UNIVERSITÀ DEGLI STUDI DI SALERNO

DIPARTIMENTO DI MATEMATICA

DOTTORATO DI RICERCA IN MATEMATICA

CICLO X (NUOVA SERIE)

DOCTORAL DISSERTATION

ANALYSIS OF A BIRTH AND DEATH PROCESS WITH ALTERNATING RATES AND OF A TELEGRAPH PROCESS WITH UNDERLYING RANDOM WALK

RELATORE - ADVISOR:

Prof. Antonio Di Crescenzo

CANDIDATA - CANDIDATE:

Dott. ssa Antonella Iuliano

COORDINATRICE - PH. D. COORDINATOR:

Prof. ssa Patrizia Longobardi

A.A. 2010 - 2011

This thesis is dedicated to my family

and to the memory of my grandfather Antonio.

Look at many things, discard safe ones and

behaved very cautiously with respect to those remaining.

You will have less probability of being wrong.

Confucio (551 a.c - 479 a.c)

CONTENTS

PREFACE

This thesis is the result of three years of research and study in probability theory and stochastic processes, under the supervision of Professor Antonio Di Crescenzo. My attention focuses on the study of some probability distributions and their related characteristics, particularly on those of birth and death processes bilaterally and telegraph processes with underlying random walk. Such processes have been largely investigated in several fields of mathematics, in particular in biologiacal applications and in mathematical finance. My goal in writing this thesis is to give my little contribution to the basic theory of stochastic processes continuing to investigate this area to obtain quantities of interest in mathematical contexts.

I would like to thank Professor Antonio Di Crescenzo for his continuous support and guidance and Barbara Martinucci for her constant help and advises. Their encouragement during these years has been fundamental and always present. I am greatful to them for all they have done for me.

Antonella Iuliano

INTRODUCTION

Stochastic processes play an essential role in various fields of science and engineering. The theory of stochastic processes is based on probability theory and is widely used in modeling phenomena subject to random perturbations. In some cases these processes have a deterministic behaviour, such as models for population growth, competition, predation, and epidemics. One of the most relevant differences between deterministic and stochastic models is that deterministic models predict an outcome with absolute certainty, whereas stochastic models give only the probability of an outcome. More precisely, in a deterministic model, described for instance by a difference equation or differential equation with initial conditions at time $t = 0$, the solution is given by the trajectory in the solution space. In the stochastic model, the process is described by a system of difference equations (transition matrix) or differential equations (forward Kolmogorov equations or stochastic differential equations). The solution of these equations is more complicated in the sense that a single solution trajectory does not describe the entire behavior of the model but represents only a single realization of the processes. In order to understand the behaviour of a stochastic model, it is important to know the entire probability distribution of the process over time. If this is not feasible, the qualitative behaviour of the process is studied by obtaining other quantities of interest, such as the moments (mean, variance, etc) of the distribution. In population models, where the population size is large, a deterministic formulation is used. Instead, when population sizes are small, and then population extinction can occur, it is more realistic to model the variation in size by a stochastic formulation. Stochastic models may be used to analyse the probability of population extinction or the expected duration of time until population extinction. Random variations associated with demography and environment can be taken into account in stochastic models.

In this thesis some mathematical techniques are introduced and stochastic models are developed according to the classical theory. In particular, we focus on analysis of a birth and death process with alternating rates and of a telegraph process with underlying random walk. The birth and death processes are special Markov chains involving only countably many states and depending on a continuous time parameter where changes may occur at any time. The Markov chains are stochastic processes in which the future development depends only on the present state, and not on the past history of the process or the manner in which the present state was reached. Indeed, to describe the past of the process we must specify the epochs at which changes have occurred, and this involves probabilities in a continuum. These process involve only countably many states $E_1, E_2, \ldots$ and depend on a discrete time parameter, that is, changes occur only at fixed epochs $t \geq 0$. The transition probability $p_{k,n}(t)$ is the conditional probability of the state E_n at epoch $t + s$ given that at epoch $s < t + s$ the system was in state E_k. Such transition probabilities are called stationary or time-homogeneous and satisfy the *Chapman–Kolmogorov equation*:

$$p_{k,n}(t+\tau) = \sum_j p_{k,j}(\tau)p_{j,n}(t). \tag{1}$$

This relation means that at epoch 0 the system is in state E_k. The j-th term on the right then represents the probability of the event of finding the system at epoch τ in the state E_n. But a transition from E_k at epoch 0 to E_n at epoch $t+\tau$ necessarily occurs through some intermediary state E_j at epoch τ and summing over all possible E_j for arbitrary fixed $\tau > 0$ and $t > 0$.

Birth and death processes were introduced by Feller in 1939 with the aim of modeling the growth of biological populations. The wide variety of dynamic behavior exhibited by plants, insects and animals justifies the great interest of scientists in the development of mathematical models and the consequent intensive study of BDPs (see Ricciardi [68] for an accurate analysis of birth and death processes in the context of population dynamics). Furthermore, such processes arise as natural descriptors of time-varying phenomena in many other applied fields, such as queueing systems, epidemiology, epidemics, optics, neurophysiology, etc. An extensive survey text on birth and death processes has been published by Parthasarathy and Lenin [61]. In this work the authors adopt standard methods of analysis (such as power series technique and Laplace transforms) to find explicit expressions for the transient and stationary distributions of BDPs and provide applications of such results to specific fields (communication systems, chemical and biological models). In particular, in Section 9 they use BDPs to describe the time changes in the concentrations of the components of a chemical reaction and discuss the role of BDPs in the study of diatomic molecular chains. The paper by StockMayer et al. [73] gives an example of application of stochastic processes in the study of chain molecular diffusion. In this work a molecule is modeled as a freely-joined chain of two regularly alternating kinds of atoms.

All bonds have the same length but the two kinds of atoms have alternating jump rates, i.e. the forward and backward jump rates for even labeled beads are α and β, respectively, and these rates are reversed for odd labeled beads. By invoking the master equations for even and odd numbered bonds, the authors obtain the exact time-dependent average length of bond vectors.
Inspired by this work, Conolly [12] studies an infinitely long chain of atoms joined by links of equal length. The links are assumed to be subject to random shocks, that force the atoms to move and the molecule to diffuse. The shock mechanism is different according to whether the atom occupies an odd or an even position on the chain. The originating stochastic model is a randomized random walk on the integers with an unusual exponential pattern for the inter-step time intervals. The authors analyze some features of this process and investigate also its queue counterpart, where the walk is confined to the non negative integers. Some results concerning this queueing system with "chemical" rules (the so-called "chemical queue") have been obtained also by Tarabia and El-Baz (see [74] and [75]).

Stimulated by the above researches, a birth and death process $N(t)$ on the integers with a transition rate λ from even states and a possibly different rate μ from odd states has been studied in Chapter 2 of this thesis. A detailed description of the model is performed, the probability generating functions of even and odd states and the transition probabilities of the process are obtained for arbitrary initial state. Certain symmetry properties of the transition probabilities are also given. Hence, the birth and death process obtained by superimposing a reflecting boundary in the zero-state is analyzed. In particular, by making use of a Laplace transform approach, the probability of a transition from state 1 to the zero-state is obtained. Formulas for mean and variance of both processes are also provided. Furthermore, some preliminary results on the process under investigation are given in the case of zero initial state.

The second part of the thesis is dedicated to the study of the telegraph process. Some aspects of the telegraph process have been analyzed by many authors in several fields, such as engineering, mathematical finance, queueing and reliability theory etc. This process has been largely investigated in mathematical physics as a model of finite-velocity random motion with alternating velocities, whose probability density satisfies a hyperbolic partial differential equation (see Goldstein [39] and Kac [43], for instance). Various results involving absorption and first passage times problems are given in Foong *et al.* [34], Orsingher [55] and Orsingher [56]. Recently, the telegraph process has been considered also in mathematical finance to describe stochastic volatility and in actuarial problems, for obtaining a *telegraph analog* of the Black-Scholes model (see, for instance, Ratanov [65]). The author introduces a new class of financial market models based on generalized telegraph processes by alternating velocities and jumps occuring at switching velocities. A further model was studied in Di Crescenzo and Pellerey [30], where a geometric telegraph process was proposed to describe price evolution of risky assets of alternating type.

The telegraph process describes the motion of a particle on the real line, traveling at constant velocity, whose direction is reversed at random times according to the arrival epochs of a Poisson counting process $N(t)$, with rate λ, during $(0,t)$. The initial velocity is given by $V(0) = \pm c$ $(c > 0)$, with equal probability. We assume that at time $t = 0$ the particle is located at the origin of the real line and then moves in a positive or negative direction. The particle position at time t is given by

$$X(t) := V(0) \int_0^t (-1)^{N(s)} \mathrm{d}s,$$

where $N(0) = 0$ and $V(t) := V(0)\,(-1)^{N(t)}$ is the particle velocity at time t. The particle moves on the real line and its speed changes direction at any epoch. Notice that, in this simple model, the length of the time periods during which the particle is traveling in the positive or negative direction are described by independent and identically distributed (i.i.d.) exponential random variables. In particular, it is interesting to note that the probability density of the process $X(t)$, say $p(x,t)$, when the particle starts at $x_0 = 0$, at time $t_0 = 0$, is a solution of the *telegraph equation*

$$c^2 \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t}$$

(see Goldstein [39], for instance). This result is investigated in many papers and books, see for example Cane [10], Orsingher [52], and Orsingher [57]. Other results on the telegraph process are recalled in Chapter 3.

Stimulated by the above results a generalized telegraph process with underlying random walk has been studied in Chapter 4. This process is characterized by random times separating consecutive changes of direction of the moving particle having a general distribution and forming a non-regular alternating renewal process. Starting from the origin, the particle performs an alternating motion with velocities c and $-v$ $(c, v > 0)$. The direction of the motion (forward and backward) is determined by the velocity sign. The particle changes the direction according to the outcome of a Bernoulli trial. Hence, the novelty of this model is the inclusion of an underlying (possibly asymmetric) random walk governing the choice of the velocity at any epoch.

We determine the general form of probability law and the mean of the process, and then investigate two instances in which the random intertimes are exponentially distributed with (i) constant rates and with (ii) linearly increasing rates. In the first case explicit expressions of the transition density and of the conditional mean of the process are given as series of Gauss hypergeometric functions. In the second case, which leads to a damped random motion, we obtain the transition density in closed form and a logistic stationary density.

The thesis is organized in four chapters.

CHAPTER 1. In this chapter we recall some definitions and properties of birth and death processes.

CHAPTER 2. In this chapter we analyze two birth and death processes on the set of integers $\mathbb{Z}$ and on the set of non-negative integers $\{0, 1, 2, \ldots\}$ with a reflecting boundary in the zero-state. They are characterized by a transition rate λ from any even state and a different transition rate μ from any odd state. Explicit expressions of the probability densities and moments are obtained in both cases.

CHAPTER 3. This chapter contains some results and definition on the telegraph process.

CHAPTER 4. In this chapter, we present a new model of telegraph process with an underlying random walk. The aim is to determine the closed-form of probability density and moments in the case in which the random times are exponentially distributed with constant rates and with linearly increasing rates.

PART 1

Analysis of a birth and death process with alternating rates

CHAPTER 1

BACKGROUND ON STOCHASTIC PROCESSES

1.1 Introduction

In this chapter we present methods for studying stochastic processes, including the *forward* and *backward Kolmogorov equations* and generating function–based techniques. Applications of these methods show that the generating functions of probabilities of stochastic processes satisfy a partial differential equation. In some cases, the partial differential equation is linear and of first order, so that a closed-form solution to the generating function can be obtained. We consider a stochastic process $\{X(t), t \geq 0\}$ for a continuous-time Markov chain when the state space is (i) the set of nonnegative integers $\{0, 1, 2, \ldots\}$, or (ii) the set of integers $\mathbb{Z}$.

In Section 1.2, some definitions and notations on stochastic processes are given in order to introduce the continuous-time Markov chains. The Markovian property may be interpreted as stating that the conditional distribution of any future state $X(t_{j+1})$, given the past states $X(t_0), X(t_1), \ldots, X(t_{j-1})$ and the present state $X(t_j)$, is independent of the past states and dependes only on the present state, for all $j \geq 0$.

In Section 1.3, we introduce a class of continuous-time Markov chains known as *birth and death processes*. These processes are used to model populations whose size changes at any time by a single unit. It is well know in the literature that the probability distribution of a *simple birth process* is negative binomial, whereas it is binomial for a *simple death process*. For *birth and death processes* explicit formulas have been derived in the past using the generating function technique. First, we discuss the general birth and death processes with infinite state space $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, for which transition from any state n can only go to either state $n-1$ or state $n+1$. If the population size is n, then the birth and death rates will be denoted respectively as λ_n and μ_n, $n \geq 0$ (see Karlin and Taylor [45], [46]). A relevant application of birth and death processes is in queueing theory, where the state of the system is the number of customers in the queue. The arrival and departure processes of a queuing system are analogous to birth and death processes, respectively. A queueing process involves three components: *arrival process*, *queue discipline*, and *service mechanism*. The arrival process involves the arrival of customers for service and specifies the sequence of arrival times for the customers. The queue discipline is a rule specifying how custumers form a queue and how they behave while waiting. The service mechanism involves how customers are serviced and specifies the sequence of service time. When the arrival process is a Poisson process with parameter λ (mean arrival or birth rate) and the service time is exponentially distributed with parameter μ (mean departure or death rate), the process is a queueing system of type $M/M/1$. In this case, if λ and μ are constant, then the process is a birth and death process, as is described in Example 1.3.2 ($\lambda = p$ and $\mu = q$).

In Section 1.4, we introduce a *bilateral birth and death process* whose space state is the whole set of integers $\mathbb{Z}$. We give same examples of bilateral birth and death processes, whereas a new stochastic model will be discussed in Chapter 2.

Finally, in Section 1.5 a classification of states, some definitions and simple sufficient conditions for boundaries are exhibited. A boundary classification at infinity and a recurrent-transient classification are also performed in the case in which the birth and death process have state space $\mathbb{N}_0$ and $\mathbb{Z}$.

1.2 Definitions and notations

A stochastic process is a collection of random variables. More specifically, the following definition holds.

Definition 1.2.1. A *stochastic process* is a collection of random variables $\{X_t(s) : t \in T, s \in S\}$, where T is the index set and S is the *common sample space* of the random variables (finite or infinite). For each fixed $s \in S$, $X_t(s)$ (or $X(t)$) corresponds to a function defined on T, and is called *sample path* or *stochastic realization* of the process.

A stochastic model is based on a stochastic process in which specific relationships among the set of random variables $\{X(t)\}$ are assumed to hold. There are different methods and techniques for analyzing a stochastic process that depend on whether the random variables and index set are discrete or continuous. Generally, the set T represents the *time*.

In the following chapters, we discuss some continuous-time stochastic models having discrete state space. We now introduce continuous time Markov chain models. In particular, these stochastic processes are used to model many types of phenomena from variety of applied areas, including biology, physics, chemistry, finance, economics, and engineering.

Definition 1.2.2. A stochastic process $\{X(t);\ t \geq 0\}$ is a *continuous time Markov chain* if for any sequence of real numbers $0 \leq t_1 < t_2 < \cdots < t_j < t_{j+1}$, and nonnegative integers $k_1, k_2, \ldots, k_{j-1}, k, n$ such that

$$P\{X(t_j) = k, X(t_{j-1}) = k_{j-1}, \ldots, X(t_1) = k_1\} > 0,$$

one has:

$$\begin{aligned} P\{X(t_{j+1}) = n \,|\, X(t_j) = k, X(t_{j-1}) = k_{j-1}, \ldots, X(t_1) = k_1\} \\ = P\{X(t_{j+1}) = n \,|\, X(t_j) = k\}. \end{aligned}$$

The latter condition in Definition 1.2.2 is the *Markovian property.* The transition to state n at time t_{j+1} depends only on the value of the state at the most recent time t_j and does not depend on the history of the process. Each random variable $X(t)$ has an associated probability distribution $\{p_k(t)\}_{k \geq 0}$, where

$$p_k(t) = P\{X(t) = k\}.$$

A relation between the random variables $X(s)$ and $X(t)$, with $s < t$, is defined by the transition probabilities. Define the *transition probabilities* of the process at time t as the functions

$$p_{k,n}(t) = P\{X(t) = n \,|\, X(s) = k\}, \qquad t \geq 0,\ s < t$$

for $k, n = 0, 1, 2, \ldots$. If the transition probabilities do not depend on s or t but depend only on the lenght $t - s$ of the time interval (s, t), then the continuous time Markov chain is said to have *stationary* or *homogeneous* transition probabilities. Unless otherwise stated, we shall assume that the transition probabilities are stationary; that is

$$p_{k,n}(t - s) = P\{X(t) = n \,|\, X(s) = k\} = P\{X(t - s) = n \,|\, X(0) = k\}$$

for $s < t$. In general the transition probabilities satisfy the following properties:

- $p_{k,n}(t) \geq 0$ for all $k, n \in S$ and

$$\begin{aligned}\sum_{n\in S} p_{k,n}(t) &= \sum_{n\in S} P\{X(t) = n \,|\, X(0) = k\} \\ &= P\{X(t) \in S \,|\, X(0) = k\} = 1\end{aligned}$$

  for any $k \in S$;

- $p_{k,n}(0) = P\{X(0) = n \,|\, X(0) = k\} = \delta_{k,n}$, where $\delta_{k,n}$ is the Kronecker's delta;

- for any $s, t \geq 0$ and $k, n \in S$, by the Markovian property we have the following identity:

$$\begin{aligned}p_{k,n}(t+s) &= P\{X(t+s) = n \,|\, X(0) = k\} \\ &= \sum_{j=0}^{\infty} P\{X(t+s) = n, X(s) = j \,|\, X(0) = k\} \\ &= \sum_{j=0}^{\infty} P\{X(t+s) = n \,|\, X(s) = j, X(0) = k\} \\ &\times\ P\{X(s) = j \,|\, X(0) = k\} \\ &= \sum_{j=0}^{\infty} p_{k,j}(s)\, p_{j,n}(t),\end{aligned}$$

that is the *Chapman-Kolmogorov equation.*

1.3 General Birth and Death Process

Let $\{X(t); t \geq 0\}$ be a continuous time Markov chain with $X(t)$ the random variable for the total population size at time t. Let the initial population size belong to the set of nonnegative integers $\mathbb{N}_0$ and assume that the transition probabilities of a general birth and death process are:

$$p_{k,n}(t) = P\{X(t) = n | X(0) = k\}, \qquad t \geq 0. \tag{1.1}$$

The transition probability $p_{k,n}(t+\Delta t)$ can be expressed as follows, by applying the Chapman–Kolmogorov equations,

$$p_{k,n}(t+\Delta t) = \sum_{j=0}^{\infty} p_{k,j}(t)\, p_{j,n}(\Delta t).$$

The transition probability (1.1) satisfies the conditions

$$\lim_{\Delta t \to 0^+} \frac{d}{dt}\, p_{k,n}(\Delta t) = \begin{cases} \lambda_k & \text{if } n = k+1, \\ -(\lambda_k + \mu_k) & \text{if } n = k, \\ \mu_k & \text{if } n = k-1, \\ 0 & \text{otherwise,} \end{cases}$$

namely,

$$p_{k,n}(\Delta t) = \begin{cases} \lambda_k\, \Delta t + o(\Delta t) & \text{if } n = k+1, \\ 1 - (\lambda_k + \mu_k)\, \Delta t + o(\Delta t) & \text{if } n = k, \\ \mu_k\, \Delta t + o(\Delta t) & \text{if } n = k-1, \\ o(\Delta t) & \text{otherwise.} \end{cases} \tag{1.2}$$

for Δt sufficiently small. We recall that $o(\Delta t)$ represents a function $\phi(\Delta t)$ such that $\phi(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$. The above processes are characterized by a birth rate $\lambda_n \geq 0$ and a possibly different death rate $\mu_n \geq 0$, for $n \geq 0$, and $\mu_0 = 0$. The quantities λ_n, μ_n determine respectively the rate of transition from state n to state $n+1$, if a birth occurs, and to state $n-1$ if a death occurs (where n is the population size). In the case in which $\lambda_0 > 0$ and $\mu_0 = 0$ we assume a reflecting state at state zero. Such processes are defined as *basic* (Callaert and Keilson [9]).

The forward Kolmogorov differential equations for $p_{k,n}(t)$ can be derived directly from the assumptions in (1.2). Assume Δt sufficiently small and consider the transition probability $p_{k,n}(t+\Delta t)$. This transition probability can be expressed in terms of the transition probabilities at time t as follows:

$$\begin{aligned} p_{k,n}(t+\Delta t) &= p_{k,n-1}(t)\,[\lambda_{n-1}\Delta t + o(\Delta t)] + p_{k,n+1}(t)\,[\mu_{n+1}\Delta t + o(\Delta t)] \\ &\quad + p_{k,n}(t)\,[1-(\lambda_n+\mu_n)\Delta t + o(\Delta t)] \;+ \sum_{m\neq -1,0,1}^{\infty} p_{k,n+m}(t)o(\Delta t) \\ &= p_{k,n-1}(t)\lambda_{n-1}\Delta t + p_{k,n+1}(t)\mu_{n+1}\Delta t \\ &\quad + p_{k,n}(t)\,[1-(\lambda_n+\mu_n)\Delta t] + o(\Delta t), \end{aligned} \tag{1.3}$$

which holds for all k, n in the state space. If $n = 0$, then

$$p_{k,0}(t+\Delta t) \;=\; p_{k,1}(t)\mu_1\Delta t + p_{k,0}(t)\mu_1\Delta t\,[1-\lambda_0\Delta t] + o(\Delta t). \tag{1.4}$$

Hence, subtracting $p_{k,n}(t)$ and $p_{k,0}(t)$ from Eqs. (1.3) and (1.4), respectively, dividing by Δt and taking the limit as $\Delta t \to 0$, the Kolmogorov differential equations are obtained for the general birth and death process,

$$\begin{cases} \dfrac{d}{dt}\,p_{k,n}(t) = \lambda_{n-1}\,p_{k,n-1}(t) - (\lambda_n+\mu_n)\,p_{k,n}(t) + \mu_{n+1}\,p_{k,n+1}(t) \\[2ex] \dfrac{d}{dt}\,p_{k,0}(t) = -\lambda_0\,p_{k,0}(t) + \mu_1\,p_{k,1}(t) \end{cases} \tag{1.5}$$

with initial condition

$$\lim_{t\to 0^+} p_{k,n}(t) = \delta_{k,n}, \tag{1.6}$$

where $\delta_{k,n}$ is the Kronecker's delta.

1.3.1 Matrix form

The system of equations (1.5) can be expressed also in matrix form. The transition probabilities $p_{k,n}(t)$ satisfy the following differential equation

$$dP(t)/dt = QP(t), \tag{1.7}$$

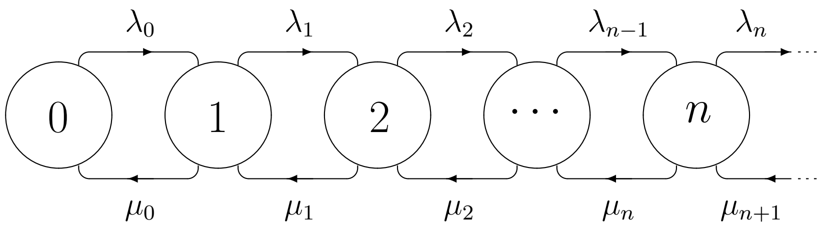


Figure 1.1: The Markov chain of a general birth and death process when $\lambda_0 > 0$ and $\lambda_n + \mu_n > 0$ for $n = 1, 2, \ldots$

where $P(t) = \{p_{k,n}(t)\}$ is the matrix of transition probabilities and $Q = \{q_{k,n}\}$ is the generator matrix. Matrix Q containes information based on the birth and death rates λ_n, μ_n of the process. In particular, we assume that the transition probabilities $p_{k,n}(t)$ are continuous and differentiable for $t \geq 0$ and for $t = 0$, they satisfy

$$p_{k,n}(0) = 0, \; k \neq n, \quad p_{k,k}(0) = 1.$$

Hence, we define the rates $q_{k,n}$ as following:

$$q_{k,n} = \lim_{\Delta t \to 0^+} \frac{p_{k,n}(t) - p_{k,n}(0)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{p_{k,n}(t)}{\Delta t}, \quad k \neq n.$$

When the state space is infinite the generator matrix Q has the following form:

$$Q = \begin{pmatrix} -\lambda_0 & \mu_1 & 0 & 0 & \ldots & \\ \lambda_0 & -\lambda_1 - \mu_1 & \mu_2 & 0 & \ldots & \\ 0 & \lambda_1 & -\lambda_2 - \mu_2 & \mu_3 & 0 & \ldots \\ 0 & 0 & \lambda_2 & \lambda_3 - \mu_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}.$$

If the initial distribution $X(0)$ is a fixed value, the state probabilities $p(t) = (p_{k,0}(t), p_{k,1}(t), \ldots)^T$ satisfy the forward Kolmogorov differential equations

$$dp(t)/dt = Qp(t). \tag{1.8}$$

These differential equations can be derived in the same manner as in Eqs. (1.3), (1.4) and (1.5). Denoting by $N = \{\nu_{k,n}\}$ the transition matrix, from the generator matrix Q, we have:

$$N = \begin{pmatrix} 0 & \mu_1/(\lambda_1+\mu_1) & 0 & 0 & \dots \\ 1 & 0 & \mu_2/(\lambda_2+\mu_2) & 0 & \dots \\ 0 & \lambda_1/(\lambda_1+\mu_1) & 0 & \mu_3/(\lambda_3+\mu_3) & \dots \\ 0 & 0 & \lambda_2/(\lambda_2+\mu_2) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

It is assumed that $\lambda_n + \mu_n > 0$ for $n = 0, 1, 2, \ldots$. If, for any n, $\lambda_n + \mu_n = 0$, then the state n is absorbing.

The embedded Markov chain can be thought of as a generalized random walk model with a reflecting boundary in zero. The probability of moving right (due to a birth) is $\nu_{n+1,n} = \lambda_n/(\lambda_n + \mu_n)$ and the probability of moving left (due to a death) is $\nu_{n-1,n} = \mu_n/(\lambda_n + \mu_n)$. See the graph in Figure 1.1. It is possible to verify from the graph that the chain is *irreducible*[1] if and only if $\lambda_n > 0$ and $\mu_{n+1} > 0$ for $n = 1, 2, \ldots$. If any $\lambda_n = 0$, then $\nu^{(m)}_{n+1,n} = 0$ for all m, and if any $\mu_n = 0$, then $\nu^{(m)}_{n-1,n} = 0$ for all m.

The Kolmogorov differential equations can be used to define a stationary probability distribution. A constant solution to system (1.5) is a stationary probability distribution. A formal definition is given next.

Definition 1.3.1. Let $\{X(t); t \geq 0\}$ be a continuous-time Markov chain with generator matrix Q. Suppose $\pi = (\pi_0, \pi_1, \ldots)^T$ is nonnegative and satisfies

$$Q\pi = 0 \qquad \sum_{i=0}^{\infty} \pi_i = 1$$

[1]A Markov chain is *irreducible* if there is only one class, i.e. if all states communicate with each other. In particular, two states k and n accessible to each other are said to *communicate*, and we write $k \leftrightarrow n$.

for $n = 0, 1, 2, \ldots$. Then π is called a *stationary probability distribution* of the continuous-time Markov chain.

For birth and death processes, there is an iterative procedure for computing the stationary probability distribution when the state space is finite or infinite. In this context it is possible to define the potential coefficients of the process $X(t)$ as follows:

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} \qquad (n = 1, 2, \ldots). \tag{1.9}$$

In particular, if the state space is infinite $\{0, 1, 2, \ldots\}$, an unique positive stationary probability distribution π exists if and only if $\mu_n > 0$ and $\lambda_{n-1} > 0$ for $n = 1, 2, \ldots$ and

$$\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} < \infty.$$

For *ergodic*[2] processes the coefficients π_n are summable and then the ergodic probabilities are given by

$$p_n = \lim_{t\to\infty} p_{k,n}(t) = \pi_n \left(\sum_{i=0}^{\infty} \pi_i\right)^{-1} \qquad (n = 0, 1, \ldots).$$

Example 1.3.2. Consider a continuous-time birth and death Markov process such that $\lambda_n = p > 0$, for $n \geq 0$ and $\mu_n = q > 0$, for $n \geq 1$ where $p + q = 1$. The embedded Markov chain is a semi-infinite random walk model with reflecting boundary conditions at zero (see Figure 1.1). We have $p = \nu_{n+1,n}$ and $q = \nu_{n-1,n}$. The chain has a unique stationary probability distribution if and only if

$$\sum_{n=1}^{\infty} \left(\frac{p}{q}\right)^n < \infty$$

[2]When the continuous-time Markov chain is irreducible and $p_n = \lim_{t\to\infty} p_{k,n}(t) > 0$ for all n, we say that the chain is *ergodic*.

with $p < q$. The stationary probability distribution is

$$\pi_n = \left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^n \qquad (n = 0, 1, \ldots).$$

In queueing theory, if $p = \lambda$ and $q = \mu$ are costant, the ratio λ/μ is referred to as the traffic intensity, with $\lambda < \mu$. If $\lambda \geq \mu$, then the queue length will tend to infinity. The mean of the stationary probability distribution represents the average number of customers C in the system (at equilibrium),

$$C = \sum_{n=1}^{\infty} n\pi_n = \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n.$$

This summation can be simplified by applying an analytic identity[3]:

$$C = \frac{\lambda/\mu}{1 - \lambda/\mu} = \frac{\lambda}{\mu - \lambda}.$$

The average amount of time W a customer spends in the system (at equilibrium) is the average number of customers divided by the average arrival rate λ:

$$W = \frac{C}{\lambda} = \frac{1}{\mu - \lambda}.$$

1.4 Bilateral Birth and Death processes

In this section we review some results on bilateral birth and death processes. Let $\{X(t); t \geq 0\}$ be a birth and death process whose state-space is the set of integers $\mathbb{Z}$. Assume that $X(t)$ has birth and death rates λ_n, μ_n for all $n \in \mathbb{Z}$, i.e.

$$\lambda_n = \lim_{h \to 0} \frac{1}{h} P\{X(t+h) = n+1 \,|\, X(t) = n\},$$

[3]The summation satisfies

$$(1-x)^2 = \sum_{i=1}^{\infty} i\; x^{i-1}.$$

$$\mu_n = \lim_{h\to 0} \frac{1}{h} P\{X(t+h) = n-1 \,|\, X(t) = n\}.$$

For every $n \in \mathbb{Z}$, the parameters λ_n, μ_n determine respectively the transition rate from state n to state $n+1$, if a birth occurs, and to state $n-1$, if a death occurs. We assume also that the rates λ_n, μ_n are positive, so that the birth and death process has no absorbing or reflecting states. This type of process is called *bilateral birth and death process.* We assume that $X(t)$ is *simple* (it means that the boundary ∞ and $-\infty$ are both non-regular), so that the set of rates uniquely determines the birth and death process. In particular, according to Callaert e Keilson [9], a bilateral birth and death process is simple if and only if the two component birth and death processes, obtained by setting at $n = 0$ a reflecting boundary in both directions, are simple. As $X(t)$ is simple, the transition probabilities $p_{k,n}(t)$ are the unique solution of the following system of equations:

$$\frac{d}{dt}\, p_{k,n}(t) = \lambda_{n-1}\, p_{k,n-1}(t) - \big(\lambda_n + \mu_n\big)\, p_{k,n}(t) + \mu_{n+1}\, p_{k,n+1}(t) \qquad (n \in \mathbb{Z}), \tag{1.10}$$

with initial condition (1.6).

We remark that a bilateral birth and death process has been introduced by Pruitt [64] as a continuous parameter Markov process with path function $X(t)$ assuming integer values and with stationary probabilities (1.1) satisfying the relations (1.2). A closed form for the Laplace transform of the general solution of (1.10) has been obtained by Pruitt [64] in terms of orthogonal polynomials. Some results on the associated system of orthogonal polynomials and their limit functions are given in order to have a careful examination of the convergence of the sequences of polynomials.

A simplest, possible unit step, linear, unrestricted random-walk in which the time intervals between steps are negative exponentially distributed has been analyzed in Conolly [11]. The position of the particle at time t is governed by two independent Poisson streams with parameters

$$\lambda_n = \lambda, \qquad \mu_n = \mu, \qquad n \in \mathbb{Z} \tag{1.11}$$

for motion to the right and left, respectively. A crucial role is played by the probability density in the queueing context. Removing the barrier at the origin and allowing departures even when the system state is zero or negative, a time-dependent random walk with interstep intervals having probability density function proporzional to $\mathrm{e}^{-(\lambda+\mu)t}$ has been obtained. This process identifies with the so-called *randomized random walk.*

An example of randomized random walk has been given in Baccelli and Massey [4], in which the closed form solutions for the transient distribution of a queue lenght and for the busy period of a $M/M/1$ queue have been provided in terms of modified Bessel functions. The authors analyse a $M/M/1$ queue length process, using an analytical approach based on Laplace transform. Hence, let $\{Z(t); t \geq 0\}$ be a process defined by setting:

$$Z(t) = Z(0) + N_\lambda(t) - N_\mu(t),$$

where $N_\lambda(t)$ and $N_\mu(t)$ are two independent Poisson processes with rate λ and μ respectively, and $Z(0) = m$. The transition probability of such process, for all integers $m, n \in \mathbb{Z}$ is

$$P_m\left\{Z(t) = n \,|\, Z(0) = m\right\} = e^{-(\lambda+\mu)t} \left(\frac{\lambda}{\mu}\right)^{(n-m)/2} I_{n-m}(2t\sqrt{\lambda\,\mu}),$$

where $I_n(\cdot)$ is the modified Bessel function.

An extention of this model has also been given in Di Crescenzo [15] in which the author shows how to construct a new birth and death process $\tilde{X}(t)$ having state space $\mathbb{Z}$, whose rates are obtained from those of $X(t)$ such that the transition probabilities of the two process are mutually related by a product-form relation. The trasformation from $X(t)$ to $\tilde{X}(t)$ can be viewed as a method to construct new stochastic models.

The case treated in [15] has been extended to another example of bilateral birth and death process with sigmoidal-type rates in Di Crescenzo and Martinucci [26]. The authors discuss the bimodality behaviour and symmetry properties of the transition probabilities when birth and death rates are respectively

$$\lambda_n = \lambda \frac{1 + c\left(\frac{\mu}{\lambda}\right)^{n+1}}{1 + c\left(\frac{\mu}{\lambda}\right)^{n}}, \qquad \mu_n = \mu \frac{1 + c\left(\frac{\mu}{\lambda}\right)^{n-1}}{1 + c\left(\frac{\mu}{\lambda}\right)^{n}}, \qquad n \in \mathbb{Z} \tag{1.12}$$

with $\lambda, \mu > 0$ and $c \geq 0$. In particular, thanks to certain symmetry properties they obtain the avoiding transition probabilities in the presence of a pair of absorbing boundaries, expressed as a series. Note that when $c = 0$ and $c \to \infty$ the rates in (1.12) became constant in n, and then in both cases the process identifies with the randomized random walk with birth and death rates given respectively by λ and μ (when $c = 0$) or μ and λ (when $c \to \infty$). In addition, the rates in (1.12) are equal and constant in n when $\lambda = \mu$.

In Chapter 2 we study a new bilateral birth and death process $N(t)$ characterized by a transition rate λ from any even state to the two neighboring states, and by a transition rate μ from any odd state to the neighboring states. Denoting by

$$\nu_{j,n} = \lim_{h \to 0} \frac{1}{h} P\{N(t+h) = n \,|\, N(t) = j\}$$

the transition rates of $N(t)$ from state j to state n, we assume that the allowed transitions are characterized by the following rates:

$$\nu_{2n,2n\pm1} = \lambda, \qquad \nu_{2n\pm1,2n} = \mu, \qquad \forall n \in \mathbb{Z}, \tag{1.13}$$

with $\lambda, \mu > 0$. Note that in the case when $\lambda = \mu$ the process $N(t)$ identifies with the randomized random walk on the integers with exponentially distributed intertimes (see Conolly [11]). We purpose to determine the transition probabilities of $N(t)$ for arbitrary initial state.

1.5 Classification of states

A boundary classification for birth and death processes based on properties of a natural scale and a canonical measure associated with these processes is given in Feller [33]. Feller's conditions have been reformulated subsequently in a more suitable form by Callaert and Keilson [9]. Hence, the following conditions according to Feller's boundary classification scheme have been defined:

$$A \Leftrightarrow \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} < \infty, \qquad B \Leftrightarrow \sum_{n=0}^{\infty} \pi_n < \infty,$$

$$C \Leftrightarrow \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=1}^{n} \pi_i < \infty, \qquad D \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{\mu_n \pi_n} \sum_{i=n}^{\infty} \pi_i < \infty.$$

Feller's boundary classification is given in the second column of the Table 1.1 in terms of conditions A, B, C and D, while the third column containes equivalent conditions proposed by Keilson and Callaert ($\overline{A}$ denotes negation of A). In particular, we observe from the third column that every birth and death process has a boundary at infinity of one of the four listed types. A reflecting or absorbing character is necessary to complete the characterization of the process. This may be achieved for example by treating the process as the limit of a sequence of birth and death processes with reflecting or absorbing boundaries.

A birth and death process may also be classified in another way. If the process $X(t)$ leaves state n, one may ask (α) whether it returns to state n with probability one and (β) whether the mean time to reach state n is finite.

Definition 1.5.1. A process is *transient* if the return to any state is not a

Classification of boundary ∞	Feller's conditions	Callaert and Keilson conditions
Regular	A, B	C, D
Exit	$A, \overline{B}, C$	$C, \overline{D}$
Entrance	$\overline{A}, B, D$	$\overline{C}, D$
Natural	$\begin{cases} \overline{A}, \overline{B} \\ \text{o } A, \overline{B}, \overline{C} \\ \text{o } \overline{A}, B, \overline{D} \end{cases}$	$\overline{C}, \overline{D}$

Table 1.1: Feller's conditions and Callaert and Keilson conditions.

certain event; it is *null-recurrent* if the return is certain in any state and the mean return is infinity; it is *positive recurrent* if the return in any state is certainly with finite mean return time.

If the process is non-regular, then necessary and sufficient conditions are given by (Karlin and McGregor [44]):

- $X(t)$ transient ($\overline{\alpha}$) $\quad\Leftrightarrow\quad A, \overline{B}$;
- $X(t)$ null-recurrent (α e $\overline{\beta}$) $\quad\Leftrightarrow\quad \overline{A}, \overline{B}$;
- $X(t)$ positive-recurrent (α e β) $\quad\Leftrightarrow\quad \overline{A}, B$.

The process has a regular boundary if both conditions A and B are satisfied. For this class of processes, denoted by $\mathcal{C}_1$, the recurrent-transient classification of such regular processes depends on the conditions imposed on the

	transient	null–recurrent	positive–recurrent
Natural	$\mathcal{C}_2$	$\mathcal{C}_4$	$\mathcal{C}_5$
Exit	$\mathcal{C}_3$	$\emptyset$	$\emptyset$
Entrance	$\emptyset$	$\emptyset$	$\mathcal{C}_6$

Table 1.2: Classification of states of a birth and death process.

boundary at infinity. If this boundary is a reflecting one, the answer to (α) and (β) above will be affirmative so that the process will be positive-recurrent. If the boundary at ∞ is absorbing (the process stops when it reaches infinity), the process will be transient.

Note that non-regular birth and death processes are uniquely determined by the initial condition and the transitions rates λ_n and μ_n. For these processes the boundary classification at infinity and the recurrent-transient classification are not independent of each other, in the sense that a combination of two properties (one property of each classification system) can be contradictory. The other classes for birth and death processes are denoted by $\mathcal{C}_i$, $2 \leq i \leq 6$, and listed in Table 1.2. The $\emptyset$ sign means that the class is empty. It is interesting to observe that for any birth and death process the implication

$$\liminf_{n\to\infty} (\lambda_n + \mu_n) < \infty \quad \Rightarrow \quad \textit{natural}$$

is true. It follows that if the boundary infinity is non–natural, then

$$\lim_{n\to\infty} \lambda_n = \infty \qquad \text{or} \qquad \lim_{n\to\infty} \mu_n = \infty.$$

More precisely, one has:

- entrance $\Rightarrow \lim_{n\to\infty}\mu_n = \infty$;
- exit $\Rightarrow \lim_{n\to\infty}\lambda_n = \infty$;
- regular $\Rightarrow \lim_{n\to\infty}\lambda_n = \infty$ and $\lim_{n\to\infty}\mu_n = \infty$.

We now establish some simple and sufficient conditions for the classification of bith and death processes. By setting $\varrho_n = \lambda_n/\mu_n$ and $\varrho_n^* = \lambda_{n-1}/\mu_n$ for $n = 1, 2, \ldots$, one has:

- $\liminf_{n\to\infty}\varrho_n > 1, \quad \limsup_{n\to\infty}\varrho_n^* < 1 \quad \Rightarrow \quad \mathcal{C}_1$;
- $\liminf_{n\to\infty}\varrho_n > 1, \quad |\{n : \varrho_n^* \geq 1\}| = \infty, \quad \sum_{n=0}^{\infty}\frac{1}{\lambda_n} = \infty \quad \Rightarrow \quad \mathcal{C}_2$;
- $\liminf_{n\to\infty}\varrho_n^* > 1, \quad \sum_{n=0}^{\infty}\frac{1}{\lambda_n} < \infty \quad \Rightarrow \quad \mathcal{C}_3$;
- $|\{n : \varrho_n \leq 1\}| = \infty, \quad |\{n : \varrho_n^* \geq 1\}| = \infty \quad \Rightarrow \quad \mathcal{C}_4$;
- $|\{n : \varrho_n \leq 1\}| = \infty, \quad \limsup_{n\to\infty}\varrho_n^* < 1, \quad \sum_{n=1}^{\infty}\frac{1}{\mu_n} = \infty \quad \Rightarrow \quad \mathcal{C}_5$;
- $\limsup_{n\to\infty}\varrho_n < 1, \quad \sum_{n=1}^{\infty}\frac{1}{\mu_n} < \infty \quad \Rightarrow \quad \mathcal{C}_6$.

It is possible to classify also the boundary ∞ and $-\infty$ of bilateral birth and death processes proceeding in a similar way to the case of basic processes. Recalling (1.9) the following sequence of positive constants π_n are also given:

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} \quad \pi_{-n} = \frac{\mu_0\mu_{-1}\cdots\mu_{-n+1}}{\lambda_{-1}\lambda_{-2}\cdots\lambda_{-n}} \qquad (n = 1, 2, \ldots). \tag{1.14}$$

Let us introduce the series:

$$S_1^+ = \sum_{n=1}^{\infty}\frac{1}{\lambda_n\pi_n}\sum_{i=1}^{n}\pi_i\,, \qquad S_2^+ = \sum_{n=1}^{\infty}\pi_n\sum_{i=1}^{n-1}\frac{1}{\lambda_i\pi_i}\,,$$

$$S_1^- = \sum_{n=-\infty}^{-1}\frac{1}{\lambda_n\pi_n}\sum_{i=n+1}^{-1}\pi_i\,, \qquad S_2^- = \sum_{n=-\infty}^{-1}\pi_n\sum_{i=n}^{-1}\frac{1}{\lambda_i\pi_i}\,.$$

Classification	boundary ∞	boundary $-\infty$
Regular	$S_1^+ < \infty, S_2^+ < \infty$	$S_1^- < \infty, S_2^- < \infty$
Exit	$S_1^+ < \infty, S_2^+ = \infty$	$S_1^- < \infty, S_2^- = \infty$
Entrance	$S_1^+ = \infty, S_2^+ < \infty$	$S_1^- = \infty, S_2^- < \infty$
Natural	$S_1^+ = \infty, S_2^+ = \infty$	$S_1^- = \infty, S_2^- = \infty$

Table 1.3: Classification of states of a bilateral birth and death process.

Table 1.3 shows the conditions that classify the boundary ∞ and $-\infty$ for bilateral birth and death processes.

CHAPTER 2

BIRTH AND DEATH PROCESS WITH ALTERNATING RATES

2.1 Introduction

Birth and death processes were introduced to describe random growth (see, for instance, Ricciardi [68] for an accurate description of birth and death processes in the context of population dynamics). Furthermore, they arise as natural descriptors of time-varying phenomena in several applied fields such as queueing, epidemiology, epidemics, optics, neurophysiology, etc. An extensive survey has been provided in Parthasarathy and Lenin [61]. In particular, in Section 9 of such paper certain birth and death processes are used to describe the time changes in the concentrations of the components of a chemical reaction, and their role in the study of diatomic molecular chains is emphasized.

Moreover, Stockmayer *et al.* [73] gave an example of application of stochastic processes in the study of chain molecular diffusion, by modeling a molecule as a freely-joined chain of two regularly alternating kinds of atoms. The two kinds of atoms have alternating jump rates, and these rates are reversed for odd labeled beads. By invoking the master equations for even and odd numbered bonds, the authors obtained the exact time-dependent average length of bond vectors.

Inspired by this work, Conolly *et al.* [12] studied an infinitely long chain of atoms joined by links of equal length. The links are assumed to be subject to random shocks, that force the atoms to move and the molecule to diffuse. The shock mechanism is different according to whether the atom occupies an odd or an even position on the chain. The originating stochastic model is a randomized random walk on the integers with an unusual exponential pattern for the inter-step time intervals. The authors analyze some features of this process and investigate also its queue counterpart, where the walk is confined to the non negative integers. Various results concerning such queueing system with "chemical" rules (the so-called "chemical queue") were obtained also by Tarabia and El-Baz [74], [75] and more recently by Tarabia *et al.* [76].

Another example arising in a chemical context where the role of parity is crucial is provided in Lente [49], where the probability of a more stable enantiomer is different according on whether the number of chiral molecules is even or odd.

Stimulated by the above investigations, in this chapter we consider a birth and death process $N(t)$ on the integers with a transition rate λ from even states and a possibly different rate μ from odd states. This model arises by suitably modifying the death rates of the process considered in the above papers. A detailed description of the model is performed in Section 2.2, where the probability generating functions of even and odd states and the

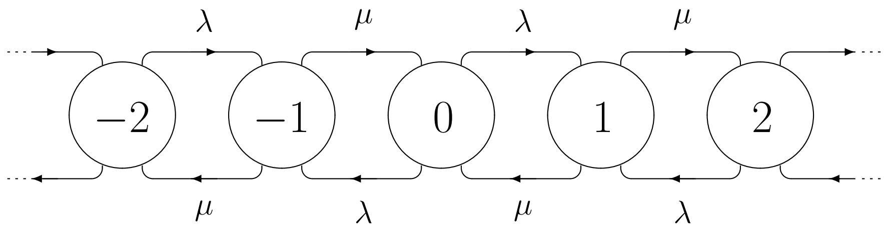


Figure 2.1: Transition rate diagram of $N(t)$.

transition probabilities of the process are obtained for arbitrary initial state. Certain symmetry properties of the transition probabilities are also given. In Section 2.3, we study the birth and death process obtained by superimposing a reflecting boundary in the zero-state. In particular, by making use of a Laplace transform-based approach, we obtain the probability of a transition from state 1 to the zero-state. Formulas for mean and variance of both processes are also provided. We remark that some preliminary results on the process under investigation are given in Iuliano and Martinucci [42] for the case of zero initial state.

It should be mentioned that closed-form results on bilateral birth and death processes have been obtained in the past only in few solvable cases, such as those in the above mentioned papers, and those given in Di Crescenzo [15], Di Crescenzo and Martinucci [26], Pollett [63]. Moreover some results on birth and death processes with alternating rates that will be given in the following sections have been presented in Di Crescenzo *et al.* [21].

2.2 TRANSIENT PROBABILITY DISTRIBUTION

We consider a birth and death process $\{N(t);\, t \geq 0\}$ with state-space $\mathbb{Z}$, and denote by

$$p_{k,n}(t) = P\{N(t) = n \,|\, N(0) = k\}, \quad t \geq 0, \quad n \in \mathbb{Z}$$

its transition probabilities, where $k \in \mathbb{Z}$ is the initial state. We assume that $N(t)$ is characterized by a transition rate λ from any even state to the two neighboring states, and by a possibly different transition rate μ from any odd state to the neighboring states. In other terms, denoting by

$$\nu_{j,n} = \lim_{h\to 0} \frac{1}{h} P\{N(t+h) = n \,|\, N(t) = j\}$$

the time-homogeneous transition rates of $N(t)$ from state j to state n, we assume that the allowed transitions are characterized by the following rates:

$$\nu_{2n,2n\pm 1} = \lambda, \qquad \nu_{2n\pm 1,2n} = \mu, \qquad \forall n \in \mathbb{Z}, \tag{2.1}$$

with $\lambda, \mu > 0$. The associated transition rate diagram of this process is given in Figure 2.1. We note that rates (2.1) are different from those of the birth and death model considered in Conolly *et al.* [12] and Tarabia *et al.* [76], where $\nu_{2n,2n+1} = \nu_{2n+1,2n} = \lambda$ and $\nu_{2n-1,2n} = \nu_{2n,2n-1} = \mu$ for any $n \in \mathbb{Z}$.

Due to assumptions (2.1), the transition probabilities of $N(t)$ satisfy the following system of Kolmogorov differential-difference equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} p_{k,2n}(t) = \mu\, p_{k,2n-1}(t) - 2\lambda\, p_{k,2n}(t) + \mu\, p_{k,2n+1}(t), \\ \\ \dfrac{\mathrm{d}}{\mathrm{d}t} p_{k,2n+1}(t) = \lambda\, p_{k,2n}(t) - 2\mu\, p_{k,2n+1}(t) + \lambda\, p_{k,2n+2}(t), \end{cases} \tag{2.2}$$

for any $t \geq 0$, $n \in \mathbb{Z}$ and for any initial state $k \in \mathbb{Z}$. The initial condition is expressed by:

$$p_{k,n}(0) = \delta_{n,k}, \tag{2.3}$$

where $\delta_{n,k}$ is the Kronecker's delta. We notice that in the special case when $\lambda = \mu$ process $N(t)$ identifies with the so-called "randomized random walk" (see, for instance, Conolly [11]).

In order to obtain the state probabilities of $N(t)$, hereafter we develop a probability generating function-based approach. We recall that this method has been used in the past to determine probabilities of interest in several stochastic models (see, for instance, Giorno and Nobile [36] and Ricciardi and Sato [69] for the distribution of the range of one-dimensional random walks). Let us define the probability generating functions of the sets of even and odd states of $N(t)$, respectively:

$$F_k(z,t) := \sum_{j=-\infty}^{+\infty} z^{2j} p_{k,2j}(t), \qquad G_k(z,t) := \sum_{j=-\infty}^{+\infty} z^{2j+1} p_{k,2j+1}(t), \tag{2.4}$$

with $z \in \mathbb{Z}$. Note that, due to (2.3), the following initial conditions hold:

$$F_k(z,0) = \begin{cases} z^k & k \text{ even} \\ 0 & k \text{ odd}, \end{cases} \qquad G_k(z,0) = \begin{cases} 0 & k \text{ even} \\ z^k & k \text{ odd}. \end{cases} \tag{2.5}$$

From system (4.22) we have that the generating functions (2.5) satisfy the following differential system:

$$\begin{cases} \dfrac{\partial}{\partial t} F_k(z,t) = \mu\, z G_k(z,t) - 2\lambda\, F_k(z,t) + \dfrac{\mu}{z}\, G_k(z,t), \\[2ex] \dfrac{\partial}{\partial t} G_k(z,t) = \lambda\, z F_k(z,t) - 2\mu\, G_k(z,t) + \dfrac{\lambda}{z}\, F_k(z,t), \end{cases}$$

so that

$$\frac{\partial}{\partial t} \begin{pmatrix} F_k(z,t) \\ G_k(z,t) \end{pmatrix} = A \cdot \begin{pmatrix} F_k(z,t) \\ G_k(z,t) \end{pmatrix}, \qquad A := \begin{pmatrix} -2\lambda & \mu \dfrac{z^2+1}{z} \\ \lambda \dfrac{z^2+1}{z} & -2\mu \end{pmatrix}.$$

Hence, by use of standard methods, due to conditions (2.5) we obtain

$$\begin{pmatrix} F_{2k}(z,t) \\ G_{2k}(z,t) \end{pmatrix} = \mathrm{e}^{At} \cdot \begin{pmatrix} z^{2k} \\ 0 \end{pmatrix}, \tag{2.6}$$

and

$$\begin{pmatrix} F_{2k+1}(z,t) \\ G_{2k+1}(z,t) \end{pmatrix} = \mathrm{e}^{At} \cdot \begin{pmatrix} 0 \\ z^{2k+1} \end{pmatrix}, \tag{2.7}$$

where

$$e^{At} = \exp\left\{\begin{pmatrix} -2\lambda & \mu\dfrac{z^2+1}{z} \\ \lambda\dfrac{z^2+1}{z} & -2\mu \end{pmatrix} t\right\}.$$

To determine the eigenvalues of matrix A, we consider the following equation

$$\det(A - vI) \equiv (2\lambda + v)(2\mu + v) - \lambda\mu\frac{(z^2+1)^2}{z^2} = 0,$$

whose roots are given by

$$v_1, v_2 = -(\lambda + \mu) \pm \frac{1}{z}h(z), \qquad v_1 < v_2,$$

with

$$h(z) = \sqrt{(\mu z^2 + \lambda)(\lambda z^2 + \mu)}.$$

In the sequel we shall denote by

$$V \equiv \begin{pmatrix} -(\lambda+\mu) - h(z)/z & 0 \\ 0 & -(\lambda+\mu) + h(z)/z \end{pmatrix} \tag{2.8}$$

the matrix eigenvalues of A, and by

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

the corresponding matrix eigenvectors. By setting

$$S_{21} = S_{22} = \lambda\frac{(z^2+1)}{z},$$

the following system holds:

$$\begin{cases} -2\lambda S_{11} + \mu\lambda\dfrac{(z^2+1)^2}{z^2} = S_{11}(-\lambda - \mu - h(z)/z), \\ \\ -2\lambda S_{12} + \mu\lambda\dfrac{(z^2+1)^2}{z^2} = S_{12}(-\lambda - \mu + h(z)/z). \end{cases} \tag{2.9}$$

By straightforward calculations we have $A = S \cdot V \cdot S^{-1}$, where

$$S = \begin{pmatrix} \mu - \lambda - \dfrac{h(z)}{z} & \mu - \lambda + \dfrac{h(z)}{z} \\ \\ \lambda \dfrac{z^2+1}{z} & \lambda \dfrac{z^2+1}{z} \end{pmatrix}, \qquad V = \begin{pmatrix} v_1 & 0 \\ \\ 0 & v_2 \end{pmatrix}. \tag{2.10}$$

$$S^{-1} = -\frac{z}{2\lambda(z^2+1)h(z)} \begin{pmatrix} \lambda(z^2+1) & z(\lambda-\mu) - h(z) \\ -\lambda(z^2+1) & z(\mu-\lambda) - h(z) \end{pmatrix}. \tag{2.11}$$

If the initial state is even ($2k$), Eqs. (2.6) and (2.10)–(2.11) give

$$\mathrm{e}^{At} \cdot \begin{pmatrix} z^{2k} \\ \\ 0 \end{pmatrix} = \begin{pmatrix} \mu - \lambda - \dfrac{h(z)}{z} & \mu - \lambda + \dfrac{h(z)}{z} \\ \\ \lambda \dfrac{(z^2+1)}{z} & \lambda \dfrac{(z^2+1)}{z} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{v_1 t} & 0 \\ \\ 0 & \mathrm{e}^{v_2 t} \end{pmatrix} \begin{pmatrix} -\dfrac{z^{2k+1}}{2h(z)} \\ \\ \dfrac{z^{2k+1}}{2h(z)} \end{pmatrix},$$

and then

$$\mathrm{e}^{At} \cdot \begin{pmatrix} z^{2k} \\ \\ 0 \end{pmatrix} = \mathrm{e}^{-(\lambda+\mu)t} \cdot \frac{z^{2k}}{h(z)} \begin{pmatrix} h(z)\cosh\left[t\,\dfrac{h(z)}{z}\right] + z(\mu-\lambda)\sinh\left[t\,\dfrac{h(z)}{z}\right], \\ \\ \lambda(z^2+1)\sinh\left[t\,\dfrac{h(z)}{z}\right] \end{pmatrix}. \tag{2.12}$$

Hence, from Eqs. (2.6) and (2.12) we obtain the explicit expression of the probability generating functions when the initial state is even:

$$F_{2k}(z,t) = \mathrm{e}^{-(\lambda+\mu)t} \frac{z^{2k}}{h(z)} \left\{ h(z)\cosh\left[t\,\frac{h(z)}{z}\right] + z(\mu-\lambda)\sinh\left[t\,\frac{h(z)}{z}\right] \right\} \tag{2.13}$$

$$G_{2k}(z,t) = \mathrm{e}^{-(\lambda+\mu)t} \frac{z^{2k}}{h(z)} \lambda\left(z^2+1\right)\sinh\left[t\,\frac{h(z)}{z}\right]. \tag{2.14}$$

Similarly, if the initial state is odd $(2k+1)$, Eqs. (2.7) and (2.10)-(2.11) give:

$$\mathrm{e}^{At}\cdot\begin{pmatrix}0\\z^{2k+1}\end{pmatrix}=\begin{pmatrix}\mu-\lambda-\dfrac{h(z)}{z} & \mu-\lambda+\dfrac{h(z)}{z}\\[2ex] \lambda\dfrac{(z^2+1)}{z} & \lambda\dfrac{(z^2+1)}{z}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{v_1t} & 0\\[2ex] 0 & \mathrm{e}^{v_2t}\end{pmatrix}\begin{pmatrix}\theta_1(z)\\[2ex]\theta_2(z)\end{pmatrix},$$

where

$$\theta_1(z)=\frac{[h(z)-z(\lambda-\mu)]}{2\lambda(z^2+1)h(z)}z^{2k+2},\qquad \theta_2(z)=\frac{[h(z)-z(\mu-\lambda)]}{2\lambda(z^2+1)h(z)}z^{2k+2}$$

and then,

$$\mathrm{e}^{At}\cdot\begin{pmatrix}0\\z^{2k+1}\end{pmatrix}=\mathrm{e}^{-(\lambda+\mu)t}\cdot\frac{z^{2k+1}}{h(z)}\begin{pmatrix}\mu(z^2+1)\sinh\left[t\,\dfrac{h(z)}{z}\right],\\[2ex] h(z)\cosh\left[t\,\dfrac{h(z)}{z}\right]+z(\lambda-\mu)\sinh\left[t\,\dfrac{h(z)}{z}\right]\end{pmatrix}. \tag{2.15}$$

Hence, the explicit expression of the probability generating functions are:

$$F_{2k+1}(z,t)=\mathrm{e}^{-(\lambda+\mu)t}\frac{z^{2k+1}}{h(z)}\mu\left(z^2+1\right)\sinh\left[t\,\frac{h(z)}{z}\right], \tag{2.16}$$

$$G_{2k+1}(z,t)=\mathrm{e}^{-(\lambda+\mu)t}\frac{z^{2k+1}}{h(z)}\left\{h(z)\cosh\left[t\,\frac{h(z)}{z}\right]+z(\lambda-\mu)\sinh\left[t\,\frac{h(z)}{z}\right]\right\}. \tag{2.17}$$

We are now able to evaluate the state probabilities of the process.

2.2.1 Zero initial state

Proposition 2.2.1. *For all $r \in \mathbb{Z}$ and $t \geq 0$ the transition probabilities of $N(t)$, when the initial state is zero, are:*

$$p_{0,0}(t) = \mathrm{e}^{-(\lambda+\mu)t} \sum_{n=0}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu - \lambda}{\lambda} \right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right] \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} \left(\frac{\lambda}{\mu} \right)^{-2k},$$

$$p_{0,2r}(t) = p_{-2r}(t) = \mathrm{e}^{-(\lambda+\mu)t} \sum_{n=r}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu - \lambda}{\lambda} \right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right]$$
$$\times \sum_{k=0}^{n-r} \binom{n}{k} \binom{n}{r+k} \left(\frac{\lambda}{\mu} \right)^{-2k-r}, \quad r \geq 1; \tag{2.18}$$

$$p_{0,2r+1}(t) = p_{-(2r+1)}(t) = \mathrm{e}^{-(\lambda+\mu)t} \left\{ \sum_{n=r}^{+\infty} \frac{(\lambda t)^{2n+1}}{(2n+1)!} \sum_{k=0}^{n-r} \binom{n}{k} \binom{n}{r+k} \left(\frac{\lambda}{\mu} \right)^{-2k-r} \right.$$
$$\left. + \sum_{n=r+1}^{+\infty} \frac{(\lambda t)^{2n+1}}{(2n+1)!} \sum_{k=0}^{n-r-1} \binom{n}{k} \binom{n}{r+k+1} \left(\frac{\lambda}{\mu} \right)^{-2k-r-1} \right\}. \tag{2.19}$$

Proof. Recalling (2.5) we have:

$$F_{2n}(z,t) = e^{-(\lambda+\mu)t} \left\{ \sum_{n=0}^{+\infty} \frac{\left[t \, \frac{h(z)}{z} \right]^{2n}}{(2n)!} - \frac{z(\mu - \lambda)}{h(z)} \sum_{n=0}^{+\infty} \frac{\left[t \, \frac{h(z)}{z} \right]^{2n+1}}{(2n+1)!} \right\}$$
$$= e^{-(\lambda+\mu)t} \left\{ \sum_{n=0}^{+\infty} \frac{(\lambda t)^{2n}}{z^{2n}(2n)!} \left(1 + \frac{\mu}{\lambda} z^2 \right)^n \left(z^2 + \frac{\mu}{\lambda} \right)^n \right.$$
$$\left. + (\mu - \lambda) \sum_{n=0}^{+\infty} \frac{(\lambda t)^{2n+1}}{z^{2n}(2n+1)!} \left(1 + \frac{\mu}{\lambda} z^2 \right)^n \left(z^2 + \frac{\mu}{\lambda} \right)^n \right\}. \tag{2.20}$$

Consider the following identity:

$$\left(1 + \frac{\mu \; z^2}{\lambda} \right)^n \left(z^2 + \frac{\mu}{\lambda} \right)^n = \sum_{x=0}^{n} z^{2x} \sum_{k=0}^{x} \binom{n}{k} \binom{n}{n-k} \left(\frac{\lambda}{\mu} \right)^{-n-2k+x}$$
$$+ z^{2n+2} \sum_{x=0}^{n-1} z^{2x} \sum_{k=0}^{n-x-1x} \binom{n}{k} \binom{n}{n+k+1} \left(\frac{\lambda}{\mu} \right)^{-x-2k-1}. \tag{2.21}$$

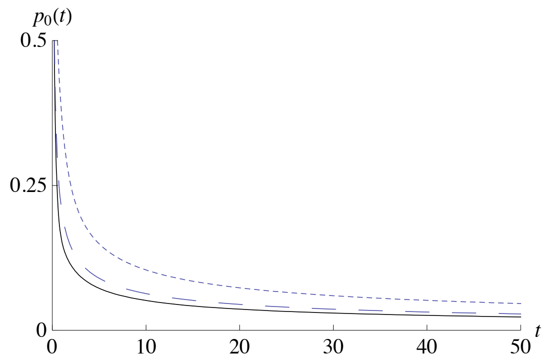


Figure 2.2: Plots of $p_0(t) = p_{0,0}(t)$ for $(\lambda, \mu) = (2, 1)$ (solid line), $(\lambda, \mu) = (2, 2)$ (dashed line) and $(\lambda, \mu) = (1, 2)$ (dotted line).

By substituting (2.21) in (2.20) and by setting $n - x = r$, with $r \geq 1$, we obtain

$$F_{2n}(z,t) = e^{-(\lambda+\mu)t} \sum_{n=r}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu - \lambda}{\lambda} \right) \frac{(\lambda\ t)^{2n+1}}{(2n+1)!} \right] \\ \times \sum_{k=0}^{n-r} \binom{n}{k} \binom{n}{r+k} \left(\frac{\lambda}{\mu} \right)^{-2k-r} \left\{ \sum_{r=0}^{+\infty} \frac{1}{z^{2r}} + \sum_{r=1}^{+\infty} z^{2r} \right\}. \tag{2.22}$$

Extracting the coefficients of z^{2r} in (2.22) the even probabilities are given. Similarly, from (2.5) and (2.16)-(2.17), for $r \geq 0$, the odd state probabilities follow. $\square$

Some plots of the transition probabilities of the process $N(t)$, when initial state is zero, are given in Figures 2.2, 2.3, 2.4 and 2.5.

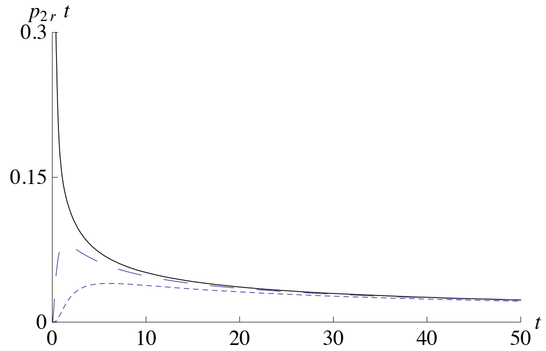


Figure 2.3: Plots of $p_{2r}(t) = p_{0,2r}(t)$ for $(\lambda, \mu) = (2, 1)$ and $r = 0$ (solid line), $r = 1$ (dashed line) and $r = 2$ (dotted line).

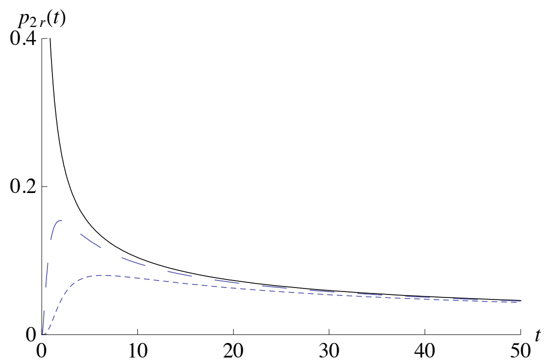


Figure 2.4: Plots of $p_{2r}(t) = p_{0,2r}(t)$ for $(\lambda, \mu) = (1, 2)$ and $r = 0$ (solid line), $r = 1$ (dashed line) and $r = 2$ (dotted line).

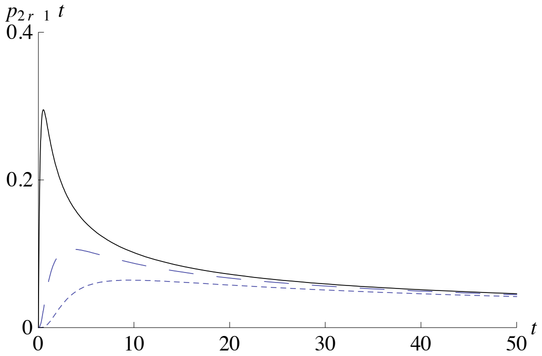


Figure 2.5: Plots of $p_{2r+1}(t) = p_{0,2r+1}(t)$ for $(\lambda, \mu) = (2, 1)$ and $r = 0$ (solid line), $r = 1$ (dashed line) and $r = 2$ (dotted line).

2.2.2 Non-zero initial state

Proposition 2.2.2. *For all $l, r \in \mathbb{Z}$ and $t \geq 0$ the transition probabilities of $N(t)$, when the initial state is arbitrary, are:*

$$
\begin{aligned}
p_{2l,2r}(t) = \mathrm{e}^{-(\lambda+\mu)t} & \sum_{n=|r-l|}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu - \lambda}{\lambda} \right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right] \\
& \times \sum_{k=0}^{n-|r-l|} \binom{n}{k} \binom{n}{k+|r-l|} \left(\frac{\lambda}{\mu} \right)^{-2k-|r-l|}, \qquad (2.23)
\end{aligned}
$$

$$
\begin{aligned}
p_{2l,2r+1}(t) = \mathrm{e}^{-(\lambda+\mu)t} & \left\{ \sum_{n=|r-l|}^{+\infty} \frac{(\lambda t)^{2n+1}}{(2n+1)!} \sum_{k=0}^{n-|r-l|} \binom{n}{k} \binom{n}{k+|r-l|} \left(\frac{\lambda}{\mu} \right)^{-2k-|r-l|} \right. \\
& \left. + \sum_{n=|r-l+1|}^{+\infty} \frac{(\lambda t)^{2n+1}}{(2n+1)!} \sum_{k=0}^{n-|r-l+1|} \binom{n}{k} \binom{n}{k+|r-l+1|} \left(\frac{\lambda}{\mu} \right)^{-2k-|r-l+1|} \right\}, \qquad (2.24)
\end{aligned}
$$

$$
\begin{aligned}
p_{2l+1,2r}(t) = \mathrm{e}^{-(\lambda+\mu)t} \Bigg\{ & \sum_{n=|r-l-1|}^{+\infty} \frac{(\mu t)^{2n+1}}{(2n+1)!} \\
& \times \sum_{k=0}^{n-|r-l-1|} \binom{n}{k} \binom{n}{k+|r-l-1|} \left(\frac{\mu}{\lambda}\right)^{-2k-|r-l-1|} \\
& + \sum_{n=|r-l|}^{+\infty} \frac{(\mu t)^{2n+1}}{(2n+1)!} \sum_{k=0}^{n-|r-l|} \binom{n}{k} \binom{n}{k+|r-l|} \left(\frac{\mu}{\lambda}\right)^{-2k-|r-l|} \Bigg\}, \\
p_{2l+1,2r+1}(t) = \mathrm{e}^{-(\lambda+\mu)t} & \sum_{n=|r-l|}^{+\infty} \left[\frac{(\mu t)^{2n}}{(2n)!} + \left(\frac{\lambda-\mu}{\lambda}\right) \frac{(\mu t)^{2n+1}}{(2n+1)!} \right] \\
& \times \sum_{k=0}^{n-|r-l|} \binom{n}{k} \binom{n}{k+|r-l|} \left(\frac{\mu}{\lambda}\right)^{-2k-|r-l|}.
\end{aligned} \tag{2.25}
$$

Proof. Recalling (2.13) and (2.21), when $2k$ is even, we consider the following generating function:

$$
\begin{aligned}
F_{2k}(z,t) = e^{-(\lambda+\mu)t} & \sum_{n=0}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right] \\
& \times \frac{z^{2k}}{z^{2n}} \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k} \binom{n}{j} \left(\frac{\lambda}{\mu}\right)^{-n-k+j} z^{2(k+j)}.
\end{aligned} \tag{2.26}
$$

By setting $k+j=r$ and $l=k$ in (2.26), when $n>l>0$ and $l \geq n \geq 0$, we obtain the generating function:

$$
\begin{aligned}
F_{2k}(z,t) = e^{-(\lambda+\mu)t} \Bigg\{ & \sum_{n=1}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right] \\
& \times \sum_{r=l+1}^{l+n} y^r \sum_{k=0}^{n-r+l} \binom{n}{k} \binom{n}{k+r-l} \left(\frac{\lambda}{\mu}\right)^{-2k-r+l} \\
& + \sum_{n=0}^{l} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{r=l-r}^{l} y^r \sum_{k=0}^{n+r-l} \binom{n}{k} \binom{n}{k+l-r} \left(\frac{\lambda}{\mu}\right)^{-2k+r-l} \\
&+ \sum_{n=l+1}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right) \frac{(\lambda t)^{2n+1}}{(2n+1)!}\right] \\
&\times \sum_{r=0}^{l} y^r \sum_{k=0}^{n+r-l} \binom{n}{k} \binom{n}{k+l-r} \left(\frac{\lambda}{\mu}\right)^{-2k+r-l} \\
&+ \sum_{n=l+1}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right) \frac{(\lambda t)^{2n+1}}{(2n+1)!}\right] \\
&\times \sum_{r=1}^{n-l} \frac{1}{y^r} \sum_{k=0}^{n-r-l} \binom{n}{k} \binom{n}{k+l+r} \left(\frac{\lambda}{\mu}\right)^{-2k-r-l} \Bigg\}, \qquad (2.27)
\end{aligned}
$$

where $y = z^2$. By Fubini's Theorem, after some calculations, we have:

$$
\begin{aligned}
F_{2k}(z,t) = e^{-(\lambda+\mu)t} \Bigg\{ &\sum_{r=0}^{l} y^r \sum_{n=l-r}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right) \frac{(\lambda t)^{2n+1}}{(2n+1)!}\right] \\
&\times \sum_{k=0}^{n-l+r} \binom{n}{k} \binom{n}{k+l-r} \left(\frac{\lambda}{\mu}\right)^{-2k+r-l} \\
&+ \sum_{r=l+1}^{+\infty} y^r \sum_{n=r-l}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right) \frac{(\lambda t)^{2n+1}}{(2n+1)!}\right] \\
&\times \sum_{k=0}^{n+l-r} \binom{n}{k} \binom{n}{k-l+r} \left(\frac{\lambda}{\mu}\right)^{-2k-r+l} \\
&+ \sum_{r=1}^{+\infty} \frac{1}{y^r} \sum_{n=r+l}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right) \frac{(\lambda t)^{2n+1}}{(2n+1)!}\right] \\
&\times \sum_{k=0}^{n-l-r} \binom{n}{k} \binom{n}{k+l+r} \left(\frac{\lambda}{\mu}\right)^{-2k-r-l} \Bigg\}. \qquad (2.28)
\end{aligned}
$$

Hence, we analyse the case in which $1 \leq |l| < n$ and $|l| \geq n \geq 0$. After some calculations we obtain:

$$\begin{aligned}
F_{2k}(z,t) = e^{-(\lambda+\mu)t} \Bigg\{ & \sum_{r=|l|}^{+\infty} \frac{1}{x^r} \sum_{n=r-|l|}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda} \right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right] \\
& \times \sum_{k=0}^{n+|l|-r} \binom{n}{k} \binom{n}{k-|l|+r} \left(\frac{\lambda}{\mu} \right)^{-2k-r+|l|} \\
& + \sum_{r=0}^{|l|-1} \frac{1}{x^r} \sum_{n=|l|-r}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda} \right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right] \\
& \times \sum_{k=0}^{n-|l|+r} \binom{n}{k} \binom{n}{k+|l|-r} \left(\frac{\lambda}{\mu} \right)^{-2k+r-|l|} \\
& + \sum_{r=1}^{\infty} x^r \sum_{n=|l|+r}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda} \right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right] \\
& \times \sum_{k=0}^{n-r-|l|} \binom{n}{k} \binom{n}{k+|l|+r} \left(\frac{\lambda}{\mu} \right)^{-2k-r-|l|} \Bigg\},
\end{aligned} \tag{2.29}$$

with $x = z^2$. Hence, Eqs. (2.23) follow by extracting the coefficients of z^{2r} in (2.28) and (2.29). Similarly, when $2k$ is even, by extracting the coefficients of z^{2r} in (2.14), Eq.(2.24) follows. Then, when $2k+1$ is odd by extracting the coefficients of z^{2r+1} in (2.16)-(2.17), Eqs. (2.25) and (2.25) finally follow. $\square$

Figure 2.6 shows some plots of transition probabilities given in Proposition 2.2.2.

2.2.3 Symmetry properties

The relevance of symmetry properties of transition functions of birth and death processes has been emphasized in Anderson and McDunnough [2] and in Di Crescenzo [16], for instance. We stress that the role of symmetry is closely connected to the analysis of the first-passage-time problem in Markov processes. See, for instance, the contributions of Giorno *et al.* [37], [38] and Di Crescenzo *et al.* [18], [19], where some relations involving the transition

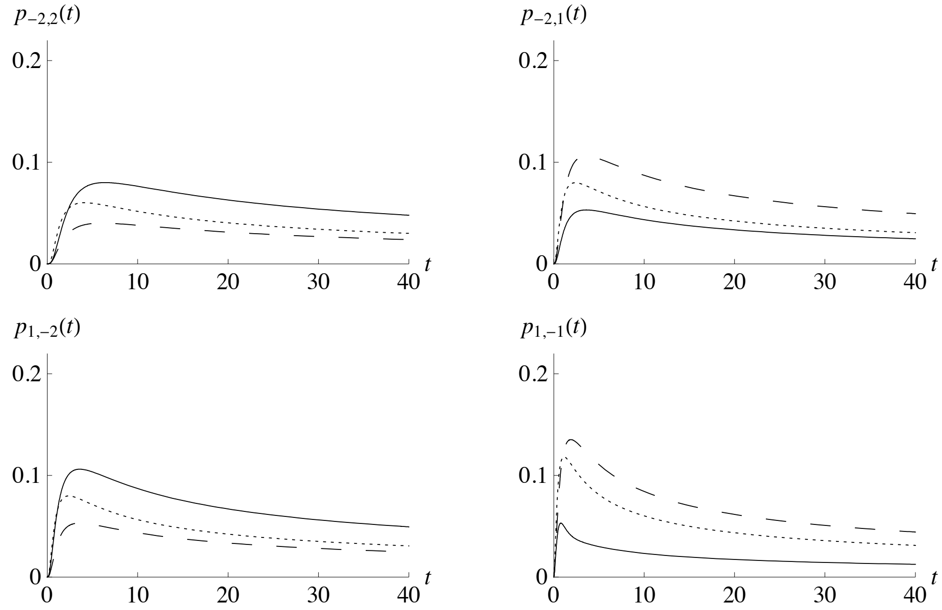


Figure 2.6: Plots of some transition probabilities for $(\lambda, \mu) = (1, 2)$ (solid line), $(\lambda, \mu) = (2, 2)$ (dotted line), $(\lambda, \mu) = (2, 1)$ (dashed line).

probability density functions and the first-passage-time density functions of symmetric diffusion processes in the presence of suitable time-varying boundaries.

Hereafter we analyze some symmetry properties of the transition probabilities obtained in Proposition 2.2.1 and Proposition 2.2.2. When necessary we emphasize the dependence on the parameters by writing $p_{k,n}(t; \lambda, \mu)$ instead of $p_{k,n}(t)$.

Remark 2.2.3. From Proposition 2.2.1, denoting by $p_{2r+1}(t; \lambda, \mu)$ the right-hand-side of (2.19) and , due to Eq. (2.14), the following symmetry property holds:

$$p_{0,2r+1}(t; \lambda, \mu) = \frac{\lambda}{\mu} p_{0,2r+1}(t; \mu, \lambda).$$

Proposition 2.2.4. *For every $t \geq 0$ and $n, k \in \mathbb{Z}$ the following symmetry relations hold:*

(i)	$p_{N-k,N-n}(t) = p_{k,n}(t),$	*if N is even*
(ii)	$p_{N-k,N-n}(t;\lambda,\mu) = p_{k,n}(t;\mu,\lambda),$	*if N is odd;*
(iii)	$p_{n,k}(t;\lambda,\mu) = p_{k,n}(t;\mu,\lambda);$	
(iv)	$p_{N+k,N+n}(t) = p_{k,n}(t),$	*if N is even*
(v)	$p_{N+k,N+n}(t;\lambda,\mu) = p_{k,n}(t;\mu,\lambda),$	*if N is odd.*

Proof. These properties follow from direct analysis of the probabilities (2.18)-(2.19) and (2.23)-(2.25). In particular, hereafter we prove the property (ii). If $N = 2m+1$ is odd, by setting $k = 2s$ (even) and $j = 2q$ (odd) and by recalling Eq. (2.25) the following relation holds:

$$\begin{aligned} p_{N-k,N-j}(t) = p_{2(m-s)+1,2(m-q)+1}(t) &= \mathrm{e}^{-(\lambda+\mu)t} \\ &\times \sum_{n=|s-q|}^{+\infty} \left[\frac{(\mu t)^{2n}}{(2n)!} + \left(\frac{\lambda-\mu}{\mu}\right) \frac{(\mu t)^{2n+1}}{(2n+1)!} \right] \\ &\times \sum_{k=0}^{n-|s-q|} \binom{n}{k} \binom{n}{k+|s-q|} \left(\frac{\mu}{\lambda}\right)^{-2k-|s-q|}, \end{aligned} \tag{2.30}$$

whereas, recalling (2.23), we have:

$$\begin{aligned} p_{k,j}(t) = p_{2s,2q}(t) = \mathrm{e}^{-(\lambda+\mu)t} \sum_{n=|q-s|}^{+\infty} \left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right) \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right] \\ \times \sum_{k=0}^{n-|q-s|} \binom{n}{k} \binom{n}{k+|q-s|} \left(\frac{\lambda}{\mu}\right)^{-2k-|q-s|}. \end{aligned} \tag{2.31}$$

The other cases can be proved similarly. $\square$

In Figure 2.6 the plots of $p_{-2,1}(t)$ and $p_{1,-2}(t)$ illustrate a case in which property (ii) of Proposition 2.2.4 holds.

2.2.4 Moments

Hereafter we express in closed form the mean and the variance of $N(t)$. In particular, we shall obtain that the mean is equal to the initial state. This result is intuitively justified by the symmetry of the Markov chain. Indeed, by choosing $N = 2k$ and $n = k - r$ in identity (i) of Proposition 2.2.4 we have $p_{k,k+r}(t) = p_{k,k-r}(t)$ $\forall k, r \in \mathbb{Z}$, and $t \geq 0$.

Proposition 2.2.5. *For $t \geq 0$ and $k \in \mathbb{Z}$ we have*

$$E[N(t)|N(0) = k] = k, \tag{2.32}$$

$$Var[N(t)|N(0) = k] = \frac{4\lambda\mu}{\lambda+\mu}\,t + (-1)^k \frac{\lambda(\lambda-\mu)}{(\lambda+\mu)^2}\left[1 - \mathrm{e}^{-2(\lambda+\mu)t}\right]. \tag{2.33}$$

Proof. The mean (2.32) easily follows from Eqs. (4.22) and (2.3). Moreover, by setting $\psi_k(t) := E[N^2(t)|N(0) = k]$ from system (4.22) we obtain:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\psi_k(t) &= 2\mu \sum_{n=-\infty}^{+\infty} p_{k,2n+1}(t) + 2\lambda \sum_{n=-\infty}^{+\infty} p_{k,2n}(t) \\
&= 2\mu\, G_k(1,t) + 2\lambda\, F_k(1,t), \qquad t \geq 0,
\end{aligned}$$

where F_k and G_k have been defined in (2.5). Hence, recalling Eqs. (2.13)–(2.17), after some calculations we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_k(t) = \begin{cases} \dfrac{4\lambda\mu}{\lambda+\mu} + \dfrac{2\lambda(\lambda-\mu)}{\lambda+\mu}\,\mathrm{e}^{-2(\lambda+\mu)t}, & k \text{ even} \\ \dfrac{4\lambda\mu}{\lambda+\mu} + \dfrac{2\mu(\mu-\lambda)}{\lambda+\mu}\mathrm{e}^{-2(\lambda+\mu)t}, & k \text{ odd} \end{cases}$$

with $\psi_k(0) = k^2$. Finally, Eq. (2.33) follows. $\square$

2.3 A reflecting boundary

In this section we consider the case in which the state-space is reduced to the set of non-negative integers. We shall denote by $\{R(t);\, t \geq 0\}$ the birth and death process having state-space $\{0, 1, 2, \ldots\}$, with 0 reflecting, whose rates are identical to those of $N(t)$. This describes, for instance, the number of customers in a queueing system with alternating rates. For $n = 0, 1, 2, \ldots$, let us introduce the transition probabilities

$$q_{k,n}(t) = P\{R(t) = n \,|\, R(0) = k\}, \quad t \geq 0.$$

The related differential-difference equations are, for $n = 1, 2, \ldots$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} q_{k,0}(t) = \mu \, q_{k,1}(t) - \lambda \, q_{k,0}(t), \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t} q_{k,2n}(t) = \mu \, q_{k,2n-1}(t) - 2\lambda \, q_{k,2n}(t) + \mu \, q_{k,2n+1}(t), \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t} q_{k,2n-1}(t) = \lambda \, q_{k,2n}(t) - 2\mu \, q_{k,2n-1}(t) + \lambda \, q_{k,2n-2}(t), \end{cases} \tag{2.34}$$

with

$$q_{k,n}(0) = \delta_{n,k}. \tag{2.35}$$

Remark 2.3.1. We point out that the steady-state distribution of $R(t)$ does not exist. Indeed, from system (2.34) it is not hard to see that

$$\lim_{t \to \infty} q_{k,n}(t) = 0, \qquad \forall k, n \in \mathbb{Z}.$$

2.3.1 Moments

Let us now set, for $k \in \mathbb{Z}$,

$$P_k(t) = P\left\{R(t) \, \text{even} \,|\, R(0) = k\right\} = \sum_{n=0}^{+\infty} q_{k,2n}(t), \qquad t \geq 0. \tag{2.36}$$

Now mean and variance of $R(t)$ will be formally expressed in terms of (2.36).

Proposition 2.3.2. *For $t \geq 0$ we have*

$$E[R(t)|R(0)=k] = \lambda \int_0^t q_{k,0}(\tau)d\tau + k, \tag{2.37}$$

$$\begin{aligned} Var[R(t)|R(0)=k] = 2(\lambda-\mu)\int_0^t P_k(\tau)d\tau - \lambda(2k+1)\int_0^t q_{k,0}(\tau)d\tau \\ -\lambda^2\left[\int_0^t q_{k,0}(\tau)d\tau\right]^2 + 2\mu t, \end{aligned} \tag{2.38}$$

where

$$P_k(t) = \frac{2\mu}{\lambda+\mu} + \frac{\lambda-\mu}{\lambda+\mu}\,\mathrm{e}^{-2(\lambda+\mu)t} + \lambda\int_0^t \mathrm{e}^{-2(\lambda+\mu)(t-\tau)} q_{k,0}(\tau)d\tau. \tag{2.39}$$

Proof. The mean (2.37) easily follows from system (2.34) and condition (2.35). Moreover, from Eqs. (2.34) we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E[R^2(t)|R(0)=k] &= 2\mu\sum_{n=1}^{+\infty} q_{k,2n+1}(t) + 2\mu q_{k,1}(t) + \lambda q_{k,0}(t) + 2\lambda\sum_{n=1}^{+\infty} q_{k,2n}(t) \\ &= 2\mu[1-P_k(t)-q_{k,1}(t)] + 2\mu q_{k,1}(t) + \lambda q_{k,0}(t) + 2\lambda\left[P_k(t)-q_{k,0}(t)\right] \\ &= 2(\lambda-\mu)P_k(t) + \lambda q_{k,0}(t) + 2\mu, \end{aligned}$$

where $P_k(t)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P_k(t) = -2(\mu+\lambda)\,P_k(t) + \lambda\,q_{k,0}(t) + 2\mu. \tag{2.40}$$

Since the solution of (2.40) is Eq. (2.39), the conditional variance (2.38) easily follows. □

2.3.2 Probabilities

In the case in which the initial state k is zero, recalling (2.34) and (2.35) and denoting by

$$\pi_{k,n}(s) := \mathcal{L}_s[q_{k,n}(t)] = \int_0^\infty \mathrm{e}^{-st}\,q_{k,n}(t)\,\mathrm{d}t, \quad s>0, \tag{2.41}$$

the Laplace transform of the transition probabilities of $R(t)$, we have:

$$\begin{cases} (\lambda + s)\,\pi_{0,0}(s) = 1 + \mu\pi_{0,1}(s), \\ \\ (2\lambda + s)\,\pi_{0,2n}(s) = \mu\pi_{0,2n-1}(s) + \mu\pi_{0,2n+1}(s), \\ \\ (2\mu + s)\,\pi_{0,2n-1}(s) = \lambda\pi_{0,2n}(s) + \lambda\pi_{0,2n-2}(s). \end{cases} \tag{2.42}$$

The solution of system (2.42) involves the roots of the biquadratic equation

$$\lambda\mu\, x^4 - \left[(\lambda + \mu + s)^2 - \lambda^2 - \mu^2\right] x^2 + \lambda\mu = 0,$$

which are given by

$$\psi_1^2(s) = \frac{(A + B)^2}{a^2 - b^2}, \qquad \psi_2^2(s) = \frac{(A - B)^2}{a^2 - b^2},$$

where

$$a = \lambda + \mu, \qquad b = \lambda - \mu.$$

and

$$A^2 = (a + s)^2 - a^2, \qquad B^2 = (a + s)^2 - b^2.$$

Since $\psi_1^2(s) > 1$ and $0 < \psi_2^2(s) < 1$, from system (2.42) we have

$$\pi_{0,2n}(s) = \frac{2(2\mu + s)}{s(2\mu + s) + AB}\left[\frac{(A - B)^2}{a^2 - b^2}\right]^n, \qquad n = 0, 1, \dots \tag{2.43}$$

and, similarly,

$$\pi_{0,2n-1}(s) = \frac{2\lambda}{s(2\mu + s) + AB}\left[\frac{(A - B)^2}{a^2 - b^2}\right]^{n-1}\left[1 + \frac{(A - B)^2}{a^2 - b^2}\right], \; n = 1, 2, \dots$$

In particular, from Eq. (2.43) we have

$$\pi_{0,0}(s) = \frac{2}{s}\left[1 + \sqrt{1 + \frac{2\lambda}{s}}\sqrt{1 + \frac{2\lambda}{2\mu + s}}\right]^{-1}. \tag{2.44}$$

To obtain the expression of the zero-state probability $q_{0,0}(t)$, we note that Eq. (2.44) can be written as

$$\pi_{0,0}(s) = \frac{1}{a+b}\left[\sqrt{\frac{2a+s}{s}} - \sqrt{\frac{a+s-b}{a+s+b}}\right] \times \left(1 - \frac{b^2}{(a+s)(\sqrt{(a+s)^2-b^2}+a+s)}\right).$$

Hence, by making use of Eqs. (19) and (24) of [32], after some calculations, we obtain

$$\begin{aligned} q_{0,0}(t) &= \frac{\mathrm{e}^{-at}}{a+b}\left\{a\Big[I_0(at)+I_1(at)\Big] + b\Big[I_0(bt)-I_1(bt)\Big]\right\} \\ &+ \frac{a\,\mathrm{e}^{-at}}{2(a+b)}\int_0^t s\,{}_1F_2\left(\frac{1}{2},\frac{3}{2},2,\frac{b^2s^2}{4}\right)\Big[I_0(a(t-s))+I_1(a(t-s))\Big]\mathrm{d}s \\ &+ \frac{b\,\mathrm{e}^{-at}}{2(a+b)}\int_0^t s\,{}_1F_2\left(\frac{1}{2},\frac{3}{2},2,\frac{b^2s^2}{4}\right)\Big[I_0(b(t-s))-I_1(b(t-s))\Big]\mathrm{d}s, \end{aligned} \tag{2.45}$$

where

$$I_n(x) = \sum_{k=0}^{+\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}$$

is the modified Bessel function of the first kind and

$${}_1F_2\left(c_1,c_2,c_3,x\right) = \sum_{k=0}^{+\infty} \frac{(c_1)_k\, x^k}{(c_2)_k\,(c_3)_k\,k!}$$

is the hypergeometric function, with $(\cdot)_k$ denoting the Pochhammer symbol. The evaluation of the integrals in Eq. (2.45) finally gives the transition probability (see Section 3 of Iuliano and Martinucci [42]).

$$\begin{aligned} q_{0,0}(t) = \frac{\mathrm{e}^{-at}}{a+b}\sum_{k=0}^{+\infty}\frac{(t/2)^{2k}}{k!^2}&\left\{(a^{2k+1}+b^{2k+1})\,{}_1F_2\left(-\frac{1}{2},k+\frac{1}{2},k+1,\frac{b^2t^2}{4}\right)\right. \\ &\left.+\frac{t(a^{2k+2}-b^{2k+2})}{2(k+1)}\,{}_1F_2\left(-\frac{1}{2},k+1,k+\frac{3}{2},\frac{b^2t^2}{4}\right)\right\}, \qquad t \geq 0, \end{aligned}$$

Some plots of the transition probability $q_0(t) = q_{0,0}(t)$ are shown in Figure 2.7 for various choices of λ and μ. Now we analyse the case in which the

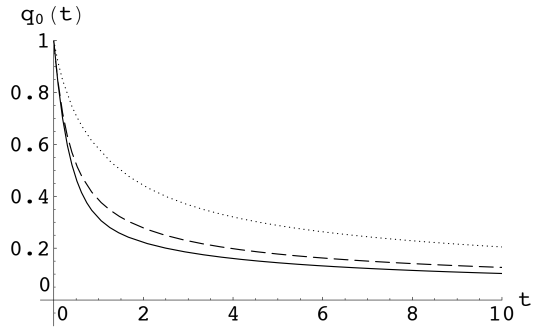


Figure 2.7: Plots of $q_0(t) = q_{0,0}(t)$ for $(\lambda, \mu) = (2, 1)$ (solid line), $(\lambda, \mu) = (2, 2)$ (dashed line) and $(\lambda, \mu) = (1, 2)$ (dotted line).

initial state is $k = 1$. By using (2.41) the transition probabilities of $R(t)$, from Eqs. (2.34) we have

$$\begin{cases} (\lambda + s)\, \pi_{1,0}(s) = \mu \pi_{1,1}(s) \\ (2\mu + s)\, \pi_{1,1}(s) = 1 + \lambda \pi_{1,2}(s) + \lambda \pi_{1,0}(s) \\ (2\lambda + s)\, \pi_{1,2n}(s) = \mu \pi_{1,2n-1}(s) + \mu \pi_{1,2n+1}(s), \qquad n \geq 1 \\ (2\mu + s)\, \pi_{1,2n-1}(s) = \lambda \pi_{1,2n}(s) + \lambda \pi_{1,2n-2}(s), \qquad n \geq 2. \end{cases} \tag{2.46}$$

After some calculation, we obtain:

$$\pi_{1,2n}(s) = \frac{(2\mu + s)(\lambda + s)\, [\psi_2^2(s)]^{n+1}}{\lambda^2\, [\mu(1 - \psi_2^2(s)) - s\, \psi_2^2(s)]}, \qquad n \geq 1, \tag{2.47}$$

and, similarly,

$$\pi_{1,2n-1}(s) = \frac{(\lambda + s)\, [\psi_2^2(s)]^n\, [1 + \psi_2^2(s)]}{\lambda\, [\mu(1 - \psi_2^2(s)) - s\, \psi_2^2(s)]}, \qquad n \geq 1. \tag{2.48}$$

In particular, when the initial state is 1 and the final state is zero, by making use of Eqs. (2.47) and (2.48) and substituting in (2.46), we have

$$\pi_{1,0}(s) = \frac{(2\lambda + s)(2\mu + s) - AB}{\lambda\, [s(2\mu + s) + AB]}. \tag{2.49}$$

By inversion of (2.49), after some calculations, we obtain

$$q_{1,0}(t) = \frac{\mathrm{e}^{-at}}{2\lambda(a+b)} \int_0^t \left[-b^2 \frac{I_1(b(t-s))}{b(t-s)} + a^2 \frac{I_1(a(t-s))}{a(t-s)} \right] h(s)\mathrm{d}s + \\ + \frac{\mathrm{e}^{-at}}{2\lambda(a+b)} (a^2 - b^2) \int_0^t \frac{b(t-s)}{2} \, {}_1F_2 \left(\frac{1}{2}, \frac{3}{2}, 2, \frac{b^2(t-s)^2}{4} \right) h(s)\mathrm{d}s, \tag{2.50}$$

where

$$h(x) := a \left[I_0(ax) + I_1(ax) \right] + b \left[I_0(bx) - I_1(bx) \right],$$

with $I_n(\cdot)$ denoting the modified Bessel function of the first kind. Evaluating of the integrals in Eq. (2.50) the following result finally follows.

Proposition 2.3.3. *For $t \geq 0$, we have*

$$\begin{aligned} q_{1,0}(t) = \frac{\mathrm{e}^{-at}}{\lambda(a+b)} \Bigg\{ & \sum_{n=0}^{+\infty} \frac{t^{2n}}{n!(n+1)!} \left[\left(\frac{a}{2}\right)^{2n+2} - \left(\frac{b}{2}\right)^{2n+2} \right] \xi \left(\frac{1}{2}, 1, a, b \right) \\ & + \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{n!(n+1)!(2n+1)} \left[\frac{a^2 - b^2}{2} \left(\frac{b}{2}\right)^{2n+1} \right] \xi \left(1, \frac{3}{2}, a, b \right) \\ & + \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{n!(n+1)!(2n+1)} \left[\left(\frac{a}{2}\right)^{2n+2} - \left(\frac{b}{2}\right)^{2n+2} \right] \eta \left(1, \frac{3}{2}, a, b \right) \\ & + \sum_{n=0}^{+\infty} \frac{t^{2n+2}}{n!(n+1)!(2n+1)(2n+2)} \left[\frac{a^2 - b^2}{2} \left(\frac{b}{2}\right)^{2n+1} \right] \eta \left(\frac{3}{2}, 2, a, b \right) \Bigg\}, \end{aligned}$$

where

$$\xi(u, v, a, b) = {}_1F_2 \left(\frac{1}{2}; n+u, n+v; \frac{a^2t^2}{4} \right) - {}_1F_2 \left(\frac{1}{2}; n+v, n+u; \frac{b^2t^2}{4} \right),$$

$$\eta(u, v, a, b) = a \; {}_1F_2 \left(\frac{1}{2}; n+u, n+v; \frac{a^2t^2}{4} \right) + b \; {}_1F_2 \left(\frac{1}{2}; n+v, n+u; \frac{b^2t^2}{4} \right).$$

Some illustrative plots of $q_{1,0}(t)$ are shown in Figure 2.8.

Stimulated by some previous works on the applications of stochastic processes to the study of chain molecular diffusion, in this chapter we have

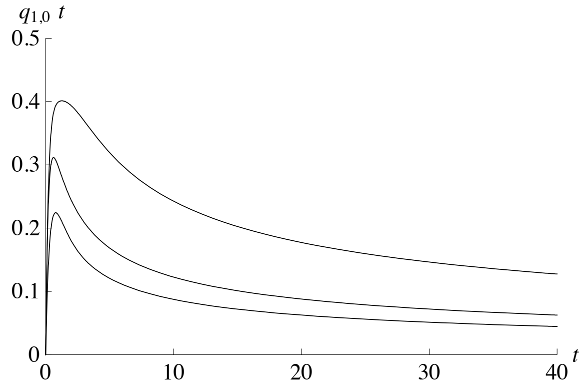


Figure 2.8: Plots of $q_{1,0}(t)$ for $(\lambda, \mu) = (1, 2), (2, 2), (2, 1)$, from top to bottom.

analyzed a birth and death process on $\mathbb{Z}$ characterized by alternating transition rates. The probability generating functions of even and odd states and the transition probabilities of the bilateral process have been obtained in two special cases: (i) when the initial and final state is zero and (ii) when the initial state is arbitrary. A preliminary investigation on the transient behavior of the birth and death process obtained by superimposing a reflecting boundary in the zero-state has also been performed in both cases.

In conclusion, the results given in this chapter deserve also special interest in the fields of chemical queueing processes and two-periodic random walks, according to the lines traced in various papers, such as Conolly *et al.* [12] and Böhm and Hornik [7], for instance.

PART 2

Analysis of a telegraph process with underlying random walk

CHAPTER 3

TELEGRAPH PROCESS

3.1 Introduction

In this chapter we analyze a random motion governed by the *telegraph equation.* Models of random evolution deserve large interest in mathematical biology, as they naturally appear in many contexts involving biological phenomena. Among those, we recall the telegraph random process, which arises also in other applied fields, such as engineering, mathematical finance, queueing and reliability theory.

The telegraph random process has been studied in the past by many authors aiming to describe the motion of a particle on the real line, traveling at constant speed, whose direction is reversed randomly according to the arrival epochs of a Poisson counting process (for instance, see Kac [43]). The telegraph random process first appeared in the literature in a work of Goldstein [39] in fluid dynamics. In this paper, a particle, starting from the origin, moves in steps of length $\Delta = v\tau$, the duration of each step being τ. At each time the particle has probability p of maintaining and the probability $q = 1-p$

of reversing the direction of the previous step (at $\tau = 0$ both directions are equiprobable). The probability distribution of the position is calculated exactly and various asymptotic expressions are found. In particular limit the solution can be obtained by solving the telegraph equation. This is remarkable since the equation is hyperbolic and one usually encounters equations of parabolic type. Some aspects of this process, including absorption and first-passage-time problems, have been studied in Foong and Kanno [34] and Orsingher [55]. Several Markovian generalizations of the telegraph random process have been proposed in the recent literature, in which the reversals of velocities of the motion are still driven by a homogeneous Poisson process, and the transition probability density function governing the process is usually a solution of a hyperbolic partial differential equation. This solution is very complicated when the number of velocities is high (see Kolesnik [47]), so that closed-form solutions have been found in the past only in very few cases. Restricting our attention to one-dimensional models, various kinds of generalizations of the telegraph process have been proposed towards motion characterized by random velocities (see Stadje and Zacks [72]); models characterized by more than two directions, for instance, see Orsingher *et al.* [58] and by Kolesnik [47]; velocities alternating at gamma or Erlang-distributed random times (see Di Crescenzo *et al.* [17], [24] and Pogorui *et al.* [62]).

The interest for this process has increased with the years and many important features obtained from different viewpoints and different techniques have been proposed. Recently, the telegraph process has been exploited in probabilistic financial fields, such as for stochastic volatility modeling and actuarial problems based on generalized telegraph process characterized by alternating velocities and jumps occurring at switching velocities (see Ratanov [65] and Ratanov [66], for instance). Among other applications these processes have been exploited for stochastic volatility models (see Di Masi *et al.* [31]). A new model based on a geometric telegraph process to study the price evolution of risky assets has been analized in Di Crescenzo and Pellerey [30].

In the following sections the standard telegraph process is introduced and the relative properties are analyzed in order to show the explicit expressions of the transition probabilities of the stochastic process. In Section 3.2, the telegraph process is formally defined. Some preliminary results of the transition probabilities of the process $V(t)$ and relative moments are introduced. In Section 3.3 the integrated telegraph process $X(t)$ is analyzed and the relative probability densities of the process $X(t)$ are defined in order to introduce the telegraph equation and moments of particle position. Then, in Section 3.4, the probability law of the process $(X(t), V(t))$ characterized by a discrete component on the extremes of the domain $[-ct, ct]$ and by an absolutely component over the domain is exhibited. Finally, in Section 3.5 moment genarating function of the telegraph process is evaluated.

3.2 Telegraph process

In this section we give some results on the telegraph process. We consider a stochastic process $\{(X(t), V(t)); t \geq 0\}$, where $X(t)$ and $V(t)$ denote respectively position and velocity at time t of a particle moving on the real line with two alternating velocities and opposite directions. The random times separating consecutive changes of direction of the motion have a general distribution and perform an alternating renewal process. In the basic model such random times have exponential distribution with parameter λ.

Starting from the origin, the particle moves following an alternating motion with velocities c and $-c$ ($c > 0$). The direction of the motion (forward and backward) is determinated by the sign of the velocity that forces the particle to change the direction. The telegraph process was introduced to represent a random motion with finite velocity, in order to superate the serious limitations of the *Brownian motion process* in the realistic representation of real random motions:

- infinite speed with which it travels the trajectories,
- the non-differentiability of trajectory (which implies total absence of inertia).

The Brownian motion process describes the motion exhibited by a small particle that is totaly immersed in a liquid or gas. Moreover, it is suitably defined as the limit of a symmetric random walk.

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson process with parameter λ, so that

$$P\{N(t) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

For a finite number of time points

$$0 \leq t_1 < t_2 < t_3 < \ldots < t_n$$

the increments

$$N(t_2) - N(t_1), N(t_3) - N(t_2), \ldots, N(t_n) - N(t_{n-1})$$

are independent. If the particle starts with positive velocity, then the velocity is c ($c > 0$). When there is a collision, the particle velocity changes becoming negative, i.e. $-c$. The velocity remains the same until another collision, and so on. Now we derive the distribution of the velocity of the particle. The number of collisions at time t is just $N(t)$ and the sign of velocity changes at any collision. We assume that at time $t = 0$ the particle is located at the origin and moves in a positive or negative direction. Hence, the velocity of the particle at time t is

$$V(t) := V(0)(-1)^{N(t)}, \qquad t \geq 0, \tag{3.1}$$

where $N(t)$ is the number of events of an homogeneous Poisson process with intensity λ during $(0, t)$ and $V(0)$ is a random variable independent from $N(t)$ such that

$$P\{V(0) = c\} = P\{V(0) = -c\} = \frac{1}{2}.$$

The process $V(t)$ can be interpreted as the velocity at time t of a particle moving on the real line. It is easy to see that the following theorem holds:

Theorem 3.2.1. *The probabilities of the process $V(t)$ conditional on initial velocity $V(0) = c$ are:*

$$\begin{aligned} P\{V(t) = c \,|\, V(0) = c\} &= \frac{1}{2}(1 + e^{-2\lambda t}), \\ P\{V(t) = -c \,|\, V(0) = c\} &= \frac{1}{2}(1 - e^{-2\lambda t}). \end{aligned} \tag{3.2}$$

Proof. The probabilities

$$\begin{aligned} p_c(t) &= P\{V(t) = c \,|\, V(0) = c\} \\ p_{-c}(t) &= P\{V(t) = -c \,|\, V(0) = c\} \end{aligned} \tag{3.3}$$

satisfy the following system:

$$\begin{cases} p_c(t + \Delta t) = p_c(t)(1 - \lambda \Delta t) + p_{-c}(t)\lambda \Delta t + o(\Delta t) \\ \\ p_{-c}(t + \Delta t) = p_{-c}(t)(1 - \lambda \Delta t) + p_c(t)\lambda \Delta t + o(\Delta t). \end{cases} \tag{3.4}$$

Hence, the following system of differential equations holds:

$$\begin{cases} \dfrac{\partial}{\partial t} p_c(t) = -\lambda p_c(t) + \lambda p_{-c}(t) \\ \\ \dfrac{\partial}{\partial t} p_{-c}(t) = -\lambda p_{-c}(t) + \lambda p_c(t) \end{cases} \tag{3.5}$$

From system (3.5) we obtain the equation

$$\frac{\partial^2}{\partial t^2} p_c(t) + 2\lambda \frac{\partial}{\partial t} p_c(t) = 0 \tag{3.6}$$

with initial conditions:

$$\begin{cases} p_c(0) = 1 \\ \\ \dfrac{\partial}{\partial t} p_{-c}(0) = -\lambda. \end{cases} \tag{3.7}$$

Finally, Eqs. (3.2) follow from (3.6) and (3.7). □

Similarly, from the symmetry of the process $V(t)$, it is possible to prove that the following probabilities conditional on $V(0) = -c$ hold:

$$P\{V(t) = \pm c \,|\, V(0) = -c\} = \frac{1}{2}(1 \pm e^{-2\lambda t}). \tag{3.8}$$

3.2.1 Moments of the particle velocity

It is easy to derive, from Eqs. (3.2), the conditional mean and the conditional variance of process $V(t)$

$$\mathrm{E}\,[V(t) \,|\, V(0) = c] = c\,[p_c(t) - p_{-c}(t)] = c\; e^{-2\lambda t}, \tag{3.9}$$

$$Var\,[V(t) \,|\, V(0) = c] = c^2 \left(1 - e^{-4\lambda t}\right). \tag{3.10}$$

The covariance is given by

$$Cov\;[V(t), V(s)] = c^2 e^{-2\lambda|t-s|} - \left(c\; e^{-2\lambda t} \cdot e^{-2\lambda s}\right), \tag{3.11}$$

with $0 < s < t$.

Symilarly, the following relations conditional on $V(0) = -c$ hold:

$$\mathrm{E}\,[V(t) \,|\, V(0) = -c] = -c\; e^{-2\lambda t}, \tag{3.12}$$

$$Var\,[V(t) \,|\, V(0) = -c] = c^2 \left(1 - e^{-4\lambda t}\right), \tag{3.13}$$

$$Cov\ [V(t), V(s)] = c^2 e^{-2\lambda|t-s|} + \left(c\ e^{-2\lambda t} \cdot e^{-2\lambda s}\right), \qquad 0 < s < t. \quad (3.14)$$

We note that when $t \to \infty$, the means (3.9) and (3.12) tend to 0 and the variances (3.10) and (3.13) tend to the constant c^2.

3.3 Integrated telegraph process

The *integrated telegraph process* is obtained as integral of Eq. (3.1). This process describes the position of a particle during its motion. Hence, the istantaneous position of the particle at time t is

$$X(t) := V(0) \int_0^t (-1)^{N(s)} \mathrm{d}s, \qquad t \geq 0. \quad (3.15)$$

We assume that $N(0) = 0$. Notice that, in this model, the length of times at which the particle is traveling in the positive or negative direction are indipendent and identically distributed (i.i.d.) exponential random variables.

Definition 3.3.1. For $t \geq 0$ and $-ct < x < ct$, we introduce the following probability densities of $X(t)$ conditional on initial velocity $V(0) = c$:

$$f(x, t\,|\,c)\mathrm{d}x = \frac{\partial}{\partial x} P\left\{X(t) \leq x\,|\,V(t) = c\right\}, \quad (3.16)$$

$$b(x, t\,|\,c)\mathrm{d}x = \frac{\partial}{\partial x} P\left\{X(t) \leq x\,|\,V(t) = c\right\}. \quad (3.17)$$

The densities $f(x, t\,|\,-c)$ and $b(x, t\,|\,-c)$, conditional on initial velocity $V(0) = -c$ can be defined similarly.

The functions f and b are defined respectively as the density that the particle occupies location near x at time t with *forward* velocity, and that the particle is located in x at time t with *backward* velocity.

Definition 3.3.2. Define for $t \geq 0$ and $-ct < x < ct$ the forward density $f(x, t)$ and backward density $b(x, t)$:

$$f(x, t) = \frac{1}{2}\left[f(x, t\,|\,c) + f(x, t\,|\,-c)\right],$$

$$b(x,t) = \frac{1}{2}\left[b(x,t\,|\,c) + b(x,t\,|\,-c)\right].$$

Hence, the probability density of $X(t)$, when the particle stars at $x_0 = 0$, at time $t_0 = 0$ is denoted as follows:

$$p(x,t) = \frac{\partial}{\partial x} P\{X(t) \leq x\} = f(x,t) + b(x,t). \tag{3.18}$$

The probability density $p(x,t)$ introduced in Eq. (3.18) is the solution of the *telegraph equation*:

$$c^2 \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} \tag{3.19}$$

(for istance, see Goldstein [39]). The explicit form of the associated flow function

$$w(x,t) = f(x,t) - b(x,t) \tag{3.20}$$

of a random motion governed by the telegraph equation is also introduced. This function represents, at each time t, the excess of forward moving particle with respect to backward moving ones near point x.

In the following sections the closed form of the density $p(x,t)$ is given. It is interesting to note that the solution of the process $X(t)$ is similar to the solution of the equation of the vibrating string:

$$G(x,t) = \begin{cases} e^{-\lambda t} I_0\left(\dfrac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right), & |x| \leq ct \\ \\ 0, \qquad \text{otherwise.} \end{cases} \tag{3.21}$$

where the time t is replaced by the *randomized time* $\int_0^t (-1)^{N(s)} \mathrm{d}s$. The function $G(x,t)$ represents the instantaneous form of a string perfoming damped vibration initiated at time $t = 0$ by a unit impulse at $x = 0$, where

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{2}x\right)^{2k}$$

is the Bessel function with imaginary argument of order zero. An expression for the probability density $p(x,t)$, based on $G(x,t)$, is obtained in Orsingher [52]. Finally, the telegraph process has been generalized in many direction for one-dimensional generalizations (see Orsingher [54]) and for two-dimensional generalizations (see Orsingher [53]).

3.3.1 Moments of particle position

The variance and the covariance of the process $X(t)$ follow from Eq. (3.15). These results are contained in the following proposition.

Proposition 3.3.3.

$$Var[X(t)] = \frac{1}{2}c^2\left[\frac{2t}{\lambda} - \frac{(1-e^{-\lambda t})}{\lambda^2}\right] \tag{3.22}$$

$$Cov[X(t),X(s)] = \frac{1}{4}c^2\left[\frac{4\ \min(t,s)}{\lambda} - \frac{(1-e^{-2\lambda \min(t,s)})(1+e^{-2\lambda|t,s|})}{\lambda^2}\right]. \tag{3.23}$$

Proof. Using Eq. (3.30), we have

$$\begin{aligned}\mathrm{E}X^2(t) &= \int_{ct}^{ct} x^2 p(x,t)\mathrm{d}x + c^2t^2e^{-\lambda t} \\ &= \frac{e^{-\lambda t}}{2c}\left[\lambda\int_{ct}^{ct} x^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x\right] + \frac{\partial}{\partial t}\int_{ct}^{ct} x^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x.\end{aligned}$$

Note that

$$\int_{ct}^{ct} x^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x = \frac{c^3t}{\lambda}\left(e^{\lambda t}-e^{-\lambda t}\right),$$

so that Eq. (3.22) follows. Instead, from Eq. (3.15) we obtain

$$\begin{aligned}\mathrm{E}X^2(t) &= \mathrm{E}\left[V^2(0)\int_0^t\int_0^t (-1)^{N(s)+N(z)}\mathrm{d}s\mathrm{d}z\right] \\ &= c^2\int_0^t\int_0^t \mathrm{E}((-1)^{N(s)+N(z)})\mathrm{d}s\mathrm{d}z.\end{aligned}$$

When $z > s$, we obtain

$$\begin{aligned}\mathrm{E}[(-1)^{N(s)+N(z)}] &= \mathrm{E}[(-1)^{N(z)-N(s)}] \\ &= P\{N(z)-N(s) = \text{even}\} - P\{N(z)-N(s) = \text{odd}\} \\ &= e^{-2\lambda(z-s)}.\end{aligned}$$

Therefore

$$\mathrm{E}X^2(t) = c^2 \int_0^t \int_0^t e^{-2\lambda|z-s|} \mathrm{d}s\mathrm{d}z 2c^2 \int_0^t \int_0^s e^{-2\lambda(s-z)} \mathrm{d}s\mathrm{d}z.$$

Then, after same calculation Eq. (3.23) is obtained. □

3.3.2 Connection with Brownian motion

Kac [43] showed that the wave equation (3.19) becomes the heat equation when λ tends to ∞ and c^2/λ tends to σ^2. Letting $\lambda \to \infty$ means that velocity changes occur continuously, while $c^2/\lambda \to \sigma^2$ implies that also the speed of the particle tends to ∞. Therefore, the limiting behaviour of the integrated telegraph process is similar to that of the Brownian motion process. In particular, the density function $p(x,t)$ defined in (3.18), when λ tends to ∞ and c^2/λ tends to σ^2, becomes the Gaussian transition function of Brownian motion:

$$\lim_{\substack{\lambda\to\infty \\ c^2/\lambda\to\sigma^2}} p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{x^2}{2\sigma^2 t}\right\},$$

for $x \in \mathbb{R}$ and $t > 0$.

3.4 The probability law of the process

The probability law of the stochastic process $(X(t), V(t))$, when at time $t \geq 0$ the particle is located in the domain $[-ct, ct]$, has a discrete component concentrated on $\{-ct, ct\}$, with

$$P\{X(t) = ct, V(t) = c\} = P\{V(0) = c, N(t) = 0\} = \frac{1}{2}e^{-\lambda t},$$
$$P\{X(t) = -ct, V(t) = -c\} = P\{V(0) = -c, N(t) = 0\} = \frac{1}{2}e^{-\lambda t}, \tag{3.24}$$

and an absolutely continuous component over the domain $(-ct, ct)$. This is expressed given by the densities $f(x, t \,|\, c)$ and $b(x, t \,|\, c)$, defined in Eqs. (3.16) and (3.17), when the initial velocity is c (resp. $f(x, t \,|\, -c)$ and $b(x, t \,|\, -c)$ when the initial velocity is $-c$). Hereafter, we provide densities $f(x, t)$ and $b(x, t)$ in closed-form. We give olso the density $f(x, t \,|\, c)$ conditonal on initial velocity c.

We start by considering the following relations:

$$f(x, t + \Delta t) = f(x - c\Delta t, t)(1 - \lambda\Delta t) + b(x + c\Delta t, t)\lambda\Delta t + o(\Delta t), \tag{3.25}$$

$$b(x, t + \Delta t) = b(x + c\Delta t, t)(1 - \lambda\Delta t) + f(x - c\Delta t, t)\lambda\Delta t + o(\Delta t). \tag{3.26}$$

Expanding the latter equations up to the second order terms with respect to x and to the first order terms with respect to t, dividing for Δt and letting $\Delta t \to 0$, the densities f and b are solution of the differential system:

$$\begin{cases} \dfrac{\partial f}{\partial t} = -c\dfrac{\partial f}{\partial x} + \lambda(b - f) \\ \\ \dfrac{\partial b}{\partial t} = c\dfrac{\partial b}{\partial x} + \lambda(f - b). \end{cases} \tag{3.27}$$

The explicit solution of the system (3.27) is proved in Cane [10] and in Orsingher [52] and [54]. It is easy to see that the latter differential system can be written in terms of the transition density $p = f + b$, Eq. (3.18) and the flow function $w = f - b$, defined in Eq. (3.20). By adding and subtracting the equations in the system (3.27), we have that the density p and the function w are solution of the system

$$\begin{cases} \dfrac{\partial p}{\partial t} = -c\dfrac{\partial w}{\partial x} \\[2ex] \dfrac{\partial w}{\partial t} = -c\dfrac{\partial p}{\partial x} - 2\lambda w. \end{cases} \tag{3.28}$$

Deriving two times equations in (3.28) and substituting the function w in the first equation of the system, we obtain equation (3.19).

Remark 3.4.1. If $\lambda = 0$ in Eq. (3.19), then the classical *wave equation*

$$\frac{\partial^2 p}{\partial t^2} = c^2\frac{\partial^2 p}{\partial x^2} \tag{3.29}$$

can be obtained.

Hence, the probability law $p(x,t)$ of the process $(X(t), V(t))$, with $t \geq 0$, is the solution of the telegraph equation. Symilarly, the functions w, f and b are solutions of Eq. (3.19). It follows directly from system (3.28), deriving the first equation with respect to t and substituting the second derivative with respect to x. In the following theorem the density $p(x,t)$ and the flow function $w(x,t)$ are given (see the proof in Section 2 of Orsingher [55], for instance).

Theorem 3.4.2. *The explicit form of $p(x,t)$ is*

$$p(x,t) = \frac{e^{\lambda t}}{2c}\left[\lambda I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right], \tag{3.30}$$

while $w(x,t)$ is given by

$$w(x,t) = \frac{1}{2}e^{-\lambda t}\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) \tag{3.31}$$

when $|x| < ct$. Furthemore

$$P\{X(t) = ct\} = P\{X(t) = -ct\} = \frac{1}{2}e^{-\lambda t}. \tag{3.32}$$

We remark that the forward density is given by:

$$f(x,t) = \frac{e^{-\lambda t}}{4c}\Bigg\{\lambda I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \\ -c\frac{\partial}{\partial x}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\Bigg\}, \tag{3.33}$$

for $|x| < ct$.

It is interesting to observe that the flow function (3.31) shows that in $(0, ct)$ the particles moving forward exceed those moving backward in $(-ct, 0)$. This is in agreement with the fact that the particles diffuse out of the real line as time proceeds.

Remark 3.4.3. We note that, when $t \to \infty$, the discrete component (3.32) decreases, the interval domain $(-ct, ct)$ increases, and the probability mass tends to zero:

$$\lim_{t\to\infty} p(x,t) = 0.$$

Some results suggest that the motion of certain micro-organisms can be approximated by trajectories which change directions at exponentially distributed random times. Let U_k and D_k denote the random duration of the k-th time period during which the particle moves forward, with velocity c and backward, with velocity $-c$, respectively. Furthermore, $\{U_k; k = 1, 2, \ldots\}$ and $\{D_k; k = 1, 2, \ldots\}$ are mutually independent sequences of non-negative and absolutely continuous independent random variables. This leads to consider the telegraph process with alternating velocity in the case when the random alternating times U_k and D_k are exponentially distributed with parameters λ. We denote by T_k the k-th random epoch at which the motion changes velocity, $T_0 = 0$, and the amplitudes of intervals $[T_n, T_{n+1})$ constitute a sequence of non-negative random variables (see Figure 3.1). In particular,

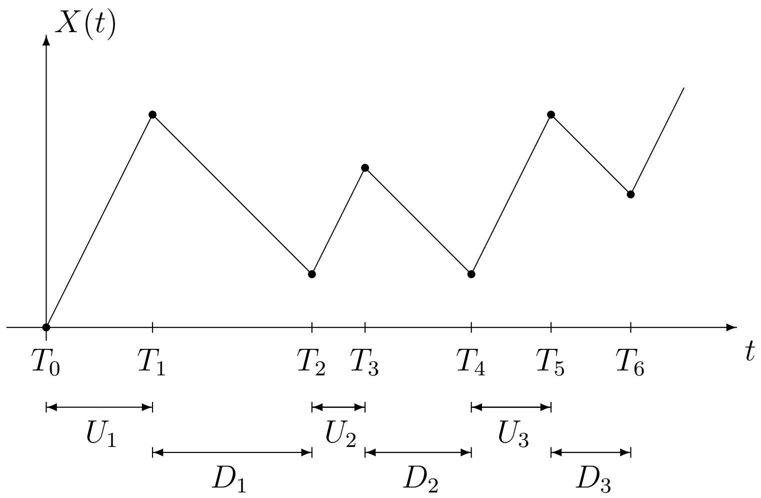


Figure 3.1: A sample-path of $X(t)$ with $V(0) = c$.

a crucial role is played by the random variables

$$U^{(k)} = U_1 + U_2 + \cdots + U_k, \qquad D^{(k)} = D_1 + D_2 + \cdots + D_k, \qquad k = 1, 2, \ldots . \tag{3.34}$$

Following an usual approach within the models of random evolution, it is possible to show hereafter that the probability densities of the motion are solution of the differential system (3.27). Conditioning on the number k of velocity reversals from $-c$ to c in $[0, t]$, and on the last instant s preceding t in which the particle changes velocity from $-c$ to c, by setting $V(0) = c$, the density $f(x, t \,|\, c)$ (resp. $b(x, t \,|\, c)$) can be determined as shown in Di Crescenzo [17] in the case of Erlang-distributed inter-renewal times.

We consider, for instance, the case in which the random times separating consecutive velocity reversals are assumed to have Erlang distribution. Hence, for $t \geq 0$, the forward density conditional on initial velocity $V(0) = c$

can be evaluated by noting that

$$f(x,t\,|\,c)\mathrm{d}x = \sum_{k=1}^{+\infty}\int_0^t P\{T_{2k}\in \mathrm{d}s, X(s)+c(t-s)=x, T_{2k+1}-T_{2k}>t-s\}.$$

Recalling $V(0)=c$, we have $T_{2k}\equiv U^{(k)}+D^{(k)}=s$ and $X(s)=cU^{(k)}-vD^{(k)}$, so that

$$\begin{aligned} f(x,t\,|\,c)\mathrm{d}x = \sum_{k=1}^{+\infty}\int_0^t & P\{U^{(k)}+D^{(k)}\in \mathrm{d}s, cU^{(k)}-cD^{(k)}=x-c(t-s),\\ & T_{2k+1}-T_{2k}>t-s\}. \end{aligned}$$

We observe that conditions $X(s)+c(t-s)=x$ and $X(s)\geq -cs$ give

$$s\geq (ct-x)/2c.$$

This implies that

$$\begin{aligned} f(x,t\,|\,c) = \frac{1}{2c}\sum_{k=1}^{+\infty}\int_{\frac{ct-x}{2c}}^t & f_U^{(k)}\left(s-\frac{ct-x}{2c}\right)\cdot f_D^{(k)}\left(\frac{ct-x}{c+v}\right)\\ & \times\, P\{T_{2k+1}-T_{2k}>t-s\}\mathrm{d}s, \end{aligned}$$

where $f_{U^{(k)}}$ and $f_{D^{(k)}}$ are Erlang densities of $U^{(k)}$ and $D^{(k)}$, and $P\{T_{2k+1}-T_{2k}>t-s\}=\mathrm{e}^{\lambda(t-s)}$. After some calculations we obtain the expression of the forward density in terms of Bessel function:

$$\begin{aligned} f(x,t\,|\,c) &= \frac{e^{-\lambda t}}{2c}\sum_{k=1}^{+\infty}\frac{\lambda^{2k}}{[(k-1)!]^2}\left(\frac{ct-x}{2c}\right)^{k-1}\int_{\frac{ct-x}{2c}}^t\left(s-\frac{ct-x}{2c}\right)^{k-1}\mathrm{d}s\\ &= \frac{e^{-\lambda t}}{ct-x}\sum_{k=1}^{+\infty}\frac{1}{k[(k-1)!]^2}\left[\frac{\lambda\sqrt{c^2t^2-x^2}}{2c}\right]^{2k}\\ &= \frac{e^{-\lambda t}}{ct-x}\left(\frac{\lambda\sqrt{c^2t^2-x^2}}{2c}\right)^2\sum_{s=0}^{+\infty}\frac{1}{(s+1)[(s)!]^2}\left[\frac{\lambda\sqrt{c^2t^2-x^2}}{2c}\right]^{2s}\\ &= \frac{e^{-\lambda t}}{2c}\left(\lambda\sqrt{\frac{ct+x}{ct-x}}\right)I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right). \end{aligned}$$

Finally, we have the explicit expression of the forward density conditional on initial velocity[1] $V(0) = c$:

$$f(x,t\,|\,c) = \frac{e^{-\lambda\, t}}{2c}\left\{\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) - c\frac{\partial}{\partial x}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right\}.$$

3.5 Moment generating function

In the following proposition (see Section 2 of Di Crescenzo and Martinucci [23], for instance) we give the moment generating function of the integrated telegraph process.

Proposition 3.5.1. *For all $s \in \mathbb{R}$ and $t \geq 0$ the moment generating function of $X(t)$ is*

$$\begin{aligned} M(s,t) &:= \mathrm{E}\left[\mathrm{e}^{s\,X(t)}\right] \\ &= \mathrm{e}^{-\lambda\, t}\left[\cosh\left(t\sqrt{\lambda^2+s^2c^2}\right) + \frac{\lambda}{\sqrt{\lambda^2+s^2c^2}}\sinh\left(t\sqrt{\lambda^2+s^2c^2}\right)\right]. \end{aligned} \tag{3.35}$$

Proof. From Eqs. (3.24) and (3.30) it follows

$$M(s,t) = \frac{\mathrm{e}^{-\lambda t}}{2}\left(\mathrm{e}^{sct}+\mathrm{e}^{-sct}\right) \tag{3.36}$$

$$\begin{aligned} &+\frac{\mathrm{e}^{-\lambda t}}{2c}\int_{-ct}^{ct}\mathrm{e}^{sx}\left[\lambda\, I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right]\mathrm{d}x \\ &= \frac{\mathrm{e}^{-\lambda t}}{2c}\left[\lambda\, Q(s,t) + \frac{\partial}{\partial t}Q(s,t)\right], \qquad s \in \mathbb{R},\ t \geq 0, \end{aligned} \tag{3.37}$$

[1] We need to know that

$$I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) = -\frac{c\sqrt{c^2t^2-x^2}}{\lambda x}\frac{\partial}{\partial x}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right),$$

$$I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) = \frac{\sqrt{c^2t^2-x^2}}{\lambda ct}\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right).$$

where we have set

$$Q(s,t) := \int_{-ct}^{ct} \mathrm{e}^{sx}\, I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \mathrm{d}x.$$

Making use of Eq. (25), from Orsingher [55] we obtain:

$$\int_{-ct}^{ct} \mathrm{e}^{sx} \frac{\partial^2}{\partial t^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \mathrm{d}x = \int_{-ct}^{ct} \mathrm{e}^{sx} c^2 \frac{\partial^2}{\partial x^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \mathrm{d}x + \lambda^2\, Q(s,t).$$

A two-fold integration by parts shows that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} Q(s,t) = (\lambda^2 + s^2c^2)\, Q(s,t).$$

Solving this equation with initial conditions $Q(s,0)=0$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q(s,t)\big|_{t=0} = 2c$ we have:

$$Q(s,t) = \frac{c}{\sqrt{\lambda^2+s^2c^2}} \left[\mathrm{e}^{t\sqrt{\lambda^2+s^2c^2}} - \mathrm{e}^{-t\sqrt{\lambda^2+s^2c^2}}\right], \qquad s\in\mathbb{R},\ t\geq 0.$$

Using this formula in the right-hand-side of (3.37), expression (3.35) finally follows. □

It should be noticed that (3.15) could also be obtained from the initial-value problem for the telegraph equation

$$\begin{cases} \frac{\partial^2}{\partial t^2}p + 2\lambda\frac{\partial}{\partial t}p = c^2\frac{\partial^2}{\partial x^2}p \\ \\ p(x,0) = \delta(x) \\ \\ \frac{\partial}{\partial t}p(x,t)\Big|_{t=0} = 0, \end{cases} \tag{3.38}$$

where $\delta(x)$ is the Dirac delta function. Indeed, M is solution of

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2}M + 2\lambda\frac{\mathrm{d}}{\mathrm{d}t}M = s^2c^2M \\ \\ M(s,0) = 1 \\ \\ \frac{\partial}{\partial t}M(s,t)\Big|_{t=0} = 0. \end{cases}$$

CHAPTER 4

TELEGRAPH PROCESS WITH UNDERLYING RANDOM WALK

4.1 Introduction

The interest for the (integrated) telegraph process as a realistic model of random motion (see Goldstein [39] and Bartlett [5]) is increased with the years and many important features are investigated from several authors since the 1950s. Such process describes a motion of a particle on the real line characterized by constant speed, the direction being reversed at the random epochs of a Poisson process. The probability density of the particle position satisfies a hyperbolic differential equation, whose probabilistic properties have been studied for example by Orsingher [55], Orsingher [56], Foong and Kanno [34], and more recently by Beghin *et al.* [6]. Other authors proposed one-dimensional generalizations of the telegraph process, where the intertimes between two consecutive changes of direction have more general distributions. We recall the papers by Di Crescenzo [17] for the case of Erlang distribution, Di Crescenzo and Martinucci [24] for that of gamma distribution, Di Crescenzo and Martinucci [27] for the case of exponential distribution with linearly increasing rate. Moreover, Stadje and Zacks [72] studied a telegraph process with random velocities, as well as De Gregorio [13] investigated a motion with finite random velocities that randomly change when a Poissonian event occurs. A generalized telegraph process governed by an alternating renewal process was analyzed by Zacks [77]. The time-fractional telegraph equation was studied by Orsingher and Beghin [59], which in a special case leads to the distribution of a telegraph process with Brownian time. An inhomogeneous telegraph process is investigated in Iacus [41], giving a rare example where an explicit law of the process has been obtained.

The aim of this chapter is to consider a generalized telegraph process with underlying random walk. This model describes a two-velocity random motion performed by a particle on the real line, such that the random intertimes separating consecutive time epochs have a general distribution. Moreover, differently from the classical telegraph process, whose positive and negative velocities alternate, we assume that at each time epoch the new velocity is determined by the outcome of a Bernoulli trial.

Hence, in the following Section 4.2, we present the description of the mathematical model of the motion giving the formal expression of the position and velocity of the particle at time t, with $t \geq 0$. Then, in Section 4.3 we study the position and the velocity of the particle. More precisely, by exploiting a suitable renewal-based procedure we obtain the general form of the probability law of stochastic process $\{(X(t), V(t)); t \geq 0\}$. Our attention is also devoted to the means of the motion conditional on the initial velocity. Specific choices of the distribution of the random intertimes U_k and D_k, with $k = 0, 1, 2, \ldots$, yield the probability law and the conditional mean of the motion. Those are explicitly achieved when the random intertimes are assumed to have exponential distributions with constant rates, in Section 4.4, and with increasing linear rates, in Section 4.5. We point out that the latter case produces a kind of damped motion, which in a special case exhibits a truncated logistic density, yielding a logistic stationary density.

4.2 THE STOCHASTIC MODEL

Let $\{(X(t), V(t)); t \geq 0\}$ be a continuous-time stochastic process, where $X(t)$ and $V(t)$ denote respectively position and velocity at time t of the moving particle. The motion is characterized by two velocities, c and $-v$, with $c, v > 0$, the direction of the motion being specified by the sign of the velocity. At time $T_0 = 0$ the particle starts from the origin, thus $X(0) = 0$. The initial velocity $V(0)$ is determined by a Bernoulli trial, such that

$$P\{V(0) = c\} = p, \qquad P\{V(0) = -v\} = 1 - p, \tag{4.1}$$

for $0 < p < 1$. At the random time $T_1 > 0$ the particle is subject to an event, whose effect possibly changes the velocity according to a Bernoulli trial independent from the previous one. This behavior is repeated ciclically at every instant of a sequence of random epochs $T_1 < T_2 < T_3 < \cdots$. We assume that the durations of time intervals $[T_n, T_{n+1})$, $n = 0, 1, 2, \ldots$, constitute a sequence of non-negative random variables. Precisely, let U_k and D_k denote the random duration of the k-th time period during which the particle moves forward, with velocity c and backward, with velocity $-v$, respectively. Furthermore, $\{U_k; k = 1, 2, \ldots\}$ and $\{D_k; k = 1, 2, \ldots\}$ are mutually independent sequences of non-negative and absolutely continuous independent random variables. Denoting by Z_n the velocity of the particle during interval $[T_n, T_{n+1})$, we assume that $\{Z_n; n = 0, 1, \ldots\}$ is a sequence of

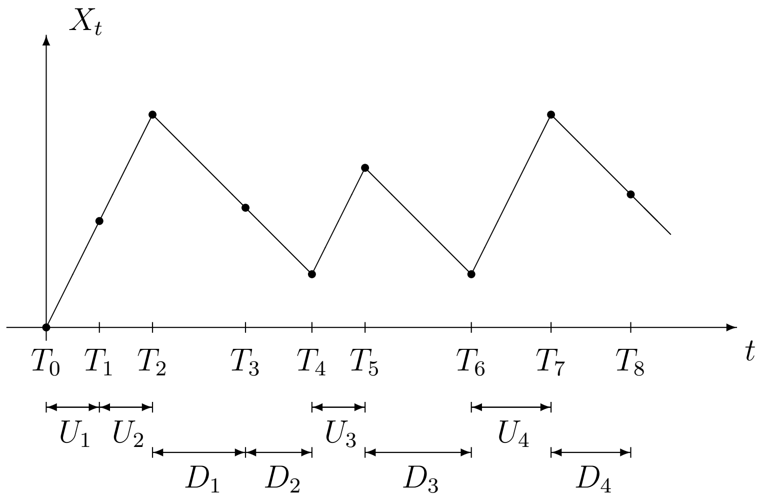


Figure 4.1: A sample-path of $X(t)$ with $V(0) = c$.

i.i.d. random variables identically distributed as the initial velocity $V(0)$ at time $t = 0$, and independent from random variables U_k and D_k. Recalling Eq. (4.1), iteratively, at the random time T_n ($n = 0, 1, 2, \dots$) we have:

$$P\{Z_n = c\} = p, \qquad P\{Z_n = -v\} = 1 - p, \qquad n = 0, 1, \dots. \tag{4.2}$$

4.2.1 Position and velocity of the particle

Position and velocity of the particle at time t can thus be formally expressed as:

$$V(t) = \sum_{k=0}^{+\infty} Z_k \,\mathbf{1}_{\{T_k \le t < T_{k+1}\}}, \qquad X(t) = \int_0^t V(s)\,\mathrm{ds}, \qquad t > 0. \tag{4.3}$$

Figure 4.1 shows a sample-path of $X(t)$, with indication of random intertimes U_k and D_k.

The novelty of the present model is the inclusion of a sequence of Bernoulli trials that regulate the velocity of the particle. This introduces an underlying (possibly asymmetric) random walk governing the random motion at epochs T_n.

Indeed, the following stochastic equation holds:

$$X_{T_{n+1}} = X_{T_n} + W_n, \qquad n = 0, 1, \dots,$$

where $\{W_n; n = 0, 1, \dots\}$ is a sequence of independent random variables such that

$$W_0 =_d \begin{cases} cU_1 & \text{if } Z_0 = c \\ -vD_1 & \text{if } Z_0 = -v, \end{cases}$$

and, for $n = 1, 2, \dots$,

$$W_n =_d \begin{cases} cU_k & \text{if } Z_n = c \text{ and } B_0 + N_{n-1} = k - 1 \\ -vD_k & \text{if } Z_n = -v \text{ and } B_0 + N_{n-1} = n - k + 1, \end{cases} \tag{4.4}$$

with '$=_d$' meaning equality in distribution. The random variables appearing in Eq. (4.4) are defined as follows: B_0 is the Bernoulli variable describing the first trial outcome, i.e. $B_0 = 1$ if $Z_0 = c$ and $B_0 = 0$ if $Z_0 = -v$; moreover N_{n-1} is the binomial variable that counts the number of Bernoulli trials yielding velocity c among the trials going from the 2-nd to the n-th one, i.e.

$$N_0 = 0, \qquad N_{n-1} = \sum_{i=1}^{n-1} \mathbf{1}_{\{Z_i = c\}} \qquad (n = 2, 3, \dots). \tag{4.5}$$

Hence, we have

$$P\{N_{n-1} = j\} = \binom{k-1}{j} p^j (1-p)^{k-1-j}, \qquad j = 0, 1, \dots, k-2. \tag{4.6}$$

Example 4.2.1. Recalling (4.6), in the case in which $k = 2$, we obtain the following probability density (see Figure 4.2)

$$P\{N_1 = j\} = \binom{1}{j} p^j (1-p)^{1-j} = 1 - p, \qquad j = 0.$$

For $k = 3$ and then, for $k = 4$, we have (see Figure 4.4 and Figure 4.3)

$$P\{N_2 = j\} = \binom{2}{j} p^j (1-p)^{2-j} = \begin{cases} (1-p)^2, & j = 0 \\ 2p(1-p), & j = 1, \end{cases}$$

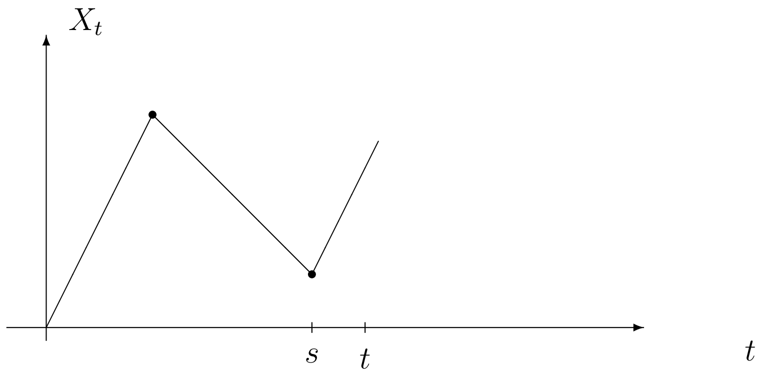


Figure 4.2: A sample-path of $X(t)$ with $k = 2$ and $j = 0$.

$$P\{N_3 = j\} = \binom{3}{j} p^j (1-p)^{3-j} = \begin{cases} (1-p)^3, & j = 0 \\ 2p(1-p)^2, & j = 1 \\ 2p^2(1-p), & j = 2. \end{cases}$$

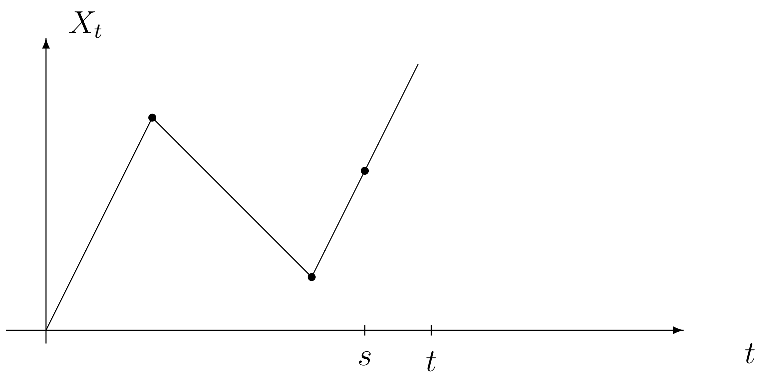


Figure 4.3: A sample-path of $X(t)$ with $k = 3$ and $j = 1$.

Another generalization of the telegraph process that includes the choice of velocities by means of random schemes have been proposed by Kolesnik [47], where the nth-order hyperbolic equation for the partial densities of the

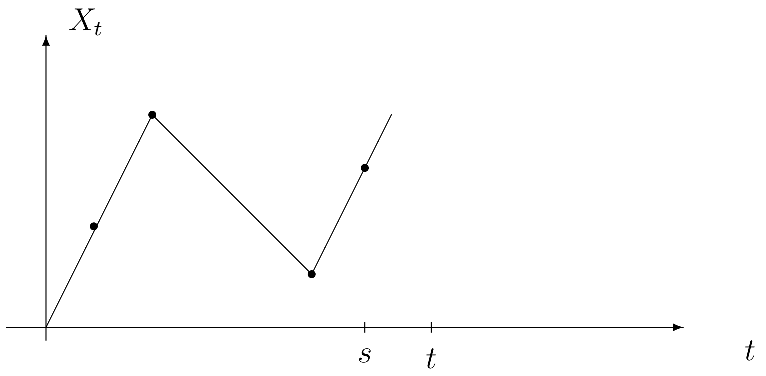


Figure 4.4: A sample-path of $X(t)$ with $k = 4$ and $j = 2$.

particle position is obtained. See also the paper by Orsingher and Bassan [58], where the telegraph process is extended to the case when cyclic changes of n velocities and changes of the sign velocity are governed by two independent Poisson processes.

We remark that an example of stochastic processes describing a motion characterized by randomly chosen directions is given in Leorato and Orsingher [50]. In that paper the authors study the motion of a particle falling on a Sierpinski gasket, where the particle at each time unit can move downwards to the 2 vertices of a triangular atom or can fall on its base with probability 1/3.

Various recent articles have pinpointed the interest on stochastic models based on functionals of stationary alternating Poisson renewal processes within applied fields such as mathematical insurance and finance, reliability theory, queueing theory, mathematical biology (see, for instance, Lachal [48]). Indeed, stochastic processes characterized by upward and downward periods are often employed to describe the evolution of (i) positive incomes and payments due to claims, (ii) working periods and repair times in repairable systems, (*iii*) busy-periods and idle-periods in service stations, (*iv*) growth periods and loss periods of prices in financial markets (see e.g. Di Crescenzo and Pellerey [30]), (*v*) sequence of alternating pauses and runs in the dispersal of cells and organisms (cf. Othmer *et al.* [60] and Garcia *et al.* [35]).

4.3 General form of the probability law

At time $t \geq 0$ the particle is located in the domain $[-vt, ct]$. The probability law of $X(t)$, $t \geq 0$, thus possesses an absolutely continuous component over $(-vt, ct)$ and a discrete component on the points $-vt$ and ct. Hence, for $y \in \{-v, c\}$ and $t \geq 0$ we define the probability density of $X(t)$ conditional on initial velocity y:

$$p(x, t \,|\, y) = \frac{\partial}{\partial x} P\{X(t) \leq x \,|\, X(0) = 0, V(0) = y\}, \qquad x \in (-vt, ct). \quad (4.7)$$

This can be expressed as

$$p(x, t \,|\, y) = f(x, t \,|\, y) + b(x, t \,|\, y), \quad (4.8)$$

where f and b denote the densities of particle position when the motion at time t is characterized by forward and backward velocity, respectively, i.e.

$$f(x, t \,|\, y) = \frac{\partial}{\partial x} P\{X(t) \leq x, V(t) = c \,|\, X(0) = 0, V(0) = y\}, \quad (4.9)$$

$$b(x, t \,|\, y) = \frac{\partial}{\partial x} P\{X(t) \leq x, V(t) = -v \,|\, X(0) = 0, V(0) = y\}. \quad (4.10)$$

We denote by f_{U_k} (resp. F_{U_k}, $\overline{F}_{U_k}$) and by f_{D_k} (resp. F_{D_k}, $\overline{F}_{D_k}$) the probability densities (resp. distribution functions, survival functions) of U_k and D_k, respectively.

In the following theorem we obtain the formal probability law of $(X(t), V(t))$

conditioned by $V(0) = c$. A crucial role is played by the partial sums

$$U^{(k)} = U_1 + U_2 + \cdots + U_k, \qquad D^{(k)} = D_1 + D_2 + \cdots + D_k, \qquad k = 1, 2, \dots, \tag{4.11}$$

whose densities will be denoted by $f_U^{(k)}$ and $f_D^{(k)}$, respectively.

Theorem 4.3.1. *For all $t \geq 0$ we have*

$$\begin{aligned} &P\{X(t) = ct, V(t) = c \,|\, X(0) = 0, V(0) = c\} \\ &\quad = \overline{F}_{U_1}(t) + \sum_{k=1}^{+\infty} p^k \int_0^t f_U^{(k)}(s) \overline{F}_{U_{k+1}}(t-s)\,\mathrm{d}s; \end{aligned} \tag{4.12}$$

moreover, for $-vt < x < ct$, the densities (4.9) and (4.10) are given by

$$\begin{aligned} f(x,t \,|\, c) = {} & \frac{1}{c+v} \sum_{k=2}^{+\infty} \sum_{j=0}^{k-2} \binom{k-1}{j} p^{j+1} (1-p)^{k-1-j} \\ & \times f_D^{(k-j-1)}(t-\tau_*) \int_{t-\tau_*}^t f_U^{(j+1)}(s-t+\tau_*) \overline{F}_{U_{j+2}}(t-s)\mathrm{d}s, \end{aligned} \tag{4.13}$$

$$\begin{aligned} b(x,t \,|\, c) = {} & \frac{1}{c+v} \Bigg\{ \sum_{k=1}^{+\infty} (1-p) p^{k-1} \overline{F}_{D_1}(t-\tau_*) f_U^{(k)}(\tau_*) + \sum_{k=1}^{+\infty} \sum_{j=0}^{k-2} \binom{k-1}{j} \\ & \times p^j (1-p)^{k-j} f_U^{(j+1)}(\tau_*) \int_{\tau_*}^t f_D^{(k-j-1)}(s-\tau_*) \overline{F}_{D_{k-j}}(t-s)\mathrm{d}s \Bigg\}, \end{aligned} \tag{4.14}$$

where

$$\tau_* = \tau_*(x,t) = \frac{vt + x}{c+v}. \tag{4.15}$$

Proof. Eq. (4.12) follows by conditioning on the number k of epochs T_i occurring before t and on the instant s when the last epoch T_k takes place before t, and by requiring that every Bernoulli trial occurring at epochs $T_1, T_2, \dots, T_k$ yields velocity c. Recalling (4.5) and (4.9) with $y = c$, for $t \geq 0$ and $-vt < x < ct$ we have

$$\begin{aligned} f(x,t \,|\, c)\mathrm{d}x = \sum_{k=2}^{+\infty} \int_0^t & P\{T_k \in \mathrm{d}s, Z_k = c, X(s) + c(t-s) \in \mathrm{d}x, \\ & T_{k+1} - T_k > t-s, 0 \leq N_{k-1} \leq k-2 \,|\, X(0) = 0, Z_0 = c\}. \end{aligned}$$

Conditioning on N_{k-1}, and taking into account the number of time periods during which the particle moved upward and backward, we obtain

$$\begin{aligned} f(x,t\,|\,c)\mathrm{d}x = \sum_{k=2}^{+\infty}\sum_{j=0}^{k-2}\int_0^t & P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s,\\ & cU^{(j+1)}-vD^{(k-j-1)}+c(t-s)\in \mathrm{d}x\}P\{Z_k=c\}\\ & \times P\{U_{j+2}>t-s\}P\{N_{k-1}=j\}. \end{aligned}$$

Note that conditions $X(s)+c(t-s)=x$ and $X(s)\geq -vs$ provide

$$s\geq (ct-x)/(c+v)\equiv t-\tau_*.$$

This inequality and the independence of $U^{(k)}$ and $D^{(k)}$ thus give

$$\begin{aligned} f(x,t\,|\,c) = \sum_{k=2}^{+\infty}\sum_{j=0}^{k-2}\int_{t-\tau_*}^t & h(s,x-c(t-s))P\{Z_k=c\}\\ & \times P\{U_{j+2}>t-s\}P\{N_{k-1}=j\}\mathrm{d}s, \end{aligned}$$

where $h(\cdot,\cdot)$ is the joint probability density of

$$(U^{(j+1)}+D^{(k-j-1)},cU^{(j+1)}-vD^{(k-j-1)}). \tag{4.16}$$

Since

$$h(s,x-c(t-s))=\frac{1}{c+v}f_U^{(j+1)}\left(s-\frac{ct-x}{c+v}\right)f_D^{(k-j-1)}\left(\frac{ct-x}{c+v}\right)$$

and N_{k-1} is binomial with parameters $k-1$ and $p\in(0,1)$, we have

$$\begin{aligned} f(x,t\,|\,c) = \frac{1}{c+v}\sum_{k=2}^{+\infty}\sum_{j=0}^{k-2}\int_{t-\tau_*}^t & f_U^{(j+1)}\left(s-\frac{ct-x}{c+v}\right)f_D^{(k-j-1)}\left(\frac{ct-x}{c+v}\right)\\ \times\overline{F}_{U_{j+2}}(t-s) & \binom{k-1}{j}p^{j+1}(1-p)^{k-j-1}\mathrm{d}s. \end{aligned} \tag{4.17}$$

Eq. (4.13) thus follows directly from (4.17). Similarly, recalling (4.5) and

(4.10) with $y = c$, for $t \geq 0$ and $-vt < x < ct$ we have

$$
\begin{aligned}
b(x,t\,|\,c)\mathrm{d}x = \frac{1}{c+v}\sum_{k=1}^{+\infty} P\Big\{ & T_k = \frac{vt+x}{c+v}, Z_k = -v, T_{k+1} - T_k > t - \frac{vt+x}{c+v}, \\
& N_{k-1} = k-1 \,|\, X(0) = 0, Z_0 = c \Big\} \\
+ \sum_{k=1}^{+\infty} \int_0^t & P\{T_k \in \mathrm{d}s, Z_k = -v, X(s) - v(t-s) \in \mathrm{d}x, \\
& T_{k+1} - T_k > t-s, 0 \leq N_{k-2} \leq k-2 \,|\, X(0) = 0, Z_0 = c\}.
\end{aligned}
$$

Hence, we obtain that

$$
\begin{aligned}
b(x,t\,|\,c)\mathrm{d}x = {} & \frac{1}{c+v}\sum_{k=1}^{+\infty}(1-p)p^{k-1} f_U^{(k)}(\tau_*)\overline{F}_{D_1}(t-\tau_*) \\
& +(1-p)\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\int_0^t P\{U^{(j+1)} + D^{(k-j-1)} \in \mathrm{d}s, \\
& cU^{(j+1)} - vD^{(k-j-1)} - v(t-s) \in \mathrm{d}x\} \\
& \times P\{D_{k-j} > t-s\} P\{N_{k-1} = j\}.
\end{aligned}
$$

Note that conditions $X(s) - v(t-s) = x$ and $X(s) \geq cs$ provide

$$
s \geq (vt+x)/(c+v) \equiv \tau_*.
$$

By recalling Eq. (4.16) we have

$$
\begin{aligned}
b(x,t\,|\,c)\mathrm{d}x = {} & \frac{1}{c+v}\Bigg\{\sum_{k=1}^{+\infty}(1-p)p^{k-1} f_U^{(k)}(\tau_*)\overline{F}_{D_1}(t-\tau_*) \\
& +\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\int_{\tau_*}^t f_U^{(j+1)}(\tau_*) f_D^{(k-j-1)}(s-\tau_*) \\
& \times \overline{F}_{U_{k-j}}(t-s)\binom{k-1}{j} p^j (1-p)^{k-j}\mathrm{d}s\Bigg\}. \qquad (4.18)
\end{aligned}
$$

Finally, Eq. (4.14) thus follows directly from (4.18). □

Remark 4.3.2. The equations in Theorem 4.3.1 are similar to some relation obtained by Masoliver *et al.* [51] based on Fourier-Laplace transforms of the transition law of continuous random walks on the real line.

Remark 4.3.3. Due to symmetry, the probability law of $(X(t), V(t))$ conditional on $V(0) = -v$ follows from Theorem 4.3.1 by interchanging f with b, U_k with D_k, c with v, x with $-x$, p with $1-p$. This would allow to evaluate the density of the particle position

$$p(x,t) := \frac{\partial}{\partial x} P\{X(t) \leq x \,|\, X(0) = 0\} = p(x,t\,|\,c)\,p + p(x,t\,|\,-v)(1-p). \tag{4.19}$$

Remark 4.3.4. Indeed, we note that, from Eq. (4.12),

$$\lim_{p \to 1^-} P\{X(t) = ct, V(t) = c \,|\, X(0) = 0, V(0) = c\} = 1.$$

It is now essential to show that the probability mass of $X(t)$ is unity over $[-vt, ct]$.

Proposition 4.3.5. *For all $t \geq 0$ we have*

$$P\{-vt \leq X(t) \leq ct \,|\, X(0) = 0, V(0) = c\} = 1, \tag{4.20}$$

$$P\{-vt \leq X(t) \leq ct \,|\, X(0) = 0, V(0) = -v\} = 1. \tag{4.21}$$

Proof. By setting $z = (vt + x)/(c + v)$, from (4.13) and (4.15) we obtain

$$\begin{aligned}
&\int_{-vt}^{ct} f(x,t\,|\,c)\mathrm{d}x = \frac{1}{c+v} \sum_{k=2}^{+\infty} \sum_{j=0}^{k-2} \binom{k-1}{j} p^{j+1}(1-p)^{k-j-1} \\
&\times \int_{-vt}^{ct} \mathrm{d}x \int_{t-\tau_*}^{t} f_U^{(j+1)}\left(s - t + \frac{vt+x}{c+v}\right) f_D^{(j+1)}\left(t - \frac{vt+x}{c+v}\right) \overline{F}_{U_{j+2}}(t-s)\mathrm{d}s \\
&= \sum_{k=2}^{+\infty} \sum_{j=0}^{k-2} \binom{k-1}{j} p^{j+1}(1-p)^{k-j-1} \\
&\times \int_0^t \mathrm{d}z \int_{t-z}^{t} f_U^{(j+1)}(s-t+z)\, f_D^{(k-j-1)}(t-z)\overline{F}_{U_{j+2}}(t-s)\mathrm{d}s
\end{aligned}$$

so that, by Fubini's theorem,

$$\int_{-vt}^{ct} f(x,t\,|\,c)\mathrm{d}x = \sum_{k=2}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}p^{j+1}(1-p)^{k-j-1} \times \int_0^t \overline{F}_{U_{j+2}}(t-s)P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s\}. \quad (4.22)$$

Similarly, from (4.14) we have

$$\int_{-vt}^{ct} b(x,t\,|\,c)\mathrm{d}x = \sum_{k=1}^{+\infty}(1-p)p^{k-1}\int_0^t \overline{F}_{D_1}(t-s)f_U^{(k)}(z)$$
$$+\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}p^j(1-p)^{k-j}\int_0^t f_U^{(j+1)}(s-z)\int_z^t f_D^{(k-j-1)}(t-z)\overline{F}_{D_{k-j}}(t-s)\mathrm{d}s\mathrm{d}z$$
$$=\sum_{k=1}^{+\infty}(1-p)p^{k-1}\int_0^t \overline{F}_{D_1}(t-s)P\{U^{(k)}\in \mathrm{d}s\}+\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}p^j(1-p)^{k-j}$$
$$\times\int_0^t \overline{F}_{D_{k-j}}(t-s)P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s\}$$
$$=\sum_{k=1}^{+\infty}\sum_{j=0}^{k-1}\binom{k-1}{j}p^j(1-p)^{k-j}\int_0^t \overline{F}_{D_{k-j}}(t-s)P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s\}. \quad (4.23)$$

Making use of Eqs. (4.8), (4.12), (4.22) and (4.23), and recalling (4.5), some calculations finally yield:

$$P\{X(t)=ct, V(t)=c\,|\,X(0)=0, V(0)=c\}+\int_{-vt}^{ct}p(x,t\,|\,c)\mathrm{d}x$$
$$=\overline{F}_{U_1}(t)+p\int_0^t f_U^{(k)}(s)\overline{F}_{U_{k+1}}(t-s)\mathrm{d}s+$$
$$+\sum_{k=2}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}p^{j+1}(1-p)^{k-j-1}\int_0^t \overline{F}_{U_{j+2}}(t-s)P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s\}$$
$$+\sum_{k=1}^{+\infty}\sum_{j=0}^{k-1}\binom{k-1}{j}p^j(1-p)^{k-j}\int_0^t \overline{F}_{D_{k-j}}(t-s)P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s\}$$

$$
\begin{aligned}
&= \overline{F}_{U_1}(t) + p\sum_{k=1}^{+\infty}\sum_{j=0}^{k-1} P\{N_{k-1}=j\}\int_0^t \overline{F}_{U_{j+2}}(t-s)P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s\}\\
&+(1-p)\sum_{k=1}^{+\infty}\sum_{j=0}^{k-1} P\{N_{k-1}=j\}\int_0^t \overline{F}_{D_{k-j}}(t-s)P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s\}\\
&= \overline{F}_{U_1}(t) + \sum_{k=1}^{+\infty}\sum_{j=0}^{k-1} P\{N_{k-1}=j\}\sum_{z\in\{-v,c\}} P\{Z_k=z\}\int_0^t P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s\}\\
&\qquad \times P\{T_{k+1}-T_k>t-s\}\\
&= \overline{F}_{U_1}(t) + \sum_{k=1}^{+\infty}\sum_{j=0}^{k-1}\sum_{z\in\{-v,c\}} P\{Z_k=z\}P\{N_{k-1}=j\}\int_0^t P\{T_k\in \mathrm{d}s\}\\
&\qquad \times P\{T_{k+1}-T_k>t-s\} = P\{U_1>t\}+P\{T_1\le t\}=1,
\end{aligned}
$$

since $U_1 =_d T_1$ when $V(0)=c$. Finally, Eq. (4.21) can be obtained similarly.

□

4.3.1 The means of particle velocity and position

Hereafter, we evaluate the conditional means of he process $V(t)$ and $X(t)$ when the initial velocity is positive.

Proposition 4.3.6. *For all $t \ge 0$ we have*

$$
\begin{aligned}
\mathrm{E}[V(t)\,|\,V(0)=c] &= c\,\overline{F}_{U_1}(t)\\
&+p\,c\left\{\sum_{k=1}^{+\infty}\int_0^t f_{U_{k+1}}(s)\mathrm{d}s\int_{t-s}^t f_{T_k}(x)\mathrm{d}x+\sum_{k=1}^{+\infty}\overline{F}_{U_{k+1}}(t)F_{T_k}(t)\right\}\\
&+(1-p)(-v)\left\{\sum_{k=1}^{+\infty}\int_0^t f_{D_{k+1}}(s)\mathrm{d}s\int_{t-s}^t f_{T_k}(x)\mathrm{d}x+\sum_{k=1}^{+\infty}\overline{F}_{D_{k+1}}(t)F_{T_k}(t)\right\}
\end{aligned}
\tag{4.24}
$$

and

$$
\begin{aligned}
&\mathrm{E}[X(t)\,|\,V(0)=c] = c\int_0^t \overline{F}_{U_1}(s)\mathrm{d}s \\
&+p\,c\left\{\sum_{k=1}^{+\infty}\int_0^t \mathrm{d}z\int_0^z f_{U_{k+1}}(s)\mathrm{d}s\int_{z-s}^z f_{T_k}(x)\mathrm{d}x+\sum_{k=1}^{+\infty}\int_0^t \overline{F}_{U_{k+1}}(y)F_{T_k}(y)\mathrm{d}y\right\} \\
&+(1-p)(-v)\left\{\sum_{k=1}^{+\infty}\int_0^t \mathrm{d}z\int_0^z f_{D_{k+1}}(s)\mathrm{d}s\int_{z-s}^z f_{T_k}(x)\mathrm{d}x+\sum_{k=1}^{+\infty}\int_0^t \overline{F}_{D_{k+1}}F_{T_k}(y)\mathrm{d}y\right\}.
\end{aligned}
\tag{4.25}
$$

Proof. For every positive integer k the following equations holds:

$$
\begin{aligned}
&\mathrm{E}[Z_k\,\mathbf{1}_{\{T_k\le t<T_{k+1}\}}] = \mathrm{E}[\mathrm{E}[Z_k\,\mathbf{1}_{\{T_k\le t<T_{k+1}\}}|Z_k]] \\
&= \mathrm{E}[Z_k\,\mathbf{1}_{\{T_k\le t<T_{k+1}\}}\,|\,Z_k=c]\,p\;+\;\mathrm{E}[Z_k\,\mathbf{1}_{\{T_k\le t<T_{k+1}\}}\,|\,Z_k=-v]\,(1-p) \\
&=\; c\,p\,\mathrm{E}[\mathbf{1}_{\{T_k\le t<T_{k+1}\}}\,|\,Z_k=c]\;+\;(1-p)(-v)\,\mathrm{E}[\mathbf{1}_{\{T_k\le t<T_{k+1}\}}\,|\,Z_k=-v] \\
&=\; c\,p\,P\{T_k\le t<T_{k+1}\,|\,Z_k=c\}\;+\;(1-p)(-v)\,P\{T_k\le t<T_{k+1}\,|\,Z_k=-v\}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\mathrm{E}[Z_k\,\mathbf{1}_{\{T_k\le t<T_{k+1}\}}] =\; p\,c\int_0^\infty P\{t-s\le T_k\le t)\}f_{U_{k+1}}(s)\mathrm{d}s \\
&+(1-p)(-v)\int_0^\infty P\{t-s\le T_k\le t\}f_{D_{k+1}}(s)\mathrm{d}s \\
&=\; p\,c\left\{\int_0^t P\{t-s\le T_k\le t\}f_{U_{k+1}}(s)\mathrm{d}s\;+\int_t^\infty f_{U_{k+1}}(s)\mathrm{d}s\,P\{T_k\le t\}\right\} \\
&+(1-p)(-v)\left\{\int_0^t P\{t-s\le T_k\le t\}f_{D_{k+1}}(s)\mathrm{d}s\;+\int_t^\infty f_{D_{k+1}}(s)\mathrm{d}s\,P\{T_k\le t\}\right\} \\
&=\; p\,c\left\{\int_0^t f_{U_{k+1}}(s)\mathrm{d}s\int_{t-s}^t f_{T_k}(x)\mathrm{d}x\;+\;\overline{F}_{U_{k+1}}(t)F_{T_k}(t)\right\} \\
&+(1-p)(-v)\left\{\int_0^t f_{D_{k+1}}(s)\mathrm{d}s\int_{t-s}^t f_{T_k}(x)\mathrm{d}x\;+\;\overline{F}_{D_{k+1}}(t)F_{T_k}(t)\right\}.
\end{aligned}
$$

Eq. (4.24) thus follows by recalling the first identity of (4.3). Moreover, the conditional mean $\mathrm{E}[X(t)\,|\,V(0)=c]$ can be easily expressed making use of the second of (4.3) and Eq. (4.24). □

In the following sections we analyse two special cases arising when the random intertimes U_k and D_k have exponential distributions with constant rates and with linear increasing rates.

4.4 INTERTIMES WITH I.I.D. EXPONENTIAL DISTRIBUTIONS

Let us assume that each of the two sequences of intertimes is formed by i.i.d. exponential random variables with parameters $\lambda, \mu > 0$. Therefore, the survival functions of U_k and D_k, $k = 1, 2, \ldots$, are

$$\overline{F}_{U_k}(x) = \mathrm{e}^{-\lambda x}, \qquad \overline{F}_{D_k}(x) = \mathrm{e}^{-\mu x}, \qquad x \geq 0, \tag{4.26}$$

respectively. The above assumptions can be interpreted as follows: the particle is subject to events arriving according to a Poisson process with alternating rate, where the rate is λ if the motion is forward, and μ if the motion is backward. We notice that an example of asymmetric random walk with exponentially distributed up and down steps is studied in Boutsikas *et al.* [8]. Some simulations of $X(t)$ for the case addressed in this section are showed in Figure 4.5. They illustrate the intuitive result that the motion drifts forward or backward when p is close to 1 or close to 0, respectively.

Under the assumptions indicated in (4.26) the following theorem gives the probability law of $(X(t), V(t))$ in terms of the Gauss hypergeometric function

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}. \tag{4.27}$$

We recall that in this case the probability densities of (4.11) are of Erlang type:

$$f_U^{(k)}(x) = \frac{\lambda^k \, x^{k-1} \, \mathrm{e}^{-\lambda x}}{(k-1)!}, \qquad f_D^{(k)}(x) = \frac{\mu^k \, x^{k-1} \, \mathrm{e}^{-\mu x}}{(k-1)!}, \qquad x \geq 0. \tag{4.28}$$

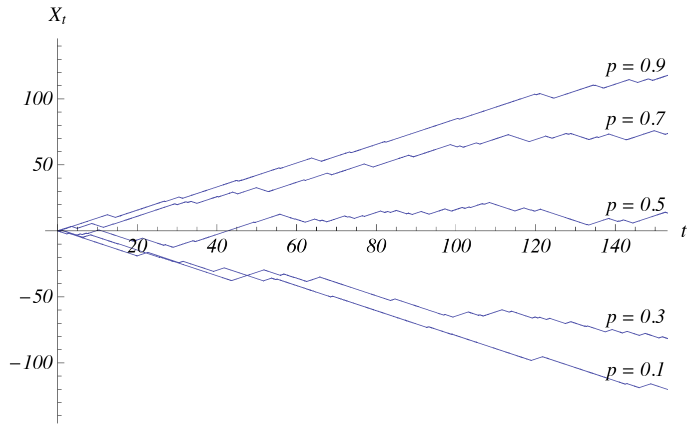


Figure 4.5: Simulated sample-paths of $X(t)$ for the exponential case (4.26), with $\lambda = \mu = c = v = 1$ and different choices of p.

Theorem 4.4.1. *Let U_k and D_k be exponentially distributed with parameter λ and μ, respectively, for $k = 1, 2, \ldots$. For all $t \geq 0$ we have*

$$P\{X(t) = ct, V(t) = c \,|\, X(0) = 0, V(0) = c\} = \mathrm{e}^{-\lambda(1-p)t}, \tag{4.29}$$

moreover, for $-vt < x < ct$,

$$\begin{aligned} f(x,t\,|\,c) &= \xi(x,t)\lambda\,\mu\,p(1-p)\,\tau_* \sum_{k=2}^{+\infty} \frac{[\mu(1-p)(t-\tau_*)]^{k-2}}{(k-2)!} \\ &\quad \times {}_2F_1\left(1-k, 2-k, 2; \frac{\lambda p \tau_*}{\mu(1-p)(t-\tau_*)}\right), \end{aligned} \tag{4.30}$$

$$\begin{aligned} b(x,t\,|\,c) &= \xi(x,t)\,\lambda(1-p) \sum_{k=1}^{+\infty} \frac{[\mu(1-p)(t-\tau_*)]^{k-1}}{(k-1)!} \\ &\quad \times {}_2F_1\left(1-k, 1-k, 1; \frac{\lambda p \tau_*}{\mu(1-p)(t-\tau_*)}\right), \end{aligned} \tag{4.31}$$

where τ_* is defined in (4.15) and

$$\xi(x,t) = \frac{1}{c+v} \exp\left\{-\lambda\tau_* - \mu(t-\tau_*)\right\}. \tag{4.32}$$

Proof. Due to (4.26) and (4.28), from Eq. (4.12), for $t \geq 0$ using direct calculations

$$\begin{aligned}
P\{X(t) = ct, V(t) = c | X(0) = 0, V(0) = c\} &= \mathrm{e}^{-\lambda t} \\
&+ \sum_{k=1}^{+\infty} p^k \int_0^t \frac{\lambda^k s^{k-1} \mathrm{e}^{-\lambda s} \mathrm{e}^{-\lambda(t-s)}}{(k-1)!} \mathrm{d}s \\
&= \mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t} \sum_{k=1}^{+\infty} \frac{(p\lambda)^k}{(k-1)!} \int_0^t s^{k-1} \mathrm{d}s \\
&= \mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t} \sum_{k=1}^{+\infty} \frac{(p\lambda t)^k}{k!} \\
&= \mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t} (\mathrm{e}^{p\lambda t} - 1) = \mathrm{e}^{-\lambda(1-p)t}.
\end{aligned} \tag{4.33}$$

Eq. (4.29) can be obtained. Moreover, Eq. (4.13) gives

$$\begin{aligned}
f(x, t \,|\, c) = &\frac{1}{c+v} \sum_{k=2}^{+\infty} \sum_{j=0}^{k-2} \binom{k-1}{j} p^{j+1} (1-p)^{k-j-1} \int_{t-\tau_*}^{t} \frac{\lambda^{j+1} (s - t + \tau_*)^j}{j!} \\
&\times \mathrm{e}^{-\lambda(s-t+\tau_*)} \frac{\mu^{k-j-1} (t-\tau_*)^{k-j-2}}{(k-j-2)!} \, \mathrm{e}^{-\mu(t-\tau_*)} \mathrm{e}^{-\lambda(t-s)} \mathrm{d}s \\
= &\xi(x,t) \sum_{k=2}^{+\infty} \sum_{j=0}^{k-2} \binom{k-1}{j} p^{j+1} (1-p)^{k-j-1} \\
&\times \int_{t-\tau_*}^{t} \frac{\lambda^{j+1} (s-t+\tau_*)^j}{j!} \frac{\mu^{k-j-1} (t-\tau_*)^{k-j-2}}{(k-j-2)!} \, \mathrm{d}s.
\end{aligned}$$

By making use of Eq. 15.4.1 of Abramowitz and Stegun [1] and recalling (4.27) Eq. (4.30) follows from

$$\begin{aligned}
f(x, t \,|\, c) = \xi(x,t) &\sum_{k=2}^{+\infty} \sum_{j=0}^{k-2} \binom{k-1}{j} p^{j+1} (1-p)^{k-j-1} \\
&\times \frac{\lambda^{j+1}}{j!} \, \frac{\tau_*^{j+1}}{j+1} \, \frac{\mu^{k-j-1} (t-\tau_*)^{k-j-2}}{(k-j-2)!}.
\end{aligned}$$

Similarly, from density (4.14) we have

$$
\begin{aligned}
b(x,t\,|\,c) &= \frac{1}{c+v}\Bigg\{\sum_{k=1}^{+\infty}(1-p)p^{k-1}\mathrm{e}^{-\mu(t-\tau_*)}\,\frac{\lambda^k {\tau_*}^{k-1}\mathrm{e}^{-\lambda\tau_*}}{(k-1)!}\\
&\quad+\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}p^j(1-p)^{k-j}\\
&\quad\times\frac{\lambda^{j+1}\tau_*^j\mathrm{e}^{-\lambda\tau_*}}{j!}\int_{\tau_*}^{t}\frac{\mu^{k-j-1}(s-\tau_*)^{k-j-2}\mathrm{e}^{-\mu(s-\tau_*)}}{(k-j-2)!}\,\mathrm{e}^{-\mu(t-s)}\mathrm{d}s\Bigg\}\\
&= \frac{1}{c+v}\Bigg\{\sum_{k=1}^{+\infty}(1-p)p^{k-1}\mathrm{e}^{-\mu(t-\tau_*)}\,\frac{\lambda^k {\tau_*}^{k-1}\mathrm{e}^{-\lambda\tau_*}}{(k-1)!}+\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}\\
&\quad\times p^j(1-p)^{k-j}\frac{\lambda^{j+1}\tau_*^j\mathrm{e}^{-\lambda\tau_*}}{j!}\,\frac{\mu^{k-j-1}\mathrm{e}^{-\mu(t-\tau_*)}}{(k-j-2)!}\int_{\tau_*}^{t}(s-\tau_*)^{k-j-2}\mathrm{d}s\Bigg\}\\
&= \frac{1}{c+v}\Bigg\{\sum_{k=1}^{+\infty}(1-p)p^{k-1}\frac{\lambda^k {\tau_*}^{k-1}}{(k-1)!}\mathrm{e}^{-\mu(t-\tau_*)-\lambda\tau_*}+\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}\\
&\quad\times p^j(1-p)^{k-j}\frac{\lambda^{j+1}\tau_*^j}{j!}\,\frac{\mu^{k-j-1}}{(k-j-2)!}\,\frac{(t-\tau_*)^{k-j-1}}{(k-j-1)!}\mathrm{e}^{-\mu(t-\tau_*)-\lambda\tau_*}\Bigg\}.
\end{aligned}
$$

Hence, after some calculations we obtain

$$
\begin{aligned}
b(x,t\,|\,c) &= \xi(x,t)\Bigg\{\lambda(1-p)\mathrm{e}^{\lambda p\tau_*}+\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}p^j(1-p)^{k-j}\frac{\lambda^{j+1}\tau_*^j}{j!}\\
&\quad\times\frac{\mu^{k-j-1}}{(k-j-2)!}\,\frac{(t-\tau_*)^{k-j-1}}{(k-j-1)!}\Bigg\}.
\end{aligned}
$$

Finally, as before, Eq. (4.31) follows by making use of Eq. 15.4.1 of Abramowitz and Stegun [1] and recalling (4.27). □

We remark that, due to Eq. (4.8), the probability density $p(x,t\,|\,c)$ can be immediately obtained from Eqs. (4.30) and (4.31). Moreover, as specified in Remark 4.3.3, the probability law of $p(x,t\,|\,-v)$, conditional on $V(0)=-v$, can be determined via Theorem 4.4.1.

Theorem 4.4.2. *Under the assumptions of Theorem 4.4.1, for $t\geq 0$ and $k=0,1,2,\ldots$ we have*

$$
P\{X(t)=ct\,|\,X(0)=0\}=p\,\mathrm{e}^{-\lambda(1-p)t}, \tag{4.34}
$$

$$P\{X(t) = -vt \,|\, X(0) = 0\} = (1-p)\,\mathrm{e}^{-\mu p t}, \tag{4.35}$$

moreover, for $-vt < x < ct$,

$$\begin{aligned} p(x,t) &= \xi(x,t)\,\lambda\,(1-p)\big\{\mu\,p\,\tau_*\Phi(x,t) + \Psi(x,t)\big\} \\ &\quad + \eta(x,t)\,\mu\,p\big\{\lambda(1-p)(t-\tau_*)\Theta(x,t) + \Gamma(x,t)\big\}, \end{aligned} \tag{4.36}$$

where $\xi(x,t)$ *is defined in (4.32) and*

$$\eta(x,t) = \frac{1}{c+v}\exp\left\{-\lambda(t-\tau_*) - \mu\tau_*\right\},$$

and

$$\begin{aligned} \Phi(x,t) &= \sum_{k=2}^{+\infty} \frac{[\mu(1-p)(t-\tau_*)]^{k-2}}{(k-2)!}\,{}_2F_1\left(1-k, 2-k, 2; \frac{\lambda p\tau_*}{\mu(1-p)(t-\tau_*)}\right), \\ \Psi(x,t) &= \sum_{k=1}^{+\infty} \frac{[\mu(1-p)(t-\tau_*)]^{k-1}}{(k-1)!}\,{}_2F_1\left(1-k, 1-k, 1; \frac{\lambda\,p\,\tau_*}{\mu(1-p)(t-\tau_*)}\right), \\ \Theta(x,t) &= \sum_{k=2}^{+\infty} \frac{[\lambda\,p\,\tau_*]^{k-2}}{(k-2)!}\,{}_2F_1\left(1-k, 2-k, 2; \frac{\mu(1-p)(t-\tau_*)}{\lambda p\tau_*}\right), \\ \Gamma(x,t) &= \sum_{k=1}^{+\infty} \frac{[\lambda\,p\,\tau_*]^{k-1}}{(k-1)!}\,{}_2F_1\left(1-k, 1-k, 1; \frac{\mu(1-p)(t-\tau_*)}{\lambda p\tau_*}\right). \end{aligned}$$

Proof. From Eq. (4.12) by using direct calculations we obtain Eq. (4.34) and (4.35), whereas density $p(x,t)$ follows immediately from (4.19). □

Some plots of such density are shown in Figure 4.6.

4.4.1 The conditional means of $V(t)$ and $X(t)$

We evaluate the conditional means of $V(t)$ and $X(t)$ on $V(0) = c$ in this special case of exponentially distributed intertimes in terms of the confluent hypergeometric function

$${}_1F_1(a,b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{z^k}{k!}$$

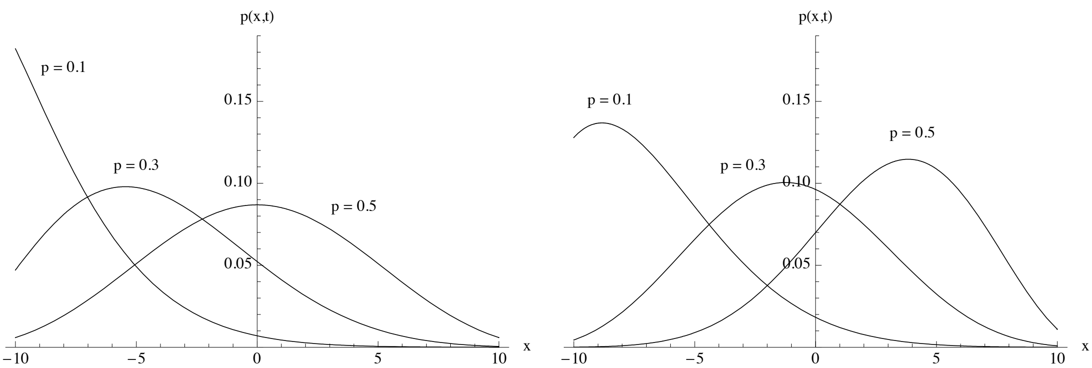


Figure 4.6: Density $p(x,t)$ for the exponential case (4.26), with $t = 10$, $\lambda = c = v = 1$, $\mu = 1$ (left) and $\mu = 2$ (right), and different choices of p.

and of the Gaussian hypergeometric function defined in (4.27).

Proposition 4.4.3. *Under the assumptions of Theorem 4.4.1, for* $t \geq 0$ *and* $k = 1, 2, \dots$ *we have*

$$\begin{aligned} \mathrm{E}[V(t)\,|\,V(0)=c] &= c\,\mathrm{e}^{-\lambda(1-p)t} + \lambda\,\mathrm{e}^{-\lambda t} \\ &\times\Bigg\{p\,c\sum_{k=2}^{+\infty}\frac{t^k}{k!}\sum_{j=0}^{k-2}\binom{k-1}{j}(\lambda p)^j[\mu(1-p)]^{k-j-1}\,{}_1F_1(k-j-1,k+1;(\lambda-\mu)t) \\ &+(1-p)(-v)\sum_{k=1}^{+\infty}\frac{t^k}{k!}\sum_{j=0}^{k-1}\binom{k-1}{j}(\lambda p)^j[\mu(1-p)]^{k-j-1}\,{}_1F_1(k-j,k+1;(\lambda-\mu)t)\Bigg\}, \end{aligned} \tag{4.37}$$

and

$$
\begin{aligned}
\mathrm{E}[X(t)\,|\,V_0=c] &= c\left(\frac{1-\mathrm{e}^{-\lambda(1-p)t}}{\lambda(1-p)}\right)+\lambda\,\mathrm{e}^{-\lambda t}\sum_{r=0}^{+\infty}(\lambda-\mu)^r \\
&\times\Bigg\{p\,c\sum_{k=2}^{\infty}\binom{k+r-2}{r}[\mu(1-p)]^{k-1}\,\frac{t^{k+r+1}}{(k+r+1)!} \\
&\times{}_1F_1\left(1,k+r+2;\lambda\,t\right)\,{}_2F_1\left(1-k,2-k,2-k-r;\frac{-\lambda p}{\mu(1-p)}\right) \\
&+(1-p)(-v)\sum_{k=1}^{\infty}\binom{k+r-1}{r}[\mu(1-p)]^{k-1}\,\frac{t^{k+r+1}}{(k+r+1)!} \\
&\times{}_1F_1\left(1,k+r+2;\lambda\,t\right)\,{}_2F_1\left(1-k,1-k,1-k-r;\frac{-\lambda p}{\mu(1-p)}\right)\Bigg\}.
\end{aligned}
\tag{4.38}
$$

Proof. Recalling (4.5), for the probability density of random epoch T_k, with $k=1,2,\ldots$, we have

$$
\begin{aligned}
f_{T_k}(s)\mathrm{d}s &= \sum_{j=0}^{k-1}P\{T_k\in \mathrm{d}s, N_{k-1}=j\} \\
&= \sum_{j=0}^{k-1}P\{U^{(j+1)}+D^{(k-j-1)}\in \mathrm{d}s\}\,P\{N_{k-1}=j\}.
\end{aligned}
$$

Thus implies, recalling (4.28), that

$$
\begin{aligned}
f_{T_k}(s) &= \sum_{j=0}^{k-1}\binom{k-1}{j}p^j(1-p)^{k-j-1}\int_0^s f_U^{(j+1)}(x)\,f_D^{(k-j-1)}(s-x)\mathrm{d}x \\
&= \frac{\lambda\,\mathrm{e}^{-\mu s}s^{k-1}}{(k-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}(\lambda\,p)^j[\mu(1-p)]^{k-j-1}{}_1F_1\left(j+1,k;(\mu-\lambda)s\right).
\end{aligned}
\tag{4.39}
$$

From Eq. (4.24), making use of (4.39), of Eq. 7.613.1 of Gradshteyn and Ryzhik [40] and of Eq. 13.1.27 of Abramowitz and Stegun [1], after some calculations we have

$$
\begin{aligned}
&\mathrm{E}[V(t)\,|\,V(0)=c] = c\,\mathrm{e}^{-\lambda t}\\
&\quad + p\,c\,\mathrm{e}^{-\lambda t}\sum_{k=1}^{\infty}\int_0^t \mathrm{e}^{-\lambda x} f_{T_k}(x)\mathrm{d}x \;+\; (1-p)(-v)\,\mathrm{e}^{-\mu t}\sum_{k=1}^{\infty}\int_0^t \mathrm{e}^{-\mu x} f_{T_k}(x)\mathrm{d}x\\
&= c\,\mathrm{e}^{-\lambda t} + \,p\,c\,\mathrm{e}^{-\lambda t}\sum_{k=1}^{\infty}\frac{t^k}{k!}\sum_{j=0}^{k-1}\binom{k-1}{j}\lambda\,(\lambda\,p)^j[\mu(1-p)]^{k-j-1}\\
&\quad \times {}_1F_1\,(k-j-1, k+1; -(\mu-\lambda)t)\\
&+ (1-p)(-v)\,\mathrm{e}^{-\lambda t}\sum_{k=1}^{\infty}\frac{t^k}{k!}\sum_{j=0}^{k-1}\binom{k-1}{j}\lambda\,(\lambda\,p)^j[\mu(1-p)]^{k-j-1}\\
&\quad \times {}_1F_1\,(k-j, k+1; -(\mu-\lambda)t)\,.
\end{aligned}
$$

Hence, Eq. (4.37) immediately can be obtained. By expressing the confluent hypergeometric functions in (4.37) as series, recalling the second of (4.3) we have

$$
\begin{aligned}
&\mathrm{E}[X(t)\,|\,V(0)=c] = c\int_0^t \mathrm{e}^{-\lambda(1-p)y}\mathrm{d}y +\\
&\quad \times p\,c\sum_{r=0}^{+\infty}\frac{(\lambda-\mu)^r}{r!}\sum_{k=2}^{+\infty}\left[\int_0^t \lambda\,\mathrm{e}^{-\lambda y}\frac{y^{k+r}}{(k+r)!}\mathrm{d}y\right][\mu(1-p)]^{k-1}\,\frac{(k+r-2)!}{(k-2)!}\\
&\qquad \times {}_2F_1\left(1-k, 2-k, 2-k-r; \frac{-\lambda p}{\mu(1-p)}\right)\\
&\quad + (1-p)(-v)\sum_{r=0}^{+\infty}\frac{(\lambda-\mu)^r}{r!}\sum_{k=2}^{+\infty}\left[\int_0^t \lambda\,\mathrm{e}^{-\lambda y}\frac{y^{k+r}}{(k+r)!}\mathrm{d}y\right][\mu(1-p)]^{k-1}\\
&\qquad \times \frac{(k+r-1)!}{(k-1)!}\,{}_2F_1\left(1-k, 1-k, 1-k-r; \frac{-\lambda p}{\mu(1-p)}\right)
\end{aligned}
$$

where,

$$
\int_0^t \lambda\,\mathrm{e}^{-\lambda y}\frac{y^{k+r}}{(k+r)!}\mathrm{d}y = \lambda\,\mathrm{e}^{-\lambda t}\frac{t^{k+r+1}}{(k+r+1)!}\,{}_1F_1\,(1, k+r+2; \lambda t)\,.
$$

Finally, Eq. (4.38) can be obtained. □

Some plots of $\mathrm{E}[X(t)\,|\,V(0)=c]$ are shown in Figure 4.7 for various choices of the involved parameters.

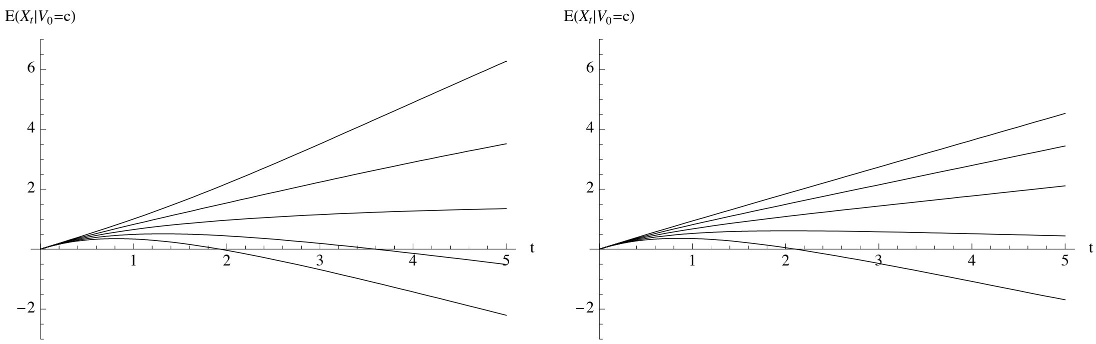


Figure 4.7: Conditional mean (4.38) on $V_0 = V(0) = c$, with $\lambda = c = v = 1$, $\mu = 1$ (left) and $\mu = 2$ (right), for $p = 0.1$, 0.3, 0.5, 0.7, 0.9 (from bottom to top).

Remark 4.4.4. Due to symmetry the conditional mean $\mathrm{E}[X(t) \,|\, V(0) = -v]$ follows from Proposition 4.4.3.

Remark 4.4.5. In the particular case when $\lambda = \mu$ and $p = \frac{1}{2}$ it is interesting to note that Eqs. (4.37) and (4.38) become

$$\mathrm{E}[V(t) \,|\, V(0) = c] = \frac{1}{2}\big\{(c-v)+(c+v)\mathrm{e}^{-\lambda t}\big\}, \quad \mathrm{E}[X(t) \,|\, V(0) = c] = \frac{t}{2}(c-v),$$

respectively (see also Eq. (39) of Di Crescenzo *et al.* [28], with $\alpha = \beta = 0$, for instance).

4.5 INTERTIMES EXPONENTIALLY DISTRIBUTED WITH LINEAR RATES

Stimulated by previous studies (see Di Crescenzo and Martinucci [27] and Di Crescenzo *et al.* [29]) involving random motions with finite velocities characterized by stochastically decreasing random intertimes. In this section we assume that the random variables U_k and D_k have exponential distribution

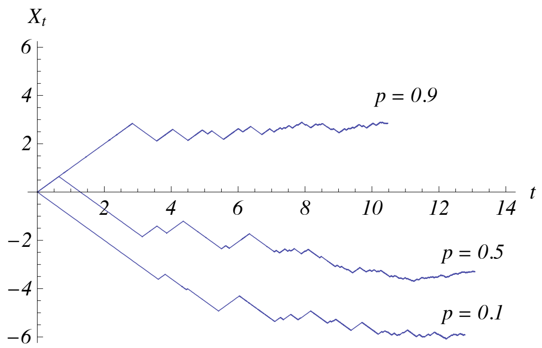


Figure 4.8: Simulated sample-paths of $X(t)$ in the exponential damped case, with $\lambda = \mu = c = v = 1$, for different choices of p.

with linear rates λk and μk. Since the parameters λk and μk are linear increasing in k, the process $X(t)$ exhibits a damped behavior, in the sense that its sample-paths are composed by line segments that become stochastically smaller and smaller. The assumption that such parameters are linear in k implies that the random times separating consecutive velocity reversals have the same distribution of the intertimes of a simple birth process (see Ricciardi [68], for instance). Hence, the survival functions are

$$\overline{F}_{U_k}(x) = \mathrm{e}^{-\lambda k\, x}, \qquad \overline{F}_{D_k}(x) = \mathrm{e}^{-\mu k\, x}, \qquad x \geq 0, \tag{4.40}$$

with $\lambda, \mu > 0$. Figure 4.8 shows some simulations of $X(t)$ in the present case, where the particle exhibits a kind of damped motion. Due to assumption (4.40), $U^{(k)}$ and $D^{(k)}$, $k \geq 1$, have generalized exponential densities

$$f_U^{(k)}(x) = k(1 - \mathrm{e}^{-\lambda x})^{k-1}\lambda \mathrm{e}^{-\lambda x}, \quad f_D^{(k)}(x) = k(1 - \mathrm{e}^{-\mu x})^{k-1}\mu \mathrm{e}^{-\mu x}, \quad x > 0. \tag{4.41}$$

Hence, $U^{(k)}$ and $D^{(k)}$ are distributed as the maximum of k i.i.d. random variables having exponential distributions with rates λ and μ, respectively.

By making use of Theorem 4.3.1 hereafter we obtain the conditional probability law of the process $(X(t), V(t))$.

Theorem 4.5.1. *Let U_k and D_k be esponentially distributed with rates λk and μk, $k = 1, 2, \cdots$, respectively. For all $t \geq 0$, we have*

$$P\{X(t) = ct, V(t) = c \,|\, X(0) = 0, V(0) = c\} = \frac{\mathrm{e}^{-\lambda t}}{(1-p) + p\,\mathrm{e}^{-\lambda t}}; \qquad (4.42)$$

moreover, for $-vt < x < ct$,

$$f(x, t \,|\, c) = \frac{\mu\, p(1-p)\mathrm{e}^{\mu(t+\tau_*)}(\mathrm{e}^{\lambda\tau*} - 1)}{(c+v)\,[(1-p)\mathrm{e}^{(\lambda+\mu)\tau_*} + p\,\mathrm{e}^{\mu t}]^2}, \qquad (4.43)$$

$$b(x, t \,|\, c) = \frac{\lambda\,\mathrm{e}^{-\mu t + (\lambda+\mu)\tau_*}(1-p)}{(c+v)\,[(1-p)\mathrm{e}^{\lambda\tau_*} + p]^2}\Big\{1 + (1-p)(e^{\mu\tau_*} - \mathrm{e}^{\mu t}) \times \frac{\big[p\,\mathrm{e}^{\mu t}(p - (1-p)\mathrm{e}^{2\lambda\tau_*} - 2\,p\,\mathrm{e}^{\lambda\,\tau_*}) - (1-p)\mathrm{e}^{(2\lambda+\mu)\tau_*}\big]}{[(1-p)\mathrm{e}^{(\lambda+\mu)\tau_*} + p\,\mathrm{e}^{\mu t}]^2}\Big\}, \qquad (4.44)$$

where τ_* is defined in (4.15).

Proof. Since U_1 and U_{k+1} are exponentially distributed with parameters λ and $\lambda(k+1)$, respectively, making use of the first of (4.41), Eq. (4.42) follows from (4.12). Furthermore, due to (4.40) and (4.41), from (4.13) we have

$$\begin{aligned} f(x, t \,|\, c) &= \frac{\mu}{c+v} \sum_{k=2}^{+\infty} \sum_{j=0}^{k-2} \binom{k-1}{j} p^{j+1}(1-p)^{k-1-j}(k-j-1)\mathrm{e}^{-\mu(t-\tau_*)} \\ &\times \big[1 - \mathrm{e}^{-\mu(t-\tau_*)}\big]^{k-j-2} \int_{t-\tau_*}^{t} \lambda(j+1)\mathrm{e}^{-\lambda(s-t+\tau_*)}\, \big[1 - \mathrm{e}^{-\lambda(s-t+\tau_*)}\big]^j\, \mathrm{e}^{-\lambda(t-s)(j+2)}\mathrm{d}s \\ &= \frac{\mu}{c+v}\mathrm{e}^{-\mu(t-\tau_*)-2\lambda\tau_*} \sum_{k=2}^{+\infty} \sum_{j=0}^{k-2} \binom{k-1}{j} p^{j+1}(1-p)^{k-1-j}(k-j-1) \\ &\times \big[1 - \mathrm{e}^{-\mu(t-\tau_*)}\big]^{k-j-2} (1 - \mathrm{e}^{-\lambda\tau_*})^j(\mathrm{e}^{\lambda\tau_*} - 1) \\ &= \frac{\mu}{c+v}\mathrm{e}^{-\mu(t-\tau_*)-2\lambda\tau_*}(\mathrm{e}^{\lambda\tau_*} - 1)\Big\{\frac{p(1-p)\mathrm{e}^{2\mu t + 2\lambda\tau_*}}{[(1-p)\mathrm{e}^{(\lambda+\mu)\tau_*} + p\,\mathrm{e}^{\mu t}]^2}\Big\}. \end{aligned}$$

The latter equation yields density (4.43). Similarly, from (4.14) we obtain the following expression

$$
\begin{aligned}
b(x,t\,|\,c) = {} & \frac{1}{c+v}\Bigg\{\frac{\lambda(1-p)\mathrm{e}^{-\mu t+(\lambda+\mu)\tau_*}}{\left[(1-p)\mathrm{e}^{\lambda\tau_*}+p\right]^2} \\
& +\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}p^j(1-p)^{k-j}\lambda(j+1)\mathrm{e}^{-\lambda\tau_*}(1-\mathrm{e}^{-\lambda\tau_*})^j \\
& \times\int_{t-\tau_*}^{t}\mu(k-j-1)\mathrm{e}^{-\mu(s-\tau_*)}\left[1-\mathrm{e}^{-\mu(s-\tau_*)}\right]^{k-j-2}\mathrm{e}^{-\mu(t-s)(k-j)}\mathrm{d}s\Bigg\} \\
= {} & \frac{1}{c+v}\Bigg\{\frac{\lambda(1-p)\mathrm{e}^{-\mu t+(\lambda+\mu)\tau_*}}{\left[(1-p)\mathrm{e}^{\lambda\tau_*}+p\right]^2} \\
& +\sum_{k=1}^{+\infty}\sum_{j=0}^{k-2}\binom{k-1}{j}p^j(1-p)^{k-j}\lambda(j+1)\mathrm{e}^{-\lambda\tau_*}(1-\mathrm{e}^{-\lambda\tau_*})^j \\
& \times\mathrm{e}^{\mu(t-\tau_*)}(1-\mathrm{e}^{\mu(t-\tau_*)})^{k-j-1}\Bigg\}.
\end{aligned}
$$

After some calculations we have

$$
\begin{aligned}
b(x,t\,|\,c) = {} & \frac{1}{c+v}\Bigg\{\frac{\lambda(1-p)\mathrm{e}^{-\mu t+(\lambda+\mu)\tau_*}}{\left[(1-p)\mathrm{e}^{\lambda\tau_*}+p\right]^2}+\lambda(1-p)^2\mathrm{e}^{(\lambda+\mu)\tau_*} \\
& \times\frac{(\mathrm{e}^{\mu(\tau_*-t)}-1)\left[p\mathrm{e}^{\mu t}(p-\mathrm{e}^{2\lambda\tau_*}(1-p)-2p\mathrm{e}^{\lambda\tau_*})-(1-p)\mathrm{e}^{(2\lambda+\mu)\tau_*}\right]}{\left[(1-p)\mathrm{e}^{\lambda\tau_*}+p\right]^2\left[\mathrm{e}^{(\lambda+\mu)\tau_*}(1-p)+p\mathrm{e}^{\mu\tau_*}\right]^2}\Bigg\}.
\end{aligned}
$$

Finally, Eq. (4.44) can be obtained. □

Let us now analyse the behavior of the densities (4.43) and (4.44) when x tends to the endpoints of the state space $[-vt, ct]$.

Corollary 4.5.2. *Under the assumptions of Theorem 4.5.1, for $t \geq 0$ we have*

$$
\lim_{x\downarrow -vt} f(x,t\,|\,c) = 0, \qquad \lim_{x\uparrow ct} f(x,t\,|\,c) = \frac{\mu\, p(1-p)\mathrm{e}^{-\lambda t}(1-\mathrm{e}^{-\lambda t})}{(c+v)\left[(1-p)+p\,\mathrm{e}^{-\lambda t}\right]^2},
$$

and

$$
\lim_{x\downarrow -vt} b(x,t\,|\,c) = \frac{\lambda(1-p)}{(c+v)\left[(1-p)+p\,\mathrm{e}^{\mu t}\right]},
$$

$$
\lim_{x\uparrow ct} b(x,t\,|\,c) = \frac{\lambda(1-p)\,\mathrm{e}^{-\lambda t}}{(c+v)\left[(1-p)+p\,\mathrm{e}^{-\lambda t}\right]^2}.
$$

Remark 4.5.3. By recalling (4.8), from densities (4.43) and (4.44) we obtain

$$p(x,t\,|\,c) = \frac{1-p}{c+v}\left\{\frac{\lambda(1-p)\mathrm{e}^{(\lambda+2\mu)\tau_*} + p\,\mathrm{e}^{\mu(t+\tau_*)}\left[(\lambda+\mu)\mathrm{e}^{\lambda\tau_*}-\mu\right]}{\left[(1-p)\mathrm{e}^{(\lambda+\mu)\tau_*}+p\,\mathrm{e}^{\mu t}\right]^2}\right\}. \tag{4.45}$$

Density $p(x,t\,|-v)$ can be expressed from (4.45) by interchanching x with $-x$, c with v, t_* with $t-\tau_*$, λ with μ, and p with $(1-p)$.

We are now able to find out the probability law of $X(t)$ in closed form.

Theorem 4.5.4. *Under the assumptions of Theorem 4.5.1, for all $t \geq 0$, we have*

$$P\{X(t)=ct\,|\,X(0)=0\} = \frac{p\,\mathrm{e}^{-\lambda t}}{(1-p)+p\,\mathrm{e}^{-\lambda t}}, \tag{4.46}$$

$$P\{X(t)=-vt\,|\,X(0)=0\} = \frac{(1-p)\,\mathrm{e}^{-\mu t}}{(1-p)+p\,\mathrm{e}^{-\mu t}}; \tag{4.47}$$

moreover, for $-vt < x < ct$,

$$p(x,t) = \frac{p}{1-p}\,\frac{1}{s}\,\frac{\exp\left\{(\mu-\frac{v}{s})t-\frac{1}{s}x)\right\}}{\left[1+\frac{p}{1-p}\exp\left\{(\mu-\frac{v}{s})t-\frac{1}{s}x\right\}\right]^2}, \tag{4.48}$$

where $s=(c+v)/(\lambda+\mu)$.

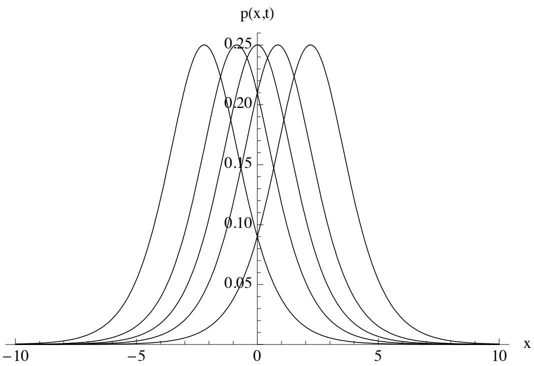
Proof. Probabilities (4.47) and () follow from (4.42) and by symmetry of the process. From (4.19) and Remark 4.5.3 we have

$$p(x,t) = \frac{p(1-p)(\lambda+\mu)\,\mathrm{e}^{\mu t+(\lambda+\mu)\tau_*}}{(c+v)[(1-p)\,\mathrm{e}^{(\lambda+\mu)\tau_*}+p\,\mathrm{e}^{\mu t}]^2}, \qquad -vt<x<ct,$$

where τ_* is defined in (4.15). Eq. (4.48) then follows. □

Plots of density (4.48) are given in Figures 4.9 and 4.10 for various choices of the parameters.

Let us now evaluate the limits of density (4.48) when x tends to $-vt$ and x to ct.



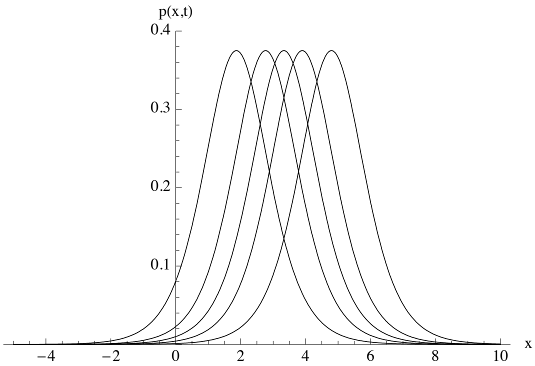


Figure 4.9: Density $p(x,t)$ for the exponential case with linear rates (4.40), with $t = 10$, $\lambda = c = v = 1$, $\mu = 1$ (left plot) and $\mu = 2$ (right plot), and $p = 0.1, 0.3, 0.5, 0.7, 0.9$ (from left to right in both plots).

Corollary 4.5.5. *From Theorem 4.5.4 we have, for $t \geq 0$,*

$$\lim_{x\downarrow -vt} p(x,t) = \frac{p}{1-p}\,\frac{1}{s}\,\frac{e^{\mu t}}{\left[1+\frac{p}{1-p}\,e^{\mu t}\right]^2}, \qquad \lim_{x\uparrow ct} p(x,t) = \frac{p}{1-p}\,\frac{1}{s}\,\frac{e^{-\lambda t}}{\left[1+\frac{p}{1-p}\,e^{-\lambda t}\right]^2}.$$

The following special case follows by straightforward calculations.

Proposition 4.5.6. *Under the assumptions of Theorem 4.5.4, if $\lambda\, v = \mu\, c$ then density (4.48) can be expressed as:*

$$p(x,t) = \frac{e^{-(x-m)/s}}{s\left[1+e^{-(x-m)/s}\right]^2}, \qquad -vt < x < ct, \tag{4.49}$$

where

$$m = s\,\ln\left(\frac{p}{1-p}\right), \qquad s = \frac{v}{\mu} = \frac{c}{\lambda}. \tag{4.50}$$

It is interesting to note that (4.49) is a truncated logistic density. Hereafter we investigate the stationary behavior of density (4.48).

Corollary 4.5.7. *Under the assumptions of Theorem 4.5.4, if $\lambda\, v = \mu\, c$ then*

$$\lim_{t\to+\infty} p(x,t) = \frac{e^{-(x-m)/s}}{s\left[1+e^{-(x-m)/s}\right]^2}, \qquad x \in \mathbb{R}, \tag{4.51}$$

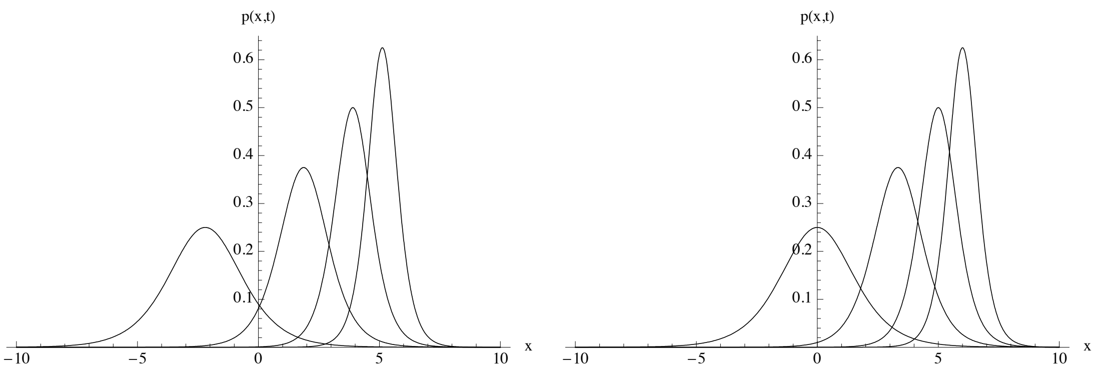


Figure 4.10: Same as Figure 4.6, for $p = 0.1$ (left plot) and $p = 0.5$ (right plot), with $\mu = 1$, 2, 3, 4 (from left to right in both plots).

where m and s are defined in (4.50); if $\lambda\, v \neq \mu\, c$ then

$$\lim_{t\to+\infty} p(x,t) = 0, \qquad x \in \mathbb{R}.$$

We note that the right-hand-side of Eq. (4.51) is a logistic density with mean m and variance $\pi^2 s^2/3$. In addition, if $p = 1/2$ then the mean m vanishes, and the density identifies with the stationary p.d.f. of a damped telegraph process, as obtained in Corollary 3.3 of Di Crescenzo and Martinucci [27].

4.5.1 The conditional means of $V(t)$ and $X(t)$

For the case under investigation, we express the conditional means of $V(t)$ and $X(t)$ in terms of hypergeometric functions

$$\tilde{F}(z) := {}_1F_1(1,2;z), \text{ and } \hat{F}(z) := {}_1F_1(2,3;z).$$

Proposition 4.5.8. *Under the assumptions of Theorem 4.5.1, for $t \geq 0$ we* *have*

$$
\begin{aligned}
&\mathrm{E}[V(t)\,|\,V(0)=c] = c\,\mathrm{e}^{-\lambda t} + (1-\mathrm{e}^{-\lambda t})\left[p\,c\,\mathrm{e}^{-2\lambda\,t} + (1-p)(-v)\mathrm{e}^{-2\mu t}\right]\\
&+\lambda\,t\sum_{k=2}^{+\infty}\sum_{j=0}^{k-1}\binom{k-1}{j}p^j(1-p)^{k-1-j}(j+1)\sum_{\ell=0}^{k-j-1}\binom{k-j-1}{\ell}(-1)^\ell\sum_{r=0}^{j}\binom{j}{r}\\
&\times(-1)^r\mathrm{e}^{-\mu\,\ell\,t}\tilde{F}((\mu\,\ell-\lambda(r+1))t)\left(p\,c\,\mathrm{e}^{-\lambda(k+1)t}+(1-p)(-v)\mathrm{e}^{-\mu(k+1)t}\right)
\end{aligned}
\tag{4.52}
$$

and

$$
\begin{aligned}
&\mathrm{E}[X(t)\,|\,V(0)=c] = \frac{c}{\lambda}(1-\mathrm{e}^{-\lambda\,t}) + p\,c\,t\left[\tilde{F}(-2\lambda)-\tilde{F}(-3\lambda\,t)\right]\\
&+(1-p)(-v)t\left[\tilde{F}(-2\mu\,t)-\tilde{F}(-(2\mu+\lambda)t)\right]\\
&+\lambda\sum_{k=2}^{\infty}\sum_{j=0}^{k-1}\binom{k-1}{j}p^j(1-p)^{k-j-1}(j+1)\sum_{\ell=0}^{k-j-1}\binom{k-j-1}{\ell}(-1)^\ell\sum_{r=0}^{j}\binom{j}{r}(-1)^r\\
&\times\Bigg\{\mathbf{1}_{\{\mu\,\ell=\lambda(r+1)\}}\frac{t^2}{2}\left[p\,c\,\hat{F}(-\left(\lambda(k+1)+\ell\,\mu\right)t)-(1-p)v\,\hat{F}(-\mu(k+\ell+1)t)\right]\\
&+\mathbf{1}_{\{\mu\,\ell\neq\lambda(r+1)\}}\left[\frac{p\,c\,t}{\mu\,\ell-\lambda(r+1)}[\tilde{F}(-\lambda\,(r+k+2)\,t)-\tilde{F}(-\left(\lambda(k+1)+\mu\,\ell\right)t)]\right]\\
&+\frac{(1-p)(-v)t}{\mu\,\ell-\lambda(r+1)}\left[\tilde{F}(-\left(\lambda(r+1)+\mu(k+1)\right)t)-\tilde{F}(-\mu(k+\ell+1)t)\right]\Bigg\}.
\end{aligned}
\tag{4.53}
$$

Proof. For every positive integer k the following equations holds:

$$
\begin{aligned}
&\mathrm{E}[Z_k\,\mathbf{1}_{\{T_k\le t<T_{k+1}\}}]\\
&=p\,c\left\{\int_0^t P\{t-s\le T_k\le t\}f_{U_{k+1}}(s)\mathrm{d}s+\int_t^\infty P\{T_k\le t\}\,f_{U_{k+1}}(s)\mathrm{d}s\right\}\\
&+(1-p)(-v)\left\{\int_0^t P\{t-s\le T_k\le t\}f_{D_{k+1}}(s)\mathrm{d}s+\int_t^\infty P\{T_k\le t\}\,f_{D_{k+1}}(s)\mathrm{d}s\right\}\\
&=p\,c\left\{\int_0^t f_{U_{k+1}}(s)\mathrm{d}s\int_{t-s}^t f_{T_k}(x)\mathrm{d}x+\overline{F}_{U_{k+1}}(t)F_{T_k}(t)\right\}\\
&+(1-p)(-v)\left\{\int_0^t f_{D_{k+1}}(s)\mathrm{d}s\int_{t-s}^t f_{T_k}(x)\mathrm{d}x+\overline{F}_{D_{k+1}}(t)F_{T_k}(t)\right\}.
\end{aligned}
\tag{4.54}
$$

Recalling (4.5) and (4.41) the probability density and the distribution function of random times T_k, for $k \geq 2$ and $t > 0$ are given by

$$
\begin{aligned}
f_{T_k}(s)\mathrm{d}s &= \sum_{j=0}^{k-1} \binom{k-1}{j} p^j (1-p)^{k-j-1} \int_0^s f_U^{(j+1)}(x)\, f_D^{(k-j-1)}(s-x)\mathrm{d}x \\
&= \sum_{j=0}^{k-2} \binom{k-1}{j} p^j (1-p)^{k-1-j} \Big\{ \lambda\,\mu\, s (j+1)(k-1-j) \sum_{\ell=0}^{k-j-2} \binom{k-j-2}{\ell} (-1)^{\ell} \\
&\times \sum_{r=0}^{j} \binom{j}{r} (-1)^r \,\mathrm{e}^{-\mu(\ell+1)s} \tilde{F}(\mu(\ell+1) - \lambda(r+1))s) \Big\} \\
&+ p^{k-1} k\,\lambda\,\mathrm{e}^{-\lambda\,s}(1-\mathrm{e}^{-\lambda\,s})^{k-1}
\end{aligned}
$$

and

$$
\begin{aligned}
F_{T_k}(s) &= \sum_{j=0}^{k-1} \binom{k-1}{j} p^j (1-p)^{k-1-j} \int_0^s f_U^{(j+1)}(x)\, F_D^{(k-j-1)}(s-x)\mathrm{d}x \\
&= \lambda\, s \sum_{j=0}^{k-1} \binom{k-1}{j} p^j (1-p)^{k-1-j} (j+1) \sum_{\ell=0}^{k-j-1} \binom{k-j-1}{\ell} (-1)^{\ell} \sum_{r=0}^{j} \binom{j}{r} (-1)^r \\
&\times \mathrm{e}^{-\mu\,\ell\,s} \tilde{F}((\mu\,\ell - \lambda(r+1))s),
\end{aligned}
$$

respectively. By recalling Eq. (4.54), we thus obtain that

$$
\mathrm{E}[Z_k \cdot \mathbf{1}_{\{T_k \leq t < T_{k+1}\}}] = \begin{cases}
(1-\mathrm{e}^{-\lambda\,t}) \left[p\,c\,\mathrm{e}^{-2\lambda\,t} + (1-p)(-v)\mathrm{e}^{-2\mu\,t} \right], \; k=1, \\[2ex]
\lambda\,t \sum_{j=0}^{k-1} \binom{k-1}{j} p^j (1-p)^{k-1-j} (j+1) \sum_{\ell=0}^{k-j-1} \binom{k-j-1}{\ell} \\
\quad \times (-1)^{\ell} \sum_{r=0}^{j} \binom{j}{r} (-1)^r \mathrm{e}^{-\mu\,\ell\,t} \tilde{F}((\mu\,\ell - \lambda(r+1))t) \\
\quad \times \left(p\,c\,\mathrm{e}^{-\lambda(k+1)t} + (1-p)(-v)\mathrm{e}^{-\mu(k+1)t} \right), \; k \geq 2.
\end{cases} \tag{4.55}
$$

Hence, by recalling the first of (4.3) and (4.55), Eqs. (4.52) can be obtained.

Moreover, making use the second of (4.3) we have

$$
\begin{aligned}
&\mathrm{E}[X(t)\,|\,V(0)=c] = c\int_0^t \mathrm{e}^{-\lambda\,s}\mathrm{d}s\\
&+\int_0^t (1-\mathrm{e}^{-\lambda\,s})\left[p\,c\,\mathrm{e}^{-2\lambda\,s}+(1-p)(-v)\mathrm{e}^{-2\mu\,s}\right]\mathrm{d}s\\
&+\lambda\sum_{j=0}^{k-1}\binom{k-1}{j}p^j(1-p)^{k-1-j}(j+1)\sum_{\ell=0}^{k-j-1}\binom{k-j-1}{\ell}(-1)^\ell\sum_{r=0}^{j}\binom{j}{r}(-1)^r\\
&\times\int_0^t s\,\mathrm{e}^{-\mu\,\ell\,s}\tilde{F}((\mu\,\ell-\lambda(r+1))s)\left(p\,c\,\mathrm{e}^{-\lambda(k+1)s}+(1-p)(-v)\mathrm{e}^{-\mu(k+1)s}\right)\mathrm{d}s
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^t (1-\mathrm{e}^{-\lambda\,s})\left[p\,c\,\mathrm{e}^{-2\lambda\,s}+(1-p)(-v)\mathrm{e}^{-2\mu\,s}\right]\mathrm{d}s\\
&=p\,c\;t\left[\tilde{F}(-2\lambda)-\tilde{F}(-3\lambda\,t)\right]+(1-p)(-v)t\left[\tilde{F}(-2\mu\,t)-\tilde{F}(-(2\mu+\lambda)t)\right].
\end{aligned}
$$

By analyzing separately the cases $\mu\,\ell=\lambda(r+1)$ and $\mu\,\ell\neq\lambda(r+1)$ we can obtained that

$$
\begin{aligned}
&\int_0^t s\,\mathrm{e}^{-\mu\,\ell\,s}\left(p\,c\,\mathrm{e}^{-\lambda(k+1)s}+(1-p)(-v)\mathrm{e}^{-\mu(k+1)s}\right)\mathrm{d}s\\
&=\;p\,c\frac{t^2}{2}\tilde{F}(-(\lambda(k+1)+\ell\,\mu)t)+(1-p)(-v)\,\frac{t^2}{2}\tilde{F}(-(\mu\,t(k+\ell+1))
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t s\,\mathrm{e}^{-\mu\,\ell\,s}\tilde{F}((\mu\,\ell-\lambda(r+1))s)\left(p\,c\,\mathrm{e}^{-\lambda(k+1)s}+(1-p)(-v)\mathrm{e}^{-\mu(k+1)s}\right)\mathrm{d}s\\
&=\;\frac{p\,c\,t}{\mu\,\ell+\lambda(r+1)}\left\{\tilde{F}(-\lambda(r+k+2)t)-\tilde{F}(-(\lambda(k+1)+\mu\,\ell)t)\right\}\\
&+\frac{(1-p)(-v)\,t}{\mu\,\ell+\lambda(r+1)}\left\{\tilde{F}(-\lambda(k+1)+\mu(k+1)t)-\tilde{F}(-\mu(k+\ell+1)t)\right\},
\end{aligned}
$$

respectively. Finally, Eq. (4.53) follows. □

In conclusion, Figure 4.11 shows some plots of $\mathrm{E}[X(t)\,|\,V(0)=c]$.

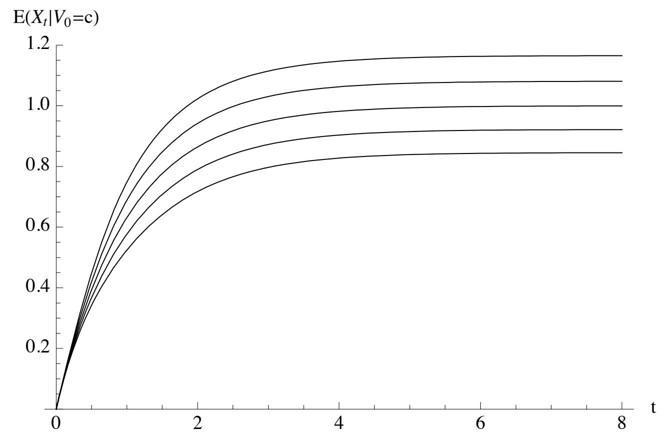


Figure 4.11: Conditional mean (4.53) on $V_0 = V(0) = c$ for $\lambda = \mu = c = v = 1$, with $p = 0.1, 0.3, 0.5, 0.7, 0.9$ (from bottom to top).

4.6 Concluding remarks

The telegraph process has attracted the attention of several mathematicians, including Bartlett [5], Kac [43], Cane [10], and Orsingher [55] just to mention a few. This process describes the motion of a particle on the real line, traveling at constant speed, whose direction is reversed at the arrival epochs of a Poisson process. Often the distribution of the process has been derived by solving Cauchy problems, defined such as in (3.38). In this chapter we have analyzed a generalized telegraph process characterized by underlying random walk. The probability law of the process has been obtained in a general form, and then in two special cases: (i) when the random times have exponential distribution with constant rates and (ii) when the intertimes have exponential distribution with increasing linear rates. In particular, the second case exhibits a kind of damped motion, which leads to a stationary logistic density. Hence, in this chapter we have applied recent results on the telegraph process to derive new properties of a more general integrated telegraph process with the inclusion of a sequence of Bernoulli trials that regulate the velocity of the particle at any epochs.

The role of the telegraph process in mathematical finance has been pinpointed in various articles, where the alternating behaviour of its sample-paths has been useful to construct suitable models. Recall, for instance, paper Di Masi *et al.* [31], where the classical Black-Scholes model is generalized to the case where the price of a stock satisfies a stochastic differential equation involving a Markov process (independent of the Wiener process) which can be viewed as the velocity process of a telegraph process. A further model was studied in Di Crescenzo *et al.* [30], where a geometric telegraph process was proposed to describe price evolutions of alternating type. Such a model has been refined in Ratanov [66] and [67], in which the inclusion of jumps in a generalized telegraph processes, occurring when the velocities are switching, allows to construct an arbitrage-free and complete market model and to obtain explicit formulas for prices of European options using perfect and quantile hedging. Other recent papers that explore properties and probability laws of generalized telegraph process with deterministic jumps and damped geometric telegraph process are Di Crescenzo *et al.* [27] and [28], respectively. An inhomogeneous telegraph process is investigated in Iacus [41], giving a rare example where an explicit law has been obtained means of statistical techniques. Moreover, in De Gregorio *et al.* [14] a parametric estimation for the standard and geometric telegraph process is observed at equidistant times in view of financial applications.

On the ground of the above mentioned researches, the future activity will be finalized to extend some of the previous results to a more general and flexible model, in which the occurrence of jumps with random amplitudes is included. Precisely, we purpose to obtain closed-form results for the probability densities of financial models based on the jump-telegraph process characterized by exponentially distributed jumps. Therefore, we aim to give a rigorous description of the process throught analitical techniques, simulation algorithms and codes, to obtain quantitative results that are expected to be of interest especially in financial contexts. Indeed, we purpose to incorporate different trends and extreme events of market evolution.

BIBLIOGRAPHY

[1] Abramowitz M. and Stegun I.A. (1992) Handbook of Mathematical Functions with Formulas, Graph, and Mathematical Tables. *Dover New York.*

[2] Anderson W.J. and McDunnough P.M. (1990) On the representation of symmetric transition functions *Adv. Appl. Prob.* **22**, 548–563.

[3] Allen L.J.S. (2003) An introduction to stochastic process with applications to biology. *Pearson Education, INC., Upper Saddle River, New Jersey*, 07458.

[4] Baccelli F. and Massey W.A. (1989) A simple path analysis of $M/M/1$ queue. *J. Appl. Prob.* **26**, 418–422.

[5] Bartlett M.S. (1957) Some problems associated with random velocity. *Publ. Inst. Statist. Univ. Paris* **6**, 261–270.

[6] Beghin L., Nieddu L. and Orsingher E. (2001) Probabilistic analysis of the telegrapher's process with drift by means of relativistic transformations. *J. Appl. Math. Stochastic Anal.* **14**, 11–25.

[7] Böhm W. and Hornik K. (2008) On two-periodic random walks with boundaries. *Research Report Series / Department of Statistics and Mathematics*, **75**. Department of Statistics and Mathematics, WU Vienna University of Economics and Business, Vienna.

[8] Boutsikas, M.V., Rakitzis, A.C. and Antzoulakos, D.L.(2011) On the relation between the distributions of stopping time and stopped sum with applications. *Preprint, arXiv:1008.0116v2.*

[9] Callaert H. and Keilson J. (1973) On exponential ergodicity and spectral structure for birth-death processes I. *North-Holland Publ. Co.*, 187–216.

[10] Cane V. (1975) Diffusion models with relativity effects. In *J.Gani, ed, Perspective in Probability and Statistics (Accademy Press Appl. Probab., Trust, Sheffield, UK)*, pag. 263–273.

[11] Conolly B. W. (1971) On randomized random walks. *SIAM Review* **13**, 81-99.

[12] Conolly B. W., Parthasarathy P. R. and Dharmaraja S. (1997) A chemical queue. *The Mathematical Scientist* **22**, 83-91.

[13] De Gregorio A. (2010) Stochastic velocity motions and processes with random time. *Adv. Appl. Prob.* **42**, 1028-1056.

[14] De Gregorio A., Iacus M.S. (2008) Parametric estimation for the standard and geometric telegraph process observed at discrete times. *Stat Infer Stoch Process*, **11**, 249-263, DOI 10:1007/s11203.007-9017-9.

[15] Di Crescenzo A. (1994) On some transformations of bilateral birth-and-death processes with applications to first passage time evaluations.

In *Sita '94-Proc. 17th Symp. Inf. Theory Appl. Hiroshima available at http://arXiv.org/pdf/0803.1413*, 739-742.

[16] Di Crescenzo A. (1998) First-passage-time densities and avoiding probabilities for birth and death processes with symmetric simple paths. *J. Appl. Prob.* **35**, 383-394.

[17] Di Crescenzo A. (2001) On random motions with velocities alternating at Erlang-distributed random times. *Adv. Appl. Prob.* **33**, 690-701.

[18] Di Crescenzo A., Giorno V., Nobile A.G. and Ricciardi L.M. (1995) On a symmetry-based constructive approach to probability densities for two-dimensional diffusion processes. *J. Appl. Prob.* **32**, 316-336.

[19] Di Crescenzo A., Giorno V., Nobile A.G. and Ricciardi L.M. (1997) On first-passage-time and transition densities for strongly symmetric diffusion processes. *Nagoya Math. J.* **145**, 143-161.

[20] Di Crescenzo A., Iuliano A., Martinucci B. (2011) On a generalized semi-Markovian telegraph process with underlying random walk. *Proceedings of Applied Stochastic Models and Data Analysis Conference*, pag. 367.

[21] Di Crescenzo A., Iuliano A., Martinucci B. (2012) On a bilateral birth-death process with alternating rates *Ricerche di Mat.*, on line first, DOI: 10.1007/s11587-011-0122-0.

[22] Di Crescenzo A., Iuliano A., Martinucci B. (2012) On a generalized telegraph process with underlying random walk. *Submitted*.

[23] Di Crescenzo A. and Martinucci B. (2006) On the effect of random alternating perturbations on hazard rates. *Sci. Math. Jpn. 64*, 381-394.

[24] Di Crescenzo A. and Martinucci B. (2007) Random motion with gamma-distributed alternating velocities in biological modeling. In *Computer Aided Systems Theory, EUROCAST 2007*; Lecture Notes in Computer Science, Vol. 4739; Moreno-Díaz, R., Pichler, F., Quesada-Arencibia, A., Eds.; Springer-Verlag: Berlin, 2007; 163-170.

[25] Di Crescenzo A. and Martinucci B. (2009) On a symmetric, non-linear birth-death process with bimodal transition probabilities. *Symmetry* **1**, 201-214.

[26] Di Crescenzo A. and Martinucci B. (2009) On a symmetry, non-linear birth-death process with bimodal transition probabilities. *Symmetry 2009.* **1**, 201-914; doi:10.3390/sym1020201.

[27] Di Crescenzo A. and Martinucci B. (2010) A damped telegraph random process with logistic stationary distribution. *J. Appl. Prob.* **47**, 84-96.

[28] Di Crescenzo A. and Martinucci B. (2012) On the generalized telegraph random process with deterministic jumps. *Meth Comput Applied Prob.*, (to appear).

[29] Di Crescenzo A., Martinucci B. and Zacks S. (2011) On the damped geometric telegrapher's process. In: *Mathematical and Statistical Methods for Actuarial Sciences and Finance, (Perna C. and Sibillo M. eds.)*, Springer: Dordrecht, p. 175-182.

[30] Di Crescenzo A. and Pellerey F. (2002) On prices' evolutions based on geometric telegrapher's process. *Appl. Stoch. Models Bus. Ind.* **18**, 171-184.

[31] Di Masi G., Kabanov Y. and Runggaldier W. (1994) Mean-variance hedging of options on stocks with Markov volatilities t. *Theory Probab. Appl.* **39**, 172-182.

[32] Erdélyi A., Magnus W., Oberhettinger F. and Tricomi F. G. (1954) Tables of Integral Transforms, Vol. I. *McGraw-Hill Publisher*, New York.

[33] Feller W. (1959) The birth-and-death processes as diffusion processes. *J.Math.Pures Appl.* **38**, 301-345.

[34] Foong S.K. and Kanno S. (1994) Properties of the telegrapher's random process with or without a trap. *Stoch. Proc. Appl.* **53**, 147-173.

[35] Garcia R., Moss F., Nihongi A., Strickler J.R., Göller S., Erdmann U., Schimansky-Geier L. and Sokolov I.M. (2007) Optimal foraging by zooplankton within patches: The case of *Daphnia*. *Math. Biosci.* **207**, 165-188.

[36] Giorno V. and Nobile A.G. (1988) On the distribution of the range of an asymmetric random walk. *Ricerche Mat.* **37**, 315-324.

[37] Giorno V., Nobile A.G. and Ricciardi L.M. (1989) A symmetry-based constructive approach to probability densities for one-dimensional diffusion processes. *J. Appl. Prob.* **26**, 707-721.

[38] Giorno V., Nobile A.G. and Ricciardi L.M. (2011) On the densities of certain bounded diffusion processes. *Ricerche Mat.*, **60**, 89-124.

[39] Goldstein S. (1951) On diffusion by discontinuous movements, and on the telegraph equation. *Quart. J. Mech. Appl. Math.* **4**, 129-156.

[40] GRADSHTEYN, I.S. AND RYZHIK, I.M.(2007) *Tables of Integrals, Series and Products*; *Academic Press: Amsterdam, 7th Ed.*

[41] IACUS S.M. (2001) Statistical analysis of the inhomogeneous telegrapher's process. *Stat. Prob. Lett.* **55**, 83-88.

[42] IULIANO A. AND MARTINUCCI B. (2010) Transient analysis of a birth-death process with alternating rates. *Cybernetics and Systems 2010 (Trappl R., ed.) Austrian Society for Cybernetic Studies, Vienna*, 187-191.

[43] KAC M. (1974) A stochastic model related to the telegrapher's equation. *Rocky Mountain J. Math.*, **4**, 497-509.

[44] KARLIN S. AND MCGREGOR J.L. (1957) The classification of bith-and-death processes *Trans. Am. Math. Soc. 86* , 366-400.

[45] KARLIN S. AND TAYLOR H.M. (1975) A first course in stochastic processes. *Second edition. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London* , xvii+557 pp.

[46] KARLIN S. AND TAYLOR H.M. (1998) An introduction to stochastic modeling. *Third edition. Academic Press, Inc., San Diego, CA, 1998* , xii+631 pp. ISBN: 0-12-684887-4.

[47] KOLESNIK A. (1998) The equations of Markovian random evolution on the line. *J. Appl. Probab.* **35**, 27-35.

[48] LACHAL A. (2006) Cyclic random motions in $\mathbb{R}^d$-space with n directions. *ESAIM Probab. Stat.* **10**, 277-316.

[49] LENTE G. (2010) The role of stochastic models in interpreting the origins of biological chirality *Symmetry.* **2**, 767-798.

[50] Leorato S. and Orsingher E. (2007) A grain of dust falling through a Sierpinski gasket. *A. Math. Sinica, English Series.* **23**, 1095-1108.

[51] Masoliver J., Lindenberg K. and Weiss G.H. (1989) A continuous–time generalization on the persistent random walk. *Phys. A* **157**, 891-898.

[52] Orsingher E. (1985) Hyperbolic equations arising in random models. *Stoch. Process. Appl.* **21**, 93-106.

[53] Orsingher E. (1986) A planar random motion governed by the two-dimensional telegraph equation *J. Appl. Probab.* **23**, 385-397.

[54] Orsingher E. (1987) Probabilistic models connected with wave equations. *Boll. Un. Mat. Ital. B(7)* **1**, 423-437.

[55] Orsingher E. (1990) Probability law, flow functions, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws. *Stoch. Process. Appl.* **34**, 49-66.

[56] Orsingher E. (1995) Motions with reflecting and absorbing barriers driven by the telegraph equation. *Random Oper. Stoch. Equat.* **1**, 9-21.

[57] Orsingher E. (1997) Elementi per il corso di calcolo delle probabiltà II. *CISU.*

[58] Orsingher E.and Bassan B. (1992) On a $2n$-valued telegraph signal and the related integrated process. *Stoch. Stoch. Rep.* **38**, 159-173.

[59] Orsingher E. and Beghin L. (2004) Time-fractional telegraph equations and telegraph processes with brownian time. *Probab. Theory Relat. Fields* **128**, 141-160.

[60] OTHMER H.G., DUNBAR S.R. AND ALT W. (1988) Models of dispersal in biological systems. *J. Math. Biol.* **26**, 263-298.

[61] PARTHASARATHY P.R. AND LENIN R.B. (2004) Birth and death process (BDP) models with applications-queueing communication systems, chemical models. Biological models: the state-of-the-art with a time-dependent perspective *American Journal of Mathematical and Management Sciences*, **24**, 1-212.

[62] POGORUI A.A. AND RODRÌGUEZ-DAGNINO R.M. (2005) One-dimensional semi-Markov evolutions with general Erlang sojourn times. *Random Operators Stoch. Equat.*, **13**, 399-405.

[63] POLLETT P.K. (2001) Similar Markov chains. *J. Appl. Prob. (Probability, Statistics and Seismology)*, **38A**, 53-65.

[64] PRUITT W. (1963) Bilateral birth and death processes. *Trans Amer Math Soc.*, **107**, 508-525.

[65] RATANOV N. (2007) Jump telegraph processes and financial markets with memory. *J. Appl. Math. Stoch. Anal, Art. ID72326*, 19pp. DOI:10.1155/2007/72326.

[66] RATANOV N. AND MELNIKOV A. (2008) On financial markets based on telegraph process. *Stochastics*, **80**, 247-268.

[67] RATANOV N. (2010) Option pricing model based on a Markov-modulated diffusion with jumps. *Brazilian Journal of Probability and Statistics,* **24**, 413-431.

[68] RICCIARDI L.M. (1986) Stochastic population theory: birth and death processes. *n Mathematical Ecology (Hallan T.G. and Levin S.A., eds.) Biomathematics. Springer* **17**, 155-190.

[69] RICCIARDI L.M. AND SATO S. (1987) On the range of a one-dimensional asymmetric random walk *Ricerche Mat.*, **36**, 153-160.

[70] ROSS S.M. (1996) Stochastic processes (Second Edition) *John Wiley and Sons. Inc.*

[71] SRIVASTAVA H.M. AND MANOCHA H.L. (1984) A treatise on generating functions *Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons*, New York.

[72] STADJE W. AND ZACKS S. (2004) Telegraph processes with random velocities. *J. Appl. Prob.* **41**, 665-678.

[73] STOCKMAYER W.H., GOBUSH W. AND NORVICH R. (1971) Local–jump models for chain dynamics. *Pure and Applied Chemistry* **26**, 555-561.

[74] TARABIA A.M.K AND EL-BAZ A.H. (2002) A new explicit solution for a chemical queue. *The Mathematical Scientist* **27**, 16-24.

[75] TARABIA A.M.K AND EL-BAZ A.H. (2002) Analysis of the busy period of the chemical queue: a series approach. *The Mathematical Scientist* **27**, 108-116.

[76] TARABIA A.M.K, TAKAGI H. AND EL-BAZ A.H. (2009) Transient solution of non–empty chemical queueing system. *Math. Meth. Oper. Res.* **70**, 78-98.

[77] Zacks S. (2004) Generalized integrated telegraph processes and the distribution of related random times. *J. Appl. Prob.* **41**, 497-507.