-year age groups. The aggregated death (and survival) probabilities have been got by means of a Poisson Log Bilinear regression (see Brouhns N. et al 2002) on the Lee Carter model (see Lee and Carter 1992).
Causes of death are defined by the International Classification of Diseases (ICD), which ensures consistencies between countries. In this section, all causes of death are considered divided by macro classes.
The ICD changed three times between 1950 and 2006, from ICD-7 to ICD-10, in order to take into account

changes in science and technology and to refine the classification. As consequence the raw data are not directly comparable for different periods.
We compare the obtained results about mortality with the more traditional forecasting approach based on ARIMA processes.
Our study is based on the U.K. population because the widest range of information is available. We consider the U.K. population divided for sex and for six different causes (Infection and Parasitic "I&P", Cancer, Circulatory System, Respiratory System, External and Other).
In the following we show that, modeling the cause of death dependence, a long-run equilibrium relationship exists among all causes (divided by six macro classes such as Cancer, Circulatory System, Respiratory System, Infection and Parasitic, External and Other causes of death) for the U.K. population. The consideration of the cointegrating relations produces positive effects on the forecasting, as shawn in Arnold, S., Passannante, V., 2014. If past trends are expected to continue in the future, including them in the model instead of modeling each cause in isolation, assists in forecasting future mortality rates (Arnold, S., Sherris M., 2013).
This work confirms then that cointegration is a worthwhile tool in understanding and improving cause-specific mortality forecasts. In what follows we will consider separately the two aspect of discontinuity and dependence in the data.

2.2 About the discontinuity: smoothing the structural breaks

Several mortality data cause changes may affect cause-specific time trends, thus altering the interpretation and the forecasting.
There are some quantitative methods that detect abrupt changes ("jumps") and estimates correction factors that may be used for further analysis. One way to smooth the jumps of the mortality index k_t in correspondence of the time of the ICD updating is represented by the spline regression. In this work, we use the new model in Haberman S. and Villegas A., 2014.
The model was presented in the IME Conference in the 2013 by Villegas A.
The model is inspired by the procedures introduced by Ray G. et al. (2011). The authors use a spline function in order to smooth the mortality time series.
The Authors assume that the number of deaths are independent Poisson responses $D_{xt} \sim Poisson(e_{xt}\mu_{xt})$
Let $S = \{s_1, s_2 \dots, s_h\}$ be the times at which coding changes occur. In order to account for the coding changes, we assume as in Haberman S. et al., 2014, that the force of mortality is given by:

$$\log \mu_{xt} = \alpha_x + \beta_x k_t + \sum_{i=1}^{h} \delta_x^{(i)} f^{(i)}(t) \qquad (21)$$

where:

- α_x, β_x and k_t are the some parameters of the standard Lee – Carter Model in equation (2);

- $f^{(i)}(t) = I_{s_{i-1}} \leq t < s_i$ is an indicator function valued at time t taking the value 0 if no new classification occurs in t and the value 1 otherwise. The index i means the numbers of ICD coming true in the considered period;

- $\delta_x^{(i)}$ measures the magnitude of coding change at age x;

$$\{\widetilde{\alpha_x}, \widetilde{k_t}\} = \{\alpha_x + b_1\beta_x, k_t - b_1\} \quad (22)$$

$$\{\widetilde{\beta_x}, \widetilde{k_t}\} = \left\{\frac{1}{b_2}\beta_x, b_2 k_t\right\} \quad (23)$$

$$\left\{\widetilde{\delta_x^{(i)}}, k_t\right\} = \left\{\delta_x^{(i)} + a_i\beta_x, k_t - a_i f^{(i)}(t)\right\}, \quad i = 1, \dots, h \quad (24)$$

Transformation (22) and (23) are the original ones from the Lee-Carter basic model, whilst the family of transformation defined by (24) are induced by the new parameters $\delta_x^{(i)}$ (Haberman S. et al., IME 2013).
In order to ensure the complete characterization of the model, the following constraints need to be imposed:

$$k_{t_n} = 0 \quad (25)$$
$$\sum_x \beta_x = 1 \quad (26)$$

where t_n is the observed last period. In the model the underlying mortality trend is captured only by k_t whilst parameters $\delta_x^{(i)}$ captures the discontinuities in mortality

trend induced by the changes in the coding system of the causes of death. In order to accomplish this, we use the family of transformation defined in formula (24).
As we said before, inspired by the procedure introduced by Ray G. et al. (2011), we set the constants $a_i, i = 1, \dots, h$, by fitting the model

$$k_t = g(t) + \sum_{i=1}^{h} \delta_x^{(i)} f^{(i)}(t) + \epsilon_t \qquad (27)$$

where $g(t)$ is a continuous function fitted by a thin plate penalized regression spline and ϵ_t is an error term.
Given constants $a_i, i = 1, \dots, h$ from model (27), we can write:

$$k_t \longrightarrow k_t - \sum_{i=1}^{h} a_i f^{(i)}(t) \qquad (28)$$

$$\delta_x^{(i)} \longrightarrow \delta_x^{(i)} f^{(i)} + a_i \beta_x \qquad i = 1, \dots, h \qquad (29)$$

The only parameter to be projected through a procedure Box – Jenkis, useful for determining an appropriate ARIMA, is k_t.According to the model, the mortality rate has a linear trend on the basis of an ARIMA (0,1,0), which is well adapted in the representation of the evolution of the index over time. k_t therefore refers to the following model (Lee R.D. and Carter L., 1992):

$$k_t = k_{t-1} - c - e_t \qquad (30)$$

Again on the basis on the date presented in the preceding section, but specializes in particular on a specific cause of death (circulatory system), we have got the following figures 1, 2, 3 and 4. In Figure 1, it is possible to observe

an example (referred to the cause of death related to the circulatory system) of fitting of the parameters α_x, β_x and k_t. Figure 2 shows the fitting trend of the mortality index k_t with respect to that observed (that has two jumps exactly in the years in which the ICD changed): as evident, no more jumps are present in the graph. Figures 3 and 4 show the trends in the case of respiratory system. In the numerical application we will place greater emphasis on this point. With these graphs we can see how the new transformation transfers the jumps in mortality due to data production changes to the $\delta_x^{(i)}$ parameters and leaves k_t representing the underlying mortality trend plus the fluctuations around this trend.

Fig. 1 - Fitting parameters Circulatory System, HV Model.

Source: Source: Arnold S. and Passannante V., Long-Run Analysis of Cause-of-Death Mortality, Presentation Maf 2014, 2-4 April 2014

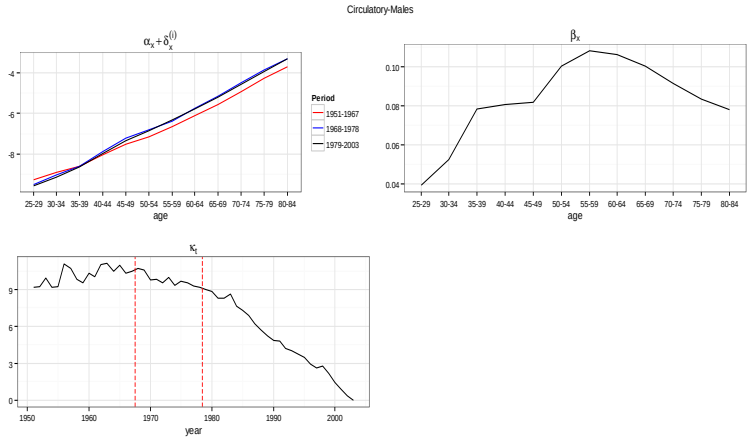


Fig. 2 - k_t with the coding changes for different group age, Circulatory System HV Model.

Source: Source: Arnold S. and Passannante V., Long-Run Analysis of Cause-of-Death Mortality, Presentation Maf 2014, 2-4 April 2014

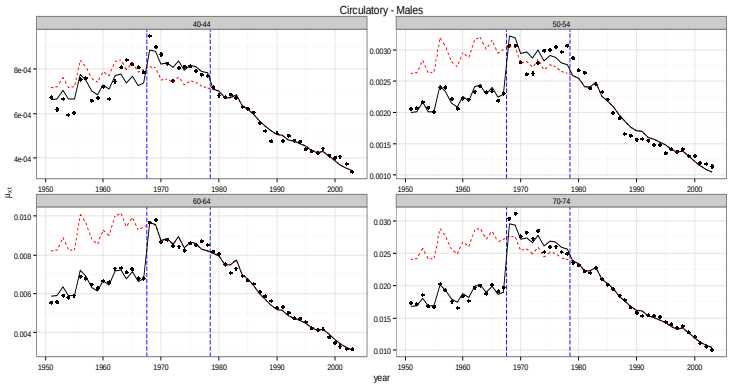


Fig. 3 - Fitting parameters Respiratory System, HV Model.

Source: Source: Arnold S. and Passannante V., Long-Run Analysis of Cause-of-Death Mortality, Presentation Maf 2014, 2-4 April 2014

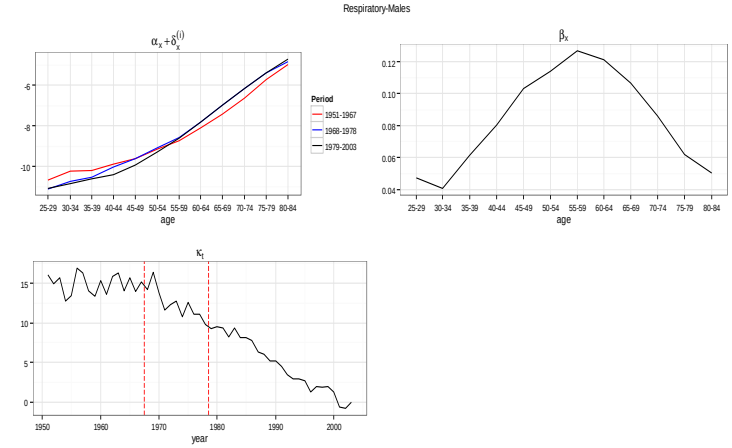
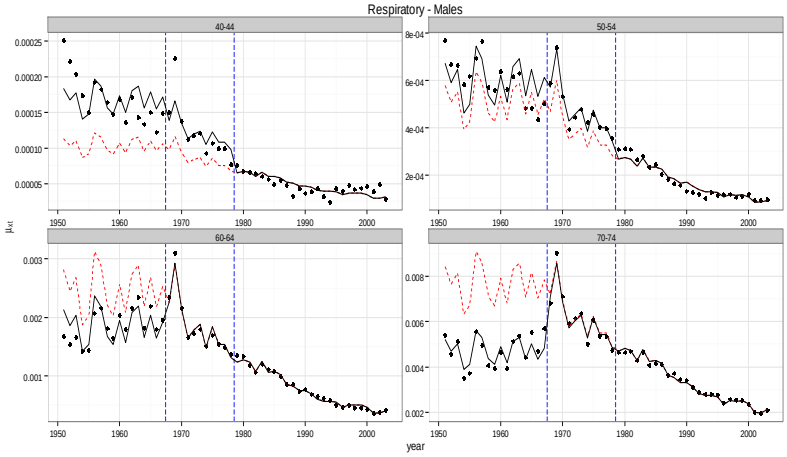


Fig. 4 - k_t with the coding changes for different group age, Respiratory System, HV Model.

Source: Source: Arnold S. and Passannante V., Long-Run Analysis of Cause-of-Death Mortality, Presentation Maf 2014, 2-4 April 2014



2.3 About the dependence: Vector Autoregressive Models and Vector Error Correction Models

The Vector Autoregressive model (VAR hereafter) is the technique used to model the multivariate time series: it focuses on the joint behavior of the vector element, $y_t = (y_{1t}, y_{1t} \dots y_{kt})'$, with dimension $(kx1)$ and the time $t = 1, \dots T$. The model represents the multivariate version of an $AR(p)$ process. The VAR model was introduced in a famous work of Sims C.A. (1980), in which he criticized the model of simultaneous equation, the most popular model in microeconomics in those years. In general a VAR model of order p, VAR(p) can be expressed as follows:

$$y_t = c + A_1 y_{t-1} + \ \ldots + A_p y_{t-p} + u_t \qquad (31)$$

where A_p are the matrices of the coefficients (KxK), $c = (c_1, \ldots, c_k)'$ is the vector of the intercepts $(Kx1)$, $\mathcal{E}_t$ is a K-dimensional white noise process, where the variance–covariance matrix is not singular.
To test the eventual stationeries of a VAR*(1)* process it is sufficient to check that all the eigenvalues of the matrix A_1 are minor then one in absolute value. Algebraically this is the stationary condition:

$$det\left(I_k - A_1\lambda - \cdots - A_p\lambda^p\right) \neq 0, \ |\lambda| < 1 \qquad (32)$$

where I_k is the identical *K*–dimensional matrix while λ is the *K* –dimensional eigenvalues vector. In particular, the VAR*(p)* process is stationary if all the eigenvalues of the matrix in companion form are in the unit circle minor then one in absolute value.
If the polynomial in (32) has a unit root (i.e., the determinant is zero), then some or all of the variables are integrated. For convenience we assume that they are at most I(1).
If the variables have a common stochastic trend, it is possible to find linear combination of them that are I(0). In that case they are *cointegrated*. The following definition holds:

Definition 1. A set of I(1) variables is cointegrated if a linear combination exists, that is I(0).
Occasionally it is convenient to consider system with both I(1) and I(0) variables.

Although the model (31) is general enough to explain variables with stochastic trends, it is not the most suitable type of model if the interest centers on the cointegration relations: this happens because the cointegrating relations do not appear explicitly in the VAR form.
The VECM form

$$\Delta y_t = \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_{p-1} \Delta y_{t-p+1} + u_t \quad (33)$$

is a more convenient model setup for cointegration analysis. In (33) (see Lutkepohl H., 2005):

$$\Pi = -(I_k - A_1 - \cdots - A_p)$$

and

$$\Gamma_i = -(A_{i+1} + \cdots + A_p) \qquad \text{for } i = 1, \dots, p-1$$

The VECM is obtained from the levels VAR form (31) by subtracting y_{t-1} from both sides and rearranging terms. Because Δy_t does not contain stochastic trends (this happens by virtue of our assumption that all the variables can be at most I(1)), the term Πy_{t-1} is the only one that includes I(1) variables.
Hence, Πy_{t-1} must also be I(0). Thus, it contains the cointegration relation. The $\Gamma_i s$ $(j = 1, \dots, p-1)$ are often referred to as the short – run or short – term parameters, and Πy_{t-1} is sometimes called the long – run or long – term part. The model in (33) will be abbreviated as $VECM(p-1)$. Of course, it is also possible to determine the A_j levels parameter matrices from the coefficients of the VECM.
More precisely:

$A_1 = \Gamma_1 + \Pi + I_k, A_i = \Gamma_i - \Gamma_{i-1}$ if $1 = 2, \dots, p-1,$

and

$A_p = -\Gamma_{p-1}.$

If the VAR(p) process has unit roots, that is:

$$det\left(I_k - A_1\lambda - \cdots - A_p\lambda^p\right) = 0 \, for \, \lambda = 1$$

the matrix $\Pi = -\left(\mathrm{I}_k - \mathrm{A}_1 - \cdots - \mathrm{A}_\mathrm{p}\right)$ is singular.
Suppose that the rank $rk(\Pi)$ of the matrix Π is equal to r. Then Π can be written as the product of the matrices α and β, both with dimension $(k \, x \, r)$, such that $rk(\alpha) = rk(\beta) = r$.
We can write:

$\Pi = \alpha\beta'.$

This equation holds if we multiply both sides by y_{t-1}. In particular, the process is I(0) (cf. Johansen 1994) because it can be obtained by premultiplying $\Pi \mathrm{y}_{\mathrm{t}-1} = \alpha\beta'\mathrm{y}_{\mathrm{t}-1}$ with $(\alpha'\alpha)^{-1}\alpha'$. Hence, $\beta'\mathrm{y}_{\mathrm{t}-1}$ contains cointegrating relations. It follows that there are $r = rk(\Pi)$ linearly independent cointegrating relations among the components of y_t. The rank of Π is therefore referred to as the cointegrating rank of the system, and β is a cointegration matrix.
For example, if there are three variables with two cointegrating relations $(r = 2)$, we have

$$\Pi y_{t-1} = \alpha\beta' y_{t-1} =$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{21} & \beta_{22} & \beta_{32} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{bmatrix} =$$

$$= \begin{bmatrix} \alpha_{11} ec_{1,t-1} + \alpha_{12} ec_{2,t-1} \\ \alpha_{21} ec_{1,t-1} + \alpha_{22} ec_{2,t-1} \\ \alpha_{31} ec_{1,t-1} + \alpha_{32} ec_{2,t-1} \end{bmatrix},$$

where

$$ec_{1,t-1} = \beta_{11} y_{1,t-1} + \beta_{21} y_{2,t-1} + \beta_{31} y_{3,t-1}$$

and

$$ec_{2,t-1} = \beta_{12} y_{1,t-1} + \beta_{22} y_{2,t-1} + \beta_{32} y_{3,t-1}$$

The matrix α is sometimes called the *loading matrix*. In fact it contains the weight attached to the cointegrating relations in the individual equations of the model. The matrices α and β are not unique, so there are many possible α and β matrices containing the cointegrating relations or linear transformations of them. In fact, using any nonsingular $(r\,x\,r)$ matrix B, we obtain a new loading matrix $\alpha\beta$ and cointegration matrix $\beta B'^{-1}$, which satisfies the following equation: $\Pi = \alpha B\left(\beta B'^{-1}\right)'$.
Consequently, cointegrating relations cannot be extracted purely from the observed time series. Some nonsample information is required to identify them uniquely.
The model (33) contains several special cases that deserve to be pointed out. If all variables are I(0), and $r = K$, the process is stationary. If $r = 0$, the term Πy_{t-1} disappears in (33). In that case, Δy_t has a stable VAR representation.

In other words, a stable VAR representation exists for the first differences of the variables rather than the levels variables.
The VECM in (33) also indicates that, for a cointegrating rank $r > 0$, the vector of first differences of the variables, Δy_t, does not have a finite order VAR representation.

2.4 Deterministic Terms

Several extensions of the basic models (31) and (33) are proposed in literature.
Usually it is necessary to represent the main characteristics of a data set of interest. A variable could include a deterministic term, such as an intercept, a linear trend term or seasonal dummy variables. A first way to include deterministic terms in the model is simply to add them to the stochastic part, as in the following expression is showed:

$$y_t = d_t + x_t \tag{34}$$

Here d_t is the deterministic component, and x_t is a stochastic process that may have a VAR or VECM representation, as in (31) or (33). In other words:

$$x_t = A_1 x_{t-1} + \cdots + A_p x_{t-p} + u_t \text{ or}$$

$$\Delta x_t = \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + \cdots + \Gamma_{p-1} \Delta x_{t-p+1} + u_t.$$

On the assumption, for instance, that u_t is a linear trend term, that is:

$$d_t = \delta_0 + \delta_1 t$$

such a model setup implies the following VAR(p) representation for y_t

$$y_t = v_0 + v_1 t + A_1 y_{t-1} + \cdots + A_p x_{t-p} + u_t \qquad (35)$$

This representation is easily derived by left-multiplying (34) with :

$$A(L) = I_k - A_1 L - \cdots - A_p L^p$$

where L is the lag operator, as usually indicated. Noting that:

$$A(L)x_t = u_t$$

and rearranging terms, we find that:

$$v_0 = A(1)\delta_0 + \left(\sum_{j=1}^{p} j A_j\right)\delta_1$$

and

$$v_1 = A(1)\delta_1.$$

Hence, v_0 and v_1 satisfy a set of restrictions implied by the trend parameters δ_0 and δ_1 and the VAR coefficients. Alternatively, one may view (35) as the basic model without restrictions for v_i $(1 = 0,1)$. In that case, the model can, in principle, generate quadratic trends if I(1) variables are included, whereas in (34), with a

deterministic term $d_t = \delta_0 + \delta_1 t$, a linear trend term is permitted only.
If we have a polynomial of q order, we can find a polynomial of $(q-1)$ order as follows (Beveridge – Nelson Decomposition)

$$C(L) = C(1) + C^*(L)u_t \tag{36}$$

Posing:

$$C(1) = [\beta_\perp H \alpha'_\perp]$$

and

$$s_t = d_t + u_t$$

we have:

$$y_t = [\beta_\perp H \alpha'_\perp] s_t + C^*(L)u_t \tag{37}$$

where s_t is a random walk I(1) and H is an invertible matrix with rank $(n - r)$.
In general, if we consider this polynomial form:

$$d_t = \delta_0 + \delta_1 t + \delta_1 t^2 + \cdots + \delta_p t^p \tag{38}$$

it is not necessary that the polynomial $\alpha'_\perp \delta_p$ has a p order.
In practice we have the following five cases:

a) $d_t = 0$. In this case there is not a trend.
b) $d_t = \delta_0, \alpha'_\perp \delta_0 = 0$. There is not a trend in the VAR but there is a drift in the VECM.

c) $d_t = \delta_0, \alpha'_\perp \delta_0 \neq 0$. There is a drift in the VAR and in the VECM.
d) $d_t = \delta_0 + \delta_1 t, \alpha'_\perp \delta_1 = 0$. There is a linear trend in the VAR and in the VECM.
e) $d_t = \delta_0 + \delta_1 t, \alpha'_\perp \delta_1 \neq 0$. There is a quadratic trend in the VAR and a linear trend in the VECM.

In what follows the aim will be to identify the type of deterministic trend to include in the process.

2.5 Likelihood ratio test in I(1) models

This section contains a description of a reduced rank regression and explains how this procedure is applied to derive estimators and test statistics for model with various restrictions on the deterministic terms.
The technique of reduced rank regression was introduced by Anderson T.W. and Rubin H. (1949) in connection with the analysis of limited information maximum likelihood and generalized to the reduced rank regression model by Anderson T.W. (1951). An excellent source of information is the monograph by Reinsel G.C. and Velu R.P. (1998), which contains a comprehensive survey of the theory and history of reduced rank regression and its many applications.
The statistical analysis of all the multivariate models is made by the same procedure, called a reduced rank regression, applied in the context of independent and identically distributed variables (Anderson T.W., 1951) and has been applied for stationary processes (Ahn S. and Reinsel G.C., 1988) and for nonstationary processes (Johansen S. and Juselius K., 1990).

Regressing U_t and V_t on Z_t to form residuals R_{ut} and $R_{vt}$and solving the reduced rank regression (Johansen S., 1988), we can write:

$$R_{u_t} = \alpha\beta' R_{v_t} + u_t \quad (39)$$

Posing:

$$S_{ij} = T^{-1} \sum_1^T R_{u_t} R'_{vt} \quad \text{, with } t = 1, \dots, T \quad (40)$$

We can solve the eigenvalue problem

$$|\lambda S_{vv} - S_{vu} S_{uu}^{-1} S_{uv}| = 0 \quad (41)$$

for eigenvalues $1 > \lambda_1 > \cdots > \lambda_p > 0$, and eigenvectors $w = (w_1, \dots, w_p)$.
The vectors w_i's satisfies the following equation:

$$\lambda_i S_{vv} w_i = S_{vu} S_{uu}^{-1} S_{uv} w_i \quad (42)$$

and are normalized as follows:

$$w' S_{vv} w = I;$$

so that

$$w' S_{vu} S_{uu}^{-1} S_{uv} w = diag(\lambda_1, \dots, \lambda_p).$$

The reduced rank estimators are given by $\hat{\beta} = (w_1, \dots, w_r)$ and $\hat{\alpha} = S_{uv}\hat{\beta}$.

Each of the five cases a, b, c, d, e mentioned in sect. 2.4 can be estimated by reduced rank regression.
Now we proceed with testing the cointegration renk under specific hypothesis on the deterministic part of the process. Following Johansen S. (1988), let's consider

$$\Pi = \alpha\beta'$$

when there is an unrestricted linear term in the VAR model written as a reduced form of an error correction model. The Gaussian errors in equation (39) gives rise to a likelihood analysis leading to a regression, and for the analysis this is seen to be a reduced rank of regression of $\mathrm{U_t} = \Delta \mathrm{X_t}$ on $\mathrm{V_t} = \mathrm{X_{t-1}}$ corrected for lagged differences, constant and linear term. The estimator of the error covariance matrix is given by (Johansen S., 1994):

$$L_{max}^{-2/t}(r) = |S_{uu}| \prod_{i=1}^{r}(1 - \lambda_i) \qquad (43)$$

Dividing (43) by a similar expression posing r = p, we find that the likelihood ratio test $\mathrm{L_{max}(r) = L_{max}(p)}$ of the model with a quadratic trend in the VAR and a linear trend in the cointegration versus the unrestricted autoregressive model, is given by:

$$-T \sum_{r+1}^{p} log(1 - \lambda_i) \qquad (44)$$

The same analysis holds for a model including trends in the variables but no trend in the cointegrating relation.
In the analysis for a model without trends we only correct for the lagged differences.
Thus in all three cases, following Johansen S. (1994), we get the test statistic (44), but with different eigenvalue.

In the model with a trend in the cointegrating relation the statistical analysis consists in a reduced rank regression of ΔX_t corrected for legged differences and the constant.

2.6 Testing the absence of trend in the trend stationary components

In this section we compare the model with a quadratic trend in the variables against the model without a quadratic trend but with a linear trend in the cointegration. Following Johansen S. (1994), the likelihood ratio test statistic is given by:

$$\frac{L^*_{max}(r)}{L_{max}(r)} = \frac{L^*_{max}(r)/L^*_{max}(p)}{L_{max}(r)/L_{max}(p)} \frac{L^*_{max}(p)}{L_{max}(p)}$$

Hence a comparison of "a model without trend in the cointegration but with trends in the variables" against "a model with a trend in the cointegrating relations" is possible. We will use the test statistic is equal to:

$$T \sum_{r+1}^{p} log\{(1-\lambda_i)/(1-\lambda_i^*\} \qquad (45)$$

wich allows for a comparison of the model without trend in the cointegration but with trends in the variables against the model with a trend both in the cointegrating relations and in the variables.
In or case the likelihho ratio test can be expressed:

$$\frac{L^1_{max}(r)}{L^*_{max}(r)} = \frac{L^1_{max}(r)/L^1_{max}(0)}{L^*_{max}(r)/L^*_{max}(0)} \frac{L^1_{max}(0)}{L^*_{max}(0)}$$

And analogously the test statistic is:

$$T \sum_1^r log\{(1 - \lambda_i^1)/(1 - \lambda_i^*\} \quad (46)$$

2.7 Model Specification

In specifying VAR models or VECMs it is necessary to specify the lag order and, for VECMs, also the cointegrating rank. To this aim, some statistical procedures are available and will be discussed in the following. Because unrestricted VAR models and VECMs usually involve a substantial number of parameters, it is desirable to impose restrictions for reducing the dimension of the parameter space and thereby improve the precision in the estimations. If the VAR has a unit root and all the variables are integrated of the same order, there could be some stationary cointegrating relations, as previously set.

2.7.1 Determining the autoregressive order

In determining the lag order of a dynamic model we can use the same procedures available for univariate models. At this stage it is useful to focus on the VAR form (31) because the cointegrating rank r is usually unknown when the choice of the lag order p is made. One possible approach is to start from a model with some prespecified maximum lag length, p_{max} and apply sequentially some tests to determine a suitable model order. Generalized versions of the criteria in the univariate case are available to that purpose. The general approach is again to fit

VAR(m) models with orders $m = 0, \dots, p_{max}$ and to choose an estimator of the order p that minimizes the preferred criterion. Many of the criteria in current use have the general form:

$$Cr(m) = log\ det\ (\widetilde{\sum}_u(m)) + c_t\varphi(m) \qquad (47)$$

where $\det(.)$ denotes the determinant, log is the natural logarithm, $\widetilde{\sum}_u(m) = T^{-1}\sum_{t=1}^{T}\hat{u}_t\hat{u}_t'$ is the residual covariance matrix for a model of order m, c_t is a sequence depending on the sample size T, and $\varphi(m)$ is a function that penalizes large VAR orders. For instance, $\varphi(m)$ may represent the numbers of parameters that have to be estimated in a VAR (m) model. The term $\log\det(\widetilde{\sum}_u(m))$ measures the fit of a model with order m. Because there is no correction for degrees of freedom in the covariance matrix estimator, the log determinant decreases when m increases. As in the univariate case, the sample size has to be held constant; hence, the number of presample values set aside for estimation is determined by the maximum order p_{max}.
The following criteria are direct generalizations of the corresponding criteria for univariate processes:

- Akaike's Information Criteria

$$AIC(m) = log\ det\ (\widetilde{\sum}_u(m)) + \frac{2}{T}mK^2$$

- Hannan-Quinn Criterion

$$HQ(m) = log\ det\ (\widetilde{\sum}_u(m)) + \frac{2loglogT}{T} mK^2$$

- Schwarz Criterion

$$SC(m) = log\ det\ (\widetilde{\sum}_u(m)) + \frac{logT}{T} mK^2$$

- Final Prediction Error

$$FPE(m) = log\ det\ (\widetilde{\sum}_u(m))(1 + \frac{2}{T} mK^2)$$

Model selection criteria can also be used for identifying coefficients that may be replaced by zero or other exclusion restrictions.

2.7.2 Dickey – Fuller test

In order to identify the order of integration of a time series there are several statistical tests. The first one considered in this work is the Dickey – Fuller test.
Consider the AR(p) model:

$$y_t = \alpha_1 y_{t-1} + \cdots + \alpha_p y_{t-p} + u_t \quad (48)$$

the process is integrated when

$$\alpha(1) = 1 - \alpha_1 - \cdots - \alpha_p = 0$$

We are interested in testing the hypotheses $\alpha_1 = 0$. To test this null hypothesis against the alternative of stationarity

of the process, it is useful to reparameterize the model. Subtracting y_{t-1} from both sides, we have:

$$\Delta y_t = \Phi y_{t-1} + \sum_{j=1}^{p-1} \alpha_j^* \Delta y_{t-j} + u_t \quad (49)$$

where $\Phi = -\alpha(1)$ and $\alpha_j^* = -(\alpha_{j+1} + \cdots + \alpha_p)$. In this model the test of the hypothesis is

$H_0: \Phi = 0$ versus $H_1: \Phi < 0$.

This test, called augmented Dickey – Fuller (ADF) test statistic is based on the t-statistic of the coefficients Φ from an OLS (Ordinary Last Square) estimation (Fuller W.A., 1976 and Dickey D.A. and Fuller W.A., 1979).
Critical values have been obtained by simulation, and they are available in Fuller W.A. (1976) and, Davidson R. and MacKinnon J.G. (1993).
If the order of integration of a time series and, hence, the number of unit roots in the AR operator, are not clear, we should calculate the first difference series so many times until the series becomes stationary. Then a unit root test is performed for the series. If the unit root is rejected, a unit root test is applied to the series, which is differenced one time less than in the previous test. If again a unit root is rejected, the procedure is repeated until a unit root cannot be rejected.

2.7.3 KPSS test

In this section we are going to investigate the integration proprieties of a series y_t testing the null hypothesis that the process is stationary ($H_0: y_t \sim I(0)$) against the

alternative that it is $I(1)$ (H_1: $y_t \sim I(1)$. Kwiatkowski D., Phillips P.C.B., Schmidt P. and Shin Y. (1992) derived a test for this pair of hypothesis. Assuming that there isn't a term representing the linear trend, the starting point is the following process:

$$y_t = x_t + z_t$$

where x_t is a random walk,

$$x_t = x_{t-1} + v_t,\ v_t \sim iid(0, \sigma_v^2),$$

and z_t is a stationary process.
In this framework the test of the hypothesis is as follows:

$$H_0: \sigma_v^2 = 0 \text{ versus } H_1: \sigma_v^2 > 0.$$

If H_0 holds, y_t is composed of a constant and the stationary process z_t: hence, y_t is also stationary. Kwiatkowski D. et al. (1992) have proposed the following test statistic:

$$KPSS = \frac{1}{T^2} \sum_{t=1}^{T} \frac{S_t^2}{\hat{\sigma}_\infty^2}$$

where $S_t = \sum_{j=1}^{t} \hat{w}_j$ with $\hat{w}_t = y_t - \bar{y}$ and $\hat{\sigma}_\infty^2$ an estimator of

$$\hat{\sigma}_\infty^2 = \lim_{T \to \infty} T^{-1} Var(\sum_{t=1}^{T} z_t)$$

that is an estimator of the long – run variance of the process $\hat{\sigma}_{\infty}^{2}$. If y_t is a stationary process, S_t is $I(0)$. If y_t is $I(1)$, the numerator will grow without bounds and the statistic becomes large for large sample sizes.

2.8 Specifying the cointegrating rank

If some of the variables are $I(1)$, a VECM is the suitable modeling and the cointegrating rank r has to be chosen in addition to the lag order. For this choice, we can use some procedures based on likelihood ratio test. The following sequence of hypothesis may be considered:

$$\begin{aligned}
&\mathrm{H}_0(0)\colon \mathrm{rk}(\Pi) = 0 \quad \text{versus} \quad \mathrm{H}_1(0)\colon \mathrm{rk}(\Pi) > 0 \\
&\mathrm{H}_0(1)\colon \mathrm{rk}(\Pi) = 0 \quad \text{versus} \quad \mathrm{H}_1(1)\colon \mathrm{rk}(\Pi) > 0 \\
&\qquad\qquad \cdots \\
&\qquad\qquad \cdots \\
&\qquad\qquad \cdots \\
&\mathrm{H}_0(\mathrm{K}-1)\colon \mathrm{rk}(\Pi) = \mathrm{K}-1 \quad \text{versus} \quad \mathrm{H}_1(1\mathrm{k}-1)\colon \mathrm{rk}(\Pi) = \mathrm{K}
\end{aligned} \tag{50}$$

The testing sequence terminates, and the corresponding cointegrating rank is selected when the null hypothesis cannot be rejected for the first time. If the first null hypothesis in this sequence, $\mathrm{H}_0(0)$, cannot be rejected, a VAR process implemented considering the first differences is considered. At the other end, if all the null hypotheses can be rejected, including $H_0(K-1)$, a levels VAR process should be considered for the analysis.
Under Gaussian assumptions, the likelihood ratio statistic under $H_0(r_0)$ is nonstandard. To present the tests, the

model (34) is a convenient point of departure. Specifically, we first consider the model:

$$y_t = \delta_0 + \delta_1 t + x_t \tag{51}$$

where x_t is a VAR process. If $\delta_1 = 0$, there is just a constant mean and no deterministic trend term. In that case, $y_t - \delta_0 = x_t$, and thus $\Delta y_t = \Delta x_t$; from the VECM form of x_t, the mean adjusted y_t has the VECM form:

$$\Delta y_t = \Pi(y_{t-1} - \delta_0) + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + u_t \tag{52}$$

or, if an intercept term is used,

$$\Delta y_t = v_0^* + \Pi y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + u_t =$$
$$= \Pi^* \begin{bmatrix} y_{t-1} \\ 1 \end{bmatrix} + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + u_t \tag{53}$$

where

$\Pi^* = [\Pi, v_0^*]$ *is* $\left(K\ X\ (K+1)\right)$ *with* $v_0^* = -\Pi\delta_0$.

Notice that, due to the absence of a deterministic trend term, the intercept can be absorbed into the cointegration relations: thus, $\Pi^* = \alpha\beta^{*\prime}$ has rank r . Both VECM versions can be used for testing the cointegrating rank. Johansen S. (1995) considers the intercept version (53) and provides critical values for the likelihood ratio test, which is known as *trace test.*
The test statistic has the following form:

$$LR(r_0) = -T \sum_{j=r_0+1}^{K} \log (1 - \lambda_j) \qquad (54)$$

where the λ_j's are the eigenvalues obtained by applying RR regression techniques to (53).

2.9 Model Checking

Many statistical tools exist for checking whether a given VAR model or VECM provides an adequate representation of the process underlying the time series of interest. As in the univariate case, many of them are based on the residuals of the final model. In what follows we show the Portmanteau test for investigate on the residuals autocorrelation and a test to check the residual's normality.

2.9.1 Portmanteau test for autocorrelation

A formal test for residual autocorrelation may be based on the Portmanteau or adjusted Portmanteau statistic. The test checks the null hypothesis

$$H_0: E(u_t, u'_{t-1}) = 0, \qquad i = 1, \dots, h > p$$

against the alternative that at least one autocovariance and, hence, one autocorrelation is nonzero. The test statistic has the form:

$$Q_h = T\sum_{j=1}^{h} tr(\hat{C}_j'\,\hat{C}_0^{-1}\hat{C}_j\hat{C}_0^{-1})$$

where
$\hat{C}_j = T^{-1}\sum_{t=i+1}^{T} u_t u'_{t-1}$.

Suppose that u_t are residuals from a stable $VAR(p)$ process. Then, under the null hypothesis, Q_h has an approximate $\chi^2\big(K^2(h-p)\big)$ – distribution. Following Hamilton J. D. (1994), a modified statistic with potentially superior small sample properties is the adjusted Portemanteau statistic:

$$Q_h^* = T^2\sum_{j=1}^{h}\frac{1}{T-j}tr(\hat{C}_j'\hat{C}_0^{-1}\hat{C}_j\hat{C}_0^{-1})$$

which is similar to the Ljung – Box statistic (Box G.E.P and Jenkins G., 1970) for univariate series. In practice, the choice of h may be critical for the test result. If h is chosen too small, the χ^2 - approximation to the null distribution may be very poor, whereas a large h may result in a loss of power.

2.9.2 Test for non normality

Multivariate tests for non normality can be constructed by generalizing the Jarque – Bera tests. The idea is to transform the joint normal distribution in order to obtain independent components and then apply the tests for the

univariate series to the independent components. Given the residuals $\hat{u}_t (t = 1, \dots, T)$ of an estimated VAR process or VECM, the residual covariance matrix is estimated as:

$$\sum_u = T^{-1} \sum_{t=1}^{T} \left(\hat{u}_t - \bar{\hat{u}}\right) \left(\hat{u}_t - \bar{\hat{u}}\right)'$$

The test on non normality can be based on the skewness and kurtosis of the standardized residuals $\hat{u}_t^s = (\hat{u}_{1t}^s, \dots, \hat{u}_{Kt}^s)$
The standardization of the residuals was proposed by Doornik J.A. and Hansen H. (1994).

$$b_1 = (b_{11}, \dots, b_{1k})'$$

with

$$b_{1k} = T^{-1} \sum_{t=1}^{T} (\hat{u}_{Kt}^s)^3$$

and

$$b_2 = (b_{21}, \dots, b_{2k})'$$

with

$$b_{2k} = T^{-1} \sum_{t=1}^{T} (\hat{u}_{Kt}^s)^4$$

Possible test statistics are:

$$s_3^2 = Tb'_1 b_1/6$$

and if we define the $(K\ X\ 1)$ vector $3_k = (3, \dots, 3)'$

$$s_4^2 = T(b_2 - 3_k)'(b_2 - 3_k)/24$$

Both statistics have asymptotic χ^2- distributions under the null hypothesis of normality.

2.10 Forecasting VAR Processes and VECMs

The forecast of the vector processes is completely analogous to the forecast of the univariate processes. The levels VAR form (31) is useful in forecasting the variable y_t. At first we assume that the process parameters are known. Suppose that the u_t 's are generated by an independent white noise process. For example, following Hamilton J.D. (1994), at the origin T of the forecast, an h - step ahead forecast is obtained as:

$$y_{T+h|T} = A_1 y_{T+h-1|T} + \cdots + A_p y_{T+h-p|T} \qquad (55)$$

where

$y_{T+j|T} = y_{T+j}$ for $j \leq 0$.

The corresponding forecast error is:

$$y_{T+h} - y_{T+h|T} = u_{T+h} + \Phi_1 u_{T+h-1} + \cdots +$$

$$+ \Phi_{h-1} u_{T+1} \tag{56}$$

and, by successive substitutions, it can be shown that:

$$\Phi_s = \sum_{j=1}^{s} \Phi_{s-j} A_j, \quad s = 1,2 \ldots, \tag{57}$$

with $\Phi_0 = I_k$ and $A_j = 0$ for $j > p$ [Lutkepohl (1991, Sec. 11.3)]. u_t is the 1 – step forecast error in period $t - 1$, and the forecast are unbiased; that is, the forecast errors have expectation 0. The MSE matrix of an h - step forecast is

$$\sum_{y}(h) = E\{(y_{T+h} - y_{T+h|T})(y_{T+h} - y_{T+h|T})'\} = \\ = \sum_{j=0}^{h-1} \Phi_j \sum_u \Phi'_j \tag{58}$$

If u_t is an uncorrelated white noise and is not necessarily independent over time, the forecasts obtained via a recursion as in (55) are just the best linear forecasts.
The forecast MSEs $\sum_y(h)$ for a stationary process converge to the unconditional covariance matrix of y_t.
If the process y_t is Gaussian, that is, $u_t \sim iid\ N(0, \sum_u)$, the forecast errors are also multivariate normal. Using this result, the following forecast intervals can be established:

$$[y_{T+h|T} - c_{1-\frac{\alpha}{2}} \sigma_k(h), y_{k,T+h|T} - c_{1-\frac{\alpha}{2}} \sigma_k(h)] \tag{59}$$

Here $c_{1-\frac{\alpha}{2}}$ is the $\left(1 - \frac{\alpha}{2}\right)100$ percentage point of the standard normal distribution, $y_{k,T+h|T}$ denotes the $k-$th

component of $y_{T+h|T}$, and $\sigma_k(h)$ is the standard deviation of the h - step forecast error for the kth component of y_t.
If the process is moldable as a VECM, to the aim of forecasting it may be rewritten in VAR form. Alternatively, equivalent forecasting equations can be obtained directly from the VECM.
If deterministic and/or exogenous variables are present, the formula must be extended. Because the future development of the deterministic variables is known, they are particularly easy to handle. They may simply be added to the stochastic part.
We have worked under the assumption that the process is known, but this assumption is unrealistic in practice. Following Hamilton J.D. (1994), denoting the optimal h - step forecast by $y_{T+h|T}$ as in (55) and providing its counterpart based on estimated coefficients by a hat, we have:

$$\hat{y}_{T+h|T} = \hat{A}_1 \hat{y}_{T+h-1|T} + \cdots + \hat{A}_p \hat{y}_{T+h-p|T} \qquad (60)$$

where

$\hat{y}_{T+j|T} = y_{T+j}$ for $j \leq 0$

and the $\hat{A}_i$s $(i = 1, \dots, p)$ are estimated parameters. The corresponding forecast error is:

$$y_{T+h} - \hat{y}_{T+h|T} = \left[y_{T+h} - y_{T+h|T}\right] + \left[y_{T+h|T} - \hat{y}_{T+h|T}\right] =$$

$$= \sum_{j=0}^{h-1} \Phi_j \, u_{T+h-j} + \left[y_{T+h|T} - \hat{y}_{T+h|T}\right] \qquad (61)$$

The first term on the right side of the equation (61) involves future residuals u_t with $t > T$ only, the second term is determined by present and past variables if only past variables have been used for the estimation. It follows that the two terms are independent if u_t is an independent white noise. Moreover, under standard assumptions, the difference $y_{T+h|T} - \hat{y}_{T+h|T}$ is small in probability as the sample size used for estimation gets large and the VAR coefficients are estimated more and more precisely. Hence, the forecast error covariance matrix is:

$$\sum_{\hat{y}}(h) = E\{(y_{T+h} - \hat{y}_{T+h|T})(y_{T+h} - \hat{y}_{T+h|T})'\} =$$
$$= \sum_y(h) + o(1)$$

The quantity $o(1)$ denotes a term tending to zero with increasing sample size. Correction factors for forecast MSEs and forecast interval may become more complicated, depending on the terms to be included in addition to the VAR part.

2.11 Empirical Application

In this chapter we will discuss about the best way to forecast the cause – specific mortality rates.
This work aim is to propose a method for mitigating the jumps caused by the reclassification ICD dropping the hypotheses of independence between all causes of death.
By means of models and processes known in literature, we build some innovative steps to follow in order to overcome these two restrictive problems. In particular, the

study is based on an extension of the standard Lee – Carter model for smoothing the mortality series (Haberman S. and Villegas A, 2014 – hereinafter mentioned as HV Model) and on the VECM (Vector Error Correction Model), analysed in order to project the long-term stationary relation between the different causes of death.
In our opinion the proposed method provides a better understanding of trends in aggregate mortality rates and implies improvements in the forecasting process, having considered the long-run relationships between causes based on historical data.
Once the specific death rates are forecasted, we can compute the resulting forecasted mortality rates and compare them with the results of the Lee Carter Model, when they are forecasted with an ARIMA processes. We will show that the proposed method produces the preferable projections in order to calculate the total central death rates.
To this aim we have to modeling the cause – specific mortality through four different steps, as follows:

1) Smoothing on the mortality rates with an extension of the Lee – Carter Model
2) Capturing the possible dependences among the cause – specific deaths with the VECM.
3) Forecasting the mortality index with the ARIMA models and the VECM (only if there are several stationary cointegrating relations between them).
4) Use the better prevision in order to calculate the future trend of mortality rates.

2.11.1 Data and purposes

Data concerning mortality, disaggregated for causes of death, are available at the Mortality Database administered by the World Health Organization [2009] (WHO) while the aggregated data (all causes) can be got from the Human Mortality Database (HMD), containing several demographic information as the number of deaths for many countries over the last 50 years for five-year age groups. The aggregated death (and survival) probabilities have been got by means of a Poisson Log Bilinear regression (see Brouhns N. et al 2002) on the Lee Carter model (see Lee R.D. and Carter L. 1992).
Causes of death are defined by the International Classification of Diseases (ICD), which ensures consistencies between countries. In this section, all causes of death are considered divided by macro classes. The ICD changed three times between 1950 and 2006, from ICD-7 to ICD-10, in order to take into account changes in science and technology and to refine the classification. As consequence the raw data are not directly comparable for different periods. Using U.K. mortality data divided by cause – specific deaths, we show that our method exhibits the best results compared with the ordinary method for modeling mortality model and the ARIMA in terms of goodness of fit and ex post forecasting performance considering the dependencies. Indeed, after the adjustment on the mortality index, we then use the VECM's to derive projections of cause - specific mortality rates and life expectancies. In particular, using selection criteria, such as Akaike's Information Criteria (AIC), Hannan-Quinn Criterion (HQ), Schwarz Criterion (SC), Final Prediction

Error (FPE), we select the lag order of the VAR (Vector Autoregressive). Several unit root tests on the variables considered are discussed; in particular a process can be defined stationary if its VAR has all its roots outside the complex unit circle (Hamilton J.D.,1994 and Lütkepohl K., 2005). Therefore, if this polynomial has a root equal to unity, some or all the variables are integrated of order one and there might be cointegrated relations among them. Unit root tests, such as the Kwiatkowski-Phillips-Schmidt-Shin test (KPSS), the Augmented Dickey-Fuller test (ADF) or the Phillips-Perron test (PP), are useful tools in order to check for the stationarity of the variables. KPSS tests the null hypothesis that the variable is level or trend stationary, while in the ADF and PP test the null hypothesis of a unit root, and thus, the null hypothesis of non-stationarity. If the variables are stationary, denoted $I(0)$ (integrated of zero order meaning that the autocovariance is decaying to 0 sufficiently quickly), a $VAR(p)$ is suitable. If the variables are $I(0)$, the Johansen's procedure is applied to find the number of cointegrated relations. Two test statistics are commonly used in order to find the number of cointegrated relations: the trace test and the maximum-eigenvalue test.

The trace test compares the null hypothesis that there are r cointegrated relations against the alternative of n cointegrated relations, where n corresponds to the number of variables under observation and $r < n$.

The maximum-eigenvalue statistic tests the null hypothesis of r cointegrated relations against the hypothesis of $r + 1$ cointegrated relations. If the variables are $I(1)$ and if there is no cointegration, a $VAR(p - 1)$ on the first difference is estimated. Otherwise, the

appropriate VECM should be found. For validating the model a test for residual autocorrelations and non-normality is used.
Once the specific death rates are forecasted, we can compute the resulting forecasted mortality rates and compare them with the results of the Poisson log – bilinear Lee Carter Model, when they are forecasted with an ARIMA process. We show that the VECM produces the best forecasts.

2.11.2 Cause–Specific Mortality smoothing

In this section we estimate the parameters of the Haberman – Villegas Model applied to the cause – specific death; in particular we will adjust the k_t series in order to prepare them for the cointegration analysis. As previously explained, the adjustment is due to the discontinuities in the data due to the reclassifications. They have took place in several Countries at different times, as reported in Table 1.

Table 1. ICD Changes

Country	ICD change	Year	Country	ICD change	Year
USA	ICD7-8	1968	Australia	ICD7-8	1968
	ICD8-9	1979		ICD8-9	1979
	ICD9-10	1999		ICD9-10	1998
Japan	ICD7-8	1968	Sweden	ICD7-8	1969
	ICD8-9	1979		ICD8-9	1987
	ICD9-10	1995		ICD9-10	1997
France	ICD7-8	1968	Switzerland	ICD7-8	1969
	ICD8-9	1979		ICD8-10	1995
	ICD9-10	2000	Singapore	ICD7-8	1969
Italy	ICD7-8	1968		ICD8-9	1979
	ICD8-9	1979	Norway	ICD7-8	1969
U.K.	ICD7-8	1968		ICD8-9	1986
	ICD8-9	1979		ICD9-10	1996
	ICD9-10	2001			

The following set of figures from 5 to 28 concerns the representation of the fitted parameters α_x, β_x, k_t and $\delta_x^{(i)}$. of the model in (21).
Each group is referred to male and female U.K. population divided for different causes of death. In clockwise, the first subplot shows the trend of the sum of the two parameters α_x and $\delta_x^{(i)}$. Being α_x constant with respect to t over the whole observed period, the four curves are referred to the 4 different values of $\delta_x^{(i)}$ got in the four intervals: 1950-1967, 1968-1978, 1979-2000, 2001-2009.
The second subplot shows the trend of the parameter β_x as function of x and the third reports the adjusted k_t trend as function of t.
The three vertical red segments point out the reclassification time: as evident, no more jumps are present in the graph.
The 4 subplots in the other figures (Adjusted k_t for different ages) highlight this trend and in particular show how the discontinuities have been mitigated. As an example they are referred to 4 different age intervals (40-44, 50-54, 60-64, 70-74): the light dots are the adjusted values of k_t, the big dots are the observed data and the continuous line represents the fitted data.
We show the same quantities in the case of both sex.

Fig. 5 - I&P (U.K., Male population) - Fitting parameter

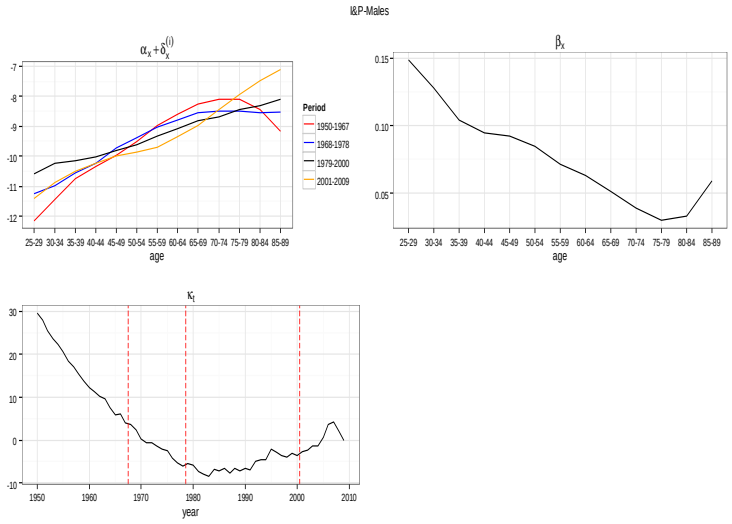


Fig. 6 – I&P (U.K., Male population) - Adjusted k_t for different ages

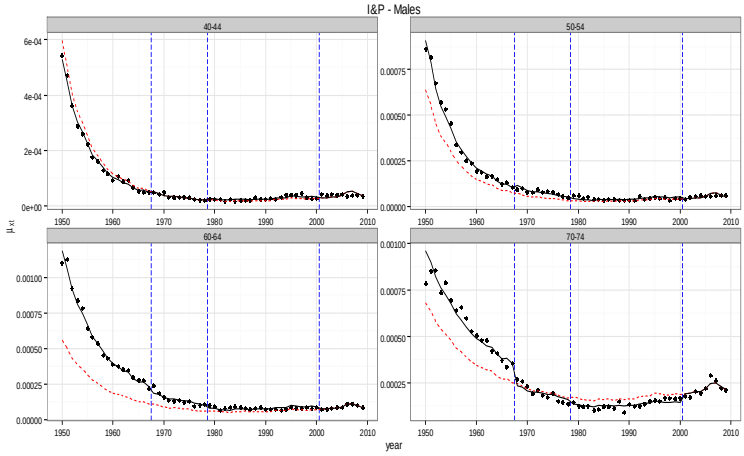


Fig. 7 - Cancer (U.K., Male population) - Fitting parameter

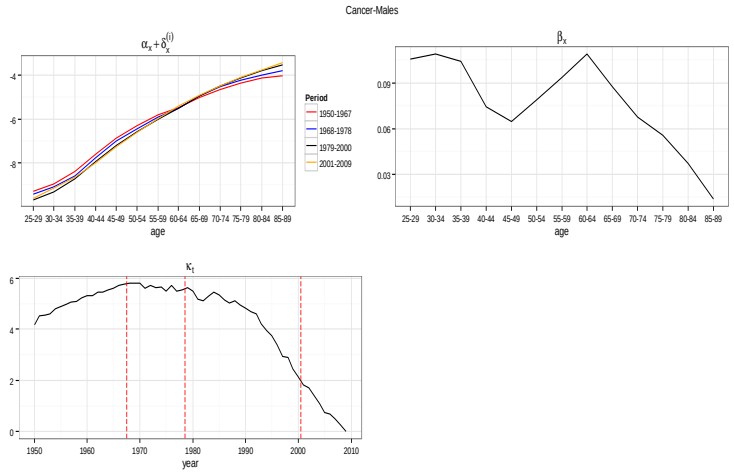


Fig. 8 - Cancer (U.K., Male population) - Adjusted $\mathbf{k_t}$ for different ages

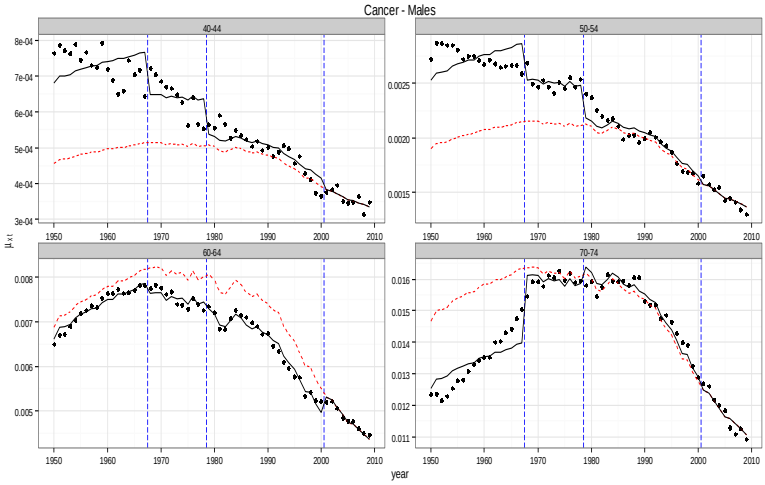


Fig. 9 - Circulatory (U.K., Male population) - Fitting parameter

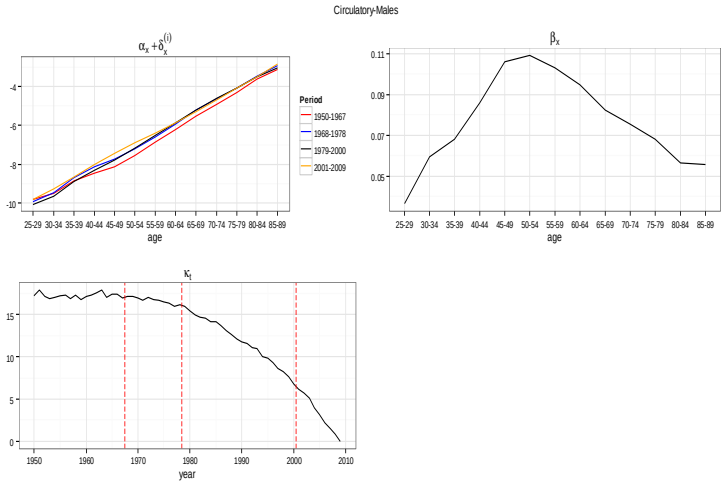


Fig. 10 - Circulatory (U.K., Male population) - Adjusted k_t for different ages

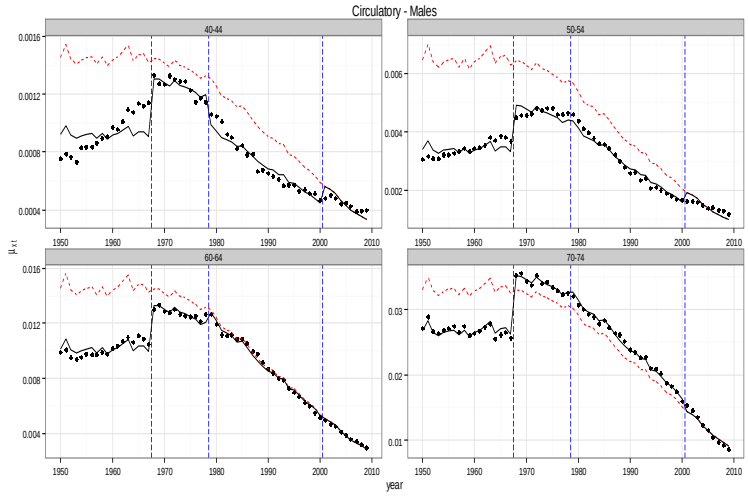


Fig. 11 - Respiratory (U.K., Male population) - Fitting parameter

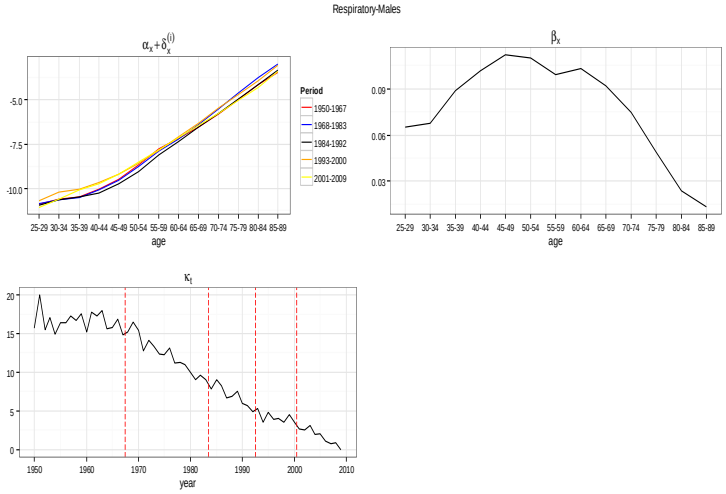


Fig. 12 - Respiratory (U.K., Male population) - Adjusted k_t for different ages

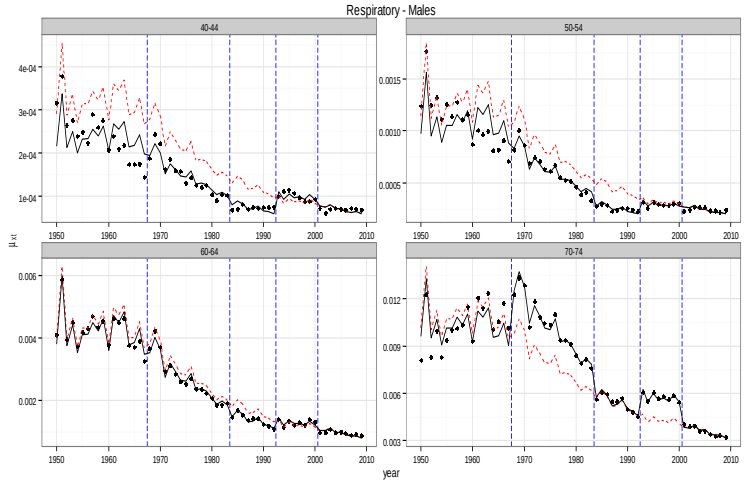


Fig. 13 - External (U.K., Male population) – Fitting parameter

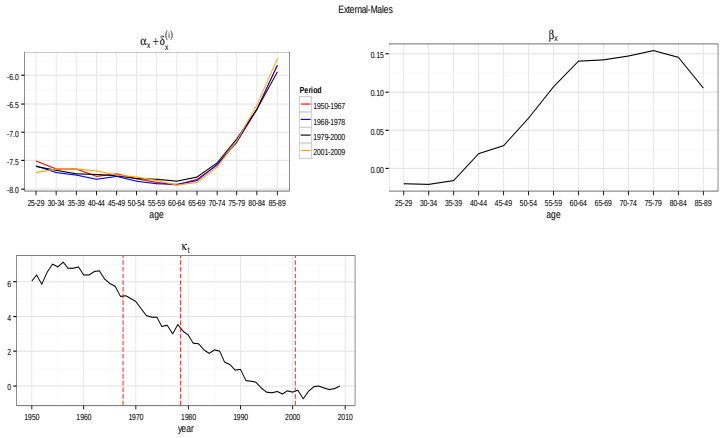


Fig. 14 - External (U.K., Male population) - Adjusted k_t for different ages

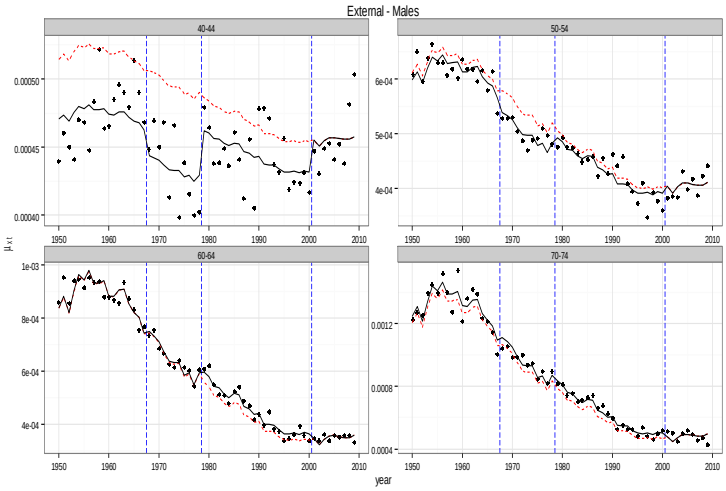


Fig. 15 - Other (U.K., Male population) - Fitting parameter

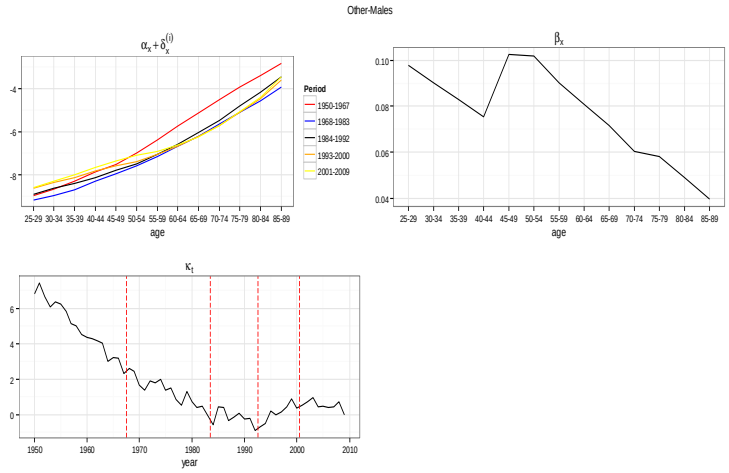


Fig. 16 - Other (U.K., Male population) - Adjusted k_t for different ages

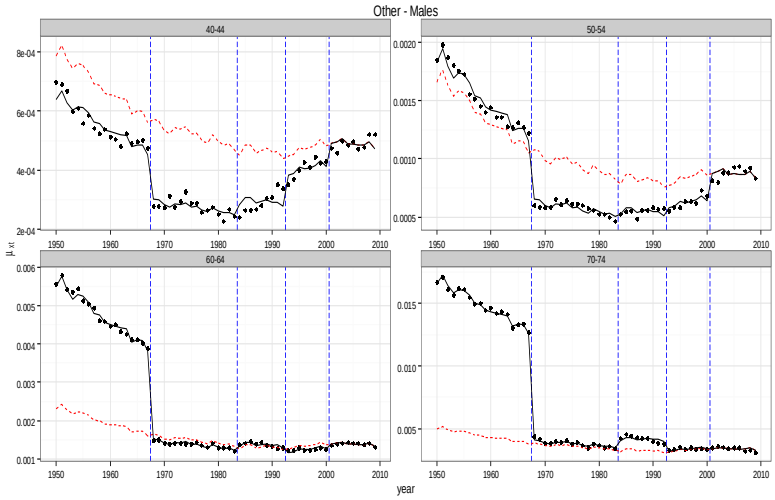


Fig. 16 – I&P (U.K., Female population) – Fitting parameter

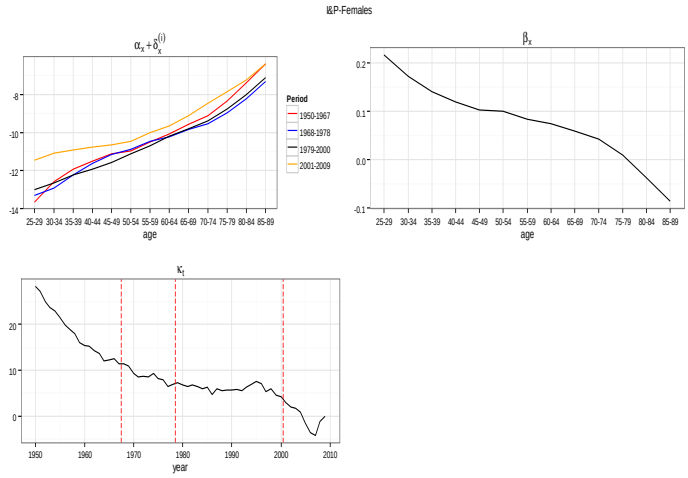


Fig. 17 – I&P (U.K., Female population) - Adjusted k_t for different ages

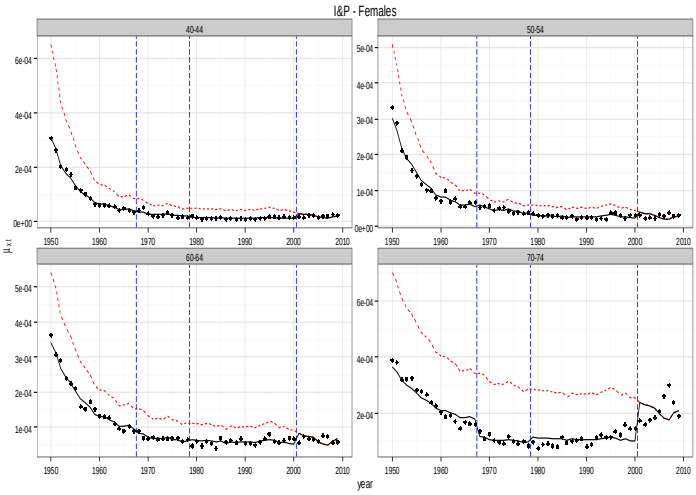


Fig. 18 – Cancer (U.K., Female population) – Fitting parameter

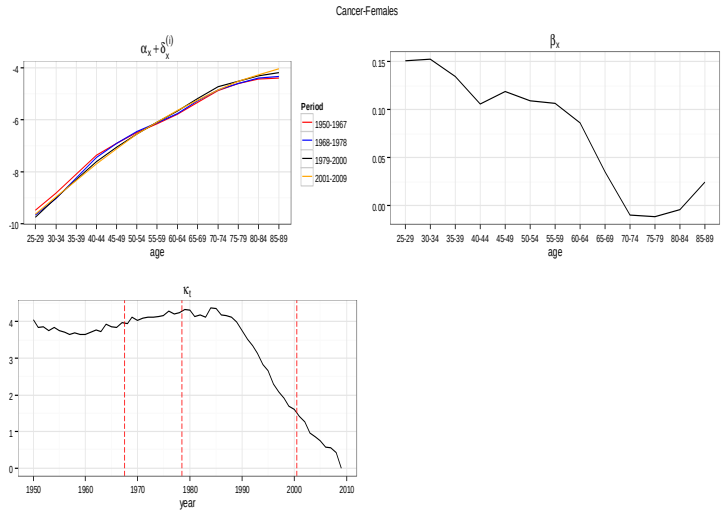


Fig. 19 – Cancer (U.K., Female population) Adjusted k_t for different ages

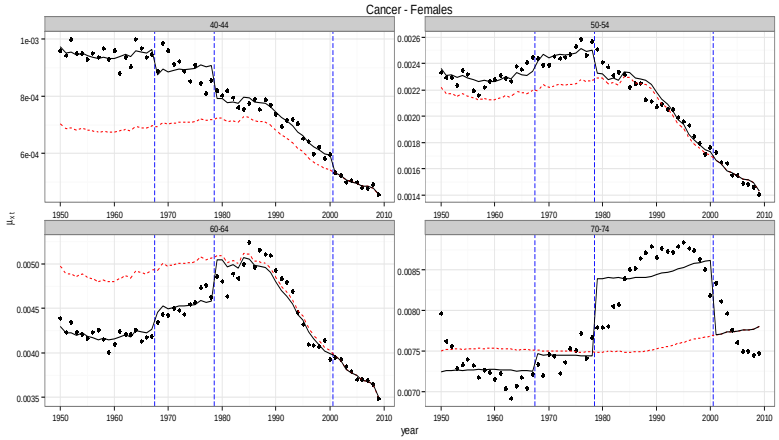


Fig. 20 – Circulatory (U.K., Female populatio) - Fitting parameter

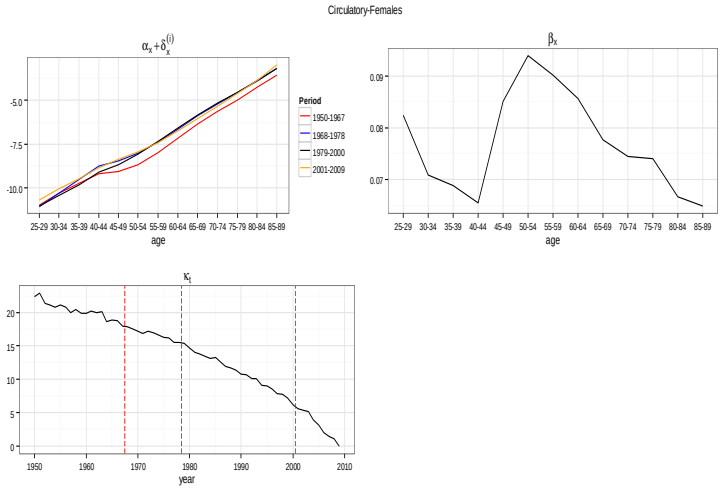


Fig. 21 – Circulatory (U.K., Female populatio) - Adjusted k_t for different ages

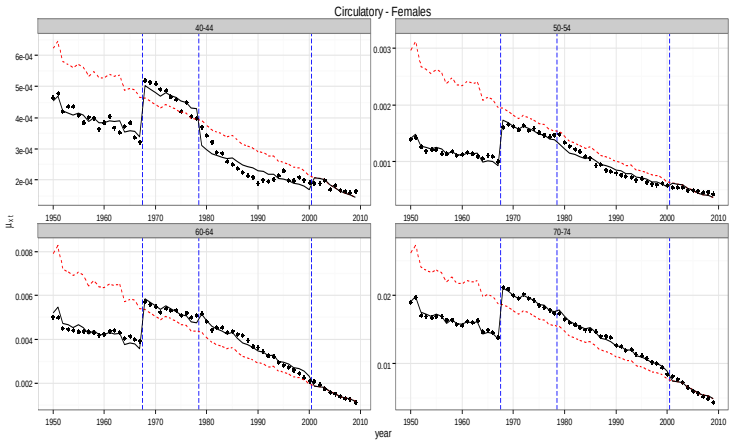


Fig. 22 – Respiratory (U.K., Female populatio) - Fitting parameter

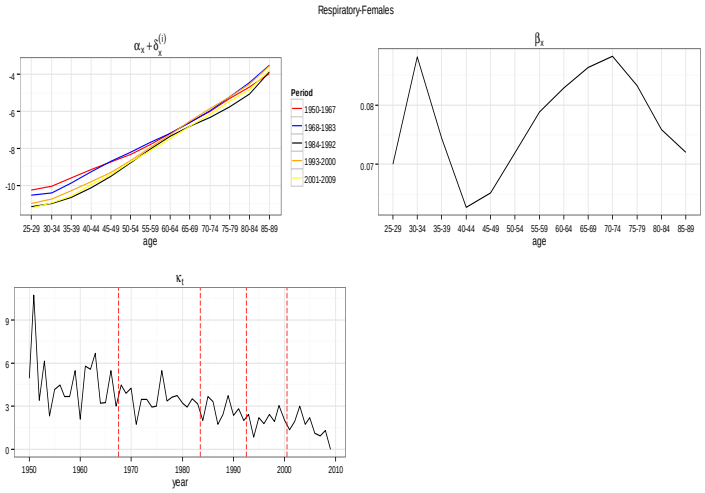


Fig. 23 – Respiratory (U.K., Female populatio) - Adjusted k_t for different ages

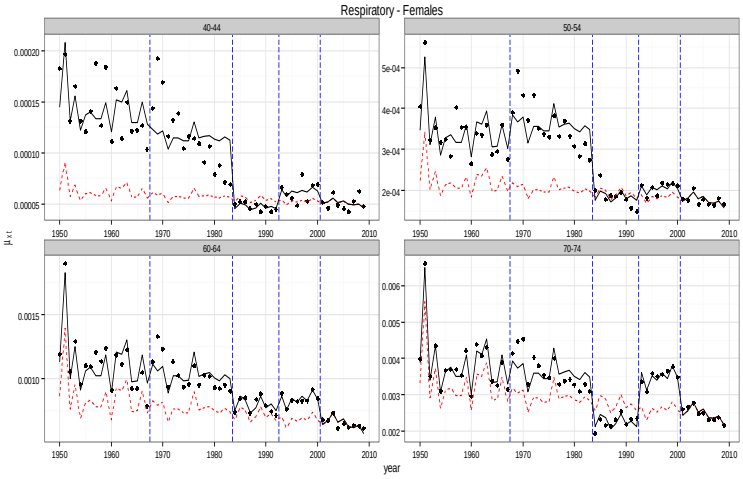


Fig. 24 – External (U.K., Female populatio) - Fitting parameter

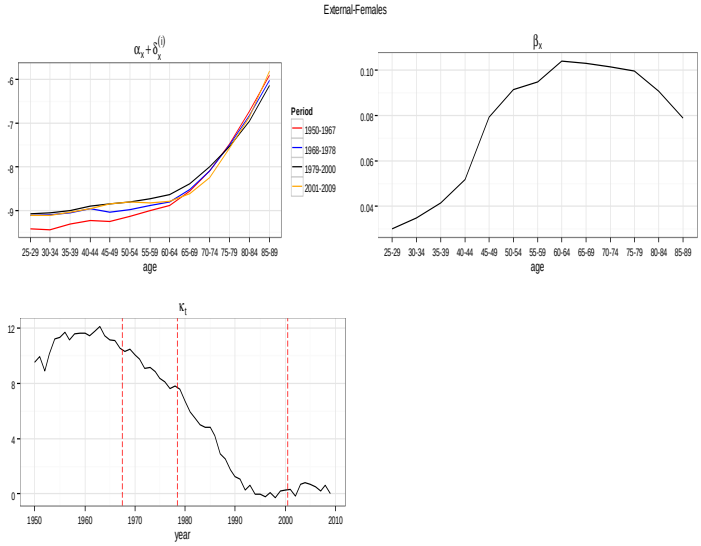


Fig. 25 – External (U.K., Female populatio) - Adjusted k_t for different ages

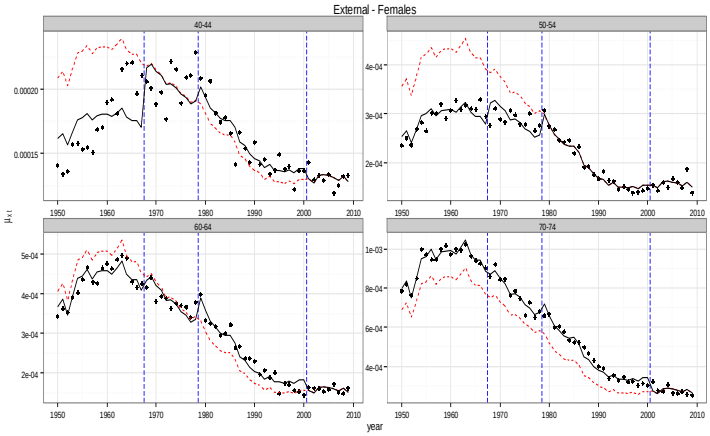


Fig. 27 – Other (U.K., Female populatio) - Fitting parameter

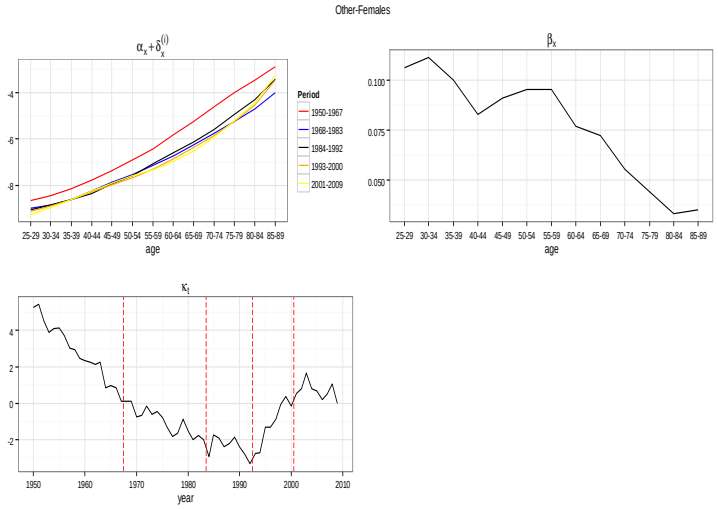
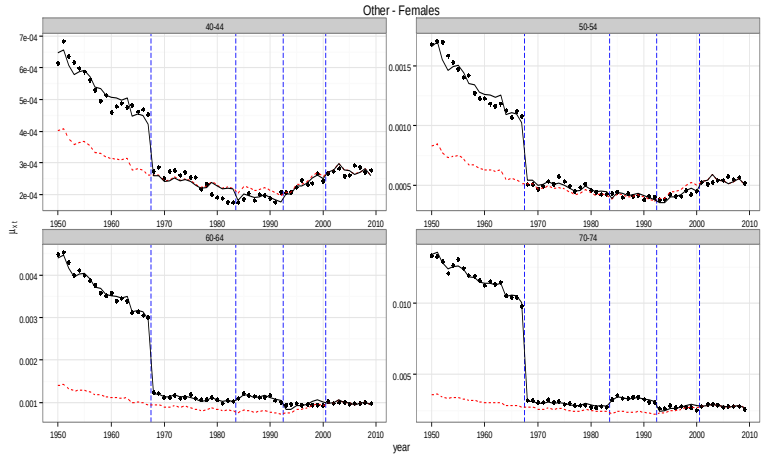


Fig. 28 – External (U.K., Female populatio) - Adjusted k_t for different ages



These figures illustrate the outputs we got throughout the procedure using the Model of Haberman S. and Villegas A. applied to the cause – specific mortality.
Data for U.K. are available from 1950 to 2009. In order to compute a back testing analysis on the k_t trends we will restrict the sample. To this aim we consider a period 1950-2001. After capturing the dependences among the causes of death using the procedure illustrated in section 2, we will forecast the mortality rates and we will compare them with the real data and with their ARIMA's forecast. In Table 2 we report the evaluation of k_t after the adjustment for the six causes considered. Table 3 shows the same quantities in the female case.

Table 2: Kt Adjusted (U.K. Male population, age 25-89)

	I&P	Cancer	Circulatory	Respiratory	External	Other
1950	336.253.815	0.9851339	160.167.506	156.624.748	529.538.742	6.814.580.077
1951	319.607.039	11.886.972	167.844.878	200.242.355	564.722.116	7.417.550.752
1952	295.987.179	12.009.202	159.258.074	155.171.448	519.598.798	6.629.722.507
1953	277.039.644	12.503.850	156.677.792	170.831.909	579.940.435	6.055.365.895
1954	264.616.438	13.831.401	158.582.222	149.173.776	629.373.491	6.336.665.779
1955	246.959.861	14.676.676	159.895.506	164.042.447	612.859.334	6.247.295.214
1956	225.770.613	15.168.594	160.109.448	164.103.402	637.843.941	5.829.558.781
1957	211.999.636	15.852.411	155.241.769	172.701.881	604.260.501	5.133.041.346
1958	195.911.310	16.415.779	159.347.845	166.583.289	602.858.469	5.019.398.865
1959	178.846.230	17.336.064	154.019.155	175.455.005	612.980.388	4.498.484.141
1960	164.534.582	17.875.804	156.878.647	151.655.742	567.517.887	4.349.075.976
1961	155.923.636	18.300.090	158.660.776	177.621.566	569.608.637	4.250.086.925
1962	144.688.875	19.623.682	161.068.255	172.848.325	591.569.146	4.132.148.774
1963	138.887.891	19.913.382	163.912.924	179.870.488	589.829.277	4.026.063.103

1964	117.639.974	20.370.748	154.900.467	155.963.241	546.704.463	3.015.523.527
1965	101.748.295	20.912.087	158.213.876	157.512.335	524.091.479	3.200.141.195
1966	103.499.355	22.286.016	158.675.597	168.200.306	510.860.433	3.157.579.360
1967	83.229.495	22.999.440	153.897.168	148.111.969	463.526.662	2.304.724.719
1968	79.326.914	23.200.692	155.899.862	151.856.457	465.905.344	2.586.610.558
1969	67.402.089	23.746.205	155.100.042	164.302.531	443.883.938	2.426.052.089
1970	47.445.450	23.940.912	152.865.006	154.351.352	430.518.552	1.672.628.048
1971	37.169.659	22.699.527	150.864.550	127.652.391	387.142.983	1.366.624.832
1972	37.908.487	23.743.947	154.068.429	141.566.197	356.420.958	1.913.494.989
1973	29.853.101	23.485.242	151.190.330	133.180.461	347.083.700	1.756.532.793
1974	22.414.986	23.925.178	150.077.413	123.821.906	350.716.366	1.997.951.769
1975	19.505.913	23.381.816	148.511.066	122.527.245	305.320.513	1.378.695.600
1976	0.3538315	24.376.293	146.864.855	131.794.233	310.877.080	1.488.252.305
1977	-0.7676349	23.604.843	143.657.478	112.027.586	263.470.328	0.841809994
1978	-16.487.370	23.941.452	145.077.615	112.933.324	308.530.505	0.514822336
1979	-14.752.314	24.052.203	143.997.535	109.639.631	293.280.667	1.284.792.636
1980	-18.464.469	23.681.645	139.231.561	99.870.847	272.173.519	0.716950875
1981	-34.224.445	22.346.253	133.819.556	90.539.462	237.947.725	0.401237318
1982	-37.705.497	22.390.622	132.093.332	95.455.457	224.473.822	0.477478870
1983	-43.307.349	23.541.096	130.433.864	89.692.575	201.238.571	-0.061986894
1984	-43.837.387	22.944.098	125.073.157	78.547.918	171.083.893	-0.580047782
1985	-47.218.298	22.817.700	126.041.602	89.844.120	190.281.461	0.420154725
1986	-40.939.658	21.922.177	121.728.971	82.576.634	182.385.071	0.397949040
1987	-53.336.032	21.443.408	116.142.165	66.124.220	125.103.820	-0.342802254
1988	-41.134.924	21.836.990	112.394.663	67.966.828	108.677.403	-0.154848099
1989	-46.957.224	21.232.599	108.611.598	75.652.028	0.84606013	0.072914775
1990	-42.294.582	20.823.205	105.070.873	59.985.791	0.76202128	-0.251098391
1991	-45.138.240	20.424.744	104.423.083	56.736.768	0.26077836	-0.222865439
1992	-24.856.106	20.214.911	99.582.112	48.892.550	0.23971824	-0.904111577
1993	-41.110.405	19.192.273	96.306.432	52.867.836	0.22689215	-0.697991076

1994	-39.854.180	18.172.083	88.054.052	34.981.664	-0.10145364	-0.493852156
1995	-16.119.934	17.317.349	86.526.780	47.536.296	-0.28953703	0.200859462
1996	-23.511.528	15.360.762	81.902.868	38.704.168	-0.33444848	-0.001449109
1997	-31.189.324	13.635.721	75.527.691	40.323.131	-0.21752818	0.148588441
1998	-34.875.554	13.478.966	72.722.232	35.309.679	-0.42444081	0.412287940
1999	-25.887.287	11.379.451	67.843.292	45.083.367	-0.25150724	0.868878441
2000	-30.951.004	0.9871767	59.666.214	34.799.406	-0.24649344	0.329306432
2001	-24.979.343	0.8013715	53.826.730	25.957.943	-0.20422115	0.510934433
2002	-21.640.843	0.7602224	49.770.640	24.933.254	-0.59723875	0.720461155
2003	-13.322.788	0.6270362	44.809.343	31.207.273	-0.19658056	0.941892092
2004	-13.975.038	0.5037456	34.886.250	19.360.007	-0.01336516	0.446392686
2005	0.7246603	0.3430953	27.521.294	19.917.497	0.07339119	0.481601064
2006	36.230.204	0.2728323	18.924.895	10.225.490	-0.01105043	0.380932392
2007	40.986.136	0.1817249	12.530.686	0.7497018	-0.08477443	0.417673975
2008	20.472.631	0.1222587	0.7731234	0.8436780	-0.11146009	0.706985951
2009	0.0000000	0.0000000	0.0000000	0.0000000	0.00000000	0.000000000

Table 3: Kt Adjusted (U.K. female population, age 25-89)

	I&P	Cancer	Circulatory	Respiratory	External	Other
1950	306.147.648	17.932.392	24.706.259	49.018.534	841.660.875	524.104.904
1951	293.799.539	15.070.223	25.302.680	107.229.961	884.522.313	543.972.326
1952	272.019.439	14.734.471	23.506.537	34.069.673	786.794.669	451.496.193
1953	259.393.937	12.887.728	23.281.824	61.506.640	884.656.908	387.930.049
1954	250.460.355	13.117.304	22.992.952	22.707.325	975.231.963	408.297.977
1955	235.657.764	12.554.384	23.281.827	41.339.169	985.944.291	416.423.970
1956	220.094.274	12.278.292	22.984.730	44.740.681	1.013.393.570	373.295.005
1957	210.895.969	10.871.747	22.097.838	36.854.835	963.624.582	303.710.043
1958	200.341.428	11.312.896	22.586.384	36.868.213	998.401.859	295.218.085

1959	181.702.856	10.664.002	21.970.470	54.551.035	1.000.622.468	247.920.637
1960	174.108.838	10.808.073	21.955.834	20.757.310	987.270.553	232.295.010
1961	172.787.595	10.841.846	22.264.875	57.449.798	971.190.568	226.711.868
1962	163.717.934	10.753.935	22.081.898	55.331.465	997.071.564	211.385.977
1963	156.814.503	10.162.353	22.213.792	66.546.255	1.021.538.352	227.097.294
1964	140.083.714	11.467.673	20.573.763	31.773.612	962.540.788	0.83075461
1965	142.032.082	10.887.962	20.772.278	32.377.586	935.133.265	0.97122888
1966	145.757.406	11.866.392	20.678.304	54.660.141	938.135.325	0.84497512
1967	133.123.141	12.232.267	19.809.896	29.964.607	891.307.438	0.11899745
1968	134.500.441	12.294.129	19.694.556	44.382.735	871.677.399	0.11720160
1969	129.161.793	13.302.927	19.335.567	38.816.411	877.841.000	0.12644362
1970	112.758.577	13.048.331	18.922.793	42.520.894	847.161.333	-0.75188241
1971	103.947.019	13.440.461	18.553.991	17.349.501	805.442.057	-0.65889027
1972	106.840.182	13.906.567	18.957.028	34.298.754	756.272.097	-0.16417524
1973	103.750.377	14.414.326	18.666.227	34.293.133	757.530.624	-0.63783931
1974	113.037.303	15.624.188	18.351.445	29.537.650	736.575.125	-0.45187338
1975	101.046.275	15.615.321	17.952.592	29.955.822	703.406.916	-0.80957752
1976	97.967.144	18.112.201	17.855.210	54.462.294	672.813.629	-131.217.432
1977	82.845.846	17.516.074	17.160.830	33.615.278	638.568.191	-183.241.378
1978	87.726.344	17.711.840	17.079.046	36.089.055	640.768.684	-166.058.498
1979	79.165.802	19.659.139	16.955.627	37.162.091	630.829.470	-0.88490175
1980	75.196.938	19.711.399	16.233.430	31.964.088	564.773.961	-152.129.684
1981	70.883.814	18.368.905	15.571.389	29.348.850	502.141.323	-199.017.632
1982	72.993.434	20.634.926	15.237.102	35.123.079	468.885.883	-178.254.968
1983	68.994.274	20.633.082	14.907.798	31.191.044	438.457.369	-200.644.489
1984	59.249.765	20.886.860	14.011.563	19.831.543	371.854.033	-293.266.455
1985	62.918.429	21.483.820	14.234.042	36.854.799	374.112.763	-175.112.520
1986	45.012.858	20.246.527	13.541.267	32.730.370	331.027.301	-191.852.983
1987	59.431.843	21.364.484	12.809.476	16.953.211	215.260.469	-238.619.539
1988	53.955.141	21.907.949	12.561.164	24.058.510	200.616.132	-219.975.156

1989	56.566.522	22.774.483	12.215.144	37.253.961	143.505.489	-187.958.863
1990	54.987.196	21.119.004	11.618.055	23.374.388	0.94781100	-238.161.948
1991	57.012.623	20.291.865	11.469.177	28.075.193	0.92979269	-280.940.405
1992	52.970.659	19.958.996	10.867.338	19.598.541	0.28617069	-331.079.635
1993	56.198.115	19.366.372	10.630.648	24.100.420	0.27241019	-275.349.809
1994	62.208.902	18.779.260	9.601.938	0.8200386	-0.19531348	-272.046.319
1995	67.873.228	17.644.446	9.424.086	21.949.246	-0.09239692	-131.960.498
1996	64.419.691	16.040.150	8.962.270	17.762.693	-0.34094589	-131.942.894
1997	48.658.051	14.855.080	8.323.541	23.923.656	-0.09849879	-0.86194271
1998	54.983.458	13.740.624	8.179.517	19.212.599	-0.25344129	-0.05639918
1999	42.015.961	12.727.202	7.590.422	30.401.252	0.13279377	0.37809699
2000	41.008.649	10.527.282	6.499.172	20.516.358	0.13829643	-0.14054475
2001	29.450.035	0.9838004	5.919.445	13.567.136	0.14697337	0.53882197
2002	20.584.806	0.8958127	5.665.381	19.353.333	-0.15433918	0.81561008
2003	16.322.333	0.6999716	5.461.873	29.634.169	0.52448562	167.014.112
2004	0.9256991	0.5116823	4.147.886	17.384.115	0.60046259	0.77920541
2005	-15.892.269	0.4914102	3.296.559	21.924.071	0.53473820	0.67295938
2006	-36.268.929	0.3611531	2.145.886	10.727.252	0.39894083	0.17880144
2007	-42.289.621	0.3259963	1.509.788	0.9540164	0.19582415	0.49368661
2008	-12.458.279	0.2564461	1.137.819	12.789.115	0.47322613	105.283.027
2009	0.0000000	0.0000000	0.000000	0.0000000	0.00000000	0.00000000

The next step will be the cointegration analysis. In particular if all the variables have unit roots and there is a stationary cointegrating relation between the k_t, we will forecast each of them in the VECM form. After that we will compare the VECM forecast with the ARIMA forecast.

2.11.3 Dependence analysis.

Going along the procedure presented in section (2.3) again, first of all we have to select the lag of the VAR through several criteria (AIC, Hannan-Quinn Criterion, Schwarz Criterion, Final Prediction Error).
With some tests (KPSS, ADF, PP) it is possible to see if the characteristic polynomial has unit root. KPSS tests the null hypothesis that the variable is trend stationary, while ADF and PP test the null hypothesis of a unit root (the null hypothesis of non-stationary). If the VAR has unit roots and all the variables are integrated of the same order, the VECM could be used. The Johansen's procedure is applied to find the number of cointegrated relations. If there is not cointegration a $VAR(p-1)$ on the first difference could be more appropriate. Finally, if all variables are I(0) a VAR(p) is suitable. In Table 4 we report the lag order of the Vector Autoregressive process obtained through the four criteria discussed is section 2.7.1

- Lag order of the VAR

Table 4: $VAR(p)$

U.K. Male Population

AIC(n)	HQ(n)	SC(n)	FPE(n)
2	1	1	1

U.K. Female Population

AIC(n)	HQ(n)	SC(n)	FPE(n)
1	1	1	1

The eigenvalues are bigger than one in absolute value. This means that the VAR could explode because its characteristic polynomial has unit roots.

- Unit roots tests

Tables from 5 to 10 show that for both sex all the variables are not stationary but I(1) except for the $k_{t,Respiratory}$ for female population. In order to make a good analysis we don't have to consider it for the cointegration analysis.
Figures 29 and 30 report the trend of k_t on the first differences for both sex and for the six causes of death.

MALES

Table 5: Augmented Dickey – Fuller test, U.K. Population

CAUSES OF DEATH	ADF	LAG ORDER	P - VALUE
I&P	-0.9381	3	0,9402
Cancer	-0.4784	3	0,9798
Circulatory System	2.0119	3	0,99
Respiratory System	-2.33	3	0,4414
External	-1.2111	3	0,8936
Other	-1.0549	3	0,9219

Table 6: Phillips - Perron, U.K. Population

CAUSES OF DEATH	PP	LAG ORDER	P - VALUE
I&P	-1.7203	3	0,9737
Cancer	-0.2694	3	0,99
Circulatory System	1.2455	3	0,99
Respiratory System	-38.1031	3	**0,01**
External	-7.3597	3	0,6793
Other	-2.5195	3	0,9523

Table 7: KPSS, U.K. Population

CAUSES OF DEATH	KPSS	LAG ORDER	P - VALUE
I&P	2.3031	3	0,01
Cancer	1.163	3	0,01
Circulatory System	2.6937	3	0,01
Respiratory System	3.0345	3	0,01
External	3.0138	3	0,01
Other	2.4834	3	0,01

FEMALES

Table 8: Augmented Dickey – Fuller test, U.K. Population

CAUSES OF DEATH	ADF	LAG ORDER	P - VALUE
I&P	-2.8242	3	0,2416
Cancer	0.1599	3	0,99
Circulatory System	1.0873	3	0,99
Respiratory System	-4.6381	3	**0,01**
External	-2.1588	3	0,5106
Other	-1.0109	3	0,9288

Table 9: Phillips - Perron, U.K. Population

CAUSES OF DEATH	PP	LAG ORDER	P - VALUE
I&P	-7.5514	3	0,6677
Cancer	1.362	3	0,99
Circulatory System	1.3336	3	0,99
Respiratory System	-74.9722	3	**0,01**
External	-5.8541	3	0,7705
Other	-2.6035	3	0,95

Table 10: KPSS, U.K. Population

CAUSES OF DEATH	KPSS	LAG ORDER	P - VALUE
I&P	2.7215	3	0,01
Cancer	0.5891	3	0.02363
Circulatory System	2.9883	3	0,01
Respiratory System	2.1376	3	0,01
External	2.9435	3	0,01
Other	1.5903	3	0,01

Fig. 29 - K_t First Difference, U.K., Male Population

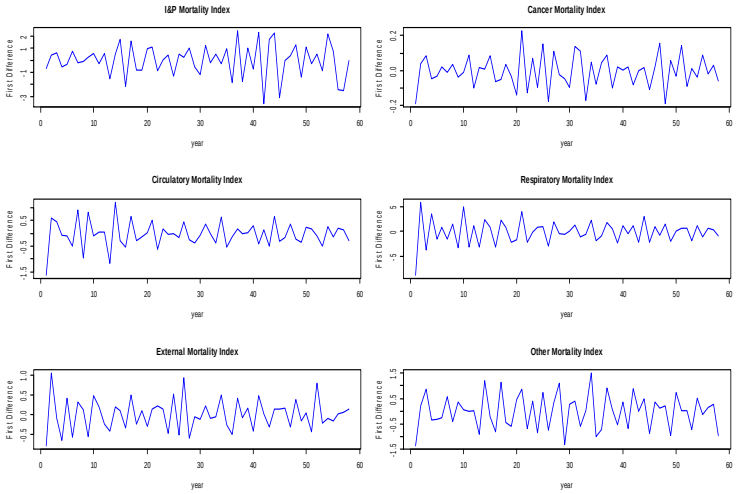
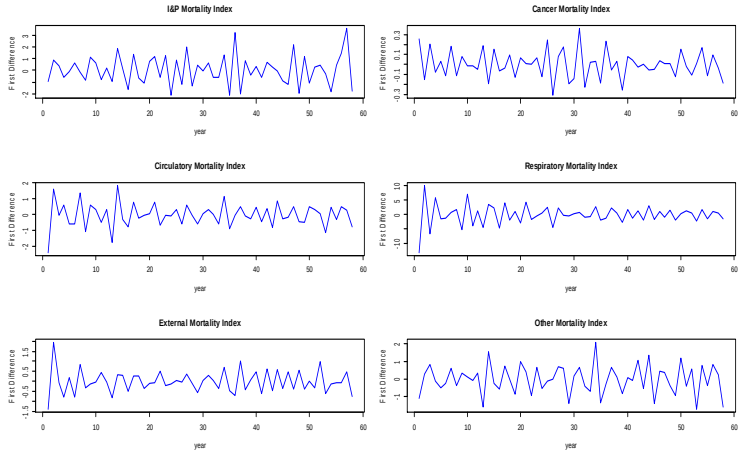


Fig. 30 - $\mathrm{K_t}$ First Difference, U.K., Female Population



- Cointegration Analysis

The Johanson's procedure shows a linear trend in the cointegration and in the variables. With trace test and maxmum eigenvalue test it is possible to know the number of cointegrating relationships.

Table 11: Trace Test, U.K. Male Population

h	*n-h*	*stat*	*10%*	*5%*	*2.5%*	*1%*
4	1	0.05041928	2.70	3.84	5.25	6.98
3	2	6,18016087	15.74	18.08	20.26	22.40
2	3	24,83020039	31.67	34.27	36.98	40.10
1	4	51,09236539	50.62	54.02	57.01	61.03
0	5	96,15921357	73.73	77.61	81.29	85.56

Table 12: Maximum – Eigenvalues Test, U.K. Male Population

h	*n-h*	*stat*	*10%*	*5%*	*2.5%*	*1%*
4	1	0.05041928	2.70	3.84	5.25	6.98
3	2	6,12974159	14.64	16.69	18.84	20.88
2	3	18,65003953	21.44	23.75	25.68	28.31
1	4	26,26216499	27.39	29.93	32.22	35.57
0	5	45,06684818	33.45	36.46	39.00	41.87

Table 13: Trace Test, U.K. Female Population

h	*n-h*	*stat*	*10%*	*5%*	*2.5%*	*1%*
4	1	0.2883353	2.70	3.84	5.25	6.98
3	2	9,5286355	15.74	18.08	20.26	22.40
2	3	39,2167916	31.67	34.27	36.98	40.10
1	4	83,8383564	50.62	54.02	57.01	61.03
0	5	155,0295236	73.73	77.61	81.29	85.56

Table 14: Maximum-Eigenvalues Test, U.K. Female Population

h	*n-h*	*stat*	*10%*	*5%*	*2.5%*	*1%*
4	1	0.2883353	2.70	3.84	5.25	6.98
3	2	9,2403001	14.64	16.69	18.84	20.88
2	3	29,6881561	21.44	23.75	25.68	28.31
1	4	44,6215648	27.39	29.93	32.22	35.57
0	5	71,1911673	33.45	36.46	39.00	41.87

Tables 11 and 12 show one cointegrating relations in the male case; tables 13 and 14 indicate two relations in the female population. In particular we can observe a quadratic trend in the variables and a linear trend in the cointegration for males and females (see Johansen's procedure). Table 15 and 16 report the result of the tests on residuals discussed in section 2.9.

Table 15: Tests on Residuals of the Fitted VECM, 1950–2001, Males in U.K.

Type of test	Name of Test	Statistic Value	P *value*
Autocorrelation	Portmanteau (15 lags)	383.38	0.02
	Portmanteau (25 lags)	624.78	0.01
Normality	Skewness	2.10	0.70
	Kurtosis	14.03	0.36
	Both	16.14	0.58

Table 16: Tests on Residuals of the Fitted VECM, 1950–2001, Females in U.K.

Type of test	Name of Test	Statistic Value	P *value*
Autocorrelation	Portmanteau (15 lags)	369.35	0,04
	Portmanteau (25 lags)	601.37	0,03
Normality	Skewness	1.98	0.88
	Kurtosis	15.20	0.09
	Both	17.19	0.34

The null hypothesis of no autocorrelation among the residuals is tested through the Portmanteau statistic, with a lag of 15 and 25, whose results are in Table 15 and 16. The skewness statistic, the kurtosis statistic, and a combination of these are used to test the normality of the

residuals. As indicated , the normality of the residuals is accepted by the three tests.
As indicated in Table 15 and 16, the normality of the residuals is accepted by the three tests.

- Forecasting

The fitted model is now used to forecast cause-specific mortality rates. Since data are available until 2009, the forecasts are compared to actual mortality, which gives us some indications on the model forecasting performance. Figures 32-34 shows the forecasted mortality rates from the fitted VECM compared with the actual data (blue lines). The curve represents the fitted model until 2001 and the resulting forecasts from 2002 to 2009. The future trend is well captured by the model for the six causes. However, to better evaluate the model performance, it is necessary to compare the results with the outcomes of a more traditional approach, the AutoRegressive Integrated Moving Average (ARIMA) process.
As for the VECM, ARIMA processes are fitted over the period 1950–2001 and used to forecast mortality until 2009. Since the approach developed in Pandit S.M. and Wu S.M. (2001) is followed, the nonstationarity in the variables is first removed by differencing the variables. In our case to ensure stationarity we will operate on the first differences of each cause-specific death rate using the tests KPSS, ADF and PP.
$ARIMA(k, 1, K-1)$ models are then successively fitted to each age-standardized cause-specific log-death rate, increasing k by one. Pandit S.M. and Wu S.M. (2001) suggest the use of the F-criterion to decide which model is the most suitable between an $ARIMA(k, 1, K-1)$ and an

$ARIMA(k+1,1,k)$, as this criterion tests the assumption that some of the coefficients in a model are restricted to zero.

Finally, noncorrelation among the residuals of the fitted model is checked. The best fitting ARIMA models resulting from this procedure are used for forecasting cause – specific rates.

The Portmanteau test indicates no significant residual autocorrelation with lags of 5, 10, 15, 20, and 25.

The forecasting performance of the two models is further evaluated through the mean absolute percentage error statistic (MAPE), the average of the absolute percentage gap between the forecasted and observed death rates. The average is made for a specific year over the five causes.

Table 17 compares the results for the VECM and ARIMA models.

The forecasts of the VECM are much closer to the actual death rates than the forecasts of the ARIMA processes, in particular in the long - run. Indeed, the MAPE is smaller for the VECM (see Arnold S. and Sherris M. 2013, 2014).

Fig. 31 - Total death rates in log scale, U.K. Male

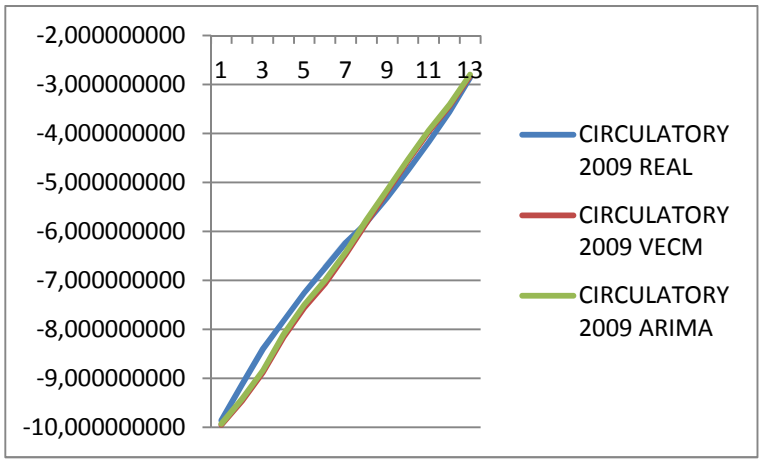


Fig. 32 - Total death rates in log scale, U.K. Male, 70-74

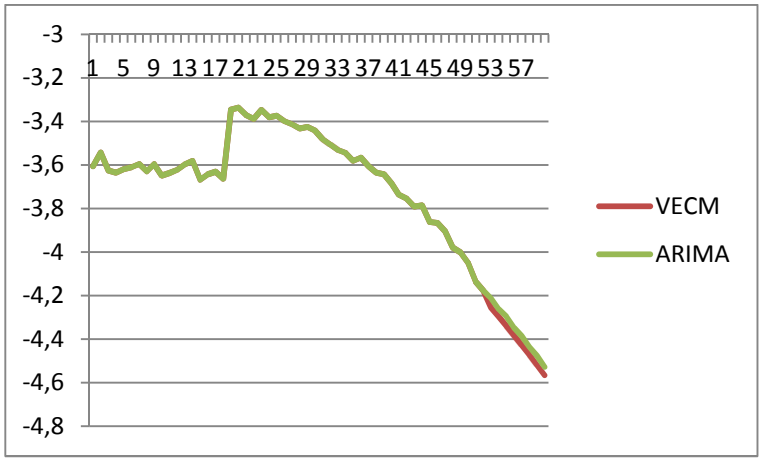


Fig. 33 - Total death rates in log scale, U.K. Male, 65-69

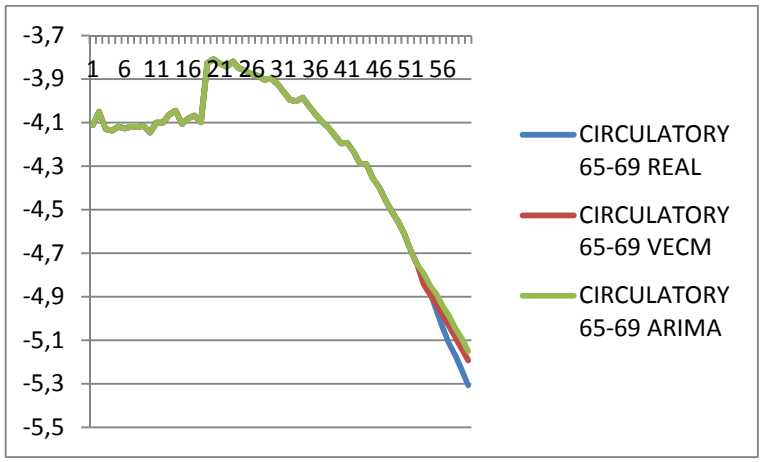
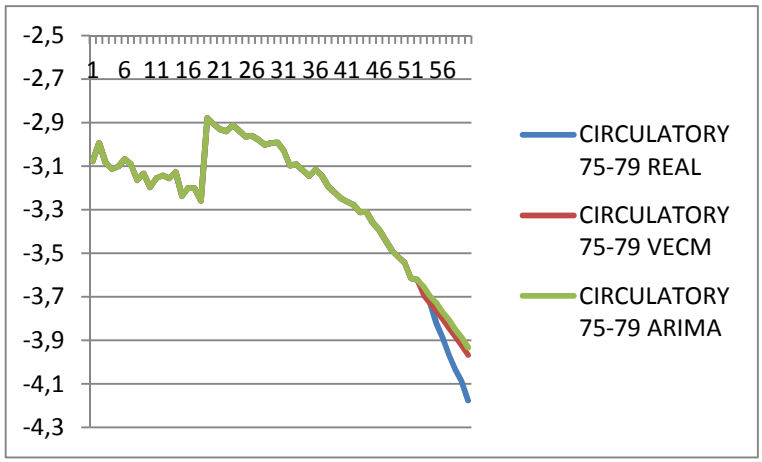


Fig. 34 - Total death rates in log scale, U.K. Male, 75-79



Figures 31 to 34 show some examples of the total deaths rates in log scale in different case. In particular in the figures 31 and 33 we have fixed the years and we have

plotted the curve taking into account only the age population, while in the figures 32 and 34, for different groups age, we have fixed the forecasted years.

Table 17: Mean Absolute Percentage Error

Male

	VECM	*ARIMA*
2002	1,32%	**0,72%**
2003	1,40%	**1,03%**
2004	1,55%	**1,42%**
2005	1,78%	**1,72%**
2006	**2,43%**	2,46%
2007	**2,53%**	2,55%
2008	**3,17%**	3,21%
2009	**3,30%**	3,35%

Female

	VECM	*ARIMA*
2002	1,08%	**0,76%**
2003	1,63%	**1,31%**
2004	1,44%	**1,47%**
2005	1,74%	**1,83%**
2006	**1,84%**	2,04%
2007	**2,17%**	2,31%
2008	**2,40%**	2,42%
2009	**2,85%**	2,93%

It shows that the forecast with the Vector error correction model is good, especially in the long run.

2.11.4 Conclusions

In this chapter the new idea introduced by Arnold S. and Sherris M., in modeling cause - specific mortality is presented. It consists in a new application of VECM to cause-of-death mortality and introduces a new modeling approach for cause-specific mortality that takes into account dependencies between causes. To this aim, first we have mitigated the discontinuity points caused by the ICD reclassifications; after we have captured the dependences among all causes of death. This innovative procedure is able to capture long-run trends and the stationary relationships between the variables. A long-run equilibrium relationship is shown to exist between the six main causes of death for U.K. females and males, providing an approach to model the cause-of-death dependence. This work confirms then that cointegration analysis, after the adjustment on the mortality index, is worthwhile in understanding and improving cause-specific mortality forecasts.

Chapter III

DREAD DISEASES AND CAUSE – SPECIFIC MORTALITY: NEW FORM OF INSURED LOAN

3.1 Introduction

In many industrialized countries, the progressive population ageing process determines a significant incidence of the diseases, which strongly increases with age. Today's individuals are expected to live longer than previous generations, but part of these extra years of life may not necessarily be healthy years. There are two outlooks of ageing and morbidity that contradict each other. The *Morbidity Compression Hypothesis* predicts that health problems will occur at higher ages, given a *rectangularization* of the health profiles (Schoder J. et al. 2011).

On the contrary, the *Morbidity Expansion Hypothesis* predicts the gradual medicalization of society as the longevity improves (Gruenberg E.M., 1977, Olshanksy J. et al., 1991, Doblhammer G. et al., 2001). Anyway the debate on the topic is ongoing.

In any of the two cases the morbidity phenomenon, both in the increasing concentration at higher ages hypothesis and in the spread effect over wider age intervals one, calls for a deep consideration, particularly in the insurance perspective.

The financial impact of the diseases due to costs for healthcare, rehabilitations, temporary and permanent assistance led to a supplementary insurance for health, current health systems, pay-as-you-go. Various classes of products have been developed by the insurance industry to specifically fulfil the needs of an ageing population facing the health risks. In particular, insurance companies start to offer coverage on financial contracts, by evaluating the exposition to long-term biometric risks such as mortality and morbidity. For instance, products protecting lenders and borrowers in the event a borrower ever stopped making payments for serious diseases or for death, are going to have great diffusion. At present, it can be observed that private insurers show more sensibility than social insurance in considering the question of life insurance coverage in case of specific cause of death and in the need of weight and balance the emerging health risks principally related to the ageing. The insurance industry traditionally proposes protection plans designed to give comprehensive financial support to the death event. A relevant case of a financial contract strongly affected by this risk is the insured loan. The contract, in its standard form, concerns the guarantee of the repayment provided by an insurance company in case of the borrower's death during the loan duration due to any cause. The insured loan is protected against default in the sense that if default occurs insurance company will pay the lender what is owed. In this chapter we are going to consider insured loans plans covering critical illnesses and cause specific death. The idea is to perform contractual schemes in which the actuarial side is tailored on the specific profile of the insured. The work focuses on the

insured loan product design when the borrower is a private person. In particular, the aim is to propose new contractual models in which the insured loan is covered in case of the borrower's death due to a specific cause and/or critical illnesses. Aim of the work is pricing the proposed products for inserting the results in the complete insured loan amortization schedule. Nevertheless it is opportune to point out that these new products have to be carefully priced. The structural breaks in the cause specific mortality time series indicate the difficulty in predicting cause specific mortality rates. Also the causes of death are competing risks. To perform this phase the mortality rates have been adjusted and predicted in the chapter 2.
The chapter is organised as follows. Section 3.2 is an outline of the main characteristics of the standard insured loan contract. Financial and actuarial details are analysed. Section 3.3 focuses on the new trend of designing contracts "specialised" according to specific death cause or specific illness. Within this section the new products are introduced and formally described. In section 3.4 the numerical application is illustrated and the new products are priced. The results are illustrated in several tables and commented in subsection 3.4.2. Section 3.5 is an outlook on some new perspective for going on in the development of the research in this subject.

3.2 Standard Insured Loan Contract

Typically in the standard amortization method the borrower refunds the lender paying instalments at periodic intervals. Usually the amortization goes on for a lot of years and this circumstance makes the operation affected

by the insolvency risk due to all the events related to the duration of the human life, whatever the age at issue of the borrower is. For these reasons it is efficient to insert in the contract an insurance policy for covering the risk that the debtor dies before having completely extinguished the debt. Broadly speaking, if the borrower dies before the contract expiry, the insurer pays to the lander the outstanding loan balance evaluated at that time. The loan becomes an insured loan and the insolvency risk due to the debtor's eventual death is cut down. In Coppola M. et al. 2009 a wide financial analysis of the insured loan is developed: formulas for single and periodic premiums, benefits and reserves are provided within the cash flow analysis. In that paper the Authors deep the risk analysis aspects, stopping in particular over the Model Risk and the Mortality Risk, the first due to the randomness in the choice of the mortality rate set and the second due to the random deviations of deaths from the expected values, considering pooling technique rather unfeasible in the specific matter in hands. Following Coppola M. et al. 2009, supposing the borrower/insured's debt is one monetary unit, we can write:

$$\sum_{k=0}^{n-1} P_k A^1_{\overline{x:k}|} = 1 \qquad (62)$$

being $A^1_{\overline{x:k}|}$ the actuarial present value of a k-year pure endowment of 1 monetary unit paid in case of life by an insured aged x, given by the following expression:

$$A^1_{\overline{x:k}|} = v(0,k)\ {}_kp_x \qquad (63)$$

with ${}_kp_x$ the survival probability of an insured aged x at inception to be alive at time k while $v(o,k)$ is the value in t=0 of 1 at time k. The annual anticipated payments P_k include the principal repaid C_k, the interest paid I_k on the outstanding balance D_{k-1} valued at time $k-1$, the actuarial premium covering the outstanding loan balance at the beginning of each year, if the death occurs before the expiration date.
In the following subsection we will concern with the financial structure of P_k, just observing that the actuarial premium and the loan instalment can be paid together to one counterparty (i.e. a bank) or separately to the lender and to an Insurance Company. These two circumstances will not have any consequence on the financial cash flows we are going to describe. The financial description will be developed in a deterministic environment, even if the stochastic approach for depicting the evolution in time of the interest rate curve could be easily implemented within a numerical application.

3.2.1 Insured loan: installment and actuarial premium analysis.

Consider that the borrower (aged x) will repay 1 monetary unit to the lender in n years by means of n constant instalments paid at the end of each year, at a given fixed annual rate of interest i or a variable one i_h. For sake of simplicity we will present the payments components in the fixed rate hypotheses. The constant annual payment amount $R_h = R$ and the outstanding loan balance D_h valued at the end of year n are respectively:

$$R_h = R = \frac{1}{a_{\overline{n}|}} \qquad D_h = \frac{a_{\overline{n-h}|}}{a_{\overline{n}|}} \tag{64}$$

having indicating with $a_{\overline{n}|}$, as usually, the present value of a periodic (annual) constant unitary income at the end of each period and for n periods, at a fixed interest rate. By means of the insurance component, if the borrower dies during the contract duration, the insurer will repay to the lender the obligations still due by the borrower at that time. We will assume that this payment operation will be done at the end of the year in which the eventual death occurs. If the death event happens at time t, $h-1 < t \leq h$, $0 < h \leq n$, what is due to the lender consists in the outstanding balance at time $h-1$ plus the annual interest on this sum. The value B_h of the benefit payable at time h ($h = 1,2,\dots,n$) if the insured-borrower aged *x* at issue dies during the *h-th* year and the probability of this event are respectively (Coppola M. et al., 2009):

$$\mathrm{B_h} = \frac{1}{a_{\overline{n}|}} \ddot{a}_{\overline{n-h+1}|} \qquad {}_{h-1/1}\mathrm{q_x} \tag{65}$$

where $\ddot{a}_{\overline{n-h+1}|}$ refers to the anticipated case.
The constant actuarial premium the borrower/insured pays at the beginning of the first m years ($0 < m \leq n, 0 \leq h < m-1$) if alive, is given by:

$${}_{/\mathrm{m}}\mathrm{P_{x,h}} = {}_{/\mathrm{m}}\mathrm{P_x} = \frac{1}{a_{\overline{n}|}} {}_{/\mathrm{m}}\pi_{\mathrm{x}} \tag{66}$$

in which:

$$ {}_{/\mathrm{m}}\pi_{\mathrm{x}} = \frac{1}{\ddot{a}_{\overline{x,m|}}} \sum_{\mathrm{j}=0}^{\mathrm{n}-1} {}_{j/}a_{\overline{n-j|}}\, {}_{\mathrm{j}/1}q_{\mathrm{x}} \qquad (67) $$

If k_x is the curtate future lifetime of the insured aged x at issue, in the case of anticipated constant annual payments, the flow X_h at the beginning of year h is given by the following scheme:

$$ X_h = \begin{cases} -{}_{/m}P_{x,h} & k_x \geq h & 0 \leq h \leq m-1 \\ 0 & k_x \geq h & h \geq m \\ \frac{1}{a_{\overline{n|}}} \ddot{a}_{\overline{n-h+1|}} & h-1 \leq k_x < h & 1 \leq h \leq n \end{cases} $$

with $h = 0, 1, \ldots, n;\ P_n = 0.$ (68)

3.3 Cause of death and diagnosis event: impact on loan repayment.

The cost of funding health care for elderly is continuously growing due to the increasing life expectancy. The topic is pregnant in the insured loan financial management. If a critical illness is diagnosed, the affected individual could not be able to completely or partially perform the engagements in his working activity and, in the specific case of the onset during the loan duration, this could involve the inability to fulfil the obligation as expected. Moreover, remaining within the traditional insured loan contract, setting the coverage in case of the borrower's death, it is interesting to study the case of a death specific cause. The result is a tailor-made contractual form providing lower costs for the insured and the insurer. Our

idea is to propose an insured loan form in which the insurance coverage involves critical illness diagnosis and/or death specific cause.
In the basic n-year term insurance, usually included in the loan amortization process, the insurer pays the benefit if the insurer dies within the n (or $h \leq n$) years of the loan duration, without specification about the death cause. Nevertheless the n-year term insurance can be specified with regard to a specific death cause. As well in the basic critical illness insurance (Pitacco E. et al., 1998) the insurer pays a lump sum upon the occurrence or diagnosis of the pre-specified dread diseases. Typically, the contractual options within the critical illness general scheme are the Stand Alone and the Accelerated. The first covers the insured just in case of diagnosis of illnesses, while the second guarantees payments in case of illness and in case of death.
The work focuses on insured loans in which such cause-specific insurance products are included, in order to explore new scenarios tending to personalize the loan contractual forms.
Aim of the analysis will be the pricing of the actuarial insurance coverage we propose and next the drawing up of the amortization schedule in which the annual instalment includes both the actuarial premium and the financial repaying process.

3.3.1 New proposals for insured loans

In what follows we pose the borrowed capital equal to 1 at time 0 while the amount the insurer will pay under the specified contractual conditions is the amount still owed,

that is the residual debt valued at the time of the benefit payment in case of the borrower's death. It is opportune to note that all the contracts in the following are designed taking into account the main aim of the operation, that is the resolution of the debt in case if the borrower's insolvency. This implies contracts built according to the amortization schedule and consequently the analysis needs a discrete approach.

Death Specific Insured Loan – SpeIL. The idea is to design a product in which the loan is saved in case of the borrower's death for a specific cause. The death cause is precisely defined in the contract, for instance ischemic heart disease, stroke, lower respiratory infections and chronic obstructive long disease. In our numerical application we consider the death for circulatory system problems.
The value of the insurer's obligations A_{SpeIL} valued at time 0 is given by:

$$A_{SpeIL} = \sum_{h=0}^{n-1} \frac{\ddot{a}_{\overline{n-h|}}}{a_{\overline{n|}}} v(0,h+1)\, {}_{h/1}q_x^{(C)} \qquad (69)$$

where ${}_{h/1}q_x^{(C)}$ is the probability that an insured aged *x* at issue dies between ages *x+h* and *x+h+1* because of a specific cause and $v(0,h+1)$ is the discount factor for valuing in *t=0* one monetary unit in *h+1*. Knowing the value A_{SpeIL} it is possible to set the equation involving the insured's obligations:

$$A_{SpeIL} = \sum_{h=0}^{n-1} P_h v(0,h)\, {}_hp_x \qquad (70)$$

where ${}_hp_x$ is the probability that x is alive at age $x+h$ and P_h is the premium the insured pays at the beginning of year h.

Standard Critical Illness Loan (Stand Alone) - SCILsa. Here the insurance protection inserted in the loan concerns the coverage of the risk to suffer a particular specified disease.
The equation concerning the insurer obligations follows:

$$A_{SCILsa} = \sum_{h=0}^{n-1} \frac{\ddot{a}_{\overline{n-h|}}}{a_{\overline{n|}}} v(0,h+1)\,{}_{h/1}w_x^{(d)} \qquad (71)$$

in which ${}_{h/1}w_x^{(d)}$ is the probability that the insured aged x at issue checks a specified diagnosis d during the year $h, h+1$.
The insured's obligations are given by the equation:

$$A_{SCILsa} = \sum_{h=0}^{n-1} P_h v(0,h)\ {}_hp_x \left(1 - {}_{h-1/1}w_x^{(d)}\right) \qquad (72)$$

in which:

$${}_{h-1/1}w_x^{(d)} = 0 \qquad if\ h = 0 \qquad (73)$$

Standard Critical Illness Loan (Accelerated) – SCILa. In this case the insurer will pay the amount if the insured suffers a specified disease or dies for any cause of death. The premium flow provides an accelerated benefit which covers the policyholder, both in case of a specified critical illness and in case of death for any cause.

The following equations hold:

$$A_{SCILa} = \sum_{h=0}^{n-1} \frac{\ddot{a}_{\overline{n-h|}}}{a_{\overline{n|}}} v(0,h+1)\left({}_{h/1}\tilde{q}_x\right) \qquad (74)$$

in which ${}_{h/1}\tilde{q}_x$ is the probability of the sum of two compatible events, both referred to the ages $x+h,\ x+h+1$, specifically to die for any cause of death and/or to suffer a specified illness.

Concerning insured's obligations we can write:

$$A_{SCILa} = \sum_{h=0}^{n-1} P_h v(0,h)\ {}_hp_x \left(1 - {}_{h-1/1}w_x^{(d)}\right) \qquad (75)$$

with the position in (73).

3.4 Numerical applications

3.4.1 Data source

The empirical analysis we are going to perform in this section aims to develop the amortization schedules for loans covered in case of death or/and critical illness of the borrower as clarified in subsection 3.3.1. We will determine the global instalment periodically due by the debtor-insured, inclusive of both the payment amount for repaying the loan and the actuarial premium for the insurance coverage. We will assume different loan durations (10-20 years) and that the debtor-insured is a 40 and 60 years old person in 2014. The study will be done referring to diverse cohorts (males, females, smokers,

non-smokers) in order to point out how the different basic characteristics impact on the contract pricing. Due to the availability of the data, we will refer to the U.K. population and will consider the circulatory system according to the diagnosis rates. Again, data for U.K. is available from 1950 to 2009. The diagnosis rates for a Stand Alone cover and for a Full Accelerated cover can be downloaded from the Continuous Mortality Investigation Bureau (CMI) and concern in particular the Circulatory System diseases (Brett P. and Du Toit J., 2007). Data concerning mortality disaggregated for causes of death are available at the Mortality Database administered by the World Health Organization [2009] (WHO), the aggregated data (all causes) can be got from the Human Mortality Database (HMD), containing demographic information as the number of deaths for many countries over the last 50 years for five-year age groups; data concerning death for any cause and/or diagnosis (accelerated from) are available in the Working Paper 14. The aggregated death (and survival) probabilities have been got by means of a Poisson Log Bilinear regression (see Brouhns N. et al 2002) on the Lee Carter model (see Lee and Carter 1992). The cause – specific mortality rates, in our case the Circulatory system, have been got using the model of Haberman S, and Villegas A., IME 2013 (see 2.11.2). Finally, we have forecasted the aggregated mortality with an ARIMA model and the mortality rate concerning to circulatory system with the VECM in order to capture the dependencies among all causes of death (see section 2.11.3).

3.4.2 Actuarial premiums

In this practical application we will take into account the specific death cause and/or illness cause "circulatory system illness", taken out from International Classification of Diseases (ICD). For developing the actuarial analysis, we have determined the adjusted mortality indexes (in chapter 2) for each cohort and the specified illness and project them along the loan duration. The procedure is quite complex and need some specifications.
Once specified the cohort under consideration, the first step is the calculation of the annual mortality rates, attainable as the ratio between the number of deaths and the number of survivors at the beginning of the year.
The diagnosis rates represent the principal end product of a program of work carried out by the CMI 2011 Critical Illness Committee to develop tables of critical illness diagnosis rates based on recent UK insured lives experience, together with sufficient supporting information to enable appropriate practical use by actuaries involved in this business. The diagnosis rates, divided by age, sex, smokers and non-smokers for durations of 5 years, are at present available only for "cancer and circulatory system illness". Coherently with this information, we consider the same specific death cause. It is crucial to observe that the ICD changed three times between 1950 and 2009, from ICD 7-8, ICD 8-9 and ICD 9-10; this happened for taking into account changes in science and technology and for refining the classification. So data are not directly comparable each other when referred to these different periods. As

Haberman et al. IME 2013 show, it is possible to smooth mortality rates across the various classifications (see chapter 2).

3.4.3 Empirical evidences and illustrations

The first step of our analysis is to evaluate α_x, β_x, k_t and $\delta_x^{(i)}$ (see Appendix A). Substituting these parameters in the characteristic equation of the model (14) it is easy to determinate and to forecast μ_{xt} trough the VECM process in order to calculate the global instalment periodically due by the debtor-insured in the 2014.
The second step is to develop the amortization schedules for loans covered in case of death or/and critical illness of the borrower as clarified in subsection 3.3.1.
The following groups in the next section of tables report the constant premiums payable in all the contractual forms considered in subsection 3.3.1, respectively in the case of UK Female and Male population. We fixed the loan annual interest rate $i = 0.07$ and the technical actuarial valuation rate $r = 0.02$. The contracts are issued in 2014. In particular in the table group 16, table 16.b refers to SpeIL and the premium is determined by formulas 69 and 70. This case is compared with the standard form SIL (Standard Insured Loan), providing the coverage in case of death for any cause, whose premium values are in table 16.a (see formula 66). As expected, premiums fell when only a specific cause of death is considered, even if "important" as the combination of cancer and circulatory system illness is.
Tables 17.a and 17.b concern the forms indicated as SCILsa and SCILa for Female non-smokers, in 18.a and

18.b there are the same results in the Female smokers case. The premiums have been calculated by formulas 71 to 75. It is self-evident how cheap the coverage is in the Specific Insured Loan case and how it increases if the population refers to smokers.
We observe in which measure the highest premium is that one referred to SCILa the contract offering the widest coverage: in this case the insurer will pay what is owed from the amortization schedule in case of death (for any cause) and in case of the specific illness diagnosis. Moreover, the diagnosis rates for female non-smokers aged 40-70 are higher than the corresponding one for the female smokers. As consequence in the case of SCILsa female non-smokers will pay more than the female smokers. This evidence works only in the Stand Alone cases. In the Accelerated coverage this effect is compensated by the expected behavior of the death probabilities and is not visible.
Tables numbered 19 – 20 – 21 contain the same values referred to the Male population. A comparison between the two groups points out the general lower premiums for the females. In spite of this, it is interesting to highlight that in the considered age interval the female non-smokers have diagnosis rates slightly higher than the male non-smokers. This circumstance turns into higher premiums for serious illness coverage in the case of female non-smokers.

3.4.4 Some final considerations

In the SCILa contract the weight's event can be estimated by quantifying the premiums payable in cases of incompatibility of events. In the below tables you can see that the premiums is affected by this hypothesis. Continuing to investigate this point, consider for example the case of a SpeCILa in cases of incompatibility between cause – specific death event and the illness events, it may be noted (see tables below) that the premium is even higher than the premium calculated for the Scila that covers a range of events much wider.

Actuarial Periodic Premium –
Female NS. Issue Time 2014, r = 2%, i = 7%,
C = 200000 – Compatible events

Specific Critical Illness Loan (Accelerated) - SpeCILa

Age at entry/ Duration	40	60
10	283.85	2095.32
20	560.70	3513.64

Standard Critical Illness Loan (Accelerated) - SCILa

Age at entry/ Duration	40	60
10	328.08	2606.21
20	663.43	4189.65

Actuarial Periodic Premium –
Female S. Issue Time 2014, r = 2%, i = 7%,
C = 200000, Compatible events

Specific Critical Illness Loan (Accelerated) - SpeCILa

Age at entry/ Duration	40	60
10	506.17	3140.46
20	966.05	5072.00

Standard Critical Illness Loan (Accelerated) - SCILa

Age at entry/ Duration	40	60
10	550.57	3656.02
20	1069.6	5756.04

Actuarial Periodic Premium –
Male NS. Issue Time 2014, r = 2%, i = 7%,
C = 200000 – Compatible events

Specific Critical Illness Loan (Accelerated) - SpeCILa

Age at entry/ Duration	40	60
10	325.18	1368.05
20	543.45	2189.11

Standard Critical Illness Loan (Accelerated) - SCILa

Age at entry/ Duration	40	60
10	352.79	1586.03
20	598.90	2531.23

Actuarial Periodic Premium –
Male S. Issue Time 2014, r = 2%, i = 7%,
C = 200000, Compatible events

Specific Critical Illness Loan (Accelerated) - SpeCILa

Age at entry/ Duration	40	60
10	375.46	1694.00
20	645.33	2672.06

Standard Critical Illness Loan (Accelerated) – SCILa

Age at entry/ Duration	40	60
10	403.07	1912.60
20	700.85	3015.47

Table 16. Actuarial Periodic Premium –
Female. Issue Time 2014, r = 2%, i = 7%, C = 200000

Table 16.a *Standard Insured Loan – SIL*

Age at entry/ Duration	40	60
10	90.19	749.04
20	175.95	1320.07

Table 16.b *Specific Insured Loan – SpeIL*

Age at entry/ Duration	40	60
10	62.65	532.64
20	120.72	981.14

Table 17. Actuarial Periodic Premium –
Female non smokers. Issue Time 2014, r = 2%, i = 7%, C = 200000

Table 17.a *Standard Critical Illness Loan (Stand Alone) – SCILsa*

Age at entry/ Duration	40	60
10	262.38	831.53
20	422.28	1198.74

Table 17.b *Standard Critical Illness Loan (Accelerated) - SCILa*

Age at entry/ Duration	40	60
10	285.62	925.37
20	456.98	1424.16

Table 18. Actuarial Periodic Premium –
Female smokers. Issue Time 2014, r = 2%, i = 7%, C = 200000

Table 18.a *Standard Critical Illness Loan (Stand Alone)- SCILsa*

Age at entry/ Duration	40	60
10	213.79	805
20	273.18	925.80

Table 18.b *Standard Critical Illness Loan (Accelerated) - SCILa*

Age at entry/ Duration	40	60
10	352.21	1304.30
20	580.70	2034.17

Table 19. Actuarial Periodic Premium –
Male. Issue Time 2014, r = 2%, i = 7%, C = 200000

Table 4.a *Standard Insured Loan – SIL*

Age at entry/ Duration	40	60
10	108.23	1251.55
20	231.06	2106.08

Table 19.b *Specific Insured Loan – SpeIL*

Age at entry/ Duration	40	60
10	64.87	746.67
20	129.59	1440.78

Table 20. Actuarial Periodic Premium –
Male non smokers. Issue Time 2014, r = 2%, i = 7%, C = 200000

Table 20.a *Standard Critical Illness Loan (Stand Alone) – SCILsa*

Age at entry/ Duration	40	60
10	218.68	1339.71
20	429.84	2049.66

Table 20.b *Standard Critical Illness Loan (Accelerated) – SCILa*

Age at entry/ Duration	40	60
10	260.27	1515.15
20	498.18	2373.22

Table 21. Actuarial Periodic Premium – Male smokers. Issue Time 2014, r = 2%, i = 7%, C = 200000

Table 21.a *Standard Critical Illness Loan (Stand Alone) - SCILsa*

Age at entry/ Duration	40	60
10	440.72	2378
20	834.17	3590.64

Table 21.b *Standard Critical Illness Loan (Accelerated) - SCILa*

Age at entry/ Duration	40	60
10	547.70	2975
20	1035.35	4686.70

3.4.5 Amortization schedule

The global obligations of the borrower/insured arise from the amortization schedule, for what concerns his financial

obligations, and from the premiums calculated in Table 16 - 21 of the preceding section, for what concerns the insurance coverage. He will pay the sum between the constant financial installment and the specific premium referred to the chosen insurance contract. As an example, in Table 7 we report the amortization schemes of a loan issued in 2014 at a fixed rate of 7% , initial debt of C=200000 and with duration 10 years (Table 7.a) and 20 years (Table 7.b).

Table 22.a. Amortization Schedule. Issue Time 2014, r = 7%, C = 200000, n = 10

Maturity	Financial Instalment	Payment due in case of insolvency
1	28475.50	214000.00
2	28475.50	198511.21
3	28475.50	181938.21
4	28475.50	164205.10
5	28475.50	145230.66
6	28475.50	124928.02
7	28475.50	103204.22
8	28475.50	79959.72
9	28475.50	55088.10
10	28475.50	28475.50

Table 22.b. Periodic Amortization Schedule.
Issue Time 2014, r = 7%, C = 200000, n = 20

Maturity	Financial Instalment	Payment due in case of insolvency	Maturity	Financial Instalment	Payment due in case of insolvency
1	18878.59	214000.00	**11**	18878.59	141876.95
2	18878.59	208779.91	**12**	18878.59	131608.26
3	18878.59	203194.41	**13**	18878.59	120620.75
4	18878.59	197217.95	**14**	18878.59	108864.10
5	18878.59	190823.10	**15**	18878.59	96284.51
6	18878.59	183980.65	**16**	18878.59	82824.33
7	18878.59	176659.19	**17**	18878.59	68421.95
8	18878.59	168825.25	**18**	18878.59	53011.41
9	18878.59	160442.95	**19**	18878.59	36961.41
10	18878.59	151473.85	**20**	18878.59	18878.59

In these Tables we report in particular the constant installment due by the borrower in case of insolvency throughout the loan duration (II column) and the payment due by the insurer in case of the borrower's insolvency, if this event happens during the year preceding the date of valuation (III column).
As an example, in the case of SCILsa, Female non-smokers, the global annual obligation is showed in Table 23.a. It is possible to appreciate the contribution of the

illness diagnosis coverage inclusion in the global amount to pay if compared with results in Table 23.b, referred to the traditional SIL contractual form.

Table 23.a. *Global annual obligation, Insured Loan and Stand Alone – SCILsa*
Female non smokers, C=200000, i=7%, r=2%

Age at entry/ Duration	40	60
10	28738.88	29307.03
20	19300.87	20077.33

Table 23.b. *Global annual obligation. Standard Insured Loan – SIL*
Female non smokers, C=200000, i=7%, r=2%

Age at entry/ Duration	40	60
10	28565.69	29224.54
20	19054.54	20198.66

Table 23.c. *Global annual obligation. Specific Insured Loan - SpeIL*
Female non smokers, C=200000, i=7%, r=2%

Age at entry/ Duration	40	60
10	28538.15	29008.14
20	18999.31	19859.73

Table 23.d. *Global annual obligation. Insured Loan and Accelerated - SCILa*
Female non smokers, C=200000, i=7%, r=2%

Age at entry/ Duration	40	60
10	28761.12	29400.87
20	19335.57	20302.75

Concluding, Table 23.c and 23.d show the global obligations in the same general conditions and in the SpeIL and SCILa cases, for which a cause - specific death and a dread disease are considered.

3.4.6 Future developments

The present chapter focused on the conjoint consideration of the financial product loan to private persons and the insurance coverage in case of specific causes (of death and illness).
On the hand the loan is very much diffused and is quite often affected by very long duration; this circumstance involves a strong insolvency risk due to critical illnesses or death of the borrower. On the other hand, we can observe the general tendency in specializing insurance contracts particularly in the more advanced Countries; this happens in order to offer products more efficient and cheaper from both the counterparty's points of view. Moreover we can add that this kind of contracts are more and more computable in light of the increasing extent of specific data. So the idea was to propose such new insurance coverage within the standard financial loan.

This aim has been followed projecting the cause specific mortality rates and the specific illness diagnosis rates. In this procedure the relevant question of the discontinuities in the mortality rates due to the reclassification of the data (ICD) has been fronted using a recent model able to mitigate the jumps in the data themselves (see the trend of k_t in chapter 2). Also, the problem of the dependencies among all causes of death has been solved using the Vector Error Correction Model. It has been possible to infer the projected data, to price the proposed contractual forms and to build the final borrower/insured payment scheme.

Our method was found to be important and innovative in order to treat the cause-specific mortality.

Future research in this topic is connected with the increasing interest in the specialization of the insurance contracts. We will propose new forms covering insolvency not due to death or illness but to other relevant circumstances as the layoff of the borrower.

Appendix A

k_t Adjusted, U.K. Male

1950	12.1977876	**1971**	12.4145334	**1992**	8.2976822
1951	12.8102253	**1972**	12.6904397	**1993**	8.1065012
1952	12.2684860	**1973**	12.4656538	**1994**	7.3969949
1953	12.1184357	**1974**	12.4076806	**1995**	7.1976506
1954	12.3434419	**1975**	12.2165739	**1996**	6.7186600
1955	12.4498179	**1976**	12.1661159	**1997**	6.0844991
1956	12.5201623	**1977**	11.8762173	**1998**	5.8743422
1957	12.2954990	**1978**	12.0086165	**1999**	5.3293180
1958	12.5672436	**1979**	11.9541044	**2000**	4.6259069
1959	12.2978883	**1980**	11.5486727	**2001**	4.2128824
1960	12.5374733	**1981**	11.0304712	**2002**	3.8810680
1961	12.6532095	**1982**	10.8691593	**2003**	3.4273460
1962	12.8941351	**1983**	10.8104845	**2004**	2.6481309
1963	13.1062926	**1984**	10.6015930	**2005**	2.0328041
1964	12.5960099	**1985**	10.6031243	**2006**	1.4724464
1965	12.8437346	**1986**	10.2144737	**2007**	0.9690608
1966	12.9091201	**1987**	9.7744505	**2008**	0.5843431
1967	12.6318339	**1988**	9.4982290	**2009**	0.0000000
1968	12.7746919	**1989**	9.1183347		
1969	12.7620518	**1990**	8.8219476		
1970	12.6047098	**1991**	8.6959803		

	β_x, U.K. male
25-29	0.05294683
30-34	0.07070276
35-39	0.07944927
40-44	0.08837821
45-49	0.10160989
50-54	0.10283334
55-59	0.09876937
60-64	0.09162840
65-69	0.07704875
70-74	0.06950730
75-79	0.06339855
80-84	0.05181474
85-89	0.05191260

$\alpha_x + \delta_x^{(i)}$, U.K. male

	1950-1967	1968-1978	1979-2000	2001-2009
25-29	0.11303276	0.040601754	-0.145563019	0
30-34	-0.12293424	-0.147652071	-0.295445332	0
35-39	-0.09762152	-0.080292606	-0.215212331	0
40-44	-0.18371761	-0.035992759	-0.184990402	0
45-49	-0.32627183	-0.131575616	-0.227023633	0
50-54	-0.30869869	-0.139023822	-0.171763923	0
55-59	-0.26081963	-0.127540138	-0.103104130	0
60-64	-0.25229734	-0.094350307	-0.057377670	0
65-69	-0.17141002	0.033268294	0.037702526	0
70-74	-0.17884584	0.049276301	0.057494502	0
75-79	-0.20438922	-0.003802946	0.010218412	0
80-84	-0.13314614	0.021280044	0.004776052	0
85-89	-0.25499646	-0.075789166	-0.143625854	0

k_t Adjusted, U.K. Female

1950	13.3375463	**1971**	9.5876877	**1992**	5.5109556
1951	13.6595987	**1972**	9.8415085	**1993**	5.3757906
1952	12.5803721	**1973**	9.6958699	**1994**	4.8537976
1953	12.4041885	**1974**	9.5207536	**1995**	4.7408648
1954	12.2233294	**1975**	9.3121571	**1996**	4.4761107
1955	12.4088270	**1976**	9.3113648	**1997**	4.1249926
1956	12.2216342	**1977**	8.8669877	**1998**	4.0494654
1957	11.6256691	**1978**	8.8257368	**1999**	3.7391840
1958	11.9706934	**1979**	8.8060686	**2000**	3.1560290
1959	11.5937203	**1980**	8.3950143	**2001**	2.8715188
1960	11.5645851	**1981**	7.9950923	**2002**	2.7722109
1961	11.7220943	**1982**	7.8211612	**2003**	2.6513063
1962	11.5965470	**1983**	7.6370548	**2004**	1.9576415
1963	11.6902031	**1984**	7.1412711	**2005**	1.5706113
1964	10.7610055	**1985**	7.3073607	**2006**	1.0105467
1965	10.8393640	**1986**	6.8944353	**2007**	0.7594275
1966	10.8161283	**1987**	6.5094327	**2008**	0.5973817
1967	10.3378527	**1988**	6.4194935	**2009**	0.0000000
1968	10.2691241	**1989**	6.2874514		
1969	10.0553331	**1990**	5.9050147		
1970	9.8260319	**1991**	5.8218406		

	β_x, U.K. Female
25-29	0.11100454
30-34	0.09460693
35-39	0.07674190
40-44	0.06682523
45-49	0.07020482
50-54	0.05079537
55-59	0.04244109
60-64	0.05290371
65-69	0.06202858
70-74	0.08302418
75-79	0.09841643
80-84	0.09784256
85-89	0.09316467

$\alpha_x + \delta_x^{(i)}$, U.K. Female

	1950-1967	1968-1983	1984-1992	1993-2000	2001-2009
25-29	-0.359994	-0.298346	-0.26892	-0.13299	0
30-34	-0.261023	-0.193566	-0.09773	-0.09332	0
35-39	-0.081888	0.010041	0.02371	-0.01741	0
40-44	0.002921	0.144483	0.04638	0.04523	0
45-49	-0.101574	0.148102	0.05075	0.01383	0
50-54	0.005708	0.268018	0.15539	0.06797	0
55-59	0.047817	0.309479	0.26768	0.11469	0
60-64	-0.057976	0.214545	0.26904	0.11936	0
65-69	-0.133052	0.144693	0.20730	0.13577	0
70-74	-0.331728	-0.009771	0.06786	0.08491	0
75-79	-0.491019	-0.141285	-0.06813	-0.02285	0
80-84	-0.483591	-0.140406	-0.09906	-0.05599	0
85-89	-0.587518	-0.210611	-0.19493	-0.20397	0

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