SIGNAL RECONSTRUCTION FROM FRAME AND SAMPLING ERASURES

A Dissertation

by

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ABSTRACT

This dissertation is concerned with the efficient reconstruction of signals from frame coefficient erasures at known locations. Three methods of perfect reconstruction from frame coefficient erasures will be discussed. These reconstructions are more efficient than older methods in the literature because they only require an $L \times L$ matrix inversion, where L denotes the cardinality of the erased set of indices. This is a significant improvement over older methods which require an $n \times n$ matrix inversion, where n denotes the dimension of the underlying Hilbert space.

The first of these methods, called *Nilpotent Bridging*, uses a small subset of the non-erased coefficients to reconstruct the erased coefficients. This subset is called the *bridge set*. To perform the reconstruction an equation, known as the *bridge equation*, must be solved. A proof is given that under a very mild assumption there exists a bridge set of size L for which the bridge equation has a solution. A stronger result is also proven that shows that for a very large class of frames, no bridge set search is required. We call this set of frames the set of *full skew-spark frames*. Using the Baire Category Theorem and tools from Matrix Theory, the set of full skew-spark frames is shown to be an open, dense subset of the set of all frames in finite dimensions.

The second method of reconstruction is called *Reduced Direct Inversion* because it provides a basis-free, closed-form formula for inverting a particular $n \times n$ matrix, which only requires the inversion of an $L \times L$ matrix. By inverting this matrix we obtain another efficient reconstruction formula.

The final method considered is a continuation of work by Han and Sun. The method utilizes an *Erasure Recovery Matrix*, which is a matrix that annihilates the range of the analysis operator for a frame. Because of this, the erased coefficients can be reconstructed

using a simple pseudo-inverse technique.

For each method, a discussion of the stability of our algorithms is presented. In particular, we present numerical experiments to investigate the effects of normally distributed additive channel noise on our reconstruction. For Reduced Direct Inversion and Erasure Recovery Matrices, using the Restricted Isometry Property, we construct classes of frames which are numerically robust to sparse channel noise. For these frames, we provide error bounds for sparse channel noise.

DEDICATION

To my parents, Dave and Sue Scholze, and to the memory of my grandmother, Lila Sackett, for their love and support, and for setting a very strong example.

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All of the work in Sections 2.1-2.3, 2.5, and 3.1 are joint work with David Larson.

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1. INTRODUCTION

1.1 Basics of Frame Theory

A *frame* for a Hilbert space, $\mathcal{H}$, is a countably indexed sequence of vectors, $F = \{f_j\}_{j\in\mathbb{J}} \subset \mathcal{H}$, for which there positive exist constants, A and B, so that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \leq B\|f\|^2. \tag{1.1}$$

The constant A is called a *lower frame bound*, and B is called an *upper frame bound*. The supremum over all lower frame bounds is called the *optimal lower frame bound*, and the infimum over all upper frame bounds is called the *optimal upper frame bound*. A frame, $F = \{f_j\}_{j\in\mathbb{J}}$, with optimal frame bounds A_0 and B_0 is *tight* if $A_0 = B_0$, and *Parseval* if $A_0 = B_0 = 1$. It is well known that a collection $\{f_j\}_{j=1}^N$ is a frame for a finite dimensional Hilbert space, $\mathcal{H}$, if and only if $\text{span}\{f_j\}_{j=1}^N = \mathcal{H}$. A collection $\{x_j\}_{j\in\mathbb{J}} \subset \mathcal{H}$ is called a *Schauder basis* for $\mathcal{H}$ if for every $x \in \mathcal{H}$, there exist unique coefficients $\{c_j\}_{j\in\mathbb{J}}$ for which

$$x = \sum_{j\in\mathbb{J}} c_j x_j.$$

(In finite dimensions, this corresponds to the conventional definition of a basis.) A frame which is also a Schauder basis is called a *Riesz basis*. A sequence $F = \{f_j\}_{j\in\mathbb{J}}$ is called a *Bessel sequence* if we can find a constant $C > 0$ so that for all $f \in \mathcal{H}$,

$$\sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \leq C\|f\|^2. \tag{1.2}$$

Thus every frame is a Bessel sequence, but not every Bessel sequence is a frame.

Remark 1.1. We will frequently use a capital letter to denote a frame whose elements are denoted by the corresponding lowercase letter. For example, we will enumerate the frame F for an n-dimensional ($n \leq \infty$) Hilbert space $\mathcal{H}$ by $\{f_j\}_{j\in\mathbb{J}}$. If $n < \infty$, the index set will be assumed to be $\mathbb{J} = \{1, 2, \cdots, N\}$. In this case, since $\{f_j\}_{j=1}^{N}$ is a spanning set, $N \geq n$.

The definition of a frame may seem a bit obscure at a first glance, however, we can think of the frame inequality (1.1) as a modified version of Parseval's inequality for orthonormal bases. With this in mind, it is reasonable to expect an expansion formula for vectors which is similar to the expansion of a vector by an orthonormal basis. Furthermore, frames are more flexible than bases because they are allowed to be redundant. Due to this redundancy, it is possible to reconstruct a signal from a frame expansion when some of its coefficient data has been erased.

In this section, we will go through some of the basics relevant to frame erasures. For a more thorough discussion of frame theory, see the books [14], [16], and [26], or the AMS memoirs [17] and [27]. To see the original paper on frames by Duffin and Schaeffer, see [19].

1.2 Operators Associated to Frames

There are several operators associated to frames. Among these are the analysis, synthesis, frame, and Gramian operators. Given a frame $F = \{f_j\}_{j\in\mathbb{J}}$ for a Hilbert space $\mathcal{H}$ and $f \in \mathcal{H}$, we call the sequence $(\langle f, f_j\rangle)_{j\in\mathbb{J}}$ the sequence of *frame coefficients* of F. The *analysis operator* $\Theta : \mathcal{H} \to \ell^2(\mathbb{J})$ maps $f \in \mathcal{H}$ to its frame coefficient sequence. That is,

$$\Theta f = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle\, e_j = (\langle f, f_j\rangle)_{j\in\mathbb{J}}\,, \tag{1.3}$$

where e_j denotes the j^{th} vector in the canonical orthonormal basis for $\ell^2(\mathbb{J})$. It is clear from equation (1.1) that if B is an upper frame bound for F, then Θ is a bounded operator with $\|\Theta\| \leq \sqrt{B}$. Notice that for $f \in \mathcal{H}$, we have

$$\langle \Theta f, e_k \rangle = \left\langle \sum_{j \in \mathbb{J}} \langle f, f_j \rangle e_j, e_k \right\rangle = \sum_{j \in \mathbb{J}} \langle f, f_j \rangle \langle e_j, e_k \rangle = \langle f, f_k \rangle .$$

Thus, $\Theta^* e_k = f_k$. Extending using linearity we define $\Theta^* : \ell^2(\mathbb{J}) \to \mathcal{H}$ by

$$\Theta^* \left(\sum_{j \in \mathbb{J}} c_j e_j \right) = \sum_{j \in \mathbb{J}} c_j f_j. \tag{1.4}$$

This operator is called the *synthesis operator* for the frame F. By composing the analysis and synthesis operators, we obtain the *frame operator* $S = \Theta^* \Theta : \mathcal{H} \to \mathcal{H}$. That is,

$$Sf = \sum_{j \in \mathbb{J}} \langle f, f_j \rangle f_j \quad \forall f \in \mathcal{H}. \tag{1.5}$$

To obtain the expansion formula we mentioned earlier, we must invert the frame operator. In order to invert the frame operator, we require the following theorem, which will also be useful later on. The series in equation (1.6) below is often called a *Neumann series* (cf. Theorem 1.2.2 in [45]). A proof is included here for the sake of completeness.

Theorem 1.2. *If T is a bounded invertible operator on a Hilbert space satisfying $\|T\| < 1$, then $I - T$ is invertible, and*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k. \tag{1.6}$$

Moreover, given $\epsilon > 0$, if $\|T\| = r < 1$, and $\kappa > \log_r(\epsilon(1-r)) - 1$, then

$$\left\| (I - T)^{-1} - \sum_{k=0}^{\kappa} T^k \right\| < \epsilon. \tag{1.7}$$

Proof. Since $\|T\| < 1$ the series $\sum_{k=0}^{\infty} T^k$ converges absolutely, and thus it converges. We have

$$(I-T)\sum_{k=0}^{\infty}T^k=\sum_{k=0}^{\infty}T^k-\sum_{k=0}^{\infty}T^{k+1}=\sum_{k=0}^{\infty}T^k-\sum_{k=1}^{\infty}T^k=I.$$

Similarly,

$$\left(\sum_{k=0}^{\infty}T^k\right)(I-T)=I.$$

Therefore, $(I-T)^{-1}=\sum_{k=0}^{\infty}T^k$.

For the moreover part, first notice that

$$\left\|(I-T)^{-1}-\sum_{k=0}^{\kappa}T^k\right\| \quad = \quad \left\|\sum_{k=\kappa+1}^{\infty}T^k\right\| \leq \sum_{k=\kappa+1}^{\infty}\|T\|^k=\sum_{k=\kappa+1}^{\infty}r^k=\frac{r^{\kappa+1}}{1-r}.$$

Thus, to guarantee that (1.7) is satisfied, it suffices to find κ for which

$$\frac{r^{\kappa+1}}{1-r}<\epsilon.$$

Taking the logarithm of base r to both sides gives

$$\kappa+1-\log_r(1-r)>\log_r(\epsilon).$$

Solving for κ yields

$$\kappa>\log_r(1-r)+\log_r(\epsilon)-1=\log_r(\epsilon(1-r))-1.$$

□

The next theorem guarantees that the frame operator is invertible, and provides a frame expansion formula (cf. Theorems 1.1.5, 5.1.5, and 5.3.4 in [16]). The proofs of these facts are included here for the sake of completeness.

Theorem 1.3. *Let $F = \{f_j\}_{j\in\mathbb{J}}$ be a frame for a Hilbert space $\mathcal{H}$ with lower frame bound A and upper frame bound B, and let S be the frame operator for F. Then,*

1. *S is a positive operator satisfying $AI \leq S \leq BI$.*

2. *S is invertible.*

3. *For all $f \in \mathcal{H}$,*
$$f = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle S^{-1} f_j = \sum_{j\in\mathbb{J}} \left\langle f, S^{-1} f_j\right\rangle f_j. \tag{1.8}$$

4. *For all $f \in \mathcal{H}$,*
$$f = \sum_{j\in\mathbb{J}} \left\langle f, S^{-\frac{1}{2}} f_j\right\rangle S^{-\frac{1}{2}} f_j. \tag{1.9}$$

Proof. To prove (1), for $f, g \in \mathcal{H}$, we have

$$S^* = (\Theta^*\Theta)^* = \Theta^*(\Theta^*)^* = \Theta^*\Theta = S.$$

Therefore, S is self-adjoint. To prove that $S \geq AI$, we have

$$\begin{aligned} \langle Sf, f\rangle &= \sum_{j\in\mathbb{J}} \langle\langle f, f_j\rangle f_j, f\rangle = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle \overline{\langle f, f_j\rangle} \\ &= \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \geq A\|f\|^2 = \langle AIf, f\rangle, \end{aligned}$$

where A denotes the lower frame bound. Similarly, $S \leq BI$.

To prove (2), since $AI \leq S \leq BI$,

$$\frac{A}{B} I \leq \frac{1}{B} S \leq I.$$

Thus,
$$I - \frac{1}{B} S \leq I - \frac{A}{B} I = \frac{B-A}{B} I.$$

So, $\left\|I - \frac{1}{B}S\right\| \leq \frac{B-A}{B} < 1$. Hence, by Theorem 1.2, $\frac{1}{B}S$ is invertible. Therefore, S is invertible.

Since S is self-adjoint, so is S^{-1}. Thus, to get the first equality in (3), we have

$$f = S(S^{-1}f) = \sum_{j \in \mathbb{J}} \left\langle S^{-1}f, f_j \right\rangle f_j = \sum_{j \in \mathbb{J}} \left\langle f, S^{-1}f_j \right\rangle f_j.$$

To prove the second equality, since S^{-1} is bounded and linear, we have

$$f = S^{-1}(Sf) = S^{-1}\left(\sum_{j \in \mathbb{J}} \langle f, f_j \rangle f_j\right) = \sum_{j \in \mathbb{J}} \langle f, f_j \rangle S^{-1}f_j.$$

To prove (4), for all $f \in \mathcal{H}$, we have

$$f = S^{-\frac{1}{2}}S(S^{-\frac{1}{2}}f) = S^{-\frac{1}{2}}\left(\sum_{j \in \mathbb{J}} \left\langle S^{-\frac{1}{2}}f, f_j \right\rangle f_j\right) = \sum_{j \in \mathbb{J}} \left\langle f, S^{-\frac{1}{2}}f_j \right\rangle S^{-\frac{1}{2}}f_j.$$

□

As we hinted at earlier, equation (1.8) gives an expansion of the vector (or signal) f which is similar to an orthonormal basis expansion. The frame $\{S^{-1}f_j\}_{j \in \mathbb{J}}$ is commonly referred to as the *canonical* or *standard dual* to F. It is easily checked that the frame $\{S^{-\frac{1}{2}}f_j\}_{j \in \mathbb{J}}$ is a Parseval frame. The following Corollary (Proposition 1.1.4 in [16]) gives a more convenient expansion formula for tight frames.

Corollary 1.4. *Assume that F is a tight frame with frame bound A. Then the frame operator is $S = AI$, and for all $f \in \mathcal{H}$,*

$$f = \frac{1}{A}\sum_{j \in \mathbb{J}} \langle f, f_j \rangle f_j. \tag{1.10}$$

In general, if F is a redundant frame, there exist many frames, G, for $\mathcal{H}$ for which

$$f = \sum_{j \in \mathbb{J}} \langle f, g_j \rangle f_j = \sum_{j \in \mathbb{J}} \langle f, f_j \rangle g_j \quad \forall f \in \mathcal{H}. \tag{1.11}$$

Any frame G which satisfies equation (1.11) is called a *dual* to F, and we will refer to (F, G) as a *dual frame pair*. For a given frame F, we will define the *dual set of* F, denoted by $\mathcal{D}(F)$, as

$$\mathcal{D}(F) = \{G : (F, G) \text{ is a dual frame pair}\}. \tag{1.12}$$

The following Proposition will be needed later on. It says that if F is a frame, G is Bessel, and equation (1.11) is satisfied, then G is also a frame.

Proposition 1.5. *Assume F is a frame for a Hilbert space $\mathcal{H}$, and G is a Bessel sequence satisfying*

$$f = \sum_{j \in \mathbb{J}} \langle f, g_j \rangle f_j \quad \forall f \in \mathcal{H}. \tag{1.13}$$

Then G is a frame.

Proof. Assume that B is an upper frame bound for F. Let $f \in \mathcal{H}$. Then,

$$\begin{aligned} \|f\|^2 &= |\langle f, f \rangle| = \left| \left\langle \sum_{j \in \mathbb{J}} \langle f, g_j \rangle f_j, f \right\rangle \right| = \left| \sum_{j \in \mathbb{J}} \langle f, g_j \rangle \langle f, f_j \rangle \right| && (1.14) \\ &\leq \left(\sum_{j \in \mathbb{J}} | \langle f, g_j \rangle |^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{J}} | \langle f, f_j \rangle |^2 \right)^{\frac{1}{2}} && (1.15) \\ &\leq \sqrt{B} \|f\| \left(\sum_{j \in \mathbb{J}} | \langle f, g_j \rangle |^2 \right)^{\frac{1}{2}}. && (1.16) \end{aligned}$$

Thus,

$$\sum_{j \in \mathbb{J}} | \langle f, g_j \rangle |^2 \geq \left(\frac{1}{\sqrt{B}} \|f\| \right)^2 = \frac{1}{B} \|f\|^2. \tag{1.17}$$

Since G is a Bessel sequence, it has an upper frame bound, and we have demonstrated that $\frac{1}{B}$ is a lower frame bound. $\square$

Remark 1.6. We will frequently use the *rank one operator* notation. For $f, g \in \mathcal{H}$, we denote by $f \otimes g : \mathcal{H} \to \mathcal{H}$ the operator defined by $(f \otimes g)(x) = \langle x, g \rangle f$ for all $x \in \mathcal{H}$. With this notation, we have

$$\begin{aligned} \Theta &= \sum_{j \in \mathbb{J}} e_j \otimes f_j, \\ \Theta^* &= \sum_{j \in \mathbb{J}} f_j \otimes e_j, \text{ and} \\ S &= \sum_{j \in \mathbb{J}} f_j \otimes f_j. \end{aligned}$$

The last operator we will introduce in this section is known as the *Gramian operator*. The Gramian is the operator $\mathcal{G} : \ell^2(\mathbb{J}) \to \ell^2(\mathbb{J})$ defined by

$$\mathcal{G} = \Theta\Theta^* = \sum_{j \in \mathbb{J}} \langle f_k, f_j \rangle \, e_j \otimes e_k. \tag{1.18}$$

The matrix representation for the Gramian with respect to the canonical basis for $\ell^2(\mathbb{J})$ is given by $\mathcal{G} = (\langle f_k, f_j \rangle)_{j,k \in \mathbb{J}}$. In matrix form this operator is commonly referred to as the *Gram matrix*. Throughout this dissertation we will be discussing minors of this matrix, and we will make several references to the following useful Lemma.

Lemma 1.7. *Assume that $\mathcal{G}$ is the Gram matrix of a frame F. Assume that Λ is a finite subset of $\mathbb{J}$, and denote by $\mathcal{G}_\Lambda$ the minor of $\mathcal{G}$ with rows and columns indexed by Λ. Then the set $\{f_j\}_{j \in \Lambda}$ is linearly independent if and only if $\mathcal{G}_\Lambda$ is invertible.*

Proof. Denote the analysis operator for the sequence $\{f_j\}_{j \in \Lambda}$ by Θ_Λ. To prove the forwards implication, we will prove the contrapositive. Assume that $\mathcal{G}_\Lambda$ is singular. Then, we can find a non-zero complex valued vector $X = (x_j)_{j\in\Lambda}$ for which

$$0 = \mathcal{G}_\Lambda X = \Theta_\Lambda \Theta_\Lambda^* X.$$

Thus,

$$0 = \langle \Theta_\Lambda \Theta_\Lambda^* X, X \rangle = \langle \Theta_\Lambda^* X, \Theta_\Lambda^* X \rangle .$$

Hence,

$$0 = \Theta_\Lambda^* X = \sum_{j\in\Lambda} x_j f_j.$$

Therefore, $\{f_j\}_{j\in\Lambda}$ is linearly dependent.

We will also prove the converse by contrapositive. Assume that $\{f_j\}_{j\in\Lambda}$ is linearly dependent. Then, we can find a non-zero complex valued vector $X = (x_j)_{j\in\Lambda}$ so that

$$\sum_{j\in\Lambda} x_j f_j = 0.$$

Thus,

$$\mathcal{G}_\Lambda X = \Theta_\Lambda \Theta_\Lambda^* X = \Theta_\Lambda \left(\sum_{j\in\Lambda} x_j f_j \right) = \Theta_\Lambda 0 = 0.$$

Therefore, $\mathcal{G}_\Lambda$ is singular. $\square$

Given a dual frame pair, (F, G) we will also be interested in minors of the *cross-Gram matrix* which we will define by

$$B = \left(\langle f_j, g_k \rangle \right)_{j,k\in\mathbb{J}} . \tag{1.19}$$

We will denote by $B(F, G, \Lambda, \Omega)$ the minor of B with rows indexed by $\Lambda \subset \mathbb{J}$, and columns indexed by $\Omega \subset \mathbb{J}$. Whenever the dual frame pair is understood, we will abbreviate this notation to $B(\Lambda, \Omega)$.

1.3 Frames for Signal and Image Processing

Frames are widely used in the analysis of signals and images due to their redundancy properties. Let (F, G) denote a dual frame pair for a Hilbert space $\mathcal{H}$. Suppose that Alice wishes to send a signal $f \in \mathcal{H}$ to Bob. To do this, Alice first encodes (or analyzes) f with respect to the *analysis frame* G to obtain the sequence $\Theta_G f = (\langle f, g_j \rangle)_{j \in \mathbb{J}}$ of frame coefficients. Alice then transmits these coefficients over some channel to Bob. On the receiving end, Bob decodes (or synthesizes) these coefficients with the *synthesis frame*, F, to obtain the original signal (via equation (1.11)). That is, Bob computes the sum

$$\Theta_F^*(\langle f, g_j \rangle)_{j \in \mathbb{J}} = \sum_{j \in \mathbb{J}} \langle f, g_j \rangle \, f_j = f.$$

1.3.1 Frame Erasures

Let (F, G) denote a dual frame pair for a Hilbert space $\mathcal{H}$. Frame erasures occur when some of the frame coefficients are erased during the transmission of a signal. We define an *erasure set*, Λ, to be a *finite* subset of $\mathbb{J}$. The following proposition discusses exactly when such a reconstruction is possible, and it provides a naive method of reconstruction from frame erasures.

Proposition 1.8. *Let (F, G) denote a dual frame pair for a Hilbert space $\mathcal{H}$.*

(1) If the set $\{g_j\}_{j \in \mathbb{J} \setminus \Lambda}$ no longer forms a frame for $\mathcal{H}$, then not every vector can be reconstructed from erasures indexed by Λ. That is, if Θ_{Λ^c} denotes the analysis operator for $\{g_j\}_{j \in \mathbb{J} \setminus \Lambda}$, then Θ_{Λ^c} is not injective.

(2) If the set $\{g_j\}_{j \in \mathbb{J} \setminus \Lambda}$ still forms a frame for $\mathcal{H}$, then every vector can be reconstructed from erasures indexed by Λ. That is, Θ_{Λ^c} is injective.

For the proof of (1), we will need the following lemma. For a proof of the lemma, see Theorem 5.4.7 in [16].

Lemma 1.9. *Let $F = \{f_j\}_{j\in\mathbb{J}}$ be a frame for a Hilbert space, $\mathcal{H}$, with frame operator S, and let $k \in \mathbb{J}$. If $\langle f_k, S^{-1} f_k\rangle = 1$, then $\overline{span}\{f_j\}_{j\in\mathbb{J}\setminus\{k\}} \neq \mathcal{H}$. However, if $\langle f_k, S^{-1} f_k\rangle \neq 1$, then $\{f_j\}_{j\in\mathbb{J}\setminus\{k\}}$ is a frame for $\mathcal{H}$.*

proof of Proposition 1.8. To prove (1), assume that $\{g_j\}_{j\in\mathbb{J}\setminus\Lambda}$ is no longer a frame for $\mathcal{H}$. Then by inductively applying Lemma 1.9, it is easily seen that $\{g_j\}_{j\in\mathbb{J}\setminus\Lambda}$ is an incomplete set. Thus, we can find a non-zero vector $f \in \mathcal{H}$ for which $\langle f, g_j\rangle = 0$ for all $j \in \Lambda^c$. Thus, its sequence of frame coefficients with respect to $\{g_j\}_{j\in\mathbb{J}\setminus\Lambda}$ is the zero vector. Therefore, f and 0 can not be differentiated by any reconstruction scheme (that is, $\Theta_{\Lambda^c} f = 0 = \Theta_{\Lambda^c} 0$ and Θ_{Λ^c} is not injective). Hence an exact reconstruction of f is impossible.

To prove (2), assume that $\{g_j\}_{j\in\mathbb{J}\setminus\Lambda}$ still forms a frame. Denote the frame operator for $\{g_j\}_{j\in\mathbb{J}\setminus\Lambda}$ by $\tilde{S}$. Note that $\tilde{S}$ is invertible by part (2) of Theorem 1.3. Thus, any vector $f \in \mathcal{H}$ can be reconstructed from its frame coefficeints indexed by $\mathbb{J} \setminus \Lambda$ with the formula

$$f = \sum_{j\in\mathbb{J}\setminus\Lambda} \langle f, g_j\rangle \tilde{S}^{-1} g_j = \tilde{S}^{-1}\Theta^*_{\Lambda^c}\Theta_{\Lambda^c} f.$$

Moreover, a left inverse of Θ_{Λ^c} is given by $\tilde{S}^{-1}\Theta^*_{\Lambda^c}$. □

Since the condition on Λ that $\{g_j\}_{j\in\mathbb{J}\setminus\Lambda}$ remains a frame is essential for reconstruction, we make the following definition. Given a dual frame pair (F, G) for a Hilbert space $\mathcal{H}$, an erasure set, Λ, is said to satisfy the *minimal redundancy condition* if $\{g_j\}_{j\in\mathbb{J}\setminus\Lambda}$ still forms a frame for $\mathcal{H}$. The reconstruction procedure in the proof of (2) above will be referred to as the *FORC method*, since we are inverting the frame operator for the remaining coefficients. In the finite dimensional case, this method produces a perfect reconstruction. However, the reconstruction is inefficient since it requires the inversion of the $n \times n$ matrix $\tilde{S}$, where

n denotes the dimension of $\mathcal{H}$. Moreover, in the infinite dimensional case this method can only be applied in certain special cases since it requires the inversion of an infinite dimensional operator.

Let (F, G) be a frame for a Hilbert space, $\mathcal{H}$. If the frame coefficients indexed by $\Lambda \subset \mathbb{J}$ are erased, the signal recipient (Bob) receives the vector

$$f_R = \sum_{j \in \mathbb{J} \setminus \Lambda} \langle f, g_j \rangle f_j. \tag{1.20}$$

We call f_R the *partial reconstruction* of the vector f. The corresponding operator $R_\Lambda : \mathcal{H} \to \mathcal{H}$ defined by

$$R_\Lambda f = f_R = \sum_{j \in \mathbb{J} \setminus \Lambda} \langle f, g_j \rangle f_j \quad \forall f \in \mathcal{H} \tag{1.21}$$

is called the *partial reconstruction operator*. Associated to the partial reconstruction operator, we call the operator $E_\Lambda : \mathcal{H} \to \mathcal{H}$ defined by

$$E_\Lambda f = f - f_R = f - \sum_{j \in \mathbb{J} \setminus \Lambda} \langle f, g_j \rangle f_j = \sum_{j \in \Lambda} \langle f, g_j \rangle f_j \tag{1.22}$$

the *error operator*. Notice that $R_\Lambda = I - E_\Lambda$, and to reconstruct from frame erasures, we can simply invert R_Λ. To do this, if $\|E_\Lambda\| < 1$, applying Theorem 1.6, we get

$$R_\Lambda^{-1} = (I - E_\Lambda)^{-1} = \sum_{k=0}^{\infty} E_\Lambda^k. \tag{1.23}$$

In practice, we must approximate the infinite Neumann series above with a finite series. This iterative method is more efficient than the FORC method in some cases, however it still requires many large matrix multiplications. We will provide better alternatives to this method in the sections to come.

There has been some research done on types of frames for which $\|E_\Lambda\|$ or the spectral radius $r(E_\Lambda)$ is small. Thus, these frames will yield a faster reconstruction. In [13], Casazza and Kovačević proved that uniform tight frames minimize $\max\{\|E_\Lambda\| : |\Lambda| = 1\}$ over the set of all tight frames. In [34], Holmes and Paulsen supplied a proof that among tight frames, equiangular tight frames minimize the quantity $\max\{\|E_\Lambda\| : |\Lambda| = 2\}$. In [40] and [41] Leng and Han, and Lopez and Han provided conditions for which the quantity $\max\{\|E_\Lambda\| : |\Lambda| = m\}$ is minimized for the standard dual. In [47], Pehlivan, Han, and Mohapatra considered the minimization of $r(E_\Lambda)$. For more on frame erasures, see [7], [8], [20], [24], [31]-[33], [36]-[39], [47], and [50].

1.4 Tight Frames and the Restricted Isometry Property

Tight frames are especially useful for signal processing because, as pointed out in Corollary 1.4, the frame operator for a tight frame is a constant multiple of the identity operator. This in turn gives the nice reconstruction given by equation (1.10). In this section, we will discuss various types of tight frames, and discuss some of their fascinating numerical properties.

As mentioned earlier, a Parseval frame is a tight frame with a tight frame bound of 1. Thus, from Corollary 1.4 the frame operator is the identity operator, and the reconstruction formula in equation (1.10) is very similar to an orthonormal basis reconstruction (or expansion) formula. The next proposition says that the Gramian operator for a Parseval frame is actually a projection onto the range of the analysis operator.

Proposition 1.10. *Assume that F is a Parseval Frame for a Hilbert space $\mathcal{H}$. Then, the Gramian operator, $\mathcal{G}$ for F is the unique orthogonal projection onto the range of the analysis operator, Θ for F.*

Proof. Recall that $\mathcal{G} = \Theta\Theta^*$, and the frame operator for F is $S = \Theta^*\Theta$. By Corollary 1.4,

$S = I$, the identity operator on $\mathcal{H}$. To show that $\mathcal{G}$ is an orthogonal projection, we have

$$\mathcal{G}^2 = \Theta\Theta^*\Theta\Theta^* = \Theta S\Theta^* = \Theta I\Theta^* = \Theta\Theta^* = \mathcal{G}$$

and

$$\mathcal{G}^* = (\Theta\Theta^*)^* = (\Theta^*)^*\Theta^* = \Theta\Theta^* = \mathcal{G}.$$

It only remains to show that range$(\Theta) =$ range$(\mathcal{G})$. Clearly,

$$\text{range}(\mathcal{G}) = \text{range}(\Theta\Theta^*) \subset \text{range}(\Theta).$$

Conversely, if $y \in$ range(Θ), then $y = \Theta x$ for some $x \in \mathcal{H}$. Since F is a Parseval frame, $x = \Theta^*\Theta x$. Thus

$$y = \Theta x = \Theta\Theta^*\Theta x = \mathcal{G}\Theta x.$$

Hence $y \in$ range$(\mathcal{G})$. Therefore, range$(\mathcal{G}) =$ range(Θ). □

A tight frame in which every vector has unit norm is called a *finite unit norm tight frame*, which is commonly abbreviated as *FUNTF* in the frame theory literature. Recently a lot of work has gone into the study of the numerical properties, and the constructions of FUNTFs (cf. [4], [24], [44], and [50]). The following is a simple computation for the frame bound of a FUNTF.

Proposition 1.11. *Let $F = \{f_j\}_{j=1}^N$ be a finite unit norm tight frame for an n-dimensional Hilbert space, $\mathcal{H}$. Then, the frame bound for F is $\frac{N}{n}$.*

Proof. Let A be the frame bound for F. Then, by Proposition 1.4, $S = AI$, and $tr(S) = An$. Since $\langle f_j, f_j\rangle = \|f_j\|^2 = 1$ for all $j = 1, \cdots, N$, we have $tr(\mathcal{G}) = N$. Thus,

$$An = tr(S) = tr(\Theta^*\Theta) = tr(\Theta\Theta^*) = tr(\mathcal{G}) = N.$$

Therefore $A = \frac{N}{n}$. $\square$

An immediate consequence of Proposition 1.11 is that for a FUNTF F, and any $f \in \mathcal{H}$,

$$f = \frac{n}{N} \sum_{j=1}^{N} \langle f, f_j \rangle f_j.$$

The most important tool we will use to study the stability of our algorithms is the famous Restricted Isometry Property, commonly abbreviated as RIP. A vector $x \in \mathbb{R}^n$ is called *s-sparse* if x has s or fewer non-zero entries. We say that an $m \times N$ matrix M satisfies the *Restricted Isometry Property of order* s with constant δ_s if for all s-sparse vectors, $x \in \mathbb{R}^N$, we have

$$(1 - \delta_s)\|x\|^2 \leq \|Mx\|^2 \leq (1 + \delta_s)\|x\|^2 \tag{1.24}$$

(cf. [22]).

It has been well studied that a matrix $M = \frac{1}{\sqrt{m}}(m_{j,k}) \in \mathbb{R}^{m \times N}$ whose entries $m_{j,k}$ are drawn independently from the standard normal distribution will satisfy the RIP for a fixed constant δ_s with high probability, provided m is sufficiently large. The following theorem is the precise statement of this. A proof of this theorem, and many other fascinating applications of the RIP can be found in [22]. For more on the RIP, including some of the original proofs, see [2], [12], [18], [43], and [48].

Theorem 1.12. *Assume that for all $1 \leq j \leq m$ and $1 \leq k \leq N$, each $m_{j,k}$ is selected independently according to the standard normal distribution, that $M = (m_{j,k})$, and that $\delta, \gamma \in (0, 1)$. Then, there exists a constant $\rho > 0$ so that the RIP constant δ_s for $\frac{1}{\sqrt{m}}M$ satisfies $\delta_s \leq \delta$ with probability $1 - \gamma$ provided*

$$m \geq \frac{\rho}{\delta^2}\left(s \ln\left(\frac{eN}{s}\right) + \ln\left(\frac{2}{\gamma}\right)\right). \tag{1.25}$$

In [5], Bodmann proved that randomly generated frames are almost tight and almost equiangular. The following is a slightly modified version of Corollary 3.5 in [5].

Theorem 1.13. *Assume F is an $n \times N$ matrix whose entries are drawn independently from the standard normal distribution. If*

$$n \leq \frac{12 \ln\left(\frac{\gamma}{2}\right) + 3\delta^2 N - 4N\delta^3}{12 \ln\left(1 + \frac{4}{\delta}\right)} \tag{1.26}$$

then with probability at least $1-\gamma$ the columns of $\frac{1}{\sqrt{n}}F$ form a frame with lower and upper frame bounds $\frac{N}{n}\frac{1}{(1+\delta)^3}$ and $\frac{N}{n}(1+\delta)^3$, respectively. That is, if f_j denotes the j^{th} column of F, then

$$\frac{N}{n}\frac{1}{(1+\delta)^3}\|f\|^2 \leq \sum_{j=1}^{N} \left| \left\langle f, \frac{1}{\sqrt{n}} f_j \right\rangle \right|^2 \leq \frac{N}{n}(1+\delta)^3 \|f\|^2 \quad \forall f \in \mathbb{R}^n. \tag{1.27}$$

In our numerical experiments, we will couple these probabilistic results together to construct tight frames which satisfy the RIP with high probability, called *TRIP frames*.

1.5 Fourier Analysis and the Shannon-Whittaker Sampling Theorem

The *Fourier Series* for a function $f \in L^2[-\pi, \pi]$ is the series

$$S_f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}, \tag{1.28}$$

where the k^{th} *Fourier coefficient*, c_k, is given by

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} \, d\theta. \tag{1.29}$$

That is, if $f_k(x) = e^{ikx}$, with the usual inner product on $L^2[-\pi, \pi]$, then

$$S_f = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \langle f, f_k \rangle \, f_k. \tag{1.30}$$

The following Theorem says that $S_f = f$ in the sense of convergence in $L^2[-\pi, \pi]$.

Theorem 1.14. *The sequence* $\frac{1}{\sqrt{2\pi}}\{e^{ikx}\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* $L^2[-\pi,\pi]$. *Moreover, if*

$$S_f^N(x) = \frac{1}{2\pi} \sum_{k=-N}^{N} \langle f, f_k \rangle \, f_k(x), \tag{1.31}$$

then S_f^N *converges to* f *in* $L^2[-\pi, \pi]$.

Furthermore, it is easily verified from Theorem 1.14 that $\sqrt{\frac{p}{2\pi}}\left\{e^{ipkx}\right\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2\left[-\frac{\pi}{p}, \frac{\pi}{p}\right]$, where $p \in (0, \infty)$. In particular for $p \in (0,1)$, $[-\pi,\pi] \subset \left[-\frac{\pi}{p}, \frac{\pi}{p}\right]$. Thus, if $f \in L^2[-\pi,\pi]$, we can extend f to $\left[-\frac{\pi}{p}, \frac{\pi}{p}\right]$, by defining it to be zero on $\left[-\frac{\pi}{p}, \frac{\pi}{p}\right] \setminus [-\pi,\pi]$. Hence, for $f \in L^2[-\pi,\pi]$

$$\frac{2\pi}{p}\|f\|^2 = \sum_{k\in\mathbb{Z}} \left|\left\langle f, e^{ipk\cdot}\right\rangle\right|^2. \tag{1.32}$$

The following Proposition summarizes these remarks.

Proposition 1.15. *The sequence* $\left\{e^{ipkx}\right\}_{k\in\mathbb{Z}}$ *is a tight frame for* $L^2[-\pi,\pi]$ *with frame bound* $\frac{2\pi}{p}$.

The *Fourier transform* of a function $f \in L^1(\mathbb{R})$ is the function

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx. \tag{1.33}$$

If $f, \hat{f} \in L^1(\mathbb{R})$, and f is continuous at x, the *Fourier Inversion Theorem* states that $f(x)$

can be recovered from its Fourier transform, $\hat{f}$, by the formula

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} \, d\xi. \tag{1.34}$$

The *Plancherel Theorem* states that for $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\left\langle \hat{f}, \hat{g} \right\rangle = 2\pi \langle f, g \rangle, \quad \text{and} \quad \|\hat{f}\|^2 = 2\pi \|f\|^2. \tag{1.35}$$

Thus, since the Fourier transform $\mathcal{F}$ is a bounded operator on the dense subset $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ of $L^2(\mathbb{R})$, $\mathcal{F}$ can be extended to a bounded invertible operator on all of $L^2(\mathbb{R})$. Moreover, $\frac{1}{\sqrt{2\pi}}\mathcal{F}$ can be extended to a unitary operator on $L^2(\mathbb{R})$.

A function $f \in L^2(\mathbb{R})$ is called *band-limited* with *band* π if spt$(\hat{f})$ (the *support* of $\hat{f}$) is contained in the interval $[-\pi, \pi]$. We denote the set of all band-limited functions with band π as PW_π (the *Paley-Weiner space*). Notice that for a given $f \in PW_\pi$ and $a \in \mathbb{R}$, the Fourier Inversion Theorem and Plancherel Theorem yield

$$f(a) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} \, d\xi = \frac{1}{2\pi} \langle \mathcal{F} f, g \rangle = \left\langle f, \mathcal{F}^{-1} g \right\rangle \tag{1.36}$$

where $g(x) = e^{ia\xi}\chi_\pi$, and χ_π is the indicator function of the set $[-\pi, \pi]$. Using the Fourier Inversion Theorem we have

$$\begin{aligned} (\mathcal{F}^{-1} g)(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} g(\xi) e^{ix\xi} \, d\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ia\xi} e^{ix\xi} \, d\xi \\ &= \int_{-\pi}^{\pi} e^{i(x-a)\xi} \, dx = \text{sinc}(\pi(x-a)), \end{aligned} \tag{1.37}$$

where by convention, we take sinc$(x) = \frac{sinx}{x}$ for $x \neq 0$, and sinc$(0) = 1$. Combining equations (1.36) and (1.37) we get

$$f(a) = \langle f, \text{sinc}(\pi(\cdot - a)) \rangle \,. \tag{1.38}$$

If $f \in PW_\pi$, then $\hat{f} \in L^2[-\pi, \pi]$. Thus, by Proposition 1.15

$$\begin{aligned}
\hat{f}(\xi) &= \frac{p}{2\pi} \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \hat{f}(\theta) e^{-ikp\theta} \, d\theta \, e^{ikp\xi} \chi_\pi \\
&= \frac{p}{2\pi} \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \hat{f}(\theta) e^{ikp\theta} \, d\theta \, e^{-ikp\xi} \chi_\pi \\
&= p \sum_{k\in\mathbb{Z}} \left[\frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\theta) e^{ikp\theta} \, d\theta \right] e^{-ikp\xi} \chi_\pi \\
&= p \sum_{k\in\mathbb{Z}} f(kp) e^{-ikp\xi} \chi_\pi .
\end{aligned}$$

Taking the inverse Fourier transform of both sides gives

$$\begin{aligned}
f(x) &= \frac{p}{2\pi} \sum_{k\in\mathbb{Z}} f(kp) \int_{\mathbb{R}} e^{-ikp\xi} \chi_\pi e^{ix\xi} \, d\xi \\
&= \frac{p}{2\pi} \sum_{k\in\mathbb{Z}} f(kp) \int_{-\pi}^{\pi} e^{i(x-kp)\xi} \, d\xi \\
&= p \sum_{k\in\mathbb{Z}} f(kp) \text{sinc}(\pi(x - kp)) \\
&= p \sum_{j\in p\mathbb{Z}} f(j) \text{sinc}(\pi(x - j)).
\end{aligned}$$

This formula is known as the *Shannon-Whittaker Sampling Theorem.* The Shannon-Whittaker Sampling Theorem is especially useful for digital signal processing because it allows us to recover a frequency (or band) limited signal from its sampled values on a lattice.

Since $f(j) = \langle f, \text{sinc}(\pi(\cdot - j)) \rangle$ for all $j \in p\mathbb{Z}$, the Shannon-Whittaker Sampling

Theorem says that

$$f = p \sum_{j \in p\mathbb{Z}} \langle f, \operatorname{sinc}(\pi(\cdot - j)) \rangle \operatorname{sinc}(\pi(\cdot - j)). \tag{1.39}$$

for all $f \in PW_\pi$. Thus, $\{\operatorname{sinc}(\pi(\cdot - j))\}_{j \in p\mathbb{Z}}$ is a frame for PW_π whose standard dual is the frame $\{p \operatorname{sinc}(\pi(\cdot - j))\}_{j \in p\mathbb{Z}}$.

For further reading on Fourier analysis, see [21] and [23]. For further reading on Sampling Theory, see [3] and [52].

2. NILPOTENT BRIDGING

2.1 The Nilpotent Bridging Algorithm[1]

If (F, G) is a dual frame pair for a Hilbert space $\mathcal{H}$, and $f \in \mathcal{H}$, equation (1.23) says that we can reconstruct the signal f from erasures indexed by Λ by computing

$$f = R_\Lambda^{-1} f_R = (I - E_\Lambda)^{-1} f_R = \sum_{k=0}^{\infty} E_\Lambda^k f_R, \tag{2.1}$$

whenever the norm (or more generally, the spectral radius of E_Λ) is less than one. Our original goal for this project was to reduce the spectral radius of E_Λ by supplementing f_R with information from the known (or non-erased) coefficients. In doing so, we discovered a method of preconditioning for which a new error operator term is nilpotent of index two. Thus, the sum in equation (2.1) for this new error term terminates after two iterations. Because of this, we called the new method *Nilpotent Bridging*.

The idea behind Nilpotent Bridging is to use information from a small collection of the non-erased coefficients to reconstruct f. That is, for $j \in \Lambda$, we will replace the erased frame coefficient $\langle f, g_j \rangle$ with $\langle f, g'_j \rangle$, where

$$g'_j = \sum_{k \in \Omega} c_{j,k} g_k \tag{2.2}$$

for some subset Ω of $\mathbb{J} \setminus \Lambda$. Any such set $\Omega \subset \mathbb{J} \setminus \Lambda$ is called a *bridge set* for Λ. The question now becomes how to choose the $c_{j,k}$ for $j \in \Lambda$ and $k \in \Omega$ in an appropriate way. To help us understand the process, we make the following definitions. The operator

[1] The statement and proof of Theorem 2.3 was reproduced from [37] and [38] With permission of Springer.

$B_\Lambda : \mathcal{H} \to \mathcal{H}$ defined by

$$B_\Lambda f = \sum_{j \in \Lambda} \left\langle f, g_j' \right\rangle f_j \quad \forall f \in \mathcal{H} \tag{2.3}$$

is called the *bridging operator*. The *bridging supplement*, denoted by f_B, is the vector

$$f_B = B_\Lambda f = \sum_{j \in \Lambda} \left\langle f, g_j' \right\rangle f_j. \tag{2.4}$$

A new, preconditioned guess for the reconstructed signal f is

$$f_{RB} = f_R + f_B = R_\Lambda f + B_\Lambda f = \sum_{j \in \mathbb{J} \setminus \Lambda} \langle f, g_j \rangle f_j + \sum_{j \in \Lambda} \left\langle f, g_j' \right\rangle f_j. \tag{2.5}$$

Thus, the new reconstruction error is given by

$$\begin{aligned} f - f_{RB} &= \sum_{j \in \mathbb{J}} \langle f, g_j \rangle f_j - \sum_{j \in \mathbb{J} \setminus \Lambda} \langle f, g_j \rangle f_j - \sum_{j \in \Lambda} \left\langle f, g_j' \right\rangle f_j \\ &= \sum_{j \in \Lambda} \langle f, g_j \rangle f_j - \sum_{j \in \Lambda} \left\langle f, g_j' \right\rangle f_j \\ &= \sum_{j \in \Lambda} \left\langle f, g_j - g_j' \right\rangle f_j. \end{aligned}$$

So, we will define the reduced error operator $\tilde{E}_\Lambda : \mathcal{H} \to \mathcal{H}$ by

$$\tilde{E}_\Lambda f = \sum_{j \in \Lambda} \left\langle f, g_j - g_j' \right\rangle f_j \quad \forall f \in \mathcal{H}. \tag{2.6}$$

Remark 2.1. At this point, the operators B_Λ and $\tilde{E}_\Lambda$ are technically not well defined since we haven't selected a choice of coefficients $C = (c_{j,k})_{j \in \Lambda,\, k \in \Omega}$. This choice will be determined next.

Notice that if

$$f_j \perp g_k - g'_k \quad \forall j, k \in \Lambda, \tag{2.7}$$

we have

$$\begin{aligned}
\tilde{E}_\Lambda^2 f &= \sum_{j\in\Lambda} \left\langle \tilde{E}_\Lambda f, g_j - g'_j \right\rangle f_j \\
&= \sum_{j\in\Lambda} \left\langle \sum_{k\in\Lambda} \langle f, g_k - g'_k \rangle f_k, g_j - g'_j \right\rangle f_j \\
&= \sum_{j,k\in\Lambda} \left\langle f_k, g_j - g'_j \right\rangle \langle f, g_k - g'_k \rangle f_j \\
&= \sum_{j,k\in\Lambda} 0 \langle f, g_k - g'_k \rangle f_j \\
&= 0.
\end{aligned}$$

Thus, to make the reduced error operator nilpotent of index two, we will choose the coefficients so that for all pairs $j, k \in \Lambda$,

$$\begin{aligned}
0 &= \langle f_j, g_k - g'_k \rangle \\
&= \left\langle f_j, g_k - \sum_{\ell\in\Omega} c_{k,\ell} g_\ell \right\rangle \\
&= \langle f_j, g_k \rangle - \sum_{\ell\in\Omega} \overline{c_{k,\ell}} \langle f_j, g_\ell \rangle .
\end{aligned}$$

Thus, for all $j, k \in \Lambda$, the following equation must be satisfied:

$$\langle f_j, g_k \rangle = \sum_{\ell\in\Omega} \overline{c_{k,\ell}} \langle f_j, g_\ell \rangle . \tag{2.8}$$

By enumerating Λ and Ω as $\Lambda = \{\lambda_j\}_{j=1}^{L}$ and $\Omega = \{\omega_j\}_{j=1}^{M}$, the following equation must be satisfied for all $1 \leq j, k \leq L$ (note the abuse of indices in the coefficient matrix):

$$\left\langle f_{\lambda_j}, g_{\lambda_k} \right\rangle = \sum_{\ell=1}^{M} \overline{c_{k,\ell}} \left\langle f_{\lambda_j}, g_{\omega_\ell} \right\rangle . \tag{2.9}$$

By fixing $k \in \{1, 2, \cdots, L\}$, the above equation is equivalent to the matrix equation given by

$$\begin{pmatrix} \langle f_{\lambda_1}, g_{\lambda_k} \rangle \\ \langle f_{\lambda_2}, g_{\lambda_k} \rangle \\ \vdots \\ \langle f_{\lambda_L}, g_{\lambda_k} \rangle \end{pmatrix} = \begin{pmatrix} \langle f_{\lambda_1}, g_{\omega_1} \rangle & \langle f_{\lambda_1}, g_{\omega_2} \rangle & \cdots & \langle f_{\lambda_1}, g_{\omega_M} \rangle \\ \langle f_{\lambda_2}, g_{\omega_1} \rangle & \langle f_{\lambda_2}, g_{\omega_2} \rangle & \cdots & \langle f_{\lambda_2}, g_{\omega_M} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_{\lambda_L}, g_{\omega_1} \rangle & \langle f_{\lambda_L}, g_{\omega_2} \rangle & \cdots & \langle f_{\lambda_L}, g_{\omega_M} \rangle \end{pmatrix} \overline{\begin{pmatrix} c_{k,1} \\ c_{k,2} \\ \vdots \\ c_{k,M} \end{pmatrix}}. \tag{2.10}$$

Combining the matrix equations for all $k \in \{1, \cdots, M\}$ yields the following matrix equation:

$$\begin{pmatrix} \langle f_{\lambda_1}, g_{\lambda_1} \rangle & \langle f_{\lambda_1}, g_{\lambda_2} \rangle & \cdots & \langle f_{\lambda_1}, g_{\lambda_L} \rangle \\ \langle f_{\lambda_2}, g_{\lambda_1} \rangle & \langle f_{\lambda_2}, g_{\lambda_2} \rangle & \cdots & \langle f_{\lambda_2}, g_{\lambda_L} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_{\lambda_L}, g_{\lambda_1} \rangle & \langle f_{\lambda_L}, g_{\lambda_2} \rangle & \cdots & \langle f_{\lambda_L}, g_{\lambda_L} \rangle \end{pmatrix} = \begin{pmatrix} \langle f_{\lambda_1}, g_{\omega_1} \rangle & \langle f_{\lambda_1}, g_{\omega_2} \rangle & \cdots & \langle f_{\lambda_1}, g_{\omega_M} \rangle \\ \langle f_{\lambda_2}, g_{\omega_1} \rangle & \langle f_{\lambda_2}, g_{\omega_2} \rangle & \cdots & \langle f_{\lambda_2}, g_{\omega_M} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_{\lambda_L}, g_{\omega_1} \rangle & \langle f_{\lambda_L}, g_{\omega_2} \rangle & \cdots & \langle f_{\lambda_L}, g_{\omega_M} \rangle \end{pmatrix} \overline{\begin{pmatrix} c_{1,1} & c_{2,1} & \cdots & c_{L,1} \\ c_{1,2} & c_{2,2} & \cdots & c_{L,2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1,M} & c_{2,M} & \cdots & c_{L,M} \end{pmatrix}}. \tag{2.11}$$

By using the notations $B(\Lambda, \Omega) = (\langle f_j, g_k \rangle)_{j \in \Lambda, k \in \Omega}$ and $C = (c_{j,k})_{1 \leq j \leq L, 1 \leq k \leq M}$, equation (2.11) can be written as

$$B(\Lambda, \Lambda) = B(\Lambda, \Omega) C^*. \tag{2.12}$$

We refer to equation (2.12) as the *bridge equation*, and we will henceforth refer to the matrix $B(\Lambda, \Omega)$ as the *bridge matrix*. We say that a bridge set Ω for Λ is a *robust bridge set* if the bridge equation has a solution.

The next proposition gives a preliminary reconstruction algorithm based on the above work. A more efficient version is given by Theorem 2.3, but its proof relies on information from Proposition 2.2.

Proposition 2.2. *Assume that (F, G) is a frame for a Hilbert space $\mathcal{H}$, that Λ is an erasure set which satisfies the minimal redundancy condition with respect to (F, G), and that Ω is a robust bridge set for Λ. Then, for any $f \in \mathcal{H}$,*

$$f = f_{RB} + \tilde{E}_\Lambda f_R = f_R + f_B + \tilde{E}_\Lambda f_R. \tag{2.13}$$

Proof. Since $R_\Lambda + B_\Lambda + \tilde{E}_\Lambda = I$, we have

$$f = R_\Lambda f + B_\Lambda f + \tilde{E}_\Lambda f. \tag{2.14}$$

Applying $\tilde{E}_\Lambda$ to both sides of equation (2.14), we get

$$\tilde{E}_\Lambda f = \tilde{E}_\Lambda R_\Lambda f + \tilde{E}_\Lambda B_\Lambda f + \tilde{E}_\Lambda^2 f = \tilde{E}_\Lambda R_\Lambda f + \tilde{E}_\Lambda B_\Lambda f. \tag{2.15}$$

Notice that the last equality follows since $\tilde{E}_\Lambda$ is nilpotent of index two. Next, we observe that $\tilde{E}_\Lambda B_\Lambda = 0$ since for all $f \in \mathcal{H}$, we have

$$\begin{aligned}
\tilde{E}_\Lambda B_\Lambda f &= \sum_{k\in\Lambda} \langle B_\Lambda f, g_k - g'_k\rangle f_k \\
&= \sum_{k\in\Lambda} \left\langle \sum_{j\in\Lambda} \langle f, g'_j\rangle f_j, g_k - g'_k \right\rangle f_k \\
&= \sum_{j,k\in\Lambda} \langle f, g'_j\rangle \langle f_j, g_k - g'_k\rangle f_k \\
&= 0.
\end{aligned}$$

The final equality above follows by equation (2.7). Thus, equation (2.15) reduces to

$$\tilde{E}_\Lambda f = \tilde{E}_\Lambda R_\Lambda f = \tilde{E}_\Lambda f_R. \tag{2.16}$$

Plugging equation (2.16) into equation (2.14) gives

$$f = R_\Lambda f + B_\Lambda f + \tilde{E}_\Lambda f_R = f_{RB} + \tilde{E}_\Lambda f_R = f_R + f_B + \tilde{E}_\Lambda f_R. \tag{2.17}$$

□

The next theorem gives a much simpler, more efficient reconstruction algorithm which we will use for our implementations later on. For the theorem, we will assume $\alpha_j = \langle f, g_j\rangle$ and $\beta_j = \langle f_R, g_j\rangle$.

Theorem 2.3. *Let (F, G) be a dual frame pair with erasure set Λ satisfying the minimal redundancy condition, and Ω be a robust bridge set. Assume $C = \left(\overline{c_j^{(k)}}\right)_{j\in\Omega,\, k\in\Lambda}$ solves the matrix equation $B(F, G, \Lambda, \Omega)C = B(F, G, \Lambda, \Lambda)$. Then,*

$$(\langle f, g_j\rangle)_{j\in\Lambda} = C^T((\alpha_j)_{j\in\Omega} - (\beta_j)_{j\in\Omega}) + (\beta_j)_{j\in\Lambda}, \tag{2.18}$$

where C^T denotes the transpose of C.

Proof. Let $\{f_j, g_j\}_{j\in\mathbb{J}}$ be a dual frame pair, Λ be an erasure set, and Ω be a corresponding robust bridge set. For $j \in \Lambda$ and $f \in \mathcal{H}$

$$\begin{aligned} \langle f, g_j\rangle &= \left\langle f, g_j'\right\rangle + \left\langle f, g_j - g_j'\right\rangle \\ &= \left\langle f, g_j'\right\rangle + \left\langle f - f_R, g_j - g_j'\right\rangle + \left\langle f_R, g_j - g_j'\right\rangle . \end{aligned}$$

Since $f - f_R \in \operatorname{span}\{f_j : j \in \Lambda\}$, equation (2.7) says that $f - f_R \perp g_j - g_j'$. So,

$$\begin{aligned} \langle f, g_j\rangle &= \left\langle f, g_j'\right\rangle + \left\langle f_R, g_j - g_j'\right\rangle \\ &= \left\langle f - f_R, g_j'\right\rangle + \langle f_R, g_j\rangle \\ &= \sum_{k\in\Omega} \overline{c_k^{(j)}} \langle f - f_R, g_k\rangle + \langle f_R, g_j\rangle . \end{aligned}$$

Therefore, we can recover the erased coefficients with the following equation:

$$(\langle f, g_j\rangle)_{j\in\Lambda} = C^T(\langle f - f_R, g_k\rangle)_{k\in\Omega} + (\langle f_R, g_j\rangle)_{j\in\Lambda}.$$

That is,

$$(\langle f, g_j\rangle)_{j\in\Lambda} = C^T((\alpha_j)_{j\in\Omega} - (\beta_j)_{j\in\Omega}) + (\beta_j)_{j\in\Lambda}.$$

□

Remark 2.4. Note that the coefficient matrix in Theorem 2.3 is slightly different here than in equation 2.12. If we use the matrix coefficient matrix, C, from 2.12, then equation 2.18 becomes

$$(\langle f, g_j\rangle)_{j\in\Lambda} = C((\langle f, g_j\rangle)_{j\in\Omega} - (\langle f_R, g_j\rangle)_{j\in\Omega}) + (\langle f_R, g_j\rangle)_{j\in\Lambda}. \tag{2.19}$$

Remark 2.5. Notice that all of the quantities in equation (2.18) are either known, or computable by the signal recipient (Bob), since Bob knows F, G, f_R, and $(\langle f, g_k\rangle)_{k\in\Omega}$. Thus, this gives a method of reconstruction from frame erasures at known locations.

Remark 2.6. Notice that if there is error in the partial reconstruction, f_R, this error will be amplified by the matrix C, and this amplification factor can be very severe as we will see in Section 2.5. However, if we overbridge (meaning we consider a bridge set whose size is larger than the set of erasures), and solve the bridge equation with the Moore-Penrose pseudo-inverse, then our reconstruction seems to be quite stable. This is likely because (in general) when $|\Omega| > |\Lambda|$, the bridge equation $B(\Lambda, \Omega)C^* = B(\Lambda, \Lambda)$ has many solutions. If we break this problem up, and let c_j and b_j denote the j^{th} columns of C^* and $B(\Lambda, \Lambda)$, respectively, then by using the Moore-Penrose pseudo-inverse, c_j will be the solution of $B(\Lambda, \Omega)c_j = b_j$ with minimal ℓ^2 norm. Thus, in some sense, the matrix C is a minimal solution of the bridge equation. Therefore, we can expect that any noise introduced to f_R will be amplified less when we overbridge. This phenomenon is discussed in more detail in Section 2.5.

The following example provides a simple formula for reconstruction from one erasure.

Example 2.7. Let (F, G) be a dual frame pair for a Hilbert space, $\mathcal{H}$. Assume $\Lambda = \{j\}$ and $\langle f_j, g_k\rangle \neq 0$ for $k \neq j$. Then $\Omega = \{k\}$ is a robust bridge set for Λ, and

$$\langle f, g_j\rangle = \langle f_R, g_j\rangle + \frac{1}{\langle f_j, g_k\rangle}\left(\langle f, g_k\rangle - \langle f_R, g_k\rangle\right). \tag{2.20}$$

We will now present a version of Theorem 2.3 for Shannon-Whittaker Sampling. Here, we will denote the Shannon-Whittaker frame on the lattice $p\mathbb{Z}$ by $F_p = \{p\,\mathrm{sinc}(\pi(x - j))\}_{j\in\mathbb{J}}$, and its standard dual by $G_p = \{\mathrm{sinc}(\pi(x - j))\}_{j\in p\mathbb{Z}}$. Since the inner products against sinc functions yield pointwise evaluations,

$$\begin{aligned} B(F_p, G_p, \Lambda, \Omega) &= p\left(\langle \mathrm{sinc}(\pi(\cdot - j)), \mathrm{sinc}(\pi(\cdot - k))\rangle\right)_{j\in\Lambda,\, k\in\Omega} \\ &= p\left(\mathrm{sinc}(\pi(k-j))\right)_{j\in\Lambda,\, k\in\Omega}. \end{aligned}$$

Thus, the Nilpotent Bridging Theorem applied to Shannon-Whittaker Sampling Theory becomes the following Theorem.

Theorem 2.8. *Let (F_p, G_p) be as above. Assume that Λ is an erasure set satisfying the minimal redundancy condition, and Ω is a robust bridge set for Λ. If $C = (c_{j,k})_{j\in\Lambda,\, k\in\Omega}$ solves the bridging equation, $B(\Lambda, \Omega)C^* = B(\Lambda, \Lambda)$, Then,*

$$(f(t_j))_{j\in\Lambda} = C((f(t_j))_{j\in\Omega} - (f_R(t_j))_{j\in\Omega}) + (f_R(t_j))_{j\in\Lambda}. \tag{2.21}$$

2.2 Existence of Robust Bridge Sets[2]

In this section, we will provide proofs for the existence of certain types of bridge sets. If Λ is an erasure set which satisfies the minimal redundancy condition with respect to a dual frame pair (F, G), the next theorem guarantees the existence of a robust bridge set $\Omega \subset \mathbb{J} \setminus \Lambda$ satisfying $|\Omega| \leq |\Lambda|$. Thus, by using our Nilpotent Bridging algorithm, inverting the $n \times n$ matrix R_Λ ($n = \dim \mathcal{H}$) simplifies to solving the $L \times L$ square system of equations given by the bridge equation. In Section 2.3, we will prove an even stronger result which says that for a sufficiently random dual frame pair (F, G), and an erasure set Λ satisfying $|\Lambda| \leq \min\{N - n, n\}$, any bridge set $\Omega \subset \Lambda^c$ for which $|\Omega| = |\Lambda|$ will be a robust bridge set for Λ. This stronger result will allow us to cut out a potentially costly search procedure in the algorithms used to implement Nilpotent Bridging.

[2]The statement and proofs of Theorems 2.9 and 2.10 were reproduced from [37] and [38] With permission of Springer.

Theorem 2.9. *Let (F, G) be a dual frame pair for a Hilbert space $\mathcal{H}$, and let Λ be an erasure set. Then there is a robust bridge set Ω for Λ if and only if Λ satisfies the minimal redundancy condition for G. In this case we can take $|\Omega| = \dim(\mathcal{F})$, where $\mathcal{F} =$ span$\{f_j : j \in \Lambda\}$.*

Proof. Assume that Λ satisfies the minimal redundancy condition. Let $\mathcal{F} = \operatorname{span}\{f_j : j \in \Lambda\}$. Let $q = \dim(\mathcal{F})$. Let $\{h_j\}_{j\in\mathbb{N}}$ be a Schauder basis for $\mathcal{F}^\perp$. Since $\mathcal{F}^\perp$ has codimension q, we can complete this set to a Schauder basis $\{h_j\}_{j\in\mathbb{N}} \cup \{g_{j_k}\}_{k=1}^q$, where each $j_k \in \Lambda^c$. Let $\Omega = \{j_k\}_{k=1}^q$. Then $|\Omega| = q$ and $\Lambda \cap \Omega = \emptyset$. For each $\ell \in \Lambda$, write

$$g_\ell = \sum_{k=1}^{q} c_{j_k}^{(\ell)} g_{j_k} + \sum_{j\in\mathbb{N}} b_j^{(\ell)} h_j.$$

Let

$$g'_\ell = \sum_{k=1}^{q} c_{j_k}^{(\ell)} g_{j_k}.$$

Then $g_\ell - g'_\ell \in \mathcal{F}^\perp$. Therefore, by (2.7), the $c_{j_k}^{(\ell)}$ solve the bridge equation (2.12) and Ω is a robust bridge set.

To prove the converse, assume that Ω is a robust bridge set. Assume that $f \perp \overline{\operatorname{span}}\{g_j : j \in \Lambda^c\}$. Then,

$$f = \sum_{j\in\mathbb{J}} \langle f, g_j \rangle f_j = \sum_{j\in\Lambda} \langle f, g_j \rangle f_j.$$

So, $f \in \operatorname{span}\{f_j : j \in \Lambda\}$. We have

$$f = \sum_{j\in\Lambda} \left\langle f, g_j - g'_j \right\rangle f_j + \sum_{j\in\Lambda} \left\langle f, g'_j \right\rangle f_j.$$

However, since $f \in \operatorname{span}\{f_j : j \in \Lambda\}$, equation (2.7) says that $\left\langle f, g_j - g'_j \right\rangle = 0$ for all $j \in \Lambda$. Since $g'_j \in \operatorname{span}\{g_j : j \in \Lambda^c\}$, $\left\langle f, g'_j \right\rangle = 0$ for all $j \in \Lambda$. Hence, $f = 0$. Therefore, $\mathcal{H} = \overline{\operatorname{span}}\{g_j : j \in \Lambda^c\}$ and Λ satisfies the minimal redundancy condition with respect to

G. $\square$

While running experiments, we wanted to determine the spectral properties of the reduced error operator when we considered bridge sets for which $|\Omega| < |\Lambda|$. In doing so, we discovered the phenomenon in the next theorem. The theorem says that (generally speaking) the spectrum of the reduced error operator contains $|\Lambda| - |\Omega|$ non-zero eigenvalues. In the following theorem, we denote the *spectrum* of an operator $T \in B(\mathcal{H})$ by

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}. \tag{2.22}$$

By $|\sigma(T) \setminus \{0\}|$ we mean the number of non-zero eigenvalues of T, counting multiplicity.

Theorem 2.10. *Let (F, G) be a dual frame pair. Assume Λ satisfies the minimal redundancy condition with respect to G, and $|\Lambda| = L$. Then, there is a bridge set Ω of any size $M \leq L$ so that $|\sigma(\tilde{E}_\Lambda) \setminus \{0\}| \leq L - M$.*

Proof. By Theorem 2.9, we can find a robust bridge set $\Omega' \subset \Lambda^c$ satisfying $|\Omega'| < L$. That is, for each $k \in \Lambda$ we can find

$$g'_k = \sum_{j \in \Omega'} c_j^{(k)} g_j$$

so that $g'_k \perp \text{span}\{f_j : j \in \Lambda\}$. Assume that $\Omega' = \{\omega_1, \cdots, \omega_{|\Omega'|}\}$. Let $\Omega = \{\omega_1, \cdots, \omega_M\}$ and

$$g''_k = \sum_{j \in \Omega} c_j^{(k)} g_j.$$

Then,

$$\tilde{E}_\Lambda = \sum_{k \in \Lambda} f_k \otimes (g_k - g''_k) = \sum_{k \in \Lambda} f_k \otimes (g_k - g'_k) + \sum_{k \in \Lambda} f_k \otimes (g'_k - g''_k).$$

Let $N = \tilde{E}_\Lambda = \sum_{k \in \Lambda} f_k \otimes (g_k - g'_k)$, and $A = \sum_{k \in \Lambda} f_k \otimes (g'_k - g''_k)$. Then, it is easily verified that N is nilpotent of index 2, and $NA = 0$. Since range$(A^*) \subset \{g_k' - g_k'' : k \in \Lambda\} \subset \{g_{\omega_k} : k = M+1, \cdots, |\Omega'|\}$, the rank of A is at most $L - M$.

Let $\lambda \in \sigma(N + A) \setminus \{0\}$. Both N and A are finite rank operators, so λ must be an eigenvalue of $N + A$. Thus, there exists $x \in \mathcal{H}$ so that

$$(N + A)x = \lambda x.$$

Multiplying by N on the left on both sides yields

$$0 = \lambda N x.$$

Since $\lambda \neq 0$, we have $Nx = 0$. Thus, $Ax = \lambda x$ and $\lambda \in \sigma(A)$. Since A can have at most $L - M$ distinct eigenvalues, it follows that $\tilde{E}_\Lambda$ has at most $L - M$ nonzero eigenvalues. $\square$

2.3 Skew-Spark Properties

In Section 2.2, we proved that if an erasure set, Λ, satisfies the minimal redundancy condition, then we are guaranteed to find a robust bridge set Ω for Λ for which $|\Omega| \leq |\Lambda|$. However, in the implementation of Nilpotent Bridging, we would like to avoid a search procedure to look for robust bridge sets. To do this we require a stronger property, known as the *skew-spark property*. In this section we will show that most frames satisfy this property. By most, we mean that the set of frames which satisfy this property is open and dense in the set of all frames. We will also show that any union of two bases cannot satisfy this property. However, we can define a new skew-spark property, known as the *block skew-spark property* which is satisfied for most unions of two bases.

2.3.1 The Skew-Spark Property for Dual Frame Pairs[3]

We open this subsection with a lemma that lays out necessary conditions for the bridge matrix, $B(\Lambda, \Omega)$ to be invertible.

Lemma 2.11. *Let (F, G) be a dual frame pair of length N in an n-dimensional Hilbert space, $\mathcal{H}$. Let Λ be an erasure set, and Ω be a bridge set satisfying $|\Lambda| = |\Omega|$. A necessary (but not sufficient) condition for $B(F, G, \Lambda, \Omega)$ to be an invertible matrix is*

$$|\Lambda| \leq \min\left\{n, N-n, \frac{N}{2}\right\} \tag{2.23}$$

Proof. If $|\Lambda| > n$, then the rows of the bridge matrix $B(F, G, \Lambda, \Omega)$ will be linearly dependent (since $\mathcal{H}$ is an n-dimensional space). Thus, $B(F, G, \Lambda, \Omega)$ will fail to be invertible.

Assume that $B(\Lambda, \Omega)$ is invertible, and $|\Lambda| > N - n$. Then, since the bridge equation $B(F, G, \Lambda, \Omega)C^* = B(F, G, \Lambda, \Lambda)$ has a solution ($C^* = B(F, G, \Lambda, \Omega)^{-1}B(F, G, \Lambda, \Lambda)$), Theorem 2.9 asserts that Λ satisfies the minimal redundancy condition with respect to G. Therefore, $|\Lambda^c| \geq n$. So, $N = |\Lambda| + |\Lambda^c| > N - n + n = N$. This is a contradiction, and therefore, if $B(F, G, \Lambda, \Omega)$ is invertible, then $|\Lambda| \leq N - n$.

If $|\Lambda| > \frac{N}{2}$, then $|\Lambda| + |\Omega| > N$. This is a contradiction since Λ and Ω are disjoint subsets of $\{1, \cdots, N\}$. □

Remark 2.12. It was pointed out by Cameron Farnsworth in a Graduate Student Seminar talk that $|\Lambda| \leq \min\{n, N-n\}$ implies that $\frac{N}{2} \geq \min\{n, N-n\}$. This is because

$$\frac{N}{2} = \frac{n}{2} + \frac{N-n}{2} \geq \frac{\min\{n, N-n\}}{2} + \frac{\min\{n, N-n\}}{2} = \min\{n, N-n\}. \tag{2.24}$$

In light of the above conditions, we make the following definitions. A dual frame pair

[3]The statements and proofs of Lemma 2.11, Theorem 2.17, Theorem 2.18, and Corollary 2.19 in this subsection were reconstructed from [37] and [38] With permission of Springer.

(F,G) is said to satisfy the *skew-spark property of order* L if for all $\Lambda \subset \mathbb{J}$, and for any subset $\Omega \subset \Lambda^c$ satisfying $|\Omega| = |\Lambda| = L$, the bridge matrix, $B(\Lambda,\Omega)$, is invertible. We say that (F,G) satisfies the *full skew-spark property* if it satisfies the skew-spark property of order $L = \min\{N-n, n\}$. (In infinite dimensions, we say (F,G) satisfies the *full skew-spark property* if it satisfies the skew-spark property of order L for every $L < \infty$.)

Remark 2.13. (1) By the definition of erasure set, $|\Lambda|$ must be finite. This will be important later on, as we will be using the Baire Category Theorem on an intersection over the set of all finite subsets of the index set $\mathbb{J}$. This set is countable whenever $\mathbb{J}$ is countable, and Λ is finite.

(2) The definition of the skew-spark property guarantees that Ω is a robust bridge set for Λ whenever $\Omega \subset \Lambda^c$ and $|\Omega| = |\Lambda| \leq \min\{n, N-n\}$. This is because the bridge equation ($B(\Lambda,\Omega)C^* = B(\Lambda,\Lambda)$) always has a solution when $B(\Lambda,\Omega)^{-1}$ exists.

We use the term "spark" in describing our property because the skew-spark property is similar to the idea of spark for frames. A frame $F = \{f_j\}_{j\in\mathbb{J}}$ is said to have *spark* k if the size of the smallest linearly dependent subcollection of $\{f_j\}_{j=1}^N$ is k. We will extend this definition to infinite frames by saying that an infinite frame for which every finite subcollection is linearly independent has spark ∞. The frame $F = \{f_j\}_{j\in\mathbb{J}}$ is said to have full spark if it has spark $n+1$ (or ∞ if F is an infinite frame). It was proven by Lu and Do in [42] that the set of finite full spark frames is dense in the set of all finite frames. In [1] Alexeev, Cahill, and Mixon proved a stronger version of this result by using the Zariski topology. For more on the full spark property, see [9], [10], and [15]. It is easily seen that if a dual frame pair has skew-spark k, then both frames in the pair must have spark at least $k+1$.

For a fixed frame F, recall that the *dual set* of F is the set of all dual frames to F, and we denote this set by $\mathcal{D}(F)$. That is,

$$\mathcal{D}(F) = \{G : (F, G) \text{ is a dual frame pair for } \mathcal{H}\}.$$

As hinted at in Remark 2.13, we will be using the Baire Category Theorem on the set $\mathcal{D}(F)$, and as such we will need to define a complete metric on $\mathcal{D}(F)$. A natural norm which will achieve this is the Bessel bound norm (cf. [28], [29], and [30]). Given a frame, $F = \{f_j\}_{j\in\mathbb{J}}$, we define the *Bessel bound norm* of F as

$$\|F\|_B = \|\Theta_F\|_{op}. \tag{2.25}$$

That is, the the Bessel bound norm of F is the operator norm of its analysis operator. It is easily verified that $\|F\|_B$ is also the square root of the optimal upper frame bound for F. To prove that the metric induced on $\mathcal{D}(F)$ by the Bessel bound norm is complete, it suffices to show that $\mathcal{D}(F)$ is a closed subspace of the Banach space $(\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B)$, where $\mathcal{H}^{\mathbb{J}}$ denotes the set of Bessel sequences equipped with the Bessel bound norm. That is,

$$(\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B) = \{F = \{f_j\}_{j\in\mathbb{J}} : \|F\|_B < \infty\}. \tag{2.26}$$

In finite dimensions, when $\mathbb{J} = \{1, 2, \cdots, N\}$, we will use the notation $\mathcal{H}^N$ in place of $\mathcal{H}^{\mathbb{J}}$. We include the proof that $(\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B)$ is a Banach space for the sake of completeness.

Proposition 2.14. $(\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B)$ *is a Banach space.*

Proof. We will only provide the proof of completeness here. Assume that $\{F_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B)$. Let Θ_k denote the analysis operator for F_k for all $k \in \mathbb{N}$. Then, $\{\Theta_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $B(\mathcal{H}, \ell^2(\mathbb{J}))$, the set of bounded linear operators from $\mathcal{H}$ to $\ell^2(\mathbb{J})$. Since $B(\mathcal{H}, \ell^2(\mathbb{J}))$ is a Banach space, we can find $\Theta \in B(\mathcal{H}, \ell^2(\mathbb{J}))$ so that $\Theta_k \to \Theta$ in operator norm. Define $F \in \mathcal{H}^{\mathbb{J}}$ by $f_j = \Theta^* e_j$ for all $j \in \mathbb{J}$. It is easily verified that the analysis operator for F is Θ, and thus $F \in (\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B)$. Since $\Theta_k \to \Theta$ in operator norm, $F_k \to F$ in $(\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B)$. Therefore, $(\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B)$ is complete. $\square$

The next proposition shows that the space $(\mathcal{D}(F), \|\cdot\|_B)$ is a complete metric space. Notice that $\mathcal{D}(F)$ is not closed under scalar multiplication, and hence it is not a vector space.

Proposition 2.15. *The space* $(\mathcal{D}(F), \|\cdot\|_B)$ *is a complete metric space.*

Proof. It suffices to show that $(\mathcal{D}(F), \|\cdot\|_B)$ is a closed subset of $(\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B)$. To do this, assume that $\{G_k\}_{k\in\mathbb{N}}$ is a sequence in $(\mathcal{D}(F), \|\cdot\|_B)$ converging to G. Since $(\mathcal{H}^{\mathbb{J}}, \|\cdot\|_B)$ is a Banach space, G is a Bessel sequence. Let Θ denote the analysis operator for G and Θ_k denote the analysis operator for G_k for all $k \in \mathbb{N}$. Let Θ_F denote the analysis operator for F. Then,

$$\Theta_F^* \Theta = \lim_{k\to\infty} \Theta_F^* \Theta_k = \lim_{k\to\infty} I = I$$

where I denotes the identity operator on $\mathcal{H}$. Hence, for all $f \in \mathcal{H}$,

$$f = \Theta_F^* \Theta f = \sum_{j\in\mathbb{J}} \langle f, g_j \rangle f_j.$$

Therefore, by Proposition 1.5, G is a frame, and $G \in (\mathcal{D}(F), \|\cdot\|_B)$. $\square$

The next proposition shows that while $(\mathcal{D}(F), \|\cdot\|_B)$ is not a vector space, it is the next best thing: a convex set.

Proposition 2.16. $\mathcal{D}(F)$ *is a convex subset of* $\mathcal{H}^{\mathbb{J}}$.

Proof. Let $G_0, G_1 \in \mathcal{D}(F)$. For any $t \in [0,1]$, define $G_t = (1-t)G_0 + tG_1$, let Θ_t denote the analysis operator for G_t, and Θ_F^* denote the synthesis operator for F. Then,

$$\Theta_F^*\Theta_t = \Theta_F^*\left((1-t)\Theta_0 + t\Theta_1\right) = (1-t)\Theta_F^*\Theta_0 + t\Theta_F^*\Theta_1 = I. \tag{2.27}$$

It remains to show that G_t is a frame. To do this, Proposition 1.5 says that we need only show that G_t is a Bessel sequence, since the computations above show that

$$f = \sum_{j\in\mathbb{J}} \left\langle f, g_j^{(t)} \right\rangle f_j \quad \forall f \in \mathcal{H}.$$

Let B and B' denote the upper frame bounds for G and G', respectively. Then,

$$\begin{aligned}
\sum_{j\in\mathbb{J}} \left|\left\langle f, g_j^{(t)} \right\rangle\right|^2 &= \sum_{j\in\mathbb{J}} |\left\langle f, (1-t)g_j + tg_j' \right\rangle|^2 \\
&= \sum_{j\in\mathbb{J}} |(1-t)\left\langle f, g_j \right\rangle + t\left\langle f, g_j' \right\rangle|^2 \\
&\leq \sum_{j\in\mathbb{J}} \left[(1-t)|\left\langle f, g_j \right\rangle| + t|\left\langle f, g_j' \right\rangle|\right]^2 \\
&= \sum_{j\in\mathbb{J}} \left[(1-t)^2|\left\langle f, g_j \right\rangle|^2 + t^2|\left\langle f, g_j' \right\rangle|^2 + 2t(1-t)|\left\langle f, g_j \right\rangle\left\langle f, g_j' \right\rangle|\right] \\
&\leq B(1-t)^2\|f\|^2 + B't^2\|f\|^2 \\
&+ 2t(1-t)\left(\sum_{j\in\mathbb{J}} |\left\langle f, g_j \right\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{j\in\mathbb{J}} |\left\langle f, g_j' \right\rangle|^2\right)^{\frac{1}{2}} \\
&\leq B(1-t)^2\|f\|^2 + B't^2\|f\|^2 + 2t(1-t)\sqrt{BB'}\|f\|^2 \\
&= \left((1-t)\sqrt{B} + t\sqrt{B'}\right)^2 \|f\|^2.
\end{aligned}$$

Therefore G_t is a Bessel sequence. This completes the proof. □

We are now ready to state the main theorem of this subsection. For the theorem, we say that A is a *generic* subset of a set B if A is an open and dense subset of B.

Theorem 2.17. *Assume that F has the full spark property. Then, $\mathcal{G} := \{G \in \mathcal{D}(F) : (F, G)$ has the full skew spark property$\}$ is generic in $\mathcal{D}(F)$.*

The proof requires a few technical lemmas, but we will give a brief overview of the results here for the sake of readability. First, for every erasure set Λ satisfying $|\Lambda| \leq \min\{n, N-n\}$ (or $|\Lambda| < \infty$ for $n = \infty$) and every bridge set $\Omega \subset \mathbb{J} \setminus \Lambda$ satisfying $|\Omega| = |\Lambda|$, we define the set

$$\mathcal{G}_{\Lambda,\Omega} = \{G \in \mathcal{D}(F) : B(F, G, \Lambda, \Omega) \text{ is invertible}\}.$$

Theorem 2.18 and Corollary 2.19 provide a proof that $\mathcal{G}_{\Lambda,\Omega}$ is non-empty for a full spark frame, F. Within the proof of Theorem 2.17, we use a convex path argument to show that $\mathcal{G}_{\Lambda,\Omega}$ is open and dense in $\mathcal{D}(F)$. The remainder of the proof of Theorem 2.17 simply says that $\mathcal{G}$ is the intersection of the sets $\mathcal{G}_{\Lambda,\Omega}$ over all possible bridge and erasure sets. Thus $\mathcal{G}$ is a dense set, by the Baire Category Theorem. The proofs here are presented for finite dimensions, however, many of the results (with the exception of openness) carry over to infinite dimensions.

The following theorem shows how an arbitrary set of vectors indexed by Λ can be extended to a dual frame to a given frame, F, provided that Λ satisfies the minimal redundancy condition with respect to F.

Theorem 2.18. *Let Λ be an erasure set for a frame F with the minimal redundancy condition and let $\{g_j\}_{j\in\Lambda}$ be assigned arbitrarily. Then, $\{g_j\}_{j\in\Lambda}$ can be extended to a dual frame $\{g_j\}_{j=1}^N \in \mathcal{D}(F)$.*

Proof. We first show that under the same conditions on F, the set $\{h_j\}_{j\in\Lambda}$ can be extended to $\{h_j\}_{j=1}^N$ so that $\sum_{j=1}^N f_j \otimes h_j = 0$. Let $A = \sum_{j\in\Lambda} f_j \otimes h_j$. Let $\{k_j\}_{j\in\Lambda^c}$ be a dual to the reduced frame $\{f_j\}_{j\in\Lambda^c}$. Then, $I = \sum_{j\in\Lambda^c} f_j \otimes k_j$. So,

$$A = \left(\sum_{j\in\Lambda^c} f_j \otimes k_j\right) A = \sum_{j\in\Lambda^c} f_j \otimes (A^* k_j).$$

For each $j \in \Lambda^c$, let $h_j = -A^* k_j$. Then,

$$\begin{aligned}
\sum_{j=1}^{N} f_j \otimes h_j &= \sum_{j\in\Lambda^c} f_j \otimes h_j + \sum_{j\in\Lambda} f_j \otimes h_j \\
&= -\sum_{j\in\Lambda^c} f_j \otimes A^* k_j + A \\
&= A - \left(\sum_{j\in\Lambda^c} f_j \otimes k_j\right) A \\
&= A - IA \\
&= 0.
\end{aligned}$$

Now, let $\{g'_j\}_{j=1}^{N} \in \mathcal{D}(F)$. Let $h_j = g_j - g'_j$ for $j \in \Lambda$. Then, as above, we can extend $\{h_j\}_{j\in\Lambda}$ to $\{h_j\}_{j=1}^{N}$ so that $\sum_{j=1}^{N} f_j \otimes h_j = 0$. For all j, let $\tilde{g}_j = g'_j + h_j$. So,

$$\sum_{j=1}^{N} f_j \otimes \tilde{g}_j = \sum_{j=1}^{N} f_j \otimes g'_j + \sum_{j=1}^{N} f_j \otimes h_j = I + 0 = I.$$

Thus, $\{\tilde{g}_j\}_{j=1}^{N} \in \mathcal{D}(F)$. Furthermore, for $j \in \Lambda$,

$$\tilde{g}_j = g'_j + h_j = g'_j + g_j - g'_j = g_j.$$

Therefore, $\{\tilde{g}_j\}_{j=1}^{N}$ is the desired extension of $\{g_j\}_{j\in\Lambda}$. $\square$

There is no problem in generalizing Theorem 2.18 to infinite dimensions.

The following corollary utilizes Theorem 2.18, and Lemma 1.7 to show that the set $\mathcal{G}_{\Lambda\Omega}$ is non-empty. As with the previous theorem, the corollary is fully generalizable to infinite dimensions.

Corollary 2.19. *Assume that $F \in \mathcal{H}^N$ satisfies the full spark property. Let Λ be an erasure set satisfying $|\Lambda| \leq \min\{n, N-n, \frac{N}{2}\}$, and Ω be a bridge set satisfying $|\Lambda| = |\Omega|$ and $\Lambda \cap \Omega = \emptyset$. Then there exists a dual frame G to F so that $B(F, G, \Lambda, \Omega)$ is invertible.*

Proof. Define a bijection $\varphi : \Omega \to \Lambda$. Let $\{g_j\}_{j\in\Omega} = \{f_{\varphi(j)}\}_{j\in\Omega}$. By Theorem 2.18, we can extend $\{g_j\}_{j\in\Omega}$ to a dual frame G for F. Then $B(F, G, \Lambda, \Omega)$ is the Gram matrix of a permutation of the finite sequence $\{f_j : j \in \Lambda\}$, which is invertible since $\{f_j : j \in \Lambda\}$ is linearly independent. $\square$

We are now ready for the proof of Theorem 2.17. As was mentioned earlier, the proof consists of three portions. The first portion (the first paragraph) says that $\mathcal{G}$ is a finite intersection of the sets $\mathcal{G}_{\Lambda,\Omega}$. Thus by the Baire Category Theorem, if each of the sets $\mathcal{G}_{\Lambda,\Omega}$ is open and dense in $\mathcal{D}(F)$, then so is $\mathcal{G}$. The second paragraph provides a proof that each of the $\mathcal{G}_{\Lambda,\Omega}$ is open in $\mathcal{D}(F)$. The third paragraph uses what we call a "where there is spark, there is fire" type argument to give a proof that $\mathcal{G}_{\Lambda,\Omega}$ is dense in $\mathcal{D}(F)$.

Proof of Theorem 2.17. Let $\Gamma = \{\Lambda \subset \{1, \cdots, N\} : |\Lambda| \leq \min\{\frac{N}{2}, N-n, n\}\}$. For a given $\Lambda \in \Gamma$, let $\Phi_\Lambda = \{\Omega \subset \{1, \cdots, N\} : |\Omega| = |\Lambda|, \Omega \cap \Lambda = \emptyset\}$. Then, $\mathcal{G} = \bigcap_{\Lambda\in\Gamma} \bigcap_{\Omega\in\Phi_\Lambda} \mathcal{G}_{\Lambda,\Omega}$, where $\mathcal{G}_{\Lambda,\Omega} = \{G \in \mathcal{D}(F) : \det(B(F, G, \Lambda, \Omega)) \neq 0\}$. Since we are intersecting over all possible erasure sets and all corresponding bridge sets, the above intersection is finite. So by the Baire category theorem, if we show that each $\mathcal{G}_{\Lambda,\Omega}$ is open and dense, then $\mathcal{G}$ will also be open and dense.

Fix an erasure set Λ, and a corresponding bridge set Ω. It is easily verified that the maps $G \overset{\alpha}{\mapsto} B(F, G, \Lambda, \Omega)$ and $B(F, G, \Lambda, \Omega) \mapsto \det(B(F, G, \Lambda, \Omega))$ are continuous. So, $\mathcal{G}_{\Lambda,\Omega} = (\det \circ\alpha)^{-1}(\mathbb{C} \setminus \{0\})$ is an open set.

To show density of $\mathcal{G}_{\Lambda,\Omega}$, let $\epsilon > 0$, and assume that $G_0 \in \mathcal{D}(F)\backslash\mathcal{G}_{\Lambda,\Omega}$. Since F satisfies the full spark property, Λ satisfies the minimal redundancy condition with respect to F.

Thus, by Corollary 5.5, there is a $G_1 \in \mathcal{D}(F)$ so that $\det(B(F, G_1, \Lambda, \Omega)) \neq 0$. Let $G_t = (1-t)G_0 + tG_1$. By proposition 2.16, $G_t \in \mathcal{D}(F)$. Furthermore, $\det(B(F, G_t, \Lambda, \Omega))$ is a polynomial in t satisfying $\det(B(F, G_t, \Lambda, \Omega))(0) = 0$ and $\det(B(F, G_t, \Lambda, \Omega))(1) \neq 0$. Thus, $\det(B(F, G_t, \Lambda, \Omega))$ has only finitely many zeros. So, we can find $0 < t_0 < \frac{\epsilon}{\|G_1 - G_0\|}$ so that $G_{t_0} \in \mathcal{G}_{\Lambda,\Omega}$. Furthermore,

$$\|G_{t_0} - G_0\| = \|(1-t_0)G_0 + t_0 G_1 - G_0\| = \|t_0(G_1 - G_0)\| \leq |t_0| \, \|G_1 - G_0\| < \epsilon.$$

Hence, $\mathcal{G}_{\Lambda,\Omega}$ is dense in $\mathcal{D}(F)$.

Therefore, by the Baire-Category theorem, $\mathcal{G}$ is generic in $\mathcal{D}(F)$. $\square$

Theorem 2.17 can be interpreted as a guarantee that a randomly generated finite dual frame pair will satisfy the full skew-spark property. This is because the set of frames which are not full spark is closed and nowhere dense, and the set of duals to a full spark frame for which the pair does not satisfy the full skew-spark property is closed and nowhere dense.

Theorem 2.17 is partially generalizable for countably infinitely indexed frames in a separable Hilbert space. However, since the intersection in the proof is no longer finite, we are no longer guaranteed openness. Thus, we get the following theorem for infinite dimensions.

Theorem 2.20. *Assume that F is a countably indexed frame for a separable Hilbert space, $\mathcal{H}$, which satisfies the full spark property. Then,*

$$\mathcal{G} := \{G \in \mathcal{D}(F) : (F, G) \textit{ satisfies the full skew spark property}\}$$

is dense in $\mathcal{D}(F)$.

2.3.2 The Skew-Spark Property for Parseval Frames[4]

In the previous subsection we proved that most dual frame pairs satisfy the full skew-spark property. In this subsection we will modify these results to show that most finite Parseval frames satisfy the full skew-spark property. To do so, let $PF_N(\mathcal{H})$ denote the set of length N frames for an n-dimensional Hilbert space, $\mathcal{H}$. In this subsection, we will require that $\dim(\mathcal{H}) = n < \infty$. The following Theorem is the main result of this subsection.

Theorem 2.21. *The set*

$$\mathcal{P} = \{F \in PF_N(\mathcal{H}) : F \textit{ has full skew-spark}\}$$

is generic in $PF_N(\mathcal{H})$.

To prove this theorem, we require two lemmas. The first lemma, Lemma 2.22, is akin to Corollary 2.19, though the proof is quite different. The lemma shows that the set

$$\mathcal{P}_{(\Lambda,\Omega)} = \{F \in PF_N(\mathcal{H}) : B(F, F, \Lambda, \Omega)^{-1} \text{ exists}\}$$

is non-empty.

Lemma 2.22. *There exists a Parseval frame F such that $B(F, F, \Lambda, \Omega)$ is $\frac{1}{2}I_L$ (the $L \times L$ identity matrix), where $L = |\Lambda| = |\Omega| \leq min\left\{n, N-n, \frac{N}{2}\right\}$ and $\Omega \cap \Lambda = \emptyset$.*

Proof. Enumerate $\Lambda = \{\lambda_j\}_{j=1}^L$, and $\Omega = \{\omega_j\}_{j=1}^L$. Let $\{e_j\}_{j=1}^L$ be an orthonormal set in $\mathcal{H}$. For each $1 \leq j \leq L$, set $f_{\lambda_j} = f_{\omega_j} = \frac{1}{\sqrt{2}}e_j$. Let $\mathcal{F} = \text{span}\{f_j : j \in \Lambda \cup \Omega\}$. Then, $\dim(\mathcal{F}^\perp) = n - L \leq N - 2L$ since $n \leq N - L$. So, since $|\{1, 2, \cdots, L\} \setminus (\Lambda \cup \Omega)| =$

[4]The statements and proofs of Theorem 2.21, Lemma 2.22, and Lemma 2.23 in this subsection were reproduced from [37] With permission of Springer.

$N-2L$, one can find a Parseval frame $\{f_j : j \in (\Lambda \cup \Omega)^c\}$ for $\mathcal{F}^\perp$. Then, $\mathcal{F} = \{f_j\}_{j=1}^N$ is a frame for $\mathcal{H}$ for which $B(F, F, \Lambda, \Omega) = \frac{1}{2} I_L$. $\square$

Lemma 2.23 is a "where there is spark, there is fire" type argument which shows that $\mathcal{P}_{(\Lambda,\Omega)}$ is a dense set in $PF_N(\mathcal{H})$. This lemma is the analogue of paragraph 3 of Theorem 2.17. However, the proof is much more difficult because the paths between Parseval frames are more complicated. Along these paths, the determinant gives rise to rational functions as opposed to polynomials in the case of dual frame pairs. However, non-zero rational functions will suffice since they only have finitely many zeros and finitely many points where they are undefined.

Lemma 2.23. *Let F be a Parseval frame for an n-dimensional Hilbert space $\mathcal{H}$. Let Λ be an erasure set with $|\Lambda| \leq \min\left\{n, N-n, \frac{N}{2}\right\}$ and Ω be a bridge set for Λ with $\Lambda \cap \Omega = \emptyset$ and $|\Lambda| = |\Omega|$. Then, given $\epsilon > 0$, there exists a Parseval frame $\tilde{F}$ with $\|F - \tilde{F}\| < \epsilon$ so that $B(\tilde{F}, \tilde{F}, \Lambda, \Omega)$ is invertible.*

Proof. Enumerate $\Lambda = \{\lambda_j\}_{j=1}^L$, and $\Omega = \{\omega_j\}_{j=1}^L$. Assume without loss of generality that $B(F, F, \Lambda, \Omega)$ is singular. By Lemma 2.22, there is a Parseval frame $F_1 = \{f_j^{(1)}\}_{j=1}^N$ such that $B(F_1, F_1, \Lambda, \Omega)$ is invertible. Let $F = F_0 = \{f_j^{(0)}\}_{j=1}^N$. For $0 < t < 1$, define $F_t = \{f_j^{(t)}\}_{j=1}^N$, where $f_j^{(t)} = (1-t)f_j^{(0)} + tf_j^{(1)}$. Then, for each $t \in [0,1]$, F_t is a set of vectors in $\mathcal{H}$, but they need not span $\mathcal{H}$, and hence aren't necessarily a frame for $\mathcal{H}$. Let S_t be the frame operator (or Bessel operator in the case that F_t is not a frame) for F_t. Then,

$$S_t = \sum_{j=1}^N f_j^{(t)} \otimes f_j^{(t)}.$$

If $\{e_j\}_{j=1}^n$ is the standard orthonormal basis for $\mathcal{H}$, the matrix coordinate functions $m_{j,k}(t) = \langle S_t e_k, e_j \rangle$ of the matrix $M(t)$ of S_t with respect to $\{e_j\}_{j=1}^n$ are quadratic functions of t. Thus, $\det M(t)$ is a polynomial function of t. Hence, the *formal inverse* matrix valued function, which we denote by $Q(t)$, that is given by the *adjoint* formula (or Cramer's rule) for the inverse of an invertible matrix has the form $Q(t) = (q_{jk}(t))_{j,k=1}^n$, where the coordinate functions $q_{jk}(t)$ are *rational* functions of t.

At points where S_t is invertible, let P_t be the Parseval frame $P_t = S_t^{-\frac{1}{2}} F_t$, where $S_t^{-\frac{1}{2}} = (S_t^{-1})^{\frac{1}{2}}$ is the positive square root of S_t^{-1}. Then, $P_0 = F_0$. We have

$$B(P_t, P_t, \Lambda, \Omega) = \left(\left\langle S_t^{-\frac{1}{2}} f_{\lambda_j}^{(t)}, S_t^{-\frac{1}{2}} f_{\omega_k}^{(t)} \right\rangle\right)_{j,k=1}^{L} = \left(\left\langle S_t^{-1} f_{\lambda_j}^{(t)}, f_{\omega_k}^{(t)} \right\rangle\right)_{j,k=1}^{L}.$$

If $[f_{\lambda_j}^{(t)}]_E$ denotes the coordinate vector of $f_{\lambda_j}^{(t)}$ with respect to $E = \{e_j\}_{j=1}^n$, and we define $[f_{\omega_k}^{(t)}]_E$ similarly, then for points $t \in [0, 1]$ for which S_t is invertible,

$$\left\langle S_t^{-1} f_{\lambda_j}^{(t)}, f_{\omega_k}^{(t)} \right\rangle = \left\langle Q(t)[f_{\lambda_j}^{(t)}]_E, [f_{\omega_k}^{(t)}]_E \right\rangle.$$

Since $Q(t)$ is an $n \times n$ matrix valued function with rational coordinate functions, and $[f_{\lambda_j}^{(t)}]_E$ and $[f_{\omega_k}^{(t)}]_E$ are matrix valued vectors with polynomial coordinate functions, this yields a formal rational matrix valued function on $[0, 1]$ with rational coordinate functions,

$$b_{j,k}(t) := \left\langle Q(t)[f_{\lambda_j}^{(t)}]_E, [f_{\omega_k}^{(t)}]_E \right\rangle.$$

Let $b(t) := (b_{j,k}(t))_{j,k=1}^L$. Let $\delta(t) := \det(b(t))$ denote the formal determinant. At points t where S_t is invertible we have $\delta(t) = \det(B(P_t, P_t, \Lambda, \Omega))$. By hypothesis, F_0 and F_1 are Parseval frames, so $S_0 = S_1 = I_n$ (the $n \times n$ identity matrix). Thus $P_0 = F_0 = F$ and $P_1 = F_1$. By hypothesis $B(F, F, \Lambda, \Omega)$ is singular, so $\delta(0) = 0$. By construction, $B(F_1, F_1, \Lambda, \Omega)$ is invertible, and so $\delta(1) \neq 0$. A nonconstant rational function can have at most finitely many points where it is undefined, and at most finitely many zeros. So, there is an $\alpha > 0$ so that $\delta(t)$ is defined and nonzero. Since S_0 is invertible and the map $t \mapsto S_t$ is continuous at $t = 0$, the map $t \mapsto S_t^{-\frac{1}{2}}$ is continuous at $t = 0$. Thus, the map

$t \mapsto P_t$ is continuous at $t = 0$. So, there exists $\alpha_1 \in (0, \alpha)$ so that $\|F - P_t\| < \epsilon$ wherever $t \in [0, \alpha_1]$. Choose $\tilde{t} \in (0, \alpha_1)$ and let $\tilde{F} = P_{\tilde{t}}$. Then, $\|F - \tilde{F}\| < \epsilon$ and $\delta(\tilde{t}) \neq 0$, so $B(\tilde{F}, \tilde{F}, \Lambda, \Omega)$ is invertible as required. □

We are now ready for the proof of Theorem 2.21. What remains of the proof is very similar to paragraphs 1 and 2 in the proof of Theorem 2.17. Thus, all that we need to show is that each of the sets $\mathcal{P}_{(\Lambda,\Omega)}$ is open and that $\mathcal{P}$ is a finite intersection of these sets.

Proof of Theorem 2.21. Let $\Gamma = \left\{\Lambda \subset \{1, 2, \cdots, N\} : |\Lambda| \leq \min\left\{n, N - n, \frac{N}{2}\right\}\right\}$, and given $\Lambda \in \Gamma$, define $\Phi_\Lambda = \{\Omega : \Omega \text{ is a bridge set for } \Lambda\}$. Then,

$$\mathcal{P} = \cap_{\Omega \in \Phi_\Lambda} \cap_{\Lambda \in \Gamma} \mathcal{P}_{(\Lambda,\Omega)}.$$

By Lemma 2.23, each $\mathcal{P}_{(\Lambda,\Omega)}$ is dense in $PF_N(\mathcal{H})$.

Define $\delta_{(\Lambda,\Omega)} : \mathcal{P}_{(\Lambda,\Omega)} \to \mathbb{C}$ by

$$\delta_{(\Lambda,\Omega)}(F) = \det(B(F, F, \Lambda, \Omega)).$$

Then since $\delta_{(\Lambda,\Omega)}$ is continuous,

$$\mathcal{P}_{(\Lambda,\Omega)} = \delta^{-1}_{(\Lambda,\Omega)}(\mathbb{C} \setminus \{0\}).$$

Thus, each $\mathcal{P}_{(\Lambda,\Omega)}$ is open.

Therefore, by the Baire Category theorem, $\mathcal{P}$ is open and dense in $PF_N(\mathcal{H})$. □

As with dual frame pairs, Theorem 2.21 should be viewed as a guarantee that any randomly selected finite Parseval frame will satisfy the full skew-spark property. This is again because the set of finite Parseval frames which do not satisfy the full skew-spark property is closed and nowhere dense.

2.3.3 The Block Skew-Spark Property[5]

In this subsection, we will fix $\mathcal{H}$ to be an n-dimensional Hilbert space. Let $E = \frac{1}{\sqrt{2}}\{e_j\}_{j=1}^{2n}$ where $\{e_j\}_{j=1}^{n}$ and $\{e_j\}_{j=n+1}^{2n}$ denote bases for a $\mathcal{H}$, with dual bases $\{\tilde{e}_j\}_{j=1}^{n}$ and $\{\tilde{e}_j\}_{j=n+1}^{2n}$, respectively. By defining $\tilde{E} = \frac{1}{\sqrt{2}}\{\tilde{e}_j\}_{j=1}^{2n}$, the pair $(\tilde{E}, E)$ is a special type of dual frame pair which we will frequently refer to as a union of two bases.

The following remark explains why special conditions must be placed on the bridge set for a given erasure set for a union of two bases.

Remark 2.24. Assume that $\Lambda = \Lambda_1 \sqcup \Lambda_2$ for $\Lambda_1 \subset \{1, \cdots, n\}$ and $\Lambda_2 \subset \{n+1, \cdots, 2n\}$. Assume that Ω is a bridge set for Λ (so $\Omega \subset \Lambda^c$) satisfying $|\Lambda| = |\Omega|$. Decompose Ω as $\Omega = \Omega_1 \sqcup \Omega_2$ where $\Omega_1 \subset \{1, \cdots, n\}$ and $\Omega_2 \subset \{n+1, \cdots, 2n\}$. Then the bridge matrix has the following block form:

$$B(\tilde{E}, E, \Lambda, \Omega) = \begin{pmatrix} B(\tilde{E}, E, \Lambda_1, \Omega_1) & B(\tilde{E}, E, \Lambda_1, \Omega_2) \\ B(\tilde{E}, E, \Lambda_2, \Omega_1) & B(\tilde{E}, E, \Lambda_2, \Omega_2) \end{pmatrix}. \tag{2.28}$$

Since $\{\tilde{e}_j\}_{j=1}^{n}$ is the dual basis to $\{e_j\}_{j=1}^{n}$, we have $\langle \tilde{e}_j, e_k \rangle = \delta_{j,k}$. But $\Lambda_1 \cap \Omega_1 = \emptyset$, and thus $B(E, \tilde{E}, \Lambda_1, \Omega_1) = 0$. Similarly, $B(\tilde{E}, E, \Lambda_2, \Omega_2) = 0$. Therefore, the bridge matrix takes the following block off-diagonal form:

$$B(\tilde{E}, E, \Lambda, \Omega) = \begin{pmatrix} 0 & B(\tilde{E}, E, \Lambda_1, \Omega_2) \\ B(\tilde{E}, E, \Lambda_2, \Omega_1) & 0 \end{pmatrix}. \tag{2.29}$$

The above matrix is invertible if and only if $B(\tilde{E}, E, \Lambda_1, \Omega_2)$ and $B(\tilde{E}, E, \Lambda_2, \Omega_1)$ are both invertible. This can only happen when $|\Lambda_1| = |\Omega_2|$ and $|\Lambda_2| = |\Omega_1|$.

In light of the previous remark, we make the following definitions to appropriately

[5] The material in this subsection is reprinted from [39], with permission from Elsevier.

restrict the class of bridge sets to a given erasure set.

Definition 2.25. Given an erasure set Λ, we will decompose Λ as the disjoint union $\Lambda = \Lambda_1 \sqcup \Lambda_2$ where $\Lambda_1 \subset \{1, \cdots, n\}$ and $\Lambda_2 \subset \{n+1, \cdots, 2n\}$.

1. We call $\Omega \subset \Lambda^c$ a *block bridge set* for Λ if when we decompose Ω as the disjoint union $\Omega = \Omega_1 \sqcup \Omega_2$ where $\Omega_1 \subset \{1, \cdots, n\}$ and $\Omega_2 \subset \{n+1, \cdots, 2n\}$, we have $|\Lambda_1| = |\Omega_2|$, and $|\Lambda_2| = |\Omega_1|$.

2. We say that the dual frame pair $(\tilde{E}, E)$ as described above satisfies the *block skew-spark property* if for any erasure set satisfying $|\Lambda| \leq n$, and any block bridge set, Ω for Λ, the matrix $B(\tilde{E}, E, \Lambda, \Omega)$ is invertible.

To begin our investigation of unions of two bases we start with a conceptual example in $\mathbb{R}^2$ which motivates the theory involved. Using the standard orthonormal basis, $E = \{e_1, e_2\}$, we can obtain any other orthonormal basis for $\mathbb{R}^2$ by rotating E with the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Thus, we can write any union of two orthonormal bases as a rotation of the following sequence.

$$E_\theta = \frac{1}{\sqrt{2}} \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, R_\theta e_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, R_\theta e_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\}. \tag{2.30}$$

Example 2.26. If θ is not a multiple of $\frac{\pi}{2}$, E_θ satisfies the block skew-spark property.

Proof. We must first show that the inner products $\langle e_1, R_\theta e_1 \rangle$, $\langle e_1, R_\theta e_2 \rangle$, $\langle e_2, R_\theta e_1 \rangle$, and $\langle e_2, R_\theta e_1 \rangle$ are non-zero (note that $\langle e_j, R_\theta e_k \rangle = \langle R_\theta e_k, e_j \rangle$). For this, we require $\cos\theta \neq 0$, and $\sin\theta \neq 0$. So, θ can not be a multiple of $\frac{\pi}{2}$. Next, we must show that $B(E_\theta, E_\theta, \{1,2\}, \{3,4\})$, $B(E_\theta, E_\theta, \{1,3\}, \{2,4\})$, and $B(E_\theta, E_\theta, \{1,4\}, \{2,3\})$ are all invertible (note that $B(E_\theta, E_\theta, \{i,j\}, \{k,l\}) = B(E_\theta, E_\theta, \{k,l\}, \{i,j\})^T$). However, it is easily verified that these are all invertible if θ is not a multiple of $\frac{\pi}{2}$. $\square$

Example 2.26 motivates the use of the group of unitaries to study the unions of two orthonormal bases. More generally, we will use this group to study the union of an orthonormal basis and a Riesz basis. Let $\mathbb{U}$ denote the set of unitary operators in $B(\mathcal{H})$. It is known that $\mathbb{U}$ is a path connected, closed subset of $B(\mathcal{H})$ (here $B(\mathcal{H})$ denotes the set of bounded operators on $\mathcal{H}$ with the operator norm topology). Thus, $\mathbb{U}$ with the induced operator norm topology forms a complete metric space.

Remark 2.27. Fix an orthonormal basis, $\{e_j\}_{j=1}^n$ for $\mathcal{H}$. Any basis for $\mathcal{H}$ can be written as $\{Te_j\}_{j=1}^n$ for some invertible operator $T \in B(\mathcal{H})$. Furthermore, using the polar decomposition, we have $T = UA$ for some unitary operator, U and some strictly positive operator, A. (In fact, we can take $A = |T|$.)

For a fixed unitary operator, U, a fixed positive operator, A, and a fixed orthonormal basis, $E = \{e_j\}_{j=1}^n$ for $\mathcal{H}$, we define the frame $E_{(U,A)} = \frac{1}{\sqrt{2}}\{e_j\}_{j=1}^{2n}$, where $e_{j+n} = UAe_j$ for $1 \leq j \leq n$. Notice that the dual basis to $\{UAe_j\}_{j=1}^n$ is given by $\{UA^{-1}e_j\}$. Thus, the frame $\tilde{E}_{(U,A)} = \frac{1}{\sqrt{2}}\{\tilde{e}_j\}_{j=1}^{2n}$ where $\tilde{e}_j = e_j$, and $\tilde{e}_{j+n} = UA^{-1}e_j$ for $1 \leq j \leq n$ is a dual to $E_{(U,A)}$. We are interested in studying the set of unitaries for which the dual frame pair $(\tilde{E}_{(U,A)}, E_{(U,A)})$ satisfies the block skew-spark property. The following theorem demonstrates that this class of unitaries is very large.

Theorem 2.28. *For a fixed positive operator, $A \in B(\mathcal{H})$, and a fixed orthonormal basis, $E = \{e_j\}_{j=1}^n$ for $\mathcal{H}$, the set*

$$\mathcal{U}_A = \{U \in \mathbb{U} : (\tilde{E}_{(U,A)}, E_{(U,A)}) \textit{ has the block skew-spark property}\}$$

is an open, dense subset of $\mathbb{U}$.

To prove this theorem, we will need the following lemmas. As with dual frame pairs and Parseval frames, the proof template will be the same, though the methods are somewhat more complex. For a fixed erasure set, Λ, and block bridge set Ω for Λ, we define

$$\mathcal{U}_{(A,\Lambda,\Omega)} = \{U \in \mathbb{U} : B(\tilde{E}_{(U,A)}, E_{(U,A)}, \Lambda, \Omega)^{-1} \text{ exists}\}.$$

Lemma 2.29 is our proof that $\mathcal{U}_{(A,\Lambda,\Omega)}$ is non-empty. In Lemma 2.31 we show that $\mathcal{U}_{(A,\Lambda,\Omega)}$ is open and dense in $\mathbb{U}$. Lastly, to prove Theorem 2.28, we invoke the Baire Category Theorem to show that $\mathcal{U}_A$ is open and dense in $\mathbb{U}$.

Lemma 2.29. *Fix Λ and Ω where $|\Lambda| \leq n$, and Ω is a block bridge set for Λ. For any strictly positive operator A and any fixed orthonormal basis $E = \{e_j\}_{j=1}^n$ for $\mathcal{H}$, there exists $U \in \mathbb{U}$ for which $B(\tilde{E}_{(U,A)}, E_{(U,A)}, \Lambda, \Omega)$ is invertible. That is, $\mathcal{U}_{(A,\Lambda,\Omega)}$ is non-empty.*

Proof. By Remark 2.24, it suffices to show that there exists a $U \in \mathbb{U}$ so that $B(\tilde{E}_{(U,A)}, E_{(U,A)}, \Lambda_1, \Omega_2)$ and $B(\tilde{E}_{(U,A)}, E_{(U,A)}, \Lambda_2, \Omega_1)$ are both invertible. We will first show that there exists $U_0 \in \mathbb{U}$ so that $B(\tilde{E}_{(U,A)}, E_{(U,A)}, \Lambda_1, \Omega_2)$ is invertible. To do this, assume that $\Lambda_1 = \{\lambda_j : 1 \leq j \leq L\}$, and $\Omega_2 = \{\omega_j + n : 1 \leq j \leq L\}$. Then, for any $U \in \mathbb{U}$, we have

$$\begin{aligned} B(\tilde{E}_{(U,A)}, E_{(U,A)}, \Lambda_1, \Omega_2) &= \frac{1}{2}\left(\langle \tilde{e}_j, e_k \rangle\right)_{j\in\Lambda_1,\, k\in\Omega_2} \\ &= \frac{1}{2}\left(\left\langle e_{\lambda_j}, UAe_{\omega_k} \right\rangle\right)_{j,k=1}^{L} \\ &= \frac{1}{2}\left(\left\langle U^{-1}e_{\lambda_j}, Ae_{\omega_k} \right\rangle\right)_{j,k=1}^{L}. \end{aligned}$$

Now, define U_0 to be any unitary mapping for which $U_0 e_{\omega_j} = e_{\lambda_j}$ for $1 \leq j \leq L$. Then,

$$\begin{aligned} B(\tilde{E}_{(U_0,A)}, E_{(U_0,A)}, \Lambda_1, \Omega_2) &= \frac{1}{2}\left(\left\langle U_0^{-1} e_{\lambda_j}, A e_{\omega_k}\right\rangle\right)_{j,k=1}^{L} \\ &= \frac{1}{2}\left(\left\langle e_{\omega_j}, A e_{\omega_k}\right\rangle\right)_{j,k=1}^{L}. \end{aligned}$$

To show this matrix is invertible, assume $c = (c_k)_{k=1}^{L} \in \ker B(\tilde{E}_{(U_0,A)}, E_{(U_0,A)}, \Lambda_1, \Omega_2)$. Then, for all $1 \leq j \leq L$,

$$\sum_{k=1}^{L} c_k \left\langle e_{\omega_j}, A e_{\omega_k}\right\rangle = \left\langle e_{\omega_j}, A\left(\sum_{k=1}^{L} \overline{c_k} e_{\omega_k}\right)\right\rangle = 0.$$

Let $x = \sum_{k=1}^{L} \overline{c_k} e_{\omega_k}$. Then,

$$\langle x, Ax\rangle = \sum_{j=1}^{L} \overline{c_j} \left\langle e_{\omega_j}, A\left(\sum_{k=1}^{L} \overline{c_k} e_{\omega_k}\right)\right\rangle = 0.$$

Since A is positive, this implies that $x = 0$. So, since $\{e_{\omega_k}\}_{k=1}^{L}$ is linearly independent, $c = 0$. Therefore,
$\ker B(\tilde{E}_{(U_0,A)}, E_{(U_0,A)}, \Lambda_1, \Omega_2) = \{0\}$, and $B(\tilde{E}_{(U_0,A)}, E_{(U_0,A)}, \Lambda_1, \Omega_2)$ is invertible. By a similar argument, we can find $U_1 \in \mathbb{U}$ so that $B(\tilde{E}_{(U_1,A)}, E_{(U_1,A)}, \Lambda_2, \Omega_1)$ is invertible.

If either $B(\tilde{E}_{(U_1,A)}, E_{(U_1,A)}, \Lambda_1, \Omega_2)$ or $B(\tilde{E}_{(U_0,A)}, E_{(U_0,A)}, \Lambda_2, \Omega_1)$ is invertible, the proof is complete. Otherwise, assume that $U_0 = e^{iD_0}$, and $U_1 = e^{iD_1}$ for self-adjoint operators D_0 and D_1 and define $\gamma : [0,1] \to \mathbb{U}$ to be the unitary path from U_0 to U_1 defined by

$$\gamma(t) = e^{i(1-t)D_0} e^{itD_1}.$$

For $j = 1, 2$, define the mappings $\delta_j : [0,1] \to \mathbb{C}$ by

$$\delta_1(t) = \det\left(B(\tilde{E}_{(\gamma(t),A)}, E_{(\gamma(t),A)}, \Lambda_1, \Omega_2)\right)$$

and

$$\delta_2(t) = \det\left(B(\tilde{E}_{(\gamma(t),A)}, E_{(\gamma(t),A)}, \Lambda_2, \Omega_1)\right).$$

To finish the proof it suffices to find a t value for which $\delta_j(t) \neq 0$ for $j = 1, 2$.

The components of $B(\tilde{E}_{(\gamma(t),A)}, E_{(\gamma(t),A)}, \Lambda_1, \Omega_2)$ are of the form

$$\left\langle e_{\lambda_j}, \gamma(t) A e_{\omega_k} \right\rangle = \left\langle e^{-i(1-t)D_0} e_{\lambda_j}, e^{itD_1} A e_{\omega_k} \right\rangle$$

Since D_0 is self-adjoint, we can find an orthonormal basis of eigenvectors for D_0, $\{w_j : 1 \leq j \leq n\}$, with corresponding eigenvalues $\{\mu_j : 1 \leq j \leq n\}$. Similarly, we can find an orthonormal basis, $\{v_j : 1 \leq j \leq n\}$ of eigenvectors for D_1 with corresponding eigenvalues $\{\nu_j : 1 \leq j \leq n\}$. Now, we can write the components of $B(\tilde{E}_{(\gamma(t),A)}, E_{(\gamma(t),A)}, \Lambda_1, \Omega_2)$ as linear combinations of the following:

$$\begin{aligned}
\left\langle e^{-i(1-t)D_0} w_j, e^{itD_1} A e_{\omega_k} \right\rangle &= \left\langle \text{diag}(e^{-i(1-t)\mu_1}, \cdots, e^{-i(1-t)\mu_n}) w_j, e^{itD_1} A e_{\omega_k} \right\rangle \\
&= e^{-i(1-t)\mu_j} \left\langle w_j, e^{itD_1} A e_{\omega_k} \right\rangle.
\end{aligned}$$

These can then be written as linear combinations of the form

$$\begin{aligned}
e^{-i(1-t)\mu_j} \left\langle v_\ell, e^{itD_1} A e_{\omega_k} \right\rangle &= e^{-i(1-t)\mu_j} \left\langle e^{-itD_1} v_\ell, A e_{\omega_k} \right\rangle \\
&= e^{-i(1-t)\mu_j} \left\langle \text{diag}(e^{-it\nu_1}, \cdots, e^{-it\nu_n}) v_\ell, A e_{\omega_k} \right\rangle \\
&= e^{-i(1-t)\mu_j} e^{-it\nu_\ell} \left\langle v_\ell, A e_{\omega_k} \right\rangle
\end{aligned}$$

Thus, the components are analytic functions of t. Hence, δ_1 is an analytic function of t. Similarly, δ_2 is an analytic function of t. Furthermore, $\delta_1(1) = \delta_2(0) = 0$, and $\delta_1(0), \delta_2(1) \neq 0$. Thus, there are only finitely many zeros of δ_1 and δ_2 in the interval $[0, 1]$. So there exists t_0 so that $\delta_1(t_0), \delta_2(t_0) \neq 0$. Thus, $B(\tilde{E}_{(\gamma(t_0),A)}, E_{(\gamma(t_0),A)}, \Lambda, \Omega)$ is invertible, and this completes the proof. $\square$

Remark 2.30. The choice of U_0 in the proof was motivated by operator theory. With this choice of U_0, the matrix $B(\tilde{E}_{(U_0,A)}, E_{(U_0,A)}, \Lambda_1, \Omega_2)$ is the transpose of the matrix for A with respect to the orthonormal basis $\{e_j\}_{j=1}^n$ compressed to the subspace span$\{e_j : j \in \Omega_2\}$. Thus, since A is positive it follows that $B(\tilde{E}_{(U_0,A)}, E_{(U_0,A)}, \Lambda_1, \Omega_2)$ is invertible.

Lemma 2.31. *Fix a positive operator, $A \in B(\mathcal{H})$ and an orthonormal basis, $E = \{e_j\}_{j=1}^n$ for $\mathcal{H}$. Let Λ be an erasure set satisfying $|\Lambda| \leq n$, and Ω be a block bridge set for Λ. Then the set $\mathcal{U}_{(A,\Lambda,\Omega)}$ is open and dense in $\mathbb{U}$.*

Proof. Assume $U_0 \in \mathbb{U} \setminus \mathcal{U}_{(A,\Lambda,\Omega)}$. By Lemma 2.29, we can find some $U_1 \in \mathcal{U}_{(A,\Lambda,\Omega)}$. Since U_0 and U_1 are unitary operators, we can find self-adjoint operators D_0 and D_1 so that $U_0 = e^{iD_0}$ and $U_1 = e^{iD_1}$. Then, the mapping $\gamma : [0,1] \to \mathbb{U}$ defined by $\gamma(t) = e^{i(1-t)D_0}e^{itD_1}$ is a continuous path of unitaries from U_0 to U_1. Define $\delta : [0,1] \to \mathbb{C}$ by $\delta(t) = \det(B(\tilde{E}_{(\gamma(t),A)}, E_{(\gamma(t),A)}, \Lambda, \Omega))$. Then δ is not a constant mapping since $\delta(0) = 0$ and $\delta(1) \neq 0$.

Next, since D_0 and D_1 are self-adjoint, we can find an orthonormal bases of eigenvectors, $\{v_j\}_{j=1}^n$ and $\{w_j\}_{j=1}^n$ for $\mathcal{H}$ with corresponding eigenvalues $\{\mu_j\}_{j=1}^n$ and $\{\nu_j\}_{j=1}^n$ for D_0 and D_1, respectively. The non-zero components of $B(\tilde{E}_{(\gamma(t),A)}, E_{(\gamma(t),A)}, \Lambda, \Omega)$ are of the form

$$\langle e_j, \gamma(t)Ae_k\rangle \text{ or } \left\langle \gamma(t)A^{-1}e_j, e_k\right\rangle.$$

Without loss of generality, we only consider the second case, which can be written as a linear combination of terms of the form

$$\begin{aligned}
\left\langle \gamma(t)A^{-1}e_j, v_k\right\rangle &= \left\langle e^{i(1-t)D_0}e^{itD_1}A^{-1}e_j, v_k\right\rangle \\
&= \left\langle e^{itD_1}A^{-1}e_j, e^{-i(1-t)D_0}v_k\right\rangle \\
&= \left\langle e^{itD_1}A^{-1}e_j, \text{diag}(e^{-i(1-t)\mu_1}\cdots, e^{-i(1-t)\mu_n})v_k\right\rangle \\
&= \left\langle e^{itD_1}A^{-1}e_j, e^{-i(1-t)\mu_k}v_k\right\rangle \\
&= e^{i(1-t)\mu_k}\left\langle e^{itD_1}A^{-1}e_j, v_k\right\rangle.
\end{aligned}$$

Similarly, each term of the form $\left\langle e^{itD_1}Ae_j, v_k\right\rangle$ can be written as a linear combination of terms of the form

$$\begin{aligned}
\left\langle e^{itD_1}A^{-1}e_j, w_\ell\right\rangle &= \left\langle A^{-1}e_j, e^{-itD_1}w_\ell\right\rangle \\
&= \left\langle A^{-1}e_j, \text{diag}(e^{-it\nu_1}\cdots, e^{-it\nu_n})w_\ell\right\rangle \\
&= \left\langle A^{-1}e_j, e^{-it\nu_\ell}w_\ell\right\rangle \\
&= e^{it\nu_\ell}\left\langle A^{-1}e_j, w_\ell\right\rangle.
\end{aligned}$$

The function $e^{i(1-t)\mu_k}e^{it\nu_\ell}\langle Ae_j, w_\ell\rangle$ is analytic as a function of t. Thus, δ is a linear combination of products of analytic functions. Hence δ is analytic, and cannot have an accumulation point of zeros in $[0,1]$. So, given $\epsilon > 0$, we can find a $t_0 \in [0,1]$ so that $\|\gamma(0) - \gamma(t_0)\| < \epsilon$, and $\gamma(t_0) \in \mathcal{U}_{(A,\Lambda,\Omega)}$. Therefore, $\mathcal{U}_{(A,\Lambda,\Omega)}$ is dense in $\mathbb{U}$.

To show that $\mathcal{U}_{(A,\Lambda,\Omega)}$ is open, let $\delta_{\Lambda,\Omega} : \mathbb{U} \to \mathbb{C}$ denote the continuous mapping defined by

$$\delta_{\Lambda,\Omega}(U) = \det(B(\tilde{E}_{(U,A)}, E_{(U,A)}, \Lambda, \Omega)).$$

Then,

$$\mathcal{U}_{(A,\Lambda,\Omega)} = \delta^{-1}_{\Lambda,\Omega}(\mathbb{C} \setminus \{0\}),$$

so it is open. $\square$

Proof of Theorem 2.28. Let $\Gamma = \{\Lambda \subset \{1, 2, \cdots, N\} : |\Lambda| \leq n\}$, and for each $\Lambda \in \Gamma$, define $\Phi_\Lambda = \{\Omega : \Omega$ is a block bridge set for $\Lambda\}$. Then

$$\mathcal{U}_A = \cap_{\Lambda \in \Gamma} \cap_{\Omega \in \Phi_\Lambda} \mathcal{U}_{(\Lambda,\Omega)}. \tag{2.31}$$

Therefore, by the Baire category theorem, $\mathcal{U}_A$ is open and dense in $\mathbb{U}$ since it is the intersection of finitely many open, dense sets in $\mathbb{U}$. $\square$

We thank the referee of [39] for suggesting the inclusion of the following result. Assume $E = \{e_j\}_{j=1}^n$ is an orthonormal basis for a Hilbert space, $\mathcal{H}$, and $T \in B(\mathcal{H})$ is an invertible operator. Let $E_T = \frac{1}{\sqrt{2}}\{e_j\}_{j=1}^{2n}$, where $e_{j+n} = Te_j$ for all $1 \leq j \leq n$. It is easily seen that $\{(T^{-1})^* e_j\}_{j=1}^n$ is the dual basis to $\{Te_j\}_{j=1}^n$. Thus, $\tilde{E}_T = \frac{1}{\sqrt{2}}\{\tilde{e}_j\}_{j=1}^{2n}$, where $\tilde{e}_j = e_j$, and $\tilde{e}_{j+n} = (T^{-1})^* e_j$ is a dual frame to E_T. The result gives a characterization of operators T for which the dual frame pair $(\tilde{E}_T, E_T)$ satisfies the block skew-spark property.

Proposition 2.32. *Assume $T \in B(\mathcal{H})$ is an invertible operator and $E = \{e_j\}_{j=1}^n$ is an orthonormal basis for $\mathcal{H}$. Then, the dual frame pair $(\tilde{E}_T, E_T)$ described above satisfies the block skew-spark property if and only if every square minor of the matrices for T and T^{-1} with respect to the orthonormal basis E is invertible.*

Proof. First observe that

$$T = \sum_{j,k=1}^{n} \langle Te_k, e_j \rangle \, e_j \otimes e_k. \tag{2.32}$$

Hence the matrix representation for T with respect to E is

$$T = \left(\langle Te_k, e_j \rangle \right)_{j,k=1}^{n}. \tag{2.33}$$

Recall from Remark 2.24 that $(\tilde{E}_T, E_T)$ satisfies the block skew-spark property if and only if $B(\tilde{E}_T, E_T, \Lambda_1, \Omega_2)$ and $B(\tilde{E}_T, E_T, \Lambda_2, \Omega_1)$ are invertible for all $\Lambda_1, \Omega_1 \subset \{1, \cdots, n\}$ and $\Lambda_2, \Omega_2 \subset \{n+1, \cdots, 2n\}$ satisfying $|\Lambda_1| = |\Omega_2|$ and $|\Lambda_2| = |\Omega_1|$. Since

$$B(\tilde{E}_T, E_T, \Lambda_1, \Omega_2) = \frac{1}{2}\left(\langle e_j, Te_k\rangle\right)_{j\in\Lambda_1,\, k\in\Omega_2 - n} = \frac{1}{2}\left(\overline{\langle Te_k, e_j\rangle}\right)_{j\in\Lambda_1,\, k\in\Omega_2-n},$$

$B(\tilde{E}_T, E_T, \Lambda_1, \Omega_2)$ is invertible if and only if $\left(\langle Te_k, e_j\rangle\right)_{j\in\Lambda_1,\, k\in\Omega_2-n}$ is invertible. From equation (2.33), this is the minor of T with respect to the basis E with rows indexed by Λ_1 and columns indexed by $\Omega_2 - n$. By considering all $\Lambda_1 \subset \{1, \cdots, n\}$ and all $\Omega_2 \subset \{n+1, \cdots, 2n\}$, we see that each matrix $B(\tilde{E}_T, E_T, \Lambda_1, \Omega_2)$ is invertible if and only if every square minor of the matrix T with respect to E is invertible. Next,

$$\begin{aligned}
B(\tilde{E}_T, E_T, \Lambda_2, \Omega_1) &= \frac{1}{2}\left(\left\langle (T^{-1})^* e_j, e_k\right\rangle\right)_{j\in\Lambda_2-n,\, k\in\Omega_1} = \frac{1}{2}\left(\left\langle e_j, T^{-1}e_k\right\rangle\right)_{j\in\Lambda_2-n,\, k\in\Omega_1} \\
&= \frac{1}{2}\left(\overline{\langle T^{-1}e_k, e_j\rangle}\right)_{j\in\Lambda_2-n,\, k\in\Omega_1}.
\end{aligned}$$

Therefore, by a similar argument, every bridge matrix of the form $B(\tilde{E}_T, E_T, \Lambda_2, \Omega_1)$ is invertible if and only if every minor of T^{-1} with respect to E is invertible. $\square$

Theorem 2.28 gives an operator theoretic density result for a union of an orthonormal basis and a Riesz basis. The next result builds on that result to give a Bessel bound norm proof that the set of all unions of two orthonormal bases which satisfy the block skew-spark property is open and dense in the set of all unions of two orthonormal bases. Let

$$\mathcal{OO} = \left\{\frac{1}{\sqrt{2}}\{e_j\}_{j=1}^{2n} : \{e_j\}_{j=1}^{n} \text{ and } \{e_j\}_{j=n+1}^{2n} \text{ are orthonormal bases}\right\}.$$

With the Bessel bound norm, $\mathcal{OO}$ is a closed subset of $\mathcal{H}^{2n}$ (sequences containing $2n$ vectors from $\mathcal{H}$). Thus, with the Bessel bound norm, $\mathcal{OO}$ is a complete metric space.

Let $\mathcal{OO}_s$ denote the set of all unions of two orthonormal bases with the block skew-spark property. For the remaining proofs, we will denote the analysis operator of the sequence $\{Te_j\}_{j=1}^n$ by $\Theta_{T\overline{E}}$. It is easy to see that $\Theta_{T\overline{E}} = \Theta_{\overline{E}}T^*$ for any operator $T \in B(\mathcal{H})$.

Theorem 2.33. *The set $\mathcal{OO}_s$ is open and dense in $\mathcal{OO}$.*

Proof. We will first show that $\mathcal{OO}_s$ is dense in $\mathcal{OO}$. Let $\epsilon > 0$ and assume $E = \{e_j\}_{j=1}^{2n} \in \mathcal{OO} \setminus \mathcal{OO}_s$. We know that $e_{j+n} = Ue_j$ for some $U \in \mathbb{U}$, and for $1 \leq j \leq n$. By Theorem 2.28, and the continuity of the adjoint mapping, we can find $U_\epsilon \in \mathcal{U}_I$ so that $\|U^* - U_\epsilon^*\| < \epsilon$. Let $E_\epsilon = \{e_j^{(\epsilon)}\}_{j=1}^{2n}$ where $e_j^{(\epsilon)} = e_j$ for $1 \leq j \leq n$ and $e_j^{(\epsilon)} = U_\epsilon e_j$ for $n+1 \leq j \leq 2n$. Then,

$$\begin{aligned} \|E - E_\epsilon\|_B &= \|\Theta_E - \Theta_{E_\epsilon}\|_{op} \\ &= \|\Theta_{U\overline{E}} - \Theta_{U_\epsilon\overline{E}}\|_{op} \\ &= \|\Theta_{\overline{E}}U^* - \Theta_{\overline{E}}U_\epsilon^*\|_{op} \\ &\leq \|\Theta_{\overline{E}}\|_{op}\|U^* - U_\epsilon^*\|_{op} \\ &< \epsilon. \end{aligned}$$

Since $U_\epsilon \in \mathcal{U}_I$, $E_\epsilon \in \mathcal{OO}_s$, and we have shown density.

To show that $\mathcal{OO}_s$ is open in $\mathcal{OO}$, it suffices to show that $\mathcal{OO}_s(\Lambda, \Omega) = \{E \in \mathcal{OO} : B(E, E, \Lambda, \Omega) \text{ is invertible}\}$ is open for all pairs (Λ, Ω) for which $|\Lambda| \leq n$ and Ω is a block bridge set for Λ. This is because $\mathcal{OO}_s$ is the intersection over all such pairs (Λ, Ω) of the sets $\mathcal{OO}_s(\Lambda, \Omega)$. To see that each $\mathcal{OO}_s(\Lambda, \Omega)$ is open, we have $\mathcal{OO}_s(\Lambda, \Omega) = \delta_{\Lambda,\Omega}^{-1}(\mathbb{C} \setminus \{0\})$, where $\delta_{\Lambda,\Omega}$ is the continuous function defined by $\delta_{\Lambda,\Omega}(E) = \det(B(E, E, \Lambda, \Omega))$. $\square$

Two orthonormal bases $\{e_j\}_{j=1}^n$ and $\{g_j\}_{j=1}^n$ are called *mutually unbiased* if

$$|\langle e_j, g_k\rangle| = \frac{1}{\sqrt{n}}$$

for all $j, k \in \{1, \cdots, n\}$ (cf. [6] and [35]). The next example shows that the union of two unbiased orthonormal bases can fail to satisfy the block skew-spark property. Thus, this property can fail to exist even in a highly structured example. We thank the referee of [39] for asking us to consider this example.

Example 2.34. Observe that

$$E = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and

$$G = \left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

are two mutually unbiased orthonormal bases. If we bridge the first two elements of the first orthonormal basis with the second two elements of the second basis, we get the bridge matrix

$$B(E \cup G, E \cup G, \{1, 2\}, \{7, 8\}) = \begin{pmatrix} \langle e_1, g_3\rangle & \langle e_1, g_4\rangle \\ \langle e_2, g_3\rangle & \langle e_2, g_4\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$

which is not invertible. Therefore, $E \cup G$ does not satisfy the block skew-spark property.

We will next prove the corresponding result for a union of an orthonormal basis and a Riesz basis. Denote by $\mathcal{OB}$ the set of all sequences in $\mathcal{H}^{2n}$ which consist of a union of an orthonormal basis and a Riesz basis. That is,

$$\mathcal{OB} = \left\{ \frac{1}{\sqrt{2}} \{f_j\}_{j=1}^{2n} : \{f_j\}_{j=1}^{n} \text{ is an orthonormal basis and } \{f_j\}_{j=n+1}^{2n} \text{ is a basis for } \mathcal{H} \right\}.$$

Denote by $\mathcal{OB}_s$ the subset of $\mathcal{OB}$ consisting of all unions of an orthonormal basis and a Riesz basis that satisfy the block skew-spark property.

Lemma 2.35. *$\mathcal{OB}_s$ is open and dense in $\mathcal{OB}$.*

Proof. We will first prove density. Let $\epsilon > 0$, and assume that $\{f_j\}_{j=1}^{2n} \in \mathcal{OB} \setminus \mathcal{OB}_s$. Then, $f_{j+n} = UAf_j$ for some unitary U and some positive operator A. By Theorem 2.28 and the continuity of the adjoint mapping, we can find some $W \in \mathbb{U}$ for which the frame $\{g_j\}_{j=1}^{2n}$ satisfies the block skew-spark property, and $\|U^* - W^*\| < \frac{\epsilon}{\|A\|_{op}}$, where $g_j = f_j$ and $g_{j+n} = WAg_j = WAf_j$ for $1 \le j \le n$. We then have

$$\begin{aligned}
\|\{f_j\}_{j=1}^{2n} - \{g_j\}_{j=1}^{2n}\|_B &= \|\Theta_F - \Theta_G\|_{op} \\
&= \|\Theta_{UA\overline{F}} - \Theta_{WA\overline{F}}\|_{op} \\
&= \|\Theta_{\overline{F}} AU^* - \Theta_{\overline{F}} AW^*\|_{op} \\
&\le \|\Theta_{\overline{F}}\|_{op} \|A\|_{op} \|U^* - W^*\|_{op} \\
&< \|A\|_{op} \frac{\epsilon}{\|A\|_{op}} \\
&= \epsilon.
\end{aligned}$$

Therefore, $\mathcal{OB}_s$ is dense in $\mathcal{OB}$.

Let $\mathcal{OB}_s(\Lambda, \Omega)$ denote the set of unions of an orthonormal basis and a Riesz basis, $F = \{f_j\}_{j=1}^{2n}$ for which $B(\tilde{F}, F, \Lambda, \Omega)$ is invertible. Define the continuous mapping δ :

$\mathcal{OB} \to \mathbb{C}$ by

$$\delta_{\Lambda,\Omega}\left(\{f_j\}_{j=1}^{2n}\right) = \det\left(B(\tilde{F}, F, \Lambda, \Omega)\right).$$

Then, $\mathcal{OB}_s(\Lambda, \Omega) = \delta_{\Lambda,\Omega}^{-1}(\mathbb{C} \setminus \{0\})$, and thus, $\mathcal{OB}_s(\Lambda, \Omega)$ is open. Since $\mathcal{OB}_s$ is the intersection over all Λ satisfying $|\Lambda| \leq n$ and all block bridge sets, Ω for Λ of the sets $\mathcal{OB}_s(\Lambda, \Omega)$, it follows that $\mathcal{OB}_s$ is open. $\square$

We say that two dual frame pairs, $(F_1, G_1) = \{f_j^1, g_j^1\}_{j=1}^N$ and $(F_2, G_2) = \{f_j^2, g_j^2\}_{j=1}^N$ are *isomorphic* if there exists a bounded, invertible operator T so that for all $1 \leq j \leq N$, we have $Tf_j^1 = f_j^2$, and $(T^*)^{-1}g_j^1 = g_j^2$. With this definition, we can make the following easy observation.

Proposition 2.36. *Assume that (F_1, G_1) and (F_2, G_2) are isomorphic. Then for all $\Lambda, \Omega \subset \{1, \cdots, N\}$, $B(F_1, G_1, \Lambda, \Omega) = B(F_2, G_2, \Lambda, \Omega)$. In particular, (F_1, G_1) satisfies the full skew-spark property if and only if (F_2, G_2) satisfies the full skew-spark property, and (F_1, G_1) satisfies the block skew-spark property if and only if (F_2, G_2) satisfies the block skew-spark property.*

We will next combine Lemma 2.35 and Proposition 2.36 to prove that the set of unions of two Riesz bases which satisfy the block skew-spark property is open and dense in the set of all unions of two Riesz bases. We define $\mathcal{BB}$ to be the set of all unions of two Riesz bases in $\mathcal{H}$. That is,

$$\mathcal{BB} = \left\{\{f_j\}_{j=1}^{2n} : \{f_j\}_{j=1}^{n} \text{ and } \{f_j\}_{j=n+1}^{2n} \text{ are bases for } \mathcal{H}\right\}.$$

We define $\mathcal{BB}_s$ to be the set of all unions of two Riesz bases which satisfy the block skew spark property.

Theorem 2.37. *$\mathcal{BB}_s$ is open and dense in $\mathcal{BB}$.*

Proof. The proof of openness is similar to the case for a union of an orthonormal basis and a Riesz basis, so we omit it here. To prove density, assume $\epsilon > 0$, and $F = \{f_j\}_{j=1}^{2n} \in \mathcal{BB} \setminus \mathcal{BB}_s$. Fix an orthonormal basis, $\{e_j\}_{j=1}^{n}$ for $\mathcal{H}$. Then, for $1 \le j \le n$, we have $f_j = U_1 A_1 e_j$ and $f_{j+n} = U_2 A_2 e_j$ for some unitary operators U_1 and U_2, and some positive operators A_1 and A_2. By defining $G = \{g_j\}_{j=1}^{2n}$ by $g_j = e_j$ and $g_{j+n} = (U_1 A_1)^{-1} U_2 A_2 e_j$ for $1 \le j \le n$, $(\tilde{G}, G)$ is isomorphic to $(\tilde{F}, F)$ (under the mapping $U_1 A_1$), and $G \in \mathcal{OB} \setminus \mathcal{OB}_s$. Thus, by Lemma 2.35, we can find $H = \{h_j\}_{j=1}^{2n} \in \mathcal{OB}_s$ with $\|G - H\|_B < \frac{\epsilon}{\|A_1\|_{op}}$. Then $U_1 A_1 H = \{U_1 A_1 h_j\}_{j=1}^{2n} \in \mathcal{BB}_s$, and

$$\begin{aligned}
\|F - U_1 A_1 H\|_B &= \|\Theta_F - \Theta_{U_1 A_1 H}\|_{op} \\
&= \|\Theta_{U_1 A_1 G} - \Theta_{U_1 A_1 H}\|_{op} \\
&= \|\Theta_G A_1 U_1^* - \Theta_H A_1 U_1^*\|_{op} \\
&\le \|\Theta_G - \Theta_H\|_{op} \|A_1\|_{op} \|U_1^*\|_{op} \\
&< \frac{\epsilon}{\|A_1\|_{op}} \|A_1\|_{op} \\
&= \epsilon.
\end{aligned}$$

Therefore, $\mathcal{BB}_s$ is dense in $\mathcal{BB}$. $\square$

The remainder of this subsection is the extension of these results to the infinite dimensional case. As with dual frame pairs, we will retain our density results, but when we use the Baire Category Theorem on countably infinite intersections, we may lose openness.

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, and $F = \{f_j\}_{j \in \mathbb{J}_1}$ and $G = \{g_j\}_{j \in \mathbb{J}_2}$ ($\mathbb{J}_1 \cap \mathbb{J}_2 = \emptyset$) be Riesz bases for $\mathcal{H}$ with dual bases $\{\tilde{f}_j\}_{j \in \mathbb{J}_1}$ and $\{\tilde{g}_j\}_{j \in \mathbb{J}_2}$, respectively. As before, we decompose any subset $\Gamma \subset \mathbb{J}_1 \sqcup \mathbb{J}_2$ as $\Gamma = \Gamma_1 \sqcup \Gamma_2$ where $\Gamma_1 \subset \mathbb{J}_1$ and $\Gamma_2 \subset \mathbb{J}_2$.

Definition 2.38. 1. We call Ω a *block bridge set* for a finite erasure set Λ if $\Omega \subset \mathbb{J}_1 \sqcup$

$\mathbb{J}_2 \setminus \Lambda$, $|\Omega_1| = |\Lambda_2|$ and $|\Omega_2| = |\Lambda_1|$.

2. We say that the union $F \cup G$ satisfies the *block skew-spark property* if for any finite erasure set Λ with $|\Lambda| < \infty$ and for any block bridge set Ω for Λ, the bridge matrix, $B(\tilde{F} \cup \tilde{G}, F \cup G, \Lambda, \Omega)$ is invertible.

Remark 2.39. Definition 2.38 is equivalent to Definition 2.25 for finite dimensions, modulo minor bookkeeping considerations.

Fortunately, many of the results generalize nicely for infinite frames. The next theorem displays these results. For the theorem, we need the following *definitions*: For a fixed orthonormal basis $\{e_j\}_{j\in\mathbb{J}_1}$, a fixed unitary operator U, a fixed positive operator A, and a fixed bijection $\alpha : \mathbb{J}_1 \to \mathbb{J}_2$, we *define* $E_{(U,A,\alpha)} = \frac{1}{\sqrt{2}}\{e_j\}_{j\in\mathbb{J}_1\sqcup\mathbb{J}_2}$, where $e_{\alpha(j)} = UAe_j$ for $j \in \mathbb{J}_1$. We also *define* $\tilde{E}_{(U,A,\alpha)} = \frac{1}{\sqrt{2}}\{\tilde{e}_j\}_{j\in\mathbb{J}_1\sqcup\mathbb{J}_2}$ where $\tilde{e}_j = e_j$ and $\tilde{e}_{\alpha(j)} = UA^{-1}e_j$ for $j \in \mathbb{J}_1$. Lastly, for the density results in (4), (5), and (6) below, we will equip $\mathcal{OO}$, $\mathcal{OB}$, and $\mathcal{BB}$ with the Bessel bound norm.

Theorem 2.40. *Let $\mathcal{H}$ be a separable Hilbert space. Let A be a fixed positive operator in $B(\mathcal{H})$, $E = \{e_j\}_{j\in\mathbb{J}_1}$ be a fixed orthonormal basis for $\mathcal{H}$, and $\alpha : \mathbb{J}_1 \to \mathbb{J}_2$ be a fixed bijection. Assume that Λ is a finite erasure set, and Ω is a block bridge set for Λ. With the terminology above:*

1. *There exists $U \in \mathbb{U}$ for which $B(\tilde{E}_{(U,A)}, E_{(U,A)}, \Lambda, \Omega)$ is invertible.*

2. *The set $\mathcal{U}_{(A,\Lambda,\Omega)} = \{U \in \mathbb{U} : B(\tilde{E}_{(U,A,\alpha)}, E_{(U,A,\alpha)}, \Lambda, \Omega)$ is invertible$\}$ is open and dense in $\mathbb{U}$.*

3. *The set*

$$\mathcal{U}_A = \{U \in \mathbb{U} : (\tilde{E}_{(U,A,\alpha)}, E_{(U,A,\alpha)}) \text{ has the block skew-spark property}\}$$

is a dense subset of $\mathbb{U}$.

4. *The set* $\mathcal{OO}_s$ *is dense in* $\mathcal{OO}$.

5. $\mathcal{OB}_s$ *is dense in* $\mathcal{OB}$.

6. $\mathcal{BB}_s$ *is dense in* $\mathcal{BB}$.

Proof. The proof of (1) and (2) directly follow from Lemmas 2.29, and 2.31, respectively.

For the proof of (3), we refer to the proof of Theorem 2.28. For the proof, in the infinite dimensional case the intersection in equation (2.31) will now be an intersection over the countably infinite index sets

$$\Gamma = \{\Lambda : |\Lambda| < \infty\} \quad \text{and} \quad \Phi_\Lambda = \{\Omega : \Omega \text{ is a block bridge set for } \Lambda\}.$$

Therefore, the Baire Category Theorem implies that $\mathcal{U}_A$ is dense in $\mathbb{U}$. (However, an intersection of countably many open sets is not necessarily open.)

Notice that in the density proofs for Theorem 2.33, Lemma 2.35, and Theorem 2.37, we only used the density result from Theorem 2.28 whose analog is item (3) above. Therefore, (4), (5), and (6) follow from (3). $\square$

2.3.4 Open Questions for Shannon-Whittaker Sampling Theory[6]

Many interesting open questions on skew-spark properties involve Shannon-Whittaker sampling theory. We denote by PW_π the set of band-limited functions with band $[-\pi, \pi]$. That is,

$$\mathrm{PW}_\pi = \left\{ f \in L^2(\mathbb{R}) : \mathrm{spt} \hat{f} \subset [-\pi, \pi] \right\},$$

[6]The material in this subsection is reprinted from [39], with permission from Elsevier.

where $\hat{f}$ denotes the Fourier transform of f using the convention

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.$$

For the space PW_π, we have the following identity:

$$f(z) = \langle f, \mathrm{sinc}(\pi(\cdot - z)) \rangle . \tag{2.34}$$

For $p \in (0, 1]$ let $g_{p,j}(t) = \mathrm{sinc}(\pi(t - pj))$. Then, $\{g_{p,j}\}_{j\in\mathbb{Z}}$ is a tight frame with a dual frame with the standard dual $\{\tilde{g}_{p,j}\}_{j\in\mathbb{Z}}$ where $\tilde{g}_{p,j}(t) = p\mathrm{sinc}(\pi(t - pj))$. The Shannon-Whittaker Sampling Theorem, states that for $f \in \mathrm{PW}_\pi$,

$$f = \sum_{j\in\mathbb{Z}} \langle f, g_{p,j} \rangle\, \tilde{g}_{p,j} = p \sum_{j\in\mathbb{Z}} f(pj)\mathrm{sinc}(\pi(\cdot - pj)).$$

Proposition 2.41. *For any rational number $p \in (0, 1)$ the dual frame pair*

$$(\{g_{p,j}\}_{j\in\mathbb{Z}}, \{\tilde{g}_{p,j}\}_{j\in\mathbb{Z}})$$

does not satisfy the <u>full</u> skew-spark property. (However, it remains open whether it satisfies the <u>block</u> skew-spark property: see questions 2.43 and 2.44, below.)

Proof. Assume p is a rational number in the interval $(0, 1)$. Then, $p = \frac{n}{m}$ for positive integers n and m with $n < m$. We have,

$$\begin{aligned} B(\{g_{p,j}\}_{j\in\mathbb{Z}}, \{\tilde{g}_{p,j}\}_{j\in\mathbb{Z}}, \{0\}, \{mp\}) &= p\,\langle \operatorname{sinc}(\pi(\cdot)), \operatorname{sinc}(\pi(\cdot - mp))\rangle \\ &= p\operatorname{sinc}(\pi mp) \\ &= p\operatorname{sinc}(\pi n) \\ &= 0. \end{aligned}$$

Therefore, $(\{g_{p,j}\}_{j\in\mathbb{Z}}, \{\tilde{g}_{p,j}\}_{j\in\mathbb{Z}})$ does not satisfy the full skew-spark property. $\square$

The case where $p \in (0, 1)$ is irrational is an open question, given below. We believe that for all irrational values of $p \in (0, 1)$, Shannon-Whittaker sampling on $p\mathbb{Z}$ satisfies the full skew-spark property.

Question 2.42. *For which irrational values of $p \in (0, 1)$ does the dual frame pair $(\{g_{p,j}\}_{j\in\mathbb{Z}}, \{\tilde{g}_{p,j}\}_{j\in\mathbb{Z}})$ satisfy the full skew-spark property?*

Notice that when $p = \frac{1}{2}$, the sets $\{g_{\frac{1}{2},j} : j \text{ is even}\}$ and $\{g_{\frac{1}{2},j} : j \text{ is odd}\}$ are complete, orthogonal sets of functions. Thus, Shannon-Whittaker sampling at the half-integers corresponds to a union of two bases, and we have the following natural question.

Question 2.43. *Let $\mathbb{J}_1$ denote the set of even integers and $\mathbb{J}_2$ denote the set of odd integers. Does the union $\frac{1}{\sqrt{2}}\{g_{\frac{1}{2},j}\}_{j\in\mathbb{J}_1} \sqcup \frac{1}{\sqrt{2}}\{g_{\frac{1}{2},j}\}_{j\in\mathbb{J}_2}$ satisfy the block skew-spark property?*

We believe the answer to Question 2.43 is yes. Notice that in the above notation, the basis $\{g_{1,j}\}_{j\in\mathbb{Z}}$ corresponds to Shannon-Whittaker sampling at the integers, and the basis $\{g_{1,j}\}_{j\in q+\mathbb{Z}}$ corresponds to Shannon-Whittaker sampling at the points $q + \mathbb{Z} = \{q + j : j \in \mathbb{Z}\}$. So the following is a more general question.

Question 2.44. *For which values of $q \in (0, 1)$ does the union $\frac{1}{\sqrt{2}}\{g_{1,j}\}_{j\in\mathbb{Z}} \cup \frac{1}{\sqrt{2}}\{g_{1,j}\}_{j\in q+\mathbb{Z}}$ satisfy the block skew-spark property?*

Again, we believe the answer to this question is yes, and we have a proof that all bridge matrices of size two or less are invertible. We also have the following partial result which states that for a dense set of q-values in $[0,1]$, we can bridge any finite subset Λ of $\mathbb{Z}$ by the corresponding shift $q+\Lambda$. More generally, we have the following proposition.

Proposition 2.45. *Assume that we can write the finite erasure set* Λ *as the disjoint union* $\Lambda = \Lambda_1 \sqcup (q+\Lambda_2)$ *where* $\Lambda_1, \Lambda_2 \subset \mathbb{Z}$ *and* $\Lambda_1 \cap \Lambda_2 = \emptyset$*. Then, for a dense set of* q*-values in* $[0,1]$*, the block bridge set defined by* $\Omega = \Omega_1 \sqcup \Omega_2$*, where* $\Omega_2 = q+\Lambda_1$ *and* $\Omega_1 = \Lambda_2$ *is a robust bridge set for* Λ *with respect to the union* $\frac{1}{\sqrt{2}}\{g_{1,j}\}_{j\in\mathbb{Z}} \cup \frac{1}{\sqrt{2}}\{g_{1,j}\}_{j\in q+\mathbb{Z}}$ *.*

Proof. Assume that Λ_1 and Λ_2 are finite sets of integers satisfying $\Lambda_1 \cap \Lambda_2 = \emptyset$. Given $q \in [0,1]$, define $\Lambda^{(q)}$ as the disjoint union $\Lambda^{(q)} = \Lambda_1^{(q)} \sqcup \Lambda_2^{(q)}$ and $\Omega^{(q)}$ as the disjoint union $\Omega^{(q)} = \Omega_1^{(q)} \sqcup \Omega_2^{(q)}$, where $\Lambda_1^{(q)} = \Lambda_1$, $\Lambda_2^{(q)} = q+\Lambda_2$, $\Omega_1^{(q)} = \Lambda_2$ and $\Omega_2^{(q)} = q+\Lambda_1$. Let $\mathcal{U}_{\Lambda^{(q)}}$ denote the set of all q-values for which $\Omega^{(q)}$ is a robust bridge set for $\Lambda^{(q)}$. To prove the proposition, by the Baire category theorem, it suffices to show that $\mathcal{U}_{\Lambda^{(q)}}$ is an open dense set, since Λ_1 and Λ_2 were chosen arbitrarily. Furthermore, it suffices to show that $\mathcal{U}^c_{\Lambda^{(q)}}$ is a finite set. To do this, first notice that using equation (2.34), we have

$$
\begin{aligned}
B(\Lambda_1^{(q)}, \Omega_2^{(q)}) &= B(\Lambda_1, q+\Lambda_1) \\
&= \frac{1}{2}\left(\langle \operatorname{sinc}(\pi(\cdot - j)), \operatorname{sinc}(\pi(\cdot - (k+q)))\rangle\right)_{j,k\in\Lambda_1} \\
&= \frac{1}{2}\left(\operatorname{sinc}(\pi(q+k-j))\right)_{j,k\in\Lambda_1}.
\end{aligned}
$$

It is easily seen that if $q=0$, then, $B(\Lambda_1^{(q)}, \Omega_2^{(q)})$ is a scalar multiple of the identity matrix. Furthermore, notice that the components of $B(\Lambda_1^{(q)}, \Omega_2^{(q)})$ are analytic functions in q. Thus, it is easily seen that the mapping $\alpha : [0,1] \to \mathbb{C}$ defined by $\alpha(q) = \det(B(\Lambda_1^{(q)}, \Omega_2^{(q)}))$ is a nonzero analytic function of q (since $\alpha(0) \neq 0$). Hence, the set of q-values in $[0,1]$ for which $B(\Lambda_1^{(q)}, \Omega_2^{(q)})^{-1}$ does not exist is finite. Similarly, the set of q-values in $[0,1]$ for which $B(\Lambda_2^{(q)}, \Omega_1^{(q)})^{-1}$ does not exist is finite. Since $B(\Lambda^{(q)}, \Omega^{(q)})$ is invertible if and only if $B(\Lambda_1^{(q)}, \Omega_2^{(q)})$ and $B(\Lambda_2^{(q)}, \Omega_1^{(q)})$ are invertible, $\mathcal{U}_{\Lambda^{(q)}}$ is finite. By the remarks above, this completes the proof. □

Lastly, we would like to know which sampling schemes have a method to choose bridge sets so that the bridge matrix is invertible. If there is such a method, we say that the sampling scheme has a modified skew-spark property.

Question 2.46. *Which sampling schemes satisfy a modified skew-spark property?*

2.4 Implementation of Nilpotent Bridging

We will next take a look at a sample Matlab implementation of the reconstruction method set forth in Theorem 2.3. We will first present the code, and afterwards present a line by line description of the program.

```
n = 2000;
N = 3000;
L = [1:100];
W = [2001:2100];

F = (1/sqrt(n))*randn(n,N);
S = F * F';
G = S \ F;

f = rand(n,1);
f = f ./ norm(f,2);

FC = G' * f;
```

```
14
15 FC(L) = zeros(size(L'));
16
17 f_R = F * FC;
18
19 FRCL = G(:,L)' * f_R;
20 FRCB = G(:,W)' * f_R;
21
22 C = (F(:,L)'*G(:,W))\(F(:,L)'*G(:,L));
23
24 FC(L) = C' * (FC(W) - FRCB) + FRCL;
25
26 g = f_R + F(:,L) * FC(L);
27
28 norm(f-g,2)
```

In lines 1-4 we define the sample parameters for an experiment. The variables n and N signify that we are using a frame of length N for $\mathbb{R}^n$. The variables L and W denote the erasure and bridge sets, respectively. In line 6, we generate an $n \times N$ matrix F, whose entries are drawn independently from the standard normal distribution. The columns of F will be our frame vectors. In other words, F is the synthesis matrix for our frame. Notice that with the normalization constant of $\frac{1}{\sqrt{n}}$, the expectation of $S = FF^*$ is $\frac{N}{n}I$. That is, F is expected to be a nearly tight frame with frame bound $\frac{N}{n}$. The matrix S in line 7 is the frame operator for F. In line 8, we compute a matrix, G, whose columns are the standard dual frame to F. In lines 10 and 11, we generate a random, unit norm vector which we will reconstruct from frame coefficient erasures indexed by L (the erasure set). In line 13, we compute the frame coefficients (denoted by FC) of f with respect to G, and in line 15, we delete the coefficients in FC indexed by L. The goal of this program is to reconstruct f from this reduced data set.

The rest of the program (lines 17-28) is the implementation of Theorem 2.3. In line 17, we compute the partial reconstruction, f_R of f by synthesizing FC with respect to F. In lines 19, 20, and 22, we compute the necessary information for equation (2.18). That is, in lines 19 and 20 we compute $(\langle f_R, g_j \rangle)_{j\in\Lambda}$ and $(\langle f_R, g_j \rangle)_{j\in\Omega}$, and in line 22 we compute the matrix C by solving the bridge equation. In line 24, we perfectly reconstruct (up to machine error) our missing coefficients by using equation (2.18) from Theorem 2.3. In line 26, we synthesize our reconstructed coefficients with $\{f_j\}_{j\in\Lambda}$ and add them to f_R to obtain a vector g. Lastly, in line 28, we compute $\|f - g\|$ to determine the error in our reconstruction. As mentioned earlier, the error term, $\|f - g\|$, is just the machine error in our reconstruction. Later on, we will examine the effects of noise on our reconstruction.

If we desire to use a tight frame which satisfies the RIP with high probability, instead of a normally distributed random frame and its corresponding dual, we can use the following construction. The frames generated using this procedure are called *TRIP frames*. We will give some analysis of these frames in Section 3.3.

Construction Algorithm 2.47.

1. *Let H denote an $n \times N$ matrix with rank n.*

2. *Compute the qr-decomposition of H^*. That is, $H^* = QR$ where Q is an $N \times n$ matrix with orthonormal columns and R is an $n \times n$ upper triangular matrix.*

3. *Let $F = \sqrt{\frac{N}{n}}Q^*$.*

4. *Let $G = \frac{n}{N}F$.*

If the entries of the matrix H in the construction are drawn independently according to the standard normal distribution, we call F a *standard normally distributed TRIP frame*. In this case, F will be a tight frame which satisfies the RIP with approximately the same RIP constant as $\frac{1}{\sqrt{n}}H$. Due to this, these frames will demonstrate a very high degree of robustness to sparse, and even normally distributed random channel noise. We call these frames TRIP frames because they are tight, and satisfy the RIP with high probability.

To use a standard normally distributed TRIP frame in place of a dual frame pair, we can use the following block of code in place of lines 6-8 above.

```
1 F = randn(N,n);
2 [F,~] = qr(F,0);
3 F = sqrt(N/n) * F';
4 G = (n/N) * F;
```

In line 1 above, we generate an $N \times n$ matrix F, whose entries are drawn independently from the standard normal distribution. In line 2, we compute the qr-factorization of F and store the Q matrix in F. That is, (essentially) we run the Gram-Schmidt orthonormalization procedure on the columns of F, and store the result back in F. In line 3, we replace F with $\sqrt{\frac{N}{n}}F$, and since F is tight with frame bound $\frac{N}{n}$, its standard dual is given by $G = \frac{n}{N}F$ (line 4).

If we desire to overbridge and use the Moore-Penrose pseudo-inverse, we should enhance the size of the bridge set (being sure that the new bridge set is still disjoint from the erasure set), as well as replace line 22 of our original code with the following line of code.

```
C = pinv(F(:,L)'*G(:,W))*(F(:,L)'*G(:,L));
```

2.5 Overbridging and Noise Mitigation

As was mentioned in Remark 2.6, if the partial reconstruction, f_R is subject to noise, then the reconstruction from erasures can be quite poor. However, we can expect less error amplification when we overbridge. In this section we run numerical experiments to gain a qualitative understanding of overbridging, and its noise mitigating effects.

In the first experiment, we generated tight frames of length $N = 1000$ for $\mathbb{R}^{250}$ according to the construction of standard normally distributed TRIP frames (Construction Algorithm 2.47), and we varied our erasure set sizes from 10 to 250 in increments of 10. For each erasure set, we ran 50 trials, each trial with a newly generated frame. In each trial we generated a unit norm standard normally distributed random vector (or signal) $f \in \mathbb{R}^{250}$. We then introduced a 5% standard normally distributed noise term to the non-erased frame coefficients. We call this type of noise term *additive channel noise*. If $\epsilon = (\epsilon_j)_{j=1}^N$ represents the additive noise term, then noisy partial reconstruction is

$$\tilde{f}_R = \sum_{j \in \Lambda^c} \left(\langle f, g_j \rangle + \epsilon_j \right) f_j, \tag{2.35}$$

and the reconstructed signal is

$$\tilde{f} = \tilde{f}_R + \sum_{j \in \Lambda} d_j f_j, \tag{2.36}$$

where C solves the bridge equation and

$$(d_j)_{j \in \Lambda} = C^* \Big((\langle f, g_k \rangle + \epsilon_k)_{k \in \Omega} - (\langle \tilde{f}_R, g_k \rangle)_{k \in \Omega} \Big) + (\langle \tilde{f}_R, g_j \rangle)_{j \in \Lambda}. \tag{2.37}$$

Notice that this is precisely the reconstruction from erasures where we use the corrupted information to bridge the erased information (see the similarities between equation (2.18) and equation (2.37)). For each trial, the norm of the error in the noisy partial reconstruction, and the norm in the error of our reconstructed signal were recorded. The results are summarized in Figure 2.1.

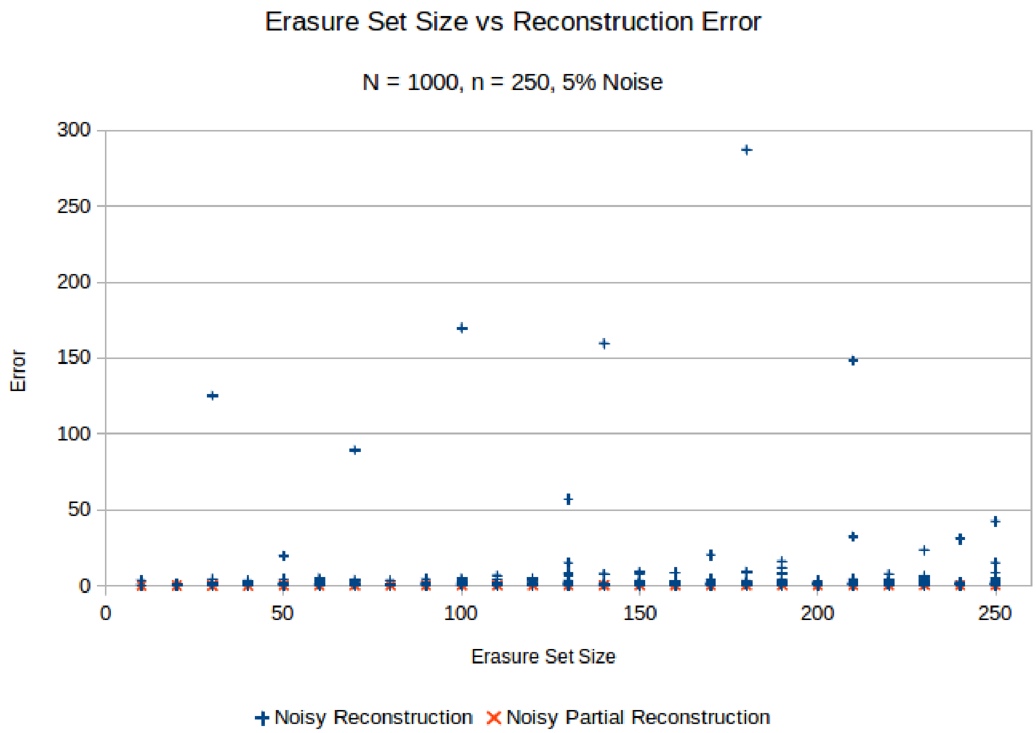

Figure 2.1: Noise amplification using Nilpotent Bridging.

Based on Figure 2.1, it is very clear that without overbridging this method is not robust to additive channel noise. However, if we enlarge our bridge set by 25%, and solve the bridge equation using a pseudo-inverse, we get Figure 2.2, which shows that this method much more robust to additive channel noise.

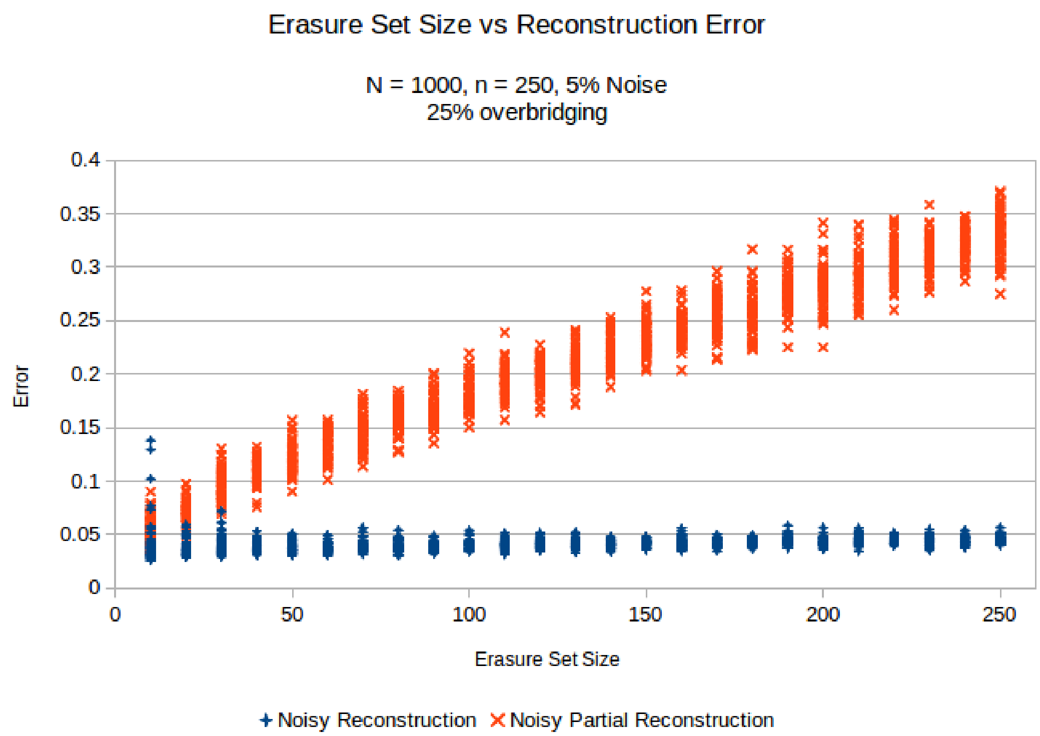

Figure 2.2: Noise amplification using 25% overbridging.

Figure 2.3 is the same experiment repeated for double bridging (i.e. $|\Omega| = 2\Lambda$), again utilizing a pseudo-inverse. In the figure, we see a further reduction in error amplification.

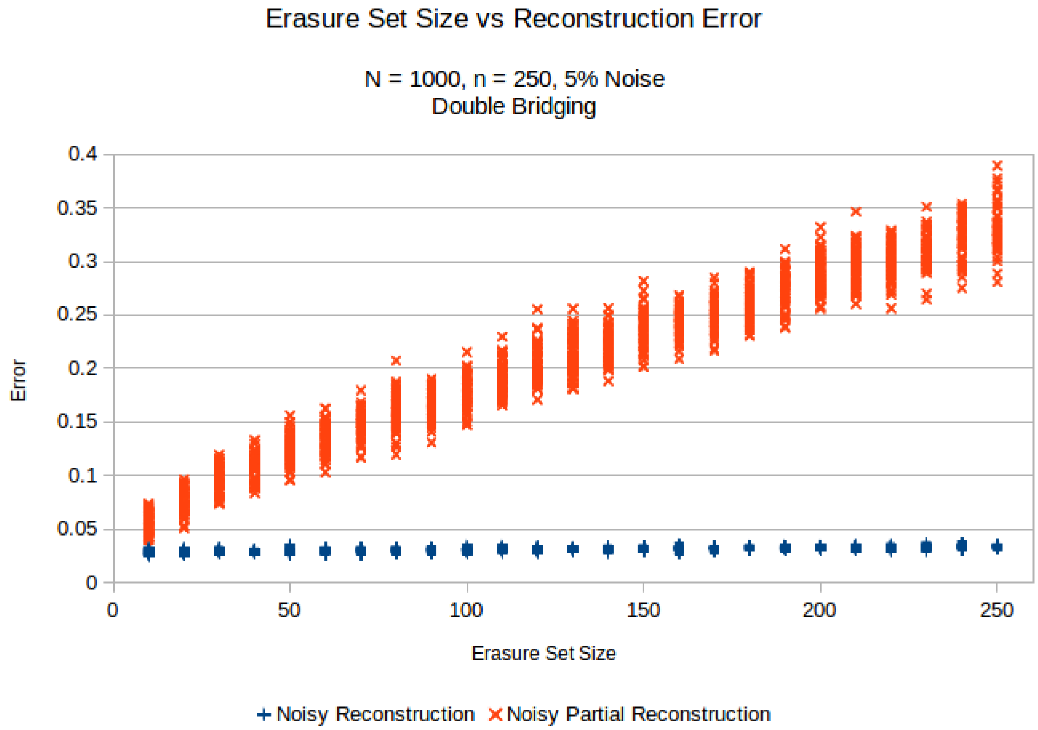

Figure 2.3: Noise amplification using double bridging.

Figure 2.4 is another good display of this behavior. To perform the experiment, we first compress a 256×256 pixel image (in this case, Lena) by deleting all but 15% of the most significant fast Fourier coefficients. For each column, we generated a new standard normally distributed TRIP frame pair (F, G) of length $N = 2n$, and used a new 5% noise term. In the first row, we display the true image corrupted only by the 5% noise term, in the second row, we display the noisy partial reconstruction, $\tilde{f}_R$, in the third row we display the noisy reconstruction where $|\Omega| = |\Lambda|$, and in the fourth row we display the noisy reconstruction from 25% overbridging. We displayed the results for 1%, 2%, 3%, 4%, and 5% erasures. In the images we see very clearly that 25% overbridging drastically improves the reconstruction, and it actually gives a very nice reconstruction of the original image which is close to the erased image subject only to noise (i.e. the first and fourth rows look quite similar). On the other hand, Nilpotent Bridging with $|\Lambda| = |\Omega|$ tends to significantly amplify the noise term.

Figure 2.4: Noise amplification for Nilpotent Bridging using Lena.

2.6 Stability Considerations for Shannon-Whittaker Sampling Theory

Throughout this section, we will denote the Shannon-Whittaker frame on the lattice $p\mathbb{Z}$ by $F_p = \{p\,\text{sinc}(\pi(x-j))\}_{j\in\mathbb{J}}$, and its standard dual by $G_p = \{\text{sinc}(\pi(x-j))\}_{j\in p\mathbb{Z}}$.

As with Nilpotent Bridging for finite frames, in the case of Shannon-Whittaker Sampling Theory, if the coefficient matrix, C, has a large norm, then we can expect our reconstruction to be unstable. Furthermore, unlike with finite dimensions, we can only compute a finite term approximation of f_R, since f_R is represented by an infinite series. Thus, our signal comes with a built in noise term. In playing around with the algorithm in Matlab, we discovered that the use of a *close* bridge set will likely ensure that the bridge matrix, $B(\Lambda, \Omega)$, is well conditioned. By a close bridge set, we mean that if $\Lambda = \{\lambda_j\}_{j=1}^{L}$, then we choose a bridge set $\Omega = \{\omega_j\}_{j=1}^{L}$ in order to minimize $\sum_{j=1}^{L} |\lambda_j - \omega_j|$. Close bridge sets typically yield well conditioned bridge matrices because they produce diagonally dominant bridge matrices. In Figure 2.5, we consider Shannon-Whittaker Sampling on the half integers, and we take $\Lambda = \{1, 2, 3, \cdots, 100\}$, and $\Omega = \frac{1}{2} + \Lambda$. For this close bridge setup, we get $\|C\| = 3.3162$. Because of this stability, we can see in the graph that the reconstruction overlaps quite nicely with the original signal ($f(t) = \text{sinc}(\pi t)$). Note that for the experiment, we are using the non-erased half integer samples on the interval $[-5000, 5000]$ to obtain our finite approximation of f_R.

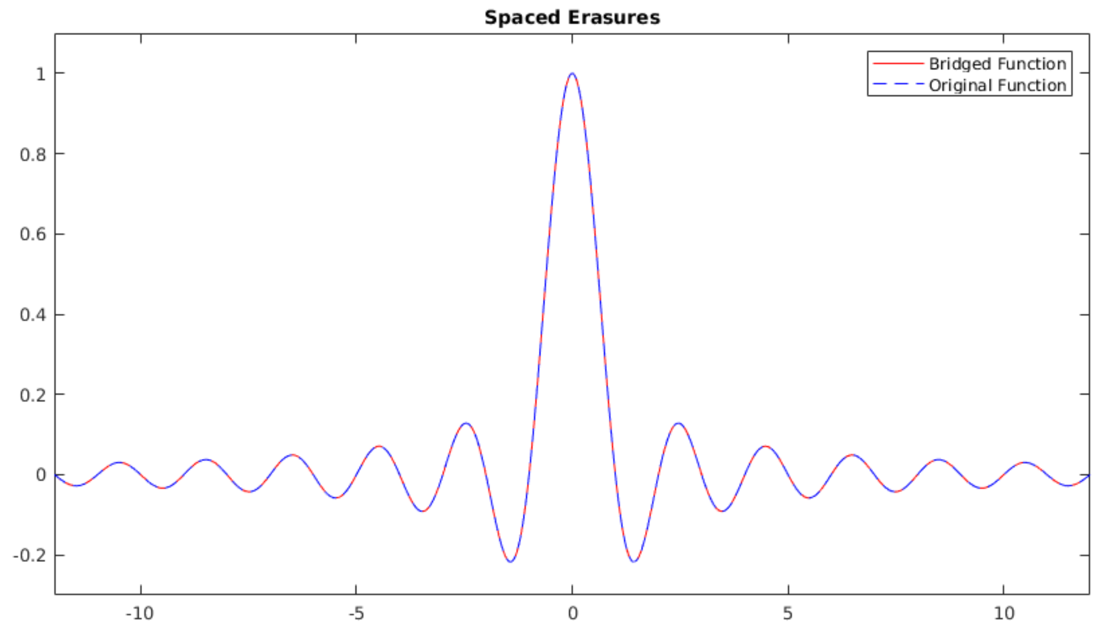

Figure 2.5: Error amplification of spaced erasures for Shannon-Whittaker Sampling on $\frac{1}{2}\mathbb{Z}$.

Figure 2.5 exhibits the good behavior of close bridge sets for erasures that are spaced out. However, if we erase only 16 consecutive samples ($\Lambda = \{1, \frac{3}{2}, 2, \cdots, 8, \frac{17}{2}\}$ and $\Omega = \{-3, -\frac{5}{2}, \cdots, 0, \frac{1}{2}, 9, \frac{19}{2}, \cdots, 12, \frac{25}{2}\}$) we get $\|C\| = 1.3921 \times 10^6$, and as we can see in the Figure 2.6, our error in the computation of f_R blows up on the erasure set. Again, we are using the non-erased samples of $f(t) = \mathrm{sinc}(\pi t)$ between $t = -5,000$ and $t = 5,000$ to approximate f_R.

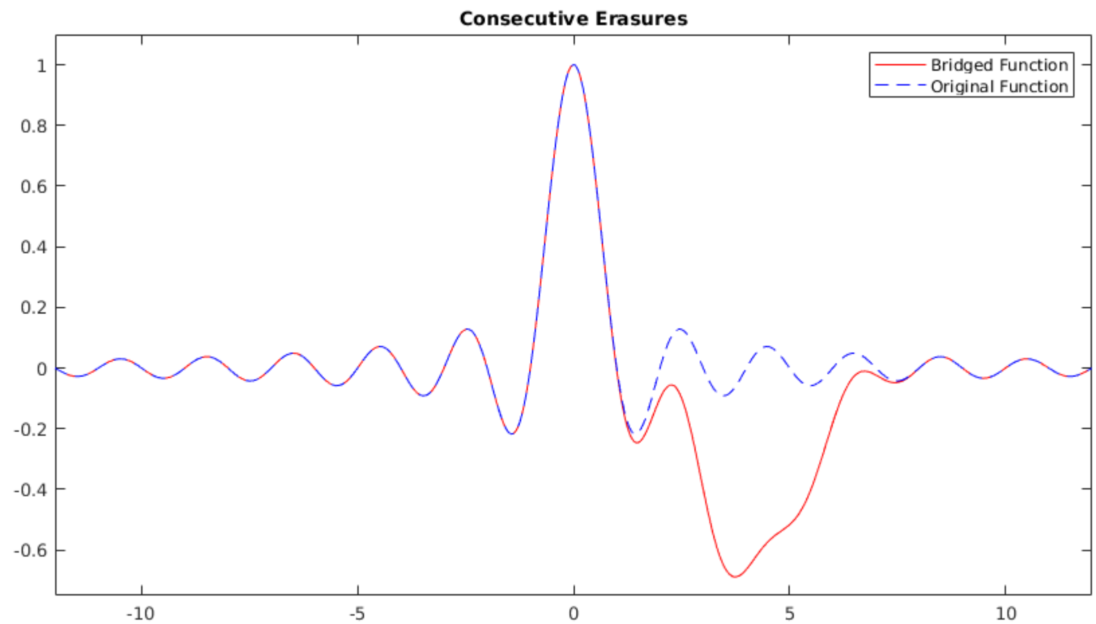

Figure 2.6: Error amplification of consecutive erasures for Shannon-Whittaker Sampling on $\frac{1}{2}\mathbb{Z}$.

3. REDUCED DIRECT INVERSION

3.1 The Reduced Direct Inversion Algorithm[1]

While studying Nilpotent Bridging, we discovered a second efficient method of inverting the partial reconstruction operator, R_Λ. The following proposition for one erasure led us to believe that something more general was true.

Proposition 3.1. *Assume that (F, G) is a dual frame pair for an n-dimensional ($n \leq \infty$) Hilbert space, $\mathcal{H}$, and $\Lambda = \{k\}$ is an erasure set such that $\langle f_k, g_k \rangle \neq 1$. Then, R_Λ is invertible, and*

$$R_\Lambda^{-1} = I_n + \frac{1}{1 - \langle f_k, g_k \rangle} f_k \otimes g_k \tag{3.1}$$

where I_n denotes the n-dimensional identity operator.

Proof. Notice that

$$\begin{aligned} R_\Lambda \left(I_n + \frac{1}{1 - \langle f_k, g_k \rangle} f_k \otimes g_k \right) &= (I_n - f_k \otimes g_k) \left(I_n + \frac{1}{1 - \langle f_k, g_k \rangle} f_k \otimes g_k \right) \\ &= I_n + \left(\frac{1}{1 - \langle f_k, g_k \rangle} - 1 \right) f_k \otimes g_k \\ &- \frac{1}{1 - \langle f_k, g_k \rangle} (f_k \otimes g_k)^2 \\ &= I_n + \frac{\langle f_k, g_k \rangle}{1 - \langle f_k, g_k \rangle} f_k \otimes g_k - \frac{\langle f_k, g_k \rangle}{1 - \langle f_k, g_k \rangle} f_k \otimes g_k \\ &= I_n. \end{aligned}$$

Similarly,

$$\left(I_n + \frac{1}{1 - \langle f_k, g_k \rangle} f_k \otimes g_k \right) R_\Lambda = I_n.$$

□

[1]The first paragraph of the proof of Theorem 3.3 was reproduced from [37] and [38] With permission of Springer.

After computing formulas for R_Λ for $|\Lambda| = 1$ and $|\Lambda| = 2$, we decided to try to determine a formula for any erasure set size. The next day, we computed the derivation which is the majority of the proof of Theorem 3.3. The theorem gives a closed-form, basis-free formula for R_Λ. For the the proof, and throughout the remainder of this section, we will use the notations laid out in the following remark.

Remark 3.2. Throughout this section, for finite frames, F will denote both the synthesis operator for the frame, and the frame itself. That is, if $F = \{f_j\}_{j=1}^N$ is a complex (or real valued) frame,

$$F = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ f_1 & f_2 & \cdots & f_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

(the matrix whose columns are the frame vectors, f_j). Similarly, F_Λ will denote both $F_\Lambda = \{f_j\}_{j\in\Lambda}$ and the synthesis operator for the sequence $\{f_j\}_{j\in\Lambda}$. For a dual frame pair, we will define $M_\Lambda = G_\Lambda^* F_\Lambda = (\langle f_k, g_j\rangle)_{j,k\in\Lambda}$. Additionally, I_d will denote the d-dimensional identity operator. Throughout, we will also use $n \leq \infty$ to denote the dimension of the Hilbert space $\mathcal{H}$, and L to denote the cardinality of the erasure set (i.e. $|\Lambda| = L$).

Theorem 3.3. *Assume that (F, G) is a dual frame pair for a Hilbert space, $\mathcal{H}$, and Λ is an erasure set. Then R_Λ is invertible if and only if $(I_L - M_\Lambda)$ is invertible. Furthermore, if $C = (c_{j,k})_{j,k\in\Lambda} = (I_L - M_\Lambda)^{-1}$, then*

$$R_\Lambda^{-1} = I_n + \sum_{j,k\in\Lambda} c_{j,k} f_j \otimes g_k. \tag{3.2}$$

For the proof, we require the following lemma, which we will isolate because we will refer to the Lemma on several occasions. Recall that the spectrum of an operator

$T \in B(\mathcal{H})$ is the set

$$\sigma(T) = \{\lambda : (\lambda I_n - T) \text{ is not invertible}\}.$$

Lemma 3.4. *Assume that (F, G) is a dual frame pair for a Hilbert space, $\mathcal{H}$, and Λ is an erasure set. Then $\sigma(E_\Lambda) \setminus \{0\} = \sigma(M_\Lambda) \setminus \{0\}$.*

Proof. We can easily see that $M_\Lambda = G_\Lambda^* F_\Lambda$, and $E_\Lambda = F_\Lambda G_\Lambda^*$. Assume that $f \in \mathcal{H}$ is an eigenvector for E_Λ with eigenvalue $\lambda \neq 0$. Then,

$$F_\Lambda G_\Lambda^* f = E_\Lambda f = \lambda f.$$

Thus, applying G_Λ^* to both sides gives

$$G_\Lambda^* F_\Lambda G_\Lambda^* f = \lambda(G_\Lambda^* f).$$

Thus, $\lambda \in \sigma(M_\Lambda) \setminus \{0\}$ provided that $G_\Lambda^* f \neq 0$. If we assume for the sake of contradiction that $G_\Lambda^* f = 0$, then

$$\lambda f = E_\Lambda f = F_\Lambda G_\Lambda^* f = 0.$$

So, either $\lambda = 0$, or $f = 0$. Either of these contradict that f is an eigenvector for E_Λ with non-zero eigenvalue λ. Thus, $G_\Lambda^* f \neq 0$, and so $\lambda \in \sigma(M_\Lambda) \setminus \{0\}$. Thus, $\sigma(E_\Lambda) \setminus \{0\} \subset \sigma(M_\Lambda) \setminus \{0\}$. A similar argument shows that $\sigma(M_\Lambda) \setminus \{0\} \subset \sigma(E_\Lambda) \setminus \{0\}$ $\square$

Proof of Theorem 3.3. Assume without loss of generality that $\Lambda = \{1, \cdots, L\}$, and that $(I_L - M_\Lambda)$ is invertible. Assume that the inverse of R_Λ has the form

$$T = I_n + \sum_{j,k=1}^{L} c_{jk} f_j \otimes g_k. \tag{3.3}$$

for some $c_{jk} \in \mathbb{C}$. Then,

$$\begin{aligned}
I_n &= TR_\Lambda \\
&= \left(I_n + \sum_{j=1}^{L}\sum_{k=1}^{L} c_{jk} f_j \otimes g_k\right)\left(I_n - \sum_{j=1}^{L} f_j \otimes g_j\right) \\
&= I_n + \sum_{j=1}^{L}\sum_{k=1}^{L} c_{jk} f_j \otimes g_k - \sum_{j=1}^{L} f_j \otimes g_j - \sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{\ell=1}^{L} c_{jk}(f_j \otimes g_k)(f_\ell \otimes g_\ell) \\
&= I_n + \sum_{j=1}^{L}\sum_{k=1}^{L} c_{jk} f_j \otimes g_k - \sum_{j=1}^{L} f_j \otimes g_j - \sum_{j=1}^{L}\sum_{\ell=1}^{L}\sum_{k=1}^{L} c_{jk} \langle f_\ell, g_k\rangle (f_j \otimes g_\ell) \\
&= I_n + \sum_{j=1}^{L}\sum_{k=1}^{L} c_{jk} f_j \otimes g_k - \sum_{j=1}^{L} f_j \otimes g_j - \sum_{\ell=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L} c_{j\ell} \langle f_k, g_\ell\rangle (f_j \otimes g_k).
\end{aligned}$$

In the last sum, we switched indices k and ℓ. Thus,

$$\sum_{j=1}^{L} f_j \otimes g_j = \sum_{j=1}^{L}\sum_{k=1}^{L} c_{jk} f_j \otimes g_k - \sum_{\ell=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L} c_{j\ell} \langle f_k, g_\ell\rangle (f_j \otimes g_k).$$

By simply setting the coefficients of the $f_j \otimes g_k$ to $\delta_{j,k}$, we obtain the following system of equations:

$$c_{jk} - \sum_{\ell=1}^{L} c_{j\ell} \langle f_k, g_\ell\rangle = \delta_{jk}. \tag{3.4}$$

For a fixed value of j, we have the system

$$(\delta_{jk})^T_{k=1,\cdots,L} = \begin{pmatrix} 1 - \langle f_1, g_1\rangle & -\langle f_1, g_2\rangle & \cdots & -\langle f_1, g_L\rangle \\ -\langle f_2, g_1\rangle & 1 - \langle f_2, g_2\rangle & \cdots & -\langle f_2, g_L\rangle \\ \vdots & \vdots & \ddots & \vdots \\ -\langle f_L, g_1\rangle & -\langle f_L, g_2\rangle & \cdots & 1 - \langle f_L, g_L\rangle \end{pmatrix} (c_{jk})^T_{k=1,\cdots,L}$$

Let $C = (c_{jk})_{j,k}$. Combining the equations for all j gives

$$I_L = (I_L - M_\Lambda^T)C^T$$

So, $C(I - M_\Lambda) = I$, and thus, $C = (I - M_\Lambda)^{-1}$. From here it is easily verified that $R_\Lambda^{-1} = T$. This establishes both the backwards implication, and the "furthermore" part.

To prove the forwards implication, assume that R_Λ is invertible. Then, since $R_\Lambda = I_n - E_\Lambda$, $1 \notin \sigma(E_\Lambda)$. Thus, by Lemma 3.4, $1 \notin \sigma(M_\Lambda)$, and so $0 \notin \sigma(I_L - M_\Lambda)$. That is, $I_L - M_\Lambda$ is injective. Therefore, since $I_L - M_\Lambda$ is a finite dimensional matrix, $I_L - M_\Lambda$ is invertible. $\square$

The next corollary provides an easily implemented algorithm for Reduced Direct Inversion. Furthermore, this algorithm will be well suited for Matlab because it uses matrix multiplications as opposed to *for* loops.

Corollary 3.5. *Assume that (F, G) is a dual frame pair for a finite dimensional Hilbert space, $\mathcal{H}$, and Λ is an erasure set. If $C = (c_{j,k})_{j,k\in\Lambda} = (I_L - M_\Lambda)^{-1}$, then for all $f \in \mathcal{H}$*

$$f = f_R + F_\Lambda C G_\Lambda^* f_R. \tag{3.5}$$

Remark 3.6. As with Nilpotent Bridging, we are also interested in noise mitigation for Reduced Direct Inversion, and we have dedicated two sections (one theoretical and one experimental) to this topic. As with bridging, errors in the partial reconstruction, f_R can be amplified by the matrix $C = (I - M_\Lambda)^{-1}$. However, the TRIP frames from Construction Algorithm 2.47 will be highly robust to noise.

Remark 3.7. Recall from equation (1.23) that we can invert R_Λ (provided $\|E_\Lambda\| < 1$) by employing a Neumann series:

$$R_\Lambda^{-1} = (I_n - E_\Lambda)^{-1} = \sum_{k=0}^{\infty} E_\Lambda^k. \tag{3.6}$$

Likewise, a Neumann series can be run to invert R_Λ by using reduced direct inversion provided $\|M_\Lambda\| < 1$. From equation (3.5), we have

$$\begin{aligned} f &= f_R + F_\Lambda (I_L - M_\Lambda)^{-1} G_\Lambda^* f_R & (3.7)\\ &= f_R + F_\Lambda \left(\sum_{k=0}^{\infty} M_\Lambda^k \right) G_\Lambda^* f_R & (3.8)\\ &= f_R + F_\Lambda \left(\sum_{k=0}^{\infty} M_\Lambda^k (G_\Lambda^* f_R) \right). & (3.9) \end{aligned}$$

Furthermore, E_Λ and M_Λ have the same spectrum, and thus both methods require the same number of iterations. Thus, since the original method (equation (3.6)) requires larger matrix multiplications, it is now obsolete whenever $L < n$. Moreover, inverting R_Λ for a tight frame is the same as the FORC method described in the Introduction. Thus, the FORC method is also obsolete provided we are using tight frames and $L < n$.

Remark 3.8. After giving a talk on this method, Dr. Foucart commented that the intuition for this method was actually quite simple, and we will give a brief exposition on this next (see the short expository article [49]).

Formally, we can write

$$
\begin{aligned}
R_\Lambda^{-1} &= (I_n - F_\Lambda G_\Lambda^*)^{-1} \\
&\sim \sum_{k=0}^{\infty} (F_\Lambda G_\Lambda^*)^k \\
&= I_n + F_\Lambda \sum_{k=1}^{\infty} (G_\Lambda^* F_\Lambda)^{k-1} G_\Lambda^* \\
&= I_n + F_\Lambda \sum_{k=0}^{\infty} (G_\Lambda^* F_\Lambda)^{k} G_\Lambda^* \\
&\sim I_n + F_\Lambda (I - G_\Lambda^* F_\Lambda)^{-1} G_\Lambda^*.
\end{aligned}
$$

This is precisely equation (3.5) in Corollary 3.5.

3.2 Implementation

In this section, we will give an implementation of our algorithm, and a brief explanation of the code.

```
n = 2000;
N = 3000;
L = [1:100];

F = (1/sqrt(n)) * randn(n,N);
S = F * F';
G = S \ F;

f = rand(n,1);
f = f ./ norm(f,2);

FC = G' * f;
```

```
13 FC(L) = zeros(size(L'));
14
15 f_R = F * FC;
16
17 M = G(:,L)' * F(:,L);
18 C = (eye(length(L)) - M) \ eye(length(L));
19
20 g = f_R + F(:,L) * (C * (G(:,L)' * f_R));
21
22 norm(f-g,2)
```

Lines 1-16 and 21-22 perform the same tasks as the Nilpotent Bridging algorithm, so for more details, see the exposition in Section 2.4. In line 17 we compute M_Λ, in line 18 we compute $C = (I_L - M_\Lambda)^{-1}$, and in line 20 we compute the reconstruction g as in equation (3.5). Lastly, in line 22, we compute the error in our reconstruction. Since we have no noise term above, this error is only machine error.

As with bridging, we can replace lines 5-7 with the following code which will create a standard normally distributed TRIP frame.

```
F = randn(N,n);
[F,~] = qr(F,0);
F = F';
G = F;
```

As was mentioned in Remark 3.7 we can use Neumann iterations to invert $I - M_\Lambda$. However, we can employ a couple of numerical shortcuts to speed up the process. We will denote by f_κ the reconstruction of f after performing κ Neumann iterations. That is,

$$f_\kappa = f_R + F_\Lambda \left(\sum_{k=0}^{\kappa} M_\Lambda^k G_\Lambda^* f_R \right). \tag{3.10}$$

We next rewrite this as

$$f_\kappa = f_R + F_\Lambda \left(\sum_{k=0}^{\kappa} M_\Lambda^k h_0 \right), \tag{3.11}$$

where $h_0 = G_\Lambda^* f_R$. Letting

$$h_n = \left(\sum_{k=0}^{n} M_\Lambda^k h_0 \right) \tag{3.12}$$

we see that

$$h_n = h_0 + M_\Lambda h_{n-1}. \tag{3.13}$$

Thus, combining equations (3.11), (3.12), and (3.13) we have

$$f_\kappa = f_R + F_\Lambda h_\kappa. \tag{3.14}$$

Using this setup gives us a more efficient computation of f_κ than directly applying equation (3.10) because it is much quicker to multiply a matrix and a vector than to multiply two matrices. An implementation of Reduced Direct Inversion with Neumann iterations can be done by replacing lines 18-20 with the following code.

```
tolerance = 10^(-10)
Mnorm = norm(M);
NumIter = round(log(tolerance*(1-Mnorm))/log(Mnorm));

h_0 = G(:,L)'*f_R;
h_k = zeros(size(L'));
for(j = 1:1:NumIter)
```

```
 8        h_k = h_0 + M * h_k;
 9    end
10
11    g = f_R + F(:,L) * h_k;
```

In the first line we specify a tolerance level for the inverse of $I_L - M_\Lambda$. In line 2 we compute the norm of M_Λ in order to determine the number of iterations required to reach our tolerance in line 3 (see Theorem 1.2 in the Introduction). In line 5 we compute h_0, and in lines 6-9 we compute h_κ using the recursion formula in equation (3.13). In line 11, we compute f_κ using equation (3.14).

3.3 An Analysis of TRIP Frames

In this section, we will give probabilistic estimates for the RIP constants for standard normally generated TRIP frames. These estimates will be of great use in Section 3.4, where we will provide bounds for the amplification of error in our reconstruction. The first lemma just verifies that the TRIP frame construction actually produces a tight frame.

Lemma 3.9. *Let H, Q, R, F, and G be as in the construction of a TRIP frame (see Construction Algorithm 2.47). Then, F is a tight frame with frame bound $\frac{N}{n}$, and G is its standard dual.*

Proof. Let S denote the frame operator for F. Then $S = FF^* = \frac{N}{n}Q^*Q = \frac{N}{n}I$ since the columns of Q form an orthonormal set. Thus, for $f \in \mathcal{H}$,

$$\frac{N}{n}\|f\|^2 = \frac{N}{n}\langle f, f\rangle = \langle Sf, f\rangle = \left\langle \sum_{j=1}^{N} \langle f, f_j\rangle f_j, f \right\rangle = \sum_{j=1}^{N} |\langle f, f_j\rangle|^2 . \tag{3.15}$$

Therefore, F is a tight frame with frame bound $\frac{N}{n}$. $\square$

The point of the next lemma is to show that if $\frac{1}{\sqrt{n}}H$ satisfies the RIP with a good constant, and $\frac{1}{\sqrt{n}}H$ is close to being a FUNTF, then, F will also satisfy the RIP with a good bound. Thus, if we were to apply Theorems 1.12 and 1.13, then, with a high probability if the entries of H are drawn from the standard normal distribution, F will satisfy the RIP with a good constant.

Lemma 3.10. *Assume $0 < \alpha \leq \beta < \infty$, and H is an $n \times N$ matrix for which*

$$\alpha\|x\|^2 \leq \left\|\frac{1}{\sqrt{n}}Hx\right\|^2 \leq \beta\|x\|^2 \tag{3.16}$$

for all s-sparse vectors $x \in \mathbb{R}^N$. Assume Q, R, and F are as in Construction Algorithm 2.47, and H has lower and upper frame bounds A and B, respectively. Then for all s-sparse vectors $x \in \mathbb{R}^N$,

$$\frac{N}{n}\frac{\alpha}{B}\|x\|^2 \leq \|Fx\|^2 \leq \frac{N}{n}\frac{\beta}{A}\|x\|^2. \tag{3.17}$$

Proof. Since $Q^*Q = I$, and $H^* = QR$, $Q^*H^* = R$. From Proposition 1.10 QQ^* is the projection onto the range of Q. Since $\sqrt{\frac{n}{N}}F = Q^*$, and $\sqrt{\frac{n}{N}}F$ is Parseval, Q is an isometry. Since $H^* = QR$, range$(H^*) \subset$ range(Q). So, $QQ^*H^* = H^*$. Thus, for any $f \in \mathcal{H}$,

$$\|Rf\|^2 = \|Q^*H^*f\|^2 = \|QQ^*H^*f\|^2 = \|H^*f\|^2. \tag{3.18}$$

Thus, $A\|f\|^2 \leq \|Rf\|^2 \leq B\|f\|^2$. So,

$$\frac{1}{B} \leq \|R^{-1}\|^2 \leq \frac{1}{A}. \tag{3.19}$$

Combining this with equation (3.16) gives

$$\|Fx\|^2 = \frac{N}{n}\|Q^*x\|^2 = \frac{N}{n}\|(R^{-1})^*Hx\|^2 \geq \frac{N}{n}\frac{1}{B}\|Hx\|^2 \geq \frac{N}{n}\frac{\alpha}{B}\|x\|^2.$$

Similarly,

$$\|Fx\|^2 \leq \frac{N}{n}\frac{\beta}{A}\|x\|^2.$$

$\square$

Remark 3.11. If H is close to a FUNTF, then $A \approx \frac{N}{n}$, and $B \approx \frac{N}{n}$. Thus, equation (3.17) becomes

$$\alpha\|x\|^2 \lesssim \|Fx\|^2 \lesssim \beta\|x\|^2.$$

The next theorem quantifies the statements that we made prior to Lemma 3.10. That is, if the entries of H are drawn independently from the standard normal distribution, then the tight frame, F, produced by our TRIP frame construction procedure is likely to satisfy the RIP.

Theorem 3.12. *Assume that H is an $n \times N$ matrix with entries drawn independently from the standard normal distribution. Let F be as in Construction Algorithm 2.47, and ρ be the constant as in Theorem 1.12. If*

$$\frac{\rho}{\delta^2}\left(s\ln\left(\frac{eN}{s}\right) + \ln\left(\frac{2}{\gamma}\right)\right) \leq n \leq \frac{12\ln\left(\frac{\gamma}{2}\right) + (3\eta^2 - 4\eta^3)N}{12\ln\left(1 + \frac{4}{\eta}\right)}, \tag{3.20}$$

then with probability at least $1 - 2\gamma$, for all s-sparse vectors $x \in \mathbb{R}^n$,

$$\frac{1}{(1+\eta)^3}(1-\delta)\|x\|^2 \leq \|Fx\|^2 \leq (1+\eta)^3(1+\delta). \tag{3.21}$$

3.4 Numerical Considerations

In this section, we will give some estimates for the number of Neumann iterations required to obtain a certain error tolerance, as well as error bounds for sparse noise. In particular, since standard normally distributed TRIP frames are easily constructed and well behaved, we will pay special attention to the stability of our algorithms for this class of frames.

The first lemma provides upper and lower bounds on the norm of M_Λ when F is a tight frame satisfying a condition similar to the RIP.

Lemma 3.13. *Assume that F is a tight frame with frame bound A, and G is the standard dual to F (i.e. $G = \frac{1}{A}F$). Suppose that for all s-sparse vectors, $x \in \mathbb{R}^N$,*

$$\alpha\|x\|^2 \leq \|Fx\|^2 \leq \beta\|x\|^2. \tag{3.22}$$

If $|\Lambda| = L \leq s$, then

$$\frac{\alpha}{A} I_L \leq M_\Lambda \leq \frac{\beta}{A} I_L. \tag{3.23}$$

In particular, for all $y \in \mathbb{R}^L$,

$$\frac{\alpha}{A}\|y\| \leq \|M_\Lambda y\| \leq \frac{\beta}{A}\|y\|. \tag{3.24}$$

Proof. For $y \in \mathbb{R}^L$,

$$\langle M_\Lambda y, y\rangle = \langle G_\Lambda^* F_\Lambda y, y\rangle = \frac{1}{A}\langle F_\Lambda^* F_\Lambda y, y\rangle = \frac{1}{A}\langle F_\Lambda y, F_\Lambda y\rangle = \frac{1}{A}\|F_\Lambda y\|^2. \tag{3.25}$$

Thus,

$$\frac{\alpha}{A} I_L \leq M_\Lambda \leq \frac{\beta}{A} I_L.$$

The "in particular" part follows from the operator inequality. □

The following corollary will give us the number of iterations we need in order to ensure that $\|f - f_\kappa\|$ meets a prescribed error bound, where f_κ is as in equation (3.10).

Corollary 3.14. *Assume that F is a tight frame with frame bound A, and G is the standard dual to F (i.e. $G = \frac{1}{A}F$). Suppose that for all s-sparse vectors, $x \in \mathbb{R}^N$,*

$$\alpha\|x\|^2 \leq \|Fx\|^2 \leq \beta\|x\|^2. \tag{3.26}$$

Assume $|\Lambda| = L \leq s$ and $\beta < A$. Given $\gamma > 0$, if $\kappa > \log_{\frac{\beta}{A}}\left(\gamma\left(\frac{A}{\beta} - 1\right)\right) - 1$, then

$$\|f - f_\kappa\| \leq \gamma\|f_R\|. \tag{3.27}$$

Proof. By Lemma 3.13 $\|M_\Lambda\| \leq \frac{\beta}{A}$. From Theorem 1.2, for $\kappa > \log_{\frac{\beta}{A}}\left(\frac{\gamma A}{\beta}\left(1 - \frac{\beta}{A}\right)\right) - 1$,

$$\|(I_L - M_\Lambda)^{-1} - \sum_{k=0}^{\kappa} M_\Lambda^k\| \leq \frac{\gamma A}{\beta}.$$

So, for $\kappa > \log_{\frac{\beta}{A}}\left(\gamma\left(\frac{A}{\beta} - 1\right)\right) - 1$, we have

$$\begin{aligned}
\|f - f_\kappa\| &= \left\|F_\Lambda(I - M_\Lambda)^{-1}G_\Lambda^* f_R - F_\Lambda\left(\sum_{k=0}^{\kappa} M_\Lambda^k\right)G_\Lambda^* f_R\right\| \\
&\leq \|F_\Lambda\|\left\|(I - M_\Lambda)^{-1} - \sum_{k=0}^{\kappa} M_\Lambda^k\right\|\|G_\Lambda\|\|f_R\| \\
&\leq \sqrt{\beta}\frac{\gamma A}{\beta}\frac{\sqrt{\beta}}{A}\|f_R\| \\
&= \gamma\|f_R\|.
\end{aligned}$$

□

Remark 3.15. By Corollary 3.14, the number of Neumann iterations required is

$$\mathcal{O}\left(\frac{\ln\left(1 - \frac{1}{A}\right)}{\ln\left(\frac{1}{A}\right)}\right).$$

However, for the TRIP frames, or a FUNTF, $A = \frac{N}{n}$, which is often referred to as the *frame excess*. Thus, the larger the frame excess, the quicker our Neumann series converges.

In the next lemma, we are interested in the amplification associated to additive channel noise. Before we are ready for the lemma, we will need to lay out some conventions. Since we are discussing channel noise, we are concerned with noise introduced to the non-erased frame coefficient sequence $(\langle f, g_j \rangle)_{j \in \Lambda^c}$. For this model, we will consider an additive sparse noise term $\epsilon = (\epsilon_j)_{j \in \Lambda^c}$. Thus, the error in the partial reconstruction, f_R, is $F_{\Lambda^c}\epsilon$. Since R_Λ^{-1} is linear, the corresponding error in the reconstruction of f is the *error term*

$$f_{err} = R_\Lambda^{-1} F_{\Lambda^c} \epsilon. \tag{3.28}$$

The next lemma gives an upper bound on $\|f_{err}\|$ for a tight frame, F, which also satisfies an RIP-style bound.

Lemma 3.16. *Assume F is a tight frame with frame bound A and there exist constants $0 < \alpha \leq \beta < A$ for which*

$$\alpha \|x\|^2 \leq \|Fx\|^2 \leq \beta \|x\|^2 \tag{3.29}$$

for all s sparse vectors, $x \in \mathbb{C}^N$. If ϵ is an s-sparse error term and $|\Lambda| \leq s$, then $\|f_{err}\| \leq \frac{A\sqrt{\beta}}{A-\beta}\|\epsilon\|$.

Proof. Since ϵ is s-sparse, and F satisfies equation (3.29), we have

$$\|F_{\Lambda^c}\epsilon\| \leq \sqrt{\beta}\|\epsilon\|. \tag{3.30}$$

Next, since $M_\Lambda \leq \frac{\beta}{A}$,

$$I_L - M_\Lambda \geq \left(1 - \frac{\beta}{A}\right) I_L = \frac{A-\beta}{A} I_L.$$

Since $0 < \beta < A$, $(I_L - M_\Lambda)^{-1}$ exists, and

$$(I_L - M_\Lambda)^{-1} \leq \frac{A}{A-\beta}. \tag{3.31}$$

Thus, combining equations (3.28), (3.30), and (3.31) gives

$$\begin{aligned}
\|f_{err}\| &= \|R_\Lambda^{-1} F_{\Lambda^c}\epsilon\| \\
&= \|(I_n + F_\Lambda (I_L - M_\Lambda)^{-1} G_\Lambda^*) F_{\Lambda^c}\epsilon\| \\
&= \left\| F_{\Lambda^c}\epsilon + \frac{1}{A} F_\Lambda (I_L - M_\Lambda)^{-1} F_\Lambda^* F_{\Lambda^c}\epsilon \right\| \\
&\leq \|F_{\Lambda^c}\epsilon\| + \frac{1}{A}\|F_\Lambda (I_L - M_\Lambda)^{-1} F_\Lambda^*\| \|F_{\Lambda^c}\epsilon\| \\
&\leq \sqrt{\beta}\|\epsilon\| + \frac{\sqrt{\beta}}{A}\|F_\Lambda\| \|(I_L - M_\Lambda)^{-1}\| \|F_\Lambda^*\| \|\epsilon\| \\
&\leq \sqrt{\beta}\|\epsilon\| + \frac{\sqrt{\beta}}{A}\beta \frac{A}{A-\beta}\|\epsilon\| \\
&= \sqrt{\beta}\left(1 + \frac{\beta}{A-\beta}\right)\|\epsilon\| \\
&= \frac{A\sqrt{\beta}}{A-\beta}\|\epsilon\|.
\end{aligned}$$

□

By combining Lemma 3.16 with Theorem 3.12, we get the following result for standard normally generated TRIP frames.

Theorem 3.17. *Assume that (F, G) is a standard normally generated TRIP frame, $s < N$, $\gamma, \delta \in (0, 1)$, $\eta > 0$, and*

$$\frac{\rho}{\delta^2}\left(s \ln\left(\frac{eN}{s}\right) + \ln\left(\frac{2}{\gamma}\right)\right) \leq n \leq \frac{12\ln\left(\frac{\gamma}{2}\right) + (3\eta^2 - 4\eta^3)N}{12\ln\left(1 + \frac{4}{\eta}\right)},$$

where ρ is the constant from Theorem 1.12. If ϵ is an s-sparse error term and $|\Lambda| \leq s$, *then with probability at least* $1 - 2\gamma$*,*

$$\|f_{err}\| \leq \frac{N\sqrt{(1+\eta)^3(1+\delta)}}{N - n(1+\eta)^3(1+\delta)}\|\epsilon\|$$

3.5 Noise Mitigation

The first experiment is the Reduced Direct Inversion equivalent of the the first experiment that we ran for Nilpotent Bridging. As with that experiment, the goal is to display how well Reduced Direct Inversion performs subject to normally distributed additive noise (not necessarily sparse). As with Nilpotent Bridging, we used standard normally distributed TRIP frames of length 1000 in $\mathbb{R}^{250}$, and varied the erasure set sizes between 10 and 250 in multiples of 10. For each erasure set size, we ran 50 trials, each with a new standard normally distributed TRIP frame. In each trial we added a new 5% Gaussian random noise term (not necessarily sparse) to the non-erased frame coefficients. In Figure 3.1, we plotted the errors in the noisy partial reconstruction $\tilde{f}_R$ (see equation (2.36)), and the noisy reconstruction,

$$\tilde{f} = \tilde{f}_R + F_\Lambda(I - M_\Lambda)^{-1}G_\Lambda^*\tilde{f}_R = f + f_{err}. \tag{3.32}$$

In every instance, Reduced Direct Inversion outperforms the noisy partial reconstruction. We also see that there is very little error amplification, even for $|\Lambda| = 250$. This is somewhat shocking as our theoretical guarantees only give error bounds for values of n on the order of

$$\mathcal{O}\left(|\Lambda| \ln\left(\frac{N}{|\Lambda|}\right)\right).$$

Figure 3.1: Noise amplification using Reduced Direct Inversion.

Figure 3.2 is meant to be a pictorial representation of the previous experiment and is similar to our Lena experiment for Nilpotent Bridging. As before, we first compressed the 256×256 pixel image Lena by 15%, and computed a Gaussian random TRIP frame of length $N = 2n$. As in the previous experiment, we added a 5% noise term to the non-erased coefficients. Our erasure set sizes for this experiment varied from 1% to 5% of the length of the frame, N. We plotted the image with only noise corruption in the first row, the erased image with noise ($\tilde{f}_R$) in the second row, and the reconstructed image ($\tilde{f}$) in the third row. As with 25% overbridging, we see a noticeable improvement in the third row over the second row, and the first and third rows are very comparable.

Figure 3.2: Noise amplification for Reduced Direct Inversion using Lena.

3.6 Applications to Shannon-Whittaker Sampling Theory

Reduced Direct Inversion can also be applied to Shannon-Whittaker Sampling Theory to reconstruct from sampling erasures. As with Nilpotent Bridging, the error in the finite term approximation of the partial reconstruction, f_R, can be highly amplified if $\|(I - M_\Lambda)^{-1}\| = \|C\|$ is too large. Again, the reconstruction is highly unstable when we erase consecutive samples, and the reconstruction is very stable when the erased data points are sufficiently scattered. For example, the first graph, Figure 3.3, shows the reconstruction of $f(x) = \text{sinc}(\pi x)$ from erasures indexed by $\Lambda = \{1, 2, \cdots, 100\}$, for Shannon-Whittaker Sampling on the half integers. We see that the reconstruction aligns very nicely with the original signal. For this example, Matlab computes $\|C\|$ to be 2.0000.

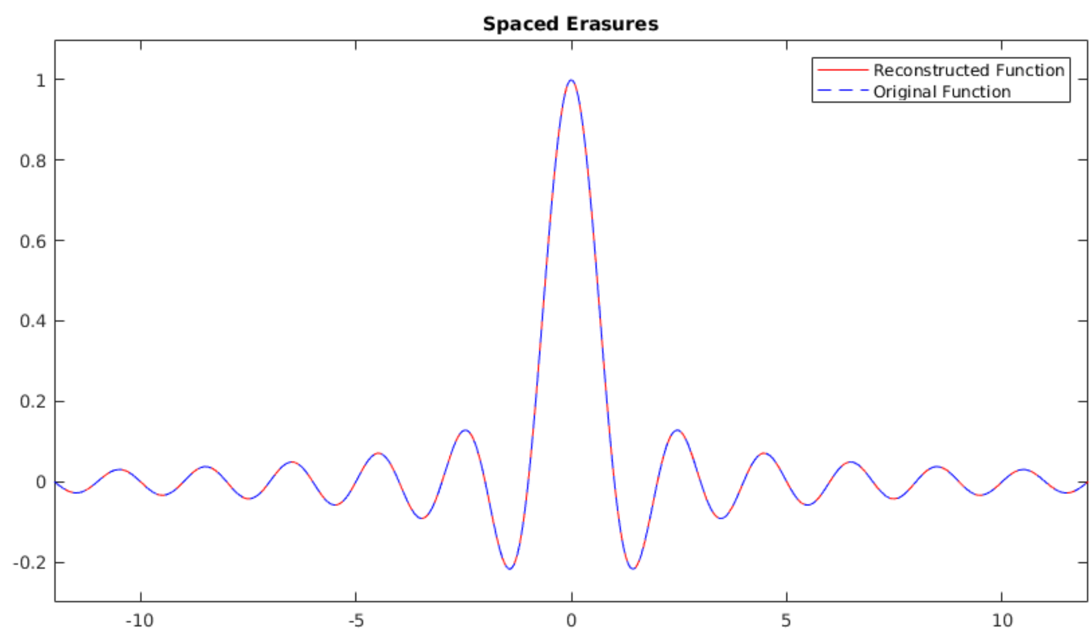

Figure 3.3: Error amplification of spaced erasures for Shannon-Whittaker Sampling on $\frac{1}{2}\mathbb{Z}$.

In stark contrast, Figure 3.4 shows the reconstruction from 12 consecutive erasures $\Lambda = \{1, \frac{3}{2}, \cdots, 6, \frac{13}{2}\}$. As with Nilpotent Bridging, we are using the non-erased sampled values between $x = -5,000$ and $x = 5,000$ to approximate f_R. For this experiment, Matlab computes $\|C\|$ to be 5.6234×10^7, and we can see a large blowup of the reconstruction on the erasure set.

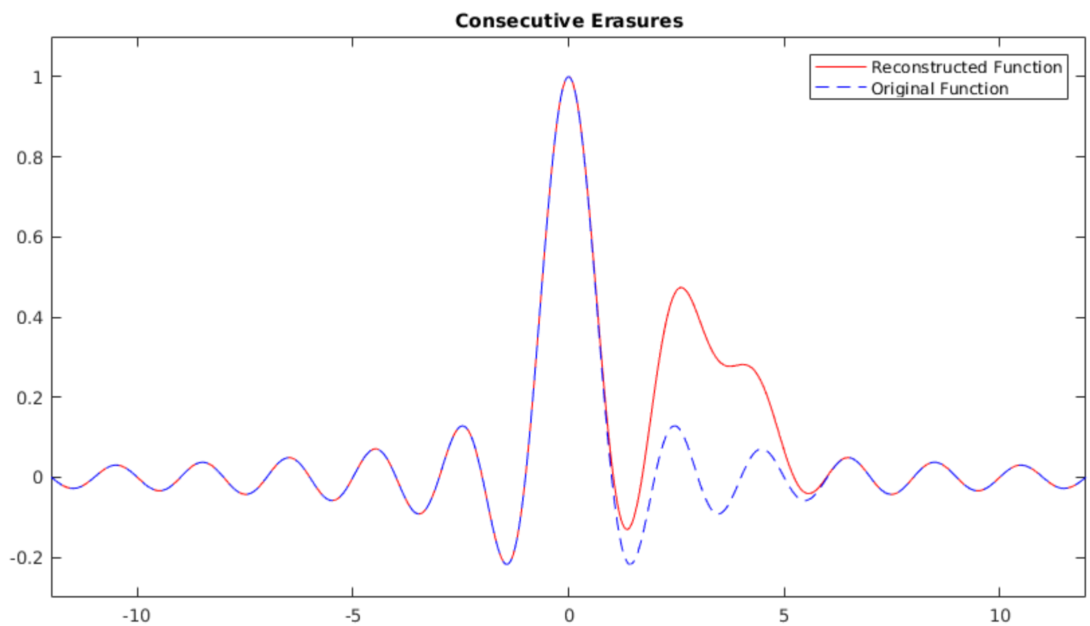

Figure 3.4: Error amplification of consecutive erasures for Shannon-Whittaker Sampling on $\frac{1}{2}\mathbb{Z}$.

Since the matrix $C = (I_L - M_\Lambda)$ is self-adjoint in the case of Shannon-Whittaker Sampling Theory, we are able to obtain some bounds on its norm by using techniques from scattered data interpolation (cf. [25], [46], and [51]).

Proposition 3.18. *Let (G_p, F_p) be the dual frame generated by Shannon-Whittaker Sam-*

pling on the lattice $p\mathbb{Z}$. Then

$$\frac{rp}{2\pi} \leq \|M_\Lambda\| \leq \frac{Rp}{2\pi}, \tag{3.33}$$

where r and R are the lower and upper Riesz bounds for the sequence $\{e^{ij\cdot}\}_{j\in\Lambda}$ in the space $L^2\,[-\pi,\pi]$.

Proof. First, notice that

$$\begin{aligned} M_\Lambda &= (\langle f_k, g_j\rangle)_{j,k\in\Lambda} \\ &= (\langle p\,\mathrm{sinc}(\pi(\cdot - k)), \mathrm{sinc}(\pi(\cdot - j))\rangle)_{j,k\in\Lambda} \\ &= p(\mathrm{sinc}(\pi(j-k)))_{j,k\in\Lambda}. \end{aligned}$$

Since $\mathrm{sinc}(\pi x)$ exhibits even symmetry, M_Λ is self-adjoint, and thus $\|M_\Lambda\|$ is bounded below by its smallest eigenvalue ($\min_{\|x\|=1} \langle M_\Lambda x, x\rangle$), and above by its largest eigenvalue ($\max_{\|x\|=1} \langle M_\Lambda x, x\rangle$). Let $\phi(x) = \mathrm{sinc}(\pi x)$. To compute these eigenvalues, we have

$$\begin{aligned} \langle M_\Lambda x, x\rangle &= \left\langle \left(p\sum_{k\in\Lambda} x_k \mathrm{sinc}(\pi(j-k))\right)_{j\in\Lambda}, x \right\rangle \\ &= p\sum_{j,k\in\Lambda} \overline{x_j} x_k \phi(\pi(j-k)) \\ &= \frac{p}{2\pi}\sum_{j,k\in\Lambda} \overline{x_j} x_k \int_{\mathbb{R}} \hat{\phi}(\xi) e^{i(j-k)\xi}\, d\xi \\ &= \frac{p}{2\pi}\int_{\mathbb{R}} \left(\sum_{j,k\in\Lambda} \overline{x_j} x_k \hat{\phi}(\xi) e^{i(j-k)\xi} \right) d\xi \\ &= \frac{p}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\xi) \left| \sum_{j\in\Lambda} x_j e^{ij\xi} \right|^2 d\xi. \end{aligned}$$

If we let $\chi_\pi(x)$ be the characteristic function of the interval $[-\pi,\pi]$, it is easily verified that $\hat{\chi}_\pi(\xi) = 2\pi \text{sinc}(\pi\xi)$. Thus, $\phi(x) = \frac{1}{2\pi}\hat{\chi}_\pi(x)$. Hence

$$\hat{\phi}(\xi) = \frac{1}{2\pi}\hat{\hat{\chi}}_\pi(\xi) = \chi_\pi(-\xi) = \chi_\pi(\xi).$$

So,

$$\langle M_\Lambda x, x\rangle = \frac{p}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{j\in\Lambda} x_j e^{ij\xi}\right|^2 d\xi = \frac{p}{2\pi}\left\|\sum_{j\in\Lambda} x_j e^{ij\cdot}\right\|^2_{L^2[-\pi,\pi]}. \tag{3.34}$$

Therefore,

$$\frac{p}{2\pi} r \leq \|M_\Lambda\| \leq \frac{p}{2\pi} R, \tag{3.35}$$

where r and R are the $L^2[-\pi,\pi]$ lower and upper Riesz bounds for $\{e^{ijx}\}_{j\in\Lambda}$, respectively.

□

Corollary 3.19. *Let (G_p, F_p) be the dual frame generated by Shannon-Whittaker Sampling on the lattice $p\mathbb{Z}$. Then*

$$\|C\| = \|(I - M_\Lambda)^{-1}\| \leq \frac{2\pi}{2\pi - Rp}, \tag{3.36}$$

where R is the upper Riesz bound for the sequence $\{e^{ijx}\}_{j\in\Lambda}$ in $L^2[-\pi,\pi]$.

4. ERASURE RECOVERY MATRICES

4.1 The Erasure Recovery Matrix Reconstruction Algorithm

This section represents ongoing joint work with Deguang Han, David Larson, and Wenchang Sun. Erasure recovery matrices were introduced by Han and Sun in [33]. Further work, particularly relating to the reconstruction from frame coefficient erasures at unknown locations can be found in the articles [31], [32], and [33]. However, in this section, we will only be concerned with frame erasures at known locations.

Throughout this section, we will denote $\mathbb{C}^n$ (or $\mathbb{R}^n$) by $\mathcal{H}$. Let $F = \{f_j\}_{j=1}^N$ be a frame for $\mathcal{H}$ and k be a positive integer. An *m-erasure recovery matrix* is a $k \times N$ matrix M whose columns have spark $m+1$, and which satisfies $Mc = 0$ for any vector $c \in \Theta(\mathcal{H})$, where Θ denotes the analysis operator for the frame F. That is,

$$M\left(\langle f, \varphi_j \rangle\right)_{j=1}^N = 0 \quad \forall f \in \mathcal{H}.$$

The next proposition lists some of the useful equivalent definitions for erasure recovery matrices.

Proposition 4.1. *Let F be a frame for $\mathcal{H}$ and Θ be its analysis operator. Suppose that $m \geq 1$ is an integer. The following are equivalent for a $k \times N$ matrix, M.*

(1) M is an m-erasure recovery matrix.

(2) The columns of M have spark $m+1$ and $\ker(M) \supseteq \Theta(\mathcal{H}_n)$.

(3) $\ker(M) \supseteq \Theta(\mathcal{H}_n)$ and for every set $\Lambda \subset \{1, 2, \cdots, N\}$ satisfying $|\Lambda| \leq m$, $(M_\Lambda^* M_\Lambda)^{-1}$ exists, where M_Λ denotes the minor of M formed by the columns indexed by Λ.

Proof. The proof of $(1) \Leftrightarrow (2)$ is quite obvious. The proof of $(1) \Leftrightarrow (3)$ follows by applying Lemma 1.7. $\square$

Remark 4.2. Assume that M is an m-erasure matrix for a frame F for $\mathcal{H}$. Assume that $f \in \mathcal{H}$, and $c = (c_j)_{j=1}^N$, where $c_j = \langle f, f_j \rangle$. Then, by definition, we have

$$Mc = 0.$$

Hence, if we let M_Λ denote the matrix with columns indexed by Λ, and c_Λ denote the vector $(c_j)_{j\in\Lambda}$ for any $\Lambda \subset \{1, \cdots, N\}$, we have

$$M_\Lambda c_\Lambda + M_{\Lambda^c} c_{\Lambda^c} = 0.$$

Rearranging the equation gives

$$M_\Lambda c_\Lambda = -M_{\Lambda^c} c_{\Lambda^c}. \tag{4.1}$$

If the goal is to reconstruct the vector c from erasures indexed by erasures at Λ, our goal is to solve equation (4.1) for c_Λ. By using part (3) of Proposition 4.1 we will be able to use a Moore-Penrose pseudo-inverse method to solve for c_Λ. Multiplying both sides of equation (4.1) by M_Λ^* gives

$$M_\Lambda^* M_\Lambda c_\Lambda = -M_\Lambda^* M_{\Lambda^c} c_{\Lambda^c}. \tag{4.2}$$

Now, simply inverting we can reconstruct c_Λ as

$$c_\Lambda = -(M_\Lambda^* M_\Lambda)^{-1} M_\Lambda^* M_{\Lambda^c} c_{\Lambda^c}, \tag{4.3}$$

whenever $|\Lambda| \leq m$.

A previous method which was used was to chop off all but $|\Lambda|$ rows of the matrices M_Λ and M_{Λ^c} in equation (4.1). In doing so, we could use a matrix inversion to solve the system of equations. However, the method of equation (4.3) turned out to be much more stable.

Remark 4.3. The matrix M_Λ in this section is not to be confused with the matrix M_Λ used for Reduced Direct Inversion, as they are not the same.

4.2 Erasure Recovery Matrix Construction Procedure

The following construction procedure will produce a pair (M, F) where M is an m-erasure recovery matrix for the Parseval frame F. Moreover, our reconstruction algorithm using this pair will be quite stable. A discussion of this stability will be provided in Sections 4.3 and 4.5.

Construction Algorithm 4.4.

1. *Generate an $m \times N$ matrix, H, whose entries are drawn independently from the standard normal distribution, and where $N \geq n + m$.*

2. *Let $M = \frac{1}{\sqrt{m}} H$.*

3. *Create an $N \times n$ random matrix T, and let A be the $N \times (n + m)$ whose first m columns are the columns are the rows of M and columns $m + 1$ through $m + n$ are the columns of the matrix T.*

4. *Compute the qr-factorization of A. That is, factor A as $A = QR$ where Q is a matrix with orthonormal columns, and R is upper triangular.*

5. *Let F be the minor of Q^* consisting of rows $m + 1$ through $m + n$.*

Proposition 4.5. *Let M and F be as constructed above. If the columns of M have spark $m+1$, and A has rank $n+m$, then F is a Parseval frame, and M is an m-erasure recovery matrix for F.*

Proof. By construction, since A has full rank the rows of F are orthonormal vectors, and are orthogonal to the rows of M. Since $FF^* = I$, F is a Parseval frame. Since $MF^* = 0$, and the columns of M have spark $m + 1$, M is an erasure recovery matrix for F. □

4.3 Numerical Considerations

Let M be an encoding frame protected m-erasure recovery matrix for a frame G for $\mathcal{H}$, and let F be a dual to G. Assume Λ is an erasure set satisfying $|\Lambda| = L$. For $f \in \mathcal{H}$, let $c_j = \langle f, g_j \rangle$ and $c = (c_j)_{j=1}^N$. From equation (4.3), we have:

$$c_\Lambda = -(M_\Lambda^* M_\Lambda)^{-1} M_\Lambda^* M_{\Lambda^c} c_{\Lambda^c}. \tag{4.4}$$

In this section, we would like to know what happens to our reconstruction when the frame coefficients indexed by Λ^c are subject to noise. Since our reconstruction operator $\Delta : \mathbb{C}^{N-L} \to \mathbb{C}^L$ defined by

$$\Delta c = -(M_\Lambda^* M_\Lambda)^{-1} M_\Lambda^* M_{\Lambda^c} c \quad \forall c \in \mathbb{C}^{N-L} \tag{4.5}$$

is linear, if we introduce a noise term $\epsilon = (\epsilon_j)_{j \in \Lambda^c}$ to the good coefficients, the corresponding error in the reconstructed coefficients is given by

$$\Delta \epsilon = -(M_\Lambda^* M_\Lambda)^{-1} M_\Lambda^* M_{\Lambda^c} \epsilon. \tag{4.6}$$

Thus, if $\|\epsilon\|$ or $\|\Delta\|$ is large, the reconstructed signal will be highly inaccurate. However, we will see that this is not the case for this situation when we use Construction Algorithm

4.4. The next lemma shows that if M satisfies a modified Restricted Isometry Property and ϵ is sparse, then the error in the coefficients is only slightly amplified.

Remark 4.6. This sparse noise model was motivated by [11]. In that paper, a similar model for erasure reconstruction was given. Their method uses a linear program to reconstruct from erasures at unknown locations. If $\epsilon = (\epsilon_j)_{j=1}^N$, is a sparse noise term, then, $M(c + \epsilon) = M\epsilon$. Since $c+\epsilon$ and M are known, to determine the noise term, they consider the minimization problem:

$$\text{argmin}\|\epsilon\|_0 \quad \text{subject to} \quad M\epsilon = M(c+\epsilon), \tag{4.7}$$

where $\|\epsilon\|_0$ denotes the number of non-zero entries of ϵ. However, this combinatorial problem is quite slow, so they solve the following much faster convex optimization problem instead:

$$\text{argmin}\|\epsilon\|_1 \quad \text{subject to} \quad M\epsilon = M(c+\epsilon). \tag{4.8}$$

Lemma 4.7. *Assume that G is a frame for $\mathcal{H}$ and M is a $k \times N$ m-erasure recovery matrix for G. Assume that for all s-sparse vectors $x \in \mathbb{C}^N$,*

$$\alpha\|x\|^2 \leq \|Mx\|^2 \leq \beta\|x\|^2.$$

If $\epsilon = (\epsilon_j)_{j\in\Lambda^c}$ is an s-sparse noise term, $|\Lambda| = L \leq s$, and Δ is the reconstruction operator as defined in equation (4.5), then,

$$\|\Delta\epsilon\| \leq \frac{\beta}{\alpha}\|\epsilon\|. \tag{4.9}$$

Proof. From equation (4.6), we have

$$\|\Delta\epsilon\| \leq \|(M_\Lambda^* M_\Lambda)^{-1}\| \|M_\Lambda\| \|M_\Lambda^c \epsilon\|. \tag{4.10}$$

Since $|\Lambda| \leq s$, whenever $\|x\| = 1$, we get

$$\langle M_\Lambda^* M_\Lambda x, x\rangle = \|M_\Lambda x\|^2 \geq \alpha.$$

Thus, if $\sigma(M_\Lambda^* M_\Lambda)$ denotes the spectrum of $M_\Lambda^* M_\Lambda$,

$$\min \sigma(M_\Lambda^* M_\Lambda) \geq \alpha.$$

Therefore

$$\|(M_\Lambda^* M_\Lambda)^{-1}\| = \frac{1}{\min \sigma(M_\Lambda^* M_\Lambda)} \leq \frac{1}{\alpha}. \tag{4.11}$$

Again, since $|\Lambda| \leq s$,

$$\|M_\Lambda\| \leq \sqrt{\beta}. \tag{4.12}$$

Since ϵ is s-sparse,

$$\|M_{\Lambda^c}\epsilon\| \leq \sqrt{\beta}\|\epsilon\|. \tag{4.13}$$

Combining equations (4.10), (4.11), (4.12), and (4.13) gives the result. □

The previous lemma gave a bound on the error of the frame coefficients. Next we will build on this error estimate for the reconstruction of a signal $f \in \mathcal{H}$. Recall that if (F, G) is a dual frame pair, then

$$f = \sum_{j=1}^{N} \langle f, g_j\rangle f_j = \sum_{j=1}^{N} c_j f_j \quad \forall f \in \mathcal{H}, \tag{4.14}$$

where $c_j = \langle f, g_j \rangle$ for all $j \in \{1, \cdots, N\}$. If the coefficients indexed by an erasure set Λ are erased, and the coefficients indexed by Λ^c are subject to an additive noise term, given by ϵ, then the corresponding error in the reconstruction of the erased coefficients is $\Delta\epsilon$. Thus the reconstructed signal, after synthesizing with $\{f_j\}_{j=1}^N$ is

$$\tilde{f} = \sum_{j \in \Lambda} (c_j + (\Delta\epsilon)_j) f_j + \sum_{j \in \Lambda^c} (c_j + \epsilon_j) f_j = f + \sum_{j \in \Lambda} (\Delta\epsilon)_j f_j + \sum_{j \in \Lambda^c} \epsilon_j f_j. \tag{4.15}$$

The following lemma gives a bound on the reconstruction error, $\|f - \tilde{f}\|$.

Lemma 4.8. *Assume that F is a Parseval dual to the frame G, and that M is a $k \times N$ m-erasure recovery matrix for $\{g_j\}_{j=1}^N$. Assume there exist constants $0 < \alpha \leq \beta \leq \infty$ so that for all s-sparse vectors $x \in \mathbb{C}^N$,*

$$\alpha \|x\|^2 \leq \|Mx\|^2 \leq \beta \|x\|^2,$$

Suppose $|\Lambda| = L \leq s$, ϵ is an s-sparse noise term, and let f and $\tilde{f}$ be defined as in equation (4.15). Then,

$$\|f - \tilde{f}\| \leq \left(1 + \frac{\beta}{\alpha}\right) \|\epsilon\|. \tag{4.16}$$

Proof. From equation (4.15),

$$\begin{aligned} \|f - \tilde{f}\| &= \left\| \sum_{j \in \Lambda} (\Delta\epsilon)_j f_j + \sum_{j \in \Lambda^c} \epsilon_j f_j \right\| = \|F_\Lambda \Delta\epsilon + F_{\Lambda^c}\epsilon\| \leq \|F_\Lambda \Delta\epsilon\| + \|F_{\Lambda^c}\epsilon\| \\ &\leq \|\Delta\epsilon\| + \|\epsilon\| \leq \left(\frac{\beta}{\alpha} + 1\right) \|\epsilon\| \end{aligned}$$

where F_Λ (resp. F_{Λ^c}) is the minor of the synthesis matrix for F with columns indexed by Λ (resp. Λ^c). $\square$

With Lemma 4.8 in mind, it should be fairly clear why Construction Algorithm 4.4

works well. In that algorithm, since M is a standard normally distributed random matrix, it will likely satisfy the RIP, as needed in the Corollary. The next theorem combines Lemma 4.8 with the RIP for standard normally distributed random matrices (Theorem 1.12) to show that for m on the order of $s \ln\left(\frac{N}{s}\right)$, the amplification of sparse additive noise will be small.

Theorem 4.9. *Assume that F and M are constructed as in Construction Algorithm 4.4. Fix $\delta, \gamma \in (0,1)$, and assume that ρ is the constant as in Theorem 1.12. If*

$$m \geq \frac{\rho}{\delta^2}\left(s \ln\left(\frac{eN}{s}\right) + \ln\left(\frac{2}{\gamma}\right)\right), \tag{4.17}$$

then with probability at least $1-\gamma$, for any s-sparse vector $x \in \mathbb{R}^N$,

$$(1-\delta)\|x\|^2 \leq \|Mx\|^2 \leq (1+\delta)\|x\|^2.$$

Moreover, with f, $\tilde{f}$, and ϵ defined as in Lemma 4.8, with probability greater than $1-\gamma$,

$$\|f-\tilde{f}\| \leq \frac{2}{1-\delta}\|\epsilon\|. \tag{4.18}$$

Remark 4.10. So far we have discussed the stability of Reduced Direct Inversion and the method of reconstruction using Erasure Recovery Matrices. For a discussion of the stability of the FORC method, see [20]. In that article, they discuss classes of frames, called *Numerically Erasure-Robust Frames* (or *NERFs*), which are able to stably reconstruct from frame erasures by using the FORC method.

4.4 Implementation

In this section, we provide an implementation for reconstruction from frame erasures by using the method of Erasure Recovery Matrices. A line by line description of the program is given below.

```
m = 250;
n = 250;
N = 1000;
L = [1:10];

M = 1 / sqrt(m) * rand(m,N);
A = [M',randn(N,n)];
[A,~] = qr(A,0);
F = A(:,m+1:m+n)';

f = randn(n,1);
f = f./norm(f);

FC = F' * f;
FC(L) = zeros(size(L'));

f_R = F * FC;

LC = setdiff(1:N,L);
FC(L) = -(M(:,L)'*M(:,L)) \ (M(:,L)'*(M(:,LC)*FC(LC)));

g = f_R + F(:,L) * FC(L);

norm(f-g)
```

In lines 1-4, we define the parameters for our experiment. The variable m is the height of the erasure recovery matrix, N is the length of our frame, L is the erasure set, and n is the dimension of the Euclidean space we are working in. In lines 6-9 we implement Construction Algorithm 4.4. In lines 11 and 12, we generate a standard normally distributed unit vector f which will serve as our test vector for the experiment. In line 14, we compute the frame coefficients of f. In line 15, we erase the coefficients indexed by the erasure set. In line 17, we compute f_R. In line 19, we compute Λ^c. In line 20, we implement equation (4.3) to recover the lost information. In line 22, we synthesize these computed coefficients with the frame vectors indexed by Λ to obtain our reconstruction, g. In the last line, we compute the norm of the error in the reconstruction. Since this method theoretically gives a perfect reconstruction, the error is just the machine error in the computation.

4.5 Noise Mitigation

In this section, as with Nilpotent Bridging and Reduced Direct Inversion, we are interested in studying the amplification of standard normally distributed additive noise for the method of reconstruction using Erasure Recovery Matrices. In the first experiment, we used Construction Algorithm 4.4 to create an erasure recovery matrix of size 250×1000 for a Parseval frame of length 1000 for $\mathbb{R}^{250}$. In this experiment, we added a 5% standard normally distributed additive noise term (not necessarily sparse) to the frame coefficients indexed by Λ^c. We varied the erasure set sizes between 10 and 250 in multiples of 10, and for each erasure set size we performed 50 trials. For each trial, new frames, erasure recovery matrices, and additive noise terms were used. In Figure 4.1, we plotted the errors in the noisy partial reconstruction, $\tilde{f}_R$ (see equation (2.35)), and the noisy reconstruction, $\tilde{f}$ (see equation 4.15). In the plot, the errors corresponding to $|\Lambda| = 250$ were omitted to avoid distortion in the graph. When $|\Lambda| = 250$, the median noisy reconstruction error was 0.52, with the maximum error at 12.97. Besides $|\Lambda| = 250$, in every instance plotted, we see that the noisy reconstruction outperforms the noisy partial reconstruction, but the noisy reconstruction continues to worsen as we get closer to $|\Lambda| = 250$. However, this is to be expected by our noise mitigation results since m should be on the order of $\mathcal{O}\left(|\Lambda| \ln\left(\frac{N}{|\Lambda|}\right)\right)$ in order to achieve a good reconstruction.

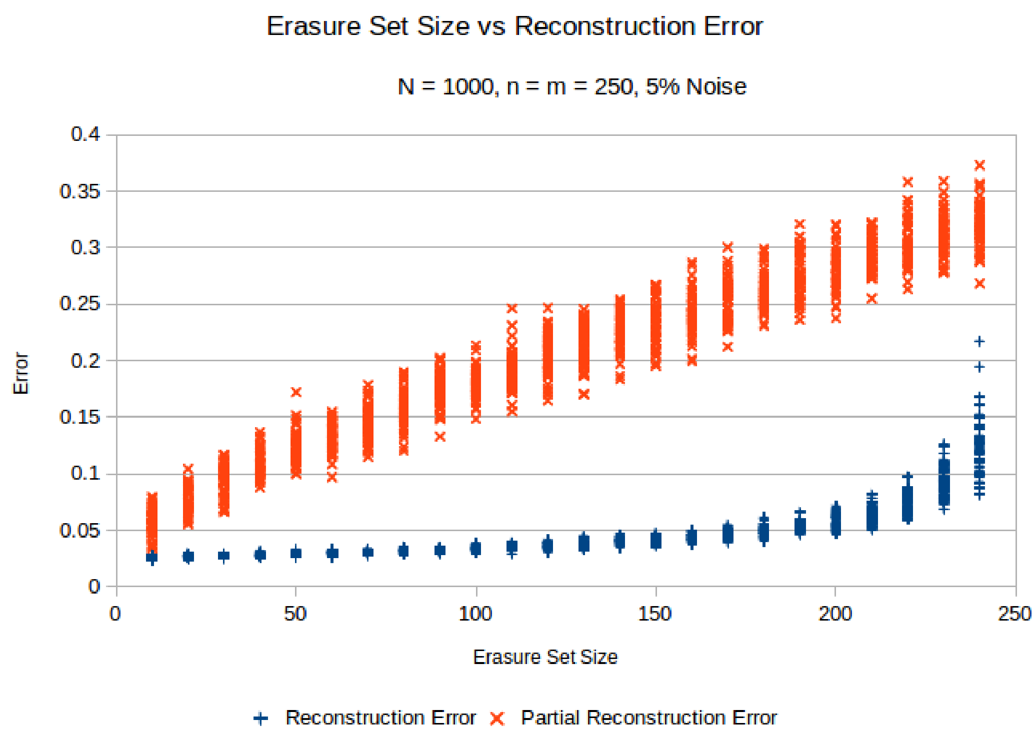

Figure 4.1: Noise amplification using Erasure Recovery Matrices.

Figure 4.2 displays a set of images which are a more visual representation of the previous experiment. As with our Nilpotent Bridging and Reduced Direct Inversion experiments with Lena, we first compress Lena (a 256×256 pixel image) by 15%. We then fixed $N = 2n + m$ for $m = 2000$. We computed an erasure recovery matrix and a Parseval frame as in Construction Algorithm 4.4. For this experiment, we used a 5% noise term, and varied our erasure sets between 1% and 5% of N. The first row of images shows what the image looks like when the good coefficients are corrupted only by our noise term. The second row shows $\tilde{f}_R$, the noisy partial reconstruction of Lena. The third row shows $\tilde{f}$, the noisy reconstruction of Lena. In the image, we see that the noisy reconstruction outperforms the noisy partial reconstruction, and the noisy image is comparable to our reconstruction.

Figure 4.2: Noise amplification for Erasure Recovery Matrices using Lena.

5. CONCLUSIONS

In this dissertation, three efficient methods of reconstruction from frame erasures were considered. These methods are more efficient than older methods in the literature because they only require an $L \times L$ matrix inversion to reconstruct from L erasures, whereas older methods require an $n \times n$ matrix inversion, where n denotes the dimension of the underlying Hilbert space.

In Section 2, the Nilpotent Bridging algorithm was presented. This method involved the use of a small collection of the non-erased frame coefficients, known as the bridge set, to reconstruct the erased frame coefficients. Using the Baire Category Theorem and tools from Matrix Theory, we were able to show that under certain mild conditions, any bridge set of size L will work to implement the Nilpotent Bridging algorithm for an open and dense collection of frames in the set of all frames in finite dimensions. We also showed this for an open and dense collection of unions of two bases (resp. orthonormal bases) in the set of all unions of two bases (resp. orthonormal bases) in finite dimensions. An implementation of Nilpotent Bridging in Matlab was provided, along with experiments to investigate the stability of Nilpotent Bridging. It was discovered that larger bridge sets can be considered to mitigate the effects of channel noise. Further work must be done to provide a quantitative analysis of this phenomenon. Additionally, in Section 2.3.4, several open questions pertaining to Shannon-Whittaker Sampling Theory were posed.

In Section 3, we discussed the Reduced Direct Inversion algorithm. This method provided a simple shortcut for inverting the partial reconstruction operator, R_Λ. A Matlab implementation for this algorithm was also provided for this method. Using the Restricted Isometry Property, we were able to provide some error bounds for the amplification of channel noise with this reconstruction technique, though there is still some room for improvement in this area. The chapter ends with a discussion on the connections between the stability of Reduced Direct Inversion for Shannon-Whittaker Sampling Theory, and Scattered Data Approximation. Further study on good Riesz bounds for finite sequences of exponentials could provide error bounds for the algorithm for Shannon-Whittaker Sampling Theory.

In Section 4, a method of reconstruction using Erasure Recovery Matrices was discussed. For this algorithm, another Matlab implementation was provided. A construction of Erasure Recovery Matrices and frames for which this method works nicely was presented. An explanation of the channel noise mitigating effects of these frames using the Restricted Isometry Property was provided. Again, there is possibly some room for improvement here. Finally, numerical experiments were presented to further display the noise mitigating effects of the frames from the construction.

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