ESSAYS ON TESTING FOR SMOOTH STRUCTURAL CHANGES IN TIME SERIES

A Dissertation

by

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ABSTRACT

This dissertation contains two essays which propose tests for smooth structural changes in dependence and volatility, respectively. In the first essay, we propose a generalized likelihood ratio test for smooth structural changes in copula parameters. Dependence between different financial assets plays a crucial role in many financial applications. The dependence structure is likely to change over time and the copula parameter also changes accordingly. Modeling the time varying nature of the copula parameter has drawn increased attention in the last decade because it has become increasingly recognized that dependence of financial assets is time-varying. In this essay, we consider the problem of testing for the time-varying copula parameter by the generalized likelihood ratio test based on the local maximum likelihood estimator. We derive the asymptotic null distribution of the proposed test. The finite sample performance of the test is illustrated by simulations and an empirical application is provided.

In the second essay, we propose a generalized Hausman test for smooth structural changes in volatility. Since volatility is central to the financial theory and its empirical applications, there is a growing interest to analyze variance stability in financial markets , and the stylized facts of financial returns like IGARCH effects or variance persistence can be well explained by structural changes in the unconditional variance. The proposed test can be viewed as a generalized Hausman's (1978) test by comparing the local linear smoothing estimator, which is an inefficient but consistent estimator under H_A, of volatility with the constrained estimator which is an efficient estimator under H_0. We show that the new test is more powerful than the CUSUM test which has been mostly used to test for structural changes in volatility.

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1. INTRODUCTION

Detecting structural changes in time series is a long standing question in econometrics. While much attention has been focused on analyzing mean shifts in variables as well as their specific sources, the recent literature in financial economics also concerns with instability on higher moments. However, most existing tests are designed for abrupt breaks in time series. Policy-regime shifts and many other factors may generate parameter instability in the underlying generating process, often leading to structural changes that usually exhibit evolutionary changes in the long term. In this dissertation, I propose two tests for smooth structural changes in volatility and dependence, respectively. In the first essay, I propose a generalized likelihood ratio test for smooth structural changes in dependence between different financial assets. The dependence structure of financial instruments or economic indexes plays a crucial role in financial applications. It is important in financial risk management when modeling the multivariate asset prices. Copula models have been widely used in handling various types of dependence between risk factors, markets, and other important financial variables. A copula is a function which connects marginal distributions of random variables to construct a multivariate distribution function. Most of the time when copulas are applied to financial time series data they are treated to be constant over time. However, the dependence structure is likely to change over time and the copula parameter also changes accordingly. Hence, based on Hafner and Reznikova (2010), I specify a semiparametric copula model where the marginals are specified as a parametric process and the copula parameter is an unknown function of time instead of assuming that the time-varying copula parameter follows any specific function. I extend Fan et al. (2001) to test for smooth structural change in the copula parameter. The test procedure employs the local constant estimator when calculating the local likelihood under the alternative hypothesis. The asymptotic distribution of the proposed test is chi-square distribution and Wilks phenomenon holds. In the second essay, I propose a generalized Hausman test for smooth structural changes in volatility. McConnell and Perez-Quirs (2000), Stock and Watson (2003), and Sensier and van Dijk (2004) among others, find strong evidence suggesting a sharp decline in the volatility of macroeconomic variables, which has important policy implications. Also, since volatility is central to the financial theory and its empirical applications, there is a growing interest to analyze variance stability in financial markets (see, among others, De Santis and Imrohoroglu (1997)), and the stylized facts of financial returns like IGARCH effects or variance persistence can be well explained by structural changes in the unconditional variance. (see Diebold (1986), Lamoureaux and Lastrapes (1990), Granger and Hyung (2004), Mikosch and Stric (2004), and Hillebrand (2005)). I propose a new consistent test for smooth structural changes as well as abrupt breaks in volatility. I estimate the volatility parameters by the local linear estimating method and compare them with the constrained estimators. The proposed test can be viewed as a generalized Hausman's (1978) test from the parametric framework to the nonparametric framework. I show that the generalized Hausman test is asymptotically more powerful than the CUSUM test.

2. TESTING FOR SMOOTH STRUCTURAL CHANGES IN COPULA PARAMETERS

Dependence between different financial assets plays a crucial role in many financial applications, e.g., pricing Collateralized Debt Obligations (CDOs) (Li (2000)), calculating the Value-at-Risk (VaR) of a portfolio(Embrechts et al., (2003); Giacomini et al., (2009)), the pricing of options with multiple underlying assets (van den Goorbergh et al. (2005)), or portfolio construction (Patton (2004)). Copula models have been widely used in handling various types of dependence between risk factors, markets, and other important financial variables. A copula is a function which connects marginal distributions of random variables to construct a multivariate distribution function. In other words, a multivariate distribution function can be decomposed into the marginal distributions that describe the individual behavior of each series and the copula that fully captures the dependence between the variables. That is, the copula provides a relatively flexible way of modeling various types of dependence structures for multivariate random variables.

Most of the time when copulas are applied to financial time series data they are treated to be constant over time. However, the dependence structure is likely to change over time and the copula parameter also changes accordingly. Modeling the time varying nature of the copula parameter has drawn increased attention in the last decade because it has become increasingly recognized that dependence of financial assets is time-varying. Erb et al. (1994), Longin and Solnik (1995), or Engle (2002) have shown that typically the dependence between financial time series varies over time. To cope with the copula parameter change, there are some previous studies on testing for structural breaks in the copula parameter. Dias and Embrechts

(2004) proposed a likelihood ratio test for the change of the copula parameter in a parametric model. Na et al. (2011) proposed a CUSUM test for the change of the copula parameter in copula-based semiparametric ARMA-GARCH models. Besides, Na et al. (2013) generalized Na et al. (2011)'s test to semiparametric copula-based multivariate dynamic models proposed by Chen and Fan (2006a). However, none of above tests are designed for time-varying nature of the dependence structure.

Patton (2006) was perhaps among the first to model the evolution of the copula parameter parametrically. Patton (2006) assumed that current dependence is relying on previous dependence and the historical average differences of the cumulative probabilities of the two series. Some other notable parametric models of time-varying correlations in multivariate volatility models are the dynamic conditional correlation generalized autoregressive conditional heteroscedasticity (DCC GARCH) model, simultaneously proposed by Engle (2002) and Tse and Tsui (2002), a stochastic volatility model with stochastic correlations by Yu and Meyer (2006) and the regime switching model for dynamic correlations by Pelletier (2006). Hafner et al. (2006) proposed a semiparametric model for correlation dynamics. Nonetheless, it is subjective to assume the nature of the evolution function of the time-varying copula parameter because of the unknown complexity of the dependence structure of financial assets. Hence, based on Hafner and Reznikova (2010), we specify a semi-parametric copula model where the marginals are specified as a parametric process and the copula parameter is an unknown function of time instead of assuming that the time-varying copula parameter follows any specific function.

We focus on the hypotheses of the form H_0 :"the copula parameter is constant over time" versus H_A : "the copula parameter is not constant over time." However, such hypotheses cannot be tested by using the canonical likelihood ratio test (LRT) because the nonparametric estimation of the copula parameter under the alternative hypothesis. Fan et al. (2001) proposed the generalized likelihood ratio test (GLRT) for testing a parametric null hypothesis versus a nonparametric alternative hypothesis. In general, maximum likelihood estimators under nonparametric regression models are hard to obtain and may not even exist. To solve these problems, Fan et al. (2001) suggested replacing the maximum likelihood estimators by any reasonable nonparametric estimator under the alternative nonparametric model. Fan et al. (2001) showed that the asymptotic null distribution of the GLRT is chi-square distribution with the degrees of freedom independent of the nuisance parameters. This result is referred to Wilks phenomenon.

We extend Fan et al. (2001)'s GLRT to test for smooth structural change in the copula parameter. The test procedure employs the local constant estimator when calculating the local likelihood under the alternative hypothesis. We show that the asymptotic distribution of the proposed test is chi-square distribution and Wilks phenomenon holds.

The organization of this paper is as follows. In Section 2, we introduce the model and estimation technique. In Section 3, we present the description of the test and the derivation of its asymptotic null distribution. The finite sample performance of the test is illustrated by simulations and an empirical application is provided in sections 4 and 5 respectively.

2.1 The Model

Hafner and Reznikova (2010) proposed a semiparametric copula model where marginals are specified as parametric processes and the copula parameter is specified as a function of time. We follow Hafner and Reznikova (2010) to specify the time-varying copula model where the copula parameter changes over time in a nonparametric way. Consider a bivariate stochastic process $\{Y_t\}$, $t = 1, 2 \ldots T$, $Y_t = (Y_{1t}, Y_{2t})'$, whose conditional means and variances are modeled as:

$$E\left[Y_{it}|\mathcal{F}_{t-1}\right] \equiv \mu_i\left(\phi_i\right), \tag{2.1}$$

$$V\left[Y_{it}|\mathcal{F}_{t-1}\right] \equiv \sigma_i^2\left(\phi_i\right) \tag{2.2}$$

where $\mathcal{F}_t$ denote the information set at time t,the sigma-field generated by $(Y_t, Y_{t-1}, \ldots)$ and μ_{it} and σ_{it}^2, $i = 1, 2$, are the finite-dimensional unknown parameter vector which are $\mathcal{F}_{t-1}$ measurable. This assumption allows for a wide variety of models, for example, ARMA-GARCH type models.

Given models for conditional means and variances, the standardized residuals are defined as

$$\varepsilon_{it} \equiv \frac{Y_{it} - \mu_{it}\left(\phi_i\right)}{\sigma_{it}\left(\phi_i\right)}. \tag{2.3}$$

Let $\{\varepsilon_{it}\}$ be an independent and identically distributed random vector with $E\left(\varepsilon_{it}\right) = 0$ and $E\left(\varepsilon_{it}^2\right) = 1$ and $E\left(|\varepsilon_{it}|^v\right) < \infty$ for some $v > 2$ and independent of $\mathcal{F}_{t-1}$ for $i = 1, 2$. Moreover, ε_{it} has a joint distribution function F^o and marginal distributions F_i^o with continuous densities f_i^o for $i = 1, 2$. Denoting $C^o\left(u, v; \theta\right) : [0, 1]^2 \to [0, 1]$ is the true copula function with the unknown copula parameter θ_t. Then by Sklar's theorem we can express $F^o\left(\varepsilon_t\right) = C^o\left(F_1^o\left(\varepsilon_{1t}\right), F_2^o\left(\varepsilon_{2t}\right); \theta_t\right)$ for $t = 1, \ldots, T$. Thus the copula function C^o maps the univariate marginal distributions F_i^o to the joint distribution F^o. The copula parameter θ_t satisfies $\theta_t = \theta\left(t/T\right)$, where $\theta\left(\cdot\right)$ is a nonstochastic càdlag̀ (right continuous with left limits) function on $(0, 1]$ with a finite number of points of discontinuity.

The log likelihood function for the model with a candidate copula function $C\left(u, v; \theta\right)$

is given by

$$
\begin{aligned}
L(\theta, \phi) &= \sum_{t=1}^{T} \ell_t(\theta, \phi) && (2.4) \\
&= \sum_{t=1}^{T} \log c(u_{t,} v_t; \theta_t) + \sum_{t=1}^{T} \sum_{i=1}^{2} \log f_i(\varepsilon_{it}; \phi_i) && (2.5) \\
&= L_C(\theta_t, \phi) + L_M(\phi) && (2.6)
\end{aligned}
$$

where $c(u, v) = \frac{\partial C(u,v)}{\partial u \partial v}$ is the copula density function associated with the copula function $C(u, v)$, and $L_C(\cdot)$, and $L_M(\cdot)$ are the copula likelihood and the likelihood of marginals, respectively. We need to estimate two sets of parameters (ϕ_1, ϕ_2), and θ_t.

We use a two-step method for parameter estimations of (ϕ_1, ϕ_2), and θ_t. In step 1, we consider the estimation of ϕ. The estimator $\tilde{\phi}$ maximizes L_M

$$
\tilde{\phi} = \left(\tilde{\phi}_1, \tilde{\phi}_2\right) = \arg\max_{\phi} L_M(\phi).
$$

Given the estimator $\tilde{\phi}$, we can compute $\tilde{\varepsilon}_{it} = \left(Y_{it} - \mu_{it}\left(\tilde{\phi}_i\right)\right) / \sigma_{it}\left(\tilde{\phi}_i\right)$. Then, we obtain the estimated marginal distribution $\tilde{u}_t = F_1(\tilde{\varepsilon}_{1t})$ and $\tilde{v}_t = F_2(\tilde{\varepsilon}_{2t})$. At the step 2, we estimate the time-varying copula parameter by applying the local likelihood estimation method. Define the local likelihood function as

$$
L(\theta_t; h, \tau) = \sum_{t=1}^{T} \log c(\tilde{u}_{t,} \tilde{v}_t; \theta_t) K_h(t/T - \tau), \tag{2.7}
$$

where $\tau \in [0, 1]$, K is a kernel function, a bandwidth $h > 0$ and $K_h(\cdot) = (1/h) K(\cdot/h)$.

Thus, the local likelihood estimator of the function $\theta(\tau)$ maximizes $L(\theta_t; h, \tau)$

$$
\hat{\theta}(\tau) = \arg\max_{\theta} L(\theta; h, \tau) \tag{2.8}
$$

2.2 Generalized Likelihood Ratio Test

It has been stylized facts that the dependence between financial asset returns are time-varying, a finding that has been documented by, among many others, Longin and Solnik (1995), Engle (2002), Patton (2006) or Rodriguez (2007). However, none of these have tested the time-varying nature of the dependence structure. This paper proposes the generalized likelihood ratio test for the time-varying copula parameter.

In what follows, for simplicity we use the notation $\ell\left(u_t, v_t; \theta\right) = \log c\left(\tilde{u}_{t,}\tilde{v}_t; \theta\right)$, $\ell'\left(u_t, v_t; \theta\right) = \partial\ell\left(u_t, v_t; \theta\right)/\partial\theta$, and $\ell''\left(u_t, v_t; \theta\right) = \partial^2\ell\left(u_t, v_t; \theta\right)/\partial\theta^2$.

The hypotheses of interest are $H_0 : \theta_t = \theta_0$ and $H_A : \theta_t$ is not constant over time. We consider the GLRT with the form of

$$\hat{\lambda} = L_T(H_A, \hat{\theta}) - L_T\left(H_0, \tilde{\theta}\right), \tag{2.9}$$

where

$$\begin{aligned} L_T\left(H_0, \tilde{\theta}\right) &= \sum_{t=1}^{T} \ell\left(u_t, v_t; \tilde{\theta}\right), \\ L_T(H_A, \hat{\theta}) &= \sum_{t=1}^{T} \ell\left(u_t, v_t; \hat{\theta}\left(\tau\right)\right), \end{aligned}$$

and $\tilde{\theta}$ is the maximum likelihood estimator under H_0. We reject the null hypothesis if $\hat{\lambda}$ is large. To derive the asymptotic distribution of $\hat{\lambda}$ under the null hypothesis, we need the following Assumptions.

Assumption 1 *The function K is symmetric and bounded.*

Assumption 2 *The function $\ell_t\left(\theta, \phi\right)$ is two times differentiable w.r.t. θ and ϕ for all u_t and v_t.*

Assumption 3 $E\left|\ell'\left(u_t, v_t;\theta\right)|t/T\right|^4 < \infty.$

Assumption 4 $E\left\{\ell''\left(u_t, v_t;\theta\right)|t/T\right\}$ *is Lipschitz continuous.*

Assumption 5 *The function* $\ell''\left(u_1, u_2;\theta\right) < 0$ *for all* $\theta \in R$*, and* $u_1,\ u_2 \in (0,1)$.

Define

$$\begin{aligned}\mu_T &= \frac{1}{h}\left(K(0) - \frac{1}{2}\int K^2(\tau)\,d\tau\right),\\ V_T &= \frac{2}{h}\int\left(K(\tau) - \frac{1}{2}K * K(\tau)\right)^2 d\tau.\end{aligned}$$

The following result shows that the asymptotic distribution of GLRT statistic is a normal distribution where mean and variance are related to μ_T and V_T, respectively.

Theorem 1 *Assume that assumptions 1-5 hold, then, as* $h \to 0$ *and* $Th^{3/2} \to \infty$,

$$V_T^{-1/2}\left(\hat{\lambda} - \mu_T\right) \xrightarrow{d} N(0,1), \tag{2.10}$$

and

$$r_K\hat{\lambda} \xrightarrow{a} \mathcal{X}^2_{r_K\mu_T} \tag{2.11}$$

where $r_K = 2\mu_T/V_T$.

One can conclude from Theorem 1 that the GLRT is fairly similar to the classical likelihood ratio test. The constant r_K is closed to 2 for commonly used kernels. When the alternative hypothesis is nonparametric, the degree of freedom of the asymptotic null distribution of the GLRT tends to infinity when $h \to 0$.

2.3 Simulation

In this section, we evaluate the finite-sample performance of the test through a simulation study. We consider Clayton, Frank, and Gumbel copula models. The data generating process (DGP) is as following:

$$\begin{aligned} Y_{it} &= \sigma_{it}\varepsilon_{it},\ \ \varepsilon_{it} \sim N(0,1), \\ \sigma_{it}^2 &= \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \\ (\varepsilon_{1t}, \varepsilon_{2t}) &\sim C(F_1(\varepsilon_{1t}), F_2(\varepsilon_{2t}); \theta_t), \end{aligned}$$

where $\theta_t = \theta(t/T)$ with $\theta(r)$ being defined on $r \in (0,1]$.The GARCH parameters are set as $\alpha_0 = 0.0001$, $\alpha_1 = 0.001$, and $\beta = 0.8$.

First, we examine the size of $\hat{\lambda}$ under H_0. For each copula models, we consider the case that each copula model is misspecified with other copula models. We use the two-side Epanechnoikov kernel function in our estimation. The Epanechnikov kernel has a smooth estimated density function and is generally regarded as a good kernel function, see Fan and Gijbels (1996). The sample sizes are set as $T \in \{100, 300, 500\}$, and the number of replications are 1000. We use three different fixed bandwidths, $h_i = 1.3^{i-1} * T^{-1/5}$. The empirical sizes are calculated at the nominal level 0.05.

Table 2.1 reports the rejection rates of the GLRT and the CUSUM test (Q) proposed by Na et al. (2013). The results show that sizes of $\hat{\lambda}$ are generally larger than those of Q. Thus the $\hat{\lambda}$ rejects more in most cases. There is no severe size distortions in most cases for both $\hat{\lambda}$ and Q. It can be seen that $\hat{\lambda}$ and Q exhibit some size distortion for small sample sizes. The empirical sizes go to the nominal 5% as the sample size increases. It can be seen that the empirical sizes of both $\hat{\lambda}$ and Q get close to the nominal level even when the true copula model is misspecified. We

		Clayton			Frank			Gumbel		
		100	300	500	100	300	500	100	300	500
h_1										
Clayton		0.099	0.069	0.053	0.088	0.078	0.062	0.077	0.071	0.061
Frank	$\hat{\lambda}$	0.080	0.064	0.056	0.029	0.069	0.057	0.036	0.039	0.042
Gumbel		0.155	0.082	0.053	0.075	0.062	0.054	0.032	0.048	0.049
Clayton		0.087	0.051	0.050	0.078	0.064	0.053	0.061	0.056	0.049
Frank	Q	0.068	0.059	0.049	0.039	0.042	0.051	0.038	0.043	0.048
Gumbel		0.101	0.061	0.051	0.068	0.056	0.050	0.031	0.042	0.052
h_2										
Clayton		0.096	0.069	0.052	0.090	0.075	0.063	0.074	0.069	0.058
Frank	$\hat{\lambda}$	0.087	0.068	0.054	0.034	0.062	0.058	0.042	0.036	0.048
Gumbel		0.134	0.078	0.053	0.079	0.064	0.055	0.035	0.048	0.049
Clayton		0.084	0.061	0.052	0.078	0.065	0.056	0.064	0.059	0.055
Frank	Q	0.077	0.062	0.051	0.068	0.056	0.049	0.043	0.046	0.052
Gumbel		0.102	0.066	0.050	0.066	0.053	0.051	0.038	0.043	0.048
h_3										
Clayton		0.100	0.069	0.053	0.078	0.064	0.059	0.067	0.073	0.057
Frank	$\hat{\lambda}$	0.090	0.058	0.060	0.034	0.067	0.062	0.034	0.037	0.048
Gumbel		0.096	0.078	0.051	0.068	0.062	0.057	0.035	0.048	0.049
Clayton		0.089	0.061	0.053	0.061	0.056	0.047	0.066	0.059	0.048
Frank	Q	0.081	0.053	0.049	0.038	0.045	0.053	0.038	0.042	0.052
Gumbel		0.012	0.066	0.051	0.062	0.057	0.053	0.040	0.045	0.051

Table 2.1: Empirical Sizes

also can see that sizes are not sensitive to the bandwidth.

Next, we consider the powers of the proposed test $\hat{\lambda}$ and the CUSUM test Q. To investigate the power, we consider the following three alternatives:

DGP 1 (single break):

$\theta(r) = a_0 + a_1 I(r > \tau), \tau \in \{0.4, 0.6\};$

DGP2 (multiple breaks):

$\theta(r) = a_0 + a_1 I(\tau \leq r \leq \tau + 0.4), \tau \in \{0.3, 0.5\};$

DGP3 (smooth change):

$\theta(r) = 5 + 4 * \sin(20r/3);$

In all DGPs, we set $a_0 = 2$, and $a_1 \in \{0, 0.2, 0.6, 1\}$. When $a_1 = 0$, the copula parameter is constant and thus the copula is static. When $a_1 \neq 0$, the copula parameter θ jumps to the level $a_0 + a_1$ from a_0 at time $[T\tau]$ and stays (GDP1), jumps to $a_0 + a_1$ from a_0 at time $[T\tau]$, and jumps back to a_0 after $[(0.4 + \tau)T]$ periods (GDP2), or changes every periods (GDP3).

The powers of the case of DGP 1 (single break) for Clayton, Frank, and Gumbel copula models are reported in Tables 2.2-2.4, respectively. It can be seen that $\hat{\lambda}$ is more powerful than Q in all cases. $\hat{\lambda}$ has powers against DGP 1 in most cases. As anticipated, the power increases either as the sample size increases or as the level of the copula parameter changes significantly. Meanwhile, even the true model is misspecified, $\hat{\lambda}$ still has good power in every cases. The powers are not sensitive either to the bandwidth or the location of the break point. Tables 2.5-2.7 show the powers of the case of DGP 2 (multiple breaks) for Clayton, Frank and Gumbel copula models, respectively. The powers of the case of DGP 3 (smooth change) are reported in Table 2.8. We have similar results to that of the case of DGP 1. The powers increase as the sample size increases and also as the copula parameter has significant changes. Also, the powers are not sensitive to the bandwidth and the location of break points.

Next, we evaluate the performance of the GLRT when the parameters other than the copula parameter experience changes. We consider the following setup:

Case 1. (only β changes in the first series):

$\beta : 0.3 \rightarrow 0.8$ at $\tau = 0.5$.

Case 2: (β changes in both series at the same point):

$(\beta_1, \beta_2) : (0.3, 0.3) \rightarrow (0.8, 0.8)$ at $\tau = 0.5$.

Case 3: (β changes in both series at the different points):

$\beta_1 : 0.3 \rightarrow 0.8$ at $\tau = 0.4$,

			Clayton							
			100	300	500	100	300	500	100	300	500
$\tau = 0.4$		a_1	h_1			h_2			h_3		
Clayton	$\hat{\lambda}$	0.2	0.194	0.228	0.372	0.210	0.251	0.380	0.198	0.243	0.384
		0.6	0.324	0.681	0.949	0.335	0.710	0.950	0.330	0.672	0.956
		1	0.551	0.949	0.999	0.581	0.952	1.000	0.589	0.959	0.999
	Q	0.2	0.185	0.219	0.354	0.223	0.267	0.397	0.187	0.241	0.365
		0.6	0.345	0.658	0.912	0.338	0.654	0.812	0.310	0.654	0.875
		1	0.555	0.854	0.923	0.521	0.821	0.859	0.559	0.812	0.890
Frank	$\hat{\lambda}$	0.2	0.150	0.225	0.280	0.167	0.243	0.291	0.143	0.240	0.296
		0.6	0.317	0.495	0.658	0.326	0.510	0.661	0.301	0.487	0.710
		1	0.452	0.715	0.874	0.431	0.732	0.885	0.467	0.729	0.899
	Q	0.2	0.210	0.243	0.268	0.157	0.221	0.250	0.110	0.227	0.275
		0.6	0.234	0.387	0.541	0.310	0.419	0.598	0.298	0.389	0.547
		1	0.341	0.543	0.679	0.375	0.509	0.798	0.444	0.650	0.778
Gumbel	$\hat{\lambda}$	0.2	0.440	0.537	0.633	0.451	0.524	0.648	0.447	0.573	0.621
		0.6	0.637	0.946	0.985	0.638	0.952	0.989	0.630	0.962	0.911
		1	0.769	0.996	1.000	0.780	0.997	1.000	0.788	0.999	1.000
	Q	0.2	0.321	0.489	0.532	0.367	0.493	0.533	0.360	0.465	0.533
		0.6	0.498	0.754	0.812	0.554	0.765	0.816	0.520	0.602	0.810
		1	0.599	0.819	0.888	0.650	0.812	0.895	0.623	0.772	0.867
$\tau = 0.6$											
Clayton	$\hat{\lambda}$	0.2	0.128	0.157	0.273	0.131	0.167	0.285	0.124	0.154	0.265
		0.6	0.226	0.475	0.875	0.234	0.488	0.849	0.216	0.510	0.889
		1	0.321	0.825	0.995	0.316	0.810	0.999	0.350	0.843	0.996
	Q	0.2	0.107	0.129	0.256	0.101	0.143	0.264	0.121	0.143	0.223
		0.6	0.198	0.375	0.698	0.210	0.387	0.778	0.210	0.466	0.765
		1	0.322	0.679	0.894	0.289	0.698	0.910	0.310	0.738	0.832
Frank	$\hat{\lambda}$	0.2	0.123	0.170	0.205	0.111	0.180	0.210	0.130	0.176	0.201
		0.6	0.243	0.453	0.579	0.239	0.448	0.569	0.256	0.495	0.587
		1	0.390	0.665	0.870	0.410	0.701	0.850	0.385	0.655	0.899
	Q	0.2	0.118	0.156	0.203	0.110	0.167	0.208	0.125	0.174	0.198
		0.6	0.232	0.398	0.532	0.215	0.389	0.476	0.224	0.401	0.538
		1	0.375	0.576	0.788	0.382	0.583	0.782	0.305	0.547	0.779
Gumbel	$\hat{\lambda}$	0.2	0.385	0.414	0.486	0.391	0.425	0.501	0.379	0.412	0.491
		0.6	0.448	0.876	0.965	0.510	0.862	0.975	0.491	0.861	0.968
		1	0.566	0.985	0.998	0.579	0.988	0.999	0.601	0.989	0.999
	Q	0.2	0.367	0.410	0.461	0.327	0.512	0.639	0.328	0.399	0.470
		0.6	0.381	0.650	0.736	0.476	0.654	0.744	0.388	0.641	0.886
		1	0.436	0.584	0.849	0.528	0.799	0.924	0.539	0.793	0.933

Table 2.2: Empirical Powers-DGP1 (Single Break). True Copula Model: Clayton

			Frank							
			100	300	500	100	300	500	100	300	500
$\tau = 0.4$		a_1	h_1			h_2			h_3		
		0.2	0.212	0.354	0.297	0.221	0.346	0.312	0.231	0.321	0.301
	$\hat{\lambda}$	0.6	0.334	0.507	0.858	0.358	0.497	0.889	0.341	0.512	0.891
Clayton		1	0.478	0.823	0.994	0.488	0.851	0.996	0.459	0.838	0.993
		0.2	0.194	0.318	0.450	0.199	0.310	0.395	0.218	0.275	0.315
	Q	0.6	0.287	0.442	0.741	0.289	0.378	0.774	0.285	0.479	0.753
		1	0.401	0.693	0.849	0.403	0.683	0.899	0.430	0.799	0.853
		0.2	0.142	0.369	0.306	0.138	0.350	0.360	0.146	0.328	0.357
	$\hat{\lambda}$	0.6	0.375	0.618	0.817	0.384	0.608	0.831	0.366	0.687	0.829
Frank		1	0.569	0.902	0.991	0.601	0.932	0.997	0.578	0.917	0.999
		0.2	0.125	0.258	0.342	0.110	0.298	0.344	0.132	0.317	0.348
	Q	0.6	0.287	0.543	0.737	0.310	0.501	0.776	0.328	0.558	0.731
		1	0.483	0.741	0.879	0.484	0.804	0.901	0.462	0.832	0.921
		0.2	0.359	0.527	0.736	0.346	0.548	0.753	0.330	0.557	0.750
	$\hat{\lambda}$	0.6	0.672	0.947	0.972	0.689	0.969	0.981	0.665	0.932	0.958
Gumbel		1	0.868	0.996	0.999	0.889	0.989	0.999	0.900	0.999	0.999
		0.2	0.228	0.410	0.742	0.301	0.489	0.632	0.288	0.439	0.642
	Q	0.6	0.445	0.560	0.859	0.333	0.532	0.699	0.587	0.639	0.843
		1	0.587	0.736	0.898	0.433	0.754	0.840	0.772	0.844	0.932
$\tau = 0.6$											
		0.2	0.147	0.348	0.235	0.149	0.339	0.340	0.139	0.350	0.348
	$\hat{\lambda}$	0.6	0.231	0.351	0.768	0.248	0.361	0.801	0.218	0.346	0.779
Clayton		1	0.300	0.655	0.984	0.316	0.684	0.975	0.318	0.674	0.991
		0.2	0.137	0.310	0.352	0.129	0.270	0.317	0.123	0.296	0.329
	Q	0.6	0.210	0.328	0.655	0.221	0.358	0.764	0.197	0.329	0.663
		1	0.296	0.582	0.853	0.298	0.611	0.875	0.310	0.578	0.894
		0.2	0.120	0.285	0.259	0.134	0.325	0.349	0.129	0.298	0.299
	$\hat{\lambda}$	0.6	0.289	0.522	0.781	0.304	0.573	0.721	0.276	0.561	0.788
Frank		16	0.468	0.832	0.984	0.429	0.865	0.991	0.432	0.879	0.983
		0.2	0.111	0.278	0.290	0.100	0.246	0.326	0.129	0.248	0.287
	Q	0.6	0.247	0.486	0.698	0.255	0.463	0.701	0.231	0.501	0.699
		1	0.432	0.744	0.914	0.338	0.775	0.863	0.411	0.749	0.894
		0.2	0.256	0.377	0.625	0.267	0.381	0.619	0.221	0.386	0.677
	$\hat{\lambda}$	0.6	0.423	0.883	0.944	0.438	0.801	0.966	0.418	0.887	0.981
Gumbel		1	0.627	0.974	0.993	0.610	0.965	0.999	0.639	0.984	0.999
		0.2	0.210	0.337	0.573	0.243	0.354	0.586	0.204	0.354	0.566
	Q	0.6	0.410	0.697	0.883	0.396	0.655	0.852	0.374	0.620	0.845
		1	0.614	0.885	0.932	0.584	0.883	0.912	0.444	0.785	0.918

Table 2.3: Empirical Powers-DGP1(Single Break). True Copula Model: Frank

		Gumbel								
		100	300	500	100	300	500	100	300	500
$\tau = 0.4$	a_1	h_1			h_2			h_3		
	0.2	0.165	0.194	0.247	0.176	0.201	0.256	0.154	0.189	0.276
$\hat{\lambda}$	0.6	0.283	0.373	0.504	0.288	0.343	0.543	0.298	0.385	0.518
Clayton	1	0.369	0.607	0.746	0.387	0.601	0.732	0.358	0.615	0.766
	0.2	0.132	0.168	0.210	0.137	0.173	0.230	0.143	0.159	0.211
Q	0.6	0.245	0.310	0.439	0.237	0.284	0.448	0.230	0.320	0.471
	1	0.321	0.523	0.622	0.327	0.459	0.599	0.274	0.432	0.576
	0.2	0.161	0.267	0.316	0.176	0.256	0.328	0.159	0.288	0.342
$\hat{\lambda}$	0.6	0.341	0.519	0.687	0.349	0.543	0.634	0.321	0.528	0.680
Frank	1	0.497	0.784	0.938	0.488	0.789	0.956	0.501	0.745	0.944
	0.2	0.115	0.217	0.269	0.147	0.243	0.318	0.137	0.247	0.316
Q	0.6	0.259	0.399	0.575	0.279	0.412	0.572	0.247	0.412	0.537
	1	0.337	0.485	0.769	0.416	0.583	0.798	0.374	0.638	0.845
	0.2	0.214	0.482	0.775	0.219	0.480	0.754	0.221	0.491	0.763
$\hat{\lambda}$	0.6	0.547	0.979	1.000	0.561	0.989	1.000	0.565	0.987	1.000
Gumbel	1	0.804	0.999	1.000	0.897	0.999	1.000	0.821	0.998	1.000
	0.2	0.185	0.379	0.557	0.148	0.285	0.464	0.156	0.297	0.442
Q	0.6	0.341	0.776	0.845	0.353	0.746	0.862	0.342	0.763	0.844
	1	0.547	0.798	0.885	0.548	0.774	0.910	0.539	0.729	0.879
$\tau = 0.6$										
	0.2	0.126	0.133	0.180	0.134	0.136	0.196	0.132	0.138	0.199
$\hat{\lambda}$	0.6	0.203	0.285	0.410	0.212	0.298	0.450	0.201	0.312	0.389
Clayton	1	0.272	0.476	0.669	0.287	0.477	0.701	0.267	0.489	0.682
	0.2	0.110	0.121	0.146	0.127	0.133	0.165	0.122	0.134	0.178
Q	0.6	0.201	0.276	0.397	0.198	0.267	0.338	0.194	0.279	0.327
	1	0.218	0.337	0.575	0.254	0.376	0.581	0.231	0.319	0.574
	0.2	0.114	0.169	0.246	0.117	0.180	0.287	0.113	0.176	0.239
$\hat{\lambda}$	0.6	0.254	0.484	0.672	0.243	0.498	0.690	0.265	0.478	0.689
Frank	1	0.373	0.746	0.928	0.332	0.756	0.936	0.385	0.765	0.902
	0.2	0.110	0.148	0.219	0.114	0.169	0.238	0.111	0.154	0.213
Q	0.6	0.221	0.418	0.535	0.221	0.355	0.572	0.214	0.348	0.561
	1	0.318	0.623	0.784	0.286	0.565	0.784	0.311	0.613	0.734
	0.2	0.151	0.286	0.615	0.165	0.278	0.638	0.149	0.276	0.602
$\hat{\lambda}$	0.6	0.308	0.895	0.996	0.331	0.866	0.999	0.309	0.901	0.999
Gumbel	1	0.798	0.994	0.999	0.887	0.990	0.999	0.832	0.998	0.999
	0.2	0.121	0.256	0.516	0.145	0.278	0.443	0.132	0.268	0.412
Q	0.6	0.324	0.756	0.855	0.368	0.735	0.892	0.336	0.745	0.825
	1	0.546	0.712	0.889	0.551	0.725	0.913	0.531	0.711	0.834

Table 2.4: Empirical Powers-DGP1(Single Break). True Copula Model: Gumbel

			Clayton							
			100	300	500	100	300	500	100	300	500
$\tau = 0.4$		a_1	h_1			h_2			h_3		
		0.2	0.214	0.200	0.579	0.221	0.220	0.588	0.210	0.219	0.576
	$\hat{\lambda}$	0.6	0.344	0.601	0.950	0.335	0.614	0.956	0.352	0.621	0.952
Clayton		1	0.450	0.879	0.996	0.439	0.889	0.997	0.451	0.867	0.999
		0.2	0.208	0.210	0.358	0.201	0.214	0.398	0.203	0.215	0.367
	Q	0.6	0.315	0.561	0.813	0.316	0.538	0.798	0.325	0.534	0.834
		1	0.331	0.739	0.845	0.357	0.752	0.867	0.329	0.712	0.884
		0.2	0.198	0.296	0.359	0.210	0.312	0.376	0.186	0.289	0.355
	$\hat{\lambda}$	0.6	0.328	0.609	0.785	0.322	0.609	0.749	0.331	0.687	0.765
Frank		1	0.449	0.801	0.935	0.451	0.812	0.921	0.441	0.789	0.954
		0.2	0.176	0.276	0.314	0.191	0.278	0.311	0.188	0.273	0.321
	Q	0.6	0.318	0.495	0.756	0.328	0.485	0.784	0.321	0.487	0.766
		1	0.367	0.673	0.778	0.385	0.635	0.791	0.376	0.624	0.776
		0.2	0.281	0.649	0.709	0.261	0.632	0.701	0.298	0.676	0.721
	$\hat{\lambda}$	0.6	0.369	0.930	0.982	0.345	0.932	0.978	0.377	0.940	0.990
Gumbel		1	0.504	0.989	0.997	0.512	0.978	0.999	0.498	0.983	0.997
		0.2	0.213	0.487	0.567	0.196	0.478	0.573	0.202	0.469	0.551
	Q	0.6	0.314	0.598	0.833	0.305	0.578	0.822	0.311	0.578	0.843
		1	0.412	0.687	0.876	0.422	0.663	0.869	0.402	0.689	0.833
$\tau = 0.6$											
		0.2	0.154	0.167	0.490	0.162	0.169	0.499	0.158	0.166	0.513
	$\hat{\lambda}$	0.6	0.271	0.588	0.945	0.287	0.564	0.956	0.277	0.598	0.943
Clayton		1	0.385	0.832	1.000	0.387	0.812	0.998	0.367	0.823	0.999
		0.2	0.132	0.154	0.367	0.128	0.147	0.349	0.122	0.156	0.385
	Q	0.6	0.243	0.512	0.768	0.251	0.549	0.788	0.241	0.561	0.779
		1	0.301	0.675	0.793	0.321	0.661	0.831	0.332	0.651	0.822
		0.2	0.139	0.217	0.310	0.147	0.219	0.340	0.129	0.209	0.319
	$\hat{\lambda}$	0.6	0.285	0.503	0.740	0.288	0.504	0.732	0.298	0.519	0.750
Frank		1	0.440	0.758	0.924	0.442	0.761	0.926	0.435	0.777	0.921
		0.2	0.123	0.208	0.307	0.121	0.210	0.313	0.119	0.203	0.318
	Q	0.6	0.251	0.385	0.611	0.261	0.375	0.604	0.265	0.367	0.606
		1	0.320	0.574	0.779	0.310	0.564	0.723	0.318	0.547	0.792
		0.2	0.311	0.580	0.645	0.332	0.560	0.651	0.320	0.589	0.666
	$\hat{\lambda}$	0.6	0.425	0.919	0.981	0.410	0.928	0.989	0.432	0.910	0.990
Gumbel		1	0.524	0.980	0.998	0.532	0.960	0.999	0.519	0.987	0.999
		0.2	0.267	0.476	0.587	0.256	0.463	0.547	0.263	0.451	0.536
	Q	0.6	0.378	0.657	0.786	0.367	0.637	0.769	0.328	0.651	0.784
		1	0.425	0.765	0.846	0.435	0.776	0.859	0.429	0.769	0.839

Table 2.5: Empirical Powers-DGP2 (Multiple Breaks). True Copula Model: Clayton

			Frank								
			100	300	500	100	300	500	100	300	500
$\tau = 0.4$		a	h_1			h_2			h_3		
		0.2	0.159	0.200	0.562	0.168	0.208	0.575	0.152	0.201	0.568
	$\hat{\lambda}$	0.6	0.287	0.468	0.932	0.286	0.478	0.942	0.265	0.438	0.938
Clayton		1	0.371	0.774	0.988	0.369	0.787	0.989	0.365	0.756	0.984
		0.2	0.132	0.178	0.367	0.129	0.165	0.358	0.138	0.175	0.386
	Q	0.6	0.211	0.312	0.776	0.201	0.306	0.753	0.218	0.326	0.786
		1	0.278	0.557	0.825	0.268	0.549	0.812	0.295	0.572	0.848
		0.2	0.157	0.291	0.492	0.167	0.301	0.487	0.148	0.287	0.502
	$\hat{\lambda}$	0.6	0.335	0.668	0.921	0.376	0.674	0.927	0.317	0.653	0.938
Frank		1	0.501	0.910	0.994	0.513	0.923	0.989	0.509	0.919	0.998
		0.2	0.132	0.263	0.351	0.131	0.257	0.328	0.142	0.271	0.361
	Q	0.6	0.184	0.231	0.384	0.181	0.226	0.374	0.194	0.225	0.364
		1	0.233	0.476	0.732	0.219	0.426	0.715	0.241	0.451	0.759
		0.2	0.287	0.628	0.800	0.298	0.616	0.812	0.289	0.633	0.801
	$\hat{\lambda}$	0.6	0.480	0.879	0.937	0.489	0.887	0.942	0.471	0.890	0.930
Gumbel		1	0.605	0.912	0.938	0.602	0.921	0.983	0.617	0.902	0.945
		0.2	0.256	0.453	0.594	0.247	0.436	0.601	0.261	0.453	0.585
	Q	0.6	0.349	0.570	0.683	0.336	0.582	0.684	0.329	0.567	0.691
		1	0.433	0.669	0.783	0.421	0.670	0.790	0.444	0.667	0.788
$\tau = 0.6$											
		0.2	0.120	0.188	0.426	0.129	0.197	0.421	0.119	0.176	0.426
	$\hat{\lambda}$	0.6	0.237	0.404	0.905	0.287	0.398	0.920	0.254	0.390	0.918
Clayton		1	0.341	0.695	0.994	0.321	0.708	0.999	0.351	0.702	0.997
		0.2	0.112	0.145	0.327	0.114	0.153	0.311	0.113	0.161	0.321
	Q	0.6	0.219	0.325	0.558	0.229	0.332	0.601	0.222	0.316	0.576
		1	0.323	0.448	0.769	0.321	0.423	0.798	0.339	0.447	0.788
		0.2	0.134	0.260	0.425	0.142	0.276	0.426	0.136	0.275	0.410
	$\hat{\lambda}$	0.6	0.315	0.601	0.900	0.331	0.598	0.910	0.308	0.609	0.921
Frank		1	0.488	0.896	0.993	0.500	0.901	0.998	0.478	0.889	0.999
		0.2	0.121	0.244	0.365	0.112	0.246	0.375	0.118	0.242	0.368
	Q	0.6	0.233	0.369	0.786	0.232	0.378	0.778	0.233	0.375	0.798
		1	0.352	0.574	0.835	0.341	0.572	0.841	0.321	0.563	0.843
		0.2	0.231	0.547	0.759	0.232	0.561	0.787	0.227	0.534	0.768
	$\hat{\lambda}$	0.6	0.475	0.853	0.948	0.487	0.843	0.952	0.456	0.888	0.950
Gumbel		1	0.596	0.906	0.928	0.610	0.903	0.921	0.601	0.912	0.930
		0.2	0.213	0.359	0.587	0.221	0.346	0.576	0.215	0.360	0.580
	Q	0.6	0.325	0.573	0.661	0.312	0.563	0.645	0.317	0.545	0.651
		1	0.433	0.694	0.856	0.421	0.685	0.861	0.431	0.699	0.873

Table 2.6: Empirical Powers-DGP2 (Multiple Breaks). True Copula Model: Frank

			Gumbel							
			100	300	500	100	300	500	100	300	500
$\tau = 0.4$		a	h_1			h_2			h_3		
		0.2	0.173	0.258	0.338	0.178	0.267	0.328	0.169	0.276	0.343
	$\hat{\lambda}$	0.6	0.235	0.498	0.667	0.243	0.510	0.673	0.228	0.502	0.658
Clayton		1	0.309	0.652	0.832	0.312	0.643	0.842	0.301	0.661	0.853
		0.2	0.144	0.215	0.301	0.132	0.211	0.311	0.137	0.221	0.310
	Q	0.6	0.212	0.355	0.532	0.211	0.348	0.531	0.221	0.351	0.518
		1	0.278	0.544	0.785	0.268	0.554	0.761	0.267	0.537	0.775
		0.2	0.130	0.270	0.387	0.134	0.263	0.348	0.148	0.269	0.378
	$\hat{\lambda}$	0.6	0.295	0.647	0.831	0.301	0.652	0.825	0.320	0.620	0.819
Frank		1	0.456	0.885	0.980	0.461	0.876	0.989	0.449	0.889	0.991
		0.2	0.121	0.223	0.318	0.117	0.214	0.305	0.111	0.204	0.315
	Q	0.6	0.216	0.453	0.685	0.215	0.447	0.673	0.221	0.428	0.673
		1	0.343	0.687	0.832	0.324	0.639	0.857	0.323	0.643	0.831
		0.2	0.166	0.488	0.845	0.178	0.478	0.835	0.187	0.501	0.857
	$\hat{\lambda}$	0.6	0.330	0.920	0.999	0.336	0.934	0.999	0.343	0.910	1.000
Gumbel		1	0.438	0.993	1.000	0.454	0.996	1.000	0.465	0.999	1.000
		0.2	0.154	0.376	0.653	0.152	0.369	0.652	0.149	0.357	0.645
	Q	0.6	0.267	0.468	0.759	0.261	0.457	0.764	0.246	0.448	0.747
		1	0.386	0.577	0.837	0.367	0.568	0.829	0.391	0.558	0.841
$\tau = 0.6$											
		0.2	0.121	0.178	0.250	0.132	0.169	0.290	0.116	0.189	0.267
	$\hat{\lambda}$	0.6	0.184	0.396	0.600	0.191	0.376	0.654	0.156	0.387	0.609
Clayton		1	0.281	0.571	0.800	0.284	0.564	0.808	0.276	0.569	0.789
		0.2	0.119	0.169	0.214	0.116	0.170	0.221	0.120	0.158	0.213
	Q	0.6	0.145	0.297	0.473	0.151	0.312	0.467	0.146	0.298	0.468
		1	0.218	0.375	0.643	0.221	0.354	0.612	0.219	0.368	0.675
		0.2	0.138	0.275	0.388	0.143	0.286	0.376	0.133	0.279	0.397
	$\hat{\lambda}$	0.6	0.283	0.638	0.834	0.289	0.645	0.826	0.287	0.633	0.831
Frank		1	0.421	0.859	0.975	0.412	0.831	0.962	0.430	0.866	0.981
		0.2	0.126	0.245	0.316	0.112	0.238	0.331	0.121	0.240	0.324
	Q	0.6	0.216	0.389	0.683	0.211	0.390	0.673	0.210	0.394	0.688
		1	0.311	0.673	0.885	0.310	0.668	0.889	0.301	0.659	0.901
		0.2	0.128	0.432	0.812	0.132	0.441	0.823	0.143	0.428	0.802
	$\hat{\lambda}$	0.6	0.305	0.908	1.000	0.304	0.902	0.999	0.298	0.917	1.000
Gumbel		1	0.411	0.992	0.999	0.423	0.999	0.999	0.402	0.997	1.000
		0.2	0.121	0.326	0.677	0.119	0.316	0.652	0.112	0.330	0.678
	Q	0.6	0.287	0.657	0.788	0.268	0.648	0.769	0.298	0.665	0.789
		1	0.321	0.765	0.921	0.311	0.733	0.911	0.331	0.738	0.901

Table 2.7: Empirical Powers-DGP2 (Multiple Breaks). True Copula Model: Gumbel

		Clayton								
		100	300	500	100	300	500	100	300	500
		h_1			h_2			h_3		
Clayton		0.304	0.672	0.971	0.319	0.682	0.974	0.301	0.668	0.969
Frank	$\hat{\lambda}$	0.393	0.740	0.900	0.401	0.765	0.899	0.382	0.751	0.910
Gumbel		0.602	0.980	0.999	0.615	0.975	0.999	0.601	0.989	0.999
Clayton		0.211	0.423	0.624	0.276	0.414	0.632	0.281	0.419	0.628
Frank	Q	0.275	0.487	0.687	0.287	0.462	0.691	0.264	0.453	0.676
Gumbel		0.421	0.573	0.712	0.416	0.563	0.703	0.419	0.559	0.718
		Frank								
Clayton		0.323	0.473	0.884	0.334	0.476	0.879	0.343	0.450	0.876
Frank	$\hat{\lambda}$	0.395	0.615	0.835	0.401	0.621	0.843	0.384	0.601	0.838
Gumbel		0.704	0.992	1.000	0.710	0.991	1.000	0.720	0.998	0.999
Clayton		0.274	0.385	0.639	0.263	0.379	0.642	0.276	0.389	0.645
Frank	Q	0.299	0.432	0.654	0.284	0.419	0.635	0.288	0.418	0.649
Gumbel		0.288	0.476	0.734	0.267	0.452	0.729	0.274	0.468	0.716
		Gumbel								
Clayton		0.713	0.989	1.000	0.702	0.979	1.000	0.743	0.990	0.999
Frank	$\hat{\lambda}$	0.457	0.823	0.968	0.589	0.843	0.978	0.456	0.812	0.965
Gumbel		0.587	1.000	1.000	0.561	0.998	1.000	0.599	0.999	1.000
Clayton		0.341	0.532	0.783	0.352	0.547	0.764	0.368	0.552	0.751
Frank	Q	0.357	0.523	0.768	0.361	0.519	0.764	0.342	0.520	0.765
Gumbel		0.338	0.558	0.746	0.329	0.562	0.756	0.312	0.564	0.758

Table 2.8: Empirical Powers-DGP3 (Smooth Change)

	Clayton			Frank			Gumbel		
	100	300	500	100	300	500	100	300	500
Case 1									
Clayton	0.078	0.044	0.053	0.041	0.071	0.062	0.039	0.043	0.049
Frank	0.083	0.049	0.540	0.029	0.078	0.064	0.017	0.070	0.043
Gumbel	0.100	0.045	0.560	0.040	0.082	0.069	0.029	0.064	0.053
Case 2									
Clayton	0.067	0.430	0.056	0.043	0.062	0.059	0.043	0.045	0.058
Frank	0.090	0.043	0.052	0.032	0.072	0.061	0.021	0.069	0.047
Gumbel	0.100	0.067	0.059	0.047	0.070	0.064	0.032	0.067	0.059
Case 3							0.031	0.059	0.048
Clayton	0.072	0.062	0.057	0.038	0.072	0.057	0.035	0.041	0.057
Frank	0.091	0.061	0.053	0.032	0.068	0.062	0.022	0.076	0.047
Gumbel	0.099	0.084	0.051	0.043	0.076	0.063	0.033	0.069	0.051

Table 2.9: Emperical Sizes-Breaks in Parameters of Marginal Distributions

$\beta_2 : 0.3 \rightarrow 0.8$ at $\tau = 0.6$.

In each case, the DGPs are the same as the above. The empirical sizes and powers are calculated at nominal level 0.05 and reported in Tables 2.9-2.12. The results are similar to the cases that GARCH parameters do not change. It can be seen that there is no severe size distortion in most cases and the empirical size gets very close to the minimal level as the sample size increases. Tables 2.10-2.12 show that $\hat{\lambda}$ produces good powers in most cases as the sample size grows.

2.4 Empirical Applications

As an empirical application, we consider the Morgan Stanley Capital International (MSCI) index for three sets of countries: G5 (France, Germany, the UK, and the US), Asia (Hong King, Korea, Singapore, Taiwan ,and Thailand), and Latin America (Argentina, Brazil, Chile, and Mexico). Weekly data are used with data coverage from 28 October 1994 to 27 December 2013 (T =1000). By using weekly data can remove the effect of different trading times for international markets, which

		100	300	500	100	300	500	100	300	500
		h_1			h_2			h_3		
	a				Clayton					
Case 1	0.2	0.095	0.157	0.086	0.102	0.168	0.104	0.098	0.176	0.143
	0.6	0.202	0.336	0.640	0.200	0.400	0.673	0.232	0.387	0.684
	1	0.388	0.727	0.972	0.347	0.776	0.987	0.398	0.716	0.969
Case 2	0.2	0.079	0.216	0.109	0.068	0.240	0.170	0.089	0.301	0.176
	0.6	0.222	0.278	0.462	0.227	0.287	0.501	0.276	0.254	0.489
	1	0.342	0.685	0.954	0.354	0.687	0.976	0.324	0.676	0.958
Case 3	0.2	0.084	0.213	0.120	0.097	0.243	0.198	0.078	0.231	0.140
	0.6	0.214	0.277	0.472	0.224	0.287	0.501	0.232	0.269	0.489
	1	0.358	0.649	0.944	0.354	0.689	0.954	0.261	0.651	0.958
					Frank					
Case 1	0.2	0.081	0.251	0.118	0.065	0.241	0.119	0.078	0.234	0.187
	0.6	0.257	0.465	0.518	0.243	0.465	0.508	0.234	0.476	0.524
	1	0.471	0.734	0.930	0.433	0.766	0.943	0.478	0.768	0.928
Case 2	0.2	0.075	0.260	0.108	0.084	0.254	0.118	0.076	0.257	0.111
	0.6	0.227	0.453	0.470	0.234	0.467	0.484	0.226	0.449	0.048
	1	0.440	0.678	0.881	0.448	0.689	0.889	0.430	0.658	0.879
Case 3	0.2	0.070	0.240	0.103	0.080	0.254	0.158	0.088	0.243	0.117
	0.6	0.263	0.465	0.458	0.265	0.468	0.478	0.234	0.476	0.480
	1	0.417	0.733	0.903	0.418	0.743	0.912	0.420	0.722	0.901
					Gumbel					
Case 1	0.2	0.143	0.250	0.175	0.134	0.232	0.187	0.154	0.261	0.199
	0.6	0.361	0.767	0.975	0.356	0.767	0.987	0.376	0.787	0.967
	1	0.630	0.988	1.000	0.634	0.987	1.000	0.627	0.990	1.000
Case 2	0.2	0.141	0.269	0.216	0.144	0.274	0.261	0.134	0.256	0.244
	0.6	0.310	0.678	0.941	0.320	0.665	0.951	0.318	0.688	0.967
	1	0.593	0.989	0.999	0.634	0.987	1.000	0.583	0.978	1.000
Case 3	0.2	0.142	0.283	0.221	0.138	0.297	0.243	0.154	0.276	0.220
	0.6	0.294	0.681	0.927	0.301	0.676	0.934	0.287	0.676	0.924
	1	0.557	0.984	0.999	0.601	0.988	1.000	0.598	0.958	0.999

Table 2.10: Emperical Powers-Breaks in Parameters of Marginal Distributions and a Single Break in the Copula Parameter

	a	h_1 100	h_1 300	h_1 500	h_2 100	h_2 300	h_2 500	h_3 100	h_3 300	h_3 500
					Clayton					
Case 1	0.2	0.130	0.109	0.248	0.157	0.117	0.259	0.134	0.120	0.239
	0.6	0.232	0.439	0.824	0.259	0.451	0.838	0.221	0.489	0.816
	1	0.336	0.731	0.990	0.340	0.714	0.999	0.329	0.750	0.996
Case 2	0.2	0.126	0.107	0.252	0.130	0.110	0.232	0.117	0.110	0.249
	0.6	0.256	0.438	0.831	0.265	0.444	0.837	0.230	0.465	0.813
	1	0.355	0.744	0.986	0.367	0.721	0.981	0.343	0.789	0.989
Case 3	0.2	0.122	0.099	0.273	0.132	0.100	0.289	0.110	0.110	0.278
	0.6	0.250	0.400	0.821	0.234	0.467	0.859	0.265	0.421	0.843
	1	0.333	0.695	0.983	0.344	0.701	0.988	0.312	0.675	0.979
					Frank					
Case 1	0.2	0.082	0.166	0.212	0.091	0.156	0.222	0.081	0.169	0.202
	0.6	0.252	0.455	0.743	0.245	0.465	0.765	0.267	0.476	0.728
	1	0.404	0.812	0.984	0.412	0.823	0.978	0.401	0.801	0.989
Case 2	0.2	0.082	0.166	0.212	0.081	0.188	0.222	0.077	0.179	0.264
	0.6	0.252	0.455	0.743	0.267	0.435	0.758	0.265	0.459	0.729
	1	0.404	0.812	0.984	0.402	0.854	0.987	0.398	0.879	0.956
Case 3	0.2	0.093	0.180	0.230	0.102	0.179	0.265	0.089	0.187	0.220
	0.6	0.235	0.476	0.754	0.234	0.498	0.712	0.294	0.418	0.802
	1	0.461	0.833	0.986	0.451	0.829	0.989	0.461	0.854	0.978
					Gumbel					
Case 1	0.2	0.085	0.175	0.348	0.078	0.170	0.365	0.980	0.167	0.330
	0.6	0.238	0.773	0.984	0.251	0.765	0.982	0.276	0.781	0.986
	1	0.384	0.963	1.000	0.367	0.971	0.999	0.387	0.968	1.000
Case 2	0.2	0.081	0.174	0.362	0.091	0.185	0.351	0.078	0.165	0.368
	0.6	0.237	0.781	0.979	0.249	0.785	0.989	0.256	0.765	0.976
	1	0.370	0.964	1.000	0.367	0.987	1.000	0.376	0.965	0.999
Case 3	0.2	0.083	0.155	0.318	0.092	0.165	0.328	0.078	0.147	0.311
	0.6	0.230	0.761	0.982	0.246	0.754	0.986	0.226	0.768	0.983
	1	0.351	0.957	0.999	0.345	0.967	1.000	0.348	0.978	0.998

Table 2.11: Emperical Powers-Breaks in Parameters of Marginal Distributions and Multiple Breaks in the Copula Parameter

	100	300	500	100	300	500	100	300	500
	h_1			h_2			h_3		
				Clayton					
Case 1	0.801	0.999	1.000	0.786	0.998	1.000	0.767	0.997	1.000
Case 2	0.799	1.000	1.000	0.802	0.998	1.000	0.788	0.988	0.999
Case 3	0.805	0.999	0.999	0.791	0.999	1.000	0.812	0.987	1.000
				Frank					
Case 1	0.405	0.652	0.881	0.410	0.645	0.886	0.402	0.630	0.892
Case 2	0.390	0.704	0.932	0.393	0.695	0.910	0.382	0.718	0.925
Case 3	0.410	0.687	0.965	0.453	0.652	0.958	0.398	0.675	0.957
				Gumbel					
Case 1	0.585	1.000	1.000	0.543	0.998	1.000	0.530	0.999	1.000
Case 2	0.709	1.000	1.000	0.765	1.000	0.999	0.701	1.000	1.000
Case 3	0.692	1.000	1.000	0.654	0.988	0.989	0.650	1.000	0.999

Table 2.12: Emperical Powers-Breaks in Parameters of Marginal Distributions and Smooth Changes in the Copula Parameter

happens when daily data are used.

Table 2.13 shows descriptive statistics of the log-returns $r_{i,t} = \log \frac{s_{it}}{s_{i,t-1}}$, where $s_{i,t}$ is a stock index of the ith country. The mean is negative for Japan, Taiwan, and Thailand and the skewness is positive for Brazil, Chile, and Mexico. There is an excess of kurtosis for all countries. Besides, the Jarque-Bera statistics for all series are significant thus we can reject normality in all cases.

For the estimation of the models for the marginals, we specify AR(1)-GARCH(1,1) models with the Student-t distribution. After fitting the models, we test for the presence of remaining autocorrelation in the error terms by using the Ljung-Box statistics.

In order to estimate the copula parameter, we first need to select an appropriate type of copula for each series. This is done by the Akaike Information Criterion (AIC). In this paper, we consider the elliptical copula: Gaussian copula, and three Archimedean copulas: Clayton which describes a greater dependence in the negative

	Mean	Med	Min	Max	Std	Skew	Kurt	Jbstat	JBPval
Fr	1.E-03	3.E-03	-0.267	0.137	0.033	-0.366	2.574	29.878	0.000
Ge	1.E-03	5.E-03	-0.261	0.152	0.035	-0.198	2.344	24.446	0.000
Ja	-2.E-04	-4.E-04	-0.164	0.110	0.029	-0.226	2.308	28.471	0.000
Uk	8.E-04	3.E-03	-0.276	0.162	0.028	-0.370	2.671	27.287	0.000
Us	1.E-03	2.E-03	-0.201	0.115	0.025	-1.109	3.897	238.366	0.000
Hk	7.E-04	2.E-03	-0.211	0.137	0.034	-0.041	2.119	32.637	0.000
Ko	8.E-04	3.E-03	-0.527	0.286	0.055	-0.487	2.618	45.671	0.000
Si	4.E-04	1.E-03	-0.258	0.185	0.034	-0.165	2.103	38.066	0.000
Ta	-2.E-05	2.E-03	-0.144	0.194	0.037	-0.381	3.272	27.322	0.000
Th	-6.E-04	4.E-04	-0.292	0.277	0.051	-0.099	1.983	44.724	0.000
Ar	4.E-04	3.E-03	-0.336	0.253	0.054	-0.524	3.363	51.314	0.000
Br	1.E-03	5.E-03	-0.331	0.256	0.054	0.161	1.606	85.249	0.000
Ch	6.E-04	2.E-03	-0.347	0.191	0.034	0.193	1.819	64.316	0.000
Me	1.E-03	5.E-03	-0.317	0.226	0.045	0.049	1.548	88.211	0.000

Table 2.13: Summary Statistics

tail than in the positive, Gumbel which captures a greater dependence in the positive tail than in the negative, and Frank copula which is a symmetric Archimedean copula. The selected copula are reported in Table 2.14. Once a type of copula is selected, we estimate the copula parameter by the procedure proposed above and calculate the generalized likelihood ratio statistics.

Table 2.15 shows the descriptive statistics of the time-varying kendall's tau, GLRT and CUSUM test statistics for each pair of countries. The GLRT rejects the constant copula parameter hypothesis at the 5% level for all cases, however the CUSUM test is not able to detect the time-varying copula parameter at the 5% significant level in practical use and it provides vague proof at the 10% significant level for some cases.

Pair	Copula	Pair	Copula	Pair	Copula
Fr-Ge	rGumbel	HK-Ko	rGumbel	Ar-Br	rGumbel
Fr-Ja	Gaussian	HK-Si	rGumbel	Ar-Ch	rGumbel
Fr-UK	rGumbel	HK-Ta	rGumbel	Ar-Me	rGumbel
Fr-US	Gaussian	HK-Th	rGumbel	Br-Ch	rGumbel
Ge-Ja	rGumbel	Ko-Si	rGumbel	Br-Me	rGumbel
Ge-UK	rGumbel	Ko-Ta	rGumbel	Ch-Me	rGumbel
Ge-US	rGumbel	Ko-Th	rGumbel		
Ja-UK	Gaussian	Si-Ta	rGumbel		
Ja-US	rGumbel	Si-Th	rGumbel		
UK-US	rGumbel	Ta-Th	rGumbel		

Table 2.14: Copula Selection

	Min	First Quartile	Mid	Third Quartile	Max	Avg	$\hat{\lambda}$	Q
Fr-Ge	0.481	0.581	0.732	0.769	0.777	0.679	0.000***	0.079*
Fr-Ja	0.176	0.210	0.289	0.329	0.358	0.257	0.001***	0.121
Fr-UK	0.453	0.519	0.639	0.668	0.689	0.600	0.000***	0.119
Fr-US	0.143	0.218	0.399	0.476	0.551	0.472	0.000***	0.057*
GE-Ja	0.182	0.203	0.297	0.318	0.343	0.267	0.000***	0.138
Ge-UK	0.444	0.466	0.591	0.631	0.638	0.557	0.001***	0.169
Ge-US	0.362	0.409	0.548	0.563	0.582	0.495	0.000***	0.168
Ja-UK	0.164	0.198	0.264	0.318	0.336	0.271	0.000***	0.154
Ja-US	0.136	0.153	0.245	0.279	0.318	0.213	0.000***	0.042**
UK-US	0.363	0.393	0.505	0.554	0.628	0.483	0.000***	0.208
HK-Ko	0.147	0.223	0.398	0.488	0.545	0.361	0.001***	0.126
HK-Si	0.350	0.369	0.493	0.571	0.593	0.475	0.000***	0.098*
HK-Ta	0.147	0.215	0.343	0.426	0.492	0.324	0.001***	0.085*
HK-Th	0.226	0.246	0.295	0.396	0.495	0.322	0.002***	0.048**
Ko-Si	0.188	0.266	0.416	0.518	0.548	0.391	0.000***	0.087*
Ko-Ta	0.050	0.200	0.438	0.529	0.609	0.370	0.001***	0.092*
Ko-Th	0.157	0.239	0.327	0.384	0.415	0.309	0.000***	0.119
Si-Ta	0.171	0.199	0.351	0.469	0.519	0.339	0.000***	0.054*
Si-Th	0.323	0.327	0.342	0.376	0.503	0.362	0.000***	0.048**
Ta-Th	0.123	0.169	0.294	0.354	0.373	0.266	0.000***	0.153
Ar-Br	0.298	0.340	0.402	0.426	0.455	0.386	0.002***	0.085*
Ar-Ch	0.269	0.283	0.303	0.317	0.338	0.302	0.001***	0.073*
Ar-Me	0.296	0.336	0.381	0.412	0.429	0.374	0.002***	0.112
Br-Ch	0.376	0.382	0.402	0.438	0.513	0.415	0.000***	0.081*
Br-Me	0.330	0.375	0.447	0.569	0.579	0.464	0.000***	0.092*
Ch-Me	0.204	0.314	0.380	0.445	0.510	0.373	0.001***	0.132

Table 2.15: p-Values of LR and CUSUM Tests and Kendall's Tau.

3. TESTING FOR SMOOTH STRUCTURAL CHANGES IN VOLATILITY

There is much evidence that economic and financial time series are non-stationary when observed over long enough periods of time. Police-regime shifts and many other factors may generate parameter instability in the underlying generating process, often leading to structural changes. While there has been an obvious interest to analyze mean shifts in variables as well as their specific sources, the recent literature in financial economics also concerns with instability on higher moments. A special interest has been directed towards addressing variance homogeneity since this moment heavily characterizes the statistical properties of economic models and their predictions. For instance, McConnell and Perez-Quirós (2000), Stock and Watson (2003), and Sensier and van Dijk (2004) among others, find strong evidence suggesting a sharp decline in the volatility of macroeconomic variables, which has important policy implications. Also, since volatility is central to the financial theory and its empirical applications, there is a growing interest to analyze variance stability in financial markets (see, among others, De Santis and Imrohoroglu (1997)), and the stylized facts of financial returns like IGARCH effects or variance persistence can be well explained by structural changes in the unconditional variance. (see Diebold (1986), Lamoureaux and Lastrapes (1990), Granger and Hyung (2004), Mikosch and Stărică (2004), and Hillebrand (2005)).

The statistical methods specially designed to estimate breaks in volatility which have been mostly used in the applied literature are based on CUSUM-type procedures. Many different types of CUSUM tests have been proposed and applied by Inclán and Tiao (1994), Loretan and Phillips (1994), Kokoszka and Leipus (1998, 2000), Kim, Cho and Lee (2000), Lee and Park (2001), Andreou and Ghysels (2002), Sansó, Aragó and Carrion (2004), Deng and Perron (2008), Cavaliere and Taylor (2008) among others. The widespread use of CUSUM-type tests is not only due to its tractability and simplicity but mostly to its statistical appeal. Most of these tests are model-free and admit a fairly general class of generating processes. Furthermore, they do not specify a particular pattern of variation and hence have power against several alternatives, such as parameter instability or distribution changes, and do not require prior knowledge of the timing of the shift. All the tests belonging to this framework adopt a similar strategy to detect breaks, although some of them differ significantly in their basic assumptions like non-normality and serial dependence. However, because of the absence of the explicit formulation of the alternative hypothesis, the CUSUM test is subject to power loss.

Juhl and Xiao (2005) proposed a nonparametric test for structural changes in a deterministic trend model, where the functional form of the trend is unknown. Chen and Hong (2012) proposed a test for smooth structural changes as well as abrupt structural breaks with known or unknown change points by comparing the fitted values of the restricted constant parameter model and an unrestricted time-varying parameter model. These two tests' target are testing for the structural change in the mean function. In this paper, we propose a new consistent test for smooth structural changes as well as abrupt breaks in volatility. We estimate the volatility parameters by the local linear estimating method and compare them with the constrained estimators. The proposed test can be viewed as a generalized Hausman's (1978) test from the parametric framework to the nonparametric framework. We show that the generalized Hausman test is asymptotically more powerful than the CUSUM test.

3.1 Model

Considering the following model for a sequence (e.g. log returns, interest rates, or GDP)

$$Y_t = g_t + \sigma_t \varepsilon_t, \qquad t = 1, 2, \ldots, T, \tag{3.1}$$

where σ_t is a deterministic function of t and accounts for the nonstationary unconditional variance. We assume that ε_t is stationary and satisfies $E(\varepsilon_t) = 0,\ E(\varepsilon_t^2) = 1$ *and* $E(\varepsilon_t^4) < \infty$. ε_t captures conditional heteroscedasticity. The mean g_t and variance σ_t satisfies $g_t = g\left(\frac{t}{T}\right)$ and $\sigma_t = \sigma\left(\frac{t}{T}\right)$ respectively. We assume $g(z)$ and $\sigma(z)$ are unknown continuous functions except for a finite number of points and at least twice differentiable at all $z \in [0, 1]$. Discontinuities of $g(z)$ and $\sigma(z)$ at a finite number of points in $[0, 1]$ allow abrupt changes. In other words, we permit $\alpha(\cdot)$ is to have finitely many discontinuities. Hence, a single structural break or multiple breaks with known or unknown breakpoints are special cases of the model (3.1). The specification that $g\left(\frac{t}{T}\right)$ and $\sigma\left(\frac{t}{T}\right)$ are functions of ratio t/T rather than time t is a common scaling scheme in the literature. The reason for this requirement is that nonparametric estimators for $g\left(\frac{t}{T}\right)$ and $\sigma\left(\frac{t}{T}\right)$ are not consistent unless the amount of data on which they depend increase, and merely increasing the sample size does not necessarily improve estimation of $g\left(\frac{t}{T}\right)$ and $\sigma\left(\frac{t}{T}\right)$ at some fixed point t.

First, we consider a simple model which assumes that the variation of the mean function is small enough to be neglected. Thus, the model is

$$Y_t = \sigma_t \varepsilon_t. \tag{3.2}$$

Rewrite the model as

$$Y_t^2 = \sigma_t^2 + e_t,$$

where $e_t = \sigma_t^2 \left(\varepsilon_t^2 - 1 \right)$. So the problem of interest is transformed to test for changing mean in Y_t^2.

3.2 Generalized Hausman Test

The hypotheses of interests are

$$H_0 : \sigma_t^2 \equiv \sigma^2.$$

$$H_A : \sigma_t^2 \text{ is not constant over } t.$$

Under H_0, the constrained estimator of the unknown constant parameter is $\widehat{\sigma^2} = \overline{Y}_t^2$, where $\overline{Y}_t^2 = \sum_{t=1}^{T} Y_t^2$. Under $H_{A,}$ $\sigma_t^2 = \sigma^2 \left(t/T \right)$ is changing over time. The constrained estimator $\widehat{\sigma^2}$ is no longer consistent. However, a nonparametric estimator can consistently estimate the time-varying parameter σ_t^2.

Many nonparametric methods can be used to estimate σ_t^2. Here, we use local linear smoothing, which includes the kernel method as a special case. Cai (2007) showed that the local linear estimators converge faster than the kernel estimators in the boundary regions near the end points of the sample period. It is quite suitable to use local linear smoothing here. Since the structural change is the local behavior of parameters, local smoothing is expected to have better approximation in many cases.

The local linear estimator of $\widehat{\sigma}_t^2$ is

$$\begin{pmatrix} \widehat{\sigma}_t^2 \left(\frac{t}{T} \right) \\ \widehat{\sigma}_t^{2\prime} \left(\frac{t}{T} \right) \end{pmatrix} = \left[\sum_{s=1}^{T} K \left(\frac{s-t}{Th} \right) \begin{pmatrix} 1 \\ s/T - t/T \end{pmatrix} (1, s/T - t/T) \right]^{-1}$$
$$\cdot \sum_{s=1}^{T} K \left(\frac{s-t}{Th} \right) \begin{pmatrix} 1 \\ s/T - t/T \end{pmatrix} (1, s/T - t/T) \, Y_s^2.$$

Cai (2007) has shown that the local linear estimator in the time varying coefficient model is consistent.

We now propose a consistent test for smooth structural changes. With $\widehat{\sigma}_t^2$, we can construct a consistent test by comparing constrained and nonconstrained nonparametric regression estimators. This can be interpreted as a generalized Hausman's (1978) test. Hausman's (1978) test compares two parameter estimators, where one is efficient but inconsistent under the alternative and the other is inefficient but consistent under the alternative. We extend Hausman's (1978) idea from a parametric regression framework to a nonparametric regression framework, where the $\widehat{\sigma}^2$ can be viewed as an efficient estimator under H_0, and the nonparametric time varying parameter regression estimator $\widehat{\sigma}_t^2$ can be viewed as an inefficient but consistent estimator under H_A. We compare these parametric and nonparametric fitted values via a sample quadratic form

The test is

$$\widehat{D} = \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\sigma}_t^2 - \widehat{\sigma}^2\right)^2.$$

The statistic $\widehat{D}$ converges to zero under H_0, and to a strictly positive constant under H_A. Any significant departure of $\widehat{D}$ from 0 is evidence of structural changes. Formally, our generalized Hausman test is a standardized version of $\widehat{D}$,

$$\widehat{H} = \left(T\sqrt{h}\widehat{D} - \widehat{M}_H\right)/\sqrt{\widehat{V}_H}, \tag{3.3}$$

where

$$
\begin{aligned}
\widehat{M}_H &= h^{-1/2} C_A \widehat{\Omega}, \widehat{V}_H = 4 C_B \widehat{\Omega}^2, \\
C_A &= T^{-1} h^{-1} \sum_{i=[-Th]}^{[Th]} \left(1 - \frac{|i|}{T}\right) k\left(\frac{i}{Th}\right) \left[k\left(\frac{i}{Th}\right) + h \int_{-1}^{1} k\left(\frac{i}{Th} + 2u\right) du\right] \\
&= \int_{-1}^{1} k^2(u)\, du + o(1), C_B = T^{-1} h^{-1} \sum_{i=1}^{T-1} \left(1 - \frac{i}{T}\right) \left[\int_{-1}^{1} k(u)\, k\left(u + \frac{i}{Th}\right) du\right]^2 \\
&= \int_0^1 \left[\int_{-1}^{1} k(u)\, k(u+v)\, du\right]^2 dv + o(1), \\
C_B &= T^{-1} h^{-1} \sum_{i=1}^{T-1} \left(1 - \frac{i}{T}\right) \left[\int_{-1}^{1} k(u)\, k\left(u + \frac{i}{Th}\right) du\right]^2 \\
&= \int_0^1 \left[\int_{-1}^{1} k(u)\, k(u+v)\, du\right]^2 dv + o(1), \\
\widehat{\Omega} &= T^{-1} \sum_{t=1}^{T} \widehat{e}_t^2.
\end{aligned}
$$

To derive the asymptotic distribution of $\widehat{H}$, we impose the following regularity conditions

Assumption 1 $\{\varepsilon_t^2 - 1\}$ *is a real zero-mean, strictly stationary* $\beta-$*mixing process with* $E\left|\varepsilon_t^2 - 1\right|^q < \infty$ *for some* $q > 2$ *and with mixing coefficients* $\{\beta(j)\}$ *satisfying* $\sum_{j=1}^{\infty} j^2 \beta(j)^{\delta/(1-\delta)} < C$ *for some* $0 < \delta < 1$.

Assumption 2 *The kernel function* $K(\cdot)$ *is a bounded symmetric density function with a support* $[-1, 1]$.

Assumption 3 *As* $T \to \infty$, $h \to 0$, and $Th \to \infty$.

We now state the asymptotic distribution of $\widehat{H}$ under H_0.

Theorem 2 *Suppose Assumptions 1-3 and* H_0 *hold. Then* $\widehat{H} \xrightarrow{d} N(0,1)$ *as* $T \to \infty$.

3.3 General Model

Now, we consider the model (3.1). If g_t is known we can regard the problem of estimating $\sigma^2(\cdot)$ as a nonparametric regression problem. However, g_t is typically unknown in practice. A natural approach is to substitute g_t by a nonparametric regression estimator.

First, the local linear estimator of g_t is

$$\begin{pmatrix} \widehat{g}\left(\frac{t}{T}\right) \\ \widehat{g}^{(1)}\left(\frac{t}{T}\right) \end{pmatrix} = \left[\sum_{s=1}^{T} K\left(\frac{s-t}{Th}\right)\begin{pmatrix} 1 \\ s/T - t/T \end{pmatrix}(1, s/T - t/T)\right]^{-1} \cdot \sum_{s=1}^{T} K\left(\frac{s-t}{Th}\right)\begin{pmatrix} 1 \\ s/T - t/T \end{pmatrix} Y_s.$$

Denote the squared residuals by $\widehat{r}_t = (Y_t - \widehat{g}_t)^2$. This leads to the residual-based estimator $\widehat{\widehat{\sigma}}_t^2$:

$$\begin{pmatrix} \widehat{\widehat{\sigma}}^2\left(\frac{t}{T}\right) \\ \widehat{\widehat{\sigma}}^{2\prime}\left(\frac{t}{T}\right) \end{pmatrix} = \left[\sum_{s=1}^{T} K\left(\frac{s-t}{Th}\right)\begin{pmatrix} 1 \\ s/T - t/T \end{pmatrix}(1, s/T - t/T)\right]^{-1} \cdot \sum_{s=1}^{T} K\left(\frac{s-t}{Th}\right)\begin{pmatrix} 1 \\ s/T - t/T \end{pmatrix} \hat{r}\left(s/T\right).$$

While the bias for $\widehat{g}_t$ itself is of order $O\left(h^2\right)$, its contribution to $\widehat{\sigma}^2(\cdot)$ is only of $o\left(h^2\right)$. This can be explained by following. Note that

$$\widehat{r}_t - r_t = 2\left\{g\left(\frac{t}{T}\right) - \widehat{g}\left(\frac{t}{T}\right)\right\} u_t + \left\{g\left(\frac{t}{T}\right) - \widehat{g}\left(\frac{t}{T}\right)\right\}^2,$$

where $u_t = Y_t - g\left(\frac{t}{T}\right)$.It is clear that the biases of the residuals are of order $O\left\{h^4 + (nh)^{-1}\right\}$ and this is the effect of the estimated regression function on the $\widehat{\widehat{\sigma}}_t^2$.

We need to show that this two-step estimator $\widehat{\widehat{\sigma}}_t^2$ is consistent, which is stated in the following lemma.

Lemma 1 *Suppose that Assumptions 1-3 hold. Then,*

$$\widehat{\widehat{\sigma}}_t^2 - \sigma_t^2 = O_p\left(h^2 + (Th)^{-1/2}\right).$$

Thus $\widehat{\widehat{\sigma}}_t^2$ is the consistent estimator of σ_t^2.

Thus, we can define the new test as

$$\widehat{\widehat{D}} = \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\widehat{\sigma}}_t^2 - \widehat{\sigma}^2\right)^2,$$

and the standard version of $\widehat{\widehat{D}}$ is

$$\widehat{\widehat{H}} = \left(T\sqrt{h}\widehat{\widehat{D}} - \widehat{M}_H\right)/\sqrt{\widehat{V}_H},$$

where $\widehat{M}_H$ and $\widehat{V}_H$ are defined as in (3.3).

Theorem 3 *Suppose that Assumptions 1-3 and H_0 hold. It follows from Lemma 1 that $\widehat{\widehat{H}} \xrightarrow{d} N(0,1)$ as $T \to \infty$.*

3.4 Numerical Examples

3.4.1 Design of the Simulation

To show the practical performance of the proposed test, a simulation study was carried out. We consider the following data generating process (DGP)

$$Y_t = g_t + u_t, u_t = \sigma_t \varepsilon_t, \varepsilon_t = \phi_t \eta_t, \phi_t^2 = \mu + \alpha\varepsilon_{t-1}^2 + \beta\phi_{t-1}^2, \eta_t \overset{iid}{\sim} N(0,1),$$

where ε_t follows a GARCH(1, 1) process with Gaussian innovations. The GARCH parameters are set as $\mu = 0.01$, $\alpha = 0.1$, $\beta \in [0.4, 0.8]$. Let $\sigma^2(z) = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2)$, $z \in (0, 1]$, $\delta = \sigma_1^2/\sigma_0^2$ and $\sigma_0^2 = 1$.Under H_0, $\delta = 1$ and $\sigma^2(z) = \sigma_0^2 = 1$.Hence, u_t follows a conventional stationary GARCH(1,1) process with Gaussian innovations. The behavior of the mean function g_t is set the same as the variance σ_t^2.

We consider three alternatives. The first case is concerned about single structure break. We let variance shifts happen at different timings, $\tau \in [0.5, 0.8]$.

$$\delta = \begin{cases} \neq 1 & \text{if } z > \tau \\ 1 & \text{otherwise} \end{cases}.$$

Hence, when $\delta \neq 1$, there is a jump in the variance from σ_0^2 to σ_1^2 at time τ.

The second case, we consider multiple structure breaks:

$$\delta = \begin{cases} 2 & \text{if } 0.1 \leq z \leq 0.2 \text{ or } 0.7 \leq z \leq 0.8, \\ 1.1 & \text{if } 0.4 \leq z \leq 0.5, \\ 1 & \text{otherwise.} \end{cases}$$

The third alternative is smooth structure change:

$$\sigma(z)^2 = F(z),$$

where $F(z) = 1.5 - 1.5\exp\left[-3(z - 0.5)^2\right]$. The simulation results depend on the bandwidth h. To check the sensitivity, we use three different fixed bandwidths $h_i = 2^{i-1}/\sqrt{12}T^{-1/5}$ for $i = 1, 2, 3$. The uniform kernel is used in the nonparametric estimation. The trimming $\Pi = [0.15, 0.85]$ is used. We generate 5000 data sets of the random samples for each of T =100, 300, 500. We compare the proposed test $\hat{H}$

	100	300	500		100	300	500
		$\beta = 0.4$				$\beta = 0.8$	
		$\hat{H}$				$\hat{H}$	
h_1	0.151	0.091	0.045	h_1	0.161	0.075	0.048
h_2	0.147	0.089	0.047	h_2	0.153	0.085	0.046
h_3	0.177	0.950	0.048	h_3	0.155	0.073	0.047
		$\hat{Q}$				$\hat{Q}$	
h_1	0.116	0.071	0.051	h_1	0.140	0.094	0.050
h_2	0.121	0.073	0.050	h_2	0.142	0.079	0.052
h_3	0.118	0.074	0.052	h_3	0.144	0.080	0.051

Empirical Sizes

Table 3.1: Empirical Sizes

with the CUSUM test $\hat{Q}$.

3.4.2 Simulation Results

Table 3.1 shows the rejection rates of the proposed test $\hat{H}$ and the CUSUM test $\hat{Q}$. The results show that sizes of $\hat{H}$ are generally larger than those of $\hat{Q}$. Thus the $\hat{H}$ rejects more in most cases. There is no severe size distortions in most cases for both $\hat{H}$ and $\hat{Q}$. It can be seen that $\hat{H}$ and $\hat{Q}$ exhibit some size distortion for small sample sizes. The empirical sizes go to the nominal 5% as the sample size increases. We also can see that sizes are not sensitive to the bandwidth.

Tables 3.2 and 3.3 show the empirical power of tests when there is a deterministic single break. Separate tables are for GARCH parameter $\beta = 0.4$ and $\beta = 0.8$. The powers are monotone. $\hat{H}$ generally has more power than $\hat{Q}$ test. The power of $\hat{H}$ test is a bit lower when the level of jumping is small especially in the case of $\beta = 0.4$, $\tau = 0.8$, and $\delta = 0.5$. $\hat{H}$ is more powerful when $\beta = 0.4$. The results are sensitive to the selection of the bandwidth h when the level of jump is small.

	$\hat{H}$									$\hat{Q}$								
	$\tau = 0.5$																	
	h_1			h_2			h_3			h_1			h_2			h_3		
	100	300	500	100	300	500	100	300	500	100	300	500	100	300	500	100	300	500
$\delta = 0.5$	0.757	1.000	1.000	0.853	1.000	1.000	0.792	1.000	1.000	0.300	0.780	0.919	0.255	0.759	0.903	0.232	0.703	0.916
$\delta = 1.5$	0.346	0.904	0.991	0.418	0.9626	0.999	0.226	0.9446	0.998	0.139	0.553	0.8138	0.106	0.522	0.776	0.084	0.429	0.768
$\delta = 2$	0.907	1.000	1.000	0.976	1.000	1.000	0.224	1.000	1.000	0.083	0.527	0.767	0.042	0.449	0.712	0.085	0.451	0.797
$\delta = 4$	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	0	0	0	0	0	0.013	0.027	0.232	0.770
	$\tau = 0.8$																	
	h_1			h_2			h_3			h_1			h_2			h_3		
	100	300	500	100	300	500	100	300	500	100	300	500	100	300	500	100	300	500
$\delta = 0.5$	0.150	0.753	0.9864	0.044	0.198	0.593	0.136	0.165	0.160	0.048	0.068	0.096	0.0282	0.044	0.052	0.039	0.282	0.030
$\delta = 1.5$	0.47	0.927	0.9932	0.7862	0.996	0.999	0.966	1.000	1.000	0.366	0.769	0.920	0.502	0.872	0.957	0.740	0.975	0.995
$\delta = 2$	0.990	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.556	0.893	0.969	0.742	0.973	0.990	0.947	0.998	1.000
$\delta = 4$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.138	0.853	0.987	0.877	0.999	0.999	1.000	1.000	1.000

Table 3.2: Empirical Power of Tests -Single Break (The GARCH parameter $\beta = 0.4$)

	$\hat{H}$									$\hat{Q}$								
	$\tau = 0.5$																	
	h_1			h_2			h_3			h_1			h_2			h_3		
	100	300	500	100	300	500	100	300	500	100	300	500	100	300	500	100	300	500
$\delta = 0.5$	0.386	0.823	0.977	0.450	0.917	0.993	0.577	0.966	0.998	0.186	0.496	0.680	0.169	0.482	0.690	0.184	0.473	0.694
$\delta = 1.5$	0.157	0.340	0.555	0.167	0.470	0.738	0.212	0.609	0.842	0.104	0.0.296	0.430	0.090	0.271	0.430	0.097	0.259	0.448
$\delta = 2$	0.343	0.812	0.974	0.380	0.914	0.994	0.324	0.932	0.999	0.133	0.420	0.628	0.100	0.387	0.621	0.098	0.378	0.626
$\delta = 4$	0.888	0.999	1.000	0.926	1.000	1.000	0.065	1.000	1.000	0.016	0.128	0.299	0.009	0.086	0.276	0.053	0.194	0.438
	$\tau = 0.8$																	
	h_1			h_2			h_3			h_1			h_2			h_3		
	100	300	500	100	300	500	100	300	500	100	300	500	100	300	500	100	300	500
$\delta = 0.5$	0.102	0.221	0.421	0.065	0.197	0.494	0.059	0.126	0.204	0.042	0.069	0.075	0.042	0.057	0.076	0.043	0.055	0.068
$\delta = 1.5$	0.183	0.343	0.508	0.179	0.449	0.692	0.246	0.541	0.769	0.158	0.57	0.495	0.162	0.389	0.547	0.235	0.517	0.690
$\delta = 2$	0.498	0.868	0.981	0.699	0.982	0.999	0.854	0.997	1.000	0.320	0.683	0.838	0.351	0.736	0.870	0.542	0.887	0.953
$\delta = 4$	0.986	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.280	0.656	0.826	0.480	0.868	0.951	0.909	0.991	0.999

Table 3.3: Empirical Power of Tests -Single Break (The GARCH parameter $\beta = 0.8$)

	$\hat{H}$			$\hat{Q}$		
	$\beta = 0.4$					
	100	300	500	100	300	500
h_1	0.233	0.495	0.993	0.118	0.106	0.464
h_2	0.373	0.590	0.999	0.119	0.135	0.559
h_3	0.511	0.632	0.997	0.121	0.157	0.623
	$\beta = 0.8$					
	100	300	500	100	300	500
h_1	0.214	0.257	0.883	0.077	0.096	0.255
h_2	0.228	0.244	0.908	0.072	0.102	0.275
h_3	0.254	0.413	0.896	0.079	0.110	0.296

Table 3.4: Empirical Power of Tests-Multiple Breaks

Tables 3.4 and 3.5 show results for cases of multiple structural breaks and smoothed structural change, respectively. We have similar results with the case of single break. $\hat{H}$ is more powerful than $\hat{Q}$. $\hat{H}$ is more powerful when the GARCH parameter β is set to 0.4 and is not sensitive to the bandwidth selection.

3.4.3 An Empirical Application

As an empirical application, we consider real GDP from, Q1, 1960 to Q4, 2012 and the monthly 4-week T-bill rate from July, 2001 to March, 2013. Stock and Watson (2003) showed that volatility of U.S. economics series have reduced since mid-1980s. We apply $\hat{H}$ and $\hat{Q}$ tests to test for volatility changes of these two series to see if the results are consistent to Stock and Watson (2003) or not. The results are shown in Table 3.6. The CUSUM test $\hat{Q}$ is not able to detect the structural break in volatility of these series at any significant level. The $\hat{H}$ test strongly reject the constant variance hypothesis when $h = \left(1/\sqrt{12}\right) T^{-1/5}$. The empirical results of $\hat{H}$ are consistent with Stock and Watson (2003).

	$\hat{H}$			$\hat{Q}$		
			$\beta = 0.4$			
	100	300	500	100	300	500
h_1	0.964	0.993	1.000	0.255	0.174	0.500
h_2	0.920	0.978	1.000	0.234	0.212	0.558
h_3	0.1152	0.131	1.000	0.230	0.246	0.620
			$\beta = 0.8$			
	100	300	500	100	300	500
h_1	0.706	0.956	1.000	0.156	0.212	0.421
h_2	0.445	0.853	0.999	0.154	0.223	0.438
h_3	0.099	0.103	0.989	0.162	0.234	0.454

Table 3.5: Empirical Power of Tests- Smooth Structural Changes

Data	n	$\hat{H}$	$\hat{Q}$
Real GDP 1960Q1-2012Q4	212	0.000	0.065
4-week T-bill rate July,2001-Mar,2013	141	0.001	0.087

Table 3.6: Volatility Break Test Results, p-values

4. CONCLUSION

In this dissertation, I propose tests for smooth structural changes in dependence and volatility, respectively. In the first essay, a generalized likelihood ratio test has been proposed for smooth structural changes in dependence between different financial assets. The test procedure employs the local constant estimator when calculating the local likelihood under the alternative hypothesis. The asymptotic distribution of the proposed test is chi-square distribution. The simulation results show that the proposed test has the good finite-sample performance. In the second essay, a generalized Hausman test has been proposed for smooth structural changes in volatility. The simulation results show that the generalized Hausman test is asymptotically more powerful than the CUSUM test.

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APPENDIX A

PROOFS OF THE THEOREM IN SECTION 2

Lemma A.1 *Under assumptions 1-5,*

$$\hat{\theta} - \theta = \frac{1}{Th} \sum_{t=1}^{T} \frac{\ell' (u_t, v_t; \theta)}{J(\tau)} K \left(\left(\frac{t}{T} - \tau \right) / h \right) + o_p(1),$$

where $J(\tau) = E\left[\ell''(u_t, v_t; \theta) \,|t/T = \tau\right].$

Define the local likelihood function using the unobserved true ranks u_t and v_t as

$$L(\theta_t, \tau) = \sum_{t=1}^{T} \ell(u_t, v_t; \theta_t) K_h \left(\frac{t}{T} - \tau \right)$$

and the corresponding estimator of θ_t by $\hat{\theta}_t = \arg\max_\theta L(\theta, \tau)$. From expansion

$$0 = \frac{\partial L\left(\hat{\theta}(t/T), \tau\right)}{\partial \theta} \approx \frac{\partial L(\theta, \tau)}{\partial \theta} + \frac{\partial^2 L(\theta, \tau)}{\partial \theta^2} \left(\hat{\theta}(t/T) - \theta \right)$$

we can obtain the bias approximation

$$\hat{\theta}(t/T) - \theta = \left[\frac{1}{T} \frac{\partial^2 L(\theta, \tau)}{\partial \theta^2} \right]^{-1} \frac{1}{T} \frac{\partial L(\theta, \tau)}{\partial \theta}.$$

We have

$$\begin{aligned} \frac{1}{T} \frac{\partial L(\theta, \tau)}{\partial \theta} &= \frac{1}{T} \sum_{t=1}^{T} \frac{\partial \ell(u_t, v_t; \theta)}{\partial \theta} K_h \left(\frac{t}{T} - \tau \right) \\ &= \frac{1}{T} \sum_{t=1}^{T} \ell\prime(u_t, v_t; \theta) K_h \left(\frac{t}{T} - \tau \right). \end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{1}{T}\frac{\partial^2 L(\theta,\tau)}{\partial\theta^2} &= \frac{1}{T}\sum_{t=1}^{T}\ell\prime\prime(u_t,v_t;\theta)K_h\left(\frac{t}{T}-\tau\right)\\ &= J(\tau)+o_p(1).\end{aligned}$$

Thus

$$\hat{\theta}-\theta=\frac{1}{Th}\sum_{t=1}^{T}\frac{\ell'(u_t,v_t;\theta)}{J(\tau)}K\left(\left(\frac{t}{T}-\tau\right)/h\right)+o_p(1).$$

Define

$$G_{T1}=\frac{1}{Th}\sum_{t=1}^{T}\sum_{r=1}^{T}\frac{\ell'(u_r,v_r;\theta)}{J(r/T)}\ell'(u_t,v_t;\theta)K\left(\left(\frac{t}{T}-\tau\right)/h\right),$$

$$\begin{aligned}G_{T2} &= \frac{1}{T^2h^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\ell\prime(u_t,v_t;\theta)\ell'(u_s,v_s;\theta)\\ &\quad\times\sum_{r=1}^{T}\frac{\ell''(u_r,v_r,\theta)}{J(r/T)^2}K\left(\left(\frac{s}{T}-\tau\right)/h\right)K\left(\left(\frac{s}{T}-\tau\right)/h\right)\end{aligned}$$

Lemma A.2 *Under assumptions 1-5, as $h\to 0$ and $Th^{3/2}\to\infty$*

$$\begin{aligned}G_{T1} &= \frac{1}{h}\frac{1}{J(\tau)}K(0)\\ &\quad+\frac{1}{Th}\sum_{r\neq t}\frac{1}{J(r/T)}\ell'(u_t,v_t;\theta)\ell'(u_r,v_r;\theta)K\left(\left(\frac{t}{T}-\frac{r}{T}\right)/h\right)\\ &\quad+o_p\left(h^{-1/2}\right),\end{aligned}$$

$$
\begin{aligned}
G_{T2} &= -h^{-1}\int K^2(\tau)\,d\tau \\
&\quad -\frac{2}{Th}\sum_{t<s}\frac{\ell'(u_t,v_t;\theta)}{\{J(t/T)\}}\ell'(u_s,v_s;\theta_s)\times K*K\left(\left(\frac{s}{T}-\frac{t}{T}\right)/h\right) \\
&\quad +o_p\left(h^{-1/2}\right)
\end{aligned}
$$

Note that

$$
\begin{aligned}
G_{T1} &= \frac{1}{Th}\sum_{r=1}^{T}\frac{[\ell'(u_r,v_r;\theta)]^2}{J(r/T)}K(0) \\
&\quad +\frac{1}{T}\sum_{r\neq t}^{T}\frac{1}{J(r/T)}\ell'(u_t,v_t;\theta)\,\ell'(u_r,v_r;\theta)\,K_h\left(\left(\frac{t}{T}-\frac{r}{T}\right)\right) \\
&= \frac{1}{h}K(0)+\frac{1}{T}\sum_{r\neq t}\frac{1}{J(r/T)}\ell'(u_t,v_t;\theta)\,\ell'(u_r,v_r;\theta)\,K_h\left(\left(\frac{t}{T}-\frac{r}{T}\right)\right) \\
&\quad +o_p\left(h^{-1/2}\right).
\end{aligned}
$$

We can decompose $G_{T2}=G_{T21}+G_{T22}$, where

$$
\begin{aligned}
G_{T21} &= \frac{1}{T^2h^2}\sum_{t=1}^{T}[\ell'(u_t,v_t;\theta)]^2\sum_{r=1}^{T}\frac{\ell''(u_r,v_r,\theta)}{J(r/T)^2}K^2\left(\left(\frac{s}{T}-\tau\right)/h\right), \\
G_{T22} &= \frac{1}{T^2}\sum_{t\neq s}^{T}\ell\prime(u_t,v_t;\theta)\,\ell'(u_s,v_s;\theta_s) \\
&\quad \sum_{r=1}^{T}\frac{\ell''(u_r,v_r,\theta)}{J(r/T)^2}K_h\left(\left(\frac{t}{T}-\frac{r}{T}\right)\right)K_h\left(\left(\frac{s}{T}-\frac{r}{T}\right)\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
G_{T21} &= \frac{1}{T^2h^2}\sum_{r=1}^{T}[\ell'(u_r,v_r;\theta)]^2\,\frac{\ell''(u_r,v_r,\theta)}{J(r/T)^2}K^2(0) \\
&\quad +\frac{1}{T^2h^2}\sum_{t\neq r}^{T}[\ell'(u_t,v_t;\theta)]^2\,\frac{\ell''(u_r,v_r,\theta)}{J(r)^2}K^2\left(\left(\frac{t}{T}-\frac{r}{T}\right)/h\right).
\end{aligned}
$$

The first sum can be shown to be $O_p\left(T^{-1}h^{-2}\right)$. Let

$$V_T = \frac{2}{T\left(T-1\right)}\sum_{t<r}\left[\frac{\ell''\left(u_r,v_r,\theta\right)}{J\left(r/T\right)^2}+\frac{\ell''\left(u_t,v_t,\theta\right)}{J\left(t/T\right)^2}\right]K_h^2\left(\left(\frac{t}{T}-\frac{r}{T}\right)\right),$$

and the second sum becomes $\left(V_T+o\left(1\right)\right)/2+O_p\left(T^{-3/2}h^{-2}\right)+o_p\left(h^{-1/2}\right)$. The decomposition theorem for U-statistics allows us to show $Var\left(V_T\right)=O\left(T^{-1}h^{-2}\right)$ as follows. First note that the leading term of V_T is $-h^{-1}\int K^2\left(\tau\right)d\tau$.Hence as $nh\to\infty$ and $h\to 0$, we have

$$G_{T21} = -h^{-1}\int K^2\left(\tau\right)d\tau + o_p\left(h^{-1/2}\right).$$

Similarly, we can decompose $G_{T22}=G_{T221}+G_{T222}$ with

$$\begin{aligned} G_{T221} &= \frac{2}{T}\sum_{t<s}\ell'\left(u_t,v_t;\theta\right)\ell'\left(u_s,v_s;\theta_s\right) \\ &\quad \frac{1}{T}\left\{\sum_{r\neq t,s}\frac{\ell''\left(u_r,v_r,\theta\right)}{J\left(r/T\right)^2}K_h\left(\left(\frac{t}{T}-\frac{r}{T}\right)\right)K_h\left(\left(\frac{s}{T}-\frac{r}{T}\right)\right)\right\}, \\ G_{T222} &= \frac{K\left(0\right)}{T^2h}\sum_{t\neq s}\ell'\left(u_t,v_t;\theta\right)\ell'\left(u_s,v_s;\theta_s\right) \\ &\quad \left\{\frac{\ell''\left(u_t,v_t,\theta\right)}{J\left(t/T\right)^2}+\frac{\ell''\left(u_s,v_s,\theta\right)}{J\left(s/T\right)^2}\right\}K_h\left(\left(\frac{t}{T}-\frac{s}{T}\right)\right). \end{aligned}$$

For $r\neq t,s$, define

$$A_{tsr,h} = \frac{\ell''\left(u_r,v_r,\theta\right)}{J\left(r/T\right)^2}K_h\left(\left(\frac{r}{T}-\frac{t}{T}\right)\right)K_h\left(\left(\frac{r}{T}-\frac{s}{T}\right)\right).$$

It can be easily shown that $Var\left(T^{-1}\sum_{r\neq t,s}A_{tsr,h}=O\left(T^{-1}h^{-2}\right)\right)$. Then,

$$G_{T221} = 2T^{-2}\left(T-2\right)\sum_{t<s}\ell'\left(u_t,v_t;\theta\right)\ell'\left(u_s,v_s;\theta_s\right)E\left(A_{tsr,h}\right)+o_p\left(h^{-1/2}\right),$$

where

$$E\left(A_{tsr,h}\right) = -\left\{hJ\left(t/T\right)\right\}^{-1}\int K\left(\tau\right)K\left(\left(\frac{s}{T}-\frac{t}{T}\right)/h\right)d\tau.$$

It is also easy to show $Var\left(G_{T222}\right) = O\left(T^{-2}h^{-3}\right)$, implying $G_{T222} = o_p\left(h^{-1/2}\right)$.

Combining G_{T21}, G_{T221}, and G_{T222} yields

$$\begin{aligned} G_{T2} &= -h^{-1}\int K^2\left(\tau\right)d\tau \\ &\quad -\frac{2}{Th}\sum_{t<s}\frac{\ell'\left(u_t,v_t;\theta\right)}{\left\{J\left(t/T\right)\right\}}\ell'\left(u_s,v_s;\theta_s\right)\times K*K\left(\left(\frac{s}{T}-\frac{t}{T}\right)/h\right)+o_p\left(h^{-1/2}\right). \end{aligned}$$

Define the following U-statistics,

$$W\left(T\right) = T^{-1}h^{1/2}\sum_{t\neq s}J\left(t\right)^{-2}\ell'\left(u_t,v_t;\theta\right)\ell'\left(u_s,v_s;\theta_s\right)A.1 \quad \text{(A.1)}$$

$$\left\{2K_h\left(\frac{s}{T}-\frac{t}{T}\right)-K_h*K_h\left(\frac{s}{T}-\frac{t}{T}\right)\right\}. \quad \text{(A.2)}$$

Lemma A.3 *Under assumptions 1-5,* $W\left(T\right)\xrightarrow{\mathcal{L}}N\left(0,V^*\right)$, *as* $h\to 0$ *and* $Th^{3/2}\to\infty$, *where* $V^* = 2\left\|2K-K*K\right\|_2^2$.

Note that

$$\begin{aligned} W\left(T\right) &= T^{-1}h^{1/2}\sum_{t\neq s}J\left(t/T\right)^{-2}\ell'\left(u_t,v_t;\theta\right)\ell'\left(u_s,v_s;\theta_s\right) \\ &\quad \left\{2K_h\left(\frac{s}{T}-\frac{t}{T}\right)-K_h*K_h\left(\frac{s}{T}-\frac{t}{T}\right)\right\}. \end{aligned}$$

Let

$$W_{ts} = T^{-1}h^{1/2}B_T\left(t,s\right)\ell'\left(u_t,v_t;\theta_t\right)\ell'\left(u_s,v_s;\theta_s\right),$$

where

$$B_T(t,s) = b_1(t,s) + b_2(t,s) - b_3(t,s) - b_4(t,s),$$

and

$$\begin{aligned} b_1(t,s) &= 2K_h\left(\frac{s}{T}-\frac{t}{T}\right)J(t/T)^{-2}, \qquad b_2(t,s) = b_1(s,t), \\ b_3(t,s) &= K_h * K_h\left(\frac{s}{T}-\frac{t}{T}\right)J(t/T)^{-2}, \qquad b_4(t,s) = b_3(s,t). \end{aligned}$$

Thus we can write $W(T) = \sum_{t<s} W_{ts}$.To apply proposition 3.2 in de Jong (1987), we need to check:

(i) $Var(W(T)) \to V^*$;

(ii) G_I is of smaller order than $Var(W(T))$;

(iii) G_{II} is of smaller order than $Var(W(T))$;

(iv) G_{IV} is of smaller order than $Var(W(T))$;

$$\begin{aligned} G_I &= \sum_{1\le t\le s\le T} E\left(W_{ts}^4\right), \\ G_{II} &= \sum_{1\le t\le s\le r\le T} \left\{E\left(W_{ts}^2 W_{tr}^2\right) + E\left(W_{st}^2 W_{sr}^2\right) + E\left(W_{rt}^2 W_{rs}^2\right)\right\}, \\ G_{IV} &= \sum_{1\le t\le s\le r\le q\le T} \\ &\quad \left\{E\left(W_{ts}W_{tr}W_{qs}W_{qr}\right) + E\left(W_{ts}W_{tq}W_{rs}W_{rq}\right) + E\left(W_{tr}W_{tq}W_{sr}W_{sq}\right)\right\}. \end{aligned}$$

To simplify the notation, let $\ell_i' = \ell'(u_i, v_i; \theta(i/T))$ and denote the m-fold convolution at t by $K\left(\frac{t}{T}, m\right) = K * \cdots * K\left(\frac{t}{T}\right)$. From direct calculation, we have

$$\begin{aligned}
E\left(b_1^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= E\left[\frac{4}{h^2}\frac{\ell_t'^2\ell_s'^2}{J(t/T)^2}\left\{K^2\left(\frac{\frac{s}{T}-\frac{t}{T}}{h}\right)\right\}\right] \\
&= \frac{4}{h^2}\int\frac{\ell_1'^2}{J(\tau_1)^2}\left\{\int \ell_2'^2 K^2\left(\frac{\tau_2-\tau_1}{h}\right)d\tau_2\right\}d\tau_1 \\
&= \frac{4}{h}\int\frac{\ell_1'^2}{J(\tau_1)^2}\int \ell_1'^2 K^2(z)\,dz\,d\tau_1\,(1+O(h)) \\
&= \frac{4}{h}K(0,2)\,(1+O(h)),
\end{aligned}$$

where $\tau_1 = 1/T$, and $\tau_2 = 2/T$. Similarly,

$$\begin{aligned}
E\left(b_2^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= 4h^{-1}K(0,2)\,(1+O(h)), \\
E\left(b_3^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= h^{-1}K(0,4)\,(1+O(h)), \\
E\left(b_4^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= h^{-1}K(0,4)\,(1+O(h)), \\
E\left(b_1^2(t,s)\,b_2^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= 4h^{-1}K(0,4)\,(1+O(h)), \\
E\left(b_1^2(t,s)\,b_3^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= 2h^{-1}K(0,3)\,(1+O(h)), \\
E\left(b_1^2(t,s)\,b_4^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= 2h^{-1}K(0,3)\,(1+O(h)), \\
E\left(b_2^2(t,s)\,b_3^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= 2h^{-1}K(0,3)\,(1+O(h)), \\
E\left(b_2^2(t,s)\,b_4^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= 2h^{-1}K(0,3)\,(1+O(h)), \\
E\left(b_3^2(t,s)\,b_4^2(t,s)\,\ell_t'^2\ell_s'^2\right) &= h^{-1}K(0,4)\,(1+O(h)).
\end{aligned}$$

Thus,

$$E\left[B_T(t,s)\,\ell_t'^2\ell_s'^2\right] = h^{-1}\left\{16K(0,2) - 16K(0,3) + 4K(0,4)\right\}(1+O(h)).$$

The leading term of $T^{-2}h\sum_{t<s}E\left[B_T(t,s)\,\ell_t'^2\ell_s'^2\right]$ yields

$$V^* = 2\{4K(0,2)-4K(0,3)+K(0,4)\} = 2\int\left[2K(\tau)-K*K(\tau)\right]^2 d\tau.$$

Note that $E\left(b_1(1,2)\,\ell_1'\ell_2'\right)^4 = E\left(b_3(1,2)\,\ell_1'\ell_2'\right)^4 = O\left(h^{-3}\right).$

Then $E\left(W_{12}^4\right) = T^{-4}h^2O\left(h^3\right),$ which implies $G_I = O\left(T^{-2}h^{-1}\right) = o(1).$ Besides, since $E\left(W_{12}^2W_{13}^2\right) = O\left(E\left(W_{12}^4\right)\right) = O\left(T^{-4}h^{-1}\right).$ Thus, $G_{II} = O\left(T^{-1}h^{-1}\right) = o(1).$ For the last condition we need to check the order of $E\left(W_{12}W_{23}W_{34}W_{41}\right).$ By some calculation, we have

$$\begin{aligned}
E\left(b_1^2(1,2)\,b_1^2(2,3)\,b_1^2(3,4)\,b_1^2(4,1)\,\ell_1'^2\ell_2'^2\ell_3'^2\ell_4'^2\right) &= O\left(h^{-1}\right)\\
E\left(b_1^2(1,2)\,b_1^2(2,3)\,b_1^2(3,4)\,b_3^2(4,1)\,\ell_1'^2\ell_2'^2\ell_3'^2\ell_4'^2\right) &= O\left(h^{-1}\right)\\
E\left(b_1^2(1,2)\,b_1^2(2,3)\,b_3^2(3,4)\,b_3^2(4,1)\,\ell_1'^2\ell_2'^2\ell_3'^2\ell_4'^2\right) &= O\left(h^{-1}\right)\\
E\left(b_1^2(1,2)\,b_3^2(2,3)\,b_3^2(3,4)\,b_3^2(4,1)\,\ell_1'^2\ell_2'^2\ell_3'^2\ell_4'^2\right) &= O\left(h^{-1}\right)\\
E\left(b_3^2(1,2)\,b_3^2(2,3)\,b_3^2(3,4)\,b_1^2(4,1)\,\ell_1'^2\ell_2'^2\ell_3'^2\ell_4'^2\right) &= O\left(h^{-1}\right).
\end{aligned}$$

Since terms with other combinations will be of the same order, we conclude that

$$E\left(W_{12}W_{23}W_{34}W_{41}\right) = T^{-4}h^2O\left(h^{-1}\right) = O\left(T^{-4}h\right),$$

and $G_{IV} = O(h) = o(1).$ This completes the proof.

Proof of Theorem 1

Denote $\tilde{\theta}$, and $\hat{\theta}(r/T)$ are maximum likelihood estimator under the null hypothesis, and the local maximum likelihood estimator under the alternative hypothesis, respectively. Then, we can express GLRT statistics as

$$\begin{aligned}\hat{\lambda} &= \sum_{r=1}^{T}\left\{\ell\left(u_r, v_r, \hat{\theta}(r/T)\right) - \ell(u_r, v_r, \theta) - \left[\ell\left(u_r, v_r, \tilde{\theta}\right) - \ell(u_r, v_r, \theta)\right]\right\} \\ &\equiv \hat{\lambda}_1 - \hat{\lambda}_2.\end{aligned}$$

Here $\hat{\lambda}_2$ is the canonical likelihood ratio statistics, thus $\hat{\lambda}_1$ governs the asymptotic distribution of $\hat{\lambda}$.

First, we approximate $\ell\left(u_r, v_r, \hat{\theta}(r/T)\right)$ around θ, then we obtain

$$\begin{aligned}\hat{\lambda}_1 &\approx \sum_{r=1}^{T}\ell'(u_r, v_r, \theta)\left(\hat{\theta}(r/T) - \theta\right) \\ &\quad + \frac{1}{2}\sum_{r=1}^{T}\ell''(u_r, v_r, \theta)\left(\hat{\theta}(r/T) - \theta\right)^2.\end{aligned}$$

Applying Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}-\hat{\lambda}_1 &= -h^{-1}\left\{K(0) - \int K^2(\tau)\,d\tau/2\right\} \\ &\quad - T^{-1}\sum_{t\neq s}\frac{\ell'(u_t, v_t;\theta)}{J(t/T)^2}\ell'(u_s, v_s;\theta_s)\,K_h\left(\left(\frac{t}{T} - \frac{s}{T}\right)\right) \\ &\quad + T^{-1}\sum_{t<s}\frac{\ell'(u_t, v_t;\theta)}{J(t/T)^2}\ell'(u_s, v_s;\theta_s)\times K_h * K_h\left(\left(\frac{s}{T} - \frac{t}{T}\right)\right) \\ &\quad + O_p\left(T^{-1}h^{-2}\right) + o_p\left(h^{-1/2}\right).\end{aligned}$$

Thus,

$$-\hat{\lambda}_1 = -\mu_T - h^{-1/2}W(T)/2 + o_p\left(h^{-1/2}\right).$$

Applying Lemma 3, we have $W(T) \xrightarrow{\mathcal{L}} N(0, V^*)$. Hence,

$$V_T^{-1/2}\left(\hat{\lambda}_1 - \mu_T\right) \xrightarrow{\mathcal{L}} N(0,1),$$

where $V_T = (4h)^{-1} V^{*.}$ Thus, the asymptotic distribution of $\hat{\lambda}$ can be expressed as

$$V_T^{-1/2}\left(\left(\hat{\lambda}_1 - \hat{\lambda}_2\right) - \mu_T + \hat{\lambda}_2\right) \xrightarrow{\mathcal{L}} N(0,1).$$

Because $\hat{\lambda}_2 = O_p(1)$ has faster rate of convergence than $\hat{\lambda}_1 = O_p(h^{-1})$, we have

$$V_T^{-1/2}\left(\hat{\lambda} - \mu_T\right) \xrightarrow{\mathcal{L}} N(0,1).$$

For the second result, note that the distribution $N(d_T, 2d_T)$ is approximately same as the chi-square distribution with degrees of freedom d_T for a sequence $d_T \to \infty$. Letting $d_T = 2\mu_T^2/V_T$ and $r_K = 2\mu_T/V_T$, we have

$$(2d_T)^{-1/2}\left(r_K\hat{\lambda} - d_T\right) \xrightarrow{\mathcal{L}} N(0,1).$$

APPENDIX B

PROOFS OF THEOREMS IN SECTION 3

Proof of Theorem 2: First we decompose $Th^{1/2}\hat{D}$:

$$\begin{aligned} Th^{1/2}\widehat{D} &= h^{1/2}\sum_{t=1}^{T}\left(\widehat{\sigma}_t^2-\sigma^2\right)^2+h^{1/2}\sum_{t=1}^{T}\left(\widehat{\sigma}^2-\sigma^2\right)^2+2h_1^{1/2}\sum_{t=1}^{T}\left(\widehat{\sigma}_t^2-\sigma^2\right)\left(\widehat{\sigma}^2-\sigma^2\right) \\ &= \widehat{J}_1+\widehat{J}_2+\widehat{J}_3. \end{aligned}$$

In the sequel, we will show that Theorem 2 follows from Theorem B1-B3 directly.

Theorem B.1 *Under the assumptions of Theorem 2,* $\widehat{H}\equiv\left(\widehat{J}_1-\widehat{M_H}\right)/\sqrt{\widehat{V}_H}\xrightarrow{d} N(0,1)$.

Theorem B.2 *Under the assumptions of Theorem 2,* $\widehat{J}_2\xrightarrow{p}0.$

Theorem B.3 *Under the assumptions of Theorem 2,* $\widehat{J}_3\xrightarrow{p}0.$

Proof of Theorem B1: To show $\widehat{H}\xrightarrow{d}N(0,1)$, it suffices to show the following two Propositions.

Proposition B.1 *Under the assumptions of Theorem 2*

$$\widehat{J}_1=\widehat{M_H}+2\widetilde{U}+o_p(1),$$

where

$$\widetilde{U}=T^{-1}h^{-1/2}\sum_{s=2}^{T}\sum_{t=1}^{s-1}e_se_t\omega_{ts}$$

and $\omega_{ts}=\int_{-1}^{1}k(u)\,k\left(u+\frac{r-s}{Th}\right)du.$

Proposition B.2 *Under the assumptions of Theorem 2,* $2\widetilde{U}/\sqrt{\widehat{V}_H}\xrightarrow{d}N(0,1).$

Proof of Proposition B.1 We first decompose

$$\begin{aligned}
\hat{J}_1 &= h^{1/2}\sum_{t=1}^{T}\left[\frac{1}{Th}\sum_{s=1}^{T}K\left(\frac{s-t}{Th}\right)e_t\right]^2 \qquad \text{(B.1)}\\
&= \frac{1}{T^{-2}h^{-3/2}}\sum_{t=1}^{T}\sum_{s=1}^{T}K^2\left(\frac{s-t}{Th}\right)e_t^2\\
&\quad +2\frac{1}{T^{-2}h^{-3/2}}\sum_{t=1}^{Th}\sum_{s=-T}^{T}K\left(\frac{t-s}{Th}\right)K\left(\frac{t+s}{Th}\right)e_t^2\\
&\quad +\frac{1}{T^{-2}h^{-3/2}}\sum_{t\neq s\neq r}e_te_sK\left(\frac{r-s}{Th}\right)K\left(\frac{r-t}{Th}\right)\\
&\quad +\frac{2}{T^{-2}h^{-3/2}}\sum_{t\neq s}e_te_sK\left(\frac{s-t}{Th}\right)K(0)\\
&\quad +\frac{2}{T^{-2}h^{-3/2}}\sum_{s=-T}^{-1}\sum_{t=-T,t\neq\pm s}^{2T}\sum_{r=1}^{T}e_te_sK\left(\frac{r-s}{Th}\right)K\left(\frac{r-t}{Th}\right)+op(1)\\
&= C_1+C_2+U_1+R_1+R_2+o_p(1)
\end{aligned}$$

The first two terms determine the asymptotic mean, the third term determines the asymptotic variance, and the remainders are higher order terms.

We further decompose the first term as

$$\begin{aligned}
C_1 &= \frac{1}{Th^{3/2}}\sum_{j=1-T}^{T-1}(1-|j|/T)\,K^2\left(\frac{j}{Th}\right)C(j) \qquad \text{(B.2)}\\
&\quad +\left[\frac{1}{T^{-2}h^{-3/2}}\sum_{t=1}^{T}\sum_{s=1}^{T}K^2\left(\frac{s-t}{Th}\right)e_t^2\right.\\
&\quad \left.-\frac{1}{Th^{3/2}}\sum_{j=1-T}^{T-1}(1-|j|/T)\,K^2\left(\frac{j}{Th}\right)C(j)\right]\\
&= C_{11}+R_{3,}
\end{aligned}$$

where $C(j)=E\left(e_{t-|j|}^2\right)=E\left(e_t^2\right)(1+o_p(1))$. Similarly, we can rewrite the second term as

$$
\begin{aligned}
C_2 &= \frac{1}{Th^{1/2}} \sum_{j=1-T}^{T-1} \left(1 - |j|/T\right) C(j) K\left(\frac{j}{Th}\right) \int_{-1}^{1} K\left(\frac{j}{Th} + 2u\right) du \qquad \text{(B.3)} \\
&\quad + \left\{ 2 \frac{1}{T^{-2}h^{-3/2}} \sum_{t=1}^{Th} \sum_{s=-T}^{T} K\left(\frac{t-s}{Th}\right) K\left(\frac{t+s}{Th}\right) e_t^2 \right. \\
&\quad \left. - \frac{1}{Th^{1/2}} \sum_{j=1-T}^{T-1} \left(1 - |j|/T\right) C(j) K\left(\frac{j}{Th}\right) \int_{-1}^{1} K\left(\frac{j}{Th} + 2u\right) du \right\} \\
&= C_{21} + R_4.
\end{aligned}
$$

We note that $C_{11} + C_{21} = h^{-1/2} C_A E\left(e_t^2\right)\left(1 + o_p(1)\right)$, where we have used the mixing inequality for the $\beta - mixing$ process.

Next, we decompose $U_{1.}$ Define

$$
\begin{aligned}
\phi\left(e_s, e_r, e_t\right) \equiv\ & e_t e_s K\left(\frac{r-s}{Th}\right) K\left(\frac{r-t}{Th}\right) + e_s e_r K\left(\frac{t-r}{Th}\right) K\left(\frac{t-s}{Th}\right) \\
& + e_r e_t K\left(\frac{s-t}{Th}\right) K\left(\frac{s-r}{Th}\right), \\
\phi_t\left(e_s, e_r\right) \equiv\ & \int \phi\left(e_s, e_r, e_t\right) df\left(e_t\right) = e_s e_r K\left(\frac{t-r}{Th}\right) K\left(\frac{t-s}{Th}\right).
\end{aligned}
$$

We can rewrite U_1 as

$$
\begin{aligned}
U_1 &= \frac{1}{3}\frac{1}{T^2h^{3/2}}\sum_{t\neq s\neq r}\phi_{srt} - 2\frac{1}{T^2h^{3/2}}\sum_{s\neq t} e_s e_t K(0)\, K\left(\frac{s-t}{Th}\right) \qquad \text{(B.4)}\\
&\quad + \frac{1}{T^2h^{3/2}}\sum_{s\neq t} e_s e_t \left(\frac{1}{Th}\sum_{r=1}^{T} K\left(\frac{r-t}{Th}\right) K\left(\frac{r-s}{Th}\right) - \omega_{ts}\right)\\
&\quad + \frac{1}{T^2h^{3/2}}\sum_{s\neq t} e_s e_t \omega_{ts}\\
&= R_5 - R_6 - R_7 + 2\tilde{U},
\end{aligned}
$$

where $\phi_{srt} = [\phi(e_s, e_r, e_t) - \phi(e_s, e_r) - \phi(e_r, e_t) - \phi(e_s, e_t)]$. Proposition A.2 follows the following lemma.

Lemma B.1 *Let R_i be defined as in (B1)-(B4), where $i = 1-7$. Then $R_i = o_p(1)$.*

Proof of Lemma B.1 The proofs of $R_i = o_p(1)$ are tedious. We only provide the proof for $i = 5$, which is the most involved to save space. Other proofs are similar. We note that

$$
ER_5^2 = C\frac{1}{T^4h^3}\sum_{1\leq i_1\leq i_2\leq i_3\leq T}\;\sum_{1\leq i_4\leq i_5\leq i_6\leq T} E\phi_{i_1i_2i_3}\phi_{i_4i_5i_6}.
$$

First we consider the case where all indices are different from each other. Given the order of i_s, there are 20 different combinations. we consider the case where $1\leq i_1\leq i_2\leq i_3\leq i_4\leq i_5\leq i_6\leq T$; other cases are similar. Let d_c be the cth largest difference among the adjacent indices. We have

$$\begin{aligned}
&\sum_{\substack{1\le i_1\le i_2\le i_3\le i_4\le i_5\le i_6\le T\\ i_2-i_1=d_1}} \left|E\phi_{i_1i_2i_3}\phi_{i_4i_5i_6}\right| \\
\le & \sum_{\substack{1\le i_1\le i_2\le i_3\le i_4\le i_5\le i_6\le T\\ i_2-i_1=d_1}} C\beta^{\delta/(1+\delta)}\left(d_1\right)W_{i_1\cdots i_6} \\
\le & \; CT\sum_{j=1}^{T} j^4\beta^{\delta/(1+\delta)}\left(j\right),
\end{aligned}$$

where

$$\begin{aligned}
W_{i_1\cdots i_6} = \; & k_{i_1i_2}k_{i_1i_3}k_{i_4i_5}k_{i_4i_6}+k_{i_1i_2}k_{i_1i_3}k_{i_6i_5}k_{i_6i_4}+k_{i_1i_2}k_{i_1i_3}k_{i_5i_6}k_{i_5i_4} \\
& +k_{i_3i_1}k_{i_3i_2}k_{i_4i_5}k_{i_4i_6}+k_{i_3i_1}k_{i_3i_2}k_{i_6i_5}k_{i_6i_4}+k_{i_3i_1}k_{i_3i_2}k_{i_5i_6}k_{i_5i_4} \\
& +k_{i_2i_3}k_{i_2i_1}k_{i_4i_5}k_{i_4i_6}+k_{i_2i_3}k_{i_2i_1}k_{i_6i_5}k_{i_6i_4}+k_{i_2i_3}k_{i_2i1}k_{i_5i_6}k_{i_5i_4i_4i_6}
\end{aligned}$$

and we have used the mixing inequality for the $\beta - mixing$ process.

Similarly,

$$\sum_{\substack{1\le i_1\le i_2\le i_3\le i_4\le i_5\le i_6\le T\\ i_6-i_5=d_1}} \left|E\phi_{i_1i_2i_3}\phi_{i_4i_5i_6}\right| \le CT\sum_{j=1}^{T} j^4\beta^{\delta/(1+\delta)}\left(j\right),$$

$$\sum_{\substack{1\le i_1\le i_2\le i_3\le i_4\le i_5\le i_6\le T\\ i_2-i_1=d_2 \; or \; i_6-i_5=d_2}} \left|E\phi_{i_1i_2i_3}\phi_{i_4i_5i_6}\right| \le CT^2h\sum_{j=1}^{T} j^3\beta^{\delta/(1+\delta)}\left(j\right),$$

$$\sum_{\substack{1\le i_1\le i_2\le i_3\le i_4\le i_5\le i_6\le T\\ i_2-i_1=d_3 \; or \; i_6-i_5=d_3}} \left|E\phi_{i_1i_2i_3}\phi_{i_4i_5i_6}\right| \le CT^3h^2\sum_{j=1}^{T} j^2\beta^{\delta/(1+\delta)}\left(j\right),$$

and

$$
\begin{aligned}
&\sum_{\substack{1\le i_1\le i_2\le i_3\le i_4\le i_5\le i_6\le T\\ \{i_2-i_1,i_6-i_5\}=\{d_4,d_5\}}} |E\phi_{i_1i_2i_3}\phi_{i_4i_5i_6}| \\
\le\;& \sum_{\substack{1\le i_1\le i_2\le i_3\le i_4\le i_5\le i_6\le T\\ \{i_2-i_1,i_6-i_5\}=\{d_4,d_5\}}} C\left[\beta^{\delta/(1+\delta)}(i_3-i_2)+\beta^{\delta/(1+\delta)}(i_4-i_3)+\beta^{\delta/(1+\delta)}(i_5-i_4)\right] \\
\le\;& CT^3h^2\sum_{j=1}^{T} j^2\beta^{\delta/(1+\delta)}(j).
\end{aligned}
$$

For the cases where indices are not distinct from each other, we have

$$
\sum_{\substack{1\le s,t,r,i,j\le T\\ s,t,r,i,j\ different}} |E\phi_{str}\phi_{sij}| \le CT^3h^2\sum_{j=1}^{T} j\beta^{\delta/(1+\delta)}(j),
$$

$$
\sum_{\substack{1\le s,t,r,i\le T\\ s,t,r,i\ different}} |E\phi_{str}\phi_{sij}| \le CT^3h^2\sum_{j=1}^{T} \beta^{\delta/(1+\delta)}(j),
$$

and $\sum_{1\le s\le t\le r\le T}|E\phi_{str}^2| = O(T^3h^2)$. Then $R_5 = o_p(1)$ follows from Chebyshev inequality.

This completes the proof of Proposition B.1.

Proof of Proposition B.2 Let

$$
R_s = \frac{1}{Th^{-1/2}}\sum_{r=1}^{s-1} e_se_r\omega_{rs}.
$$

We apply Brown's (1971) limit theorem, which states $var\left(2\widetilde{U}\right)^{-1/2}2\widetilde{U}\xrightarrow{d}N(0,1)$ if

$$
var\left(2\widetilde{U}\right)^{-1}\sum_{s=1}^{T}(2R_s)^2\,\mathbf{1}\left[|2R_s|>\eta\cdot var\left(2\widetilde{U}\right)^{1/2}\right]\to 0\quad \forall\eta>0, \tag{B.5}
$$

$$var\left(2\widetilde{U}\right)^{-1}\sum_{s=1}^{T}E\left[\left(2R_s\right)^2|\mathcal{F}_{s-1}\right]\xrightarrow{p}1.$$

(B.6)

First, we compute the variance

$$\begin{aligned} var\left(2\widetilde{U}\right) &= \frac{4}{T^2h}\sum_{s=1}^{T}\sum_{r=1}^{s-1}E\left(e_s^2e_t^2\omega_{ts}^2\right) \\ &\quad +\frac{4}{T^2h}\sum_{s=1}^{T}\sum_{r_1=1}^{s-1}\sum_{r_2=1,r_1\neq r_2}^{s-1}E\left(e_s^2e_{r_1}e_{r_2}\omega_{r_1s}\omega_{r_2s}\right) \\ &= V_1+V_{2.} \end{aligned} \tag{B.7}$$

For the first term, we have

$$\begin{aligned} V_1 &= \frac{4}{T^2h}\sum_{s=1}^{T}\sum_{r=1}^{s-1}\omega_{ts}^2\Omega^2 \\ &= 4C_B\Omega^2, \end{aligned} \tag{B.8}$$

where we have used the change of variables and the mixing inequality.

For the second term,

$$\begin{aligned} V_2 &= \frac{4}{Th}\sum_{j=1}^{T-1}\sum_{l=1,j\neq l}^{T-1}\left(1-j/T\right)\left(1-l/T\right)C\left(j,l\right) \\ &\times\int_{-1}^{1}k\left(u\right)k\left(u+\frac{j}{Th}\right)du\int_{-1}^{1}k\left(u\right)k\left(u+\frac{l}{Th}\right)du \\ &= o\left(1\right) \end{aligned} \tag{B.9}$$

where $C\left(j,l\right)=E\left[\left(e_s^2-\Omega\right)e_{s-j}e_{s-l}\right]$, and we have used the change of variables and the mixing inequality. Given (B8) and (B9), we have $var\left(2\widetilde{U}\right) = O(1)$.

We now verify equation (B5). Since we have

$$\begin{aligned}\sum_{s=2}^{T} E\left(R_s^4\right) &= \frac{1}{T^4h^2}\sum_{s=2}^{T} E\left(\sum_{i=1}^{s-1} e_i^4 e_s^4 \omega_{is}^4 + 6\sum_{1\leq i\leq j\leq s} e_i^2 e_s^2 e_j^2 e_s^2 \omega_{is}^2 \omega_{js}^2\right.\\ &\quad +4\sum_{t=1}^{s-1}\sum_{1\leq i\leq j\leq s} e_t^2 e_s^2 e_i e_s e_j e_s \omega_{ts}^2 \omega_{is}\omega_{js}\\ &\quad \left. +4\sum_{1\leq i\leq j\leq s, 1\leq t\leq e\leq s} e_i e_s e_j e_s e_t e_s e_r e_s \omega_{is}\omega_{js}\omega_{ts}\omega_{rs}\right)\\ &= O\left(T^{-2}h^{-1}\right) + O\left(T^{-1}\right) + O(h) + O(h),\end{aligned}$$

$\left[var\left(2\widetilde{U}\right)\right]^{-2} \sum_{s=1}^{T} E\left(R_s^4\right) \to 0$ and (B5) holds.

Next we verify condition (B6). Let $W_s = \sum_{r=1}^{s-1} e_r \omega_{ts}$. Then we have

$$\begin{aligned}E\left(R_s^2|\mathcal{F}_{s-1}\right) &= \frac{1}{T^2h} W_s^2 E\left(e_s^2|\mathcal{F}_{s-1}\right)\\ &= \frac{1}{T^2h} W_s^2\left[E\left(e_s^2|\mathcal{F}_{s-1}\right) - \Omega\right]\\ &\quad +\frac{1}{T^2h} W_s^2 \Omega\\ &= V_{1s} + R_{1s.}\end{aligned} \tag{B.10}$$

We further decompose

$$\begin{aligned}R_{1s} &= \frac{1}{T^2h}\left[W_s^2\Omega - E\left(W_s^2\Omega\right)\right]\\ &\quad +\frac{1}{T^2h} E\left(\sum_{r=1}^{s-1} e_r^2 \Omega \omega_{rs}^2\right)\\ &= R_{2s} + \frac{1}{T^2h}\Omega^2 \sum_{r=1}^{s-1}\omega_{rs}.\end{aligned} \tag{B.11}$$

Then we write

$$
\begin{aligned}
R_{2s} &= \frac{1}{T^2 h}\sum_{r=1}^{s-1}\left[e_r^2\Omega - E\left(e_r^2\right)\right]\omega_{rs}^2 \\
&\quad + \frac{2}{T^2 h}\sum_{r_1=1}^{s-1}\sum_{r_2=1}^{r_1} e_{r_1}\Omega e_{r_2}\omega_{r_1 s}\omega_{r_2 s} \\
&= V_{2s} + V_{3s.}
\end{aligned} \tag{B.12}
$$

It follows from (B10)-(B12) that $\sum_{s=1}^{T}\left\{E\left[(2R_s)^2|\mathcal{F}_{s-1}\right] - E\left[(2R_s)^2\right]\right\}$

$= \sum_{i=1}^{3}\sum_{s=1}^{T}4V_{is} + V_{1.}$ It suffices to show Lemmas A.2-A.4 below, which imply $E\left|\sum_{s=1}^{T}E\left[(2R_s)^2|\mathcal{F}_{s-1}\right] - E\left[(2R_s)^2\right]\right|^2 = o(1)$. Thus, condition (B6) holds and so $2\widetilde{U}/\sqrt{\widehat{V}_h} \xrightarrow{d} N(0,1)$ by Brown's (1971) theorem.

This completes the proof of the Theorem B.1.

Lemma B.2 *Let V_{1s} be defined as in (B10). Then* $E\left(\sum_{s=1}^{T}V_{1s}\right)^2 = o(1)$.

Lemma B.3 *Let V_{2s} be defined as in (B12). Then* $E\left(\sum_{s=1}^{T}V_{2s}\right)^2 = o(1)$.

Lemma B.4 *Let V_{3s} be defined as in (B12). Then* $E\left(\sum_{s=1}^{T}V_{3s}\right)^2 = o(1)$.

Proof of Lemma B.2 Let

$$
\begin{aligned}
D\left(e_{r_1}, e_s, e_{r_2}\right) &= e_{r1}\left[E\left(e_s^2|\mathcal{F}_{s-1}\right) - \Omega\right]e_{r_2}\omega_{r_1 s}\omega_{r_2 s} \\
&\quad + e_{r2}\left[E\left(e_{r1}^2|\mathcal{F}_{r_1-1}\right) - \Omega\right]e_s\omega_{r_1 r_2}\omega_{r_1 s} \\
&\quad + e_s\left[E\left(e_{r_2}^2|\mathcal{F}_{r_2-1}\right) - \Omega\right]e_{r_1}\omega_{r_2 s}\omega_{r_1 r_2.}
\end{aligned}
$$

Then we have

$$
E\left(\sum_{s=1}^{T}V_{1s}\right)^2 = CT^{-4}h^{-2}E\left[\sum_{s\neq r_1\neq r_2}D\left(e_{r_1}, e_s, e_{r_2}\right)\right]^2 = O\left(T^{-1}\right) = o(1),
$$

where we have used a similar argument as that for $R_5 = o_p(1)$.

Proof of Lemma B.3 We have

$$
\begin{aligned}
E\left(\sum_{s=1}^{T} V_{2s}\right)^2 &= T^{-4}h^{-2}E\left\{\sum_{r=1}^{T}\sum_{r=1}^{s-1}\left[e_r^2\Omega - E\left(e_r^2\Omega\right)\right]\omega_{rs}^2\right\}^2 \\
&= T^{-2}E\left\{\sum_{r=1}^{T}\left[e_r^2\Omega - E\left(e_r^2\Omega\right)\right]T^{-1}h^{-1}\sum_{j=1}^{T-1}\left(1-\frac{j}{T}\right)\omega^2\left(\frac{j}{Th}\right)\right\}^2 \\
&= T^{-2}E\sum_{r=1}^{T}\left[e_r^2\Omega - E\left(e_r^2\Omega\right)\right]^2\int\omega(u)\,du \\
&\quad +T^{-2}\sum_{r\neq s}E\left[e_r^2\Omega - E\left(e_r^2\Omega\right)\right] \\
&\quad \times\left[e_s^2\Omega - E\left(e_s^2\Omega\right)\right] + o(1) \\
&= O\left(T^{-1}\right) = o(1).
\end{aligned}
$$

Proof of Lemma B.4 We have

$$
\begin{aligned}
E\left(\sum_{s=1}^{T} V_{3s}\right)^2 &= T^{-4}h^{-2}E\left(\sum_{s=1}^{T}\sum_{r_1=1}^{s-1}\sum_{r_2=1}^{r_1-1}e_{r_1}\Omega e_{r_2}\omega_{r_1 s}\omega_{r_2 s}\right)^2 \\
&= T^{-2}E\left(\sum_{r_1=1}^{T-1}\sum_{r_2=1}^{r_1-1}e_{r_1}\Omega e_{r_2}b_{r_1 r_2}\right)^2 + o(1) \\
&= T^{-2}E\sum_{r_1=1}^{T-1}\sum_{r_2=1}^{r_1-1}e_{r_1}^2\Omega^2 e_{r_2}^2 b_{r_1 r_2}^2 \\
&\quad +T^{-2}E\sum_{r_1=1}^{T-1}\sum_{r_2=1}^{r_1-1}\sum_{r_3=1, r_3\neq r_2}^{r_1-1}e_{r_1}^2\Omega^2 e_{r_2}e_{r_3}b_{r_1 r_2}b_{r_1 r_3} \\
&= O(h) + O\left(T^{-1}\right) = o(1),
\end{aligned}
$$

where $b_{r_1 r_2} = \int\int k(u)\,k(u+v)\,du\int k(u)\,k\left(u+v+\frac{r_1-r_2}{Th}\right)dudv.$

Proof of Theorem B.2 We have

$$\begin{aligned}\widehat{J}_2 &= h^{1/2}\sqrt{T}\left(\widehat{\sigma^2}-\sigma^2\right)\sqrt{T}\left(\widehat{\sigma^2}-\sigma^2\right)\\ &= o_p(1),\end{aligned}$$

where we have used the fact $\sqrt{T}\left(\widehat{\sigma 2}-\sigma^2\right)=O_p(1)$.

Proof of Theorem B.3 We have

$$\begin{aligned}\widehat{J}_3 &= -2h^{1/2}\sqrt{T}\left(\widehat{\sigma^2}-\sigma^2\right)\left(\frac{1}{Th}\sum_{t=1}^{T}K\left(\frac{s-t}{Th}\right)\right)\\ &\quad \left(\frac{1}{Th}\sum_{s=1}^{T}K\left(\frac{s-t}{Th}\right)\right)^{-1}\frac{1}{\sqrt{T}}\sum_{s=1}^{T}e_t\\ &= o_p(1),\end{aligned}$$

where we have used the fact $\sqrt{T}\left(\widehat{\sigma^2}-\sigma^2\right)=O_p(1)$ and $\sum_{t=1}^{T}K\left(\frac{s-t}{Th}\right)=O_p(1)$.

Proof of Lemma 1

We have

$$\begin{aligned}\widehat{\widehat{\sigma^2}}(z)-\sigma^2(z) &= \left\{\frac{1}{Th}\sum_{i=1}^{T}K\left(\frac{Z_i-z}{h}\right)\left(\hat{r}(Z_i)-\sigma^2(z)-\sigma^{2^{(1)}}(z)(Z_i-z)\right)\right\}\\ &\quad \{1+o_p(1)\}. \qquad \text{(B.13)}\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}\hat{g}(z)-g(z) &= \left\{\frac{1}{Th}\sum_{i=1}^{T}K\left(\frac{Z_i-z}{h}\right)\left(Y_i-g(z)-g^{^{(1)}}(z)(Z_i-z)\right)\right\}\{1+o_p(1)\}\\ &= \frac{1}{Th}\sum_{i=1}^{T}K\left(\frac{Z_i-z}{h}\right)\sigma(Z_i)\varepsilon_i+\frac{1}{2}h^2\mu_2 g^{(2)}(z)+o_p(1), \qquad \text{(B.14)}\end{aligned}$$

where $\mu_2=\int_{-1}^{1}z^2K(z)\,dz$.

Note that

$$\begin{aligned} \hat{r}(Z_i) &= (Y_i - \hat{g}(Z_i))^2 = (\sigma(Z_i)\varepsilon_i + g(Z_i) - \hat{g}(Z_i))^2 \\ &= \sigma^2(Z_i)\varepsilon_i^2 + 2\sigma(Z_i)\varepsilon_i(g(Z_i) - \hat{g}(Z_i)) + (g(Z_i) - \hat{g}(Z_i))^2 \end{aligned}$$

It follows from (B13) that

$$\hat{\sigma}^2(z) - \sigma^2(z) = [I_1 + I_2 - I_3 + I_4] + O(h),$$

where

$$\begin{aligned} I_1 &= \frac{1}{Th}\sum_{i=1}^{T} K\left(\frac{Z_i - z}{h}\right)\left(\sigma^2(Z_i) - \sigma^2(z) - \sigma^{2^{(1)}}(z)(Z_i - z)\right), \\ I_2 &= \frac{1}{Th}\sum_{i=1}^{T} K\left(\frac{Z_i - z}{h}\right)\sigma^2(Z_i)\left(\varepsilon_i^2 - 1\right), \\ I_3 &= \frac{2}{Th}\sum_{i=1}^{T} K\left(\frac{Z_i - z}{h}\right)\sigma(Z_i)\varepsilon_i(\hat{g}(Z_i) - g(Z_i)), \\ I_4 &= \frac{1}{Th}\sum_{i=1}^{T} K\left(\frac{Z_i - z}{h}\right)(\hat{g}(Z_i) - g(Z_i))^2. \end{aligned}$$

In the sequel, we will show:

(a) $I_1 = \frac{1}{2}\sigma^{2^{(2)}}(z)h^2\mu_2 + O(h^3)$.

(b) $I_2 = O(1/Th)$.

(c) $I_3 = o_p\left(\frac{1}{Th^2}\right) + O_p\left(h^2/\sqrt{Th}\right) + O_p\left(\frac{1}{Th}\right)$.

(d) $I_4 = o_p\left(\frac{1}{T^{3/2}h^2} + \frac{1}{T} + \frac{1}{Th^2}\right) + O(h^4) + O\left(\frac{h^3}{T^2}\right)$.

It is easy to see that the Lemma 1 follows from (a)–(d) directly.

(a) For I_1, we can show that

$$
\begin{aligned}
I_1 &= \frac{1}{Th}\sum_{i=1}^{T} K\left(\frac{Z_i - z}{h}\right)\left(\sigma^2\left(Z_i\right) - \sigma^2\left(z\right) - \sigma^{2^{(1)}}\left(z\right)\left(Z_i - z\right)\right) \\
&\approx \frac{1}{h}\int_0^1 \left(\sigma^2\left(u\right) - \sigma^2\left(z\right) - \sigma^{2^{(1)}}\left(z\right)\left(u - z\right)\right) K\left(\frac{u-z}{h}\right) du \\
&= \int_{-z/h}^{(1-z)/h} \left(\sigma^2\left(z + hv\right) - \sigma^2\left(z\right) - \sigma^{2^{(1)}}\left(z\right) hv\right) K\left(v\right) dv \\
&= \int_{-1}^{1} \left(\frac{1}{2}\sigma^{2^{(2)}}\left(z\right) h^2 v^2 + O\left(h^3\right)\right) K\left(v\right) dv \\
&= \frac{1}{2}\sigma^{2^{(2)}}\left(z\right) h^2 \mu_2 + O\left(h^3\right),
\end{aligned}
$$

(b) Define $R_k =$cov$(\varepsilon_i^2 - 1, \varepsilon_k^2 - 1)$ for any i and k. For I_2, we know that $\mathrm{E}(I_2) = 0$, and

$$
\begin{aligned}
\mathrm{Var}\left(I_2\right)^2 &= \frac{1}{T^2h^2}\sum_{i=1}^{T}\sum_{j=1}^{T} K\left(\frac{Z_i - z}{h}\right) K\left(\frac{Z_j - z}{h}\right)\sigma^2\left(Z_i\right)\sigma^2\left(Z_j\right) R_j \\
&= \frac{1}{T^2h^2} R_0 \sum_{i=1}^{T} K^2\left(\frac{Z_i - z}{h}\right)\sigma^4\left(Z_i\right) \\
&\quad + \frac{2}{Th^2}\sum_{j=1}^{T}\left(1 - j/T\right) R_j \left(K\left(\frac{z_1 - z}{h}\right)\sigma^2\left(z_1\right) K\left(\frac{Z_j - z}{h}\right)\sigma^2\left(Z_j\right)\right) \\
&= A_1 + A_2
\end{aligned}
$$

$$\begin{aligned}
A_1 &= \frac{1}{T^2h^2} R_0 \sum_{i=1}^{T} K^2\left(\frac{Z_i - z}{h}\right)\sigma^4(Z_i) \\
&\approx \frac{1}{Th^2} R_0 \int_0^1 \sigma^4(z) K^2\left(\frac{u-t}{h}\right) du \\
&= \frac{1}{Th} R_0 \int \sigma^4(z+hv) K^2(v)\, dv \\
&= \frac{1}{Th} R_0 \int \left(\sigma^4(z) + \sigma^{4^{(1)}}(z) hv + O\left(h^2\right)\right) K^2(v)\, dv \\
&= \frac{1}{Th} R_0 \sigma^4(z) \nu_0 + O(h),
\end{aligned}$$

where $\nu_0 = \int_{-1}^{1} K^2(z)\, dz$.

It follows from assumption 4 and Lemma 1 Yoshihara (1976) that

$$\begin{aligned}
&\left| \mathrm{E}\left(K\left(\frac{z_1 - z}{h}\right)\sigma^2(z_1)\left(\varepsilon_1^2 - 1\right) K\left(\frac{Z_j - z}{h}\right)\sigma^2(Z_j)\left(\varepsilon_j^2 - 1\right)\right)\right| \\
\leq\ & 4M_T^{1/(1+\delta)} \left|\beta(z)\right|^{\delta/(1+\delta)},
\end{aligned}$$

where $M_T = \mathrm{E}\Big(K\left(\frac{z_1-z}{h}\right)\sigma^2(z_1)\left(\varepsilon_1^2-1\right) K\left(\frac{Z_j-z}{h}\right)\sigma^2(Z_j)\left(\varepsilon_j^2-1\right)\Big)^{1+\delta} = O\left(h^2\right) = o(h)$. Thus $A_2 = o\left((Th)^{-1}\right)$. Now, we know that

$$\mathrm{Var}(I_2) = \frac{1}{Th}\sigma^4(z) R_0 \nu_0 (1 + o(1)).$$

(c) For I_3, it follows from (B14) that

$$\begin{aligned}
I_3 &\sim \frac{2}{T^2h^2} \sum_{i=1}^{T} \sum_{j=1}^{T} K\left(\frac{Z_i - z}{h}\right) K\left(\frac{Z_j - Z_i}{h}\right) \\
&\quad \sigma(Z_i)\varepsilon_i \left\{\sigma(Z_j)\varepsilon_j + g(Z_j) - g(Z_i) - g^{^{(1)}}(Z_i)(Z_j - Z_i)\right\} \\
&= \frac{2}{T^2h^2} \sum_{i=1}^{T-1} \sum_{j>i}^{T} \varphi_{ij} + O_p\left(\frac{1}{Th}\right),
\end{aligned}$$

where $\varphi_{ij} = \psi_{ij} + \psi_{ji}$, and

$$\begin{aligned}\psi_{ij} &= K\left(\frac{Z_i - z}{h}\right) K\left(\frac{Z_j - Z_i}{h}\right) \\ &\quad \sigma(Z_i)\,\varepsilon_i \left\{\sigma(Z_j)\,\varepsilon_j + g(Z_j) - g(Z_i) - g^{(1)}(Z_i)(Z_j - Z_i)\right\}.\end{aligned}$$

Performing Hoeffding's projection decomposition of U-statistics, we express

$$I_3 \sim \frac{2}{T^2h^2}\sum_{i=1}\sum_{j>i}\{\varphi_{ij} - \varphi_i - \varphi_j\} + \frac{2}{Th^2}\sum_{i=1}^{T}\varphi_i + O_p\left(\frac{1}{Th}\right), \tag{B.15}$$

where

$$\begin{aligned}\varphi_i &= \sigma(Z_i)\,\varepsilon_i K\left(\frac{Z_i - z}{h}\right)\int K\left(\frac{a - Z_i}{h}\right)\left\{g(a) - g(Z_i) - g^{(1)}(Z_i)(a - Z_i)\right\} da \\ &= h^3\sigma(Z_i)\,\varepsilon_i K\left(\frac{Z_i - z}{h}\right) g^{(2)}(Z_i)\,\nu_2 + o_p\left(h^3\right),\end{aligned}$$

where $\nu_2 = \int_{-1}^{1} z^2K^2(z)\,dz$.

By assumption 2, we know that the second term of (B15) is $O_p\left(h^{3/2}/\sqrt{T}\right)$.

It follows from lemma A(ii) of Hjellvik et al. (1996) that for any $\varepsilon > 0$ and $\varepsilon > 0$,

$$P\left(\sum_{i=1}\sum_{j>i}\{\varphi_{ij} - \varphi_i - \varphi_j\} > \varepsilon\right) \le cT^2E\left(\sum_{i=1}\sum_{j>i}\{\varphi_{ij} - \varphi_i - \varphi_j\}\right)^2 = O\left(T^2\right).$$

Thus, the first term on the right hand side of I_3 is $o_p\left(\frac{1}{Th^2}\right)$.

(d) For I_4, we apply asymptotic expression (B14) directly.

$$\begin{aligned}
I_4 &\approx \frac{1}{Th}\sum_{i=1}^{T} K\left(\frac{Z_i - z}{h}\right)\left\{\frac{1}{Th}\sum_{j=1}^{T} K\left(\frac{Z_j - Z_i}{h}\right)\sigma\left(Z_j\right)\varepsilon_j + \frac{1}{2}h^2\mu_2 g^{(2)}\left(z_i\right)\right\}^2 \\
&= \frac{1}{T^3h^3}\sum_{i=1}^{T}\sum_{j=1}^{T}\sum_{k=1}^{T} K\left(\frac{Z_i - z}{h}\right)K\left(\frac{Z_j - Z_i}{h}\right)K\left(\frac{Z_k - Z_i}{h}\right)\sigma\left(Z_j\right)\varepsilon_j\sigma\left(Z_k\right)\varepsilon_k \\
&\quad + \frac{h^3}{4T}\sum_{i=1}^{T} K\left(\frac{Z_i - z}{h}\right)\left[\mu_2 g^{(2)}\left(Z_i\right)\right]^2 \\
&\quad + \frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T} K\left(\frac{Z_i - z}{h}\right)W\left(\frac{Z_j - Z_i}{h}\right)\sigma\left(Z_j\right)\varepsilon_j\mu_2 g^{(2)}\left(Z_i\right)
\end{aligned}$$

By the similar argument as above, the first term on the right hand side of I_4 is $o_p\left(\frac{1}{T^{3/2}h^2} + \frac{1}{T} + \frac{1}{Th^2}\right)$. By Riemann sum approximation of integral, we know the second term on the right hand side of I_4 is $O\left(h^4\right)$. It is easy to show that the third term on the right hand side of I_4 is $O\left(\frac{h^3}{T^2}\right)$.

Proof of Theorem 3 The proof of Theorem 3 is similar to that of Theorem 1. We only need to replace $\widehat{\sigma^2}$ by $\widehat{\widehat{\sigma^2}}$ to prove Theorem 3.