SYMMETRIC PROJECTIONS OF THE ENTROPY REGION

A Dissertation

by

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ABSTRACT

Entropy inequalities play a central role in proving converse coding theorems for network information theoretic problems. This thesis studies two new aspects of entropy inequalities. First, inequalities relating average joint entropies rather than entropies over individual subsets are studied. It is shown that the closures of the average entropy regions where the averages are over all subsets of the same size and all sliding windows of the same size respectively are identical, implying that averaging over sliding windows always suffices as far as unconstrained entropy inequalities are concerned. Second, the existence of non-Shannon type inequalities under partial symmetry is studied using the concepts of Shannon and non-Shannon groups. A complete classification of all permutation groups over four elements is established. With five random variables, it is shown that there are no non-Shannon type inequalities under cyclic symmetry.

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1. INTRODUCTION

1.1 Motivation

Entropy inequalities play a central role in proving converse coding theorems for network information-theoretic problems. An entropy inequality which has found many applications [3, 4] in network information theory is an inequality first proved by Han [1]. Let $(\mathsf{X}_i : i \in \mathcal{N}_n)$ be a collection of n jointly distributed discrete random variables, where $\mathcal{N}_n := \{1, \ldots, n\}$. For any $\alpha \in \mathcal{N}_n$, let

$$\overline{h}_\alpha := \frac{1}{\begin{pmatrix} n \\ \alpha \end{pmatrix}} \sum_{S \subseteq \mathcal{N}_n : |S| = \alpha} H(\mathsf{X}_S) \tag{1.1}$$

be the average joint entropy, where the average is over all subsets of $\mathcal{N}_n$ of size α. Han's inequality [1] states that for any collection of n jointly distributed discrete random variables $(\mathsf{X}_i : i \in \mathcal{N}_n)$, we have

$$\frac{\overline{h}_n}{n} \leq \frac{\overline{h}_{n-1}}{n-1} \leq \cdots \leq \overline{h}_1 \tag{1.2}$$

i.e., the average joint entropy per element decreases monotonically with the size of the subsets.

Another entropy inequality, which bears striking similarity to Han's inequality, is the so called *sliding-window inequality* first discovered in [2]. As shown in Figure 1.1, consider placing the integers from $\mathcal{N}_n$ clockwise on a circle according to their natural ordering. For any $i \in \mathcal{N}_n$ and $\alpha \in \mathcal{N}_n$, the *sliding window* $W_i^{(\alpha)}$ is defined as the set of α consecutive integers starting from i and going clockwise. (So there are a total of n sliding windows for each $\alpha \in \mathcal{N}_n$.) For any $\alpha \in \mathcal{N}_n$, let

$$\overline{h}_\alpha := \frac{1}{n} \sum_{i=1}^{n} H(\mathsf{X}_{W_i^{(\alpha)}}) \tag{1.3}$$

be the average joint entropy, where the average is over all sliding windows of size α. The sliding-window inequality [2] states that for any collection of n jointly distributed discrete random variables $(\mathsf{X}_i : i \in \mathcal{N}_n)$, we have

$$\frac{\overline{h}_n}{n} \leq \frac{\overline{h}_{n-1}}{n-1} \leq \cdots \leq \overline{h}_1 \tag{1.4}$$

i.e., the average joint entropy per element decreases monotonically with the size of the sliding windows.

As noted in [2], the total averages (1.1) can be obtained from the sliding-window averages (1.3) via a further averaging over all permutations of $\mathcal{N}_n$. Therefore, if a (linear) entropy inequality holds for the sliding-window averages, it must also hold for the total averages. The sliding-window inequality (1.4), however, shows that averaging over sliding windows is both necessary and *sufficient* for achieving the monotonicity of the average entropy per element. A question that remains to be answered is whether the above sufficiency is an isolated coincidence or a universal truth that applies to *all* entropy inequalities.

A central concept for systematic studies of entropy inequalities is *entropy region*, which was first introduced by Yeung [5, Chapter13.1]. A length-$(2^n - 1)$ vector $\boldsymbol{h} = (h_S : \emptyset \neq S \subseteq \mathcal{N}_n)$ is said to be *entropic* if

$$h_S = H(\mathsf{X}_S), \quad \forall \emptyset \neq S \subseteq \mathcal{N}_n. \tag{1.5}$$

The collection of all entropic vectors is called the *entropy region* (over n variables) and is usually denoted by Γ_n^*. As discussed in [5, Chapter13.3], a length-$(2^n - 1)$ vector $\boldsymbol{b} = (b_S : \emptyset \neq S \subseteq \mathcal{N}_n)$ identifies a valid entropy inequality

$$\sum_{\emptyset \neq S \subseteq \mathcal{N}_n} b_S H(X_S) \geq 0 \tag{1.6}$$

if and only if $\boldsymbol{b}^t \boldsymbol{h} \geq 0$ is a valid inequality for every $\boldsymbol{h} \in cl(\Gamma_n^*)$, the closure of Γ_n^*. In literature, this is known as the *geometric view* of entropy inequalities.

For $n \geq 4$, the problem of characterizing $cl(\Gamma_n^*)$ is very challenging (and remains open) due to the existence of the so-called *non-Shannon type inequalities* [6]. Fortunately, the entropy inequalities that we consider here are concerned with average joint entropies rather than entropies over individual subsets of $\mathcal{N}_n$. Towards studying inequalities for average joint entropies, we introduce the concepts of *total-average entropy region* and *sliding-window-average entropy region* below.

A length-n vector $\overline{\boldsymbol{h}} = (\overline{h}_\alpha : \alpha \in \mathcal{N}_n)$ is said to be *total-average entropic* if

$$\overline{h}_\alpha = \frac{1}{\begin{pmatrix} n \\ \alpha \end{pmatrix}} \sum_{S \subseteq \mathcal{N}_n : |S| = \alpha} H(\mathsf{X}_S), \quad \forall \alpha \in \mathcal{N}_n \tag{1.7}$$

for some collection of n jointly distributed discrete random variables $(\mathsf{X}_i : i \in \mathcal{N}_n)$. The collection of all total-average entropic vectors is called the *total-average entropy region*. Mathematically, it is given by the *total-average projection* P_T of Γ_n^*.

Similarly, a length-n vector $\overline{\boldsymbol{h}} = (\overline{h}_\alpha : \alpha \in \mathcal{N}_n)$ is said to be *sliding-window-average entropic* if

$$\overline{h}_\alpha = \frac{1}{n} \sum_{i=1}^{n} H(\mathsf{X}_{W_i^{(\alpha)}}), \quad \forall \alpha \in \mathcal{N}_n \tag{1.8}$$

for some collection of n jointly distributed discrete random variables $(\mathsf{X}_i : i \in \mathcal{N}_n)$. The collection of all sliding-window-average entropic vectors is called the *sliding-window-average entropy region* and is given by the *sliding-window-average projection* P_S of Γ_n^*.

A main result of this thesis is to show that the closures of the above two average entropy regions are, in fact, *identical*, which implies that averaging over sliding windows *always* suffices as far as unconstrained entropy inequalities are concerned. As an application of our result, the sliding-window inequality is immediately implied by Han's inequality.

1.2 Thesis Organization

The rest of the thesis is organized as follows. In Chapter 2, we show that the closures of the total-average entropy region and the sliding-window-average entropy regions are identical. Our proof is based on the general concept of *group-induced symmetric projection*. As a side result, we also show that there are *no* non-Shannon type inequalities for average entropies. Note that this is in sharp contrast to entropies over individual subsets of $\mathcal{N}_n$, which admit an infinite collection of independent non-Shannon type inequalities for $n \geq 4$ [7].

Motivated by the concept of group-induced symmetric projection introduced in Chapter 2, the existence of non-Shannon type inequality under *partial symmetry* is discussed in Chapter 3. This naturally leads to a classification criterion for all permutation groups. We present complete classification results on permutation groups over $n = 4$ and cyclic groups C_4 and C_5.

Finally, in Chapter 4, we conclude the thesis with some remarks on possible future directions.

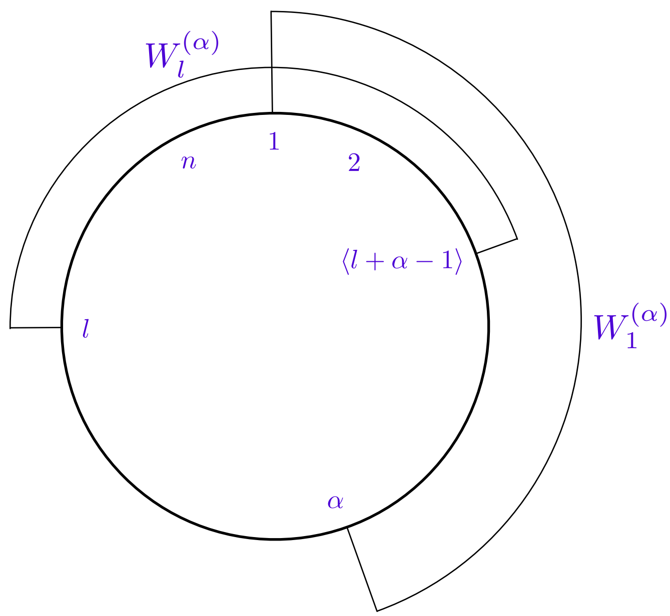

Figure 1.1: An illustration of the sliding windows of length α when the integers $1 \dots n$ are circularly placed based on their natural order.

2. ON THE AVERAGE ENTROPY REGIONS

The main result of this chapter is summarized in the following theorem.

Theorem 1. *Let Γ_n^* and Γ_n be the entropy region and the polymatroid region over n variables, respectively, and let P_T and P_S be the total-average projection and the sliding-window-average projection defined by the linear mappings:*

$$\overline{h}_\alpha = \frac{1}{\binom{n}{\alpha}} \sum_{S \subseteq \mathcal{N}_n : |S| = \alpha} h_S \tag{2.1}$$

and

$$\overline{h}_\alpha = \frac{1}{n} \sum_{i=1}^{n} h_{W_i^{(\alpha)}} \tag{2.2}$$

respectively. Then, for any integer n we have

$$cl(P_T \Gamma_n^*) = cl(P_S \Gamma_n^*) = P_T \Gamma_n. \tag{2.3}$$

As mentioned in the Introduction, the fact that the total-average entropy region and the sliding-window-average entropy regions are identical implies that averaging over sliding windows *always* suffices as far as unconstrained entropy inequalities are concerned. The fact that both average entropy regions are identical to the total-average projection of the polymatroid region implies that there are *no* non-Shannon type inequalities for average entropies. Note that this is in sharp contrast to entropies over individual subsets of $\mathcal{N}_n$, which admit an *infinite* collection of independent non-Shannon type inequalities for $n \geq 4$ [7].

The rest of the chapter is devoted to the proof of the above result. We shall begin with the concept of *group-induced symmetric projections*.

2.1 Group-Induced Symmetric Projections

Let G be a group of permutations over $\mathcal{N}_n$. Consider the group action on the nonempty subsets of $\mathcal{N}_n$ induced by that on the elements of $\mathcal{N}_n$:

$$g(S) = \{g(a) : a \in S\}$$

for any $g \in G$ and $\emptyset \neq S \subseteq \mathcal{N}_n$. Then, the orbits of G forms a partition of all $2^n - 1$ nonempty subsets of $\mathcal{N}_n$. For example, when $G = S_n$, the *symmetry group* over $\mathcal{N}_n$, two subsets S and S' are in the same orbit if and only if $|S| = |S'|$.

Let $O_1, \ldots, O_m$ be the collection of all distinct orbits of G. For any length-$(2^n - 1)$ vector $(h_S : \emptyset \neq S \subseteq \mathcal{N}_n)$, the *orbit averages* can be defined as

$$\overline{h}_\alpha := \frac{1}{|O_\alpha|} \sum_{S \in O_\alpha} h_S \tag{2.4}$$

for any $\alpha \in \mathcal{N}_m$. We call the above projection from $\boldsymbol{h} = (h_S : \emptyset \neq S \subseteq \mathcal{N}_n)$ to $\overline{\boldsymbol{h}} = (\overline{h}_\alpha : \alpha \in \mathcal{N}_m)$ the projection *induced* by G and denote it by P_G.

A set Θ of length-$(2^n - 1)$ vectors $\boldsymbol{h} = (h_S : \emptyset \neq S \subseteq \mathcal{N}_n)$ is said to be *permutation symmetric* if $\boldsymbol{h}_g \in \Theta$ for any $\boldsymbol{h} \in \Theta$ and $g \in S_n$, where $\boldsymbol{h}_g := (h_{g(S)} : \emptyset \neq S \subseteq \mathcal{N}_n)$. We note here that both Γ_n^* (and hence $cl(\Gamma_n^*)$) and Γ_n are *permutation symmetric* for any $n \in \mathcal{N}$. The following result is a simple consequence of the well-known Lagrange's theorem for group actions [8, Chapter 7, Theorem 7.1]:

Lemma 1. *For any convex, permutation symmetric set Θ of length-$(2^n - 1)$ vectors $\boldsymbol{h} = (h_S : \emptyset \neq S \subseteq \mathcal{N}_n)$ and any permutation group G over $\mathcal{N}_n$, we have $P_G\Theta = P_G\Theta'$*

where

$$\Theta' := \{\boldsymbol{h} \in \Theta : h_S = h_{S'}\ \forall S, S' \textit{ in the same orbit of } G\}. \tag{2.5}$$

Proof. Clearly, we have $P_G\Theta \supseteq P_G\Theta'$ since $\Theta \supseteq \Theta'$. To show the opposite inclusion, let $\overline{\boldsymbol{h}} = P_G\boldsymbol{h}$ for some $\boldsymbol{h} \in \Theta$. By assumption the set Θ is permutation symmetric, so we have $\boldsymbol{h}_g \in \Theta$ for any $g \in G$. By the convexity of Θ, the group average $\frac{1}{|G|}\sum_{g\in G}\boldsymbol{h}_g \in \Theta$. Furthermore, for any $k \in \mathcal{N}_m$ and any $S \in O_k$, by the Lagrange's theorem [8, Chapter 7, Theorem 7.1] we have

$$\frac{1}{|G|}\sum_{g\in G} h_{g(S)} = \frac{1}{|O_k|}\sum_{S\in O_k} h_S = \overline{h}_k. \tag{2.6}$$

We thus conclude that $\frac{1}{|G|}\sum_{g\in G}\boldsymbol{h}_g \in \Theta'$ and $P_G\left(\frac{1}{|G|}\sum_{g\in G}\boldsymbol{h}_g\right) = \overline{\boldsymbol{h}}$, i.e., $\overline{\boldsymbol{h}} \in P_G\Theta'$. This completes the proof of the opposite inclusion $P_G\Theta \subseteq P_G\Theta'$. □

For a given permutation group G, directly characterizing $cl(P_G\Gamma_n^*)$ might be difficult. The following simple inner and outer bounds are readily available.

Lemma 2. *For any permutation group G over n variables, we have*

$$P_G cl(\Gamma_n^*) \subseteq cl(P_G\Gamma_n^*) \subseteq P_G\Gamma_n.$$

Proof. The fact that $P_G cl(\Gamma_n^*) \subseteq cl(P_G\Gamma_n^*)$ follows from standard topological arguments [12]. The fact that $cl(P_G\Gamma_n^*) \subseteq P_G\Gamma_n$ follows from the fact that $\Gamma_n^* \subseteq \Gamma_n$ so $cl(P_G\Gamma_n^*) \subseteq cl(P_G\Gamma_n)$ and that Γ_n is *polyhedral* [5, Chapter 14.1] so $cl(P_G\Gamma_n) = P_G\Gamma_n$. □

The polymatroid region Γ_n is polyhedral and fully characterized by the *elemental* *inequalities* [5, Chapter 14.1]:

$$h_{S\cup\{i\}} + h_{S\cup\{j\}} - h_{S\cup\{i,j\}} - h_S \geq 0, \quad \forall i \neq j \in \mathcal{N}_n, S \subseteq \mathcal{N}_n \setminus \{i,j\} \tag{2.7}$$

$$h_{\mathcal{N}_n} - h_{\mathcal{N}_n\setminus\{i\}} \geq 0, \quad \forall i \in \mathcal{N}_n. \tag{2.8}$$

Since Γ_n is convex and permutation symmetric, by By Lemma 1 the outer region $P_G\Gamma_n$ can be obtained by setting $h_S = \overline{h}_\alpha$ for any $S \in O_\alpha$ in the elemental inequalities.

For the cases where we can further show that $P_G\Gamma_n \subseteq P_G cl(\Gamma_n^*)$, the inner and outer bounds in Lemma 2 will match, leading to a precise characterization of $cl(P_G\Gamma_n^*)$. Since both $P_G\Gamma_n$ and $P_G cl(\Gamma_n^*)$ are *convex cones*, to see whether $P_G\Gamma_n \subseteq P_G cl(\Gamma_n^*)$, it suffices to see whether all the *extreme rays* of $P_G\Gamma_n$ are in $P_G cl(\Gamma_n^*)$.

2.2 The Total-Average Projection

When $G = S_n$, the *symmetry group* over $\mathcal{N}_n$, two subsets S and S' are in the same orbit if and only if $|S| = |S'|$. We thus have

$$P_T = P_{S_n} \tag{2.9}$$

i.e., the total-average projection is precisely the group-induced symmetric projection where the underlying group is S_n.

A precise characterization of the total-average projection of the polymatroid region is summarized in the following lemma.

Lemma 3. *For any $n \in \mathcal{N}$, the total-average projection of the polymatroid region*

$P_T\Gamma_n$ is given by the set of length-n vectors $(\overline{h}_\alpha : \alpha \in \mathcal{N}_n)$ satisfying:

$$2\overline{h}_\alpha - \overline{h}_{\alpha-1} - \overline{h}_{\alpha+1} \geq 0, \quad \forall \alpha \in \mathcal{N}_{n-1} \tag{2.10}$$

$$\overline{h}_n - \overline{h}_{n-1} \geq 0 \tag{2.11}$$

where $\overline{h}_0 := 0$. Alternatively, $P_T\Gamma_n$ is the convex polyhedral cone generated by the vectors $\{\boldsymbol{r}_i = (r_{i,1}, \ldots, r_{i,n}) : i \in \mathcal{N}_n\}$, where

$$r_{i,k} = \begin{cases} k, & \text{if } k \leq i \\ i, & \text{if } k > i. \end{cases} \tag{2.12}$$

Proof. Fix $n \in \mathcal{N}$. The polymatroid region Γ_n is the set of length-$(2^n - 1)$ vectors $\boldsymbol{h} = (h_S : \emptyset \neq S \subseteq \mathcal{N}_n)$ satisfying the elemental inequalities (2.7) and (2.8). The polymatroid region Γ_n is convex and permutation symmetric. By Lemma 1, to obtain the projection $P_T\Gamma_n$, we can simply set $h_S = \overline{h}_\alpha$ for any $S \subseteq \mathcal{N}_n$ such that $|S| = \alpha$ in the elemental inequalities in (2.7) and (2.8). Removing the repeated inequalities, we may conclude that $P_T\Gamma_n$ is given by the set of length-n vectors $(\overline{h}_\alpha : \alpha \in \mathcal{N}_n)$ satisfying the inequalities in (2.10) and (2.11).

Denote the convex polyhedral cone generated by the set of vectors $\{\boldsymbol{r}_i : i \in \mathcal{N}_n\}$ by C. It is straightforward to verify that for any $i \in \mathcal{N}_n$, the vector $\boldsymbol{r}_i$ satisfies every inequality from (2.10) and (2.11). We therefore have $C \subseteq P_S\Gamma_n$.

To prove the opposite inclusion, let $\overline{\boldsymbol{h}} \in P_S\Gamma_n$. Since the set of vectors $\{\boldsymbol{r}_i : i \in \mathcal{N}_n\}$ spans the entire $\mathcal{R}^n$, we may write $\overline{\boldsymbol{h}} = \sum_{i=1}^n a_i \boldsymbol{r}_i$ for some real scalars $a_1, \ldots, a_n$. It remains to show that any real scalars $a_1, \ldots, a_n$ such that $\overline{\boldsymbol{h}} = \sum_{i=1}^n a_i \boldsymbol{r}_i$ satisfies every inequality from (2.10) and (2.11) must satisfy $a_i \geq 0$ for all $i \in \mathcal{N}_n$.

Note that by the definition of $\boldsymbol{r}_i$ for $i \in \mathcal{N}_n$, we can write $\overline{h}_\alpha$ explicitly as:

$$\overline{h}_\alpha = \sum_{j=1}^{\alpha} j a_j + \sum_{j=\alpha+1}^{n} \alpha a_j, \quad \forall \alpha \in \mathcal{N}_n. \tag{2.13}$$

By (2.10) and (2.11), we have

$$a_i = 2\overline{h}_i - \overline{h}_{i-1} - \overline{h}_{i+1} \geq 0, \quad \forall i \in \mathcal{N}_{n-1} \tag{2.14}$$

$$a_n = \overline{h}_n - \overline{h}_{n-1} \geq 0. \tag{2.15}$$

This completes the proof that $P_T\Gamma_n \subseteq C$ and hence the entire lemma. □

Note that the extreme rays $\{\boldsymbol{r}_i = (r_{i,1}, \ldots, r_{i,n}) : i \in \mathcal{N}_n\}$ of $P_T\Gamma_n$ can all be realized by a total-average projection of *uniform matroids* [13]. Since all matroids are known to be entropic, we conclude that

$$P_T\Gamma_n \subseteq P_T cl(\Gamma_n^*) \tag{2.16}$$

and hence

$$cl(P_T\Gamma_n^*) = P_T\Gamma_n. \tag{2.17}$$

2.3 The Sliding-Window-Average Projection

When $G = C_n$, the *cyclic group* generated by the permutation $(1\ 2\ 3 \cdots n)$, all sliding windows of the same size form an orbit of G. However, *not* all orbits of C_n are formed by sliding windows. For example, when $n = 4$, the cyclic group C_4 has a total of five orbits:

$$O_1 = \{\{1\}, \{2\}, \{3\}, \{4\}\} \tag{2.18}$$

$$O_2 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\} \tag{2.19}$$

$$O_3 = \{\{1, 3\}, \{2, 4\}\} \tag{2.20}$$

$$O_4 = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 1\}, \{4, 1, 2\}\} \tag{2.21}$$

$$O_5 = \{\{1, 2, 3, 4\}\}. \tag{2.22}$$

While the orbits O_1, O_2, O_4 and O_5 are formed by sliding windows of the same size, the orbit O_3 is not. Therefore, the sliding-window-average projection P_S is given by

$$P_S = P'_S P_{C_n} \tag{2.23}$$

where P'_S is the projection that keeps only the orbits formed by sliding windows of the same size.

Next, we show that the total-average projection of the polymatroid region is, in fact, an outer bound to the sliding-window-average projection of the polymatroid region.

Lemma 4. *For any $n \in \mathcal{N}$, we have*

$$P_S\Gamma_n \subseteq P_T\Gamma_n. \tag{2.24}$$

Proof. By Lemma 3, to show that $P_S\Gamma_n \subseteq P_T\Gamma_n$, it suffices to show that any $\overline{\boldsymbol{h}} = (\overline{h}_\alpha : \alpha \in \mathcal{N}_n) \in P_S\Gamma_n$ must satisfy all n inequalities in (2.10) and (2.11).

Let $\overline{\boldsymbol{h}} = (\overline{h}_\alpha : \alpha \in \mathcal{N}_n) \in P_S\Gamma_n$. Note that for any $\alpha \in \mathcal{N}$, $\overline{h}_\alpha$ is the orbit average of the cyclic group C_n where the orbit is formed by the sliding windows of size α.

The inequality (2.11) can be proved by setting $h_{\mathcal{N}_n} = \overline{h}_n$ and $h_{\mathcal{N}_n \setminus \{i\}} = \overline{h}_{n-1}$ in the elemental inequality (2.8). To prove the inequalities in (2.10), we note that for any sliding window S of size $|S| \leq n-2$ and elements i and j *just* outside of S, the sets $S \cup \{i\}$, $S \cup \{j\}$ and $S \cup \{i,j\}$ are once again sliding windows (of size $|S|+1$, $|S|+1$ and $|S|+2$, respectively). With this simple fact, the inequalities in (2.10) can be proved by setting S to be a sliding window and i and j to be just outside of S in the elemental inequality (2.7). See Figure 2.1 for an illustration of this choice of S and the elements i and j. $\square$

Note that the extreme rays $\{\boldsymbol{r}_i = (r_{i,1}, \ldots, r_{i,n}) : i \in \mathcal{N}_n\}$ of $P_T\Gamma_n$ can all be realized by a sliding-window-average projection of *uniform matroids* [13] as well. Since all matroids are known to be entropic, we conclude that

$$P_T\Gamma_n \subseteq P_S cl(\Gamma_n^*) \tag{2.25}$$

and hence

$$cl(P_S\Gamma_n^*) = P_T\Gamma_n. \tag{2.26}$$

Combining (2.24) and (2.26) completes the proof of Theorem 1.

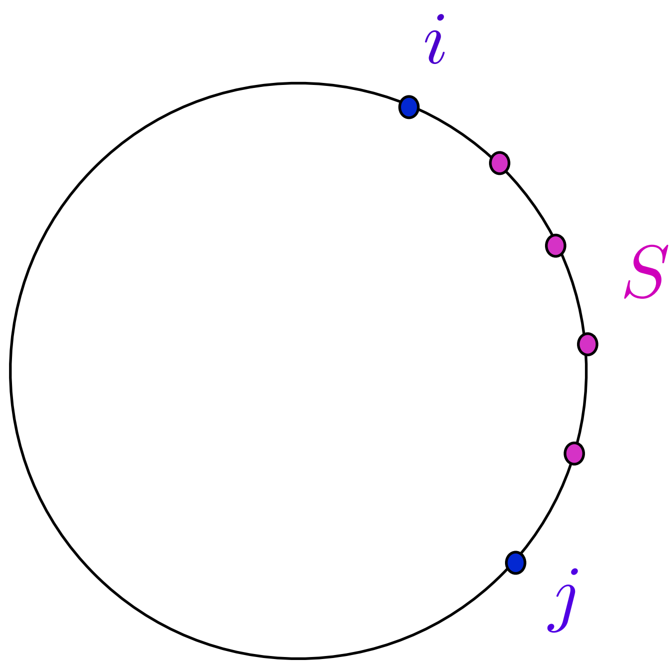

Figure 2.1: Proof of the inequalities in (2.10) by choosing S to be a sliding window and i and j to be just outside of S in the elemental inequality (2.7).

3. EXISTENCE OF NON-SHANNON TYPE INEQUALITIES UNDER PARTIAL SYMMETRY

3.1 Shannon and Non-Shannon Groups

As discussed in Chapter 2, when $G = S_n$ (the *largest* permutation group over $\mathcal{N}_n$), we have

$$P_G cl(\Gamma_n^*) = P_G \Gamma_n \tag{3.1}$$

implying that there are *no* non-Shannon type inequalities under *total* symmetry. On the other hand, when $G = \{(1)\}$ (the *smallest* permutation group over $\mathcal{N}_n$), we have

$$P_G cl(\Gamma_n^*) \subsetneq P_G \Gamma_n \tag{3.2}$$

for $n \geq 4$ due to the existence of non-Shannon type inequalities [6] (when there is no symmetry at all). Between S_n and the identity group $\{(1)\}$, there are many proper subgroups of S_n that represent various types of *partial* symmetry. Our goal in this chapter is to examine the existence of non-Shannon type inequalities under partial symmetry.

Towards this goal, we introduce the following key definition of Shannon and non-Shannon groups for permutation groups.

Definition 1. *Let G be a group of permutations over $\mathcal{N}_n$. We say that G is* Shannon *if*

$$P_G cl(\Gamma_n^*) = P_G \Gamma_n \tag{3.3}$$

and non-Shannon *if*

$$P_G cl(\Gamma_n^*) \subsetneq P_G \Gamma_n. \tag{3.4}$$

As discussed in Chapter 2, when G is a Shannon group, we have

$$cl(P_G \Gamma_n^*) = P_G \Gamma_n \tag{3.5}$$

implying that there are *no* non-Shannon type inequalities for the orbit averages induced by G.

The following simple fact is useful for classifying the proper subgroups of S_n into Shannon and non-Shannon groups.

Fact 1. *All supergroups of a Shannon group is Shannon. Conversely, all subgroups of a non-Shannon group is non-Shannon.*

3.2 The Subgroups of S_4

There are 30 subgroups of S_4, as listed in Table 3.1 and also depicted in Figure 3.1 as in the style of a Hasse diagram. We have the following results on the classification of subgroups of S_4 into Shannon and non-Shannon groups.

Theorem 2. *For symmetry group S_4, its subgroups $V4$, $P1$, $P2$, $P3$, $P4$, d, d' and d'' are Shannon; its subgroups A, B and C are non-Shannon.*

Since the subgroup $V4$ is Shannon, its supergroups D, D', D'', $A4$ and S_4 are all Shannon. Similarly, since the subgroups $P1$, $P2$, $P3$ and $P4$, their supergroups $H1$, $H2$, $H3$ and $H4$ are also Shannon. Conversely, since the subgroups A, B and C are non-Shannon, their subgroups $a1$, $a2$, $b1$, $b2$, $c1$, $c2$, $V1$, $V2$, $V3$ and $\{(1)\}$

are all non-Shannon. Therefore, Theorem 2 provides a *complete* classification of the subgroups of S_4 into Shannon and non-Shannon groups.

To show that the subgroups $V4$, $P1$, $P2$, $P3$, $P4$, d, d' and d'' are Shannon, we use the fact that the *linear rank space* $\mathcal{L}_n$ is an inner bound to the entropy region Γ_n^* [9]. For $n = 4$, the linear rank space $\mathcal{L}_4$ is *completely* characterized by the Shannon type inequalities and the Ingleton inequalities [10]:

$$h_{\{1,2\}} + h_{\{1,3\}} + h_{\{2,3\}} + h_{\{1,4\}} + h_{\{2,4\}} - h_{\{1\}} - h_{\{2\}} - h_{\{3,4\}} - h_{\{1,2,3\}} - h_{\{1,2,4\}} \geq 0 \tag{3.6}$$

$$h_{\{1,3\}} + h_{\{1,2\}} + h_{\{2,3\}} + h_{\{1,4\}} + h_{\{3,4\}} - h_{\{1\}} - h_{\{3\}} - h_{\{2,4\}} - h_{\{1,2,3\}} - h_{\{1,3,4\}} \geq 0 \tag{3.7}$$

$$h_{\{1,4\}} + h_{\{1,2\}} + h_{\{2,4\}} + h_{\{1,3\}} + h_{\{3,4\}} - h_{\{1\}} - h_{\{4\}} - h_{\{2,3\}} - h_{\{1,2,4\}} - h_{\{1,3,4\}} \geq 0 \tag{3.8}$$

$$h_{\{2,3\}} + h_{\{1,2\}} + h_{\{1,3\}} + h_{\{2,4\}} + h_{\{3,4\}} - h_{\{2\}} - h_{\{3\}} - h_{\{1,4\}} - h_{\{1,2,3\}} - h_{\{2,3,4\}} \geq 0 \tag{3.9}$$

$$h_{\{2,4\}} + h_{\{1,2\}} + h_{\{2,3\}} + h_{\{1,4\}} + h_{\{3,4\}} - h_{\{2\}} - h_{\{4\}} - h_{\{1,3\}} - h_{\{1,2,4\}} - h_{\{2,3,4\}} \geq 0 \tag{3.10}$$

$$h_{\{3,4\}} + h_{\{1,3\}} + h_{\{1,4\}} + h_{\{2,3\}} + h_{\{2,4\}} - h_{\{3\}} - h_{\{4\}} - h_{\{1,2\}} - h_{\{1,3,4\}} - h_{\{2,3,4\}} \geq 0 \tag{3.11}$$

$$\tag{3.12}$$

We use the commercial software *Polymake* [11] to compute the extreme rays of the polyhedral cones $P_G\mathcal{L}_4$ and $P_G\Gamma_4$ for $G = V4, P1, d$. For each one of these three cases, the results are given by two *identical* sets of vectors, implying that $P_G\mathcal{L}_4 = P_G\Gamma_4$ and hence $P_G cl(\Gamma_4^*) = P_G\Gamma_4$ in these cases. By symmetry, the cases for $G = P2, P3, P4$

follow from that for $G = P1$ and the cases for $G = d', d''$ follow from that for $G = d$.

To show that the subgroups A, B and C are non-Shannon, we first use *Polymake* to compute the extreme rays of the polyhedral cone $P_G\Gamma_4$ for $G = A$. We then add the well-known Yeung-Zhang non-Shannon type inequalities [5, Chapter 15, Theorem 15.7], [6]:

$$\begin{aligned} -2h_{\{1\}} - 2h_{\{2\}} + 3h_{\{1,2\}} - h_{\{3\}} - h_{\{3,4\}} + 3h_{\{1,3\}} + 3h_{\{2,3\}} \\ +h_{\{1,4\}} + h_{\{2,4\}} - 4h_{\{1,2,3\}} - h_{\{1,2,4\}} \geq 0 \end{aligned} \tag{3.13}$$

$$\begin{aligned} -2h_{\{1\}} - 2h_{\{3\}} + 3h_{\{1,3\}} - h_{\{2\}} - h_{\{2,4\}} + 3h_{\{1,2\}} + 3h_{\{2,3\}} \\ +h_{\{1,4\}} + h_{\{3,4\}} - 4h_{\{1,2,3\}} - h_{\{1,3,4\}} \geq 0 \end{aligned} \tag{3.14}$$

$$\begin{aligned} -2h_{\{1\}} - 2h_{\{4\}} + 3h_{\{1,4\}} - h_{\{2\}} - h_{\{2,3\}} + 3h_{\{1,2\}} + 3h_{\{2,4\}} \\ +h_{\{1,3\}} + h_{\{3,4\}} - 4h_{\{1,2,4\}} - h_{\{1,3,4\}} \geq 0 \end{aligned} \tag{3.15}$$

$$\begin{aligned} -2h_{\{2\}} - 2h_{\{3\}} + 3h_{\{2,3\}} - h_{\{1\}} - h_{\{1,4\}} + 3h_{\{1,2\}} + 3h_{\{1,3\}} \\ +h_{\{2,4\}} + h_{\{3,4\}} - 4h_{\{1,2,3\}} - h_{\{2,3,4\}} \geq 0 \end{aligned} \tag{3.16}$$

$$\begin{aligned} -2h_{\{2\}} - 2h_{\{4\}} + 3h_{\{2,4\}} - h_{\{1\}} - h_{\{1,3\}} + 3h_{\{1,2\}} + 3h_{\{1,4\}} \\ +h_{\{2,3\}} + h_{\{3,4\}} - 4h_{\{1,2,4\}} - h_{\{2,3,4\}} \geq 0 \end{aligned} \tag{3.17}$$

$$\begin{aligned} -2h_{\{3\}} - 2h_{\{4\}} + 3h_{\{3,4\}} - h_{\{1\}} - h_{\{1,2\}} + 3h_{\{1,3\}} + 3h_{\{1,4\}} \\ +h_{\{2,3\}} + h_{\{2,4\}} - 4h_{\{1,3,4\}} - h_{\{2,3,4\}} \geq 0 \end{aligned} \tag{3.18}$$

to the Shannon type inequalities to form a new *outer* region Γ'_4 to the entropy region Γ^*_4. We again use *Polymake* to compute the extreme rays of the polyhedral cone $P_G\Gamma'_4$ for $G = A$. The result gives a *different* set of extreme rays than those of $P_G\Gamma_4$ for $G = A$. This shows that the subgroup A is non-Shannon. By symmetry, the cases for $G = B, C$ follow from that for $G = A$.

The details of the computation are deferred to the Appendix.

3.3 The Cyclic Group C_5

The cyclic group C_4 generated by the permutation (1 2 3 4) is the subgroup in the Hasse diagram (3.1) and was shown to be Shannon from the previous discussion.

The orbits of the cyclic group C_5 generated by the permutation (1 2 3 4 5) are given by:

$$O_1 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\} \tag{3.19}$$

$$O_2 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\} \tag{3.20}$$

$$O_3 = \{\{1, 3\}, \{2, 4\}, \{3, 5\}, \{4, 1\}, \{5, 2\}\} \tag{3.21}$$

$$O_4 = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 1\}, \{5, 1, 2\}\} \tag{3.22}$$

$$O_5 = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 1\}, \{4, 5, 2\}, \{5, 1, 3\}\} \tag{3.23}$$

$$O_6 = \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{3, 4, 5, 1\}, \{4, 5, 1, 2\}, \{5, 1, 2, 3\}\} \tag{3.24}$$

$$O_7 = \{\{1, 2, 3, 4, 5\}\}. \tag{3.25}$$

Setting $h_S = \overline{h}_k$ for any $S \in O_k$ and $k \in \mathcal{N}_7$ in the elemental inequalities (2.7) and (2.8), the projection of the polymatroid region $P_{C_5}\Gamma_5$ is given by the set of vectors

$(\overline{h}_k : k \in \mathcal{N}_7)$ satisfying the following 17 inequalities:

$$2\overline{h}_1 - \overline{h}_2 \geq 0 \tag{3.26}$$

$$2\overline{h}_1 - \overline{h}_3 \geq 0 \tag{3.27}$$

$$2\overline{h}_2 - \overline{h}_1 - \overline{h}_4 \geq 0 \tag{3.28}$$

$$2\overline{h}_3 - \overline{h}_1 - \overline{h}_4 \geq 0 \tag{3.29}$$

$$\overline{h}_2 + \overline{h}_3 - \overline{h}_1 - \overline{h}_4 \geq 0 \tag{3.30}$$

$$2\overline{h}_2 - \overline{h}_1 - \overline{h}_5 \geq 0 \tag{3.31}$$

$$2\overline{h}_3 - \overline{h}_1 - \overline{h}_5 \geq 0 \tag{3.32}$$

$$\overline{h}_2 + \overline{h}_3 - \overline{h}_1 - \overline{h}_5 \geq 0 \tag{3.33}$$

$$2\overline{h}_4 - \overline{h}_3 - \overline{h}_6 \geq 0 \tag{3.34}$$

$$2\overline{h}_5 - \overline{h}_3 - \overline{h}_6 \geq 0 \tag{3.35}$$

$$\overline{h}_4 + \overline{h}_5 - \overline{h}_3 - \overline{h}_6 \geq 0 \tag{3.36}$$

$$2\overline{h}_4 - \overline{h}_3 - \overline{h}_4 \geq 0 \tag{3.37}$$

$$2\overline{h}_5 - \overline{h}_3 - \overline{h}_4 \geq 0 \tag{3.38}$$

$$\overline{h}_4 + \overline{h}_5 - \overline{h}_3 - \overline{h}_4 \geq 0 \tag{3.39}$$

$$2\overline{h}_6 - \overline{h}_6 - \overline{h}_4 - \overline{h}_7 \geq 0 \tag{3.40}$$

$$2\overline{h}_6 - \overline{h}_6 - \overline{h}_5 - \overline{h}_7 \geq 0 \tag{3.41}$$

$$\overline{h}_7 - \overline{h}_6 \geq 0. \tag{3.42}$$

Using *Polymake* [11], the extreme rays of $P_{C_5}\Gamma_5$ can be computed as:

$$\boldsymbol{r}_1 = (1, 1, 1, 1, 1, 1, 1) \tag{3.43}$$

$$\boldsymbol{r}_2 = (1, 2, 2, 2, 2, 2, 2) \tag{3.44}$$

$$\boldsymbol{r}_3 = (1, 2, 2, 3, 3, 3, 3) \tag{3.45}$$

$$\boldsymbol{r}_4 = (1, 2, 2, 3, 3, 4, 4) \tag{3.46}$$

$$\boldsymbol{r}_5 = (1, 2, 2, 3, 3, 4, 5) \tag{3.47}$$

$$\boldsymbol{r}_6 = (2, 3, 4, 4, 4, 4, 4) \tag{3.48}$$

$$\boldsymbol{r}_7 = (2, 4, 3, 4, 4, 4, 4) \tag{3.49}$$

$$\boldsymbol{r}_8 = (2, 4, 4, 5, 6, 6, 6) \tag{3.50}$$

$$\boldsymbol{r}_9 = (2, 4, 4, 6, 5, 6, 6). \tag{3.51}$$

Theorem 3. *The cyclic group C_5 is Shannon.*

Proof. To show that the cyclic group C_5 is Shannon, it suffices to show that all nine extreme rays of $P_{C_5}\Gamma_5$ are in $P_{C_5}cl(\Gamma_5^*)$. It is clear that the extreme rays $\boldsymbol{r}_i$, $i \in \mathcal{N}_5$, can be realized by a cyclic projection of *uniform matroids* [13] and are hence in $P_{C_5}cl(\Gamma_5^*)$. So we only need to show that $\boldsymbol{r}_i$, $i = 6, 7, 8, 9$, are in $P_{C_5}cl(\Gamma_5^*)$.

To show that $\boldsymbol{r}_7 \in P_{C_5}cl(\Gamma_5^*)$, let U_i, $i \in \mathcal{N}_4$, be four independent uniform variables over a finite field $\mathbb{F}$ and

$$\mathsf{X}_1 := (\mathsf{U}_1, \mathsf{U}_2 + \mathsf{U}_3) \tag{3.52}$$

$$\mathsf{X}_2 := (\mathsf{U}_2, \mathsf{U}_3 + \mathsf{U}_4) \tag{3.53}$$

$$\mathsf{X}_3 := (\mathsf{U}_3, \mathsf{U}_1) \tag{3.54}$$

$$\mathsf{X}_4 := (\mathsf{U}_4, \mathsf{U}_2 + \mathsf{U}_3) \tag{3.55}$$

$$\mathsf{X}_5 := (\mathsf{U}_4 + \mathsf{U}_1, \mathsf{U}_3 + \mathsf{U}_4). \tag{3.56}$$

It is straightforward to verify that

$$H(\mathsf{X}_S) = \begin{cases} 2\log|\mathbb{F}|, & \text{for } S \in O_1 \\ 4\log|\mathbb{F}|, & \text{for } S \in O_2 \\ 3\log|\mathbb{F}|, & \text{for } S \in O_3 \\ 4\log|\mathbb{F}|, & \text{for } S \in O_4 \\ 4\log|\mathbb{F}|, & \text{for } S \in O_5 \\ 4\log|\mathbb{F}|, & \text{for } S \in O_6 \\ 4\log|\mathbb{F}|, & \text{for } S \in O_7 \end{cases} \tag{3.57}$$

completing the proof that $\boldsymbol{r}_7 \in P_{C_5} cl(\Gamma_5^*)$.

To show that $\boldsymbol{r}_9 \in P_{C_5} cl(\Gamma_5^*)$, let U_i, $i \in \mathcal{N}_6$, be six independent uniform variables over a finite field $\mathbb{F}$ and

$$\mathsf{X}_1 = (\mathsf{U}_1, \mathsf{U}_6) \tag{3.58}$$

$$\mathsf{X}_2 = (\mathsf{U}_2, \mathsf{U}_4 + \mathsf{U}_5) \tag{3.59}$$

$$\mathsf{X}_3 = (\mathsf{U}_3, \mathsf{U}_5 + \mathsf{U}_6) \tag{3.60}$$

$$\mathsf{X}_4 = (\mathsf{U}_4, \mathsf{U}_1 + \mathsf{U}_5) \tag{3.61}$$

$$\mathsf{X}_5 = (\mathsf{U}_2 + \mathsf{U}_3, \mathsf{U}_3 + \mathsf{U}_5). \tag{3.62}$$

It is straightforward to verify that

$$H(\mathsf{X}_S) = \begin{cases} 2\log|\mathbb{F}|, & \text{for } S \in O_1 \\ 4\log|\mathbb{F}|, & \text{for } S \in O_2 \\ 4\log|\mathbb{F}|, & \text{for } S \in O_3 \\ 6\log|\mathbb{F}|, & \text{for } S \in O_4 \\ 5\log|\mathbb{F}|, & \text{for } S \in O_5 \\ 6\log|\mathbb{F}|, & \text{for } S \in O_6 \\ 6\log|\mathbb{F}|, & \text{for } S \in O_7 \end{cases} \tag{3.63}$$

completing the proof that $\boldsymbol{r}_9 \in P_{C_5} cl(\Gamma_5^*)$.

By symmetry, the cases for $\boldsymbol{r}_6$ and $\boldsymbol{r}_8$ follows from that for $\boldsymbol{r}_7$ and $\boldsymbol{r}_9$, respectively. We have thus completed the proof of the theorem. □

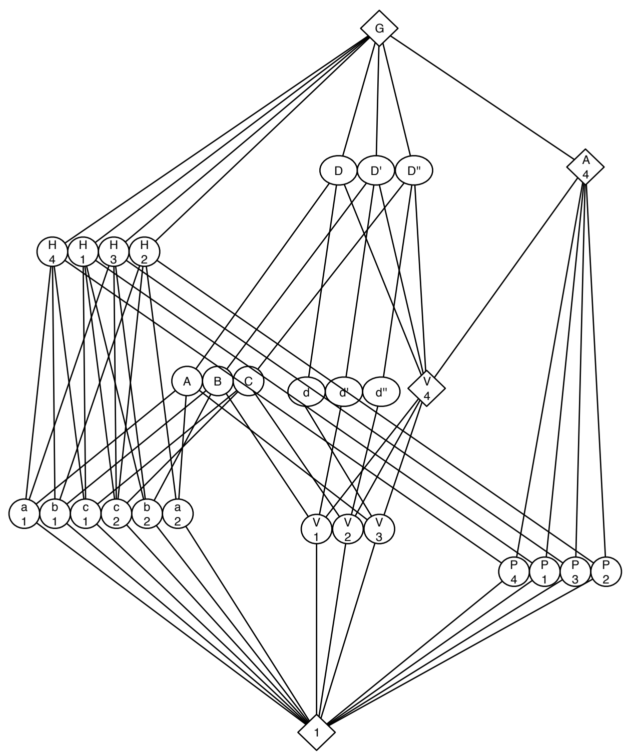

Figure 3.1: Hasse diagram of S_4.

Table 3.1: Elements of subgroups of S_4

label	elements	order	isomorphic to
A4	{e,(12)(34),(13)(24),(14)(23),(123),(124),... (132),(134),(142),(143),(234),(243)}	12	A4
V4	{e, (12)(34), (13)(24),(14)(23)}	4	V4
v1, v2, v3	{e, (12)(34)}, {e, (13)(24)}, {e, (14)(23)}	2, 2, 2	Z2
P1	{e, (123), (132)}	3	Z3
P2	{e, (124), (142)}	3	Z3
P3	{e, (134), (143)}	3	Z3
P4	{e, (234), (243)}	3	Z3
D	{e, (12), (12)(34), (13)(24), (14)(23), (34), (1324), (1423)}	8	D4
d	{e, (12)(34), (1324), (1423)}	4	Z4
D'	{e, (13), (12)(34), (13)(24), (14)(23), (24), (1234), (1432)}	8	D4
d'	{e, (13)(24), (1234), (1423)}	4	Z4
D"	{e, (14), (12)(34), (13)(24), (14)(23), (23), (1243), (1342)}	8	D4
d"	{e, (14)(23), (1243), (1342)}	4	Z4
H1	{e, (12), (13), (23), (123), (132)}	6	S3
H2	{e, (12), (14), (24), (124), (142)}	6	S3
H3	{e, (13), (14), (34), (134), (143)}	6	S3
H4	{e, (23), (24), (34), (234), (243)}	6	S3
A	{e, (12), (12)(34), (34)}	4	V4
a1, a2	{e, (12)}, {e, (34)}	2, 2	Z2
B	{e, (13), (13)(24), (24)}	4	V4
b1, b2	{e, (13)}, {e, (24)}	2, 2	Z2
C	{e, (14), (14)(23), (23)}	4	V4
c1, c2	{e, (14)}, {e, (23)}	2, 2	Z2

4. SUMMARY AND FUTURE DIRECTIONS

Entropy inequalities play a central role in proving converse coding theorems for network information theoretic problems. This thesis studied two new aspects of entropy inequalities. First, inequalities relating average joint entropies rather than entropies over individual subsets were studied. Motivated by the curious fact that the monotonicity of average joint entropy per element holds when the averaging is over both all subsets of the size [1] and the sliding window of the same size [2], it was shown that the closures of the average entropy regions where the averages are over all subsets of the same size and all sliding windows of the same size respectively are identical. This implies that that averaging over sliding windows always suffices as far as unconstrained entropy inequalities are concerned. Therefore, the aforementioned fact on the monotonicity of average joint entropy per element is a universal truth rather than an isolated curious observation.

Second, the existence of non-Shannon type inequalities [6] was one of the most significant discoveries in information theory during the last twenty years. Under total symmetry, however, it was known that all non-Shannon type inequalities are implied by Shannon type inequalities [5]. Mathematically, the total symmetry can be represented using the symmetry groups S_n. In the second part of this thesis, the existence of non-Shannon type inequalities under partial symmetry was studied, where the partial symmetry was represented using the subgroups of S_n. This naturally led to the notion of Shannon and non-Shannon groups, based on which a complete classification of all permutation groups over four elements was established. With five random variables, it was shown that there are no non-Shannon type inequalities under cyclic symmetry.

There are several directions that one may consider exploring in the future. Perhaps the most straightforward extension is to consider the cyclic groups C_n for $n \geq 6$. It is our belief that the cyclic group C_n is Shannon for *any* $n \in \mathcal{N}$. Note that even though the cases where $n = 4$ and 5 have been resolved in this thesis, the techniques that we used rely on a "brute-force" calculation of the extreme rays of $P_{C_n}\Gamma_n$ and have a complexity that grows exponentially with n. A new representation which can further expose the structure of $P_{C_n}\Gamma_n$ may be needed in order to make progress.

Another direction of interest is to understand which partial symmetry is particularly relevant to engineering and whether non-Shannon type inequalities exist under those partial symmetry. The modern development of distributed storage systems provides several examples [14,15] where there is symmetry built into the design principles and requirements.

Finally, note that with symmetry not only non-Shannon type inequalities may completely disappear (dominated by the Shannon type inequalities), the number of independent Shannon type inequalities may also be substantially reduced. For example, without any symmetry the total number of independent Shannon type inequalities (elemental inequalities in (2.7) and (2.8)) over n variables is

$$n + \binom{n}{2} 2^{n-2}.$$

By comparison, under total symmetry the total number of independent Shannon type inequalities (the inequalities in (2.10) and (2.11)) over n variables is only n. Therefore, partial symmetry can potentially provide *huge* advantages when a computational approach is utilized for characterizing the fundamental limits of complex information systems [16].

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APPENDIX A

APPENDIX FOR SHANNON AND NON-SHANNON GROUPS OF S_4

A.1 Subgroup A

Shannon-type Inequalities:

```
$inequalities=new Matrix<Rational>([[0 ,2 ,0 ,-1 ,0 ,0 ,0 ,0, 0],[0, 1 ,1 ,0 ,-1, 0 ,0 ,0 ,0],[0, 0, 2, 0, 0, -1 ,0 ,0 ,0],[0 ,-1 ,0 ,1 ,1, ,0, ,-1, 0, 0],[0, -1 ,0 ,0 ,2 ,0 ,0, -1, 0],[0, 0, -1, 0 ,2 ,0 ,-1, 0 ,0],[0 ,0 ,-1 ,0 ,1 ,1 ,0 ,-1, 0],[0, 0 ,0 ,-1, 0 ,0 ,2 ,0 ,-1],[0, 0, 0 ,0 ,-1, 0, 1, 1, -1],[0, 0, 0 ,0 ,0 ,-1, 0, 2, -1],[0 ,0 ,0 ,0, 0 ,0 ,0 ,-1, 1],[0 ,0 ,0 ,0 ,0, 0, -1, 0 ,1]]);

polytope > $p=new Polytope<Rational>(INEQUALITIES=>$inequalities);
```

polytope > print $p− >VERTICES;

0	1	0	1	1	0	1	1	1
0	1	1	1	2	2	2	2	2
0	1	1	3/2	3/2	2	2	2	2
0	1	1/2	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
0	1	1	2	3/2	3/2	2	2	2
0	1	2	2	2	2	2	2	2
0	1	1	2	2	1	2	2	2
0	1	1/2	2	3/2	1	2	2	2
0	1	1	2	3/2	2	2	2	2
0	1	1	2	2	2	2	2	2
0	1	2	2	3	4	4	4	4
0	0	1	0	1	1	1	1	1
0	1	1	2	2	2	3	3	3
1	0	0	0	0	0	0	0	0
0	0	1	0	1	2	1	2	2
0	1	0	2	1	0	2	1	2

Shannon-type Inequalities+ Ingleton Inequality:

$inequalities=new Matrix<Rational>([[0 ,2 ,0 ,-1 ,0 ,0 ,0 ,0, 0],[0, 1 ,1 ,0 ,-1, 0 ,0 ,0 ,0],[0, 0, 2, 0, 0, -1 ,0 ,0 ,0],[0 ,-1 ,0 ,1 ,1, ,0, ,-1, 0, 0],[0, -1 ,0 ,0 ,2 ,0 ,0, -1, 0],[0, 0, -1, 0 ,2 ,0 ,-1, 0 ,0],[0 ,0 ,-1 ,0 ,1 ,1 ,0 ,-1, 0],[0, 0 ,0 ,-1, 0 ,0 ,2 ,0 ,-1],[0, 0, 0 ,0 ,-1, 0, 1, 1, -1],[0, 0, 0 ,0 ,0 ,-1, 0, 2, -1],[0 ,0 ,0 ,0, 0 ,0 ,0 ,-1, 1],[0 ,0 ,0 ,0 ,0, 0, -1, 0 ,1],[0,-2,0,1,4,-1,-2,0,0]]);

polytope > $p=new Polytope<Rational>(INEQUALITIES=>$inequalities);

polytope > print $p− >VERTICES;

0	1	0	1	1	0	1	1	1
0	1	1	1	2	2	2	2	2
0	1	1/2	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
0	1	1	2	3/2	3/2	2	2	2
0	1	2	2	2	2	2	2	2
0	1	1	2	2	1	2	2	2
0	1	1/2	2	3/2	1	2	2	2
0	1	1	2	3/2	2	2	2	2
0	1	1	2	2	2	2	2	2
0	1	2	2	3	4	4	4	4
0	0	1	0	1	1	1	1	1
0	1	1	2	2	2	3	3	3
1	0	0	0	0	0	0	0	0
0	0	1	0	1	2	1	2	2
0	1	0	2	1	0	2	1	2

Shannon-type Inequalities+ YZ Inequality:

$inequalities=new Matrix<Rational>([[0 ,2 ,0 ,-1 ,0 ,0 ,0 ,0, 0],[0, 1 ,1 ,0 ,-1, 0 ,0 ,0 ,0],[0, 0, 2, 0, 0, -1 ,0 ,0 ,0],[0 ,-1 ,0 ,1 ,1, ,0, ,-1, 0, 0],[0, -1 ,0 ,0 ,2 ,0 ,0, -1, 0],[0, 0, -1, 0 ,2 ,0 ,-1, 0 ,0],[0 ,0 ,-1 ,0 ,1 ,1 ,0 ,-1, 0],[0, 0 ,0 ,-1, 0 ,0 ,2 ,0 ,-1],[0, 0, 0 ,0 ,-1, 0, 1, 1, -1],[0, 0, 0 ,0 ,0 ,-1, 0, 2, -1],[0 ,0 ,0 ,0, 0 ,0 ,0 ,-1, 1],[0 ,0 ,0 ,0 ,0, 0, -1, 0 ,1],[0, -1, -4, -1, 8, 3, 0, -5, 0],[0, -4, -1, 3, 8, -1, -5, 0, 0],[0, -3, -2, 3, 6, 1, -4, -1, 0],[0,

-2, -3, 1, 6, 3, -1, -4, 0]]);

polytope > $p=new Polytope<Rational>(INEQUALITIES=>$inequalities);

polytope > print $p− >VERTICES;

0	1	0	1	1	0	1	1	1
0	1	1	1	2	2	2	2	2
0	1	3/4	5/4	5/4	3/2	3/2	3/2	3/2
0	1	1	5/3	3/2	2	2	2	2
0	1	1	1	1	1	1	1	1
0	1	1	2	3/2	5/3	2	2	2
0	1	2	2	2	2	2	2	2
0	1	4/3	2	5/3	5/3	2	2	2
0	1	1	2	2	1	2	2	2
0	1	1	2	3/2	2	2	2	2
0	1	1	2	2	2	2	2	2
0	0	1	0	1	1	1	1	1
0	1	1/2	2	3/2	1	2	2	2
0	1	1	2	2	2	3	3	3
0	1	2	2	3	4	4	4	4
1	0	0	0	0	0	0	0	0
0	0	1	0	1	2	1	2	2
0	1	0	2	1	0	2	1	2

A.2 Subgroup V_4

Shannon-type Inequalities:

$inequalities=new Matrix<Rational>([[0, 0, 0, 0, 0, -1, 1],[0, 2, -1, 0, 0, 0, 0],[0, 2, 0, -1, 0, 0, 0],[0, 2, 0, 0, -1, 0, 0],[0, -1, 1, 1, 0, -1, 0],[0, -1, 0, 1, 1, -1, 0],[0, -1, 1, 0, 1, -1, 0],[0, 0, -1, 0, 0, 2, -1],[0, 0, 0, -1, 0, 2, -1],[0, 0, 0, 0, -1, 2, -1]]);

polytope > $p=new Polytope<Rational>(INEQUALITIES=>$inequalities);

polytope > print $p− >VERTICES;

0	1	2	2	1	2	2
0	1	1	1	1	1	1
0	1	2	1	2	2	2
0	1	1	2	2	2	2
0	1	2	2	2	3	3
0	1	2	2	2	2	2
1	0	0	0	0	0	0
0	1	2	2	2	3	4

Shannon-type Inequalities+ Ingleton Inequality:

$inequalities=new Matrix<Rational>([[0, 0, 0, 0, 0, -1, 1],[0, 2, -1, 0, 0, 0, 0],[0, 2, 0, -1, 0, 0, 0],[0, 2, 0, 0, -1, 0, 0],[0, -1, 1, 1, 0, -1, 0],[0, -1, 0, 1, 1, -1, 0],[0, -1, 1, 0, 1, -1, 0],[0, 0, -1, 0, 0, 2, -1],[0, 0, 0, -1, 0, 2, -1],[0, 0, 0, 0, -1, 2, -1],[0, -2, 0, 2, 2, -2, 0]]);

polytope > $p=new Polytope<Rational>(INEQUALITIES=>$inequalities);

polytope > print $p− >VERTICES;

0	1	2	2	1	2	2
0	1	1	1	1	1	1
0	1	2	1	2	2	2
0	1	1	2	2	2	2
0	1	2	2	2	3	3
0	1	2	2	2	2	2
1	0	0	0	0	0	0
0	1	2	2	2	3	4

A.3 Subgroup P_1

Shannon-type Inequalities

$inequalities=new Matrix<Rational>([[0, 0, 0, 0, 0, 0, -1, 1],[0, 0, 0, 0, 0, -1, 0, 1],[0, 2, 0, -1, 0, 0, 0, 0],[0, 1, 1, 0, -1, 0, 0, 0],[0, -1, 0, 2, 0, -1, 0, 0],[0, -1, 0, 1, 1, 0, -1, 0],[0, 0, -1, 0, 2, 0, -1, 0],[0, 0, 0, -1, 0, 1, 1, -1],[0, 0, 0, 0, -1, 0, 2, -1]]);

polytope > $p=new Polytope<Rational>(INEQUALITIES=>$inequalities);

polytope > print $p− >VERTICES;

0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	2	2	2	2	2
0	1	2	2	2	2	2	2
0	1	2	2	3	3	3	3
0	1	3	2	3	3	3	3
0	1	1	2	2	3	3	3
0	1	0	2	1	2	2	2
1	0	0	0	0	0	0	0
0	0	1	0	1	0	1	1
0	1	0	2	1	3	2	3

Shannon-type Inequalities+ Ingleton Inequality:

$inequalities=new Matrix<Rational>([[0, 0, 0, 0, 0, 0, -1, 1],[0, 0, 0, 0, 0, -1, 0, 1],[0, 2, 0, -1, 0, 0, 0, 0],[0, 1, 1, 0, -1, 0, 0, 0],[0, -1, 0, 2, 0, -1, 0, 0],[0, -1, 0, 1, 1, 0, -1, 0],[0, 0, -1, 0, 2, 0, -1, 0],[0, 0, 0, -1, 0, 1, 1, -1],[0, 0, 0, 0, -1, 0, 2, -1],[0, -2, 0, 3, 1, -1, -1, 0],[0, -1, -1, 1, 3, 0, -2, 0]]);

polytope > $p=new Polytope<Rational>(INEQUALITIES=>$inequalities);

polytope > print $p− >VERTICES;

0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	2	2	2	2	2
0	1	2	2	2	2	2	2
0	1	2	2	3	3	3	3
0	1	3	2	3	3	3	3
0	1	1	2	2	3	3	3
0	1	0	2	1	2	2	2
1	0	0	0	0	0	0	0
0	0	1	0	1	0	1	1
0	1	0	2	1	3	2	3

A.4 Subgroup d

Shannon-type Inequalities:

$inequalities=new Matrix<Rational>([[0, 0, 0, 0, -1, 1],[0, 2, -1, 0, 0, 0],[0, 2, 0, -1, 0, 0],[0, -1, 1, 1, -1, 0],[0, -1, 0, 2, -1, 0],[0, 0, -1, 0, 2, -1],[0, 0, 0, -1, 2, -1]]);

polytope > $p=new Polytope<Rational>(INEQUALITIES=>$inequalities);

polytope > print $p− >VERTICES;

0	1	1	1	1	1
0	1	2	3/2	2	2
0	1	1	2	2	2
0	1	2	2	3	3
0	1	2	2	2	2
1	0	0	0	0	0
0	1	2	2	3	4

Shannon-type Inequalities+ Ingleton Inequality:

$inequalities=new Matrix<Rational>([[0, 0, 0, 0, -1, 1],[0, 2, -1, 0, 0, 0],[0, 2, 0, -1, 0, 0],[0, -1, 1, 1, -1, 0],[0, -1, 0, 2, -1, 0],[0, 0, -1, 0, 2, -1],[0, 0, 0, -1, 2, -1],[0, -2, 0, 4, -2, 0],[0, -2, 2, 2, -2, 0]]);

polytope > $p=new Polytope<Rational>(INEQUALITIES=>$inequalities);

polytope > print $p− >VERTICES;

0	1	1	1	1	1
0	1	2	3/2	2	2
0	1	1	2	2	2
0	1	2	2	3	3
0	1	2	2	2	2
1	0	0	0	0	0
0	1	2	2	3	4