PRIMARY COMPONENTS OF BINOMIAL IDEALS

A Dissertation

by

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ABSTRACT

Binomials are polynomials with at most two terms. A binomial ideal is an ideal generated by binomials. Primary components and associated primes of a binomial ideal are still binomial over algebraically closed fields. Primary components of general binomial ideals over algebraically closed fields with characteristic zero can be described combinatorially by translating the operations on binomial ideals to operations on exponent vectors. In this dissertation, we obtain more explicit descriptions for primary components of special binomial ideals. A feature of this work is that our results are independent of the characteristic of the field.

First of all, we analyze the primary decomposition of a special class of binomial ideals, lattice ideals, in which every variable is a nonzerodivisor modulo the ideal. Then we provide a description for primary decomposition of lattice ideals in fields with positive characteristic.

In addition, we study the codimension two lattice basis ideals and we compute their primary components explicitly.

An ideal $I \subseteq \Bbbk[x_1, ..., x_n]$ is cellular if every variable is either a nonzerodivisor modulo I or is nilpotent modulo I. We characterize the minimal primary components of cellular binomial ideals explicitly. Another significant result is a computation of the Hull of a cellular binomial ideal, that is the intersection of all of its minimal primary components.

Lastly, we focus on commutative monoids and their congruences. We study properties of monoids that have counterparts in the study of binomial ideals. We provide a characterization of primary ideals in positive characteristic, in terms of the congruences they induce.

DEDICATION

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1. INTRODUCTION: MAIN CONCEPTS

1.1 Primary Decomposition Basics

Primary decomposition is a cornerstone of ideal theory. It is a generalization of the factorization of a number into prime powers. From a geometric point of view, primary decomposition is based on the idea of decomposing a variety into a union of simpler varieties. Indeed, primary decomposition of radical ideals corresponds to the decomposition of an affine variety into its irreducible components. When we want to take multiplicity into account, primary ideals become necessary. (They describe the multiplicity of irreducible components.) In this section, we make an introductory review to recall the basic concepts and to fix the notation we use.

Throughout this dissertation R denotes a commutative Noetherian ring with an identity element, and $\Bbbk$ denotes an algebraically closed field. We denote $S = \Bbbk[x_1, x_2, ..., x_n]$. Also, we assume $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{Z}_{>0} = \{1, 2, ...\}$.

Definition 1.1. Suppose that R is a ring. An ideal Q in R is *primary* if $Q \neq R$ and if for every $ab \in Q$ we have that either $a \in Q$ or $b^n \in Q$ for some $n > 0$.

The following reformulation is more symmetric : for Q primary, if $ab \in Q$ and $a, b \notin Q$, then some powers of a and b belong to Q.

Let Q be a primary ideal in R, then it is easy to see that $\sqrt{Q}$ is the smallest prime ideal containing Q. If $P = \sqrt{Q}$, then Q is said to be *P-primary.*

The intersection of primary ideals need not to be primary, however, we have the following

Proposition 1.2. *If Q_i are P-primary for all $i = 1, ..., n$, then $\bigcap_{i=1}^{n} Q_i$ is P-primary.*

Proof. We know that $\sqrt{\bigcap_{i=1}^{n} Q_i} = \bigcap_{i=1}^{n} \sqrt{Q_i} = P$. Suppose $xy \in \bigcap_{i=1}^{n} Q_i$ and $y \notin \bigcap_{i=1}^{n} Q_i$. For some ℓ, $xy \in Q_\ell$ with $x^k \in Q_\ell$. This implies that $x \in \sqrt{Q_\ell} = P$. But then there exists $m > 0$ such that $x^m \in Q_i$ for all i, so $x^m \in \bigcap_{i=1}^{n} Q_i$. □

The variety of $I \subseteq \mathbb{k}[x_1, ..., x_n]$, denoted $V(I)$, is the set of common zeroes of I.

$$V(I) := \{x \in \mathbb{k}^n \mid f(x) = 0, \ \textit{for all } f \in I\}.$$

We define some operations on ideals: ideal quotient and saturation are important constructions in ideal theory. Saturation removes the zerodivisors in $\mathbb{k}[x_1, ..., x_n]/I$.

Definition 1.3. Let $Q \subset R$ be an ideal and let $f \in R$. We define the *ideal quotient*

$$(I : f) = \{g \in R \mid gf \in I\},$$

and the *saturation of I by f*

$$(I : f^\infty) = \{g \in R \mid gf^n \in I, \ \text{for some positive integer n }\}.$$

The sets above are actually ideals. We are not attaching any value to f^∞. Since R is a Noetherian ring, the ascending chain $(I : f) \subseteq (I : f^2) \subseteq (I : f^3) \subseteq ...$ eventually stops. The stabilized ideal is denoted by $(I : f^\infty)$. Geometrically, the components of $V((I : f^\infty))$ are those components of $V(I)$ which do not lie on the hypersurface $V(\langle f \rangle)$.

The following is clear by definition of the quotient ideal.

Corollary 1.4. *Let $I \subseteq k[x_1, ..., x_n]$ be an ideal. Then*

$$((((I : x_1) : x_2) : \cdots) : x_n) = (I : x_1 x_2 \cdots x_n).$$

A *primary decomposition* of an ideal I in R is an expression of I as a finite intersection of primary ideals

$$I = \bigcap_{i=1}^{n} Q_i.$$

For univariate polynomials, primary decomposition is factorization of polynomials. For general rings, primary decomposition need not exist, but in a Noetherian ring R every ideal has a primary decomposition.

Theorem 1.5. *Let R be a Noetherian ring. Every ideal $I \subset R$ can be written as a finite intersection of primary ideals.*

Proof. An ideal I is *irreducible* if $I = I_1 \cap I_2$ implies that $I = I_1$ or $I = I_2$. Since R is a Noetherian ring, every ideal is an intersection of finitely many irreducible ideals. If we show that an irreducible ideal is primary, this completes the proof. Suppose I is an irreducible ideal and let $fg \in I$ with $f \notin I$. We have an ascending chain of ideals

$$(I : g) \subset (I : g^2) \subset \cdots$$

which has to stabilize since R is Noetherian. That means there exists an integer n such that $(I : g^n) = (I : g^{n+1})$. We claim that $I = (I + \langle g^n \rangle) \cap (I + \langle f \rangle)$. It is obvious that $I \subseteq (I + \langle g^n \rangle) \cap (I + \langle f \rangle)$. Let $h \in (I + \langle g^n \rangle) \cap (I + \langle f \rangle)$, so $h = a_1 + b_1 g^n$ and $h = a_2 + b_2 f$ for some $a_1, a_2 \in I$ and $b_1,\ b_2 \in R$. If we multiply both sides of the second equation by g, we obtain $hg \in I$. If we multiply both sides of the first equation by g, we obtain $b_1 g^{n+1} \in I$. This implies $b_1 \in (I : g^{n+1}) = (I : g^n)$ and so $h \in I$. □

Theorem 1.6. *(Hilbert Basis Theorem) If R is a commutative Noetherian ring with unity, then so is $R[x_1, ..., x_n]$.*

Proof. See Theorem 4.9 in [19]. □

Corollary 1.7. *Let* $\Bbbk$ *be a field, then* $S = \Bbbk[x_1, x_2, ..., x_n]$ *is a Noetherian ring.*

A primary decomposition $I = \bigcap_{i=1}^{n} Q_i$ is *irredundant*(*minimal*) if the ideals $\sqrt{Q_i}$ are distinct and $\bigcap_{j \neq i}^{n} Q_j \nsubseteq Q_i$ for all $i = 1, ..., n$. Thus I cannot be written as an intersection consisting of a proper subset of the ideals Q_i.

Theorem 1.8. *(Theorem 4.5 in [2].) Let* $I = \bigcap_{i=1}^{n} Q_i$ *be an irredundant primary decomposition of* I*. Define* $P_i := \sqrt{Q_i}$ *for* $i = 1, ..., n$*. The* P_i*'s are precisely the prime ideals which occur in the set of ideals* $\sqrt{I : r}$ *for some* $r \in R$*. Thus the* P_i*'s are independent of the particular primary decomposition of* I*.*

Definition 1.9. The prime ideals P_i in Theorem 1.8 are said to be *associated* to I. The minimal elements of the set $\{P_1, ..., P_n\}$ are called *minimal prime ideals associated to* I. The non-minimal prime ideals associated to I are called *embedded prime ideals.*

Note that Theorem 1.8 shows the uniqueness of associated primes. The names, embedded and minimal, arose from geometry: if the ideal $I \in S$ corresponds to the variety $V(I)$, the minimal primes correspond to the irreducible components of $V(I)$, the embedded primes correspond to varieties embedded in the irreducible components. In fact, the varieties corresponding to prime and primary ideals are irreducible. If P is an associated prime of I by Theorem 1.8, $P = (I : f)$ for some $f \in R$. This implies that $V(P) = V(I : f) \subseteq V(I)$, hence the irreducible variety $V(P)$ forms a part of $V(I)$.

Note that any prime ideal $I \subseteq P$ contains a minimal prime associated to I, so the set of minimal prime ideals associated to I are precisely the minimal elements in the set of all prime ideals containing I. To be consistent with the literature, we denote the set associated primes of I by Ass(S/I).

For a proof of the following commutative algebraic fact, see Lemma 3.6 in [13].

Lemma 1.10. *Let I, J and J' be ideals of S. If we have an exact sequence*

$$0 \to S/J \to S/I \to S/J' \to 0$$

then $\mathrm{Ass(S/I)} \subseteq \mathrm{Ass(S/J)} \cup \mathrm{Ass(S/J')}$.

Not only the associated primes but also the minimal primary components of an ideal are unique, which is stated in the following theorem.

Theorem 1.11. *(Theorem 4.10 in [2]). The primary components corresponding to minimal prime ideals are uniquely determined by I.*

Remark 1.12. Primary decomposition is not unique due to embedded primary components. Here is an example; $\langle x^2, xy\rangle = \langle x^2, xy, y^2\rangle \cap \langle x\rangle = \langle x^2, y\rangle \cap \langle x\rangle$.

Primary decomposition of ideals is generalized to finitely generated modules over Noetherian rings. This analogous theory is not developed only to obtain a general perspective, some of the results for ideals use the theory of primary decomposition for modules; for example is the proof of Lemma 1.10.

There are different algorithms for computing primary decompositions of polynomial ideals. The most famous one was designed by Gianni, Trager and Zacharias, (see [35]) which computes primary decomposition by reducing to the univariate case. Another important technique for primary decomposition was introduced by Eisenbud, Huneke and Vasconcelos in [36]. This is mainly based on homological methods. The work of Shimoyama and Yokoyama in [37] offers a third approach that relies on the decomposition of ideal into "pseudo" primary ideals. A pseudo-primary ideal is an ideal whose radical is a prime ideal. A detailed comparison of algorithms for primary decomposition is given in [38].

1.2 Binomial Ideals

A *binomial* is a polynomial with at most two terms and a *binomial ideal* is an ideal generated by binomials. Binomial ideals form an important link between commutative algebra and combinatorics. Beyond its intrinsic mathematical interest, binomial commutative algebra has varied applications [28], for instance in the dynamics of chemical reactions under mass-action kinetics, algebraic statistics and combinatorial game theory, see [33], [11] for references and details. Binomial ideals are also very important for the study of hypergeometric differential equations, these applications can be found in [9] and [10].

Definition 1.13. Let $S = \Bbbk[x_1, ..., x_n]$. A binomial in S is defined as the difference of two terms, $\alpha x^a - \beta x^b$, where $\alpha, \beta \in \Bbbk$ and $a, b \in \mathbb{Z}^n_{>0}$. (Here we use the multi-index notation: $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$.)

Definition 1.14. A *binomial ideal* of S is an ideal whose generators can be chosen as binomials.

Before jumping to the properties of binomial ideals, let us first consider the primary decomposition of monomial ideals. Monomial ideals are ideals generated by monomials, and are therefore also binomial ideals. Most ideal theoretic operations are far simpler for monomial ideals than in general. One of those operations is primary decomposition. We first describe what a primary monomial ideal looks like. See [17] for more details about monomial ideals.

Proposition 1.15. *A monomial ideal $I \in \Bbbk[x_1, ..., x_n]$ is primary if and only if*

$$I = \langle x_{i_1}^{m_1}, ..., x_{i_1}^{m_1} \mid \textit{ some other monomials in } x_{i_1}, ..., x_{i_1} \rangle.$$

An approach to decomposition of monomial ideals is finding the irreducible decomposition. An irreducible monomial ideal is generated by pure powers of variables. This decomposition can be found by using that for a minimal generator $m = m_1 m_2 \in I$ where m_1 and m_2 are relatively prime monomials we can write $I = (I + \langle m_1 \rangle) \cap (I + \langle m_2 \rangle)$. Irreducible decomposition of monomial ideals is unique. This follows from the uniqueness of irreducible resolutions, see Theorem 2.4 in [26]. Another approach is based on Alexander duality, for definitions and algorithms see [27].

If we look at the irreducible decomposition of binomial ideals the components are not necessarily binomial as was shown in [23].

The variety of a monomial ideal is a union of coordinate planes. Any affine variety can be defined using trinomials [12], simply by adding new variables. This means that the geometry coming from trinomial ideals is general, we cannot hope for special algebraic properties in this context. In between these two, we have binomial ideals whose geometry is special and we have effective combinatorial tools to apply to their study. Varieties associated to binomial ideals are unions of toric varieties.

The important article of Eisenbud and Sturmfels [12] can be seen as the starting point for all research related with primary decomposition of binomial ideals. They proved that the associated primes, the primary components and the radical of a binomial ideal are binomial when $\Bbbk$ is algebraically closed. The fundamental fact about binomial ideals and the key ingredient in Eisenbud and Sturmfels' arguments is that every reduced Gröbner basis of a binomial ideal consists of binomials. Indeed, this fact gives us operations which preserve binomiality.

We first review some of the results of [12] and recall some of their tools. Let us start describing "binomial friendly" operations.

The following facts can be easily proved. The sum of two binomial ideals is a binomial ideal. Every monomial can be considered as a binomial, so every monomial ideal is a binomial ideal. On the other hand, the intersections of binomial ideals need not to be a binomial ideal. For example: $\langle x-1\rangle \cap \langle x-3\rangle = \langle x-1\rangle \cdot \langle x-3\rangle = \langle x^2-4x+3\rangle$. Also, if $\Bbbk$ is not algebraically closed, the primary components of a binomial ideal need not to be binomial. Let $I = \langle x^3-1\rangle \subseteq \mathbb{R}[x]$ where $\mathbb{R}$ denotes the real numbers. Then I has a unique primary component which is not binomial: $\langle x^3-1\rangle = \langle x-1\rangle \cap \langle x^2+x+1\rangle$. For the rest of this dissertation, unless otherwise stated, we assume $\Bbbk$ is algebraically closed. There are several results for which the characteristic zero hypothesis is necessary, but that will be stated explicitly when necessary. We assume that $\Bbbk$ has arbitrary characteristic unless otherwise stated.

Definition 1.16. A *monomial order* on $\Bbbk[x_1, ..., x_n]$ is any relation $>$ on $\mathbb{N}^n$, or equivalently, any relation on the set of monomials $x^\alpha, \alpha \in \mathbb{N}^n$ satisfying

- $>$ is a total order on $\mathbb{N}^n$.
- If $\alpha > \beta$ and $\gamma \in \mathbb{N}^n$, then $\alpha + \gamma > \beta + \gamma$.
- Every nonempty subset of $\mathbb{N}^n$ has a smallest element under $>$.

Theorem 1.17. *Let $>$ be a monomial order on $S = \Bbbk[x_1, ..., x_n]$ and let I be an ideal of S. I is a binomial ideal if and only if the reduced Gröbner basis G of I with respect to $>$ consists of binomials.*

Proof. This is Corollary 1.2 in [12], here is the sketch of the proof. Let I be a binomial ideal. If we take the binomial generating set of I, the S-polynomials of these generators as needed in the Buchberger algorithm are again binomial. The converse follows from the fact that reduced Gröbner basis is unique with respect to given order $>$ and it is also a basis for the ideal. For more explanation about these concepts, see [5]. $\square$

Given $I \subseteq \Bbbk[x_1, ..., x_n]$, the *$r$-th elimination ideal* is the ideal of $\Bbbk[x_1, ..., x_r]$ defined by $I \cap \Bbbk[x_1, ..., x_r]$. Let G be a reduced Gröbner basis with respect to a monomial order $<$. Since the r-th elimination ideal is generated by a subset of the reduced Gröbner basis of I (see Theorem 2 in [5]), namely $G \cap \Bbbk[x_1, ..., x_r]$, we have the following proposition.

Proposition 1.18. *If $I \subseteq S$ is a binomial ideal, then the elimination ideal $I \cap S$ is a binomial ideal for every $r \leqslant n$.*

Here is another useful fact.

Proposition 1.19. *Let I be a binomial ideal in S and let M be a monomial ideal in S. If $f \in I + M$ and f' is the sum of the terms of f that are not individually contained in $I + M$, then $f' \in I$.*

Proof. See Proposition 1.10 in [12]. □

We have mentioned that the operation of intersection of ideals does not in general preserve binomiality, but here is one of the exceptions.

Proposition 1.20. *If I is a binomial ideal and M is a monomial ideal in S, then $I \cap M$ is a binomial ideal.*

Proof. Introduce a new variable t. We know that $I \cap M = (It + M(1-t))S[t] \cap S$. The ideal $(It + M(1-t))S[t]$ is a binomial ideal in $S[t]$. By Proposition 1.18, $I \cap M$ is binomial. □

We now review a commutative algebra fact.

Proposition 1.21. *Let I be an ideal in $S = \Bbbk[x_1, ..., x_n]$ and $g \in S$. If $\{f_1, ..., f_s\}$ is a basis of $I \cap \langle g \rangle$, then $(I : g) = \langle \frac{f_1}{g}, ..., \frac{f_s}{g} \rangle$.*

Proof. Let $q \in \langle \frac{f_1}{g}, ..., \frac{f_s}{g} \rangle$, then $q \cdot g \in \langle f_1, ..., f_s \rangle = I \cap \langle g \rangle \subseteq I$. This implies that $q \in (I : g)$.

Let $q \in (I : g)$, and therefore $q \cdot g \in I$, also $q \cdot g \in I \cap \langle g \rangle$ which implies $q \cdot g = a_1 f_1 + ... + a_s f_s$. Thus $q = a_1 \frac{f_1}{g} + ... + a_s \frac{f_s}{g}$ where each $\frac{f_i}{g}$ is a polynomial since $f_i \in I \cap \langle g \rangle$. This implies that $q \in \langle \frac{f_1}{g}, ..., \frac{f_s}{g} \rangle$. □

Using the Proposition 1.21, it is not difficult to derive the fact that the ideal quotient of a binomial ideal by a monomial is binomial. Note that quotients of binomial ideals by monomial ideals or a binomial are generally not binomial.

Proposition 1.22. *If I is a binomial ideal and m is a monomial then,* $(I : m)$ *and* $(I : m^\infty)$ *are binomial.*

Proof. By Proposition 1.20, $I \cap \langle m \rangle$ has binomial generators $\{f_1, ..., f_s\}$. Then $(I : m) = \langle \frac{f_1}{m}, ..., \frac{f_s}{m} \rangle$ is also binomial. By Corollary 1.4, the proof for $(I : m^\infty)$ is easy. □

There are algorithms to compute the saturation of any ideal in $\Bbbk[x_1, ..., x_n]$. One such algorithm is described in Chapter 4 in [5] and is based on the same ideas as in the proof of Proposition 1.22. The main tool is a Gröbner basis computation in $n + 1$ variables. Another useful approach is given by Sturmfels in Algorithm 12.3 in [32]. These algorithms are implemented, so we can compute saturation by using a computer algebra system such as *Macaulay* 2 [7] or *Singular* [16].

Definition 1.23. Let $<$ be a fixed monomial order. Assume m_1 and m_2 are monomials with $m_1 < m_2$. Let $b = m_1 - m_2$, we define $b^{[d]} = m_1^d - m_2^d$ and call this binomial the *d-th quasi-power* of b. $I^{[d]}$ is the ideal generated by d-th quasi-powers of elements of I.

The ordinary powers of a binomial are not binomials and taking quasi-power of an ideal is a natural operation which preserves binomiality. There can be an ambiguity with the sign of the quasi-power. For example, let $d = 2$ and consider the binomial $-x^3 - (-y^2) = y^2 - x^3$. The second quasi-power of the binomial in the left hand side is $x^6 - y^4$, on the other hand the second quasi-power of the binomial on the right hand side is $y^4 - x^6$. To remove that ambiguity we choose a monomial order.

1.3 Lattice Ideals

We define a special type of binomial ideals, lattice ideals, which have nice combinatorial features. In Theorem 1.30, we determine primary decomposition of lattice ideals.

Definition 1.24. A *lattice* $\mathcal{L} \subseteq \mathbb{Z}^n$ is a finitely generated free abelian subgroup.

The saturation of $\mathcal{L}$ is the lattice

$$\mathrm{Sat}(\mathcal{L}) = \{\ell \in \mathbb{Z}^n : k\ell \in \mathcal{L} \text{ for some } k \in \mathbb{Z}\}.$$

A lattice is *saturated* if it satisfies $\mathrm{Sat}(\mathcal{L}) = \mathcal{L}$.

Definition 1.25. A *partial character* is a pair $(\mathcal{L}, \rho)$ consisting of a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$ and a group homomorphism $\rho : \mathcal{L} \to \Bbbk^*$ from the additive group $\mathcal{L}$ to the multiplicative group $\Bbbk^* = \Bbbk - \{0\}$.

A partial character is *saturated* if its lattice is a saturated lattice. A partial character $(\mathcal{L}, \rho)$ is a *saturation* of $(\hat{\mathcal{L}}, \hat{\rho})$ if $\mathcal{L} = \mathrm{Sat}(\hat{\mathcal{L}})$ and $\hat{\rho} = \rho_{|\hat{\mathcal{L}}}$.

We can associate an ideal to each partial character

$$I_+(\rho) = \langle x^{u_+} - \rho(u)x^{u_-} \mid u \in \mathcal{L} \rangle \subseteq \Bbbk[x_1, ..., x_n]$$

which is called *lattice ideal.* We have a nice characterization of lattice ideals.

Lemma 1.26. *(Corollary 2.5 in [12].) A proper binomial ideal $I \subseteq S$ not containing any monomial is a lattice ideal if and only if $I = (I : (\prod x_i)^\infty)$, in other words, every variable is a nonzerodivisor modulo I.*

The arithmetic properties of partial characters are used to provide characterizations of prime binomial ideals. The following statement will describe the form of prime binomial ideals.

Theorem 1.27. *(Corollary 2.6 in [12].) Let P be a binomial ideal in S. Set $\{y_1, ..., y_\ell\} := \{x_1, ..., x_n\} \cap P$ and let $\{z_1, ..., z_k\} := \{x_1, ..., x_n\} - P$. The ideal P is prime if and only if*

$$P = I_+(\rho) + \langle y_1, ..., y_\ell \rangle$$

for a saturated partial character $(\mathcal{L}, \rho)$ on $\mathbb{Z}^k$ corresponding to $z_1, ..., z_k$.

A commutative *semigroup* $\mathcal{Q}$ is a set with an associative, commutative binary operation. If $\mathcal{Q}$ has an identity, it is called a *monoid*. The semigroup algebra is $\Bbbk[\mathcal{Q}] = \sum_{q \in \mathcal{Q}} \Bbbk \cdot t^q$ with multiplication given by $t^a \cdot t^b = t^{a+b}$. Let us fix a subset $A \subseteq \mathbb{Z}^d$ and define a semigroup algebra homomorphism $\alpha : \Bbbk[x_1, ..., x_n] \to \Bbbk[\mathbb{Z}^n]$ that maps x_i to t^{a_i}. The ideal $\ker\alpha$ is denoted I_A and called the *toric ideal* associated to A. In fact, all affine toric varieties arise in this way. We have $\Bbbk[x_1, ..., x_n]/I_A \cong \Bbbk[\mathbb{N}A]$, so I_A is a prime ideal, since $\Bbbk[\mathbb{N}A]$ is an integral domain as a subring of $\Bbbk[\mathbb{Z}^n]$. Moreover, I_A is equal to a binomial ideal as follows

$$I_A = \langle x^u - x^v \mid \alpha(x^u) = \alpha(x^v) \rangle.$$

See Lemma 4.1 in [32] for more explanation. All prime binomial ideals are translated toric ideals as follows. If ρ is not the trivial character, then we define an isomorphism between $I_+(\rho)$ and I_A by rescaling the variables $x_i \mapsto \rho(e_i)x_i$, which induces a rescaling $x^u \mapsto \rho(v)x^v$ on general monomials.

Toric geometry has many applications, these ideals encode the combinatorics of polytopes and give interactions among algebra, geometry and combinatorics. For more information and for details about toric varieties, see the classical text [14] or the newer one [6].

Definition 1.28. Let p be a prime number. We define $\mathrm{Sat}_p(\mathcal{L})$ and $\mathrm{Sat}'_p(\mathcal{L})$ to be the largest sublattices of $\mathrm{Sat}(\mathcal{L})$ containing $\mathcal{L}$ such that $\mid \mathrm{Sat}_p(\mathcal{L})/\mathcal{L} \mid = p^k$ for some $k \in \mathbb{Z}$ and $\mid \mathrm{Sat}'_p(\mathcal{L})/\mathcal{L} \mid = g$ where $(p, g) = 1$.

Then we can write

$$\mathrm{Sat}_p(\mathcal{L}) = \{m \in \mathrm{Sat}(\mathcal{L}) \mid p^e m \in \mathcal{L} \text{ for some } e \in \mathbb{N}\}.$$

$$\mathrm{Sat}'_p(\mathcal{L}) = \{m \in \mathrm{Sat}(\mathcal{L}) \mid dm \in \mathcal{L} \text{ for some } d \in \mathbb{N} \text{ such that } (d, p) = 1\}.$$

Remark 1.29. If $p = 0$, we adopt the convention that $\mathrm{Sat}'_p(\mathcal{L}) = \mathrm{Sat}(\mathcal{L})$ and $\mathrm{Sat}_p(\mathcal{L}) = \mathcal{L}$.

The following result describes the associated primes and the corresponding primary components of lattice ideals by using the saturations of lattices.

Theorem 1.30. *(Corollary 2.2 in [12]). Let* $\Bbbk$ *be a field and* $\mathrm{char}(\Bbbk) = p \geqslant 0$*. Let* $(\mathcal{L}, \rho)$ *be a partial character. If* $|\mathrm{Sat}'_p(\mathcal{L})/\mathcal{L}| = g$*, then there are* g *distinct characters* $(\mathrm{Sat}'_p(\mathcal{L}), \rho_1)$*, ...,* $(\mathrm{Sat}'_p(\mathcal{L}), \rho_g)$ *that extend* $(\mathcal{L}, \rho)$*, and for each* $(\mathrm{Sat}'_p(\mathcal{L}), \rho_i)$ *there exists a unique character* $(\mathrm{Sat}(\mathcal{L}), \hat{\rho}_i)$ *that extends* $(\mathrm{Sat}'_p(\mathcal{L}), \rho_i)$*. There is a unique partial character* $(\mathrm{Sat}_p(\mathcal{L}), \rho')$ *that extends* $(\mathcal{L}, \rho)$*. The radical, associated primes and minimal primary decomposition of* $I_+(\rho) \subset \Bbbk[x_1, ..., x_n]$ *are*

$$\sqrt{I_+(\rho)} = I_+(\rho'),$$

$$\mathrm{Ass(S/I_+(\rho)) = \{I_+(\hat{\rho}_i) \mid i = 1, ..., g\}}$$

and

$$I_+(\rho) = \bigcap_{i=1}^{g} I_+(\rho_i)$$

where $I_+(\rho_i)$ is $I_+(\hat{\rho}_i)$- primary. In particular, the associated primes $I_+(\hat{\rho}_i)$ of $I_+(\rho)$ are all minimal.

All binomial ideals in the Laurent polynomial ring $k[x_1^{\pm}, ..., x_n^{\pm}]$ are lattice ideals. Theorem 1.30 is also true for binomial ideals in $k[x_1^{\pm}, ..., x_n^{\pm}]$. In fact, since it is easier to work in the Laurent polynomial ring, the proof of this theorem was done first in $k[x_1^{\pm}, ..., x_n^{\pm}]$, then it was finished by taking the contraction from the Laurent polynomial ring.

Remark 1.31. If $p = 0$, $I_+(\tilde{\rho}_i) = I_+(\rho_i)$ in Theorem 1.30, which implies that lattice ideals are radical in $\mathrm{char}(\Bbbk) = 0$, as they are equal to the intersection of prime ideals. Note that lattice ideals do not have embedded associated primes, which means that primary decomposition of lattice ideals is unique. Also, lattice ideals do not contain monomials. This follows from, for instance, Lemma 1.42.

We know that some quasi-power of the lattice ideal is contained in its radical in $\mathrm{char}(\Bbbk) = p > 0$. The next proposition describes this power.

Proposition 1.32. *Let* $\Bbbk$ *be an algebraically closed field and* $\mathrm{char}(\Bbbk) = p > 0$*. Let $I_+(\rho)$ be a lattice ideal with partial character (L_ρ, ρ) and let q be the order of the group* $\mathrm{Sat}_p L_\rho / L_\rho$*, then*

$$\left(\sqrt{I_+(\rho)}\right)^{[q]} \subseteq I_+(\rho).$$

Proof. Since $\sqrt{I_+(\rho)}$ is a lattice ideal, we can write $\sqrt{I_+(\rho)} = I_+(\tilde{\rho})$ for some partial character $(L_{\tilde{\rho}}, \tilde{\rho})$. If $x^{\mu_+} - \tilde{\rho}(\mu)x^{\mu_-} \in I_+(\tilde{\rho})$ then $\mu \in \mathrm{Sat}_p(L_\rho)$. We want to show that the q-th quasi-power of $x^{\mu_+} - \tilde{\rho}(\mu)x^{\mu_-}$ is in $I_+(\rho)$. By the definition of $\mathrm{Sat}_p(L_\rho)$, $p^r\mu \in L_\rho$ for some r. Since q is the order of the group $\mathrm{Sat}_p L_\rho / L_\rho$, $q\mu \in L_\rho$. Then

$$x^{q\mu_+} - (\tilde{\rho}(\mu))^q x^{q\mu_-} = x^{q\mu_+} - \rho(q\mu)x^{q\mu_-} \in I_+(\rho),$$

since $\tilde{\rho}(q\mu) = \rho(q\mu)$.

□

1.4 Primary Decomposition of Binomial Ideals

Theorem 1.33. *(Theorem 6.1 in [12]). Let $\Bbbk$ be an algebraically closed field and I be a binomial ideal of $S = \Bbbk[x_1, ..., x_n]$. Then every associated prime of I is a binomial ideal.*

Proof. If $I = \langle x_1, ..., x_n\rangle$, then it is a prime ideal, so suppose I does not contain all the variables. If I is a lattice ideal $I = I_+(\rho)$ then by Theorem 1.30, the associated primes of I are generated by binomials. So assume $(I : x_i) \neq I$ for some i. We may assume $x_i \notin I$, for if $x_i \in I$, by reducing modulo x_i, we can find another variable x_j satisfying $(I : x_j) \neq I$ and $x_j \notin I$ by the assumption above. We do Noetherian induction, assuming that every binomial ideal of S strictly larger than I has binomial associated prime ideals. Then we use the exact sequence

$$0 \to S/(I : x_i) \to S/I \to S/(I + \langle x_i\rangle),$$

by Lemma 1.10, we know that $\mathrm{Ass(S/I)} \subseteq \mathrm{Ass(S/(I : x_i))} \cup \mathrm{Ass(S/(I + \langle x_i\rangle))}$. Note that both $(I : x_i)$ and $(I + \langle x_i\rangle)$ are binomial ideals so their associated primes are binomial, so by Noetherian induction, I has binomial associated primes. □

Definition 1.34. Let I be an ideal. The intersection of minimal primary components of I is denoted by $\mathrm{Hull}(I)$.

Let us introduce a new kind of binomial ideals. A binomial ideal is *cellular* if every variable is either a nonzerodivisor modulo I or is nilpotent modulo I. We look at the features and primary components of cellular binomial ideals from a general perspective in Section 3, but now we need the following property.

Theorem 1.35. *(Theorem 6.4 in [12]). If I is a cellular binomial ideal then* $\mathrm{Hull}(I)$ *is also a binomial ideal.*

The following theorem is a core result of [12].

Theorem 1.36. *Let I be a binomial ideal in $S = \Bbbk[x_1, ..., x_n]$, where $\Bbbk$ is algebraically closed. Let Ω be a finite set. Suppose $\{P_i \mid \in \Omega\}$ is the set of associated primes of I. Let δ_i be the set of nonzerodivisor variables of P_i and M_i be the maximal monomial ideal contained in P_i. We denote $\prod_{j\in\delta_i} x_j = x_{\delta_i}$*

1. If $\mathrm{char}(\Bbbk) = p > 0$*, then for sufficiently large powers $q = p^e$,*

$$I = \bigcap_{i\in\Omega} \mathrm{Hull}((I + P_i^{[q]}) : x_{\delta_i}^{\infty})$$

is a minimal primary decomposition of I into binomial ideals.

2. If $\mathrm{char}(\Bbbk) = 0$ *and e_i is a sufficiently large integer, then*

$$I = \bigcap_{i\in\Omega} \mathrm{Hull}((I + M_i^{e_i} + (P_i \cap \Bbbk[\delta_i])) : x_{\delta_i}^{\infty})$$

is a minimal primary decomposition of I into binomial ideals.

The main step of the proof of Theorem 1.36 is to show that the intersection of the proposed ideals are equal to I. The rest easily follows from the facts that these ideals are primary, and that the ideals inside the Hull operation are cellular, which preserves binomiality by Theorem 1.35.

Remark 1.37. Although the Hull operation is not explicit, we still derive the fact that in $\text{char}(\mathbb{k}) = 0$ the binomial parts of the associated primes are contained in the corresponding primary component of I. This feature in characteristic zero will be used to reduce the characterization of primary component to a monomial ideal problem which is more manageable and combinatorial see Theorem 2.2 in Section 2 for more detail. In the case that $\text{char}(\mathbb{k}) = p > 0$ on the other hand, every binomial in the associated primes has a Frobenius power that belongs to the corresponding primary component.

The following result can be derived from Theorem 1.36.

Theorem 1.38. *Let $\mathbb{k}$ be an algebraically closed field of arbitrary characteristic. Let $I \subseteq \mathbb{k}[x_1, ..., x_n]$ be a binomial ideal. Then the associated primes and corresponding primary components of I can be chosen binomial.*

Thomas Kahle has developed a Macaulay 2 package called *Binomials*, see [7], [20], which specializes well known algorithms to the case binomial ideals, namely primary decomposition, minimal primes, computations of the radical, etc. Computer algebra can implement operations in finite extensions of $\mathbb{Q}$. A *pure difference* ideal is an ideal whose generators are all differences of monic monomials. The binomial package is restricted to pure difference binomial ideals, since for them binomial primary decomposition exists in cyclotomic extensions of $\mathbb{Q}$. Let ω_ℓ denote an ℓ-th root of unity. Cyclotomic fields $\mathbb{Q}(\omega_\ell)$ can be constructed by taking the quotient of $\mathbb{Q}(x)$ modulo the principal ideal generated by the minimal polynomial of ω_ℓ. For further information about cyclotomic fields, see [19].

1.5 Primary Decomposition of Lattice Ideals in Positive Characteristic

The order of the group $\mathrm{Sat}'_p(\mathcal{L})/\mathcal{L}$ plays a key role in primary decomposition of lattice ideals. As we know in Theorem 1.30, the number of distinct saturations of the partial character defining the lattice ideal equals the order of the finite group $\mathrm{Sat}'_p(\mathcal{L})/\mathcal{L}$. We now review an algorithm for computing the distinct saturations of a partial character that was developed in [20].

Algorithm 1.39. Saturation of partial characters

Input: $(\mathcal{L}, \rho)$ a partial character where $\mathcal{L}$ is generated by the columns of a matrix L.

Output: All distinct saturations $(\mathrm{Sat}(\mathcal{L}), \rho_i)$, $i = 1, ..., n$.

1. Compute the saturation $\mathcal{L}' = Sat(\mathcal{L})$, for example, find the Hermite normal form of the matrix L and divide each column by the greatest common divisor of its elements. Let L' be the matrix for $\mathcal{L}'$.

2. Express the generators of $\mathcal{L}$ in terms of the generators of $\mathcal{L}'$, by solving the matrix system.

$$L = L'D$$

for the $r \times r$ square matrix $D = (d_{ij})$ where $r = rank(L) = rank(L')$.

3. Let ℓ_j, ℓ'_j and d_j be the columns of L, L' and A respectively. Let ρ' be one of the saturations. For the values that ρ takes on the columns of L', define a new variable; $y_i = \rho(\ell'_i)$ for $i = 1, ..., r$. Compute the following Laurent binomial ideal in $\mathbb{Q}[y_1, ..., y_r]$

$$J = \langle \rho(\ell_j) - \prod_{i=1}^{r} y_i^{d_{ij}} \mid j = 1, ..., r \rangle.$$

4. Compute

$$J' = J \cap \mathbb{Q}[y_1, ..., y_r] = \left(\langle y^{d^+} - \rho(\ell_j) y^{d^-} \mid j = 1, ..., r \rangle : (\prod_{i=1}^{r} y_i)^\infty \right).$$

5. J' is a zero dimensional ideal. Solve J' over a suitable extension of $\mathbb{Q}$ and output L' together with the list of solutions of J'.

Remark 1.40. We point out that saturations of partial characters exist only when $\Bbbk$ is algebraically closed. The proof of the correctness of this algorithm was provided originally in [20]. We constructed J, since for each generator ℓ of L, we have a relation $\ell = L'k$, and ρ' and ρ must take the same values on the generators of L. Thus, the solutions of J' give us the saturation of the partial character $(\mathcal{L}, \rho)$. The degree is equal to $\mid \mathcal{L}'/\mathcal{L} \mid$. We can find the solutions over a cyclotomic field $\mathbb{Q}(\xi_\ell)$ because the ideal is a zero-dimensional pure difference binomial ideal. For more details, see [20].

Definition 1.41. Let $B = (b_{ij})$ be an $n \times r$ integer matrix. If $b \in \mathbb{Z}^n$, define $b^+, b^- \in \mathbb{N}^n$ via $(b^+)_i = \max(b_i, 0)$ and $(b^-)_i = \max(-b_i, 0)$, so that $b = b^+ - b^-$. Form the ideal $I(B)$ from the columns $b_1, ..., b_r$ of B

$$I(B) = \langle x^{b_1^+} - x^{b_1^-}, ..., x^{b_r^+} - x^{b_r^-} \rangle.$$

The binomial ideal $I(B)$ is called the *lattice basis ideal* associated to B.

Lemma 1.42. *Let $B \in \mathbb{Z}^{n \times k}$ be an integer matrix and let $\mathcal{L}$ be the lattice generated by the columns of B. Let $(\mathcal{L}, \rho)$ be the trivial partial character. The lattice ideal $I_+(\rho)$ is computed from $I(B)$ by taking the saturation with respect to the product of all the variables*

$$I_+(\rho) = (I(B) : (x_1 ... x_n)^\infty).$$

Proof. We know that the lattice ideal $I_+(\rho)$ is saturated by Lemma 1.26 and it contains $I(B)$, so the ideal on the right hand side is contained in $I_+(\rho)$. For the converse, let $u \in \mathcal{L}$, then $u = \sum_{i=1}^r a_i b_i$ for some $a_i \in \mathbb{Z}$. This implies that

$$\frac{x^{u^+}}{x^{u^-}} - 1 = \prod \left(\frac{x^{b_i^+}}{x^{b_i^-}} \right)^{a_i} - 1.$$

If we clear the denominators we get that

$$\prod_{i=1}^r (x^{b_i^-})^{a_i} (x^{u^+} - x^{u^-}) = x^{u^-} \left(\prod_{i=1}^r (x^{b_i^+})^{a_i} - \prod_{i=1}^r (x^{b_i^-})^{a_i} \right).$$

We want to show that the right hand side lies in $I(B)$. If $x^{c^+} - x^{c^-}$ and $x^{d^+} - x^{d^-}$ lie in an ideal J then $x^{d^+}(x^{c^+} - x^{c^-}) + x^{d^-}(x^{c^+} - x^{c^-}) = x^{c^+}x^{d^+} - x^{c^-}x^{d^-}$ also lies in the ideal J. If we apply this argument to the generators of $I(B)$, we obtain that $\prod_{i=1}^r (x^{b_i^+})^{a_i} - \prod_{i=1}^r (x^{b_i^-})^{a_i}$ also lies in $I(B)$. This means that a monomial multiple of $x^{u^+} - x^{u^-}$ lies in $I(B)$ as well. □

We denote I_B the lattice ideal $(I(B) : (x_1 \ldots x_n)^\infty)$ induced by the matrix B, with trivial partial character.

As we observe in Theorem 1.30, the order of the group $\mathrm{Sat}'_p(\mathcal{L})/\mathcal{L}$ plays a key role in primary decomposition of lattice ideals in positive characteristic. Indeed, the number of distinct saturations of partial character equal to the order of the finite group $\mathrm{Sat}'_p(\mathcal{L})/\mathcal{L}$.

Let B be a nonzero $(m \times n)$- matrix over $\mathbb{Z}$, which defines the lattice ideal I_B. There exist invertible $(m \times m)$ and $(n \times n)$- matrices, $U = (u_{ij})$ and $V = (v_{ij})$

respectively, so that $U \cdot B \cdot V$ is the product

$$D = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & s_r \\ 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and the diagonal elements s_i satisfy s_i/s_{i+1} for all $1 \leqslant i \leqslant r$. This is the *Smith Normal Form* of the matrix B. Factor $s_i = p_1^{a_1^i} \cdot p_2^{a_2^i} \cdots p_k^{a_k^i}$ and let $a_l^i \leqslant a_l^j$ if $i < j$.

The lattice basis ideal of D is

$$I(D) = \langle y_1^{s_1} - 1, y_2^{s_2} - 1, ..., y_r^{s_r} - 1 \rangle.$$

The lattice basis ideal of D is equal to the lattice ideal corresponding to D, since $I_D = (I(D) : (\prod_{i=1}^n y_i)^\infty) = I(D)$. We show the last equality. By Corollary 1.4, it is sufficient to check $(I(D) : y_i) \subseteq I(D)$. $I(D)$ and $\langle y_i \rangle$ are comaximal, since $I(D) + \langle y_i \rangle = \mathbb{k}[y_1, ..., y_n]$, consequently the intersection is equal to $\langle y_1^{s_1} y_i - y_i, ..., y_r^{s_r} y_i - y_i \rangle$. By Proposition 1.21, $(I(D) : y_i) = \langle \frac{y_1^{s_1} y_i - y_i}{y_i}, ..., \frac{y_r^{s_r} y_i - y_i}{y_i} \rangle$ which is $I(D)$ again.

To prove the fact $I_D = I(D)$, we can also use the following fact which is Theorem 2.9 in [34].

Lemma 1.43. *Let B be an integer matrix whose entries are non-negative and whose columns are linearly independent. Then the lattice basis ideal corresponding to B is a lattice ideal.*

Fix p_ℓ to be the characteristic of our field as where $\ell \in \{1, ..., k\}$. Note that when $\ell \notin \{1, ..., k\}$, the primary decomposition of I_D is similar to the primary decomposition in characteristic 0 and we are not interested in it. We shorten the notation $p_\ell^{a_\ell^j} = \alpha_j$ for convenience. Let C_j be the set of $c_j = s_j/\alpha_j$-th roots of unity for all $1 \leqslant j \leqslant r$. Thus, $C_j = \{\omega_{c_j}, \omega_{c_j}^2, ..., \omega_{c_j}^{c_j-1}, 1\}$. Let $\rho_{i_1,...i_r}$ be the partial characters of the form

$$\rho_{i_1,...,i_r} := \begin{cases} e_1 \to i_1 \\ e_2 \to i_2 \\ \vdots \\ e_r \to i_r \end{cases}$$

where i_j runs over all elements of C_j. By Corollary 2.2 in [12] , each partial character induces the following associated prime ideal of I_D

$$I_{i_1,...,i_r} = \langle y_1 - i_1, ..., y_r - i_r \rangle.$$

Hence there are $s_1 \cdot s_2 \cdots s_r/\alpha_1 \cdots \alpha_r$- many associated primes of I_D.

Proposition 1.44. *Let I_D and $I_{i_1,...,i_r}$ be as above where* $\mathrm{char}(\Bbbk) = p_\ell$. *If $q = \alpha_1 \cdots \alpha_r$, then the $I_{i_1,...,i_r}$-primary component of I_D is*

$$I'_{i_1,...,i_r} = I_D + (I_{i_1,...,i_r})^{[q]}.$$

Proof. We claim that the ideals $I'_{i_1,...,i_r}$ are primary and their intersection is equal to I_D. Note that the set of ideals we constructed are pairwise comaximal. Thus the intersection of those ideals is equal to their product.

$$\begin{aligned}
\bigcap_{i_t \in C_t} I'_{i_1,\dots,i_r} &= \bigcap_{\substack{i_t \in C_t \\ t \neq r}} (I'_{i_1,\dots,i_{r-1},1} \cap I'_{i_1,\dots,i_{r-1},\omega_{c_r}} \cap \dots \cap I'_{i_1,\dots,i_{r-1},\omega_{c_r}^{c_r-1}}) \\
&= \bigcap_{\substack{i_t \in C_t \\ t \neq r}} I_D + \langle y_1^q - i_1, \dots, y_{r-1}^q - i_{r-1}, y_r^{qc_r} - 1 \rangle \\
&= \bigcap_{\substack{i_t \in C_t \\ t \neq r-1}} \bigcap_{i_{r-1} \in C_{r-1}} I_D + \langle y_1^q - i_1, \dots, y_{r-1}^q - i_{r-1}, y_r^{qc_r} - 1 \rangle \\
&= \bigcap_{\substack{i_t \in C_t \\ t \neq r-2}} \bigcap_{i_{r-2} \in C_{r-2}} I_D + \langle y_1^q - i_1, \dots, y_{r-1}^{qc_{r-1}} - 1, y_r^{qc_r} - 1 \rangle \\
&\;\vdots \\
&= I_D + \langle y_1^{qc_1} - 1, \dots, y_{r-1}^{qc_{r-1}} - 1, y_r^{qc_r} - 1 \rangle \\
&= I_D.
\end{aligned}$$

Now we need to show that each $I'_{i_1,\dots,i_r}$ is primary. Indeed, if we show that $I'_{i_1,\dots,i_r} = \langle y_1^{\alpha_1} - t_1, \dots, y_r^{\alpha_r} - t_r \rangle$ for some $t_j \in \Bbbk$, we are done by Theorem 1.30. Remember that $p_\ell^{a_\ell^j} = \alpha_j$.

Let us fix $j \in \{1, \dots, r\}$. Since α_j is the greatest common divisor of s_j and q, we have $m, n \in \mathbb{Z}$ such that $ms_j + qn = \alpha_j$. We wish show that

$$\langle y_j^{s_j} - 1, y_j^q - i_j^q \rangle = \langle y_j^{\alpha_j} - i_j^{qn} \rangle.$$

Since $s_j, q > 0$, without loss of generality $m < 0$ and $n > 0$. Then $y_j^{-ms_j} - 1$ and $y_j^{qn} - i_j^{qn}$ are elements of the ideal on the left hand side. The binomials $y_j^{ms_j+qn}(y_j^{-ms_j} - 1)$ and $y_j^{qn} - y_j^{ms_j+qn}$ are elements of the ideal on the left hand side, so is the binomial $y_j^{\alpha_j} - i_j^{qn}$.

On the other hand, recall that $c_j = s_j/\alpha_j$ and let $d_j = q/\alpha_j$. Thus, the binomials

$y_j^{\alpha_j c_j} - i_j^{qnc_j}$, $y_j^{s_j} - 1$, $y_j^{\alpha_j d_j} - i_j^{qnd_j}$ and $y_j^q - i_j^{(1-mc_j)q}$ lie in the ideal on the right hand side, so is the binomial $y_j^q - i_j^q$. This shows that

$$I'_{i_1,\dots,i_r} = I_D + \langle y_1^q - i_1^q, \dots, y_r^q - i_r^q \rangle = \langle y_1^{\alpha_1} - t_1, \dots, y_r^{\alpha_r} - t_r \rangle$$

for some $t_j \in \Bbbk$. □

When characteristic of $\Bbbk$ is $p \neq p_i$ for all $i \in \{1, \dots, k\}$, the primary decomposition of I_D is equal to the primary decomposition in $\mathrm{char}(\Bbbk) = 0$ and the primary components are $I_{i_1,\dots,i_r}$.

The following example illustrates how the operations defined above work. All the computations are performed using the computer algebra system *Singular*, [16].

Example 1.45. *Let*

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Let $\mathrm{char}(\Bbbk) = 2$ *and consider the lattice ideal* I_D *in* $\Bbbk[y_1, y_2, y_3]$

$$I_D = \langle y_1^2 - 1, y_2^6 - 1, y_3^6 - 1 \rangle.$$

There are $|\mathrm{Sat}'_2(\mathcal{L})/\mathcal{L}| = 9-$*many associated primes. The partial characters for associated primes are*

$$\rho'_{1,1,1} = \begin{cases} e_1 \to 1 \\ e_2 \to 1 \\ e_3 \to 1 \end{cases} \quad \rho'_{1,1,\omega_3^1} = \begin{cases} e_1 \to 1 \\ e_2 \to 1 \\ e_3 \to \omega_3^1 \end{cases} \quad \rho'_{1,1,\omega_3^2} = \begin{cases} e_1 \to 1 \\ e_2 \to 1 \\ e_3 \to \omega_3^2 \end{cases}$$

$$\rho'_{1,\omega_3^1,1} = \begin{cases} e_1 \to 1 \\ e_2 \to \omega_3^1 \\ e_3 \to 1 \end{cases} \quad \rho'_{1,\omega_3^1,\omega_3^1} = \begin{cases} e_1 \to 1 \\ e_2 \to \omega_3^1 \\ e_3 \to \omega_3^1 \end{cases} \quad \rho'_{1,\omega_3^1,\omega_3^2} = \begin{cases} e_1 \to 1 \\ e_2 \to \omega_3^1 \\ e_3 \to \omega_3^2 \end{cases}$$

$$\rho'_{1,\omega_3^2,1} = \begin{cases} e_1 \to 1 \\ e_2 \to \omega_3^2 \\ e_3 \to 1 \end{cases} \quad \rho'_{1,\omega_3^2,\omega_3^1} = \begin{cases} e_1 \to 1 \\ e_2 \to \omega_3^2 \\ e_3 \to \omega_3^1 \end{cases} \quad \rho'_{1,\omega_3^2,\omega_3^2} = \begin{cases} e_1 \to 1 \\ e_2 \to \omega_3^2 \\ e_3 \to \omega_3^2 \end{cases}$$

where ω_3 is a primitive cubic root of unity. The corresponding primary components are

$$I'_{1,1,1} = I_D + \langle y_1^8 - 1, y_2^8 - 1, y_3^8 - 1\rangle,$$

$$I'_{1,1,\omega_3^1} = I_D + \langle y_1^8 - 1, y_2^8 - 1, y_3^8 - \omega_3^2\rangle,$$

$$I'_{1,1,\omega_3^2} = I_D + \langle y_1^8 - 1, y_2^8 - 1, y_3^8 - \omega_3^1\rangle,$$

$$I'_{1,\omega_3^1,1} = I_D + \langle y_1^8 - 1, y_2^8 - \omega_3^2, y_3^8 - 1\rangle,$$

$$I'_{1,\omega_3^1,\omega_3^1} = I_D + \langle y_1^8 - 1, y_2^8 - \omega_3^2, y_3^8 - \omega_3^1\rangle,$$

$$I'_{1,\omega_3^1,\omega_3^2} = I_D + \langle y_1^8 - 1, y_2^8 - \omega_3^2, y_3^8 - \omega_3^2\rangle,$$

$$I'_{1,\omega_3^2,1} = I_D + \langle y_1^8 - 1, y_2^8 - \omega_3^1, y_3^8 - 1\rangle,$$

$$I'_{1,\omega_3^2,\omega_3^1} = I_D + \langle y_1^8 - 1, y_2^8 - \omega_3^1, y_3^8 - \omega_3^1\rangle,$$

$$I'_{1,\omega_3^2,\omega_3^2} = I_D + \langle y_1^8 - 1, y_2^8 - \omega_3^1, y_3^8 - \omega_3^2\rangle.$$

The intersection of first three ideals is

$$J_1 = I_D + \langle y_1^8 - 1, y_2^8 - 1, y_3^{24} - 1\rangle.$$

Intersecting the other three ideals I_{ρ_4}, I_{ρ_5} and I_{ρ_6}, we obtain

$$J_2 = I_D + \langle y_1^8 - 1, y_2^8 - \omega_3^2, y_3^{24} - 1\rangle.$$

The intersection of I_{ρ_7}, I_{ρ_8} and I_{ρ_9} is

$$J_3 = I_D + \langle y_1^8 - 1, y_2^8 - \omega_3^1, y_3^{24} - 1\rangle.$$

Finally, take the intersection of J_1, J_2 and J_3

$$J_1 \cap J_2 \cap J_3 = I_D + \langle y_1^8 - 1, y_2^{24} - 1, y_3^{24} - 1\rangle \ = I_D.$$

This example illustrates that it is easy to find the primary components of a lattice ideal whose defining matrix is a concatenation of a diagonal matrix and a zero matrix. We would like to compute the primary components of general lattice ideals in positive characteristic by using the ideas above. Computing the Smith normal form of the matrix corresponding to a given lattice ideal, we obtain the isomorphisms to find the primary components in Laurent polynomial ring. Consequently the only thing we need to do is to take the contraction from the Laurent polynomial ring.

Let $B = (b_{ij})$ be the matrix defining the lattice ideal I_B. Let

$$D_{(m\times n)} = U_{(m\times m)} \cdot B_{(m\times n)} \cdot V_{(n\times n)}$$

be the Smith normal form of B. The columns of U induce a map

$$\varphi : \Bbbk[x_1^{\pm}, ..., x_m^{\pm}] \to \Bbbk[y_1^{\pm}, ..., y_m^{\pm}]$$

that sends $x_j \mapsto \prod_{u_{ij}>0} y_i^{u_{ij}} / \prod_{u_{ij}<0} y_i^{-u_{ij}}$. Note that the inverse matrix U^{-1} of U, gives the inverse map; $\varphi^{-1} : \Bbbk[y_1^{\pm}, ..., y_m^{\pm}] \to \Bbbk[x_1^{\pm}, ..., x_m^{\pm}]$.

The next example will clarify how to construct the isomorphisms mentioned above and how to obtain the primary components.

Example 1.46. *Let*

$$B = \begin{bmatrix} 2 & 0 \\ -2 & 3 \\ 1 & -6 \end{bmatrix}$$

be a matrix and consider the corresponding lattice ideal $I_B = \langle x_1^2x_3 - x_2^2, x_3^5 - x_1^2x_2, x_1^4 - x_2x_3^4, x_1^6 - x_2^3x_3^3 \rangle \subseteq \Bbbk[x_1, x_2, x_3]$ *where* $\mathrm{char}(\Bbbk) = 2$.

The Smith normal form of B, $D = U \cdot B \cdot V$ *is equal to*

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ -2 & 3 \\ 1 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The primary components of the lattice ideal corresponding to D, $I_D = \langle y_1 - 1, y_2^3 - 1 \rangle$, *are*

$$J_1 = I_D + \langle y_1 - 1, y_2 - 1 \rangle = \langle y_1 - 1, y_2 - 1 \rangle,$$

$$J_2 = I_D + \langle y_1 - 1, y_2 - \omega_3^1 \rangle = \langle y_1 - 1, y_2 - \omega_3^1 \rangle,$$

$$J_3 = I_D + \langle y_1 - 1, y_2 - \omega_3^2 \rangle = \langle y_1 - 1, y_2 - \omega_3^2 \rangle,$$

where ω_3 *is a primitive cubic root of unity in* $\Bbbk$.

The inverse of the matrix U

$$U^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ -5 & -2 & -1 \end{bmatrix},$$

gives us the map $\varphi^{-1} : \Bbbk[y_1^{\pm}, y_2^{\pm}, y_3^{\pm}] \to \Bbbk[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}]$ *that sends* $y_1 \mapsto x_1^2x_2/x_3^5$, $y_2 \mapsto x_2/x_3^2$ *and* $y_3 \mapsto x_1/x_3$*. Thus the primary components of* I_B *are*

$$I_1 = (\varphi^{-1}(J_1) : (x_1x_2x_3)^\infty) = (\langle x_1^2x_2 - x_3^5, x_2 - x_3^2\rangle : (x_1x_2x_3)^\infty)$$

$$= \langle x_2 - x_3^2, x_1^2 - x_2x_3\rangle,$$

$$I_2 = (\varphi^{-1}(J_2) : (x_1x_2x_3)^\infty) = (\langle x_1^2x_2 - x_3^5, x_2 - \omega_3^1x_3^2\rangle : (x_1x_2x_3)^\infty),$$

$$I_3 = (\varphi^{-1}(J_3) : (x_1x_2x_3)^\infty) = (\langle x_1^2x_2 - x_3^5, x_2 - \omega_3^2x_3^2\rangle : (x_1x_2x_3)^\infty).$$

If $\Bbbk$ *has characteristic* 3*,* I_D *is a primary ideal and so is* $(\varphi^{-1}(I_D) : (x_1x_2x_3)^\infty) = I_B$.

Example 1.47. *Let* C *be a* 4×3*-matrix with the Smith Normal form,* $D = U\cdot C\cdot V$*, is*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 1 \\ 2 & -2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 0 \\ 3 & -2 & 3 \\ 0 & 0 & 6 \\ -3 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & -2 \\ -2 & 0 & -3 \\ 0 & 1 & 0 \end{bmatrix}.$$

The corresponding lattice ideal is $I_C = \langle x_1^3x_2^3 - x_4^3, x_1^2x_2^2 - 1, x_1^2x_2, x_1^4 - x_2x_3^4, x_1^6 - x_2^3x_3^3\rangle \subseteq \Bbbk[x_1, x_2, x_3, x_4]$ *where* $\mathrm{char}(\Bbbk) = 2$.

The primary components of $I_D = \langle y_1 - 1, y_2^3 - 1, y_3^6 - 1\rangle$ *in* $\Bbbk[y_1, y_2, y_3]$ *are*

$$J_1 = I_D + \langle y_1^2 - 1, y_2^2 - 1, y_3^2 - 1\rangle = \langle y_1 - 1, y_2 - 1 y_3^2 - 1\rangle,$$
$$J_2 = I_D + \langle y_1^2 - 1, y_2^2 - 1, y_3^2 - \omega_3^1\rangle,$$
$$\vdots$$
$$J_9 = I_D + \langle y_1^2 - 1, y_2^2 - \omega_3^2, y_2^2 - \omega_3^2\rangle.$$

The inverse of the matrix U *gives us the map* $\varphi^{-1} : \Bbbk[y_1^{\pm}, y_2^{\pm}, y_3^{\pm}] \to \Bbbk[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}, x_3^{\pm}]$ *that sends* $y_1 \mapsto x_1x_2x_4^3$, $y_2 \mapsto x_2x_3^2x_4$ *and* $y_3 \mapsto x_4$. *Hence the primary components of* I_B *are*

$$I_1 = (\varphi^{-1}(J_1) : (x_1x_2x_3x_4)^\infty) = \langle x_1x_2x_4^3 - 1, x_2x_3^2x_4 - 1, x_4^2 - 1\rangle$$
$$= \langle x_1x_2 - x_4, x_4^2 - 1, x_3^2 - x_1\rangle,$$
$$I_2 = (\varphi^{-1}(J_2) : (x_1x_2x_3)^\infty) = (\langle x_1x_2x_4^3 - 1, x_2x_3^2x_4 - 1, x_4^2 - \omega_3^1\rangle : (x_1x_2x_3)^\infty),$$
$$\vdots$$
$$I_9 = (\varphi^{-1}(J_9) : (x_1x_2x_3)^\infty) = (\langle x_1x_2x_4^3 - 1, x_2x_3^2x_4 - \omega_3^2, x_4^2 - \omega_3^2\rangle : (x_1x_2x_3)^\infty).$$

When $\mathrm{char}(\Bbbk) = 3$, I_D *has two primary components*

$$\tilde{J}_1 = I_D + \langle y_1^9 - 1, y_2^9 - 1, y_3^9 - 1\rangle = \langle y_1 - 1, y_2^3 - 1 y_3^3 - 1\rangle,$$

$$\tilde{J}_1 = I_D + \langle y_1^9 - 1, y_2^9 - 1, y_3^9 + 1\rangle = \langle y_1 - 1, y_2 - 1 y_3^2 + 1\rangle.$$

The primary components of I_B *are*

$$I_1 = (\varphi^{-1}(\tilde{J}_1) : (x_1x_2x_3x_4)^\infty) = \langle x_1^2x_2 - 1, x_3^6 - x_1^3, x_4^3 - 1\rangle,$$

$$I_2 = (\varphi^{-1}(\tilde{J}_2) : (x_1x_2x_3x_4)^\infty) = \langle x_1^2x_2 + 1, x_3^6 - x_1^3, x_4^3 + 1\rangle.$$

The following statements are consequences of Theorem 1.30.

Corollary 1.48. *Assume* $\mathrm{char}(\Bbbk) = p > 0$. *Let* B *be an* $n \times n$ *integer matrix with* $\mathrm{detB} = \mathrm{p^e}$ *for some* $e > 0$. *Let the columns of* B *span the lattice* $\mathcal{L}$, *and let* $I_B = \langle x^{\ell_i^+} - x^{\ell_i^-} \mid \ell_i \in \mathcal{L}\rangle$ *be the corresponding lattice ideal. Then* I_B *is primary.*

Proof. Since B has full rank, $\mathrm{Sat}(\mathcal{L}) \cong \mathbb{Z}^n$ and $\mid \mathrm{Sat}(\mathcal{L})/\mathcal{L} \mid = \mid \mathbb{Z}^n/\mathcal{L} \mid = \mathrm{detB} = p^e$. We know that $\mathcal{L} \subset \mathrm{Sat}_p'(\mathcal{L}) \subset \mathrm{Sat}(\mathcal{L})$ where $\mathrm{Sat}_p'(\mathcal{L})$ is the largest sublattice of $\mathrm{Sat}(\mathcal{L})$ such that $\mid \mathrm{Sat}_p'(\mathcal{L})/\mathcal{L} \mid = g$ where $(p, g) = 1$. $\mathrm{Sat}_p'(\mathcal{L})/\mathcal{L}$ is a subgroup of $\mathrm{Sat}(\mathcal{L})/\mathcal{L}$, thus $\mid \mathrm{Sat}_p'(\mathcal{L})/\mathcal{L} \mid = g$ must divide $\mid \mathrm{Sat}(\mathcal{L})/\mathcal{L} \mid = p^e$. Hence g must be 1, which means that I_B is primary when the characteristic of $\Bbbk$ is p, by Theorem 1.30. $\square$

Corollary 1.49. *Let* $I \subset \Bbbk[x_1, ..., x_n]$ *be a lattice ideal, where* $\Bbbk$ *is an algebraically closed field. There exists a prime number p such that for all prime numbers* p' *bigger than p, the primary decomposition of* I *in a field of characteristic* p' *is same as the one in the field of characteristic 0.*

Proof. After a certain prime, the order of the group $\mid \mathrm{Sat}_p(\mathcal{L})/\mathcal{L} \mid$ turns to be 1, which is compatible with our convention when the characteristic of the field is equal to 0. $\square$

2. LATTICE BASIS IDEALS

2.1 Combinatorial Characterization of Primary Components of Binomial Ideals

Computing primary components explicitly is difficult, as the Hull operation which discards the embedded primary components and appears in Theorem 1.36, is not explicit. Dickenstein, Matusevich and Miller [8] provided a characterization of the primary components of an arbitrary binomial ideal in a polynomial ring over an algebraically closed field of characteristic zero. They translate the operations of binomial ideals to operations on exponent vectors and associated partial characters and formulate the primary components of binomial ideals as sums of binomial and monomial ideals. They describe those monomial ideal using congruences induced by binomial ideals as in the following definition.

Definition 2.1. Let $\mathcal{Q}$ be a monoid. A *congruence* $\sim$ on $(\mathcal{Q}, +)$ is an additively closed equivalence relation: $a \sim b \;\Rightarrow a + c \sim b + c$ for all $a, b, c \in Q$. A binomial ideal $I \subset \Bbbk[\mathcal{Q}]$ induces a congruence $\sim$, which we denote by $\sim_I$, in which :

$$u \sim v \text{ if } \; t^u - \lambda t^v \in \mathrm{I} \text{ for some} \, \lambda \in \Bbbk \, \text{and} \, \lambda \neq 0.$$

Congruences give us a strong connection between combinatorics and commutative algebra of binomial ideals.

We use the following notation for the next theorem. For $\delta \subset \{1, ..., n\}$ let $\bar{\delta}$ be the complement of δ in $\{1, ..., n\}$. Let $\mathbb{N}^\delta = \{u \in \mathbb{N}^n \mid u_i = 0 \text{ for } i \in \bar{\delta}\}$. Thus $\mathbb{N}^n = \mathbb{N}^\delta \times \mathbb{N}^{\bar{\delta}}$. Including additive inverses for elements of $\mathbb{N}^\delta$, we obtain $\mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}}$ with corresponding semigroup $\Bbbk[\mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}}] = \Bbbk[x_i \mid i \in \bar{\delta}][x_i^{\pm} \mid i \in \delta]$. This is a mixed Laurent and ordinary polynomial ring. Let $\mathcal{L}$ be a saturated sublattice of $\mathbb{Z}^\delta$. The image of $\mathbb{N}^\delta$ in the torsion free group $\mathbb{Z}^\delta/\mathcal{L}$ is denoted by $\mathbb{N}^\delta/\mathcal{L}$. We denote the ideal $I[\mathbb{Z}\delta]$ as the extension of the ideal I to the ring $\Bbbk[x_j^{\pm} \mid j \in \delta][x_i \mid i \in \bar{\delta}]$.

Theorem 2.2. *(Theorem 3.2 in [8]) Let $\Bbbk$ be an algebraically closed field with characteristic 0. Let $I \subseteq \Bbbk[x_1, ..., x_n]$ be a binomial ideal with an associated prime $P = I_\rho + \langle x_i \mid i \in \bar{\delta}\rangle$ such that $\rho : \mathcal{L} \to \Bbbk^*$ is a saturated partial character with a saturated lattice $\mathcal{L} \subseteq \mathbb{Z}^\delta \subseteq \mathbb{Z}^n$. Let $\sim$ be the congruence defined by the ideal $(I + I_\rho)[\mathbb{Z}\delta]$ on $\mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}}$. Set $\mathbb{Z}\Phi = \mathbb{N}^\delta/\mathcal{L}$*

1) *If P is a minimal prime of I and U is the set of elements $\mu \in \mathbb{N}^n$ whose congruence class under $\sim$ has an infinite image in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{\delta}}$, then the P-primary component of I is*

$$Q = ((I + I_\rho) : \prod_{i \in \delta}(x_i)^\infty) + \langle x^\mu \mid \mu \in U\rangle.$$

2) *The only monomials in Q are those in $\langle x^\mu \mid, \mu \in U\rangle$.*

3) *Let K be a monomial ideal containing a sufficiently high power of $\langle x_i \mid i \in \bar{\delta}\rangle$ and let $\cong$ be the congruence on $\mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}}$ determined by $(I + I_\rho + K)[\mathbb{Z}\delta]$. Let U_K be the set of elements $\mu \in \mathbb{N}^n$ whose congruence class under $\cong$ have an infinite image in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{\delta}}$. If P is an embedded prime of I then*

$$\tilde{Q} = ((I + I_\rho + K) : \prod_{i \in \delta}(x_i)^\infty) + \langle x^\mu \mid \mu \in U_K\rangle$$

is a valid choice for the P-primary component of I.

4) *The only monomials in $\tilde{Q}$ are those in $\langle x^\mu \mid \mu \in U_K\rangle$.*

Here, we are trying to construct a binomial primary component of the ideal I starting from I itself. We start the construction by adding the binomial part of P which is contained in the lattice ideal part, $I_+(\rho)$. This is a consequence of Theorem 1.36. Then we continue extending the ideal $I + I_+(\rho)$ by taking the saturation with respect to the nonzerodivisor variables of P, specifically, x_i where $i \in \delta$. This follows from Proposition 4.8 and 4.9 in [2] : that state that the P-primary component of an ideal J is equal to the P-primary component of $(J : \prod_{j\in\delta}(x_j)^\infty)$. One of the critical steps of the construction is determining the monomials we need to add. This is achieved using congruences. For more explanation about the monomials, we refer to Lemma 2.8 in [8]. The other crucial step is to show that the constructed ideal Q is primary. To show that, Dickenstein, Matusevich and Miller reduced the problem to a monomial ideal problem by taking quotient modulo $I_+(\rho)$ and they use the characterization for primary ideals whose radical is a monomial associated prime. (See Theorem 2.23 in [28].) In order to compute embedded primary components replace I by $I + K$ and the result follows as above.

The characterization of primary components of binomial ideals is still not complete since we assume that we know which primes are associated and we assume that $\Bbbk$ is algebraically closed with characteristic zero.

Recall that we denote the ideal $I[\mathbb{Z}\delta]$ as the extension of the ideal I to the ring $\Bbbk[x_j^{\pm} \mid j \in \delta][x_i \mid i \in \bar{\delta}]$. The following lemmas characterize the monomials that belong to the primary components of binomial ideals when $\mathrm{char}(\Bbbk) = 0$ and when $\mathrm{char}(\Bbbk) = p > 0$, respectively.

Lemma 2.3. *Let I be a binomial ideal in $\Bbbk[x]$ where $\Bbbk$ is an algebraically closed field with characteristic zero. Let $P = I_+(\rho) + \langle x_i \mid i \in \bar{\delta}\rangle$ be an associated prime of I. Let Γ be a congruence class determined by $(I + (I_+(\rho)))[\mathbb{Z}\delta]$. If Γ has two distinct elements u, v such that $v - u \in \mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}}$ but $v - u \notin L_\rho$, then for all $u \in \Gamma$, t^u is in the P-primary component of I.*

Proof. Follows from Lemma 2.8 in [8]. $\square$

If we mimic the proof of the lemma above, we can determine the monomials of the primary component in fields with positive characteristic.

Lemma 2.4. *Let I be a binomial ideal in $\Bbbk[x]$ where $\Bbbk$ is an algebraically closed field with characteristic $p > 0$. Let $P = I_+(\rho) + \langle x_i \mid i \in \bar{\delta}\rangle$ be an associated prime of I. Let Γ be a congruence class determined by $(I + (I_+(\rho))^{[p^e]})[\mathbb{Z}\delta]$ for some $e \gg 0$. If Γ has two distinct elements u, v such that $v - u \in \mathbb{Z}^{\delta} \times \mathbb{N}^{\bar{\delta}}$ but $v - u \notin L_\rho$, then for all $u \in \Gamma$, t^u is in the P-primary component of I.*

Proof. Let $u \neq v \in \Gamma$ with $v - u \in \mathbb{Z}^{\delta} \times \mathbb{N}^{\bar{\delta}}$. This means that $x^u - \lambda x^v \in (I + (I_+(\rho))^{[p^e]})[\mathbb{Z}\delta]$ for some $\lambda \in \Bbbk$. We claim that $1 - \tilde{\lambda}x^{v-u} \notin P[\mathbb{Z}\delta]$ for all $\tilde{\lambda} \in \Bbbk$. By contradiction, assume there exists $\tilde{\lambda} \in \Bbbk$ such that $1 - \tilde{\lambda}x^{v-u} \in P[\mathbb{Z}\delta]$. If $v_i - u_i \neq 0$ for some $i \in \bar{\delta}$, this implies that $1 \in P$, since $x_i \in P$. Thus $v_i - u_i = 0$ for all $i \in \bar{J}$, but then $1 - \tilde{\lambda}x^{v-u} \in I_+(\rho)[\mathbb{Z}\delta]$ and therefore $v - u \in L_\rho$ which gives a contradiction by definition.

We conclude that $1 - \lambda x^{v-u}$ maps to a unit in $(\Bbbk[x]/I + (I_+(\rho))^{[p^e]})_P$. Since $x^u(1 - \lambda x^{v-u}) = x^u - \lambda x^v$ maps to zero in $\Bbbk[x]/I + (I_+(\rho))^{[p^e]}$, x^u maps to zero in $(\Bbbk[x]/I + (I_+(\rho))^{[p^e]})_P$. As the elements of Γ arise from the monomials that are scalar multiples of x^u modulo $(I + (I_+(\rho))^{[p^e]})[\mathbb{Z}\delta]$, they also map to zero. So for all $u \in \Gamma$, $x^u \in \ker\alpha : \Bbbk[x]/I + (I_+(\rho))^{[p^e]} \to (\Bbbk[x]/I + (I_+(\rho))^{[p^e]})_P$ which is also in the primary component of I. $\square$

The graph of binomial ideals is another combinatorial tool for binomial ideals. Graphs of binomial ideals provide a better way to visualize the congruence classes, which in this setting correspond to connected components. We define the graph of binomial ideals in certain submonoids of $\mathbb{Z}^n$.

Definition 2.5. Let P be a submonoid of $\mathbb{Z}^n$. A binomial ideal I in the monoid ring $\Bbbk[P]$ defines a graph $\mathscr{G}_P(I)$ whose vertices are the elements of P and whose edges are pairs $(u, v) \in P \times P$ such that $x^u - \lambda x^v \in I$ for some $\lambda \in \Bbbk^*$. A connected component of $\mathscr{G}_P(I)$ is said to be *infinite* if it consists of infinitely many vertices; otherwise it is called *finite*. A vertex of $\mathscr{G}_P(I)$ is called an *infinite vertex* if it belongs to an infinite connected component, otherwise it is called a *finite vertex*. If $P = \mathbb{N}^n$, we write $\mathscr{G}(I)$ instead of $\mathscr{G}_P(I)$.

The graph of a binomial ideal can be very difficult to draw. We illustrate them as simply as possible, for instance, the graph will have many more edges than those depicted in figures since any connected component of $\mathscr{G}_P(I)$ is a complete graph.

Theorem 2.6 (Theorem 2.15, [8]). *Let $\Bbbk$ be an algebraically closed field and $I \subset \Bbbk[x_1, ..., x_n]$ a binomial ideal. Let $\delta \subseteq \{1, \ldots, n\}$, and set $P = \mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}}$. If $\langle x_i \mid i \in \bar{\delta}\rangle$ is a minimal prime of I, its corresponding primary component is*

$$\Big(I : (\prod_{j \in \delta} x_j)^\infty\Big) + \langle x^u \mid u \in \mathbb{N}^n \text{ is an infinite vertex of } \mathscr{G}_P(\Bbbk[P] \cdot I)\rangle. \tag{2.1}$$

Moreover, the only monomials in these primary components are those of the form x^u such that $u \in \mathbb{N}^n$ is an infinite vertex of $\mathscr{G}_P(\Bbbk[P] \cdot I)$.

Remark 2.7. Since we now want to compute the primary components corresponding to monomial associated primes, we do not need to worry about the translation of the infinite image which is described in Theorem 2.2. Note that the monomials which belong to the primary component described above are precisely coming from infinite vertices of the corresponding graph. Moreover the Theorem 2.6 is true for every characteristic.

Remark 2.8. Note that the monomial ideal in (2.1) is generated by monomials x^u where $u_j = 0$ if $j \in \delta$. Indeed, if $u \in \mathbb{N}^n$ is an infinite vertex of $\mathscr{G}_P(\Bbbk[P] \cdot I)$, so is $u - \hat{u}$, where $\hat{u}_i = 0$ if $i \in \delta$ and $\hat{u}_j = u_j$ if $j \notin \delta$. This is because monomials in the variables x_j for $j \in \delta$ are units in $\Bbbk[P]$.

We use the following criterion to identify the infinite components of $\mathscr{G}_P(\Bbbk[P] \cdot I)$; this is a special case of Lemma 2.10 in [8].

Lemma 2.9. *Let $I \subseteq k[x_1, \ldots, x_n]$ be a binomial ideal, $\delta \subseteq \{1, \ldots, n\}$, and $P = \mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}}$. A connected component of $\mathscr{G}_P(\Bbbk[P] \cdot I)$ is infinite if and only if it contains two distinct vertices $u, v \in P$ such that $u - v \in P$.*

If I is a binomial ideal, there exists a (multi)grading of the polynomial ring that makes I a homogeneous ideal. In fact, it is often the case that a binomial ideal is given together with a specified grading. Depending on their behavior with respect to a given grading, the associated primes and primary components of a binomial ideal are called *toral* or *Andean*.

Definition 2.10. Let G be a commutative semigroup and let $M = \bigoplus_{g \in G} M_g$ be a G-graded module over the polynomial ring $\Bbbk[x_1, ..., x_n]$. The G-graded *Hilbert function* H_M of M is the set map $G \to \mathbb{N}$ whose value at each group element $g \in G$ is the vector space dimension $dim_{\Bbbk}(M_g)$; explicitly $H_M(g) = \mathrm{dim}_{\Bbbk}(\mathrm{M_g})$.

Definition 2.11. Let M be a finitely generated G graded module over the polynomial ring $\Bbbk[x_1, ..., x_n]$, we say M is *toral* if H_M is bounded above. A G-graded ideal I is called *toral* if $\Bbbk[x_1, ..., x_n]/I$ is a toral module. On the other hand if H_M is not bounded above, M is called an *Andean* module. A graded ideal I is called a *Andean* if $\Bbbk[x_1, ..., x_n]/I$ is an Andean module.

Example 2.12. *The toric ideal* $I_A = \langle x^u - x^v \mid u, v \in \mathbb{N}^n$ *and* $Au = Av\rangle$ *is an* $\mathbb{N}A$*-graded toral prime since* $\dim_{\Bbbk}(\Bbbk[x_1, ..., x_n]/I_A)_a$ *is either* 0 *or* 1*, see Lemma 4.3.*

In general, the toral primary components of a binomial ideal are more easily described combinatorially than the Andean ones, as their graphs can actually be drawn in much lower dimension than the number of variables. Indeed, we will see that the monomial part of the primary component is easier to compute. The following is a characterization of a toral component, if we compare to Theorem 2.2, the congruence classes we are looking at are simpler.

Theorem 2.13. *(Theorem 4.13, [8]) Let* $I \in \Bbbk[x_1, ..., x_n]$ *be an* A*-graded ideal and let* $P = I_+(\rho) + \langle x_i \mid i \in \bar{\delta}\rangle$ *be a toral minimal associated prime of* I*. We define* $\bar{I} = I \cdot \Bbbk[x]/\langle x_i - 1 \mid i \in \delta\rangle$.

1) *If* U *is the set of elements* μ *whose congruence class in* $\mathbb{N}^{\bar{\delta}}$ *under* $\sim_{\bar{I}}$ *is infinite, then* P*-primary component of* I *is*

$$Q = ((I + I_\rho + K) : \prod_{i \in \delta}(x_i)^\infty) + \langle x^\mu \mid \mu \in U\rangle.$$

2) *Let* P *be a toral embedded prime of* I*. Let* K *be a monomial ideal containing a sufficiently high power of* $\langle x_i \mid i \in \bar{\delta}\rangle$ *and let* $\bar{U}_K$ *be the set of* $u \in \mathbb{N}^{\bar{\delta}}$ *whose congruence class under* $\sim_{\bar{I}+K}$ *is infinite. Then*

$$\tilde{Q} = ((I + I_\rho) : \prod_{i \in \delta}(x_i)^\infty) + \langle x^\mu \mid \mu \in U_K\rangle$$

is a valid choice for the P*-primary component of* I*.*

The only monomials in Q *and* $\tilde{Q}$ *are those in* $\langle x^\mu \mid \mu \in U\rangle$ *and* $\langle x^\mu \mid \mu \in U_K\rangle$*, respectively.*

2.2 Lattice Basis Ideals

In this section we review important facts about lattice basis ideals and especially how their associated primes arise.

Recall that lattice basis ideal was introduced in Definition 1.41. Let M be an $n \times m$ integer matrix with rank m. Remember that we can write $\mu \in \mathbb{Z}^n$ as $\mu = \mu^+ - \mu^-$ where μ^+ and μ^- are non-negative and have disjoint support. The lattice basis ideal associated to M is

$$I(M) = \langle x^{\mu^+} - x^{\mu^-} \mid \mu \text{ is a column of } M \rangle \subseteq \Bbbk[x_1, \ldots, x_n] = \Bbbk[\mathbb{N}^n].$$

Hoşten and Shapiro studied the associated primes of lattice basis ideals in [18]. They show that the minimal primes of such an ideal are determined by the sign patterns of the entries of the corresponding matrix. On the other hand, they illustrate the fact that embedded primary components are not uniquely determined by the sign patterns of the matrix.

Convention 2.14. *$A = (a_{ij}) \in \mathbb{Z}^{d \times n}$ denotes an integer $d \times n$ matrix of rank d whose columns $A_1, \ldots, A_n$ all lie in a single open linear half space of $\mathbb{R}^d$. We also assume that the column of A span $\mathbb{Z}^d$ as a lattice.*

Convention 2.15. *Let $B = (b_{jk}) \in \mathbb{Z}^{n \times m}$ be an integer matrix of full rank $m \leqslant n$. The rows of B are denoted by $b_1, \ldots, b_n$ and its columns by $B_1, \ldots, B_m$. Assume every nonzero element of column span of B over integers is mixed, meaning that it has both strictly positive and strictly negative entries. Having chosen B, we set $d = n - m$ and pick a matrix $A \in \mathbb{Z}^{d \times n}$ of full rank such that $\mathbb{Z}A = \mathbb{Z}^d$ and $AB = 0$.*

Fix matrices A and B as in Convention 2.15, and let $I(B)$ be the lattice basis ideal corresponding to B

$$I(B) = \langle x^{B_i^+} - x^{B_i^-} \mid B_i \text{ is a column of } B \rangle \subseteq \Bbbk[x_1, \dots, x_n].$$

By using Corollary 2.1 and Theorem 2.1 in [18], we can list the minimal associated primes of $I(B)$. We first consider the associated primes of $(I(B) : \prod {x_i}^\infty)$. They arise from the saturations of the partial character of the lattice ideal $(I(B) : \prod {x_i}^\infty)$. Indeed, all of the associated primes of $(I : (\prod_{i=1}^n x_i)^\infty)$ are isomorphic, by rescaling the variables, to $I_A = \langle x^{v_+} - x^{v_-} \mid v \in \mathbb{Z}^n, Av = 0 \rangle$, where A is as in Convention 2.14. Recall that the prime ideal I_A is called the *toric ideal* associated to A. When the characteristic of the underlying field $\Bbbk$ is zero, it is shown in [12] that the primary components of $I(B)$ corresponding to these associated primes are the associated primes themselves. The case of positive characteristic is considered in Section 1.5.

Another kind of associated primes of $I(B)$ arises after row and column permutations of B. In fact, these associated primes of $I(B)$ are described as in the form of $P = \langle x_{i_1}, ..., x_{i_s}, I_+(\rho_{\mathcal{L}}) \rangle$ where $\mathcal{L}$ is the lattice generated by columns of $B_{\mathcal{L}}$ comes from the block decomposition of the matrix B of the form

$$\left[\begin{array}{c|c} N & B_{\mathcal{L}} \\ \hline M & 0 \end{array} \right]$$

where M is mixed of size $s \times t$ with no zero rows. $B_{\mathcal{L}}$ is the $(n-s) \times (m-t)$ matrix whose columns generate the lattice $\mathcal{L}$. Also M has to satisfy another block decomposition property called *irreducibility* which is the following criterion.

Definition 2.16. A matrix M is called *irreducible* if

1) M is a mixed $s \times t$ matrix where $s \leqslant t$, and

2) One cannot bring M into the following form after permuting its rows and columns

$$M = \left[\begin{array}{c|c} N' & B' \\ \hline M' & 0 \end{array}\right]$$

where M' is mixed $m \times p$ matrix where $m > p$.

Remark 2.17. In the case of lattice basis ideals, the toral associated primes arise when M which is the submatrix in the decomposition above is square and invertible. If M is not square or not invertible, then the corresponding associated prime is Andean. One can verify this fact by Lemma 4. 9 and Example 4. 11 in [8].

2.3 Codimension Two Lattice Basis Ideals

We study lattice basis ideals arising from $n \times 2$ integer matrices, known as *codimension two lattice basis ideals*. In collaboration with L. F. Matusevich [24] we gave explicit descriptions for primary components of codimension two lattice basis ideals especially for those whose corresponding associated prime is monomial and Andean.

Convention 2.18. *From now on, $B = [b_{ij}]$ is an $n \times 2$ integer matrix of rank 2. The rows of B are denoted by $b_1, \dots, b_n$, and its columns by B_1, B_2. Fix an integer $(n-2) \times n$ matrix A such that $AB = 0$, and whose columns span $\mathbb{Z}^{n-2}$ as a lattice. $I(B)$ is the lattice basis ideal corresponding to B.*

Since B has rank two, the lattice basis ideal $I(B)$ corresponding to B is a complete intersection. Therefore all of its associated primes are minimal. In the Section 2.2, we discussed how the associated primes of lattice basis ideals arise in general. In the case of codimension two, the set of associated primes of $I(B)$ consists of the associated primes of $(I : (\prod_{i=1}^{n} x_i)^\infty)$ and the monomial primes $\langle x_i, x_j \rangle$ if b_i and b_j

lie in opposite open quadrants of $\mathbb{Z}^2$. Using the following decomposition of B, we obtain the associated prime $\langle x_i, x_j \rangle$

$$B = \left[\begin{array}{cc} \multicolumn{2}{c}{N} \\ \hline b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{array}\right]$$

where the last two rows correspond to x_i and x_j respectively. Note that we do not have a binomial $I_+(\rho_{\mathcal{L}})$ in the associated prime since we do not have a $B_{\mathcal{L}}$ submatrix in the decomposition above. Lastly, we point out that $\left(\begin{smallmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{smallmatrix}\right)$ satisfies the irreducibility condition.

We now turn our attention to the primary components of $I(B)$ arising from monomial associated primes.

We mentioned that for a binomial ideal I, there exists a (multi)grading of the polynomial ring that makes I a homogeneous ideal and in general, I is given together with a specified grading. We use the matrix A in Convention 2.18 to define a $\mathbb{Z}^{n-2}$-grading of $\Bbbk[x]$, where $\deg(x_i)$ is defined to be the ith column of A. The ideal $I(B)$ is homogeneous with respect to this A-*grading*, its associated primes and primary components are homogeneous as well. By Definition 2.11, we classify the associated primes and primary components of an A-graded ideal according to their A-graded behavior.

The monomial primes $\langle x_i, x_j \rangle$ such that the corresponding rows of B, b_i and b_j, are linearly dependent and in opposite open quadrants of $\mathbb{Z}^2$ correspond to Andean components. It is easy to see that the A-graded Hilbert function of $\Bbbk[x]/I_A$ takes only the values zero and one and this holds in the same way for the other non-monomial associated primes of $I(B)$.

On the other hand, the Hilbert function of $\Bbbk[x]/\langle x_i, x_j \rangle$ is bounded if the rank of the submatrix of A indexed by $\{1,\ldots,n\}-\{i,j\}$ has full rank, and this occurs when b_i and b_j are linearly independent, see Remark 2.17.

By Theorem 2.13, we know that the toral primary components of a binomial ideal are simpler than the Andean ones, as their graphs can actually be drawn in much lower dimension.

The following fact is a consequence of Theorem 2.13 which also applies to the case when the associated prime is monomial.

Theorem 2.19. *Let I be an A-graded binomial ideal in $\Bbbk[x]$, where $\Bbbk$ is algebraically closed, and assume that $P = \langle x_i \mid i \in \bar{\delta}\rangle$ is a toral minimal prime of I. Define the binomial ideal $\bar{I} = I \cdot \Bbbk[x]/\langle x_j - 1 \mid j \in \delta\rangle \subset \Bbbk[\mathbb{N}^{\bar{\delta}}]$ by setting $x_j = 1$ for $j \in \delta$. The P-primary component of I is*

$$\left(I : (\prod_{j\in\delta} x_j)^{\infty}\right) + \langle x^u \mid u \in \mathbb{N}^{\bar{\delta}} \textit{ is an infinite vertex of } \mathscr{G}(\bar{I})\rangle.$$

Remark 2.20. The main feature of the Theorem 2.19 is that $\mathscr{G}(\bar{I})$ has vertices in $\mathbb{N}^{\bar{\delta}}$, and the cardinality of $\bar{\delta}$ can be much smaller than n. For the case of codimension two lattice basis ideals, if $\langle x_i, x_j\rangle$ is a toral associated prime of $I(B)$ $\mid \bar{\delta} \mid= 2$ always, and we can compute the monomials by the graph $\mathscr{G}(\overline{I(B)})$ where

$$\overline{I(B)} = \langle x_i^{|b_{i1}|} - x_j^{|b_{j1}|}, x_i^{|b_{i2}|} - x_j^{|b_{j2}|}\rangle \subset \Bbbk[x_i, x_j].$$

Example 2.21. *Let*

$$B = \begin{bmatrix} 2 & 4 \\ -4 & -6 \\ 2 & 3 \\ -1 & 3 \\ -1 & -2 \\ 2 & -6 \\ -8 & -12 \\ -3 & -6 \end{bmatrix},$$

so that $I(B) = \langle x_1^2x_3^2x_6^2 - x_2^4x_4x_5x_7^8x_8^3, x_1^4x_3^3x_4^3 - x_2^6x_5^2x_6^6x_7^{12}x_8^6\rangle$.

The rows $(2,4)$ *and* $(-4,-6)$ *are linearly independent and lie in opposite open quadrants, so* $\langle x_1, x_2\rangle$ *is a toral associated prime of* $I(B)$. *We can compute the monomials of the* $\langle x_1, x_2\rangle$*-primary component of* $I(B)$ *by looking the graph of* $\mathscr{G}(\overline{I(B)})$ *where* $\overline{I(B)} = \langle x_1^2 - x_2^4, x_1^4 - x_2^6\rangle$. *It is sufficient to add the monomials coming from the infinite connected components are* $\{x_1^4, x_2^6, x_1^2x_2^2\}$, *see Figure 2.1. Thus the other monomials corresponding to infinite vertices can be generated by these monomials, as can be seen in the staircase diagram in dashed lines.*

By Theorem 2.19, the $\langle x_1, x_2\rangle$*-primary component of* $I(B)$ *is*

$$(I(B) : (\prod_{\ell\neq 1,2} x_\ell)^\infty) + \langle x_1^4, x_2^6, x_1^2x_2^2\rangle = \langle x_1^4, x_2^6, x_1^2x_2^2, x_2^4x_4x_5x_7^8x_8^3 - x_1^2x_3^2x_6^2\rangle.$$

For a precise result in this case, see Proposition 2.24 and Lemma 2.33.

2.4 Graphs Associated to Matrices

In this section, we study graphs arising from 2×2 integer matrices, whose vertices are elements of $\mathbb{N}^2$. We see that these graphs are sufficient to control the primary

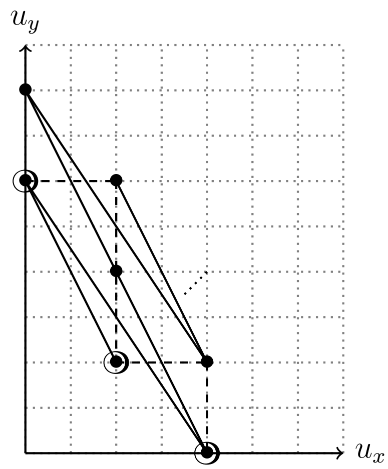

Figure 2.1: The graph $\mathscr{G}(\overline{I(B)})$

components of the codimension two lattice basis ideals.

Definition 2.22. Let Q be a subset of $\mathbb{Z}^n$ and let M be an $n \times m$ integer matrix. We define $G_Q(M)$ the graph of M whose vertices are the elements of Q, and where two vertices $u, v \in Q$ are connected by an edge if and only if $u - v$ or $v - u$ is a column of M.

Definition 2.23. A connected component of $G_Q(M)$ is called *infinite*, if it contains infinitely many vertices; otherwise it is called *finite*. A *finite* (or *infinite*) vertex of $G_Q(M)$ is one that belongs to a finite (or infinite) connected component.

If $Q = \mathbb{N}^n$, we omit Q from the notation, and write $G(M)$ instead of $G_{\mathbb{N}^n}(M)$.

We consider graphs $G_Q(M)$ where Q is a submonoid of $\mathbb{Z}^n$ such as $\mathbb{N}^n$ or $\mathbb{N}^k \times \mathbb{Z}^{n-k}$ or a subset of $\mathbb{Z}^n$, such as $\{u \in \mathbb{N}^n \mid \lambda_1 u_1 + \cdots + \lambda_n u_n = \lambda_0\}$, for fixed given $\lambda_0, \ldots, \lambda_n \in \mathbb{Q}$ (Lemma 2.44), or $\{u \in \mathbb{N}^2 \mid u_1 \leqslant \ell\}$, for fixed given $\ell \in \mathbb{N}$.

We are interested in determining whether the connected components are infinite or finite. We will derive the infinite vertices which are mentioned in Theorem 2.6

from the infinite vertices of $G_Q(M)$. Indeed, $G_Q(M)$ carries the algebraic information we need to find the monomial part of the associated prime and it is much easier to analyze. The following result describes the connected components of $G(M)$, where M is a 2×2 nonsingular matrix whose rows lie in non adjacent open quadrants of $\mathbb{Z}^2$.

Proposition 2.24 (Lemma 6.5 in [10])**.** *Let $M = [\mu_{ij}]_{i,j\in\{1,2\}} \in \mathbb{Z}^{2\times 2}$ of rank two, and assume that $\mu_{11}, \mu_{12} > 0$ and $\mu_{21}, \mu_{22} < 0$. Set*

$$\mathscr{R} = \begin{cases} \{u \in \mathbb{N}^2 \mid u_1 < \mu_{12}, u_2 < -\mu_{21}\} & \text{if } |\mu_{11}\mu_{22}| > |\mu_{12}\mu_{21}|, \\ \{u \in \mathbb{N}^2 \mid u_1 < \mu_{11}, u_2 < -\mu_{22}\} & \text{if } |\mu_{11}\mu_{22}| < |\mu_{12}\mu_{21}|. \end{cases}$$

Every finite connected component of $G(M)$ contains exactly one vertex in $\mathscr{R}$. In particular, the number of finite connected components of $G(M)$ is the cardinality of $\mathscr{R}$, which is $\min(|\mu_{11}\mu_{22}|, |\mu_{12}\mu_{21}|)$.

Example 2.25. *Let $M = \begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}$. $G(M)$ has $min(|-4|, |-6|) = 4$ finite connected components, which are shown in Figure 2.2.*

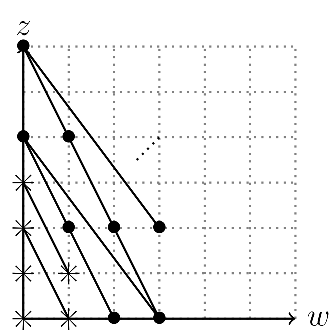

Figure 2.2: The graph of M

If the entries of $M \in \mathbb{Z}^{2\times 2}$ are all positive integers, then $G(M)$ has no finite connected components. In this case, we focus on a family of subgraphs of $G(M)$, as follows.

Definition 2.26. Let $M \in \mathbb{Z}^{2\times 2}$ of rank two, all of whose entries are positive. For $\ell \in \mathbb{N}$ let $Q_\ell = \{u \in \mathbb{N}^2 \mid u_1 \leqslant \ell\}$. We denote $G_\ell(M) := G_{Q_\ell}(M)$, and call these graphs the *band graphs* of M. Note that $G_\ell(M)$ is the induced subgraph of $G(M)$ whose vertices lie in Q_ℓ, and consequently if $\ell \leqslant \ell'$, then $G_\ell(M)$ is a subgraph of $G_{\ell'}(M)$.

Before proving our results about band graphs, we need a few more definitions.

Definition 2.27. Let M be a rank two integer matrix

$$M = \begin{bmatrix} r & s \\ a & b \end{bmatrix}$$

such that $r \geqslant s > 0$, $0 < a \leqslant b$, and $\gcd(r, s) = 1$. The graphs $G_\ell(M)$ have two types of edges: those parallel to the first column of M are called the *r-edges* of $G_\ell(M)$, and those parallel to the second column of M are called the *s-edges* of $G_\ell(M)$. If $r = s$, we could refer to these edges as a-edges and b-edges. Consider $\mathbb{N}^2$ with coordinates w, z. A vertex of $G_\ell(M)$ is called a *turn* if it is adjacent to both an r-edge and an s-edge of $G_\ell(M)$. A turn (w_0, z_0) is called a *left turn* if there is a vertex adjacent to (w_0, z_0) whose w-coordinate is smaller than w_0. Turns that are not left turns are called *right turns*.

Intuitively, when we walk along a connected component of $G_\ell(M)$ in the direction that increases z, we turn left at a left turn, and right at a right turn. For an illustration of the definitions above, see Figure 2.3.

The following result, joint with L. F. Matusevich, characterizes the situation when a band graph has an infinite connected component.

Proposition 2.28. *Let M be as in Definition 2.27. If $\ell < r + s - 1$, then every connected component of $G_\ell(M)$ is finite. If $\ell \geqslant r + s - 1$, then for every $w_0 \in \{0, 1, \ldots, \ell\}$ there exists $z_0 \in \mathbb{N}$ such that (w_0, z_0) belongs to an infinite component of $G_\ell(M)$.*

Proof. Write $r = sq_1 + q_2$ where $0 \leqslant q_2 < s$. We claim that any connected component of $G_r(M)$ contains at most $2q_1 + 2$ vertices (implying that $G_r(M)$, and therefore $G_\ell(M)$ for $\ell \leqslant r$, has no infinite connected components). A connected component of $G_r(M)$ can only contain one r-edge since the w-coordinates of vertices in $G_r(M)$ are bounded by r. Thus, we can have at most two turns in such a connected component. We can connect at most q_1-many s-edges at each turn. Including the turns, the number of vertices in a connected component of $G_r(M)$ is at most equal to $2q_1 + 2$.

We observe that a modification of the argument above shows that a connected component of $G_\ell(M)$ is infinite if and only if it contains infinitely many left turns.

Now consider $G_{r+t}(M)$ where $0 \leqslant t < s$. We show that $G_{r+t}(M)$ has an infinite connected component if and only if $t = s - 1$. Note that not all components of $G_{r+t}(M)$ have left turns, for instance, the vertex $(r + t, 0)$ is itself a connected component, which therefore has no turns. In what follows, we study how many left turns a connected component can have.

The ordering $\succ$ on the elements of $\mathbb{N}^2$ defined by $(w, z) \succ (w', z')$ if $z > z'$, or $z = z'$ and $w' > w$, induces a total ordering on the set of left turns of a given component of $G_{r+t}(M)$.

Let C be a connected component of $G_{r+t}(M)$, and suppose that (w_0, z_0) is a left turn in C. We wish to produce the next left turn of C according to $\succ$, if it exists.

Since (w_0, z_0) is a left turn in C, we have $(w_0 - r, z_0 - a) \in C$. This is a right turn, because $G_{r+t}(M)$ cannot contain two adjacent r-edges, as the w-coordinates of the vertices of $G_{r+t}(M)$ are bounded by $r + t$, and $t < s$. We attach s-edges to $(w_0 - r, z_0 - a)$, to produce a vertex $(w_0 - r, z_0 - a) + (qs, qb) \in C$, where $q > 0$ is as large as possible. The integer q is produced by writing $r + t - (w_0 - r) = qs + [r + t - (w_0 - r) \mod s]$, where $[\alpha \mod \beta]$ denotes the remainder of α upon division by β, for $\alpha, \beta \in \mathbb{Z}$, $\alpha > 0$.

If $(w_0 - r, z_0 - a) + (qs, qb)$ is coordinatewise greater than or equal to (r, a), then $(w_0 - r, z_0 - a) + (qs, qb)$ is a left turn of C which is greater according to $\succ$ than (w_0, z_0). Now, $z_0 - a \geqslant 0$ and $b \geqslant a$ imply that $z_0 - a + qb \geqslant a$. Therefore, in order for $(w_0 - r, z_0 - a) + (qs, qb)$ to be a left turn, we need $r \leqslant w_0 - r + qs = r + t - [r + t - (w_0 - r) \mod s]$, or equivalently, $t \geqslant [2r + t - w_0 \mod s]$.

Replacing (w_0, z_0) by $(w_0 - r, z_0 - a) + (qs, qb)$, we see that the condition needed for the existence of a left turn which is greater according to $\succ$ than $(w_0 - r, z_0 - a) + (qs, qb)$ is $t \geqslant [2r + t - (w_0 - r + qs) \mod s] = [3r + t - w_0 \mod s]$.

Continuing in this manner, the existence of infinitely many left turns in C is equivalent to requiring $t \geqslant [\ell r + t - w_0 \mod s]$ for all $\ell > 0$. However, since $\gcd(r, s) = 1$, there exists $\ell > 0$ such that $[\ell r + t - w_0 \mod s] = s - 1$. Therefore, if $t < s - 1$, C has finitely many left turns, and is finite, and if $t = s - 1$, C has infinitely many left turns, and is infinite.

If $t < s - 1$, a component of $G_{r+t}(M)$ either has no left turns or finitely many left turns, which shows that $G_{r+t}(M)$ has no infinite components.

Let $t = s - 1$ and $w_0 \in \{0, \ldots, r + s - 1\}$. If $w_0 \geqslant r$, then for large enough z_0, (w_0, z_0) is a vertex of both an r- and an s-edge whose other vertex has lower z-coordinate, and is therefore a left turn in its connected component, which is thus infinite. If $w_0 < r$, we can choose z_0 sufficiently large such that attaching as many

s-edges to (w_0, z_0) as possible yields a left turn, which implies that the component of (w_0, z_0) is infinite.

Finally, if $\ell > r + s - 1$, for each $0 \leqslant t \leqslant \ell - (r + s - 1)$, $G_\ell(M)$ contains as a subgraph the image of $G_{r+s-1}(M)$ under the translation $(w, z) \mapsto (w + t, z)$. This implies that for each $w_0 \in \{0, \ldots, \ell\}$, there is $z_0 > 0$ such that (w_0, z_0) is an infinite vertex of $G_\ell(M)$. $\square$

Example 2.29. *Let*

$$M = \begin{bmatrix} 7 & 4 \\ 1 & 1 \end{bmatrix}$$

The band graphs $G_4(M)$ and $G_7(M)$ are illustrated in Figure 2.3.

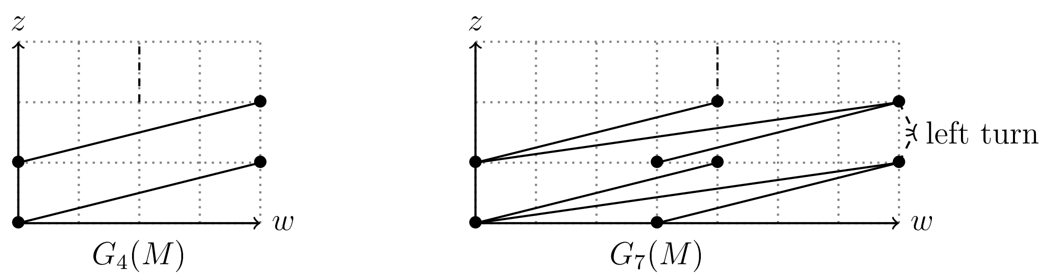

Figure 2.3: Examples of band graphs

All of the connected components of $G_4(M)$ and $G_7(M)$ are finite. The minimum $\ell \in \mathbb{N}$ such that $G_\ell(M)$ has an infinite connected component is $\ell = 10$ (see Figure 2.4).

Remark 2.30. In Proposition 2.28, we assumed that the entries in the top column of M were relatively prime. In the following result, we remove that assumption.

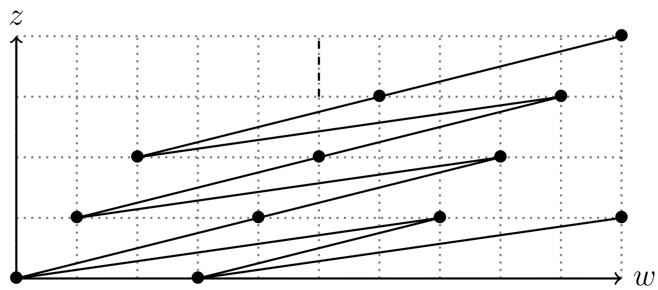

Figure 2.4: A band graph with an infinite component

Theorem 2.31. *Consider a rank two matrix*

$$M = \begin{bmatrix} r & s \\ a & b \end{bmatrix}$$

where r, s, a, b *are positive integers,* $r \geqslant s$*,* $a \leqslant b$ *and* $\gcd(r, s) = d \geqslant 1$*. The minimal* $\ell \in \mathbb{N}$ *such that* $G_\ell(M)$ *has an infinite connected component is* $\ell = r + s - d$*. If* $0 \leqslant t < d$ *and* $w_0 \in \{0, \ldots, r + s - d + t\}$*, there exists* z_0 *such that* (w_0, z_0) *is an infinite vertex of* $G_{r+s-d+t}(M)$ *if and only if* w_0 *is divisible by* d*. If* $\ell > r + s$*, for each* $w_0 \in \{0, \ldots, \ell\}$*, there exists* z_0 *such that* (w_0, z_0) *is an infinite vertex of* $G_\ell(M)$*.*

Proof. Let $\hat{M}$ be the (integer) matrix obtained from M by dividing r and s by d, so that Proposition 2.28 applies to the band graphs of $\hat{M}$.

Let $\ell \in \mathbb{N}$ and set $\hat{\ell} = \lfloor \ell/d \rfloor$, the integer part of ℓ/d. We show that $G_\ell(M)$ is a disjoint union of graphs isomorphic to $G_{\hat{\ell}}(\hat{M})$ or $G_{\hat{\ell}-1}(\hat{M})$.

Let $(w_0, z_0) \in \mathbb{N}^2$ such that $w_0 \leqslant \ell$, so that (w_0, z_0) is a vertex of $G_\ell(M)$. Write $w_0 = \hat{w}_0 d + t_0$ where t_0 is an integer with $0 \leqslant t_0 < d$.

If $t_0 = 0$, then (w_0, z_0) belongs to the image of the map $\varphi_{\ell,0} : G_{\hat{\ell}}(M) \to G_\ell(M)$ defined on vertices by $(w, z) \mapsto (dw, z)$. Since r and s are divisible by d, any vertex in the connected component of $G_\ell(M)$ that contains (w_0, z_0) also has its w-coordinate divisible by d. This implies that the connected component of (w_0, z_0) in $G_\ell(M)$ is the image under $\varphi_{\ell,0}$ of the connected component of $(w_0/d = \hat{w}_0, z_0)$ in $G_{\hat{\ell}}(\hat{M})$.

If $t_0 > 0$, consider the map $\varphi_{\ell,t_0} : G_{\hat{\ell}-1}(M) \to G_\ell(M)$ defined on vertices by $(w, z) \mapsto (dw + t_0, z)$. Since r and s are divisible by d, the w-coordinates of all the vertices of $G_\ell(M)$ connected to (w_0, z_0) are congruent to t_0 modulo d. This implies that the connected component of (w_0, z_0) in $G_\ell(M)$ is the image under φ_{ℓ,t_0} of the connected component of $(\hat{w}_0, z_0)$ in $G_{\hat{\ell}-1}(\hat{M})$.

Note that the images of the maps $\varphi_{\ell,i}$ have no common vertices, and their union is $G_\ell(M)$. Now use Proposition 2.28 to obtain the desired conclusions. $\square$

Example 2.32. *Let* $M = \begin{bmatrix} 2 & 6 \\ 1 & 2 \end{bmatrix}$. *When* $\ell = 6$*, the band graph* $G_6(M)$ *has an infinite connected component. However, the vertices* (w, z) *where* w *is odd are finite vertices for all* z*; see Figure 2.5.*

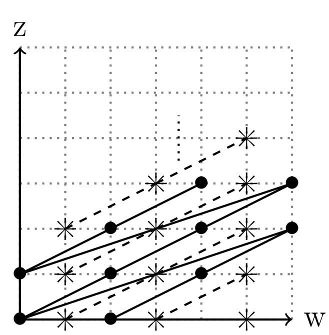

Figure 2.5: The band graph $G_6(M)$

The following lemma relates the graphs of lattice basis ideals and those associated to matrices.

Lemma 2.33. *Let M be an $n \times m$ integer matrix of rank m, and $I(M)$ its corresponding lattice basis ideal as in Definition 1.41. Let $\tau \subseteq \{1, \dots, n\}$ and let $P = \mathbb{N}^\tau \times \mathbb{Z}^{\bar{\tau}}$. Then $u, v \in P$ are connected in $\mathscr{G}_P(\Bbbk[P] \cdot I(M))$ if and only if they are connected in $G_P(M)$.*

Proof. Assume that $u, v \in P$ are connected in $G_P(M)$. We show that $x^u - x^v \in \Bbbk[P] \cdot I(M)$ by induction on the length of the path connecting u to v. If this path has length one, then u and v are connected by an edge of $G_P(M)$, meaning that $u - v$ or $v - u$, say $u - v$, equals a column μ of M. Then $u - v = \mu = \mu_+ - \mu_-$, so that $v - \mu_- = u - \mu_+ =: \nu$. Since $v - \mu_- + \mu_+ = u \in P$ and for all i, $(\mu_+)_i$ and $(\mu_-)_i$ are not simultaneously nonzero, we see that $\nu \in P$. But then $x^u - x^v = x^\nu(x^{\mu_+} - x^{\mu_-}) \in \Bbbk[P] \cdot I(M)$, as we wished.

Now assume that u and v are connected in $G_P(M)$ by a path of length $\ell > 1$. This means that there are vertices $u = \nu^{(0)}, \nu^{(1)}, \dots, \nu^{(\ell)} = v$ of $G_P(M)$ such that $(\nu^{(i)}, \nu^{(i+1)})$ is an edge of $G_P(M)$ for $i = 0, \dots, \ell$. By inductive hypothesis, since $\nu^{(1)}$ and v are connected in $G_P(M)$ by a path of length $\ell - 1$, we have $x^{\nu^{(1)}} - x^v \in \Bbbk[P] \cdot I(M)$. But we also know $x^u - x^{\nu^{(1)}} \in \Bbbk[P] \cdot I(M)$. We conclude that $x^u - x^v \in \Bbbk[P] \cdot I(M)$, and therefore u and v are connected in $\mathscr{G}_P(I(M))$.

For the converse, we start by noting that a lattice basis ideal (and its extension to $\Bbbk[P]$) contains no monomials. This follows, for instance, from Lemma 7.6 in [30], which implies that the saturation $(I(M) : \langle x_1 \cdots x_n \rangle^\infty) \subseteq \Bbbk[x]$ is not the unit ideal.

Since every connected component of $\mathscr{G}_P(I(M))$ is a complete graph, if u and v are connected in $\mathscr{G}_P(I(M))$, then (u, v) is an edge in $\mathscr{G}_P(I(M))$. Thus, there exists nonzero $\rho \in \Bbbk$ such that $x^u - \rho x^v \in \Bbbk[P] \cdot I(M)$, and if $\mu^{(1)}, \dots, \mu^{(m)}$ are the columns of M, we can write $x^u - \rho x^v = F_1(x)(x^{\mu_+^{(1)}} - x^{\mu_-^{(1)}}) + \cdots + F_m(x)(x^{\mu_+^{(m)}} - x^{\mu_-^{(m)}})$ for certain $F_1, \dots, F_m \in \Bbbk[P]$. We can represent this expression as a subgraph K of

$G_P(M)$: for every term λx^ν in F_i, K contains the edge $(\nu + \mu_+^{(i)}, \nu + \mu_-^{(i)})$ and its corresponding vertices. We label this edge by the coefficient λ, and we label each vertex by the combination of the labels of the edges adjacent to it, with a positive sign if we look at the vertex $\nu + \mu_+^{(i)}$ of $(\nu + \mu_+^{(i)}, \nu + \mu_-^{(i)})$, and a negative sign for the vertex $\nu + \mu_-^{(i)}$. Thus, the only two vertices with nonzero labels are u and v.

Let K_u be the connected component of K containing u. We wish to show that v is a vertex in K_u, as this implies that u and v are connected in $G_P(M)$. But if this is not the case, we can use K_u to form a polynomial expression with a summand $\lambda x^\nu(x^{\mu_+^{(i)}} - x^{(\mu_-^{(i)}})$ for each edge $(\nu + \mu_+^{(i)}, \nu + \mu_-^{(i)})$ labeled by λ in K_u, and this expression equals the sum over the vertices in K_u of the label of each vertex times the corresponding monomial. Since the only vertex with a nonzero label in K_u is u (that label is 1), then we obtain an expression for x^u as a combination of the generators of $\Bbbk[P] \cdot I(M)$. This contradicts the fact that $\Bbbk[P] \cdot I(M)$ contains no monomials. $\square$

Remark 2.34. We can construct $\mathscr{G}_P(I(M))$ by adding edges to $G_P(M)$ until each connected component becomes a complete graph. This is correct by the hypotheses and notation of the previous lemma. For an arbitrary binomial ideal $I \subseteq \Bbbk[P]$, it is always possible to construct a subgraph of $\mathscr{G}_P(I)$ using a generating set of I, so that the vertex sets of their connected components are the same. This implies that saturating the connected components of this subgraph with edges yields $\mathscr{G}_P(I)$. Note that not every generating set of I contains sufficient information, what we need is a generating set of I that contains all the generators of the maximal monomial ideal in I. A different perspective can be found in the Lemmas 1 and 2 in [29].

Remark 2.35. We can construct $\mathscr{G}(\overline{I(B)})$ from the graph $G(M)$ associated to the 2×2 matrix M whose rows are b_i and b_j by Lemma 2.33. Also, since we have already characterized the connected components of $G(M)$ in Proposition 2.24, we can describe the corresponding primary component by applying Theorem 2.19. This yields a very nice picture of the toral components of a codimension two lattice basis ideals in characteristic zero.

2.5 Codimension Two Lattice Basis Ideals in Three Variables

In this section, we study the Andean components of codimension two lattice basis ideals. We first look at the case of three variables.

Convention 2.36. *Let B be 3×2 matrix of full rank 2 as follows*

$$B = \begin{bmatrix} r & s \\ -\lambda r & -\lambda s \\ a & b \end{bmatrix}$$

where $r, s, a, b \in \mathbb{Z}_{>0}$, $a \leqslant b$, $r \geqslant s$, $\gcd(r, s) = d$, and $0 < \lambda = p/q$ in lowest terms. We work in the polynomial ring $\Bbbk[x, y, z]$. The lattice basis ideal associated to B is $I(B) = \langle x^r z^a - y^{\lambda r}, x^s z^b - y^{\lambda s}\rangle \subseteq \Bbbk[x, y, z]$. We let $P = \mathbb{N}^2 \times \mathbb{Z}$, and work with $\Bbbk[P] = \Bbbk[z^{\pm}][x, y]$.

Remark 2.37. Note that for a codimension two lattice basis ideal $I(B)$ corresponding to a matrix B as in Convention 2.36, a vertex $u = (u_x, u_y, u_z)$ of $\mathscr{G}(I(B))$ that lies on a hyperplane $u_y = -\lambda u_x + \lambda \ell$, for $\ell \in \mathbb{Q}$, can only be connected to other vertices on that hyperplane. The reason is that the columns of B which are the building blocks of the graph $G(B)$ are parallel to the hyperplane $u_y = -\lambda u_x$.

Let B be as in Convention 2.36. In [24], we compute the primary component of $I(B)$ corresponding to the Andean associated prime $\langle x, y\rangle$. The only ingredient we need to understand this component is the graph arising from the extension of $I(B)$ to

$\Bbbk[z^{\pm}][x, y] = \Bbbk[P]$. Indeed, the following proposition illustrates that it is sufficient to understand the infinite connected components of $\mathscr{G}(I(B))$.

Proposition 2.38. *Let B and $P = \mathbb{N}^2 \times \mathbb{Z}$ as in Convention 2.36. Then*

$$\{(u_x, u_y) \in \mathbb{N}^2 \mid \exists u_z \in \mathbb{Z} \text{ such that } (u_x, u_y, u_z) \text{ is an infinite vertex of } \mathscr{G}_P(\Bbbk[P]\cdot I(B))\}$$

$$= \{(u_x, u_y) \in \mathbb{N}^2 \mid \exists u_z \in \mathbb{N} \text{ such that } (u_x, u_y, u_z) \text{ is an infinite vertex of } \mathscr{G}(I(B))\}.$$

Proof. If $u = (u_x, u_y, u_z) \in \mathbb{N}^3$ is an infinite vertex of $\mathscr{G}(I(B))$, then it is clear that it is also an infinite vertex of $\mathscr{G}_P(\Bbbk[P] \cdot I(B))$.

Let $u = (u_x, u_y, u_z) \in P$ be an infinite vertex of $\mathscr{G}_P(\Bbbk[P] \cdot I(B))$. By Lemma 2.9 there exists $v = (v_x, v_y, v_z), \tilde{v} = (\tilde{v}_x, \tilde{v}_y, \tilde{v}_z) \in \mathbb{N}^2 \times \mathbb{Z}$ connected to u such that $\tilde{v}_x \geqslant v_x$ and $\tilde{v}_y \geqslant v_y$. Since u is connected to v, we can find a nonzero $\rho \in k$ such that $x^{u_x}y^{u_y}z^{u_z} - \rho x^{v_x}y^{v_y}z^{v_z} \in \Bbbk[P] \cdot I(B)$, and by clearing denominators, we can produce $\mu \in \mathbb{N}$ such that $z^{\mu}(x^{u_x}y^{u_y}z^{u_z} - \rho x^{v_x}y^{v_y}z^{v_z}) \in I(B)$; in particular, $\mu + u_z$ and $\mu + v_z$ are non negative. Thus, the vertices $(u_x, u_y, u_z + \mu), (v_x, v_y, v_z + \mu) \in \mathbb{N}^3$ are connected in $\mathscr{G}(I(B))$. Enlarging μ as needed, we may assume that $(u_x, u_y, u_z + \mu), (v_x, v_y, v_z + \mu), (\tilde{v}_x, \tilde{v}_y, \tilde{v}_z + \mu)$ are coordinatewise non negative and connected in $\mathscr{G}(I(B))$.

By Remark 2.37, there exists $\ell \in \mathbb{Q}$ such that $v_y = -\lambda v_x + \lambda\ell$ and $\tilde{v}_y = -\lambda\tilde{v}_x + \lambda\ell$, so that $\tilde{v}_y - v_y = -\lambda(\tilde{v}_x - v_x)$. Since $\lambda > 0$ and $\tilde{v}_x - v_x$, $\tilde{v}_y - v_y$ are non negative, we see that $v_x = \tilde{v}_x$ and $v_y = \tilde{v}_y$.

In conclusion, the vertices $(v_x, v_y, v_z + \mu), (v_x, v_y, \tilde{v}_z + \mu) \in \mathbb{N}^3$ are connected in $G(I(B))$, since either $(v_x, v_y, v_z + \mu) - (v_x, v_y, \tilde{v}_z + \mu)$ or $(v_x, v_y, \tilde{v}_z + \mu) - (v_x, v_y, v_z + \mu)$ belongs to $\mathbb{N}^3$, we see that these vertices belong to an infinite component of $\mathscr{G}(I(B))$ by Lemma 2.9. As these vertices are connected to $(u_x, u_y, u_z + \mu)$, we conclude that this is an infinite vertex of $\mathscr{G}(I(B))$. $\square$

Corollary 2.39. *Let B and $P = \mathbb{N}^2 \times \mathbb{Z}$ as in Convention 2.36. The primary component of $I(B)$ corresponding to the associated prime $\langle x, y\rangle$ is*

$$(I : z^\infty) + \langle x^{u_x} y^{u_y} \rangle \mid \exists u_z \in \mathbb{N} \text{ such that } (u_x, u_y, u_z) \text{ is an infinite vertex of } G(B)\rangle.$$

Proof. The result follows from Theorem 2.6, Remark 2.8 and Lemma 2.33. □

The fact in Remark 2.37 motivates the following definition.

Definition 2.40. Let B be as in Convention 2.36 and $\ell \in (1/p)\mathbb{N}$. Set $S(\ell) = \mathbb{N}^n \cap \{(u_x, u_y, u_z) \mid u_y = -\lambda u_x + \lambda\ell\}$. The graph $G_{S(\ell)}(B)$ is called the *ℓ-th slice graph of B*. $G(B)$ equals the disjoint union $\bigcup_{\ell\in(1/p)\mathbb{N}} G_{S(\ell)}(B)$ as a graph.

Example 2.41. *Let $I(B) = \langle x^4z - y^4, x^7z - y^7\rangle$, the slice graphs of $I(B)$ are illustrated in Figure 2.6.*

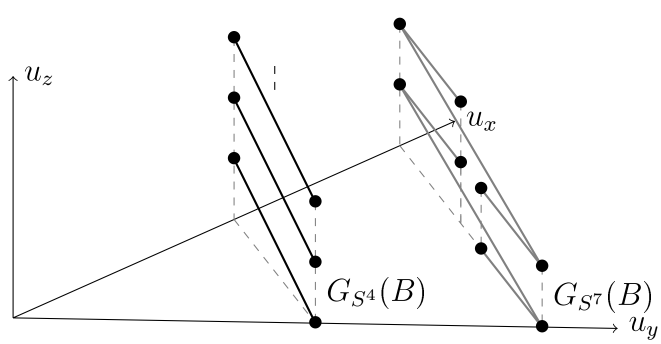

Figure 2.6: Slice graphs for $I(B) = \langle x^4z - y^4, x^7z - y^7\rangle$

We group the slice graphs of B according to isomorphism in the following result.

Lemma 2.42. *Let B as in Notation 2.36. Suppose $(c_x, c_y, c_z) \in \mathbb{N}^3$ is a vertex of $G_{S(\ell)}(B)$, where $\ell \in (1/p)\mathbb{N}$. Write $c_x = q\bar{c}_x + i$, $c_y = p\bar{c}_y + j$, where $\bar{c}_x, \bar{c}_y, i, j \in \mathbb{N}$ and $0 \leqslant i < q$, $0 \leqslant j < p$. Then $G_{S(\ell)}(B)$ is isomorphic to the slice graph $G_{S(\ell+i+j/\lambda)}(B)$ that contains $(q\bar{c}_x, p\bar{c}_y, c_z)$ as a vertex. Consequently, in order to understand the (connected components of) all the slice graphs of B, it is enough to understand $G_{S(\ell)}(B)$ for $\ell \in q\mathbb{N}$.*

Proof. The desired isomorphism φ_{ij} between $G_{S(\ell)}(B)$ and $G_{S(\ell+i+j/\lambda)}(B)$ is defined by

$$(u_x, u_y, u_z) \mapsto (u_x + i, u_y + j, u_z).$$

Note that a vertex of the form $(q\bar{c}_x, p\bar{c}_y, c_z) \in \mathbb{N}^3$, where $\bar{c}_x, \bar{c}_y \in \mathbb{N}$ belongs to a slice graph $G_{S(\bar{\ell})}(B)$ where $\bar{\ell} \in q\mathbb{N}$. $\square$

Example 2.43. *To better understand how the ideas above work, let us illustrate them for the ideal $I = \langle x^{12}z^6 - y^{18}, x^{18}z^6 - y^{27}\rangle$. Thus, $r = 12, s = 18, \lambda = p/q = 3/2$ and $d = 6$. We know that an infinite connected component first occurs in $G_{S(24)}(B)$. By Lemma 2.44 and Theorem 2.31, we realize that $(24, 0, z), (18, 9, z), (12, 18, z)$, $(6, 27, z)$ and $(0, 36, z)$ are infinite vertices for some z. The projections onto the xy-plane of the slice graphs of B are shown in Figure 2.7. The monomials induced by these infinite vertices give a staircase diagram that can be seen in the picture in the left hand side.*

Let us consider the vertices under the stair starting with $(0, 36)$ and ending with $(6, 27)$. We know that the vertices shown with circles are finite by Theorem 2.31. To understand the connected components of all the slice graphs of B that passes through the vertices in this figure, it is enough to understand $G_{S(24)}(B), G_{S(26)}(B)$ and $G_{S(28)}(B)$. For example the vertices depicted with stars belong to the slice graphs which are isomorphic to $G_{S(24)}(B)$ and they are all finite. Also, the vertices shown

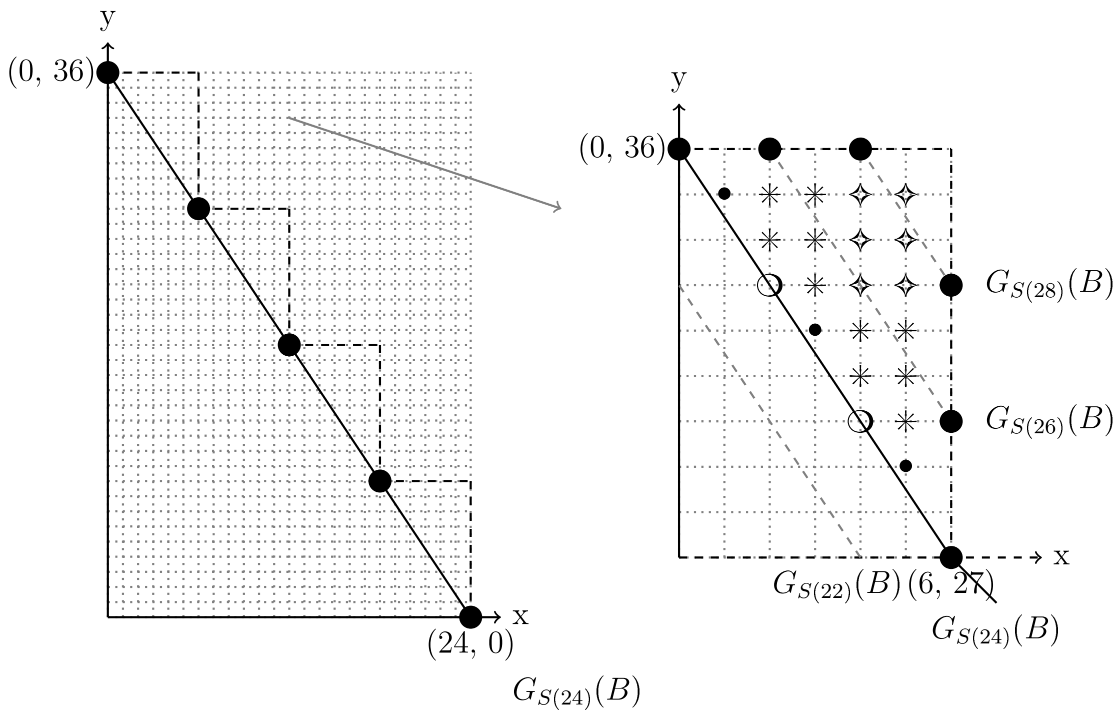

Figure 2.7: Slice graphs for $I(B) = \langle x^{12}z^6 - y^{18}, x^{18}z^6 - y^{27} \rangle$

with a square belong to the slice graphs which are isomorphic to $G_{S(26)}(B)$ and they are all finite too. The vertices depicted with bullets on the slice graphs $G_{S(26)}(B)$ and $G_{S(28)}(B)$ correspond to infinite vertices, but note that the monomials induced by these can be divided by x^6y^{27} or y^{36}. Thus the minimal generating set of the monomial ideal we are looking for does not contain the aforementioned monomials. Lastly, the vertices which are shown with small bullets are related to the slice graph $G_{S(22)}(B)$, and they are all finite, since infinite vertices first appear in $G_{S(24)}(b)$.

In order to draw the graphs in a simpler way, we "straighten out" the slice graphs of B. Recall the notation in Convention 2.36. Given $\ell \in \mathbb{N}$, let $\phi_\ell : \mathbb{N}^2 \to \mathbb{Z}^3$ be the injective function defined by $(w, k) \mapsto (qw, \lambda(q\ell - qw), k) = (qw, p(\ell - w), k)$. Note that the image $\phi_\ell(\{(w, k) \in \mathbb{N}^2 \mid 0 \leqslant w \leqslant \ell, 0 \leqslant k\})$ is the intersection with $\mathbb{N}^3$ of the hyperplane given by $u_y = -\lambda u_x + \lambda q\ell$.

Lemma 2.44. *Let B be as in Convention 2.36 and let $M = \left[\begin{smallmatrix} r/q & s/q \\ a & b \end{smallmatrix}\right]$. Given $\ell \in \mathbb{N}$, define ϕ_ℓ as above. The image under ϕ_ℓ of the band graph $G_\ell(M)$ is the slice graph $G_{S(q\ell)}(B)$.* □

We have already studied the connected components of the band graphs $G_\ell(M)$, thus we can compute the primary component of $I(B)$ associated to $\langle x, y\rangle$.

Theorem 2.45. *Let B as in Notation 2.36. The primary component of $I(B)$ corresponding to the associated prime $\langle x, y\rangle$ is*

$$(I(B) : z^\infty) + \langle y^{\lambda(r+s-d)}, x^d y^{\lambda(r+s-2d)}, x^{2d} y^{\lambda(r+s-3d)}, \ldots, x^{r+s-d}\rangle.$$

Proof. By Theorem 2.6, the desired component is $(I(B) : z^\infty) + \mathscr{M}$, where

$$\mathscr{M} = \langle x^{u_x} y^{u_y} \mid \exists u_z \in \mathbb{Z} \text{ with } u = (u_x, u_y, u_z) \text{ an infinite vertex of } \mathscr{G}_{\mathbb{N}^2\times\mathbb{Z}}(I(B))\rangle.$$

By Proposition 2.38, we can use $\mathscr{G}(I(B))$ instead of $\mathscr{G}_{\mathbb{N}^2\times\mathbb{Z}}(I(B))$ in the definition of $\mathscr{M}$, and by Lemma 2.33, we can use $G(B)$ instead of $\mathscr{G}(I(B))$.

Now using Remark 2.37, Lemmas 2.42 and 2.44 and Theorem 2.31 we obtain the desired result. □

Theorem 2.45 is an important for understanding the primary decomposition of $I(B)$ in the general $n \times 2$ case. Indeed, we reduce the general case to 3×2 case. We are now ready to state and prove the main result in [24], which gives an explicit expression for the Andean primary components of a codimension two lattice basis ideal.

Theorem 2.46. *Let B as in Convention 2.18, and suppose that b_i, b_j are linearly dependent rows of B lying in opposite open quadrants of $\mathbb{Z}^2$. Without loss of generality, assume that $b_{i1}, b_{i2} > 0$, let $d = \gcd(b_{i1}, b_{i2})$, and write $\lambda = -b_{j1}/b_{i1} = -b_{j2}/b_{i2} > 0$. The primary component of $I(B)$ corresponding to the associated prime $\langle x_i, x_j\rangle$ is*

$$(I(B) : (\prod_{k\neq i,j} x_k)^\infty) + \langle x_j^{\lambda(b_{i1}+b_{i2}-d)}, x_i^d x_j^{\lambda(b_{i1}+b_{i2}-2d)}, x_i^{2d} x_j^{\lambda(b_{i1}+b_{i2}-3d)}, \ldots, x_i^{b_{i1}+b_{i2}-d}\rangle.$$

The only monomials in this ideal are those in the ideal

$$\langle x_j^{\lambda(b_{i1}+b_{i2}-d)}, x_i^d x_j^{\lambda(b_{i1}+b_{i2}-2d)}, \ldots, x_i^{b_{i1}+b_{i2}-d}\rangle.$$

Proof. Let $\sigma = \{i, j\}$, and set $P = \mathbb{N}^\sigma \times \mathbb{Z}^{\bar{\sigma}}$. Choose $0 < a \leqslant b \in \mathbb{Z}$ such that the matrix $\hat{B} = \left[\begin{smallmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \\ a & b \end{smallmatrix}\right]$ has rank 2. Our result follows from Theorems 2.45 and 2.6 if we show that

$$\{(u_i, u_j) \in \mathbb{N}^\sigma \mid \exists u \in P \text{ an infinite vertex of } \mathscr{G}_P(I(B))\} =$$
$$\{(c_1, c_2) \in \mathbb{N}^2 \mid \exists c_3 \in \mathbb{Z} \text{ with } (c_1, c_2, c_3) \text{ an infinite vertex of } \mathscr{G}_{\mathbb{N}^2\times\mathbb{Z}}(I(\hat{B})).$$

By Lemma 2.33, it is enough to show that

$$\{(u_i, u_j) \in \mathbb{N}^\sigma \mid \exists u \in P \text{ with } u_i \text{ and } u_j \text{ an infinite vertex of } G_P(B)\} =$$
$$\{(c_1, c_2) \in \mathbb{N}^2 \mid \exists c_3 \in \mathbb{Z} \text{ with } (c_1, c_2, c_3) \text{ an infinite vertex of } G_{\mathbb{N}^2\times\mathbb{Z}}(\hat{B}).$$

We show $\subseteq$; the other inclusion is similar.

Let $(u_i, u_j) \in \mathbb{N}^\sigma$, such that there is $u \in \mathbb{N}^\sigma \times \mathbb{Z}^{\bar{\sigma}}$ (whose i-th and j-th coordinate are u_i and u_j) that is an infinite vertex of $G_P(B)$. By Lemma 2.9, there are $v, \tilde{v} \in P$ connected to u such that $\tilde{v} - v \in P$. Since v and $\tilde{v}$ are connected, there is a sequence of vertices $u = \mu^{(1)}, \ldots, \mu^{(\ell_1)} = v, \mu^{(\ell_1+1)}, \ldots, \mu^{(\ell_2)} = \tilde{v} \in P$ such that $(\mu^{(k)}, \mu^{(k+1)})$ is an edge in $G_P(B)$ for $k = 1, \ldots \ell_2 - 1$.

Recall that B_1 and B_2 are the columns of B, and denote $\hat{B}_1$ and $\hat{B}_2$ the columns of $\hat{B}$. Define

$$v^{(k)} = \begin{cases} \hat{B}_1 & \text{if } \mu^{(k+1)} - \mu^{(k)} = B_1, \\ -\hat{B}_1 & \text{if } \mu^{(k+1)} - \mu^{(k)} = -B_1, \\ \hat{B}_2 & \text{if } \mu^{(k+1)} - \mu^{(k)} = B_2, \\ -\hat{B}_2 & \text{if } \mu^{(k+1)} - \mu^{(k)} = -B_2. \end{cases}$$

Choose any $c \in \mathbb{Z}$ and let $\nu^{(1)} = (u_i, u_j, c)$. Set also $\nu^{(k+1)} = \nu^{(k)} + v^{(k)}$ for $k = 1, \ldots, \ell_2 - 1$. Then the first and second coordinates of $\nu^{(k)}$ are equal to the i-th and j-th coordinates of $\mu^{(k)}$ respectively, which implies that $\nu^{(1)}, \ldots, \nu^{(\ell_2)} \in \mathbb{N}^2 \times \mathbb{Z}$. By construction, $(\nu^{(k)}, \nu^{(k+1)})$ is an edge of $G_{\mathbb{N}^2\times\mathbb{Z}}(\hat{B})$ for $k = 1, \ldots, \ell_2 - 1$, so that in particular, $\nu^{(1)}$, $\nu^{(\ell_1)}$ and $\nu^{(\ell_2)}$ belong to the same connected component of $G_{\mathbb{N}^2\times\mathbb{Z}}(\hat{B})$. Moreover, $\tilde{v} - v \in P$ implies that $\nu^{(\ell_2)} - \nu^{(\ell_1)} \in \mathbb{N}^2 \times \mathbb{Z}$, so by Lemma 2.9, $\nu^{(1)}$ is an infinite vertex of $G_{\mathbb{N}^2\times\mathbb{Z}}(\hat{B})$, and we conclude that (u_i, u_j) belongs to the set in the right hand side. □

Example 2.47. *We continue to compute the primary components left off in Example 2.21 Recall that* $I(B) = \langle x_1^2x_3^2x_6^2 - x_2^4x_4x_5x_7^8x_8^3, x_1^4x_3^3x_4^3 - x_2^6x_5^2x_6^6x_7^{12}x_8^6\rangle$.

- $(2,3)$ *and* $(-4,-6)$ *are linearly dependent rows of* B *and they are in opposite open quadrants. The* $\langle x_2, x_3\rangle$*- primary component of* $I(B)$ *is*

$$((I(B) : (\prod_{\ell\neq 2,3} x_\ell)^\infty) + \langle x_2^8, x_2^6x_3, x_2^4x_3^2, x_2^2x_3^3, x_3^4\rangle)$$

$$= \langle x_2^8, x_2^6x_3, x_2^4x_3^2, x_2^2x_3^3, x_3^4, x_2^2x_3^2x_6^{10} - x_2^4x_3x_4^5x_7^4,$$

$$x_2^4x_4x_5x_7^8x_8^3 - x_1^2x_3^2x_6^2, x_2^2x_3^2x_5x_6^8x_7^4x_8^3 - x_1^2x_3^3x_4^4,$$

$$x_2^6x_5x_6^8x_7^4x_8^3 - x_1^2x_2^4x_3x_4^4, x_2^6x_5^2x_6^6x_7^{12}x_8^6 - x_1^4x_3^3x_4^3\rangle$$

- *If we look at the* $\langle x_3, x_7\rangle$*- primary component of* $I(B)$

$$((I(B) : (\prod_{\ell\neq 3,7} x_\ell)^\infty) + \langle x_3^4, x_3^3x_7^4, x_3^2x_7^8, x_3x_7^{12}, x_7^{16}\rangle)$$

$$= \langle x_3^4, x_3^3x_7^4, x_3^2x_7^8, x_3x_7^{12}, x_7^{16}, x_3^2x_6^{10}x_7^4 - x_2^2x_3x_4^5x_7^8,$$

$$x_2^4x_4x_5x_7^8x_8^3 - x_1^2x_3^2x_6^2, x_2^2x_3^2x_5x_6^8x_7^4x_8^3 - x_1^2x_3^3x_4^4,$$

$$x_2^6x_5x_6^8x_7^4x_8^3 - x_1^2x_2^4x_3x_4^4, x_2^6x_5^2x_6^6x_7^{12}x_8^6 - x_1^4x_3^3x_4^3\rangle$$

- $(-1, 3)$ *and* $(2, -6)$ *are linearly dependent rows of* B *and they lie in opposite open quadrants of* $\mathbb{Z}^2$*. The* $\langle x_4, x_6\rangle$*- primary component of* $I(B)$ *is*

$$((I(B) : (\prod_{\ell\neq 4,6} x_\ell)^\infty) + \langle x_4^3, x_4^2x_6^2, x_4x_6^4, x_6^6\rangle) = \langle x_4^3, x_4^2x_6^2, x_4x_6^4, x_6^6, x_2^4x_4x_5x_7^8x_8^3 - x_1^2x_3^2x_6^2\rangle.$$

- *If we look at the* $\langle x_1, x_8\rangle$*- primary component of* $I(B)$

$$((I(B) : (\prod_{\ell\neq 1,8} x_\ell)^\infty) + \langle x_1^4, x_1^2x_8^3, x_8^6\rangle) = \langle x_1^4, x_1^2x_8^3, x_8^6, x_2^4x_4x_5x_7^8x_8^3 - x_1^2x_3^2x_6^2\rangle.$$

To conclude this section we remark that the techniques we use to compute the Andean component of codimension two lattice basis ideal are hard to generalize. The first obstacle towards a generalization is that for codimension two lattice basis ideals, we know which primes are associated to the ideal. For general lattice basis ideals on the other hand, we do not have an efficient combinatorial method to find the minimal associated primes. In other words, we do not have an efficient algorithm to test whether the matrix has the wanted block decomposition which is mentioned in [18]. The second difficulty is that while we are able to pass from the monoid

$\mathbb{N}^2 \times \mathbb{Z}^{n-2}$ to $\mathbb{N}^n$, by using the linear dependence of the rows under consideration (Proposition 2.38.), but this does not have to hold in the general case. The third difficulty can be about finding the infinite vertices of the graph of binomial ideals. In the codimension two case, we have two edges that are building blocks. This gives us a combinatorial control to detect the infinite vertices, as in Theorem 2.38. When the number of columns increases, we cannot hope to compute the finite or infinite connected components of the graphs we construct by simply drawing them.

3. CELLULAR BINOMIAL IDEALS

In this section, we find a characterization for the minimal primary components of a cellular binomial ideal. We have already considered some properties of cellular binomial ideals in previous sections. Let us recall the central definition of this section and fix the corresponding notations.

3.1 Cellular Decomposition of Binomial Ideals

Definition 3.1. An ideal $I \subseteq \Bbbk[x_1, ..., x_n]$ is *cellular* if every variable is either a nonzerodivisor modulo I or is nilpotent modulo I.

The study of cellular decomposition was motivated by analyzing the intersection of a variety corresponding to a binomial ideal with coordinate cells $(\Bbbk^*)^\delta$

$$(\Bbbk^*)^\delta = \{(a_1, ..., a_n) \mid a_i \neq 0, i \in \delta \text{ and } a_j = 0, j \notin \delta\},$$

where δ runs over all subsets of $\{1, ..., n\}$. The closure of the intersection of the variety corresponding to a binomial ideal $V(I)$ with $(\Bbbk^*)^\delta$ is the variety corresponds to the ideal

$$I_\delta = (I + \langle x_i \mid i \notin \delta \rangle) : (\prod_{i \in \delta} x_i)^\infty.$$

If I is a radical ideal then $I = \bigcap_\delta I_\delta$. A more refined version of this statement is given in the following theorem.

Theorem 3.2. *(Theorem 6.2 in [12].) Let I be a binomial ideal then*

$$I = \bigcap_{\delta \subseteq \{1,...,n\}} \left((I + \langle x_i^{d_i} \mid i \notin \delta \rangle) : (\prod_{i \in \delta} x_i)^\infty \right),$$

for d_i is sufficiently large.

To be more precise about the numbers d_i, we can use the fact that for some primary decomposition of $I = \bigcap_j Q_j$, $x_i^{d_i} \in Q_j$ if and only if $x_i \in \sqrt{Q_j}$ for all i and j. There is also a partial criterion to determine the numbers d_i without using a primary decomposition, that can be found in Theorem 2.8 in [31].

The ideals $((I + \langle x_i^{d_i} \mid i \notin \delta\rangle) : (\prod_{i\in\delta} x_i)^\infty)$ we manufactured from I are obviously cellular binomial ideals. In fact, I is cellular if and only if $I = I_\delta^{(d)}$ for some $\delta \subseteq \{1, ..., n\}$ and $d \in \mathbb{Z}^n_{>0}$. From now on, we usually specify δ and b when we work with a cellular binomial ideal. If the index set for the nonzerodivisor variables is $\delta \subseteq \{1, ..., n\}$, then we call I a δ-*cellular* binomial ideal. We now look at some features of these ideals.

Proposition 3.3. *Let $I = I_\delta^{(b)} \subseteq \Bbbk[x_1, ..., x_n]$ be a cellular binomial ideal .*

1. *There exists a partial character (ρ, L_ρ) such that $I \cap \Bbbk[x_i \mid i \in \delta] = I_+(\rho)$ and $\sqrt{I} = \sqrt{I_+(\rho)} + \langle x_i \mid i \in \bar{\delta}\rangle$.*

2. *Let $I = I_\delta^{(b)}$ be a cellular binomial ideal, and let $P = I_+(\tilde{\rho}) + M(P)$ be an associated prime of I, where $M(P)$ is the largest monomial ideal contained in P. Then $M(P) = \langle x_i \mid i \in \bar{\delta}\rangle$.*

Proof. This is a combination of Propositions 2.2 and 2.3 in [31]. □

Proposition 3.4. *Let $I \subseteq \Bbbk[x_1, ..., x_n]$ be a binomial ideal and $g \in \Bbbk[x_1, ..., x_n]$. If $(I : g) = (I : g^\infty)$ then $I = (I : g) \cap (I + \langle g\rangle)$.*

Proposition 3.4 is the key ingredient in an algorithm for computing cellular decomposition of a binomial ideal. If I is not cellular, we can find a variable x_ℓ which is a zerodivisor but not nilpotent modulo I. For sufficiently large r, $(I : x_\ell^r) = (I : x_\ell^\infty)$.

Now say $x_\ell^r = g$ and apply the proposition above. Note that both $(I : g)$ and $(I + \langle g \rangle)$ are binomial ideals that strictly contain I. If we repeat the same argument with the new ideals we constructed, we obtain a chain of ideals that has to stop by Noetherianity. At the end, we obtain a cellular decomposition of the original ideal. The following algorithm is due to Ojeda and Sanchez [31].

Algorithm 3.5. Cellular decomposition of a binomial ideal

Input : A binomial ideal I.

Output : A cellular decomposition of I.

1. If I is cellular return I.

2. Choose x_i that is zero divisor but not nilpotent modulo I.

3. Determine the power m such that $(I : x_i^m) = (I : x_i^\infty)$.

4. Repeat the algorithm for $(I : x_i^m)$ and $I + \langle x_i^m \rangle$.

To compute the integer in the third step, one can use the algorithm given by Becker and Weispfenning in [4].

Cellular decompositions of binomial ideals are not canonical; they can be considered as an approximation for primary decomposition. Eisenbud and Sturmfels proved the binomiality of the primary decomposition by using this intermediate cellular decomposition step, for example in Theorem 1.36 and Theorem 1.35. An irredundant primary decomposition of a binomial ideal I can be obtained from the given primary decompositions of the cellular components of I by deleting redundant components. In [1] and [12], the authors discussed cases where a cellular decomposition is a primary decomposition.

3.2 Computing Associated Primes and Primary Components of a Cellular Binomial Ideal

Theorem 3.6. *Let $I = I_\delta^{(b)}$ be a cellular binomial ideal in $S = \Bbbk[x_1, ..., x_n]$, where $\mathrm{char}(\Bbbk) = p \geqslant 0$. If $P = I_+(\tilde{\tau}) + M(P)$ where $M(P)$ is the maximal monomial ideal in P is an associated prime of I, then there exists a monomial m in the variables $\{x_i\}_{i \in \bar{\delta}}$ and a partial character τ such that $\tilde{\tau}$ is a saturation of τ and*

$$(I : m) \cap \Bbbk[x_i \mid i \in \delta] = I_+(\tau)$$

Proof. See Theorem 8.1 in [12]. □

Theorem 3.6 states that the associated primes of cellular binomial ideals have partial characters supported on different lattices. In the light of theorem above, we review the algorithm for computing associated primes of cellular binomial ideals due to Eisenbud and Sturmfels [12].

Algorithm 3.7. Associated primes of a cellular binomial ideal $I = I_\delta^{(b)}$

Input: A cellular binomial ideal I.

Output: The list of associated primes $P_1, ..., P_s$ of I.

1. Compute a Gröbner basis of I with respect to a term order $<$.

2. Set U to be the set of standard monomials of I in the variables $\{x_i \mid i \in \bar{\delta}\}$.

3. For each m in U,

   3.1. Compute the partial character τ that satisfies $I_+(\tau) = (I : m) \cap \Bbbk[x_i \mid i \in \delta]$.

   3.2. For each saturation τ_i of τ, output the prime ideal $I_+(\tau_i) + \langle x_i \mid i \in \bar{\delta}\rangle$.

Using Theorem 3.6, Eisenbud and Sturmfels gave an alternative decomposition of a binomial ideal into unmixed binomial ideals when $\mathrm{char}(\Bbbk) = 0$. Recall that an ideal I is called *unmixed* if its height is equal to the heights of its associated primes. In particular, for an unmixed ideal the associated primes have the same height, which means that there are no embedded primes of I.

Corollary 3.8. *Let $\Bbbk$ be a field of characteristic zero. Let I be a δ-cellular binomial ideal in $\Bbbk[x_1, ..., x_n]$. Then I can be written as a finite intersection of unmixed binomial ideals as follows*

$$I = \bigcap_{m \text{ a monomial in } \{x_i\}_{i \in \bar{\delta}}} \mathrm{Hull}(I + ((I : m) \cap \Bbbk[\delta])),$$

where m is a monomial in $k[\bar{\delta}]$.

Example 3.9. *Let $I = \langle x^2zt^2 - y^2zt^2, z^2, t^3, t^2v^2 - t^2w^2 \rangle \subseteq \Bbbk[x, y, z, t, v, w]$. The ideal I is a $\delta = \{x, y, z, w\}$-cellular binomial ideal. The monomials we need to check in order to obtain an unmixed decomposition of I as above are $\{1, z, zt, t^2, zt^2\}$. We use Singular [16] in our computations*

$$(I : 1) \cap \Bbbk[\delta] = \varnothing, I_1 = \mathrm{Hull}(I + \varnothing) = \langle z^2, t^2 \rangle,$$

$$(I : t^2) \cap \Bbbk[\delta] = \langle v^2 - w^2 \rangle, I_2 = \mathrm{Hull}(I + \langle v^2 - w^2 \rangle) = \langle v^2 - w^2, z^2, t^3, zt^2 \rangle,$$

$$(I : zt^2) \cap \Bbbk[\delta] = \langle x^2 - y^2, v^2 - w^2 \rangle, I_3 = \mathrm{Hull}(I + \langle x^2 - y^2, v^2 - w^2 \rangle)$$

$$= \langle v^2 - w^2, z^2, t^3, x^2 - y^2 \rangle.$$

Then,

$$I = I_1 \cap I_2 \cap I_3.$$

Note that the computations for monomials z and zt give the ideal I_1 again, which is redundant. In the case of $\operatorname{char}(\Bbbk) = 2$, *the computations are exactly same for I. Indeed, Eisenbud and Sturmfels have a conjecture that Corollary 3.8 is true for positive characteristic as well.*

Definition 3.10. A lattice L is *potentially associated* to a cellular binomial ideal $I = I_\delta^{(b)}$ if there exists a witness monomial $m \in k[x_i \mid i \in \bar{\delta}]$ such that $(I : m) \cap \Bbbk[x_i \mid i \in \delta] = I_+(\tau)$ for some character $\tau : L \to \Bbbk^*$.

The lattice ideals $(I : m) \cap k[x_i \mid i \in \delta]$ are partially ordered by inclusion.

Definition 3.11. Let $\operatorname{char}(\Bbbk) = p \geqslant 0$. Let $I = I_\delta^{(b)}$ be a cellular binomial ideal such that $I \cap \Bbbk[x_i \mid i \in \delta] = I_+(\rho)$. A potentially associated lattice L is called *embedded* if it properly contains L_ρ and if $\operatorname{Sat}(L_\rho) \neq \operatorname{Sat}(L)$.

Definition 3.12. Let I be a cellular binomial ideal. We define $M_{\text{emb}}(I)$ to be the monomial ideal generated by all witness monomials of embedded lattices of I.

Note that 1 is not in $M_{\text{emb}}(I)$.

The definition of the $M_{\text{emb}}(I))$ implies the following result.

Lemma 3.13. *Let I be a δ-cellular binomial ideal, and let (ρ, L_ρ) such that $I \cap \Bbbk[x_j \mid j \in \delta] = I_+(\rho)$. A monomial $m \in \Bbbk[x_i \mid i \in \bar{\delta}] - I$ belongs to $M_{\text{emb}}(I)$ if and only if there exists a binomial $x^a - \lambda x^b \in \Bbbk[x_j \mid j \in \delta]$ such that*

1. *$\lambda \neq 0$,*

2. *$a - b \notin \operatorname{Sat}(L_\rho)$, and*

3. *$m(x^a - \lambda x^b) \in I$.*

Moreover, in this case, $mx^a, mx^b \notin I$, and we may assume $\gcd(x^a, x^b) = 1$.

Proof. If $m \in M_{\mathrm{emb}}(I)$, $(I : m) \cap \Bbbk[x_j \mid j \in \delta] = I_+(\tau)$, and $\mathrm{Sat}(L_\tau) \neq \mathrm{Sat}(L_\rho)$. Pick $x^a - \lambda x^b \in I_+(\tau) \subseteq \Bbbk[x_j \mid j \in \delta]$ such that $a - b \notin \mathrm{Sat}(L_\rho)$. We can do this, because if we had $a - b \in \mathrm{Sat}(L_\rho)$ for all $x^a - \lambda x^b \in I_+(\tau)$, then we would have $L_\tau \subseteq \mathrm{Sat}(L_\rho)$, and consequently $\mathrm{Sat}(L_\tau) \subseteq \mathrm{Sat}(L_\rho)$, which together with the fact that L_τ properly contains L_ρ would imply $\mathrm{Sat}(L_\tau) = \mathrm{Sat}(L_\rho)$, a contradiction.

Since $I_+(\tau)$ contains no monomials, we see that $\lambda \neq 0$. Moreover, if $mx^a, mx^b \in I$, then we would have $m \in I$, since I is δ-cellular, and the monomials x^a and x^b only involve the variables indexed by δ. The fact that we may assume that $\gcd(x^a, x^b) = 1$ also follows from the fact that I is δ-cellular, as $\gcd(x^a, x^b)$ can only involve the variables indexed by δ.

For the converse, let $m \notin I$ be a monomial involving only the variables indexed by $\bar{\delta}$, and suppose there is a binomial $x^a - \lambda x^b \in \Bbbk[x_i \mid i \in \delta]$ satisfying the three conditions required above.

Now $x^a - \lambda x^b \in (I : m) \cap \Bbbk[x_i \mid i \in \delta]$. Since $(I : m)$ is δ-cellular, we know that $(I : m) \cap \Bbbk[x_i \mid i \in \delta] = I_+(\tau)$ for some character $\tau : L_\tau \to \Bbbk^*$. Since $(I : m) \supseteq I$, we have $I_+(\tau) \supseteq I_+(\rho)$, and therefore $L_\tau \supseteq L_\rho$. However, $a - b \in L_\tau - \mathrm{Sat}(L_\rho)$, which implies that L_τ is an embedded potentially associated lattice to I and therefore $m \in M_{\mathrm{emb}}(I)$. $\square$

Example 3.14. $I = \langle x^3z - x^3, x^4, x^2yt - x^2y, y^2, t^3 - 1\rangle \subseteq \Bbbk[x, y, z, t]$ *is a cellular binomial ideal such that* $\{z, t\}$ *are the cellular variables and* $\{x, y\}$ *are the nilpotent variables. In this case* $M_{\mathrm{emb}}(I) = \langle x^3\rangle$. *Observe that* x^2y *is not in* $M_{emb}(I)$, *although* $(I : x^2y) \cap k[z, t]$ *contains the binomial* $t - 1$, *since we already have* $t^3 - 1$ *in* I, *and the corresponding lattices have the same saturation.*

Remark 3.15. A slightly different definition of $M_{\mathrm{emb}}(I)$ first appears in [21]. Kahle introduces this notion to improve the implementations in the package *Binomials*.

In particular, he designed $M_{\text{emb}}(I)$ to reduce the number of colon operations in that algorithm, see the algorithm in [21]. In Proposition 6, he gave the minimal primary component of a cellular binomial ideal which has exactly one minimal prime : $I + M_{\text{emb}}(I)$ in $\text{char}(\mathbb{k}) = 0$. We modified the definition of $M_{\text{emb}}(I)$ by adding the extra condition that $\text{Sat}(L_\rho) \neq \text{Sat}(L)$. Moreover, we gave a description in [25] for minimal primary components of general cellular binomial ideals in any characteristic, which is reproduced Theorem 3.22.

We now give an algorithm to find $M_{\text{emb}}(I)$ for I a cellular binomial ideal. This will be useful to find the primary components of cellular binomial ideals.

Algorithm 3.16. The monomial ideal $M_{\text{emb}}(I)$.

Input : A δ-cellular binomial ideal I.

Output : The monomial ideal $M_{\text{emb}}(I) \subseteq \mathbb{k}[\bar{\delta}]$ where $x_i, i \in \bar{\delta}$, are nilpotent variables.

1. Compute the lattice ideal $I \cap \mathbb{k}[\delta] = I_+(\rho)$.
2. Initialize a to-do list with all monomials in a $\mathbb{k}$-basis of $\mathbb{k}[\bar{\delta}]/(I \cap \mathbb{k}[\bar{\delta}])$.
3. Iterate the following until the to-do list is empty.
   - 3.1 Choose a monomial m in a $\mathbb{k}$-basis of $\mathbb{k}[\bar{\delta}]/(I \cap \mathbb{k}[\bar{\delta}])$. Compute the lattice ideal $(I : m) \cap \mathbb{k}[\delta] = I_\tau$ and check if $\text{Sat}(\text{L}_\tau) \neq \text{Sat}(\text{L}_\rho)$.
     - 3.1.1 If yes, then add m to $M_{\text{emb}}(I)$ and remove all monomials which can be divided by m from the to-do list.
     - 3.1.2 If no, then remove m and every monomial that can divide m from the to-do list and return to step 3.1.

Proof of Correctness. Computing the intersection of I with nonzerodivisor variables can be computed by using Gröbner bases and elimination. The intersection ideal is a lattice ideal by Proposition 3.3. If a monomial $m \in \mathbb{k}[\bar{\delta}]/(I \cap \mathbb{k}[\bar{\delta}])$ satisfy $\mathrm{Sat}(\mathrm{L}_\tau) \neq \mathrm{Sat}(\mathrm{L}_\rho)$, we do not need to check the monomials that can be divided by m. Since m can generate them, so they also belong to $M_{\mathrm{emb}}(I)$. Let $\tilde{n}$ be a monomial in to-do list and let $\tilde{n}$ divide a monomial $\tilde{m}$ in the list where $(I : \tilde{n}) \cap \mathbb{k}[x_i \mid i \in \delta] = I_+(\tau) \subseteq (I : \tilde{m}) \cap \mathbb{k}[x_i \mid i \in \delta] = I_+(\tilde{\tau})$ with $\mathrm{Sat}(\mathrm{L}_{\tilde{\tau}}) = \mathrm{Sat}(\mathrm{L}_\rho)$. This implies that $\mathrm{Sat}(\mathrm{L}_\tau) = \mathrm{Sat}(\mathrm{L}_\rho)$, so $\tilde{n} \notin M_{\mathrm{emb}}(I)$. There are finitely many steps since there are finitely many monomials $m \in \mathbb{k}[\bar{\delta}]/(I \cap \mathbb{k}[\bar{\delta}])$ to check, as I is cellular and I contains pure powers of variables x_i for $i \in \bar{\delta}$. □

Proposition 3.17. *Let I be a δ-cellular binomial ideal with $M_{\mathrm{emb}}(I) = \varnothing$. Then I does not have any embedded associated primes.*

Proof. By Theorem 3.6, the embedded primes of I are of the form $I_+(\tilde{\tau}) + \langle x_i | i \in \bar{\delta} \rangle$, where $\tilde{\tau}$ is the saturation of the partial character τ given by

$$(I : m) \cap \mathbb{k}[x_i \mid i \in \delta] = I_+(\tau),$$

for some monomial $m \in M_{\mathrm{emb}}(I)$.

Such a monomial m must belong to $M_{\mathrm{emb}}(I)$. Since $M_{\mathrm{emb}}(I) = \varnothing$, we see that I has no embedded primes. □

Remark 3.18. Note that if I has one minimal primary component and if $M_{\mathrm{emb}}(I) = \varnothing$ then I is a primary ideal.

Theorem 3.19. *Let I be an δ-cellular binomial ideal with $I \cap \mathbb{k}[\delta] = I_+(\rho)$. Then the ideal $I + M_{\mathrm{emb}}(I)$ is δ-cellular binomial ideal whose primary components are all* *minimal and*

$$(I + M_{\mathrm{emb}}(I)) \cap \Bbbk[\delta] = I_+(\rho).$$

Proof. For all x_i $i \in \bar{\delta}$, there exists a $d_i \in \mathbb{Z}_{>0}$ such that $x_i^{d_i} \in I \subseteq I + M_{\mathrm{emb}}(I)$, which implies that the variables indexed by $\bar{\delta}$ are nilpotent modulo $I + M_{\mathrm{emb}}(I)$.

We now show that the variables indexed by δ are nonzerodivisors modulo $I + M_{\mathrm{emb}}(I)$. It is sufficient to show $((I + M_{\mathrm{emb}}(I)) : x_j) = I + M_{\mathrm{emb}}(I)$ for all $j \in \delta$. The ideal $I + M_{\mathrm{emb}}(I)$ is binomial and so is $((I + M_{\mathrm{emb}}(I)) : x_j)$. The equality $((I + M_{\mathrm{emb}}(I)) : x_j) = I + M_{\mathrm{emb}}(I)$ follows if we show that the generators of $((I + M_{\mathrm{emb}}(I)) : x_j)$ are contained in $I + M_{\mathrm{emb}}(I)$.

Let x^μ be a monomial generator of $((I + M_{\mathrm{emb}}(I)) : x_j)$ where $j \in \delta$ then $x^\mu x_j \in I + M_{\mathrm{emb}}(I)$. If $x^\mu x_j \in I$, then $x^\mu \in I$ as I is δ-cellular. Assuming $x^\mu x_j \notin I$, we claim that there exists a monomial $x^v \in M_{\mathrm{emb}}(I)$ such that $x^\mu x_j - \lambda x^v \in I$.

The claim is true if $x^\mu x_j \in M_{\mathrm{emb}}(I)$, so suppose not. We know that $x^\mu x_j \in I + M_{\mathrm{emb}}(I))$, so we may write

$$x^\mu x_j = F_1 g_1 + \cdots F_k g_k + H_1 t_1 + \cdots H_r t_r + H_{r+1} t_{r+1} + \cdots + H_N t_N, \tag{3.1}$$

where $g_1, \ldots, g_k$ are binomials in I, neither of whose monomials belong to I, $t_1, \ldots, t_r$ are monomials in I, and $t_{r+1}, \ldots, t_N$ are monomials in $M_{\mathrm{emb}}(I)$. We visualize (3.1) as a graph G whose vertices are the exponent vectors of the monomials in the right hand side of (3.1); for instance, if F_1 contains a monomial x^ν with nonzero coefficient, and $g_1 = \lambda_1 x^a - \lambda_2 x^b$, then $\nu + a$ and $\nu + b$ are vertices of G. The edges of G come from the binomials and monomials $g_1, \ldots, g_k$, $t_1, \ldots, t_N$. For instance, $(\nu + a, \nu + b)$ is an edge in G arising from the binomial g_1. The monomials give rise to loops in G: if $t_i = x^v$ and $x^{\nu'}$ is a monomial with nonzero coefficient in H_i, $(v + \nu', v + \nu')$

is a loop in G. Note that G may have multiple edges and multiple loops. Each vertex in G receives a label as follows. The initial label of the vertex u is zero. For each edge (u, v) where $v \neq u$ arises from a binomial $\lambda_1 x^u + \lambda_2 x^v$, add λ_1 to the label of u. For every loop (u, u), which arises from a term γx^u, add γ to the label of x^u. Note that this process labels each vertex by its coefficient in (3.1), so that only the vertex $\mu + e_j$ receives a nonzero label (which equals 1); here e_j denotes the vector in $\mathbb{Z}^n$ whose only nonzero coordinate is the j-th one, which equals 1. A connected component Γ of G corresponds to an element of $I + M_{\text{emb}}(I)$, namely $f(\Gamma) = \sum_{w \text{ vertex in } \Gamma} \text{label}(w) x^w$. If Γ is the connected of $\mu + e_j$, then $f(\Gamma) = x^\mu x_j$. Since $x^\mu x_j \notin I$, the connected component Γ of $\mu + e_j$ in G must contain a vertex v coming from $M_{\text{emb}}(I)$. As $x^\mu x_j \notin M_{\text{emb}}(I)$, $\mu + e_j$ and v must be connected by a sequence of edges corresponding to binomials in I. Now the claim follows by induction on the length of this path.

We have shown, since $M_{\text{emb}}(I)$ is generated by monomials in the variables indexed by $\bar{\delta}$, there exists $x^w \in M_{\text{emb}}(I) \cap k[x_i \mid i \in \bar{\delta}]$ that divides x^v. We write $x^v = x^u x^w$ for some monomial x^u.

Since $x^w \in M_{\text{emb}}(I)$, $x^w(x^a - \tilde{\lambda} x^b) \in I$ for some $a - b \in \delta$ which is not in $\text{Sat}(L_\rho)$. Note that $(x^\mu x_j - \lambda x^v)(x^a - \tilde{\lambda} x^b) \in I$. The binomial $\lambda x^v(x^a - \tilde{\lambda} x^b)$ belongs to I, so does $x^\mu x_j(x^a - \tilde{\lambda} x^b)$. This implies that $x^\mu(x^a - \tilde{\lambda} x^b) \in I$ and $x^\mu \in M_{\text{emb}}(I)$.

Let $x^\alpha - \lambda x^\beta$ be a generator of $((I + M_{\text{emb}}(I) : x_i)$, we want to show that it is also in $I + M_{\text{emb}}(I)$. By definition $x_i(x^\alpha - \lambda x^\beta) \in I + M_{\text{emb}}(I)$. Assume $x^\alpha x_i \notin I + M_{\text{emb}}(I)$, this implies $x^\beta x_i \notin I + M_{\text{emb}}(I)$. By Proposition 1.19, $x_i(x^\alpha - \lambda x^\beta) \in I$, so also $x^\alpha - \lambda x^\beta \in I$. If $x_i x^\alpha, x_i x^\beta \in I + M_{\text{emb}}(I)$, then we use the same argument as above to conclude that $x^\alpha, x^\beta \in M_{\text{emb}}(I)$.

Assume that $x^\mu \in M_{\text{emb}}(I + M_{\text{emb}}(I))$ is not a monomial in $I + M_{\text{emb}}(I)$. Hence, there exists a binomial $x^\alpha - \lambda x^\beta \in \mathbb{k}[\delta]$ where $\alpha - \beta \notin \text{Sat}(\mathrm{L}_\rho)$ and $x^\mu(x^\alpha - \lambda x^\beta) \in$

$I + M_{\mathrm{emb}}(I)$. We use the same argument as before; if $x^\mu x^\alpha$ is in $I + M_{\mathrm{emb}}(I)$, then x^μ is in $I + M_{\mathrm{emb}}(I)$, which is a contradiction. If $x^\mu x^\alpha$ and $x^\mu x^\beta$ are not in $I + M_{\mathrm{emb}}(I)$, then by Proposition 1.19, $x^\mu(x^\alpha - \lambda x^\beta) \in I$, so $x^\mu \in M_{\mathrm{emb}}(I)$, which is a contradiction. □

Before stating the main theorem in this section, we want to give an important feature of cellular binomial ideals.

Proposition 3.20. *Let I be a δ-cellular binomial ideal. Let $x^a - \lambda x^b \in I$ such that $a_i \geqslant b_i$ for all $i \in \bar{\delta}$. If $x^b \notin I$ (which also implies $x^a \notin I$) then $a_i = b_i$ for all $i \in \bar{\delta}$.*

Proof. Let $i \in \bar{\delta}$ such that $a_i > b_i$. Since I is a cellular binomial ideal and x_i is nilpotent, there exists a pure power monomial $x_i^{c_i}$ in I (here $c_i \in \mathbb{Z}_{>0}$). Since $x^a \notin I$, we see that $c_i > a_i$.

The binomial $x_i^{c_i - a_i}(x^a - \lambda x^b)$ belongs to I, and so does the monomial $x_i^{c_i - a_i} x^a$. This implies that $x_i^{c_i - a_i + b_i} \prod_{\ell \in \bar{\delta}, \ell \neq i} x_\ell^{b_\ell}$ also belongs to I (saturating out the cellular variables). Since $a_\ell \geqslant b_\ell$ for $\ell \in \bar{\delta}$, we see that $x_i^{c_i - a_i + b_i} \prod_{\ell \in \bar{\delta}, \ell \neq i} x_\ell^{a_\ell} \in I$. Using the fact that $x^a \notin I$, we conclude that $c_i - a_i + b_i > a_i$, and so $c_i - 2a_i + b_i > 0$.

We repeat the previous argument, using the product $x_i^{c_i - 2a_i + b_i}(x^a - \lambda x^b) \in I$, to see that $x_i^{c_i - 2a_i + 2b_i} \prod_{\ell \in \bar{\delta}, \ell \neq i} x_\ell^{b_\ell} \in I$, which implies that $x_i^{c_i - 2a_i + 2b_i} \prod_{\ell \in \bar{\delta}, \ell \neq i} x_\ell^{b_\ell} \in I$, and as before, $c_i - 2a_i + 2b_i > a_i$.

Continuing in this manner, we conclude that $c_i - k(a_i - b_i) > a_i$ for all $k \in \mathbb{Z}_{>0}$, a contradiction, since $a_i > b_i$. □

Recall our notation. Let $\mathrm{char}(\Bbbk) = p \geqslant 0$ and let $I = I_\delta^{(b)} \in \Bbbk[x_1, ..., x_n]$ be a cellular binomial ideal, where $I \cap \Bbbk[x_i \mid i \in \delta] = I_+(\rho)$. We define $\mathrm{Sat_p}(L_\rho)$ and $\mathrm{Sat'}_p(L_\rho)$ to be the largest sublattices of $\mathrm{Sat}(L_\rho)$ containing L_ρ such that $|\mathrm{Sat_p}(L_\rho)/L_\rho| = p^k$ for some $k \in \mathbb{Z}$ and $|\mathrm{Sat'}_p(L_\rho)/L_\rho| = g$ where $(p, g) = 1$. There are g distinct partial characters $\tilde{\rho}_i$ that extends (ρ, L_ρ) to $\mathrm{Sat}'_p(L_\rho)$. For each $\tilde{\rho}_i$, there exists a unique partial character ρ_i that extends $\tilde{\rho}_i$ to $\mathrm{Sat}(L_\rho)$ by Theorem 1.30. When $\mathrm{char}(\Bbbk) = 0$, $I_+(\rho_i) = I_+(\tilde{\rho}_i)$ for all i.

Lemma 3.21. *Let I be a δ-cellular binomial ideal in $\Bbbk[x]$ where $\Bbbk$ is algebraically closed with characteristic $p > 0$. Let $P = I_+(\rho_i) + \langle x_i \mid i \in \bar{\delta}\rangle$ be an associated prime of I. Let Γ be a congruence class determined by $(I + (I_+(\tilde{\rho}_i))[\mathbb{Z}\delta]$. If Γ has two distinct elements u, v such that $v - u \in \mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}}$ but $v - u \notin L_{\rho_i}$, then for all $u \in \Gamma$, t^u is in the P-primary component of I.*

Proof. The proof is same as the one in Lemma 2.4.

□

Theorem 3.22. *Use the notation introduced above. Let I be a cellular binomial ideal with $I \cap \Bbbk[x_i \mid i \in \delta] = I_+(\rho)$. The minimal associated primes of I are*

$$P_i = I_+(\rho_i) + \langle x_i \mid i \in \bar{\delta}\rangle \text{ for } i = 1, ..., g.$$

Let $J_i = ((I + I_+(\tilde{\rho}_i)) : (\prod_{\ell\in\delta} x_\ell)^\infty)$, the P_i-primary component of I is

$$Q_i = J_i + M_{\mathrm{emb}}(J_i).$$

Proof. By Theorem 3.6, the minimal associated primes of I are of the form $I_+(\rho_i) + \langle x_i \mid i \in \bar{\delta}\rangle$.

Fix i, we claim that Q_i is P_i-primary. First, we show that Q_i has a unique minimal associated prime. If we look at the radical of the ideal, we see $\sqrt{(J_i) + M_{\mathrm{emb}}(J_i)} \subseteq P_i$. Note that $\sqrt{I + I_+(\tilde{\rho}_i)} \subseteq \sqrt{J_i + M_{\mathrm{emb}}(J_i)}$, which implies that $P_i = \sqrt{\sqrt{I} + \sqrt{I_+(\tilde{\rho}_i)}}$ $\subseteq \sqrt{Q_i} \subseteq P_i$. Hence $\sqrt{Q_i} = P_i$. Thus Q_i has a unique minimal associated prime. By

Theorem 3.19, Q_i does not have any embedded associated primes. Consequently P_i is the only associated prime of Q_i which implies that Q_i is P_i-primary.

We show that Q_i is the P_i-primary component of I. It is enough to show that $M_{\text{emb}}(J_i) \subseteq \ker\alpha$, where $\alpha : \Bbbk[x] \to (\Bbbk[x]/I)_{P_i}$, since J_i is already in $\ker\alpha$. The ideal $I+I_+(\tilde{\rho}_i)$ must be in the P_i-primary component of I, so is $((I+I_+(\tilde{\rho}_i)) : (\prod_{\ell\in\delta} x_\ell)^\infty) = J_i$ since we know that $\ker\alpha$ is the intersection of P_i-primary ideals that contain I by Corollary 10.21 in [2]. Let $x^\mu \in M_{\text{emb}}(J_i) \subseteq \Bbbk[\bar{\delta}]$. Let τ be a partial character such that $(J_i : x^\mu) \cap \Bbbk[\delta] = I_+(\tau)$ and $\text{Sat}(L_\tau) \neq \text{Sat}(L_\rho)$. Hence there exists an element $b \in L_\tau$ such that $db \notin L_\rho$ for all $d \in \mathbb{Z}_{>0}$. This implies that $x^{b_+} - \tau(b)x^{b_-} \in I_+(\tau)$ and $x^{b_+} - \lambda x^{b_-} \notin I_+(\rho_i)$ for all $\lambda \in \Bbbk$. Thus, $x^\mu(x^{b_+} - \tau(b)x^{b_-}) \in J_i$. For some $x^\nu \in \Bbbk[\delta]$, $x^\mu x^\nu(x^{b_+} - \tau(b)x^{b_-}) \in I + I_+(\tilde{\rho}_i)$. When $\text{char}(\Bbbk) = p > 0$, the congruence class Γ containing $\mu+\nu+b_+ \sim \mu+\nu+b_-$ determined by $I+I_+(\tilde{\rho}_i)[\mathbb{Z}\delta]$ has a pair of elements mentioned in Lemma 3.21, thus x^μ belongs to $\ker\alpha$.

When $\text{char}(\Bbbk) = 0$, let Γ' be the congruence class containing $\mu+\nu+b_+ \sim \mu+\nu+b_-$ determined by $I + (I_+(\rho_i))[\mathbb{Z}\delta]$ and note that $b \notin L_{\rho_i}$. By Lemma 2.3, x^μ belongs to $\ker\alpha$. □

Remark 3.23. When $\text{char}(\Bbbk) = 0$, there is another way to show that Q_i is P_i-primary component of I based on the characterization in Theorem 2.2. Let U be the set defined in Theorem 2.2. The only thing we need to show is that the monomials in J_i and $M_{\text{emb}}(J_i)$ are the same as the monomials in $M = \langle x^u \mid u \in U\rangle$. It is obvious that monomials in J_i are in M, and we showed that $M_{\text{emb}}(J_i) \subseteq M$ in the proof of Theorem 3.22. Now we need to show the other containment. Let $x^\mu \in M$, we want to show that $x^\mu \in J_i + M_{\text{emb}}(J_i)$. Suppose $x^\mu \notin J_i$. Since $x^\mu \in M$, μ is in a congruence class Γ induced by $(I + (I_+(\rho_i)))[\mathbb{Z}\delta]$ which has an infinite image in $\mathbb{N}^\delta/L_{\rho_i} \times \mathbb{N}^{\bar{\delta}}$. By Lemma 2.9 in [8], there exists $v, w \in \Gamma$ which satisfy that $v_i \geqslant w_i$ for all i and $x^v x^a - x^w x^b \in (I + (I_+(\rho_i)))[\mathbb{Z}\delta]$ where $a, b \in \delta, v, w \in \bar{\delta}$. For some $x^\nu \in \Bbbk[\delta]$, $x^v x^{a+\nu} - x^w x^{b+\nu} \in I + I_+(\rho_i) \subseteq J_i$.

J_i is a δ-cellular binomial ideal, so by Proposition 3.20, $v_i = w_i$ for all i. Thus x^v belongs to $M_{\mathrm{emb}}(J_i)$. Since μ and v are in the same congruence class, there is a binomial $x^\mu x^a - x^v x^b \in I + (I_+(\rho_i))[\mathbb{Z}\delta]$, which implies that $x^\mu x^a x^\nu - x^v x^b x^\nu \in I + (I_+(\rho_i))$ for some $x^\nu \in \Bbbk[\delta]$. Since $x^v \in M_{\mathrm{emb}}(J_i)$, it follows that $x^\mu x^a x^\nu \in J_i + M_{\mathrm{emb}}(J_i)$.

Example 3.24. *(Example 3.14 continued.) If* $\mathrm{char}(\Bbbk) \neq 3$*, there are three associated primary components of the lattice ideal* $I \cap \Bbbk[z, t] = \langle t^3 - 1\rangle$*. They are* $\langle t-1\rangle$, $\langle t-\omega\rangle$ *and* $\langle t - \omega^2\rangle$ *where* ω *is a primitive cubic root of unity in* $\Bbbk$*. Hence the minimal primary components of* I *are*

$$J_1 = ((I + \langle t - 1\rangle) : (zt)^\infty),\ M_{\mathrm{emb}}(J_1) = \varnothing,\ Q_1 = J_1 + M_{\mathrm{emb}}(J_1) = \langle t - 1, x^3, y^2\rangle,$$

$$J_2 = ((I+\langle t-\omega\rangle) : (zt)^\infty),\ M_{\mathrm{emb}}(J_2) = \varnothing,\ Q_2 = J_2+M_{\mathrm{emb}}(J_2) = \langle t-\omega, x^3, y^2, x^2y\rangle,$$

$$J_3 = ((I+\langle t-\omega^2\rangle) : (zt)^\infty),\ M_{\mathrm{emb}}(J_3) = \varnothing,\ Q_3 = J_3+M_{\mathrm{emb}}(J_3) = \langle t-\omega^2, x^3, y^2, x^2y\rangle.$$

The monomial x^2y *belongs to* Q_2 *since* $(x^2yt - x^2y\omega)$ *and* $(x^2yt - x^2y)$ *are in* J_2*, so* $x^2y\omega - x^2y = x^2y(\omega - 1)$ *also belongs to* J_2*. Note that* $(\omega - 1)$ *has an inverse in* $\Bbbk$*.*

Example 3.25. *The saturation operation cannot be omitted from the description of the minimal primary components of cellular binomial ideals. Let* $I = \langle x^2 - y^2, xu - yv, u^3, v^3, u - v\rangle \subseteq \Bbbk[x, y, u, v]$*.* I *is a cellular binomial ideal with nilpotent variables* $\{u, v\}$*. Note that* $I \cap \Bbbk[x, y] = \langle x^2 - y^2\rangle = \langle x - y\rangle \cap \langle x + y\rangle$ *if* $\mathrm{char}(\Bbbk) \neq 2$*. When* $\mathrm{char}(\Bbbk) = 2$, $\langle x^2 - y^2\rangle$ *is itself primary. Note that* $M_{emb}(I) = \varnothing$*. Thus, the minimal* *primary components of I in characteristic $p \neq 2$ are*

$$J_1 = ((I + \langle x - y\rangle) : (xy)^\infty),\ M_{\mathrm{emb}}(J_1) = \varnothing,\ Q_1 = J_1 + M_{\mathrm{emb}}(J_1) = \langle x - y, u - v, v^3\rangle,$$

$$J_2 = ((I + \langle x + y\rangle) : (xy)^\infty),\ M_{\mathrm{emb}}(J_2) = \varnothing,\ Q_2 = J_2 + M_{\mathrm{emb}}(J_2) = \langle x + y, u, v\rangle.$$

$I + \langle x + y\rangle$ contains the monomial yv without containing v, which shows that we need saturation operation in this description.

When $\mathrm{char}(\Bbbk) = 2$, *$J = (I + \langle x^2 - y^2\rangle) : (xy)^\infty)$ and $J + M_{\mathrm{emb}}(J) = I$ is already primary.*

The following is another useful characterization for the minimal primary components of a cellular ideal.

Theorem 3.26. *Let I be a δ-cellular binomial ideal, and suppose $I \cap \Bbbk[x_i \mid i \in \delta] = I_+(\rho)$. Let $I_+(\tilde{\rho}_i)$ be a primary component of $I_+(\rho)$. Let $J_i = \Big((I + I_+(\tilde{\rho}_i)) : \big(\prod_{i\in\delta} x_i\big)^\infty\Big)$. Then*

$$J_i + M_{\mathrm{emb}}(J_i) = \Big(\big(I + I_+(\tilde{\rho}_i) + M_{\mathrm{emb}}(I)\big) : \big(\prod_{i\in\delta} x_i\big)^\infty\Big).$$

Proof. We first claim that $M_{\mathrm{emb}}(I) \subseteq M_{\mathrm{emb}}(J_i)$. To see this, note that J_i is δ-cellular and $J_i \cap \Bbbk[x_i \mid i \in \delta] = I_+(\tilde{\rho}_i)$, with $\mathrm{Sat}(L_{\tilde{\rho}_i}) = L_{\tilde{\rho}_i} = \mathrm{Sat}(L_\rho)$. If m is a monomial in $M_{\mathrm{emb}}(I)$, the binomial produced in Lemma 3.13 can also be used to show that $m \in M_{\mathrm{emb}}(J_i)$, since $I \subseteq J_i$.

It is now enough to show that $M_{\mathrm{emb}}(J_i) \subseteq J_i + M_{\mathrm{emb}}(I)$. Since $M_{\mathrm{emb}}(J_i)$ is generated by monomials in the variables indexed by $\bar{\delta}$, let $\mu \in \mathbb{Z}^n$ such that $\mu_\delta = 0$ and $x^\mu \in M_{\mathrm{emb}}(J_i)$. Note that $\mu_\delta = 0$ denote that $\mu_i = 0$ for all $i \in \delta$.

If $x^\mu \in J_i$, then $x^\mu \in J_i + M_{\mathrm{emb}}(I)$, so we may assume that $x^\mu \notin J_i$, and pick a binomial $x^a - \lambda x^b \in \Bbbk[x_i \mid i \in \delta]$ as in Lemma 3.13. In particular, $x^\mu(x^a - \lambda x^b) \in J_i$, and therefore, we can find a monomial x^ν such that $\nu_{\bar{\delta}} = 0$ and $x^\nu x^\mu(x^a - \lambda x^b) \in I + I_+(\tilde{\rho}_i)$.

As $x^\mu \notin J_i$, the monomials $x^\nu x^\mu x^a, x^\nu x^\mu x^b$ do not belong to $I + I_+(\tilde{\rho}_i)$.

Since $x^\nu x^\mu(x^a - \lambda x^b) \in I + I_+(\tilde{\rho}_i)$, we can write

$$x^\nu x^\mu(x^a - \lambda x^b) = F_1 f_1 + \cdots + F_r f_r + F_{r+1} f_{r+1} + F_s f_s + H_1 t_1 + \cdots + H_k t_k, \quad (3.2)$$

where $F_1, \ldots, F_s, H_1, \ldots, H_k, f_1, \ldots, f_s, t_1, \ldots, t_k$ are polynomials,

$$\langle f_1, \ldots, f_r \rangle = I_+(\tilde{\rho}_i), \quad \langle f_{r+1}, \ldots, f_s, t_1, \ldots, t_k \rangle = I,$$

the polynomials $f_1, \ldots, f_r$ are binomials arising from $L_{\tilde{\rho}_i}$ (in particular, they are not monomials, and involve only the variables indexed by δ), the polynomials $f_{r+1}, \ldots, f_s$ are binomials that are not monomials, and $t_1, \ldots, t_k$ are monomials.

As we did in the proof of Theorem 3.19, we visualize the expression (3.2) as a graph G with labeled vertices. Since $x^\nu x^\mu x^a$ and $x^\nu x^\mu x^b$ do not belong to $I + I_+(\tilde{\rho}_i)$, the vertices $\nu + \mu + a$ and $\nu + \mu + b$ belong to the same connected component of G and therefore there is a path in G connecting $\nu + \mu + a$ to $\nu + \mu + b$, that is, there exists a sequence of edges $\varepsilon_1 = (\alpha_1, \beta_1), \ldots, \varepsilon_\ell = (\alpha_\ell, \beta_\ell)$ arising from the binomials $f_1, \ldots, f_s$ such that $\alpha_1 = \nu + \mu + a$, $\beta_i = \alpha_{i+1}$ for $i = 1, \ldots, \ell - 1$ and $\beta_\ell = \nu + \mu + b$.

Each edge ε_j arises from a binomial $\lambda_{\varepsilon_j,1} x^{\alpha_j} - \lambda_{\varepsilon_j,2} x^{\beta_j}$ which is a multiple of one of the binomials f_i, and therefore belongs to either I or $I_+(\tilde{\rho}_i)$.

If we have an edge ε_j such that $(\alpha_j)_\ell = (\beta_j)_\ell$ for all $\ell \in \bar{\delta}$ and $\alpha_j - \beta_j \notin \mathrm{Sat}(L_\rho)$, then the associated polynomial must belong to I. By Lemma 3.13, either $x^{(\alpha_j)_{\bar{\delta}}}$ is an element of I, or it is an element of $M_{\mathrm{emb}}(I)$. In both cases, we can do induction on the length j of the path that connects $\nu+\mu+a$ to α_j to conclude that $x^{\nu+\mu+a}$ belongs to $I+I_+(\tilde{\rho}_i)$ in the first case (which is a contradiction), and to $I+I_+(\tilde{\rho}_i)+M_{\mathrm{emb}}(I)$ in the second case. Then $x^{\nu+\mu+a} \in I+I_+(\tilde{\rho}_i)+M_{\mathrm{emb}}(I)$ implies that $x^\mu \in \Big(\big(I + I_+(\tilde{\rho}_i) + M_{\mathrm{emb}}(I)\big) : \big(\prod_{i\in\delta} x_i\big)^\infty\Big)$.

Thus we may assume that for every edge ε_j such that $(\alpha_j)_{\bar{\delta}} = (\beta_j)_{\bar{\delta}}$, we have $\alpha_j - \beta_j \in \mathrm{Sat}(L_\rho)$.

Let $\varepsilon_{i_1}, \ldots, \varepsilon_{i_q}$ be the subsequence of $\varepsilon_1, \ldots, \varepsilon_\ell$ consisting of edges ε_j such that $(\alpha_j)_{\bar{\delta}} \neq (\beta_j)_{\bar{\delta}}$, and observe that each of these edges is therefore associated to a polynomial $\lambda_{\varepsilon_j,1}x^{\alpha_j} - \lambda_{\varepsilon_j,2}x^{\beta_j}$ that lies in I.

Note that $(\alpha_{i_1})_{\bar{\delta}} = \mu = (\beta_{i_q})_{\bar{\delta}}$.

Consider $\lambda_{\varepsilon_{i_1},1}x^{\alpha_{i_1}} - \lambda_{\varepsilon_{i_1},2}x^{\beta_{i_1}}$ and $\lambda_{\varepsilon_{i_2},1}x^{\alpha_{i_2}} - \lambda_{\varepsilon_{i_2},2}x^{\beta_{i_2}}$. Since the edges ε_j for $i_1 < j < i_2$ are parallel to elements of $\mathrm{Sat}(L_\rho)$, we see that $\beta_{i_1} - \alpha_{i_2} \in \mathrm{Sat}(L_\rho)$, and in particular $(\beta_{i_1})_{\bar{\delta}} = (\alpha_{i_2})_{\bar{\delta}}$. Let $x^v = \mathrm{lcm}(x^{\beta_{i_1}}, \alpha_{i_2})$, and consider the following element of I

$$\frac{\lambda_{\varepsilon_{i_1},2}x^v}{x^{\beta_{i_1}}}\big(\lambda_{\varepsilon_{i_1},1}x^{\alpha_{i_1}} - \lambda_{\varepsilon_{i_1},2}x^{\beta_{i_1}}\big) - \frac{\lambda_{\varepsilon_{i_2},1}x^v}{x^{\alpha_{i_2}}}\big(\lambda_{\varepsilon_{i_2},1}x^{\alpha_{i_2}} - \lambda_{\varepsilon_{i_2},2}x^{\beta_{i_2}}\big)$$
$$= \frac{\lambda_{\varepsilon_{i_1},2}\lambda_{\varepsilon_{i_1},1}x^v}{x^{\beta_{i_1}}}x^{\alpha_{i_1}} - \frac{\lambda_{\varepsilon_{i_2},1}\lambda_{\varepsilon_{i_2},2}x^v}{x^{\alpha_{i_2}}}x^{\beta_{i_2}}.$$

We write the preceding binomial as $\Lambda_{1,1}x^{\nu_{1,1}}x^{\alpha_{i_1}} - \Lambda_{1,2}x^{\nu_{1,2}}x^{\beta_{i_2}} \in I$, where $\Lambda_{1,1}, \Lambda_{1,2} \in \Bbbk^*$, $x^{\nu_{1,1}}$ and $x^{\nu_{1,2}}$ are relatively prime monomials involving only the variables indexed by δ, and $\nu_{1,1} - \nu_{1,2} \in \mathrm{Sat}(L_\rho)$.

Repeating this procedure, we find nonzero $\Lambda_{q-1,1}, \Lambda_{q-1,2} \in \Bbbk^*$ and relatively prime monomials $x^{\nu_{q-1,1}}$ and $x^{\nu_{q-1,2}}$ involving only the variables indexed by δ such that $\nu_{q-1,1} - \nu_{q-1,2} \in \mathrm{Sat}(L_\rho)$, and $\Lambda_{q1,1}x^{\nu_{q-1,1}}x^{\alpha_{i_1}} - \Lambda_{q-1,2}x^{\nu_{q-1,2}}x^{\beta_{i_q}} \in I$. We recall that $(\alpha_{i_1})_{\bar{\delta}} = \mu = (\beta_{i_\ell})_{\bar{\delta}}$; since I is δ-cellular, we may assume that $\gcd(x^{\nu_{q-1,1}}x^{\alpha_{i_1}}, x^{\nu_{q-1,2}}x^{\beta_{i_q}}) = x^\mu$. Moreover, $\alpha_{i_1} - \beta_{i_q}$ is congruent to $a-b$ modulo $\mathrm{Sat}(L_\rho)$, since $a+\nu+\mu-\alpha_{i_1} \in$

$\mathrm{Sat}(L_\rho)$, $b + \nu + \mu - \beta_{i_q} \in \mathrm{Sat}(L_\rho)$ and $\nu_{q-1,1} - \nu_{q-1,2} \in \mathrm{Sat}(L_\rho)$. We conclude that $\alpha_{i_1} - \beta_{i_q} \notin \mathrm{Sat}(L_\rho)$, and therefore $\Lambda_{q1,1} x^{\nu_{q-1,1}} x^{\alpha_{i_1}} - \Lambda_{q-1,2} x^{\nu_{q-1,2}} x^{\beta_{i_q}} \in I$ implies that $x^\mu \in M_{\mathrm{emb}}(I)$, as we wanted.

□

Proposition 3.27. *Let* $\mathrm{Hull}(I)$ *denote the intersection of minimal primary components of* I*. If* I *is a* δ*-cellular binomial ideal then*

$$\mathrm{Hull}(I) = I + M_{\mathrm{emb}}(I).$$

Proof. Recall that $I + M_{\mathrm{emb}}(I)$ is δ-cellular, and $(I + M_{\mathrm{emb}}(I)) \cap \Bbbk[x_j \mid j \in \delta] = I_+(\rho) = I \cap \Bbbk[x_j \mid j \in \delta]$.

We claim that $M_{\mathrm{emb}}(I + M_{\mathrm{emb}}(I)) = \varnothing$. By contradiction, let $x^\mu \in M_{\mathrm{emb}}(I + M_{\mathrm{emb}}(I))$. By definition, $x^\mu \notin I + M_{\mathrm{emb}}(I)$, and we may assume $\mu_\delta = 0$. Then there exists a binomial $x^a - \lambda x^b \in \Bbbk[x_j \mid j \in \delta]$ such that $\lambda \neq 0$, $a - b \notin \mathrm{Sat}(L_\rho)$ and $x^\mu(x^a - \lambda x^b) \in I + M_{\mathrm{emb}}(I)$. If $x^\mu x^a \in I + M_{\mathrm{emb}}(I)$, then $x^\mu \in I + M_{\mathrm{emb}}(I)$, since $I + M_{\mathrm{emb}}(I)$ is δ-cellular, giving a contradiction. Similarly, we see $x^\mu x^b \notin I + M_{\mathrm{emb}}(I)$.

As before, we write an expression for $x^\mu(x^a - \lambda x^b) \in I + M_{\mathrm{emb}}(I)$ as a combination of binomials and monomials in I, and monomials in $M_{\mathrm{emb}}(I)$, and think of this as a labeled graph Γ. Since $x^\mu x^a, x^\mu x^b \notin I + M_{\mathrm{emb}}(I)$, $\mu + a$ and $\mu + b$ are in the same connected component of this graph. If a vertex in this component is the exponent vector of a monomial in $I + M_{\mathrm{emb}}(I)$, then every vertex in that component is an exponent vector of a monomial in $I + M_{\mathrm{emb}}(I)$, which contradicts $x^\mu x^a, x^\mu x^b \notin I + M_{\mathrm{emb}}(I)$.

This means that the connected component of $\mu + a$ and $\mu + b$ in Γ contains edges arising from I (and not from $M_{\mathrm{emb}}(I)$), and we conclude that $x^\mu(x^a - \lambda x^b) \in I$. But then $x^\mu \in M_{\mathrm{emb}}(I)$, which is also a contradiction.

We conclude that $I + M_{\text{emb}}(I)$ has no embedded associated primes by Proposition 3.17.

The second step we show that minimal primes of $I+M_{\text{emb}}(I)$ are the same as those of I. The primary component of $I + M_{\text{emb}}(I)$ associated to $P_i = I_+(\tilde{\rho}_i) + \langle x_j \mid j \notin \delta \rangle$ is $\left(\big(I + M_{\text{emb}}(I) + I_+(\tilde{\rho}_i)\big) : \big(\prod_{j\in\delta}\big)^{\infty} \right) + M_{\text{emb}} \left(\big(I + M_{\text{emb}}(I) + I_+(\tilde{\rho}_i)\big) : \big(\prod_{j\in\delta}\big)^{\infty} \right)$ by Theorem 3.17.

By Theorem 3.17, $\left(\big(I + M_{\text{emb}}(I) + I_+(\tilde{\rho}_i)\big) : \big(\prod_{j\in\delta}\big)^{\infty} \right) = J_i + M_{\text{emb}}(J_i)$, where $J_i = \left(\big(I + I_+(\tilde{\rho}_i)\big) : \big(\prod_{j\in\delta} x_j\big)^{\infty} \right)$, so the same argument that proved $M_{\text{emb}}(I + M_{\text{emb}}(I)) = \varnothing$ shows that $M_{\text{emb}}(J_i + M_{\text{emb}}(J_i)) = \varnothing$.

We conclude that the primary component of $I+M_{\text{emb}}(I)$ associated to P_i is $\left(\big(I + M_{\text{emb}}(I) + I_+(\tilde{\rho}_i)\big) : \big(\prod_{j\in\delta}\big)^{\infty} \right) = J_i + M_{\text{emb}}(J_i)$, which is the primary component of I associated to P_i.

Since $I+M_{\text{emb}}(I)$ has the same minimal primes and minimal primary components as I, and $I + M_{\text{emb}}(I)$ has no embedded primes, we see that $I + M_{\text{emb}}(I) = \text{Hull}(I)$.

$\square$

4. COMMUTATIVE MONOID CONGRUENCES AND BINOMIAL IDEALS

In this section, we focus on congruences in commutative monoids to derive some results about primary decomposition of binomial ideals. The decomposition of congruences in commutative monoids is an analogous theory. But, this theory does not reflect all the features of primary decomposition of binomial ideals and does not truly lead to the corresponding combinatorics. Kahle and Miller in [22] define a new type of intersection decomposition which is called *mesoprimary* decomposition by using congruences on monoids. This new decomposition is finer than cellular decomposition, but not as fine as primary decomposition. A good feature of this new theory is that it allows significant speed-ups in computations [21].

Primary decomposition of binomial ideals can be recovered from mesoprimary decomposition which is more advantageous in terms of combinatorial clarity and computational efficiency. Also, we are not supposed to assume some properties about the base field, for example being algebraically closed and about its characteristics.

In characteristic zero, primary components contain the binomial part of their radicals (see Remark 1.37), which reflects the combinatorial features more accurately. In characteristic $p > 0$, on the other hand, an additional problem arises from the fact that binomials of the radical of a primary ideal I are not necessarily contained in the ideal I itself.

Our aim in this section is to review some results in [22] and try to characterize the primary binomial ideals in positive characteristic in terms of congruences.

4.1 Congruences on Monoids

Recall from Definition 2.1, a congruence is an additively closed equivalence relation on a monoid. For example, equality satisfies the definition of congruence. This is called *identity* congruence. Also recall that a binomial ideal I of a monoid algebra $\Bbbk[\mathcal{Q}]$ induces a congruence $\sim$, which we denote by $\sim_I$, in which

$$u \sim v \text{ if } t^u - \lambda t^v \in \text{I for some} \, \lambda \neq 0.$$

Remark 4.1. As binomial ideals induce congruences, indeed by Theorem 9.12 in [22], we know that every congruence is induced by some canonical unital binomial ideal. Recall that a unital binomial ideal (pure difference binomial) in $\Bbbk[\mathcal{Q}]$ is an ideal which does not have any monomials and is generated by difference of monic monomials $t^a - t^b$.

Definition 4.2. Let $\mathcal{Q}$ be a commutative monoid and $\sim$ be a congruence on $\mathcal{Q}$. The quotient $\mathcal{Q}/\sim$ is the set of equivalence classes under addition. We denote $\mathcal{Q}/_{\sim} =: \bar{\mathcal{Q}}$.

A congruence on $\mathcal{Q}$ induces a $\bar{\mathcal{Q}}$-grading on the semigroup algebra $\Bbbk[\mathcal{Q}]$, in which the monomial t^u has degree $\bar{u} \in \mathcal{Q}$ whenever the image of u under the quotient map $\mathcal{Q} \to \bar{\mathcal{Q}}$ is $\bar{u}$. Under this grading, it is easy to define the Hilbert function of the semigroup algebra.

Lemma 4.3. *The Hilbert function $H_M : \mathcal{Q} \to \mathbb{N}$ satisfies*

$$H_M(\bar{q}) := dim_{\Bbbk}(\Bbbk[\mathcal{Q}])_{\bar{q}} = \begin{cases} 0, & \text{if } \bar{q} = \{u \in \mathcal{Q} \mid t^u \in I\}, \\ 1, & \text{otherwise.} \end{cases}$$

Remark 4.4. Although the statements about binomial primary decomposition in Theorem 1.33 and Theorem 1.38 are for polynomial rings, they can be extended to hold for binomial ideals in general monoid algebras. One can start by choosing a presentation $\mathbb{N}^n \to \mathcal{Q}$. The kernel of the induced surjection $\Bbbk[\mathbb{N}^n] \to \Bbbk[\mathcal{Q}]$ is a binomial ideal as is proved in Theorem 7.11 in [15]. Thus the preimage $\tilde{I}$ of the binomial ideal $I \subseteq \Bbbk[\mathcal{Q}]$ is a binomial ideal such that $\Bbbk[\mathbb{N}^n/_{\tilde{I}}] = \Bbbk[\bar{\mathcal{Q}}]$. If we replace I by $\tilde{I}$, we may assume $\mathcal{Q} = \mathbb{N}^n$ and the result follows.

Definition 4.5. An element ∞ in $(\mathcal{Q}, +)$ is called *nil* if $q + \infty = \infty$ for all $q \in \mathcal{Q}$.

- An element $q \in \mathcal{Q}$ is called *nilpotent* if $nq = \infty$ for some $n \in \mathbb{N}$.
- An element $q \in \mathcal{Q}$ is called *cancellative* if addition by q is injective : $q+a = q+b \Rightarrow a = b$ in $\mathcal{Q}$.
- An element $q \in \mathcal{Q}$ is called *partly cancellative* if $q + a = q + b \neq \infty \Rightarrow a = b$ for all cancellative $a, b \in \mathcal{Q}$.

Definition 4.6. An *affine semigroup* is a monoid that is isomorphic to a finitely generated submonoid of a free abelian group. In other words, an affine semigroup is isomorphic to

$$\mathbb{N}A = \{c_1a_1 + ... + c_na_n \mid c_i \in \mathbb{N}\}$$

for some $A = \{a_1, ..., a_n\} \subset \mathbb{Z}^d$.

Definition 4.7. A set F of elements in $\mathcal{Q}$ is *torsion free* if $na = nb \Rightarrow a = b$ for all $n \in \mathbb{N}$, whenever $a, b \in F$. Fix a prime number p, F is called *p-torsion free* if $na = nb$ and $(n, p) = 1 \Rightarrow a = b$ for all $a, b \in F$ and $n \in \mathbb{N}$.

Definition 4.8. Use the same notation as in Definition 4.2.

- The congruence $\sim$ is called *primary* if every element of $\bar{\mathcal{Q}}$ is either nilpotent or cancellative.
- The congruence $\sim$ is called *mesoprimary* if it is primary and every element of $\bar{\mathcal{Q}}$ is partly cancellative.

- The congruence $\sim$ is called *primitive* if it is mesoprimary and the subset of $\bar{Q}$ consisting of cancellative elements is torsion free.

- The congruence $\sim$ is called *toric* if the non-nil elements of $\bar{Q}$ form an affine semigroup.

Example 4.9. *Giving a congruence on $\mathbb{N}^n$ is equal to giving a unital binomial ideal in $\Bbbk[\mathbb{N}^n]$. The generators of $\mathbb{N}^n$ corresponding to the variables $x, y, ...$ will be denoted by $e_x, e_y, ...$ for simplicity.*

1. The congruence induced by the binomial ideal $I = \langle zx^2 - zy^2, z^2 \rangle \subseteq \Bbbk[x, y, z]$ is primary. The elements $\overline{e_x}$ and $\overline{e_y} \in \overline{\mathbb{N}^3}$ generate the cancellative class and $\overline{e_z}$ generates the nilpotent class.

2. The congruence induced by the binomial ideal $I = \langle x^2 - y^2, z^2 \rangle \subseteq \Bbbk[x, y, z]$ is mesoprimary. The elements $\overline{e_x}$ and $\overline{e_y} \in \overline{\mathbb{N}^3}$ generate the cancellative class and $\overline{e_z}$ generates the nilpotent class. Observe that all elements of $\mathbb{N}/{\sim_I}$ satisfy the partly cancellative property.

3. The congruence induced by the binomial ideal $I = \langle x - y, z^2 \rangle \subseteq \Bbbk[x, y, z]$ is primitive. This congruence is mesoprimary and the cancellative subset consisting of the elements $\overline{e_x}$ and $\overline{e_y}$ is torsion-free.

4. The congruence induced by the binomial ideal $I = \langle x^2 - y \rangle \subseteq \Bbbk[x, y]$ is toric, since $\bar{Q}$ is isomorphic to $\mathbb{N}A$ where $A = \{1, 2\}$.

The difference between a primary congruence and a mesoprimary congruence is that in a mesoprimary congruence injectivity is required of addition by a nilpotent element. The following example illustrates this distinction.

Example 4.10. *The binomial ideal $I = \langle yx^2 - y, y^2 \rangle \subseteq \Bbbk[x, y]$ induces a primary congruence. The connected components in the Figure 4.1 exhibit the congruence classes of the congruence $\sim_I$.*

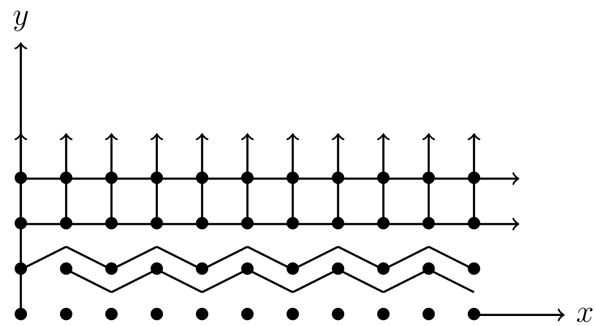

Figure 4.1: Cogruence classes of $\sim_I$

The translation of two dots in different connected components, for instance $(1,0)$ *and* $(3,0)$*, upwards by one unit, are connected.*

Figure 4.2 shows the mesoprimary congruence induced by $J = \langle x^2 - 1, y^2 \rangle \subseteq \Bbbk[x,y]$*. In fact, the pictures for mesoprimary congruences are homogeneous. As shown in the figure, any translation of two dots from different classes cannot be connected except in the connected component corresponding to the nil class.*

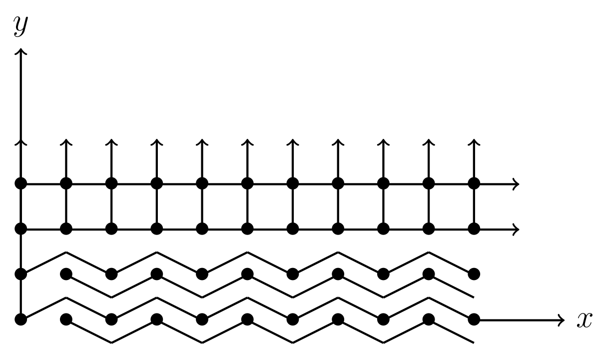

Figure 4.2: Cogruence classes of $\sim_J$

The notions introduced above for the elements of monoids have counterparts for binomial ideals; these counterparts will be used for the characterizations of special types of binomial ideals.

Definition 4.11. A binomial ideal is called *mesoprimary* if it is maximal among the ideals inducing a given mesoprimary congruence.

Definition 4.12. Let $\mathcal{Q}$ be a monoid. The ideal

$$\tilde{I}_{\text{aug}} = \langle t^q - 1 \mid q \in \mathcal{Q} \rangle$$

generated by all monomial differences is called *unital augmentation ideal* of $\mathcal{Q}$. The ideal

$$I_{\text{aug}} = \langle t^q - \lambda_q \mid q \in \mathcal{Q}, \lambda_q \in \Bbbk^* \rangle$$

is an augmentation ideal for a given binomial ideal $I \subseteq \Bbbk[\mathcal{Q}]$ if $I \cap I_{\text{aug}}$ is a binomial ideal.

Remark 4.13. The ideals $\langle x^2 - x^3 \rangle$ and $\langle x^2 \rangle \subseteq \Bbbk[x]$ induce the same congruence on $\mathbb{N}$. Observe that the first ideal is not cellular, and the second ideal is primary. To characterize the binomial ideals with respect to the congruences they induce, we need another condition which is described in the following theorem.

Theorem 4.14. *(Theorem 9.12 in [22].) Let $I_0 \subset I_1$ be binomial ideals in $\Bbbk[\mathcal{Q}]$ inducing the same congruence on $\mathcal{Q}$, then I_1 contains monomials and I_0 does not, also $I_0 = I_1 \cap I_{\text{aug}}$ for an augmentation ideal I_{aug} compatible with I_1. If I_2 is a binomial ideal that contains I_1 and induces the same congruence as I_1 then $I_2 = I_1$.*

By the theorem above, if $I_0 \subset \cdots \subset I_n$ is a chain of distinct binomial ideals in $\Bbbk[\mathcal{Q}]$ inducing the same congruence on $\mathcal{Q}$, then $n \leqslant 1$. Moreover, I is maximal among the ideals inducing the same congruence if I contains a monomial in the case that $\bar{\mathcal{Q}}$ has a nil ∞.

Theorem 4.15. *$I \subseteq \Bbbk[\mathcal{Q}]$ is a cellular binomial ideal if and only if $\mathcal{Q}/_{\sim_I}$ is primary and I is maximal among proper ideals inducing that congruence.*

Proof. This is Theorem 10.6.1 in [22]. Let us discuss how the maximality condition arises. If a monomial $t^q \in \Bbbk[\mathcal{Q}]$ is a nonzerodivisor or nilpotent modulo I then the image $\bar{q} \in \bar{\mathcal{Q}}$ of q is cancellative or nilpotent respectively. Now we need to find why I is maximal among the ideals inducing the same congruence. By Theorem 4.14, if $\bar{\mathcal{Q}}$ has a nil then I contains a monomial. Let $\bar{a} = \infty$ then $a \sim_I \ell a$ for all $\ell \in \mathbb{Z}_{>0}$ by the definition of nil. This implies that for all $\ell \in \mathbb{Z}_{>0}$, there exists a $\lambda \in \Bbbk^*$ such that $t^a - \lambda_\ell t^{\ell a} \in I$. Since I is cellular, t^a is nilpotent. Let $t^{ra} \in I$ for some $r \in \mathbb{Z}_{>0}$. Taking $\ell = r$ for the binomial above, this implies that $t^a \in I$. The converse of the statement is clear.

□

Our next result is Theorem 10.6.5 in [22] which shows the relation between a prime binomial ideal I and the congruence induced by I.

Theorem 4.16. *Let $I \subseteq \Bbbk[\mathcal{Q}]$ be a binomial ideal, where $\Bbbk$ is algebraically closed. The ideal $I \subseteq \Bbbk[\mathcal{Q}]$ is a prime binomial ideal if and only if $\mathcal{Q}/_{\sim_I}$ is toric and I is maximal among ideals inducing that congruence.*

Theorem 4.17. *Let $I \subseteq \Bbbk[\mathcal{Q}]$ be a binomial ideal. If $\mathcal{Q}/_{\sim_I}$ is primitive and I is maximal among proper ideals inducing that congruence, then I is primary. The converse holds if $\Bbbk$ is algebraically closed with characteristic zero.*

Proof. This is Theorem 10.6.3 in [22]. □

The converse of Theorem 4.17 is not necessarily true in positive characteristic. When $\mathrm{char}(\Bbbk) = 0$, binomial primary ideals have to be mesoprimary. Essentially this is because of the fact that binomial primary ideals contain the binomial part of its radical, in other words, the binomial part of the corresponding associated prime. On the other hand, binomial primary ideals need not to be mesoprimary in $\Bbbk[\mathcal{Q}]$ when $\Bbbk$ is algebraically closed with characteristic $p > 0$. The example below demonstrates this fact.

Example 4.18. *Let $I = \langle z^2, x^2z - y^2z, x^4 - y^4\rangle \subseteq k[x, y, z, t]$. In $\mathrm{char}(\Bbbk) = 2$, I is a primary ideal. But it is not mesoprimary. In $\bar{Q}$, e_z is nilpotent since $2e_z = \infty$, but it is not partly cancellative*

$$2e_x + e_z \sim 2e_y + e_z \neq \infty$$

but $2e_x \nsim 2e_y$, or in other words, $x^2 - \lambda y^2 \notin I$ for all $\lambda \neq 0$.

The Example 4.18 justifies that we can say that the mesoprimary decomposition in [22] is more helpful to find the primary decomposition of binomial ideals when $\mathrm{char}(\Bbbk) = 0$ than for the positive characteristic case.

Our next theorem is the main result of this section, which discusses what a primary binomial ideal looks like when $\mathrm{char}(\Bbbk) = p > 0$. Note that we replace the condition of being mesoprimary with condition iii).

Theorem 4.19. *Let $\Bbbk$ be algebraically closed with characteristic $p > 0$. Let I be a binomial ideal in $\Bbbk[\mathbb{N}^n]$. I is a primary ideal if and only if it satisfies the following conditions*

i) the congruence $\sim_I$ induced by I is a primary congruence and I is maximal among proper ideals inducing that congruence,

ii) the cancellative subset of $\mathbb{N}^n/_{\sim_I}$ is p-torsion free and

iii) if q is a nilpotent element that is not partly cancellative with $q + a \sim q + b \neq \infty$ and $a \nsim b$, then $p^e a \sim p^e b$ for some $e \in \mathbb{Z}_{>0}$.

Proof. First, we assume that I is primary and we show that $\sim_I$ satisfies the required conditions. Since I is primary, it is cellular, so $\sim_I$ is primary by Corollary 4.15. Let us assume $I = I_\delta^{(b)}$ for some $\delta \subseteq \{1, ..., n\}$ and $b \in \mathbb{Z}^n_{>0}$ and $I \cap k[x_i \mid i \in \delta] = I_+(\rho)$ for some partial character. Note that $I_+(\rho)$ is a primary ideal with unique associated prime $I_+(\tilde{\rho})$, where $\tilde{\rho}$ is a saturation of ρ. Thus the unique associated prime of I is $I_+(\tilde{\rho}) + \langle x_i \mid i \notin \delta \rangle$.

Now we show that $\sim_I$ satisfies condition ii). Let a and b be elements in the subset of $\mathbb{N}^n/_{\sim_I}$ consisting of the cancellative elements. In other words, x^a and x^b are nonzerodivisors modulo I. Let $ka \sim kb$ such that $gcd(k, p) = 1$. Then there exists a binomial $x^{ka} - \lambda x^{kb} \in I$ for some λ, which implies $x^{ka} - \lambda x^{kb} \in I_+(\rho)$. Since $(ka - kb) \in L_\rho$ and $gcd(k, p) = 1$, by definition $(a - b) \in \mathrm{Sat}'_p(L_\rho)$. Recall that $I_+(\rho)$ is a primary ideal, which means $\mathrm{Sat}'_p(L_\rho) = L_\rho$ and $(a - b) \in L_\rho$. This implies $x^a - \lambda' x^b \in I_+(\rho) \subseteq I$ for some λ' so, $a \sim_I b$ in $\mathbb{N}^n/_{\sim_I}$. Thus, the cancellative subset of $\mathbb{N}^n/_{\sim_I}$ is p-torsion free.

In order to prove iii), we assume there exists a nilpotent element c which is not partly cancellative, which means there exist cancellative elements a and b such that $c + a \sim c + b \neq \infty$ and $a \nsim b$. In other words, there exists a binomial $x^c x^a - \lambda x^c x^b \in I$ for some $\lambda \in \mathbb{k}^*$ and $a - b \notin L_\rho$. Since $c + a \sim c + b \neq \infty$, which means that x^c is a standard monomial of I in the nilpotent variables. By Theorem 3.22, $a - b \in L_{\tilde{\rho}}$ since if not $x^c \in M_{emb}(I)$ and since I is primary $x^c \in M_{emb}(I) \subseteq I$. But we assume that $x^c \notin I$; this implies that there exists a λ' such that $x^a - \lambda' x^b \in I_+(\tilde{\rho})$, and for some p^e, p^e-th quasi power of $x^a - \lambda' x^b$ is in I, which implies that $p^e a \sim p^e b$.

For the proof of the converse, we assume $\sim_I$ satisfies the three conditions stated above, we claim that I has one associated prime, so it is primary. By Corollary 4.15, I is cellular and assume $I \cap k[x_i \mid i \in \delta] = I_+(\rho)$. We want to show that $\mathrm{Sat}'_p(L_\rho) = L_\rho$. Let $u \in \mathrm{Sat}'_p(L_\rho)$ then $nu \in L_\rho$ for some $n \in \mathbb{Z}_{>0}$ such that $gcd(n, p) = 1$. So

$x^{nu_+} - \rho(nu)x^{nu_-} \in I$, this implies that $nu_+ \sim_I nu_-$. Since $\sim_I$ is p-torsion free, $u_+ \sim_I u_-$. This implies $u \in L_\rho$ and $\mathrm{Sat'}_p(L_\rho) \subseteq L_\rho$. Remember that the index of the group $\mathrm{Sat'}_p(L_\rho)/L_\rho$ determine the saturations of ρ and this index is one in here. Thus there is one extension of ρ to $\mathrm{Sat'}_p(L_\rho)$ which is itself, I has one minimal associated prime; $P = I_+(\tilde{\rho}) + \langle x_i \mid i \in \bar{\delta}\rangle$ where $\tilde{\rho}$ is the saturation of ρ.

We claim that the only associated prime of I is P. Assume there exists an embedded associated prime $\tilde{P}$ of I which is different than P. By Algorithm 3.7, $\tilde{P}$ is in the form $I_+(\tilde{\tau}) + \langle x_i \mid i \in \bar{\delta}\rangle$ where $(I : m) \cap k[x_i \mid i \in \delta] = I_+(\tau)$ for some monomial $m = x^\mu \in \{x_i \mid i \in \bar{\delta}\}$ where $m = x^\mu \notin I$ and $\tilde{\tau}$ is the saturation of τ. Since $\tilde{P} \neq P$, $L_{\tilde{\tau}}$ strictly contains $L_{\tilde{\rho}}$. This means that there exists an element $u \in L_{\tilde{\tau}}$ and $u \notin L_{\tilde{\rho}}$. Thus, $nu \in L_\tau$ for some $n \in \mathbb{Z}$ and $tu \notin L_\rho$, for all $t \in \mathbb{Z}$. So, $x^{nu_+} - x^{nu_-} \in (I : x^\mu) \cap k[x_i \mid i \in \delta]$. This implies that $x^\mu(x^{nu_+} - x^{nu_-}) \in I$. μ is a nilpotent element and it is not partly cancellative since $nu_+ \not\sim nu_-$ in $\mathbb{N}^n/\sim_I$. By the condition iii), we deduce that $p^e nu_+ \sim p^e nu_-$ in $\mathbb{N}^n/\sim_I$ for some $e \in \mathbb{Z}_{>0}$. This implies $x^{p^e nu_+} - \lambda x^{p^e nu_-} \in I_+(\rho)$ for some λ, contradicting the fact that $tu \notin L_\rho$, for all $t \in \mathbb{Z}$.

□

This description can be used to characterize the primary components of binomial ideals in positive characteristic fields, which is missing in the literature.

Remark 4.20. We assume the notation in Theorem 4.19. If the congruence induced by I has a nilpotent element c that is not partly cancellative, then I is primary only when $\mathrm{char}(\mathbb{k}) = p$. In fields with different characteristics, I cannot be primary since the binomial $x^{p^e a} - \lambda x^{p^e b}$ which occurs in the proof of the above theorem can be factored over fields with characteristics $\tilde{p} \neq p$ and implies two different associated primes of I.

5. CONCLUSION AND FURTHER QUESTIONS

We have discussed primary decompositions of special binomial ideals and have given descriptions for primary components. One of the main motivations for this dissertation is to understand how the combinatorics arising from congruences or graphs of binomial ideals governs the primary decomposition. These geometric combinatorial techniques first appeared in [8]. Dickenstein, Matusevich and Miller provided a characterization of the primary components of an arbitrary binomial ideal in a polynomial ring over an algebraically closed field of characteristic zero.

In characteristic zero, the main idea is that, because a primary ideal contains binomial part of its radical, we can take the whole situation modulo the binomial part of the associated prime ideal. The monomial part, on the other hand, can be computed by using the infinite vertices of the graphs (or elements of infinite congruence classes induced by binomial ideals) of some localizations of the binomial ideals. On the other hand, in positive characteristic the primary component contains a Frobenius power of the binomial part of its associated prime. This blocks the use of known techniques that reduce the problem to a manageable monomial ideal problem. We have provided a partial answer to this open question in the case of cellular binomial ideals. A natural continuation of the theoretical part of this work is to investigate the description for primary components of general binomial ideals in positive characteristic fields. We propose the following conjecture

Conjecture 5.1. *Let $I \in k[\mathbb{N}^n]$ be a binomial ideal where* $\operatorname{char}(\Bbbk) = p > 0$. *Let $P = I_+(\rho) + \langle x_i \mid i \in \bar{\delta}\rangle$ be a minimal prime of I. Let $\sim$ be the congruence on $\mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}}$ induced by the ideal $I + (I_+(\rho))^{p^e}[\mathbb{Z}^\delta]$. Let $\tilde{U}$ be the set of $u \in \mathbb{N}^n$ whose congruence classes contain two elements v, w such that $v - w \in (\mathbb{Z}^\delta \times \mathbb{N}^{\bar{\delta}})$ and $v - w \notin L_\rho$. Then* *the P-primary component of I is*

$$Q = ((I + I_+(\rho)^{p^e} : (\prod_{i\in\delta} x_i)^\infty) + \langle x^u \mid u \in \tilde{U}\rangle).$$

for some $e \in \mathbb{Z}_{>0}$.

It is straightforward to check that $(I + I_+(\rho)^{p^e} : (\prod_{i\in J} x_i)^\infty)$ is contained in the P-primary component of I. One can also show that $\langle x^u \mid u \in \tilde{U}\rangle$ is contained in the primary component by following the steps in Lemma 2.4. The missing step is to show that the ideal Q is primary. This can be achieved by providing combinatorial conditions to describe a primary ideal in positive characteristics. This question can be answered by using the Theorem 4.19.

Another question we formulate for further research is the analog of the unmixed decomposition of cellular binomial ideals in positive characteristic which is mentioned in Example 3.9.

One last closing remark is that we can adapt the techniques and combinatorial methods we improve in here to other special types of binomial ideals which already have combinatorial flavor, for instance circuit ideals. A circuit ideal is a subideal of a (prime) toric ideal. Eisenbud and Sturmfels proved that the embedded primes of a circuit ideal are indexed by certain faces of a cone. Bogart, Jensen and Thomas in [3] gave a characterization for these faces. But it is still an open problem to characterize the embedded primary components of a circuit ideal, a question that was posed by Eisenbud and Sturmfels. A combinatorial characterization of embedded primary components of circuit ideals might be valuable for applications of binomial ideals to integer programming and statistics.

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