ANALYSIS OF THE THREE-DIMENSIONAL SUPERRADIANCE PROBLEM AND SOME GENERALIZATIONS

A Dissertation

by

INDRANIL SEN GUPTA

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2010

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Goong Chen
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	Francis Narcowich
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ABSTRACT

Analysis of the Three-dimensional Superradiance Problem and Some Generalizations. (August 2010)

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Chair of Advisory Committee: Dr. Goong Chen

We study the integral equation related to the three and higher dimensional superradiance problem. Collective radiation phenomena has attracted the attention of many physicists and chemists since the pioneering work of R. H. Dicke in 1954. We first consider the three-dimensional superradiance problem and find a differential operator that commutes with the integral operator related to the problem. We find all the eigenfunctions of the differential operator and obtain a complete set of eigensolutions for the three-dimensional superradiance problem. Generalization of the three-dimensional superradiance integral equation is provided. A commuting differential operator is found for this generalized problem. For the three dimensional superradiance problem, an alternative set of complete eigenfunctions is also provided. The kernel for the superradiance problem when restricted to one-dimension is the same as appeared in the works of Slepian, Landau and Pollak. The uniqueness of the differential operator commuting with that kernel is indicated. Finally, a concentration problem for the signals which are bandlimited in disjoint frequency-intervals is considered. The problem is to determine which bandlimited signals lose the smallest fraction of their energy when restricted in a given time interval. A numerical algorithm for solution and convergence theorems are given. Orthogonality properties of analytically extended eigenfunctions over $L^2(-\infty, \infty)$ are also proved. Numerical computations are carried out in support of the theory.

To Jahar Lal Sen Gupta, Saswati Sen Gupta and Dr. Sankar C. Ghosh

ACKNOWLEDGMENTS

I would like to express my sincere thanks to my advisor Prof. Goong Chen for his guidance and encouragement during my graduate studies and in the preparation of this dissertation. He not only led me into the magical world of mathematical physics and special functions, but also gave me valuable advice which will be useful in my future research.

I wish to express my sincere gratitude to the members of my advisory committee, Prof. Harold Boas, Prof. Dudley Herschbach, and Prof. Francis Narcowich. I consider it an honor to work with Prof. Dudley Herschbach (Nobel laureate in Chemistry 1986) who gave me many valuable suggestions on this dissertation.

I also wish to thank Dr. Bo Sun and Dr. Weisheng Jiang for their valuable suggestions and comments on my dissertation.

I am very grateful to Ms. Angela Allen, for her help with LaTeX. I also wish to acknowledge the websites *Wikipedia* and *Google*, as I frequently used those for searching various documents for my research.

I sincerely thank Prof. Paulo Lima-Filho for his encouragement and valuable advice. I must also thank Ms. Monique Stewart, Assistant for Graduate Programs, for her assistance in processing much of the paper-works, that helped me to complete everything in time.

I am greatly indebted to my Mathematics/Statistics teacher in the past: Dr. Sankar C. Ghosh. He is an inspiration to me for doing mathematical research.

Last, but not the least, I wish to thank my parents Mr. Jahar Lal Sen Gupta and Mrs. Saswati Sen Gupta, who inspired, encouraged and fully supported me for every trial that came on my way.

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CHAPTER I

INTRODUCTION

A. Collective radiation and superradiance

The collective radiation phenomenon has been an interesting subject since the pioneering work of Dicke [12] in 1954. In that classic paper, Dicke considered two types of collective radiation phenomena: superradiance and subradiance in a collection of two-level atoms when all atoms are confined inside a volume much smaller than radiation wavelength. Dicke introduced the notion of super-radiance when discussing the formation of a short-lived state in a radiant gas of N identical two level atoms. When the gas is confined to a volume with a size smaller than the wave length of the radiation, the atoms are coherently coupled through the common radiation field. By considering the entire collection of atoms as a single quantum-mechanical system, Dicke found that under certain conditions the individual atoms cooperate to emit radiation at a rate which is much greater than their incoherent emission rate. A system which exhibits cooperative effects of this nature is said to be "superradiant".

In quantum mechanics, superradiance refers to a class of radiation effects (or enhanced radiation effects) typically associated with the acceleration or motion of a nearby body (which supplies the energy and momentum for the effect). Superradiance allows a body with concentration of angular or linear momentum to move towards a lower energy state, even when there is no obvious classical mechanism for this to happen. In this sense, the effect has some similarities with quantum tunnelling.

The theory of superradiance was further developed by the improvement of experimental ability to manipulate coherently large collections of optically resonant

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This dissertation follows the style of Journal of Functional Analysis.

atoms. The experimental ability to manipulate coherently large collections of optically resonant atoms is provided by the experimental observations of photon echoes, self-induced transparency, optical nutation and optical adiabatic inversion ([1, 17, 21, 33]) etc. It is clear that in the optical region of the spectrum it is no longer practical to assume that all of the active atoms are confined to a region which has linear dimensions smaller than a wavelength. Rehler and Eberly [34] generalized Dicke's description of superradiance to an extended system. They considered an arbitrary number of atoms coupled to all radiation modes with the atoms contained in a volume which may be large or small compared with the cube of the average emitted wavelength.

The research on cooperative emission was further developed by Bonifacio and Lugiato [4]. They brought up the concept of superfluorescence which describes the cooperative emission from a system of uncorrelated excited atoms. This process is usually started by normal spontaneous emission but later develops correlation among the system. This has been studied extensively by Skribanowitz, Hermann, MacGillivray and Feld [45] and MacGillivray and Feld [27].

Collective spontaneous radiation is interesting for both developing a mathematical theory for a physical model and applications to many other related problem. From the physical standpoint cooperative spontaneous emission is an example of a many-body quantum problem of N atoms collectively interacting with an electromagnetic field. Cummings and Dorri [9] showed that interaction of N atoms, in the equivalent mode position, with the single-mode resonant field leads to the radiation suppression. Successively Cummings ([10, 11]) presented the exact solution for the spontaneous emission of a single atom which is initially excited in the presence of the $N-1$ initially unexcited atoms, interacting with the M modes of the field. Buzek [7] studied the dynamics of the system of N identical, but distinguishable, two-level atoms in free space interacting with the radiation field, when at the initial time ($t = 0$) only one of the atoms is in the excited state and all others are in the ground state. The problem of cooperative spontaneous emission of N atoms reduces to finding all eigenstates of a related integral equation. From these eigenstates the evolution of an arbitrary initial state is obtained by expanding the initial condition in terms of the set of eigenstates. A similar eigenvalue problem occurs for a spherical atomic cloud in Weisskopf and Wigner theory. This problem was studied by Ernst [14]. Ressayre and Tallet [35] and many others studied the same problem in various geometries.

Superradiance is very useful as one of the methods for producing coherent emission without coherent pumping. This is especially important in x-ray or γ-ray where there are no effective mirrors which limits the use of ordinary stimulated emission process. Besides, with the recent advances of quantum informatics, decoherence free subspace [22] has been proposed to be one of the strategies to combat the effects of decoherence in quantum computation and quantum communication. *Decoherence* is the process whereby the quantum-mechanical state of any macroscopic system is rapidly correlated with that of its environment in such a way that no measurement on the system alone (without a simultaneous measurement of the complete state of the environment) can demonstrate any interference between two quantum states of the system. *Decoherence-free subspace*(DFS) is a special set of quantum states which is insensitive to some particular noise. These subspaces prevent destructive environmental interactions by isolating *quantum information*. They are important in *quantum computing*, where coherent-control of quantum systems is the desired goal. Since *quantum computers* cannot be isolated from their environment and information can be lost, the study of DFSs is important for the implementation of quantum computers into the real world. A collective system of many two-level particles is one of the ideal candidates to realize decoherence free subspace. An ensemble of N two-level atoms with one excitation also plays an important role in quantum memory and quantum networking [8].

Recent quantum optical computations and experiments study the problem in which a single photon is stored in a gas and then retrived at a later time ([40, 41]). In [53] the correlated spontaneous emission from N atoms in free-space is studied. In very recent works of Svidzinsky et. al. [54, 55] the problem of single photon spontaneous emission is discussed in detail. The paper clarifies many issues of recent interest namely the effect of virtual processes and situations when the quantum N atom problem is analogous to the radiation of a system of N harmonic oscillators. The mathematical motivation of our work is to find a complete set of eigenfunctions of the superradiance integral equation omitting the contribution of virtual photons. This turns out to be a problem of solving the integral equation

$$\int_V \frac{\sin(k_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|}\beta(\mathbf{r}')dr' = \lambda\beta(\mathbf{r}), \tag{I.1}$$

where $\mathbf{r}, \mathbf{r}' \in V \subset \mathbb{R}^3$ and the integral is taken over the volume V of a sphere of radius R. Our work will focus on several concepts and results involving the solution of the above integral equation. We will also show that this three-dimensional superradiance problem is a special case of a more generalized integral equation for dimension ≥ 2.

B. Mathematical motivations

A similar problem to (I.1) in one-dimension has been in the literature since the pioneering work by Slepian, Landau and Pollak (cf. [46, 47, 48, 49, 50]) in communication theory. Suppose $f(t) \in L^2(-\infty, \infty)$ and denote the Fourier transform of $f(t)$ by $F(\omega)$. We write

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

and

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Define $\Omega = 2\pi W$ where W is a positive real number. The bandlimited version of $f(t)$ is defined by

$$Bf(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F(\omega)e^{i\omega t}d\omega$$

and the timelimited version of $f(t)$ is defined by

$$Df(t) = \begin{cases} f(t), & |t| \leq T/2 \\ \\ 0, & |t| > T/2 \end{cases}$$

The problem considered by Slepian, Landau and Pollak (cf. [46, 47, 48, 49, 50]) was to maximize μ for

$$\mu = ||BDf||_{\infty}^{2}/||f||_{\infty}^{2}$$

for $f(t) \in L^2(-\infty, \infty)$. This reduces to a problem of solving the integral equation

$$\lambda f(t) = \int_{-T/2}^{T/2} \frac{\sin\Omega(t-s)}{\pi(t-s)} f(s)ds, \tag{I.2}$$

where $|t| \leq T/2$. The maximum μ equals the largest eigenvalue λ_0 of (I.2). Finding the eigenfunction of a finite convolution integral operator is not easy, even if one is satisfied with quite good numerical approximations. Of all the strategies one can dream of to solve this problem, none sounds so appealing as that of finding a second order differential operator with simple spectrum which will commute with the given integral operator and thus will have same eigenfunctions as the integral operator. This is exactly what Slepian, Landau and Pollak did in their work. The prolate spheroidal wave function pops up as a solution of the related differential equation and thus solves (I.1). Spheroidal wave functions are solutions of the Laplace equation that are found by writing the equation in spheroidal coordinates and applying the technique of separation of variables, just like the use of spherical coordinates leads to spherical harmonics. They are called oblate spheroidal wave functions or oblate harmonics if oblate spheroidal coordinates are used and prolate spheroidal wave functions or prolate harmonics if prolate spheroidal coordinates are used. Originally, the spheroidal wave functions were introduced by C. Niven [32] in 1880 when studying the conduction of heat in an ellipsoid of revolution, which lead to a Helmholtz equation in spheroidal coordinates.

Slepian [49] has found that a similar situation holds in higher dimensional Euclidean space $\mathbb{R}^n$. Slepian [50] also showed that a similar situation holds for the case of the integers or the circle. Grünbaum [19] proved a similar result for the group of roots of unity. Grünbaum et. al. [18] gave many analogous commutation results with the real line replaced by either a non-Abelian group or a symmetric space. Their work includes topological groups, the rotational group $SO(n)$, the sphere S^2, real two-dimensional projective space, higher dimensional spheres, hyperbolic Minkowski space etc. The analysis depends on special functions and the theory of Group Representations [57]. Grünbaum [20] also gave an account of commutator of convolution integral operator that contains differential operator of fourth order. Simons et. al. [43, 44] gave an analogue of Slepian's time-frequency concentration problem on the surface of the unit sphere to determine an orthogonal family of strictly bandlimited functions that are optimally concentrated within a closed region of the sphere. They posed and solved the spherical spatiospectral concentration problem for a geographical region of arbitrary shape. Remarkably there is also a differential operator that commutes with the relevant integral operator which was extremely helpful in solving the eigenvalue problem. The solution for this problem is useful in a variety of geophysical, planetary, cosmological and other applications.

Motivated by all these works we will approach the three dimensional superradiance problem by finding a differential operator that commutes with the related integral operator. We will find the eigenfunctions of the differential operator and derive many properties of such commuting operators which will enable us to find the solution for (I.1). We will generalize the kernel of (I.1) in dimensions ≥ 2 and find a commuting differential operator. We will show that this kernel is exactly $\frac{\sin(k_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|}$ in three-dimensions where $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^3$, and thus we have a generalization of the integral equation related to the superradiance problem.

CHAPTER II

MOTIVATION AND ALREADY KNOWN RESULTS

A. Physical background and model from superradiance

Svidzinsky, Chang and Scully studied the correlated spontaneous emission from N atoms in free-space. In [53] Svidzinsky and Chang considered the following problem:

- Consider a system of two level (a and b) atoms (at time $t = 0$) one of which is in excited state a and all other atoms are in the ground state b.
- The energy difference between level a and b is $E_a - E_b = \hbar\omega$.
- Atoms are located at positions $\mathbf{r}_j$ ($j = 1, 2, \ldots, N$).
- At $t > 0$ the initial state starts to decay by emitting a photon. The problem we consider is to find how the state decays with time.

As observed in [53], the Hamiltonian for this system can be given by

$$\hat{H}_{int} = \sum_{\mathbf{k}} \sum_{j=1}^{N} g_k [\hat{\sigma}_j \hat{a}^\dagger_{\mathbf{k}} E_1(t) + \hat{\sigma}^\dagger_j \hat{a}_{\mathbf{k}} E_2(t)]$$

where,

- $E_1(t) = \exp(i(\nu_k - \omega)t - i\mathbf{k} \cdot \mathbf{r}_j)$
- $E_2(t) = \exp(-i(\nu_k - \omega)t + i\mathbf{k} \cdot \mathbf{r}_j)$
- $\hat{\sigma}_j$ is the lowering operator for atom j
- $\hat{a}_{\mathbf{k}}$ is the photon operator
- $\nu_k = ck$ is the frequency of the photon with wave vector $\mathbf{k}$

- g_k = atom-photon coupling constant for k mode = $\omega \frac{d_{ab}}{\hbar}\sqrt{\frac{\hbar}{\epsilon_0 \nu_k V_{ph}}}$

- d_{ab} is the electric-dipole transition matrix element between level a and b

- V_{ph} is the photon volume

Svidzinsky and Chang [53] looked for a solution of the Schrödinger equation as a linear combination of Fock states

$$\Psi = \sum_{j=1}^{N} \beta_j(t)|b_1 b_2 \ldots a_j \ldots b_N 0\rangle + \sum_{\mathbf{k}} \gamma_{\mathbf{k}}(t)|b_1 b_2 \ldots b_N 1_{\mathbf{k}}\rangle$$

where,

- $|b_1 b_2 \ldots a_j \ldots b_N 0\rangle$ are states corresponding to zero number of photons and one atom j is in the excited state a

- $|b_1 b_2 \ldots b_N 1_{\mathbf{k}}\rangle$ are states in which there is one photon and all N atoms are in ground state b

Substituting this Ψ in the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

and using

- $\hat{\sigma}_j \hat{a}^{\dagger}_{\mathbf{k}} |b_1 b_2 \ldots a_j \ldots b_N 0\rangle = |b_1 b_2 \ldots b_j \ldots b_N 1_{\mathbf{k}}\rangle$

- $\hat{\sigma}^{\dagger}_j \hat{a}_{\mathbf{k}} |b_1 b_2 \ldots b_j \ldots b_N 1_{\mathbf{k}}\rangle = |b_1 b_2 \ldots a_j \ldots b_N 0\rangle$

we end up with the following equation for functions $\beta_j(t)$ and $\gamma_k(t)$

- $$\dot{\beta}_j(t) = -i \sum_{\mathbf{k}} g_k \gamma_{\mathbf{k}} \exp(-i(\nu_k - \omega)t + i\mathbf{k}\cdot\mathbf{r}_j), \tag{II.1}$$

•

$$\dot{\gamma}_{\mathbf{k}}(t) = -i\sum_{j=1}^{N} g_k\beta_j(t)\exp(i(\nu_k-\omega)t - i\mathbf{k}\cdot\mathbf{r}_j). \tag{II.2}$$

Integrating (II.2) over time we obtain

$$\gamma_{\mathbf{k}}(t) = \gamma_{\mathbf{k}}(0) - i\int_0^t dt'\sum_{j=1}^{N}\exp(i(\nu_k-\omega)t' - i\mathbf{k}\cdot\mathbf{r}_j)g_k\beta_j(t'). \tag{II.3}$$

Assuming no photon at $t=0$ and substituting (II.3) into (II.1) gives

$$\dot{\beta}_j(t) = -\sum_{\mathbf{k}}\sum_{l=1}^{N}\int_0^t dt' g_k^2\beta_l(t')\exp[i(\nu_k-\omega)(t'-t) + i\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_l)]. \tag{II.4}$$

(II.4) can be reduced to

$$\dot{\beta}_j(t) = -\gamma\sum_{l=1}^{N} f_{jl}\beta_l(t), \tag{II.5}$$

where $f_{jl} = \frac{\sin(k_0|\mathbf{r}_j-\mathbf{r}_l|)}{(k_0|\mathbf{r}_j-\mathbf{r}_l|)}$, $k_0 = \frac{\omega}{c}$ and and γ is the single atom spontaneous decay rate given by

$$\gamma = \frac{V_{ph}k_0^2}{\pi c}g_{k_0}^2.$$

Now (II.5) can be written in the following matrix form

$$\dot{B} = -\gamma F B, \tag{II.6}$$

where $B = \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \\ . \\ . \\ . \\ \beta_N(t) \end{bmatrix}$, $F = \begin{bmatrix} 1 & f_{12} & \ldots & f_{1N} \\ f_{21} & 1 & \ldots & f_{2N} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ f_{N1} & f_{N2} & \ldots & 1 \end{bmatrix}$

The matrix F is symmetric because $f_{jl} = f_{lj}$. Let $|\lambda_i\rangle$ be eigenvectors of F and λ_i ($i = 1, 2, \ldots, N$) the corresponding eigenvalues. Then a general solution of the Schrödinger equation can be expressed as a superposition of eigenstates $\Psi = C_1 e^{-\gamma\lambda_1 t}|\lambda_1\rangle + C_2 e^{-\gamma\lambda_2 t}|\lambda_2\rangle + \cdots + C_N e^{-\gamma\lambda_N t}|\lambda_N\rangle + \sum_{\mathbf{k}} \gamma_{\mathbf{k}}(t)|b_1 b_2 \ldots b_N 1_{\mathbf{k}}\rangle$, where $C_1, C_2, \ldots, C_N$ are constants determined by the initial conditions.

Dense Atomic Cloud: We calculate the eigenvalues and eigenvectors for a dense spherically symmetric atomic cloud. The equation we need to solve is

$$\begin{pmatrix} 1 & f_{12} & \ldots & f_{1N} \\ f_{21} & 1 & \ldots & f_{2N} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ f_{N1} & f_{N2} & \ldots & 1 \end{pmatrix} \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ . \\ . \\ . \\ \beta_N(t) \end{pmatrix} = \lambda_n \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ . \\ . \\ . \\ \beta_N(t) \end{pmatrix}$$

This can be written in a more compact form as follows

$$\sum_{m=1}^{N} \frac{\sin(k_0|\mathbf{r}_j - \mathbf{r}_m|)}{k_0|\mathbf{r}_j - \mathbf{r}_m|}\beta_m = \lambda_n \beta_j.$$

For a dense cloud with uniform atom density N/V (V is the volume of the cloud) we can replace the sum by an integral and treat β_j as a continuous function. Then we obtain

$$\frac{N}{V}\int d\mathbf{r}' \frac{\sin(k_0|\mathbf{r} - \mathbf{r}'|)}{k_0|\mathbf{r} - \mathbf{r}'|}\beta(\mathbf{r}') = \lambda_n \beta(\mathbf{r}). \tag{II.7}$$

It is well known that a set of solutions for (II.7) is given by

$$\beta_{nm}(\mathbf{r}) = j_n(k_0 r) Y_{nm}(\theta, \phi)$$

with

$$\lambda_n = \frac{3N}{2}[j_n^2(k_0 R) - j_{n-1}(k_0 R) j_{n+1}(k_0 R)],$$

where

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z),$$

and

$$Y_{nm}(\theta,\phi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos\theta) e^{im\phi}$$

are spherical functions, $P_n^m(\cos\theta)$ are associated Legendre polynomials.

When $z << 1$, $j_n(z) \approx \frac{z^n}{(2n+1)!!}$ and therefore $\lambda_n \approx \frac{3N}{(2n+3)[(2n+1)!!]^2}(k_0 R)^{2n}$. Thus if $k_0 R << 1$ only one eigenvalue with $n = 0$ is large and approximately equal to N (Dicke superradiance [12]) while eigenvalues with $n > 0$ are suppressed by a factor $(k_0 R)^{2n}$. Those states are trapped. *But this set of eigenfunctions is NOT complete! Our goal is to find a complete set of solutions for (II.7).* For simplicity we will take $R = 1$.

B. Spherical and hyperspherical harmonics

In mathematics, the spherical harmonics are the angular portion of a set of solutions to Laplace's equation. Represented in a system of spherical coordinates, Laplace's spherical harmonics Y_ℓ^m (or $Y_{\ell m}$) are a specific set of spherical harmonics that forms an orthogonal system, first introduced by Pierre Simon de Laplace. Spherical harmonics are important in many theoretical and practical applications, particularly in the computation of atomic orbital electron configurations, representation of gravitational fields, geoids, and the magnetic fields of planetary bodies and stars, and characterization of the cosmic microwave background radiation. In 3D computer graphics, spherical harmonics play a special role in a wide variety of topics including indirect lighting (ambient occlusion, global illumination, precomputed radiance transfer, etc.)

and in recognition of 3D shapes (cf. Wikipedia). They are given by

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi},$$

where $P_n^m(\cos\theta)$ are associated Legendre polynomials. θ and ϕ represent colatitude and longitude, respectively. In particular, the colatitude θ, or polar angle, ranges from 0 at the North Pole to π at the South Pole, assuming the value of $\pi/2$ at the Equator, and the longitude ϕ, or azimuth, may assume all values with $0 \leq \phi < 2\pi$. When Laplace's equation is solved on the surface of the sphere, the periodic boundary conditions in ϕ, as well as regularity conditions at both the north and south poles, ensure that the degree ℓ and order m are integers that satisfy $\ell \geq 0$ and $|m| \leq \ell$. ℓ known as the orbital angular momentum quantum number, and m the magnetic quantum number.

Spherical harmonics satisfy some remarkable properties (cf. [3]). For example

- $$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} Y_\ell^m\, Y_{\ell'}^{m'*}\, d\Omega = \delta_{\ell\ell'}\,\delta_{mm'},$$

- $$\nabla^2 Y_\ell^m(\theta,\phi) = -\ell(\ell+1)Y_\ell^m(\theta,\phi),$$

  where

  $$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

- Spherical harmonics form a complete set of orthonormal functions and thus form an orthonormal basis of the Hilbert space of square-integrable functions. On the unit sphere, any square-integrable function can thus be expanded as a linear combination of these

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m \, Y_\ell^m(\theta,\phi).$$

This expansion holds in the sense of mean-square convergence in L^2 of the sphere which is to say that

$$\lim_{N\to\infty} \int_0^{2\pi} \int_0^{\pi} \left| f(\theta,\phi) - \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\theta,\phi) \right|^2 \sin\theta \, d\theta d\phi = 0.$$

Let us now define a point in $n = p + 2$ dimensional Euclidean space ($p = 1, 2, 3, \ldots$) by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_{p+2})$. We shall use hyperspherical polar coordinates $\mathbf{x} = (r, \theta_1, \theta_2, \ldots, \theta_p, \phi)$ defined by

$$x_1 = r\cos\theta_1,$$

$$x_2 = r\sin\theta_1\cos\theta_2,$$

$$x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3,$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$x_p = r\sin\theta_1\sin\theta_2\ldots\sin\theta_{p-1}\cos\theta_p,$$

$$x_{p+1} = r\sin\theta_1\sin\theta_2\ldots\sin\theta_p\cos\phi,$$

$$x_{p+2} = r\sin\theta_1\sin\theta_2\ldots\sin\theta_p\sin\phi,$$

where $r \geq 0$, $0 \leq \theta_j \leq \pi$ ($j = 1, 2, \ldots, p$), $0 \leq \phi \leq 2\pi$.

In these coordinates, the $(p+2)$-dimensional volume element is given by

$$dV = r^{p+1}(\sin\theta_1)^p(\sin\theta_2)^{p-1}\dots(\sin\theta_p)dr d\theta_1\dots d\theta_p d\phi$$

and the surface element $d\Omega$ becomes

$$d\Omega = (\sin\theta_1)^p(\sin\theta_2)^{p-1}\dots(\sin\theta_p)d\theta_1\dots d\theta_p d\phi.$$

Denote the components of unit-vector by $\xi = (\theta_1, \theta_2, \dots, \theta_p, \phi)$. Then the hyperspherical harmonics of degree ℓ are denoted by S_ℓ^m. Here $m = 1, 2, \dots, h(\ell, p)$ is the number of linearly independent surface harmonics of degree ℓ where $h(\ell, p) = (2\ell + p)\frac{(\ell+p-1)!}{p!\ell!}$. They satisfy

$$\int_\Omega S_\ell^m \, S_{\ell'}^{m'*} \, d\Omega = \delta_{\ell\ell'} \, \delta_{mm'}$$

and S_ℓ^m, $\ell = 0, 1, 2, \dots$, $m = 1, 2, \dots, h(\ell, p)$ form a complete orthonormal basis in S^{n-1}.

C. Prolate spheroidal wave function

The *prolate spheroidal wave functions* are extremely important tool for studying the *concentration problem* in communication theory. The technique used in [46, 47, 48, 49, 50] is a major motivation for our work in the next chapter. Prolate spheroidal wave functions are extensively studied in [15]. In this section for the sake of completeness we draw freely from [15, 46].

When c is real, the differential equation

$$(1-t^2)\frac{d^2u}{dt^2} - 2t\frac{du}{dt} + (\chi - c^2t^2)u = 0$$

has continuous solutions in the closed t interval $[-1, 1]$ only for certain discrete real positive values $0 < \chi_0(c) < \chi_1(c) < \chi_2(c) < \dots$ of the parameter χ. Corresponding to each eigenvalue $\chi_n(c)$, $n = 0, 1, 2, \ldots$ there is a unique solution $S_{0n}(c,t)$ such that $S_{0n}(c,0) = P_n(0)$ where $P_n(t)$ is the n-th Legendre polynomial. The functions $S_{0n}(c,t)$ are called angular prolate spheroidal functions. As shown in [15] prolate spheroidal functions are real for real t, are continuous functions of c for $c \geq 0$, and can be extended to entire functions of the complex variable t. They have many remarkable properties. They are orthogonal in $(-1,1)$ and $S_{0n}(c,t)$ has exactly n zeros in $(-1,1)$, reduces to $P_n(t)$ uniformly in $[-1,1]$ as $c \to 0$. For a fixed c, $S_{0n}(c,t)$ with $n = 0,1,2,\ldots$ are complete in $L^2(-1,1)$. They are even or odd according as n is even or odd, $n = 0,1,2,\ldots$. The eigenvalues $\chi_n(c)$ are continuous functions of c and $\chi_n(0) = n(n+1)$, $n = 0,1,2,\ldots$.

As mentioned in [46], a second set of solutions $R_{0n}^{(1)}(c,t)$, $n = 0,1,2,\ldots$, called radial prolate spheroidal functions, which differ from the angular functions only by a real scale factor,

$$R_{0n}^{(1)}(c,t) = k_n(c) S_{0n}(c,t),$$

are of use in many applications. They also satisfy the following equations

$$\frac{2c}{\pi}[R_{0n}^{(1)}(c,1)]^2 S_{0n}(c,t) = \int_{-1}^{1} \frac{\sin c(t-s)}{\pi(t-s)} S_{0n}(c,s)ds, \tag{II.8}$$

$$2i^n R_{0n}^{(1)}(c,1) S_{0n}(c,t) = \int_{-1}^{1} e^{icts} S_{0n}(c,s)ds. \tag{II.9}$$

$n = 0,1,2,\ldots$. These relations are valid for all t, real or complex. Equation (II.8) shows that when $|t| < 1$ and $\rho_c(\tau) = \frac{1}{2\pi}\int_{-c}^{c} e^{i\omega\tau}d\omega$, $S_{0n}(c,t)$ is a solution of the integral equation

$$\lambda f(t) = \int_{-1}^{1} \rho_c(t-s) f(s)ds, \tag{II.10}$$

corresponding to the eigenvalue

$$\lambda_n(c) = \frac{2c}{\pi}[R_{0n}^{(1)}(c,1)]^2, \quad n = 0,1,2,\ldots. \tag{II.11}$$

Remark II.1. A very important observation is made in [46] related to the dimension of eigenspace of (II.10). The completeness of S_{0n} in $L^2(-1,1)$ gives that the quantities given in (II.11) are the only eigenvalues of (II.10). If these quantities are distinct then the S_{0n} are (apart from multiplicative constants) the unique $L^2(-1,1)$ solutions of (II.10). If several of the quantities (II.11) are equal for different values of n, then linear combinations of the corresponding S_{0n} will also satisfy (II.10). Within the sense of this degeneracy, the S_{0n} are unique solutions of (II.10). It has been shown in [46] that this degeneracy does not occur.

From the equation $\rho_c(\tau) = \frac{1}{2\pi}\int_{-c}^{c} e^{i\omega\tau} d\omega$ and Bochner's theorem we observe the kernel of (II.10) is positive definite. The quantities (II.11) are therefore strictly positive. Then

$$||S_{0n}(c,t)||^2 = \int_{-1}^{1} [S_{0n}(c,t)]^2 dt.$$

Since all the eigenvalues are nonnegative, we finally define

$$\psi_n(c,t) = \sqrt{\lambda_n}\frac{S_{0n}(c,t)}{||S_{0n}(c,t)||}.$$

We will drop the parameter c whenever there is no confusion in doing so. In the next section we will prove some properties of $S_{0n}(t)$ and $\psi_n(t)$.

D. Properties of prolate spheroidal functions

Prolate Spheroidal Functions have many remarkable properties. In this section for the sake of completeness we draw freely from [46]. Define

$$\psi_n(t) = \sqrt{\lambda_n}\frac{S_{0n}(t)}{||S_{0n}(t)||}$$

where $S_{0n}(t)$ is angular prolate spheroidal function.

Lemma II.2.

$$\int_{-1}^{1} \psi_n(t)\psi_m(t)dt = \lambda_n \delta_{mn}.$$

Proof. We have

$$\psi_n(t) = \sqrt{\lambda_n}\frac{S_{0n}(t)}{||S_{0n}(t)||}. \tag{II.12}$$

where $S_{0n}(t)$ is angular prolate spheroidal function. It is known for real t that the $S_{0n}(t)$ are real. The $S_{0n}(t)$ are orthogonal in $(-1, 1)$ and complete in $L^2(-1, 1)$. Since $S_{0n}(t)$ are orthogonal in $(-1, 1)$ therefore

$$\int_{1}^{1} \frac{S_{0n}(t)}{||S_{0n}(t)||}\frac{S_{0m}(t)}{||S_{0n}(t)||}dt = \delta_{mn}.$$

Using (II.12) thus we obtain

$$\int_{-1}^{1} \frac{\psi_n(t)}{\sqrt{\lambda_n}}\frac{\psi_m(t)}{\sqrt{\lambda_m}}dt = \delta_{mn},$$

i.e.,

$$\int_{-1}^{1} \psi_n(t)\psi_m(t)dt = \sqrt{\lambda_n\lambda_m}\delta_{mn}.$$

Hence

$$\int_{-1}^{1} \psi_n(t)\psi_m(t)dx = \lambda_n\delta_{mn}.$$

□

Lemma II.3.

$$\int_{-\infty}^{\infty} \rho_c(t-u)\rho_c(u-s)du = \rho_c(t-s),$$

where

$$\rho_c(\tau) = \frac{1}{2\pi}\int_{-c}^{c} e^{i\omega\tau}d\omega.$$

Proof. Since $\rho_c(\tau)$ is even,

$$\rho_c(t-u) = \rho_c(u-t) = \frac{1}{2\pi}\int_{-c}^{c} e^{i\omega(u-t)}d\omega.$$

If we take $f(u) = \rho_c(u-t)$, then the Fourier transform is given by $F(\omega) = e^{-i\omega t}$, $-c < \omega < c$. Then taking $g(u) = \rho_c(u-s)$ we have $G(\omega) = e^{-i\omega s}$, $-c < \omega < c$. We use Parseval's Theorem to get

$$\langle f(u), g(u)\rangle = \frac{1}{2\pi}\langle F(\omega), G(\omega)\rangle.$$

i.e.,

$$\int_{-\infty}^{\infty} \rho_c(t-u)\rho_c(u-s)du = \frac{1}{2\pi}\int_{-c}^{c} e^{-i\omega t}e^{i\omega s}d\omega = \frac{1}{2\pi}\int_{-c}^{c} e^{i\omega(s-t)}d\omega = \rho_c(s-t) = \rho_c(t-s).$$

□

Lemma II.4.

$$\int_{-\infty}^{\infty} \psi_n(t)\psi_m(t)dt = \delta_{mn}.$$

Proof. We know that for all n, m that $\psi_n(t)$, $\psi_m(t)$ are solutions of

$$\lambda f(t) = \int_{-1}^{1} \rho_c(t-s)f(s)ds, \tag{II.13}$$

$|t| < 1$ where $\rho_c(\tau) = \frac{\sin(c\tau)}{\pi\tau} = \frac{1}{2\pi}\int_{-c}^{c} e^{i\omega\tau}d\omega$. Now using (II.13) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_n(t)\psi_m(t)dt &= \int_{-\infty}^{\infty}\left(\frac{1}{\lambda_n}\int_{-1}^{1}\rho_c(t-s)\psi_n(s)ds\right)\left(\frac{1}{\lambda_m}\int_{-1}^{1}\rho_c(t-u)\psi_m(u)du\right)dt \\ &= \frac{1}{\lambda_n\lambda_m}\int_{-1}^{1}du\int_{-1}^{1}\psi_n(s)\psi_m(u)ds\int_{-\infty}^{\infty}\rho_c(u-t)\rho_c(t-s)dt.\end{aligned}$$

Use Lemma II.3 to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_n(t)\psi_m(t)dt &= \frac{1}{\lambda_n\lambda_m}\int_{-1}^{1}du\int_{-1}^{1}\psi_n(s)\psi_m(u)\rho_c(u-s)ds \\ &= \frac{1}{\lambda_n\lambda_m}\int_{-1}^{1}\psi_m(u)du\int_{-1}^{1}\rho_c(u-s)\psi_n(s)ds.\end{aligned}$$

Finally using (II.13) twice we obtain

$$\int_{-\infty}^{\infty} \psi_n(t)\psi_m(t)dx = \delta_{mn}.$$

□

Lemma II.5. *$\psi_n(t)$ is even or odd according as n even or odd.*

Proof. We know from [26], $S_{mn}(\eta) = \begin{cases} (1-\eta^2)^{m/2}\sum_{k=0}^{\infty} C_{2k}^{mn}(1-\eta^2)^k & (n-m)-even \\ \\ \eta(1-\eta^2)^{m/2}\sum_{k=0}^{\infty} C_{2k}^{mn}(1-\eta^2)^k & (n-m)-odd \end{cases}$

where C_{2k}^{mn} are independent of η. In our case $m = 0$ and therefore

$$S_{0n}(t) = \begin{cases} \sum_{k=0}^{\infty} C_{2k}^{0n}(1-t^2)^k & n-even \\ \\ t\sum_{k=0}^{\infty} C_{2k}^{0n}(1-t^2)^k & n-odd \end{cases}$$

Clearly $S_{0n}(t)$ is even or odd if n is even or odd. Consequently $\psi_n(t)$ is even or odd according as n even or odd. □

Lemma II.6. *$\psi_n(x)$ has exactly n zeros in $(-1, 1)$.*

Proof. We know $\psi_n(x)$ satisfies

$$\frac{d}{dx}\left((1-x^2)\frac{d\psi_n(x)}{dx}\right) + (\chi_n - c^2x^2)\psi_n(x) = 0. \tag{II.14}$$

$\psi_n(x)$ reduces to $P_n(x)$ uniformly in $[-1, 1]$ as $c \to 0$ where $P_n(x)$ is the n-th degree Legendre polynomial. The eigenfunction $\psi_n(x)$ corresponding to the eigenvalue χ_n depends smoothly on parameter c. Suppose for $c = c_1$, $\psi_n(x)$ has k zeros in $(-1, 1)$ and for $c = c_2$, $\psi_n(x)$ has at least $(k+1)$ zeros in $(-1, 1)$. Without loss of generality let us assume $c_1 < c_2$. Since $\psi_n(x)$ depends smoothly on c there must exist $c_3 \in (c_1, c_2)$ such that for arbitrarily small $\epsilon > 0$ if $c \leq c_3 - \epsilon$, $\psi_n(x)$ has k zeros and if $c \geq c_3$ $\psi_n(x)$ has $(k + 1)$ zeros (at least).

When $c \leq c_3 - \epsilon$, let us assume the positions of zeros of $\psi_n(x)$ are $x_1(c), x_2(c), \ldots, x_k(c) \in (-1, 1)$. When $c = c_3$ we must have $x_{k+1}(c_3) \in (-1, 1)$ such that $\psi_n(x_{k+1}(c_3)) = 0$

(also, $x_{k+1}(c_3)$ is different from $x_1(c_3), x_2(c_3), \ldots, x_k(c_3)$). We claim $\psi_n'(x_j(c_3)) = 0$ for some $x_j(c_3) \in \{x_1(c_3), x_2(c_3), \ldots, x_{k+1}(c_3)\}$. [NOTE: $x_1(c_3), x_2(c_3), \ldots, x_{k+1}(c_3)$ are not necessarily ordered.]

To prove this, first of all, we observe since $\psi_n(x)$ when viewed as a complex function is entire therefore the zeros are isolated. If $\psi_n'(x_j(c_3)) \neq 0$ for all $j \in \{1, 2, \ldots, k+1\}$ then for all $x_j(c_3)$ there exists a neighborhood $(x_j(c_3) - \delta, x_j(c_3) + \delta)$ such that

$\psi_n(x) > 0$ or < 0 in $(x_j(c_3), x_j(c_3) + \delta)$

$\psi_n(x) < 0$ or > 0 in $(x_j(c_3) - \delta, x_j(c_3))$

This is possible as we are given $\psi_n(x_j(c_3)) = 0$, $j \in \{1, 2, \ldots, k+1\}$.

We can take δ sufficiently small so that no two consecutive neighborhoods overlap. If we change c_3 to $c_3 - \epsilon$, as $\psi_n(x)$ depends continuously on c, therefore in each of the intervals $(x_j(c_3) - \delta, x_j(c_3) + \delta)$ we still have $\psi_n(x) > 0$ for some x and $\psi_n(x) < 0$ for some other x in the interval. Since $\psi_n(x)$ is continuous therefore we must still have $(k + 1)$ zeros (at least) when c_3 changes to $c_3 - \epsilon$. But this contradicts that when $c \leq c_3 - \epsilon$, $\psi_n(x)$ has k zeros. Thus we must have $\psi_n'(x_j(c_3)) = 0$ for some $x_j(c_3) \in \{x_1(c_3), x_2(c_3), \ldots, x_{k+1}(c_3)\}$ and this proves the claim.

Now let us denote the $x_j(c_3)$ for which $\psi_n(x_j(c_3)) = 0$ and $\psi_n'(x_j(c_3)) = 0$ by $x_j(c_3) = u$. So for $c = c_3$ we have $u \in (-1, 1)$ such that

$$\psi_n(u) = \psi_n'(u) = 0. \tag{II.15}$$

Using (II.15) we obtain from (II.14) that $\psi_n''(u) = 0$. Repeated differentiation (which is possible since $\psi_n(x)$ is entire) shows if $\psi_n(u) = 0$, $u \in (-1, 1)$ then $\psi_n(x) \equiv 0$ in $(-1, 1)$ which is a contradiction. Hence we conclude the number of zeros of $\psi_n(x)$ is independent of the parameter c. Thus it is sufficient to consider the case when $c = 0$. In that case spheroidal wave function $(\psi_n(x))$ reduces to Legendre polynomials ($P_n(x)$). But it is well known that $P_n(x)$ has exactly n zeros in $(-1, 1)$. As a consequence $\psi_n(x)$ has exactly n zeros in $(-1, 1)$. □

The proof of the following two lemmas can be found in [46].

Lemma II.7. *There cannot be two distinct S_{0n} belonging to the same eigenvalue λ in (II.10) when $c > 0$.*

Lemma II.8. *Eigenvalues of*

$$\int_{-1}^{1} \rho_c(t-s)\psi_n(s)ds = \lambda_n \psi_n(t), \quad |t| < 1 \tag{II.16}$$

where $\rho_c(\tau) = \frac{\sin(c\tau)}{\pi\tau}$ are nondegenerate.

E. Similar problem in communication theory

In [49] the problem considered is to find eigenfunctions and eigenvalues of integral equation

$$\alpha_j \psi_j(x) = \int_{-1}^{1} e^{icxy} \psi_j(y)dy, \quad |x| \leq 1. \tag{II.17}$$

The eigenfunctions can be analytically continued throughout the complex plane. They possess many special properties that make them most useful for studying the bandlimited functions. The ψ_j are also the eigenfunctions of the integral equation

$$\lambda \psi(x) = \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)} \psi(y)dy, \tag{II.18}$$

and the relation between λ and α's is given by

$$\lambda = \frac{c}{2\pi}|\alpha|^2.$$

For the sake of completeness of this discussion we draw freely from [49]. Denote points in the Euclidean space of D dimension, $\mathbb{R}^D$, by vectors $\mathbf{x} = (x_1, x_2, \ldots, x_D)$.

A square-integrable function of D variables, $f(\mathbf{x})$, is said to be R-limited if it can be represented as a Fourier integral

$$f(\mathbf{x}) = (2\pi)^{-D} \int_R \exp(i\mathbf{x} \cdot \mathbf{y}) F(\mathbf{y}) d\mathbf{y}, \tag{II.19}$$

over the bounded region R. By Parseval's theorem

$$\int_{\mathbb{R}^D} |f(\mathbf{x})|^2 d\mathbf{x} = (2\pi)^{-D} \int_R |F(\mathbf{y})|^2 d\mathbf{y}, \tag{II.20}$$

whereas the energy of f in the bounded region S is

$$\begin{aligned} \int_S |f(\mathbf{z})|^2 d\mathbf{z} &= \int_S d\mathbf{z} (2\pi)^{-2D} \int_R d\mathbf{x} \exp(i\mathbf{z} \cdot \mathbf{x}) F(\mathbf{x}) \int_R d\mathbf{y} \exp(-i\mathbf{z} \cdot \mathbf{y}) \bar{F}(\mathbf{y}) \\ &= (2\pi)^{-D} \int_R d\mathbf{x} \int_R d\mathbf{y} K_S(\mathbf{x} - \mathbf{y}) F(\mathbf{x}) \bar{F}(\mathbf{y}), \end{aligned}$$

where

$$K_S(\mathbf{x} - \mathbf{y}) = (2\pi)^{-D} \int_S \exp(i\mathbf{z} \cdot (\mathbf{x} - \mathbf{y})) d\mathbf{z}. \tag{II.21}$$

The largest fraction of energy that an R-limited function can have in the region S is therefore the maximum value of the fraction

$$\int_R d\mathbf{x} \int_R d\mathbf{y} K_S(\mathbf{x} - \mathbf{y}) F(\mathbf{x}) \bar{F}(\mathbf{y}) / \int_R |F(\mathbf{y})|^2 d\mathbf{y}$$

taken over all functions F square-integrable through R. This maximum is the largest eigenvalue of the integral equation

$$\lambda \psi(\mathbf{x}) = \int_R K_S(\mathbf{x} - \mathbf{y}) \psi(\mathbf{y}) d\mathbf{y}, \tag{II.22}$$

where $\mathbf{x} \in R$. This is analogous to (II.18). The kernel (II.21) of (II.22) is positive definite, since

$$\int_R d\mathbf{x} \int_R d\mathbf{y} K_S(\mathbf{x} - \mathbf{y}) f(\mathbf{x}) \bar{f}(\mathbf{y}) = (2\pi)^{-D} \int_S d\mathbf{z} |\int_R d\mathbf{x} e^{i\mathbf{z} \cdot \mathbf{x}} f(\mathbf{x})|^2 > 0$$

whenever

$$\int_R |f(\mathbf{x})|^2 d\mathbf{x} > 0.$$

By well-known theorems the eigenvalues of (II.22) are real and positive and the eigenfunctions, orthogonal on R, are complete in the class of functions square-integrable in R.

We can extend the domain of definition of ψ by defining

$$\psi(\mathbf{x}) = \frac{1}{\lambda} \int_R K_S(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})d\mathbf{y}$$

where $\mathbf{x} \in \mathbb{R}^D$. Then for two different eigenvalues in (II.22) we have

$$\int_{E_D} \psi_i(\mathbf{x})\bar{\psi}_j(\mathbf{x})d\mathbf{x} = \frac{1}{\lambda_i\lambda_j} \int_R d\mathbf{x} \int_R d\mathbf{y}\psi_i(\mathbf{x})\bar{\psi}_j(\mathbf{y}) \int_{E_D} d\mathbf{z}K_S(\mathbf{z} - \mathbf{x})\bar{K}_S(\mathbf{z} - \mathbf{y}).$$

But

$$\int_{E_D} K_S(\mathbf{z} - \mathbf{x})\bar{K}_S(\mathbf{z} - \mathbf{y})d\mathbf{z} = \bar{K}_S(\mathbf{x} - \mathbf{y}).$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^D} \psi_i(\mathbf{x})\bar{\psi}_j(\mathbf{x})d\mathbf{x} &= \frac{1}{\lambda_i\lambda_j} \int_R d\mathbf{x}\psi_i(\mathbf{x}) \int_R d\mathbf{y}\bar{K}_S(\mathbf{x} - \mathbf{y})\bar{\psi}_j(\mathbf{y}) \\ &= \frac{1}{\lambda_i} \int_R d\mathbf{x}\psi_i(\mathbf{x})\bar{\psi}_j(\mathbf{x}).\end{aligned}$$

Thus the orthogonality of the ψ_i over R implies the orthogonality over $\mathbb{R}^D$ as well.

1. Symmetry considerations

We will be considering the solution of (II.22). Simplification occurs when the region R is symmetric, that is $\mathbf{x} \in R$ implies $-\mathbf{x} \in R$ and S is the scaled version of R. We write $S = cR$ where $\mathbf{x} \in cR$ if and only if $\mathbf{x}/c \in R$ with c a positive constant. We start off with

$$\alpha\psi(\mathbf{x}) = \int_R e^{ic\mathbf{x}\cdot\mathbf{y}}\psi(\mathbf{y})d\mathbf{y}, \tag{II.23}$$

where $\mathbf{x} \in R$. As shown in [49] the solution of (II.23) is completely equivalent to solution of (II.22) when R is symmetric. Also when R is symmetric then if $\psi(\mathbf{x})$ is a solution of (II.22) then so is $\psi(-\mathbf{x})$, so that both $\psi_e(\mathbf{x}) = \psi(\mathbf{x}) + \psi(-\mathbf{x})$ and $\psi_o(\mathbf{x}) = \psi(\mathbf{x}) - \psi(-\mathbf{x})$ are solutions as well. The eigenfunctions of (II.23) can be chosen either even or odd functions of $\mathbf{x}$. Also the eigenvalues of (II.23) associated with even eigenfunctions are real, the eigenvalues of (II.23) associated with odd eigenfunctions are pure imaginary.

2. The case $D = 2$, when R is a circle

We now treat in detail the equation

$$\alpha\psi(x_1, x_2) = \int_R e^{ic(x_1y_1+x_2y_2)}\psi(y_1, y_2)dy_1dy_2, \tag{II.24}$$

where R is the unit circle $y_1^2 + y_2^2 \le 1$. In polar coordinates (II.24) becomes

$$\alpha\psi(r, \theta) = \int_0^1 dr'r' \int_0^{2\pi} d\theta' e^{icrr'\cos(\theta-\theta')}\psi(r', \theta'). \tag{II.25}$$

But using the generating formula for Bessel functions

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{-\infty}^{\infty} J_n(x)z^n,$$

we obtain by taking $z = ie^{i\theta}$

$$e^{ix\cos\theta} = i^n \sum_{-\infty}^{\infty} J_n(x)e^{in\theta}.$$

So (II.25) becomes

$$\alpha\psi(r, \theta) = \sum_{-\infty}^{\infty} i^m e^{im\theta} \int_0^1 dr'r'J_m(crr') \int_0^{2\pi} d\theta' e^{-im\theta'}\psi(r', \theta'). \tag{II.26}$$

Eigenfunctions of (II.26) and their corresponding eigenvalues can be written as $\psi_{0,n}(r,\theta) = R_{0,n}(r)$ with $\alpha_{0,n} = 2\pi\beta_{0,n}$ and $\psi_{N,n}(r,\theta) = R_{N,n}(r)\cos N\theta$ or $\psi_{N,n}(r,\theta) = R_{N,n}(r)\sin N\theta$ with $\alpha_{N,n} = 2\pi i^N \beta_{N,n}$, $N = 1, 2, \ldots$, $n = 0, 1, 2, \ldots$ where

$$\beta R(r) = \int_0^1 J_N(crr')R(r')r'dr', \tag{II.27}$$

where $0 \le r \le 1$. With $\gamma = \sqrt{c}\beta$ and $\phi(r) = \sqrt{r}R(r)$, (II.27) becomes

$$\gamma\phi(r) = \int_0^1 J_N(crr')\sqrt{crr'}\phi(r')dr', \tag{II.28}$$

with $0 \le r \le 1$. Also we observe $\phi(0) = 0$.

Denote

$$K_N(x) = J_N(x)\sqrt{x}.$$

Let the operator M be defined by $[M\phi](x) = \int_0^1 K_N(cxy)\phi(y)dy$. Suppose

$$L_x = \frac{d}{dx}(1-x^2)\frac{d}{dx} + \left(\frac{\frac{1}{4} - N^2}{x^2} - c^2x^2\right)$$

and C is the class of functions square-integrable in $(0, 1)$ and twice differentiable there that vanish at the origin. Then it is shown in [49] that on C, the operators M and L commute. It follows that the solutions of

$$L_x\phi(x) = -\chi\phi(x)$$

in C are also solutions of (II.28). Solutions of

$$(1-x^2)\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} + \left(\frac{\frac{1}{4} - N^2}{x^2} - c^2x^2 + \chi\right)\phi = 0 \tag{II.29}$$

are explicitly studied in [49].

3. The case $D > 2$, when R is unit sphere

In treating this general case we first assume

$$D = p + 2$$

where $p = 1, 2, \ldots$. Let $\mathbf{x} = r\xi$ and $\mathbf{y} = r'\eta$ where ξ and η are unit vectors in $\mathbb{R}^{p+2}$. Therefore (II.23) becomes

$$\alpha\psi(r,\xi) = \int_0^1 dr' r'^{p+1} \int_\Omega e^{icrr'\xi\cdot\eta}\psi(r',\eta)d\Omega(\eta), \tag{II.30}$$

where Ω is the surface of the unit sphere in $\mathbb{R}^{p+2}$.

Theorem II.9. *(Funk-Hecke Theorem) Let $F(x)$ be a function of the real variable x which is continuous for $-1 \leq x \leq 1$ and let $S_n(\xi)$ be any surface harmonic of degree n. Then for any unit-vector η*

$$\int_{\Omega(\xi)} F(\xi,\eta)S_n(\xi)d\Omega(\xi) = \lambda_n S_n(\eta),$$

where the integral is taken over the whole area of the unit hypersphere Ω and where

$$\lambda_n = \frac{\omega'}{C_n^{p/2}(1)} \int_{-1}^1 F(x)C_n^{p/2}(x)(1-x^2)^{\frac{p}{2}-\frac{1}{2}}dx.$$

Here ω' denotes the total area of the unit-hypersphere in the $(p+1)$-dimensional space

$$\omega' = \frac{2\pi^{\frac{p}{2}+\frac{1}{2}}}{\Gamma(\frac{p}{2}+\frac{1}{2})},$$

and $C_n^\nu(x)$ is the Gegenbauer polynomial of degree n and order ν. □

Now let

$$h(N,p) = (2N+p)\frac{(N+p-1)!}{p!N!}$$

where $N = 0, 1, 2, \ldots$ and let $S_N^l(\xi)$, $l = 1, 2, \ldots, h(N,p)$, be a complete set of orthonormal surface harmonics of degree N. Then by Funk-Hecke theorem

$$\int_{\Omega} e^{icrr'\xi\cdot\eta} S_N^l(\eta) d\Omega(\eta) = H_N(crr') S_N^l(\xi), \tag{II.31}$$

where

$$H_N(crr') = \frac{2\pi^{(p+1)/2} N!(p-1)!}{\Gamma((p+1)/2)(N+p-1)!} \int_{-1}^{1} e^{icrr'u} C_N^{p/2}(u)(1-u^2)^{(p+1)/2} du \tag{II.32}$$

is independent of l. Expanding ψ in surface harmonics

$$\psi(r,\xi) = \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,p)} R_{N,l}(r) S_N^l(\xi)$$

we can obtain from (II.30) and (II.31)

$$\alpha_{N,l} R_{N,l}(r) = \int_0^1 dr' r'^{p+1} H_N(crr') R_{N,l}(r') \tag{II.33}$$

from which it is seen that $R_{N,l}(r)$ and α_{Nl} are independent of l. As shown in [49] $H_N(crr')$ can be simplified to

$$H_N(crr') = i^N (2\pi)^{1+p/2} J_{N+\frac{p}{2}}(crr')/(crr')^{p/2}.$$

The solution of (II.30) is thus given by

$$\psi_{N,l,n}(r,\xi) = R_{N,n}(r) S_N^l(\xi),$$

where $l = 1, 2, \ldots, h(N,p)$ and $\alpha_{N,n} = i^N (2\pi)^{1+p/2} \beta_{N,n}$ and $N, n = 0, 1, 2, \ldots$ where

$$\beta_{N,n} R_{N,n}(r) = \int_0^1 \frac{J_{N+\frac{p}{2}}(crr')}{(crr')^{p/2}} r'^{p+1} R_{N,n}(r') dr'. \tag{II.34}$$

Setting $\gamma = \beta c^{(p+1)/2}$ and $\phi = r^{(p+1)/2} R$, (II.34) becomes

$$\gamma\phi(r) = \int_0^1 J_{N+\frac{p}{2}}(crr') \sqrt{crr'} \phi(r') dr'. \tag{II.35}$$

But (II.35) is (II.28) with N replaced by $N + p/2$. Consequently we can completely solve (II.30). All these above computations are given in more details in [13] and [49].

CHAPTER III

SOLUTION AND GENERALIZATION OF SUPERRADIANCE PROBLEM

A. Introduction

Our main objective is to find a complete set of eigenfunctions for the problem

$$\alpha\psi(\mathbf{x}) = \int_{\mathbf{B}(0,1)} \frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|}\psi(\mathbf{y})d\mathbf{y}, \quad \mathbf{x},\mathbf{y} \in \mathbf{B}(0,1) \tag{III.1}$$

where $\mathbf{B}(0,1)$ denotes the unit ball in $\mathbb{R}^3$. As mentioned in the *Introduction* the idea here is to find a differential operator which commutes with the given integral operator and then to solve the eigenvalue problem for the differential operator.

We will first generalize this problem in such a way that the kernel of the integral equation takes the same form as the kernel of (III.1) in three dimensions (up to some constant multiple). We will then find a differential operator that commutes with the generalized integral operator. Restricting the differential operator in 3-dimension we will get the required operator that commutes with the integral operator in (III.1). Then we will solve the eigenvalue problem for the differential operator that corresponds to the 3-dimensional case. We will have a singular Sturm-Liouville problem and finally we will show that the solution indeed forms a complete orthogonal set. Throughout our work we will reserve the letter n for the dimension of the related space.

B. Generalization to dimensions ≥ 2

Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^n$ with $||\mathbf{x}|| = r$ and $||\mathbf{y}|| = r'$. Suppose the angle between two vectors be ϕ. That is $\mathbf{x}\cdot\mathbf{y} = ||\mathbf{x}||||\mathbf{y}||\cos\phi$. We define $\varpi = \sqrt{r^2 + r'^2 - 2rr'\cos\phi}$. Suppose J_ν be the Bessel function of order ν. First we will state some of the well known results for Bessel functions. Proofs of the following two theorems can be found in [59].

Theorem III.1. *(Lommel's expansion formula)*

$$\frac{J_\nu(\sqrt{(\zeta+h)})}{(\zeta+h)^{\frac{\nu}{2}}} = \sum_{p=0}^{\infty} \frac{(-\frac{1}{2}h)^p}{p!} \frac{J_{\nu+p}(\sqrt{\zeta})}{\zeta^{\frac{1}{2}(\nu+p)}}.$$

Theorem III.2. *For $Z \in \mathbb{C}$*

$$\frac{J_{\nu+p+q}(Z)}{Z^q} = \sum_{k=0}^{q} \frac{q!}{k!(q-k)!} \frac{\nu+p+2k}{2^q} \frac{\Gamma(\nu+p+k)}{\Gamma(\nu+p+q+k+1)} J_{\nu+p+2k}(Z).$$

Gegenbauer Polynomials: The Gegenbauer Polynomials $C_n^{(\alpha)}(x)$ can be defined as the coefficients of t^n in the expansion of $(1-2xt+t^2)^{-\alpha}$.
i.e.,

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x) t^n.$$

Gegenbauer Polynomials satisfy the following theorem.

Theorem III.3.

$$C_m^\nu(\cos\phi) = \sum_{k=0}^{\leq\frac{1}{2}m} \frac{(-1)^k 2^{m-2k}\Gamma(\nu+m-k)\cos^{m-2k}\phi}{(m-2k)!k!\Gamma(\nu)}.$$

Next we will state and prove the theorems which will be very relevant for our work. They are taken from [59].

Theorem III.4. *Suppose $Z, z \in \mathbb{C}$. Then*

$$J_0(\varpi) = \sum_{m=-\infty}^{\infty} J_m(r) J_m(r') e^{im\phi}.$$

Proof. We consider Parseval's integral formula

$$J_0(\varpi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\varpi\cos\theta} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\varpi\cos(\theta-\alpha)} d\theta$$

which is valid for all (complex) values of ϖ and α, the integrand being a periodic analytic function of θ with period 2π.

Suppose we choose α as a solution of the equations

$$\varpi \sin\alpha = r - r'\cos\phi$$

and

$$\varpi \cos\alpha = r'\sin\phi.$$

Then

$$\begin{aligned} J_0(\varpi) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(r-r'\cos\phi)\sin\theta + ir'\sin\phi\cos\theta} d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\sum_{m=-\infty}^{\infty} J_m(r)e^{im\theta}\} e^{ir'\sin(\phi-\theta)} d\theta \\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} J_m(r)\int_{-\pi}^{\pi} e^{im\theta + ir'\sin(\theta-\phi)} d\theta \\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} J_m(r)\int_{-\pi}^{\pi} e^{im(\theta+\phi) - ir'\sin\theta} d\theta \\ &= \sum_{m=-\infty}^{\infty} J_m(r)J_m(r')e^{im\phi}. \end{aligned}$$

The interchange of the order of summation and integration follows from the uniformity of convergence of the series. □

Theorem III.5. *Suppose $Z, z \in \mathbb{C}$. Then*

$$\frac{J_\nu(\varpi)}{\varpi^\nu} = 2^\nu\Gamma(\nu)\sum_{m=0}^{\infty}(\nu+m)\frac{J_{\nu+m}(Z)}{Z^\nu}\frac{J_{\nu+m}(z)}{z^\nu}C_m^\nu(\cos\phi).$$

Proof. In Theorem III.1 if we take $\zeta = Z^2 + z^2$ and $h = -2Zz\cos\phi$ then we have

$$\begin{aligned}\frac{J_\nu(\varpi)}{\varpi^\nu} &= \sum_{p=0}^{\infty} \frac{(Zz\cos\phi)^p}{p!} \frac{J_{\nu+p}(\sqrt{Z^2+z^2})}{(Z^2+z^2)^{\frac{1}{2}(\nu+p)}} \\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^q z^{p+2q}\cos^p\phi}{2^q p! q!} \frac{J_{\nu+p+q}(Z)}{Z^{\nu+q}},\end{aligned}$$

where in the last step we used Theorem III.1 with $\zeta = Z^2$ and $h = z^2$. Next, using Theorem III.2 we obtain

$$\frac{J_\nu(\varpi)}{\varpi^\nu} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{q} \frac{(-1)^q(\nu+p+2k)\Gamma(\nu+p+k)z^{p+2q}\cos^p\phi}{2^{2q}p!k!(q-k)!\Gamma(\nu+p+q+k+1)} \frac{J_{\nu+p+2k}(Z)}{Z^\nu}. \quad \text{(III.2)}$$

The triple series on the right hand side is absolutely convergent by comparison with

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{q} \left| \frac{\Gamma(\nu+p+k)z^{p+2q}Z^{\nu+p+2k}}{2^{p+2q+2k}p!k!(q-k)!\Gamma(\nu+p+2k)\Gamma(\nu+p+q+k+1)} \right|.$$

But for an absolutely convergent series we have

$$\sum_{q=0}^{\infty}\sum_{k=0}^{q} u_{k,q} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} u_{k,k+n}.$$

Hence from (III.2) we obtain

$$\begin{aligned}\frac{J_\nu(\varpi)}{\varpi^\nu} &= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{k+n}(\nu+p+2k)\Gamma(\nu+p+k)z^{p+2k+2n}\cos^p\phi}{2^{2k+2n}p!k!n!\Gamma(\nu+p+2k+n+1)} \frac{J_{\nu+p+2k}(Z)}{Z^\nu} \\ &= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k 2^{\nu+p}(\nu+p+2k)\Gamma(\nu+p+k)\cos^p\phi}{p!k!} \frac{J_{\nu+p+2k}(Z)}{Z^\nu} \frac{J_{\nu+p+2k}(z)}{z^\nu} \\ &= \sum_{k=0}^{\infty}\sum_{m=2k}^{\infty} \frac{(-1)^k 2^{\nu+m-2k}(\nu+m)\Gamma(\nu+m-k)\cos^{m-2k}\phi}{(m-2k)!k!} \frac{J_{\nu+m}(Z)}{Z^\nu} \frac{J_{\nu+m}(z)}{z^\nu} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\le\frac{1}{2}m} \frac{(-1)^k 2^{\nu+m-2k}(\nu+m)\Gamma(\nu+m-k)\cos^{m-2k}\phi}{(m-2k)!k!} \frac{J_{\nu+m}(Z)}{Z^\nu} \frac{J_{\nu+m}(z)}{z^\nu}.\end{aligned}$$

Applying Theorem III.3 we thus obtain

$$\frac{J_\nu(\varpi)}{\varpi^\nu} = 2^\nu \Gamma(\nu) \sum_{m=0}^{\infty} (\nu+m) \frac{J_{\nu+m}(Z)}{Z^\nu} \frac{J_{\nu+m}(z)}{z^\nu} C_m^\nu(\cos\phi). \tag{III.3}$$

This is valid for all values of Z, z and ϕ and for all ν with the exception of $0, -1, -2, \ldots$.

□

Corollary III.6.

$$\frac{\sin\varpi}{\varpi} = \pi \sum_{m=0}^{\infty} (m+\frac{1}{2}) \frac{J_{m+\frac{1}{2}}(Z)}{\sqrt{Z}} \frac{J_{m+\frac{1}{2}}(z)}{\sqrt{z}} P_m(\cos\phi),$$

where $P_m(x)$ is the Legendre polynomial of m-th order.

Proof. We take $\nu = \frac{1}{2}$ in Theorem III.5 and observe that $C_m^\nu(\cos\phi) = P_m(\cos\phi)$. □

Zonal spherical harmonics: In the mathematical study of rotational symmetry, the zonal spherical harmonics are special spherical harmonics that are invariant under the rotation through a particular fixed axis. The zonal spherical functions are a broad extension of the notion of zonal spherical harmonics to allow for a more general symmetry group. The zonal harmonics appear naturally as coefficients of the Poisson kernel for the unit ball in $\mathbb{R}^n$. For ξ and η unit vectors

$$\frac{1}{\omega_{n-1}} \frac{1-r^2}{|\xi - r\eta|^n} = \sum_{k=0}^{\infty} r^k Z_\xi^{(k)}(\eta),$$

where ω_{n-1} is the surface area of $(n-1)$ dimensional sphere. If we define $C_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ then $\omega_{n-1} = nC_nR^{n-1} = 2\frac{\pi^{n/2}}{\Gamma(n/2)}R^{n-1}$ where R is the radius of the hypersphere. For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ zonal spherical harmonics are related to Gegenbauer Polynomials as

$$Z_{\mathbf{x}}^{(l)}(\mathbf{y}) = c_{n,l} C_l^\alpha(\mathbf{x}\cdot\mathbf{y}), \tag{III.4}$$

where $\alpha = \frac{(n-2)}{2}$ and $c_{n,k} = \frac{1}{\omega_{n-1}} \frac{2k+n-2}{(n-2)}$. Zonal spherical harmonics satisfy a remarkable property which will be used in our later work.

Theorem III.7. *Let S_l^k be an arbitrary orthonormal basis of the space $\mathbf{H}_l$ of degree l spherical harmonics on the n-sphere. Then $Z_{\mathbf{x}}^{(l)}(\mathbf{y})$ of degree l zonal harmonic corresponding to unit vector $\mathbf{x}$ decomposes as*

$$Z_{\mathbf{x}}^{(l)}(\mathbf{y}) = \sum_{k=1}^{\dim(\mathbf{H}_l)} S_l^k(\mathbf{x}) S_l^{k^*}(\mathbf{y}).$$

Let us rewrite (III.3) as

$$\begin{aligned}
\frac{J_\nu(\varpi)}{\varpi^\nu} &= \frac{2^{3\nu}}{\pi^{2\nu}}\Gamma(\nu)\sum_{m=0}^{\infty}(\nu+m)\left(\frac{\pi}{2Z}\right)^\nu J_{\nu+m}(Z)\left(\frac{\pi}{2z}\right)^\nu J_{\nu+m}(z)C_m^\nu(\cos\phi) \\
&= \frac{2^{3\nu+1}}{\pi^{\nu-1}}\sum_{m=0}^{\infty} c_{n,k}\left(\frac{\pi}{2Z}\right)^\nu J_{\nu+m}(Z)\left(\frac{\pi}{2z}\right)^\nu J_{\nu+m}(z)C_m^\nu(\cos\phi), \qquad \text{(III.5)}
\end{aligned}$$

where $c_{n,k}$ is defined in (III.4) and $\nu = \frac{(n-2)}{2}$. Writing $\Delta_m(\nu, r) = \left(\frac{\pi}{2r}\right)^\nu J_{\nu+m}(r)$ we thus have

$$\frac{J_\nu(\varpi)}{\varpi^\nu} = \frac{2^{3\nu+1}}{\pi^{\nu-1}}\sum_{m=0}^{\infty} c_{n,k}\Delta_m(\nu, Z)\Delta_m(\nu, z)C_m^\nu(\cos\phi). \qquad \text{(III.6)}$$

Theorem III.8. *Suppose $\mathbf{x} = (r, \xi)$ and $\mathbf{y} = (r', \eta)$ are in $\mathbb{R}^2$ where ξ and η are angular parts of $\mathbf{x}$ and $\mathbf{y}$ respectively. Then*

$$\int_{S^1} J_0(c|\mathbf{x}-\mathbf{y}|)e^{ik\eta}d\eta = 2\pi J_k(cr)J_k(cr')e^{ik\xi}.$$

Proof. Applying Theorem III.4 we obtain

$$\begin{aligned}
\int_{S^1} J_0(c|\mathbf{x}-\mathbf{y}|)e^{ik\eta})d\eta &= \sum_{t=-\infty}^{\infty} J_t(cr)J_t(cr')e^{it\xi}\int_{\eta=0}^{2\pi} e^{i\eta(k-t)}d\eta \\
&= 2\pi\sum_{t=-\infty}^{\infty} J_t(cr)J_t(cr')e^{it\xi}\delta_{kt} \\
&= 2\pi J_k(cr)J_k(cr')e^{ik\xi}.
\end{aligned}$$

□

Theorem III.9. *Suppose $\mathbf{x} = (r, \xi)$ and $\mathbf{y} = (r', \eta)$ are in $\mathbb{R}^n$ where where ξ and η*

are angular parts of $\mathbf{x}$ *and* $\mathbf{y}$ *respectively and* $\nu = \frac{(n-2)}{2}$*. Then*

$$\int_{S^{n-1}} \frac{J_\nu(c|\mathbf{x}-\mathbf{y}|)}{(c|\mathbf{x}-\mathbf{y}|)^\nu} S_k^s(\eta)d\eta = \frac{2^{3\nu+1}}{\pi^{\nu-1}} \Delta_n(\nu, cr)\Delta_n(\nu, cr')S_k^s(\xi).$$

Proof. Applying Theorem III.5 we obtain

$$\int_{S^{n-1}} \frac{J_\nu(c|\mathbf{x}-\mathbf{y}|)}{(c|\mathbf{x}-\mathbf{y}|)^\nu} S_k^s(\eta)d\eta = \frac{2^{3\nu+1}}{\pi^{\nu-1}} \sum_{t=0}^{\infty} c_{t,m}\Delta_t(\nu, cr)\Delta_t(\nu, cr') \int_{S^{n-1}} C_t^\nu(\cos\phi)S_k^s(\eta)d\eta.$$

But

$$c_{t,m}C_t^\nu(\cos\phi) = \sum_{m=1}^{\dim(\mathbf{H}_t)} S_t^m(\xi)S_t^{m*}(\eta).$$

Hence

$$\int_{S^{n-1}} \frac{J_\nu(c|\mathbf{x}-\mathbf{y}|)}{(c|\mathbf{x}-\mathbf{y}|)^\nu} S_k^s(\eta)d\eta = \frac{2^{3\nu+1}}{\pi^{\nu-1}} \sum_{t=0}^{\infty} \sum_{m=1}^{\dim(\mathbf{H}_t)} \Delta_t(\nu, cr)\Delta_t(\nu, cr')S_t^m(\xi) \int_{S^{n-1}} S_t^{m*}(\eta)S_k^s(\eta)d\eta$$

$$= \frac{2^{3\nu+1}}{\pi^{\nu-1}} \sum_{t=0}^{\infty} \sum_{m=1}^{\dim(\mathbf{H}_t)} \Delta_t(\nu, cr)\Delta_t(\nu, cr')S_t^m(\xi)\delta_{tk}\delta_{ms} = \frac{2^{3\nu+1}}{\pi^{\nu-1}} \Delta_k(\nu, cr)\Delta_k(\nu, cr')S_k^s(\xi).$$

□

1. Dimension $n = 2$

Suppose $n = 2$ and $\mathbf{x} = (r, \xi)$ and $\mathbf{y} = (r', \eta)$ are in $\mathbb{R}^2$. Let $\mathbb{D}$ denote the unit disk in 2-dimension centered at the origin. We want to solve for $\mathbf{x}, \mathbf{y} \in \mathbb{D}$ the eigenvalue problem:

$$\alpha\psi(\mathbf{x}) = \int_{\mathbb{D}} J_0(c|\mathbf{x}-\mathbf{y}|)\psi(\mathbf{y})d\mathbf{y}. \tag{III.7}$$

We write

$$\psi(\mathbf{y}) = \psi(r', \eta) = \sum_{N=-\infty}^{\infty} R_N(r')e^{iN\eta}, \tag{III.8}$$

where $R_N(r')$ are to be determined. Hence (III.7) becomes

$$\begin{aligned}\alpha \sum_{N=-\infty}^{\infty} R_N(r)e^{iN\xi} &= \sum_{N=-\infty}^{\infty} \int_{\mathbb{D}} J_0(c|\mathbf{x}-\mathbf{y}|)R_N(r')e^{iN\eta}d\eta \\ &= \sum_{N=-\infty}^{\infty} \int_0^1 r'dr' \int_{S^1} J_0(c|\mathbf{x}-\mathbf{y}|)R_N(r')e^{iN\eta}d\eta) \\ &= 2\pi \sum_{N=-\infty}^{\infty} \int_0^1 r'dr' R_N(r')J_N(cr)J_N(cr')e^{iN\xi},\end{aligned}$$

where in the last step we used Theorem III.8. So it is sufficient to solve

$$\alpha_N R_N(r) = 2\pi \int_0^1 r' R_N(r')J_N(cr)J_N(cr')dr',$$

i.e.

$$\alpha'_N R_N(r) = \int_0^1 J_N(cr)J_N(cr')R_N(r')r'dr', \tag{III.9}$$

where $\alpha'_N = \frac{\alpha_N}{2\pi}$. Let

$$\phi_N(r) = r^{\frac{1}{2}} R_N(r). \tag{III.10}$$

Then (III.9) becomes

$$\gamma'_N \phi_N(r) = \int_0^1 J_N(cr)J_N(cr')c\sqrt{rr'}\phi_N(r')dr', \tag{III.11}$$

where $\gamma'_N = c\alpha'_N$. So the eigenfunctions of (III.7) are

$$\psi(\mathbf{x}) = \psi_{N,k}(r,\xi) = R_N(r)e^{iN\xi} \tag{III.12}$$

and eigenvalues are given by

$$\alpha_N = \frac{2\pi\gamma'_N}{c}, \tag{III.13}$$

$N, k = 0, 1, 2, \ldots$.

2. Dimension $n \geq 3$

Suppose $n = p + 2$ where $p = 1, 2, 3, \ldots$ and $\mathbf{x} = (r, \xi)$ and $\mathbf{y} = (r', \eta)$ are in $\mathbb{R}^n$. Let $\mathbf{B}(0,1)$ denote the unit ball in n dimensions. First we observe $p = n - 2$. But $\nu = \frac{(n-2)}{2}$. Hence

$$\nu = \frac{p}{2}. \tag{III.14}$$

We want to solve for $\mathbf{x}, \mathbf{y} \in \mathbf{B}(0,1)$ the eigenvalue problem:

$$\alpha\psi(\mathbf{x}) = \int_{\mathbf{B}(0,1)} \frac{J_\nu(c|\mathbf{x}-\mathbf{y}|)}{(c|\mathbf{x}-\mathbf{y}|)^\nu} \psi(\mathbf{y}) d\mathbf{y}. \tag{III.15}$$

It is known [13] that $\dim(\mathbf{H}_N) = h(N,p) = (2N+p)\frac{(N+p-1)!}{p!N!}$, $N = 0, 1, 2, \ldots$. Let $S_N^l(\xi)$, $l = 1, 2, \ldots, h(N,p)$ be a complete set of orthonormal surface harmonics of degree N. Then we can write

$$\psi(\mathbf{y}) = \psi(r', \eta) = \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,p)} R_{Nl}(r') S_N^l(\eta) \tag{III.16}$$

where $R_{Nl}(r')$ are to be determined. Hence (III.15) becomes

$$\begin{aligned}
\alpha \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,p)} R_{Nl}(r) S_N^l(\xi) &= \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,p)} \int_{\mathbf{B}(0,1)} \frac{J_\nu(c|\mathbf{x}-\mathbf{y}|)}{(c|\mathbf{x}-\mathbf{y}|)^\nu} R_{Nl}(r') S_N^l(\eta) d\mathbf{y} \\
&= \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,p)} \int_0^1 r'^{p+1} dr' \int_{S^{n-1}} \frac{J_\nu(c|\mathbf{x}-\mathbf{y}|)}{(c|\mathbf{x}-\mathbf{y}|)^\nu} R_{Nl}(r') S_N^l(\eta) d\eta \\
&= \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,p)} \int_0^1 r'^{p+1} dr' \frac{2^{3\nu+1}}{\pi^{\nu-1}} \Delta_N(\nu, cr) \Delta_N(\nu, cr') R_{Nl}(r') S_N^l(\xi),
\end{aligned}$$

where in the last step we used Theorem III.9. So, it is sufficient to solve

$$\alpha_{Nl} R_{Nl}(r) = \int_0^1 r'^{p+1} dr' \frac{2^{3\nu+1}}{\pi^{\nu-1}} \Delta_N(\nu, cr) \Delta_N(\nu, cr') R_{Nl}(r')$$

from which it is seen that $R_{Nl}(r)$ and α_{Nl} are independent of l.

The last equation can be written as

$$\alpha'_{Nl}R_{Nl}(r) = \int_0^1 \Delta_N(\nu, cr)\Delta_N(\nu, cr')R_{Nl}(r')r'^{p+1}dr'$$

where $\alpha'_{Nl} = \frac{\alpha_{Nl}\pi^{\nu-1}}{2^{3\nu+1}}$. The last equation can be written as

$$\alpha'_{Nl}R_{Nl}(r) = \int_0^1 \left(\frac{\pi}{2cr}\right)^\nu \left(\frac{\pi}{2cr'}\right)^\nu J_{N+\nu}(cr)J_{N+\nu}(cr')R_{Nl}(r')r'^{p+1}dr',$$

i.e.,

$$\beta'_{Nl}R_{Nl}(r) = \int_0^1 (rr')^{-\nu} J_{N+\nu}(cr)J_{N+\nu}(cr')R_{Nl}(r')r'^{p+1}dr', \qquad \text{(III.17)}$$

where $\beta'_{Nl} = \alpha'_{Nl}\frac{(2c)^{2\nu}}{\pi^{2\nu}}$. Let

$$\phi_{Nl}(r) = r^{\frac{p+1}{2}}R_{Nl}(r). \qquad \text{(III.18)}$$

Then (III.17) becomes (with the use of $\nu = \frac{p}{2}$)

$$\gamma'_{Nl}\phi_{Nl}(r) = \int_0^1 J_{N+\nu}(cr)J_{N+\nu}(cr')c\sqrt{rr'}\phi_{Nl}(r')dr'. \qquad \text{(III.19)}$$

where $\gamma'_{Nl} = c\beta'_{Nl}$. So the eigenfunctions and eigenvalues of (III.15) are

$$\psi(\mathbf{x}) = \psi_{N,l,k}(r, \xi) = R_{Nk}(r)S_N^l(\xi) \qquad \text{(III.20)}$$

and

$$\alpha_{Nk} = \frac{(2\pi)^{\nu+1}\gamma'_{Nk}}{c^{2\nu+1}} \qquad \text{(III.21)}$$

where $N, k = 0, 1, 2, \ldots$ and $l = 1, 2, \ldots, h(N, p)$ and γ'_{Nk} and $R_{Nk}(r)$ are given by (III.18)-(III.19).

Combination of results for $n = 2$ and $n \geq 3$: We observe (III.19) takes the form of (III.11) when $\nu = 0$. But $\nu = 0$ also corresponds to $n = 2$. Therefore from now on we will consider the equation (III.19) for $\nu = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ which corresponds to dimensions $n = 2, 3, 4, 5, \ldots$ respectively.

3. Commuting differential operator

Denote the kernel of (III.19) by $Ker(r,r') = cJ_{N+\nu}(cr)J_{N+\nu}(cr')\sqrt{rr'}$. Denote $K_N(x) = \sqrt{x}J_N(x)$ therefore

$$Ker(r,r') = K_{N+\nu}(cr)K_{N+\nu}(cr'). \tag{III.22}$$

We will now deduce some standard results

Lemma III.10.

$$\frac{d^2}{dr^2}\left(K_N(cr)\right) = -\left(c^2 + \frac{\frac{1}{4} - N^2}{r^2}\right)K_N(cr).$$

Proof. With $y = cr$, we have $J_N(cr) = J_N(y)$. Now $J_N(y)$ satisfies

$$y^2\frac{d^2J_N(y)}{dy^2} + y\frac{dJ_N(y)}{dy} + (y^2 - N^2)J_N(y) = 0. \tag{III.23}$$

But $\frac{dJ_N(y)}{dy} = \frac{1}{c}\frac{dJ_N(cr)}{dr}$ and $\frac{d^2J_N(y)}{dy^2} = \frac{1}{c^2}\frac{d^2J_N(cr)}{dr^2}$. Hence (III.23) gives

$$r^2\frac{d^2J_N(cr)}{dr^2} + r\frac{dJ_N(cr)}{dr} + (c^2r^2 - N^2)J_N(cr) = 0. \tag{III.24}$$

Now $J_N(cr) = \frac{K_N(cr)}{\sqrt{cr}}$. Substituting this in (III.24) we have

$$\frac{d^2}{dr^2}\left(K_N(cr)\right) = -\left(c^2 + \frac{\frac{1}{4} - N^2}{r^2}\right)K_N(cr).$$

□

Lemma III.11.

$$2rc\frac{K_N'(cr)}{K_N(cr)} = 1 + 2rc\frac{J_N'(cr)}{J_N(cr)}.$$

Proof.

$$K_N(cr) = \sqrt{cr}J_N(cr)$$

Differentiating with respect to r, we have

$$cK'_N(cr) = \frac{\sqrt{c}}{2\sqrt{r}} J_N(cr) + \sqrt{cr}\, c J'_N(cr).$$

So

$$2rc\frac{K'_N(cr)}{K_N(cr)} = 1 + 2rc\frac{J'_N(cr)}{J_N(cr)}.$$

□

Theorem III.12. *Suppose*

$$L_x \equiv \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

is a differential operator with $p(1) = 0$ *and* K *is an integral operator such that*

$$Kg = \int_0^1 Ker(x,y)g(y)dy$$

for $g \in \Lambda$ *where* Λ *is the class of functions square integrable in* $(0,1)$ *and twice differentiable there that vanish at the origin. Suppose* $p(x)$ *is twice differentiable. Also, suppose* $Ker(x,y)$ *is twice differentiable with respect to* x *and* y*. Assume that* $Ker(x,y) = Ker(y,x)$ *and* $Ker(0,y) = Ker(x,0) = 0$*. Then the necessary and sufficient condition for two operators* L *and* K *to commute on the functions in* Λ *is*

$$L_x Ker(x,y) = L_y Ker(x,y).$$

Proof. We have for $g \in \Lambda$

$$L_x[Kg] = L_x \int_0^1 Ker(x,y)g(y)dy = \int_0^1 g(y)\{L_x Ker(x,y)\}dy. \tag{III.25}$$

On the other hand

$$\begin{aligned}
K[L_x g] = \int_0^1 Ker(x,y)\{L_y g(y)\}dy &= \int_0^1 Ker(x,y)\left[\frac{\partial}{\partial y}\left(p(y)\frac{\partial}{\partial y}\right) + q(y)\right] g(y)dy \\
&= \int_0^1 Ker(x,y)\frac{\partial}{\partial y}\left(p(y)\frac{\partial g}{\partial y}\right)dy + \int_0^1 Ker(x,y)q(y)g(y)dy \\
&= \int_0^1 Ker(x,y)\frac{\partial}{\partial y}\left(p(y)\frac{\partial g}{\partial y}\right)dy + \int_0^1 Ker(x,y)q(y)g(y)dy \\
&= \left[Ker(x,y)p(y)\frac{\partial g}{\partial y}\right]_{y=0}^{y=1} - \int_0^1 \frac{\partial Ker(x,y)}{\partial y}p(y)\frac{dg}{dy}dy \\
&\qquad + \int_0^1 Ker(x,y)q(y)g(y)dy.
\end{aligned}$$

But the first term drops as $Ker(x,0) = 0$ and $p(1) = 0$. Therefore

$$\begin{aligned}
K[L_x g] &= -\int_0^1 \left[\frac{\partial Ker(x,y)}{\partial y}p(y)\right]\frac{dg}{dy}dy + \int_0^1 Ker(x,y)q(y)g(y)dy \\
&= -\left[\frac{\partial Ker(x,y)}{\partial y}p(y)g(y)\right]_{y=0}^{y=1} + \int_0^1 \frac{\partial}{\partial y}\left(p(y)\frac{\partial Ker(x,y)}{\partial y}\right)g(y)dy \\
&\qquad + \int_0^1 Ker(x,y)q(y)g(y).
\end{aligned}$$

Again, the first term drops as $p(1) = 0$ and for the class of functions we are considering we have $g(0) = 0$. Hence

$$K[L_x g] = \int_0^1 \left[\frac{\partial}{\partial y}\left(p(y)\frac{\partial Ker(x,y)}{\partial y}\right) + q(y)Ker(x,y)\right] g(y)dy.$$

That is,

$$K[L_x g] = \int_0^1 g(y)\{L_y Ker(x,y)\}dy. \tag{III.26}$$

Now (III.25) and (III.26) gives

$$L_x K g - K L_x g = \int_0^1 g(y)\left[L_x Ker(x,y) - L_y Ker(x,y)\right]dy, \tag{III.27}$$

for all $g \in \Lambda$. Hence the necessary and sufficient condition for two operators L and

K to commute is

$$L_x Ker(x,y) = L_y Ker(x,y).$$

$\square$

Now with $Ker(r,r')$ given in (III.22) we want to find a self-adjoint differential operator M_r such that

$$M_r Ker(r,r') = M_{r'} Ker(r,r').$$

Let $M_r = \frac{d}{dr}\left(p(r)\frac{d}{dr}\right) + q(r)$. Then

$$\begin{aligned} M_r Ker(r,r') = p'(r)\frac{d}{dr}\left(K_{N+\nu}(cr)\right)K_{N+\nu}(cr') + \\ p(r)\frac{d^2}{dr^2}\left(K_{N+\nu}(cr)\right)K_{N+\nu}(cr') + q(r)K_{N+\nu}(cr)K_{N+\nu}(cr'). \end{aligned} \tag{III.28}$$

Using Lemma III.10 we obtain from (III.28)

$$\begin{aligned} M_r Ker(r,r') = p'(r)\frac{d}{dr}\left(K_{N+\nu}(cr)\right)K_{N+\nu}(cr') \\ + \left(-p(r)\left(c^2 + \frac{\frac{1}{4}-(N+\nu)^2}{r^2}\right) + q(r)\right)K_{N+\nu}(cr)K_{N+\nu}(cr'). \end{aligned} \tag{III.29}$$

Now, let $q(r) = q_1(r) + q_2(r)$. Suppose we choose $p(r) = 1 - r^2$,

$$q_1(r) = -p'(r)\frac{\frac{d}{dr}\left(J_{N+\nu}(cr)\right)}{J_{N+\nu}(cr)} = -cp'(r)\frac{J'_{N+\nu}(cr)}{J_{N+\nu}(cr)}$$

and

$$q_2(r) = \left(\frac{\frac{1}{4}-(N+\nu)^2}{r^2} - c^2r^2\right).$$

Then

$$M_r Ker(r,r') = M_{r'} Ker(r,r').$$

Hence the solutions of

$$\frac{d}{dr}\left((1-r^2)\frac{d\phi(r)}{dr}\right)+\left(\frac{\frac{1}{4}-(N+\nu)^2}{r^2}-c^2r^2\right)\phi(r)+2rc\frac{J'_{N+\nu}(cr)}{J_{N+\nu}(cr)}\phi(r)=-\chi\phi(r), \tag{III.30}$$

with $\phi(0) = 0$, are the solutions of (III.19). Solutions of (III.30) gives $\phi_{Nl}(r)$ and then we can find $R_{Nl}(r)$ from (III.18) and hence we can find ψ of (III.16). Thus we solve (III.15).

4. Special case

When $n = 3$, then $p = 1$ and $\nu = \frac{1}{2}$.

Then

$$\frac{J_{\frac{1}{2}}(c|\mathbf{x}-\mathbf{y}|)}{\sqrt{c|\mathbf{x}-\mathbf{y}|}} = \sqrt{\frac{2}{\pi}}j_0(c|\mathbf{x}-\mathbf{y}|),$$

where $j_N(cr) = \sqrt{\frac{\pi}{2cr}}J_{N+\frac{1}{2}}(cr)$.

Therefore

$$\frac{J_{\frac{1}{2}}(c|\mathbf{x}-\mathbf{y}|)}{\sqrt{c|\mathbf{x}-\mathbf{y}|}} = \sqrt{\frac{2}{\pi}}\frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|}.$$

This is the kernel required for the study of three dimensional superradiance problem and the corresponding differential operator is identical to the operator we will obtain in next section. □

C. Commutative operator in three dimensional case

We will just restrict to three-dimensions our discussion of the previous section.

1. Finding the operator

To begin we reproduce (up to a constant multiple) Theorem III.9 for $n = 3$.

Lemma III.13.

$$\int_{\Omega} \frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|} Y_{ks}(\eta) d\Omega(\eta) = 4\pi j_k(cr) j_k(cr') Y_{ks}(\xi),$$

where $\mathbf{x} = (r, \theta, \phi)$ *and* $\mathbf{y} = (r', \theta', \phi')$ *are in* $\mathbb{R}^3$, $\xi = (\theta, \phi)$ *and* $\eta = (\theta', \phi')$ *and where* Ω *is the surface of the unit sphere in three dimensions and* $j_N(cr) = \sqrt{\frac{\pi}{2cr}} J_{N+\frac{1}{2}}(cr)$.

Proof. We use the expansion of $\frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|}$ from [2].

$$\begin{aligned}
\int_{\Omega} \frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|} Y_{ks}(\eta) d\Omega(\eta) &= \int_{\Omega} 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_n(cr) j_n(cr') Y_{nm}(\xi) Y_{nm}^*(\eta) Y_{ks}(\eta) d\Omega(\eta) \\
&= 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_n(cr) j_n(cr') Y_{nm}(\xi) \int_{\Omega} Y_{nm}^*(\eta) Y_{ks}(\eta) d\Omega(\eta) \\
&= 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_n(cr) j_n(cr') Y_{nm}(\xi) \delta_{nk} \delta_{ms} \\
&= 4\pi j_k(cr) j_k(cr') Y_{ks}(\xi).
\end{aligned}$$

□

Next, we shall solve for $\mathbf{x}, \mathbf{y} \in \mathbf{B}(0,1)$ the eigenvalue problem:

$$\alpha\psi(\mathbf{x}) = \int_{\mathbf{B}(0,1)} \frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|} \psi(\mathbf{y}) d\mathbf{y}. \tag{III.31}$$

Let $h(N,1) = 2N+1$, $N = 0, 1, 2, \ldots$ and $Y_{Nl}(\xi)$, $l = 1, 2, \ldots, h(N,1)$ be a complete set of orthonormal surface (here same as spherical) harmonics of degree N. Then we can write

$$\psi(\mathbf{y}) = \psi(r', \eta) = \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,1)} R_{Nl}(r') Y_{Nl}(\eta), \tag{III.32}$$

where $R_{Nl}(r')$ are to be determined. Hence (III.31) becomes

$$\begin{aligned}\alpha \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,1)} R_{Nl}(r) Y_{Nl}(\xi) &= \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,1)} \int_{\mathbf{B}(0,1)} \frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|} R_{Nl}(r') Y_{Nl}(\eta) d\mathbf{y} \\ &= \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,1)} \int_0^1 r'^2 dr' \int_{\Omega} \frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|} R_{Nl}(r') Y_{Nl}(\eta) d\Omega(\eta) \\ &= \sum_{N=0}^{\infty} \sum_{l=1}^{h(N,1)} \int_0^1 r'^2 dr' 4\pi j_N(cr') j_N(cr) R_{Nl}(r') Y_{Nl}(\xi).\end{aligned}$$

where in the last step we used Lemma III.13. So, it is sufficient to solve

$$\alpha_{Nl} R_{Nl}(r) = \int_0^1 r'^2 dr' 4\pi j_N(cr') j_N(cr) R_{Nl}(r')$$

from which it is seen that $R_{Nl}(r)$ and α_{Nl} are independent of l. The last equation can be written as

$$\alpha_{Nl} R_{Nl}(r) = \int_0^1 r'^2 dr' 4\pi \sqrt{\frac{\pi}{2cr'}} \sqrt{\frac{\pi}{2cr}} J_{N+\frac{1}{2}}(cr') J_{N+\frac{1}{2}}(cr) R_{Nl}(r'),$$

i.e.,

$$\beta_{Nl} R_{Nl}(r) = \int_0^1 r'^2 \frac{c}{\sqrt{rr'}} J_{N+\frac{1}{2}}(cr') J_{N+\frac{1}{2}}(cr) R_{Nl}(r') dr'. \tag{III.33}$$

where $\beta_{Nl} = \frac{c^2 \alpha_{Nl}}{2\pi^2}$. Let

$$\phi_{Nl}(r) = r R_{Nl}(r). \tag{III.34}$$

Then (III.33) becomes

$$\beta_{Nl} \phi_{Nl}(r) = \int_0^1 J_{N+\frac{1}{2}}(cr') J_{N+\frac{1}{2}}(cr) c\sqrt{rr'} \phi_{Nl}(r') dr'. \tag{III.35}$$

So the eigenfunctions and eigenvalues of (III.31) are

$$\psi(\mathbf{x}) = \psi_{N,l,k}(r,\xi) = R_{Nk}(r) Y_{Nl}(\xi) \tag{III.36}$$

and

$$\alpha_{Nk} = \frac{2\pi^2 \beta_{Nk}}{c^2}, \tag{III.37}$$

where $N, k = 0, 1, 2, \ldots$ and $l = 1, 2, \ldots, h(N, 1)$ and β_{Nk} and $R_{Nk}(r)$ are given by (III.33). Denote the kernel of (III.35) by $Ker(r, r') = cJ_{N+\frac{1}{2}}(cr')J_{N+\frac{1}{2}}(cr)\sqrt{r'r}$. Denote $K_N(x) = J_N(x)\sqrt{x}$. Therefore

$$Ker(r, r') = K_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr'). \tag{III.38}$$

Now, with $Ker(r, r')$ given in (III.38) we want to find a self-adjoint differential operator M_r such that

$$M_r Ker(r, r') = M_{r'} Ker(r, r').$$

Let $M_r = \frac{d}{dr}\left(p(r)\frac{d}{dr}\right) + q(r)$. Then

$$\begin{aligned} M_r Ker(r, r') = p'(r)\frac{d}{dr}\left(K_{N+\frac{1}{2}}(cr)\right)K_{N+\frac{1}{2}}(cr') + \\ p(r)\frac{d^2}{dr^2}\left(K_{N+\frac{1}{2}}(cr)\right)K_{N+\frac{1}{2}}(cr') + q(r)K_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr'). \end{aligned} \tag{III.39}$$

But it is known that

$$\frac{d^2}{dr^2}\left(K_{N+\frac{1}{2}}(cr)\right) = -\left(c^2 + \frac{\frac{1}{4} - (N + \frac{1}{2})^2}{r^2}\right)K_{N+\frac{1}{2}}(cr). \tag{III.40}$$

Using (III.40) we obtain from (III.39)

$$\begin{aligned} M_r Ker(r, r') = p'(r)\frac{d}{dr}\left(K_{N+\frac{1}{2}}(cr)\right)K_{N+\frac{1}{2}}(cr') \\ + \left(-p(r)\left(c^2 + \frac{\frac{1}{4} - (N + \frac{1}{2})^2}{r^2}\right) + q(r)\right)K_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr'). \end{aligned} \tag{III.41}$$

Now, let $q(r) = q_1(r) + q_2(r)$. Suppose we choose $p(r) = 1 - r^2$,

$$q_1(r) = -p'(r)\frac{\frac{d}{dr}\left(J_{N+\frac{1}{2}}(cr)\right)}{J_{N+\frac{1}{2}}(cr)} = -cp'(r)\frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)}$$

and

$$q_2(r) = \left(\frac{\frac{1}{4} - (N + \frac{1}{2})^2}{r^2} - c^2 r^2 \right).$$

Then (III.41) gives (by using Lemma III.11)

$$M_r Ker(r, r') = \left(-1 - c^2 + \frac{1}{4} - (N + \frac{1}{2})^2 \right) K_{N+\frac{1}{2}}(cr) K_{N+\frac{1}{2}}(cr').$$

Similarly,

$$M_{r'} Ker(r, r') = \left(-1 - c^2 + \frac{1}{4} - (N + \frac{1}{2})^2 \right) K_{N+\frac{1}{2}}(cr) K_{N+\frac{1}{2}}(cr').$$

Hence the solutions of

$$\frac{d}{dr}\left((1 - r^2) \frac{d\phi(r)}{dr} \right) + \left(\frac{\frac{1}{4} - (N + \frac{1}{2})^2}{r^2} - c^2 r^2 \right) \phi(r) + 2rc \frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)} \phi(r) = -\chi \phi(r). \tag{III.42}$$

with $\phi(0) = 0$, are the solutions of (III.35). Solutions of (III.42) give $\phi_{Nl}(r)$ and then we can find $R_{Nl}(r)$ from (III.34) and hence we can find ψ of (III.32). Thus we solve (III.31). For notational convenience we will now consider the problem

$$\frac{d}{dr}\left((1 - r^2) \frac{d\phi(r)}{dr} \right) + \left(\frac{\frac{1}{4} - N^2}{r^2} - c^2 r^2 \right) \phi(r) + 2rc \frac{J'_N(cr)}{J_N(cr)} \phi(r) = -\chi \phi(r), \tag{III.43}$$

with $\phi(0) = 0$.

Remark III.14. $r = 0$ is a regular singular point for (III.43) (also of (III.42)). To check this it is sufficient to show

$$\lim_{r \to 0} r^2 \left(2cr \frac{J'_N(cr)}{J_N(cr)} \right)$$

exists. But we know when r is near 0 we have

$$J_N(cr) \approx \frac{(cr)^N}{2^N \Gamma(N + 1)} \tag{III.44}$$

and differentiating with respect to r we have

$$cJ'_N(cr) \approx N\frac{c^N r^{N-1}}{2^N\Gamma(N+1)}. \tag{III.45}$$

Hence

$$\lim_{r\to 0} r^2\left(2cr\frac{J'_N(cr)}{J_N(cr)}\right) = 0$$

and thus the limit exists and hence $r = 0$ is a regular singular point.

Remark III.15. Let us now compute (for $r \in (0,1)$) the limit

$$\lim_{c\to 0} 2cr\frac{J'_N(cr)}{J_N(cr)}.$$

Since r is bounded therefore when c is tending to 0 using (III.44) and (III.45) we have

$$\lim_{c\to 0} 2cr\frac{J'_N(cr)}{J_N(cr)} = \lim_{c\to 0} 2r\frac{N\frac{c^{N+\frac{1}{2}}r^{N-\frac{1}{2}}}{2^N\Gamma(N+1)}}{\frac{(cr)^{N+\frac{1}{2}}}{2^N\Gamma(N+1)}} = 2N.$$

2. Solution of the eigenvalue problem for three dimensions

Eigenvalues and eigenfunctions of (III.43):

Case I: Consider first the case $c = 0$ in (III.43).

Then (III.43) becomes

$$\frac{d}{dr}\left((1-r^2)\frac{d\phi(r)}{dr}\right) + \left(\frac{\frac{1}{4}-N^2}{r^2}\right)\phi(r) + (2N+\chi)\,\phi(r) = 0. \tag{III.46}$$

Let $\tilde{\chi} = 2N + \chi$. Then (III.46) takes the form

$$\frac{d}{dr}\left((1-r^2)\frac{d\phi(r)}{dr}\right) + \left(\frac{\frac{1}{4}-N^2}{r^2} + \tilde{\chi}\right)\phi(r) = 0. \tag{III.47}$$

We substitute

$$\phi(r) = \sum_{j=0}^{\infty} a_{2j} r^{\alpha+2j}$$

into (III.47). Then we have $\alpha = \frac{1}{2} \pm N$. If $N \neq 0$, the negative sign leads to solutions having a singularity at $r = 0$. If $N = 0$, a second solution can be found, but it has a logarithmic singularity at $r = 0$. We must have therefore

$$\alpha = \frac{1}{2} + N$$

The coefficients are given by the recurrence relation

$$a_{2j+2} = a_{2j} \frac{(\alpha + 2j)(\alpha + 2j + 1) - \tilde{\chi}}{(\alpha + 2j + 2)(\alpha + 2j + 1) + \frac{1}{4} - N^2}.$$

Substituting the value of α thus we have

$$a_{2j+2} = a_{2j} \frac{(N + 2j + \frac{1}{2})(N + 2j + \frac{3}{2}) - \tilde{\chi}}{4(j+1)(N + j + 1)}. \tag{III.48}$$

For large j, $\frac{a_{2j+2}}{a_{2j}} \rightarrow 1$, so unless the series terminates, this solution becomes unbounded as $r \rightarrow 1$. Choosing $\tilde{\chi}$ to terminate the series at $r^{\alpha+2l}$, we have

$$\tilde{\chi} = \tilde{\chi}_{N,l}(0) = (N + 2l + \frac{1}{2})(N + 2l + \frac{3}{2}), \tag{III.49}$$

where $l = 0, 1, 2, \ldots$ and the series solution [i.e., eigenfunctions of (III.47)] becomes (choosing $a_0 = 1$):

$$\phi(r) = r^{N+\frac{1}{2}} R_{N,l}(r), \tag{III.50}$$

$$R_{N,l}(r) = F(-l, l + N + 1; N + 1; r^2), \tag{III.51}$$

where

$$F(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots$$

is the usual *Gaussian hypergeometric function*. Clearly from (III.49) we get the eigenvalues for (III.43) with $c = 0$ are given by

$$\chi = \chi_{N,l}(0) = (N + 2l + \frac{1}{2})(N + 2l + \frac{3}{2}) - 2N \tag{III.52}$$

where $l = 0, 1, 2, \ldots$.

Case II: We consider (III.43) with arbitrary values of c. We consider a series solution of (III.43) of the form

$$\phi(r) = \sum_{j=0}^{\infty} a_j r^{\alpha+j}, \tag{III.53}$$

where the index α and the coefficients a_j's are to be determined.

We write

$$2rc\frac{J'_N(cr)}{J_N(cr)} = \sum_{j=0}^{\infty} b_j r^j, \tag{III.54}$$

where b_j's are Taylor coefficients of the expansion. We can find

$$b_{2k+1} = 0, \tag{III.55}$$

$k = 0, 1, 2, 3, \ldots$ and

$$b_0 = 2N, \tag{III.56}$$

$$b_2 = \frac{-4}{N+1}\left(\frac{c}{2}\right)^2, \tag{III.57}$$

$$b_4 = \frac{-8}{2!(N+1)^2(N+2)}\left(\frac{c}{2}\right)^4, \tag{III.58}$$

$$b_6 = \frac{-48}{3!(N+1)^3(N+2)(N+3)}\left(\frac{c}{2}\right)^6, \tag{III.59}$$

$$b_8 = \frac{2(-240N-528)}{4!(N+1)^4(N+2)^2(N+3)(N+4)}\left(\frac{c}{2}\right)^8, \tag{III.60}$$

$\ldots$, etc.

In general, we have

$$b_{2k} = \frac{2}{k!g(N, 2k-1)}\left(\frac{c}{2}\right)^{2k}, \tag{III.61}$$

where $g(N, 2k-1)$ is a function of N of order $(2k-1)$ and $k = 0, 1, 2, 3, \ldots$. Then

$$2rc\frac{J'_N(cr)}{J_N(cr)}\phi(r) = \left(\sum_{j=0}^{\infty} b_j r^j\right)\left(\sum_{j=0}^{\infty} a_j r^{\alpha+j}\right) = \sum_{j=0}^{\infty} \gamma_j r^{\alpha+j}, \tag{III.62}$$

where

$$\gamma_j = \sum_{k=0}^{j} a_k b_{j-k}.$$

(Note: The coefficients b_{k-j} are known. Therefore γ_k is actually dependent on the set $\{a_0, a_1, a_2, \ldots, a_k\}$.)

We will now substitute (III.53) in the following equation (equivalent to (III.43)):

$$(1-r^2)\frac{d^2\phi(r)}{dr^2} - 2r\frac{d\phi(r)}{dr} + \left(\frac{\frac{1}{4}-N^2}{r^2} - c^2r^2 + 2rc\frac{J_N'(cr)}{J_N(cr)} + \chi\right)\phi(r) = 0. \quad \text{(III.63)}$$

We observe that

$$\begin{aligned}
&(1-r^2)\frac{d^2\phi(r)}{dr^2} - 2r\frac{d\phi(r)}{dr} \\
&\quad = \alpha(\alpha-1)a_0r^{\alpha-2} + \sum_{j=1}^{\infty} a_j(\alpha+j)(\alpha+j-1)r^{\alpha+j-2} - \alpha(\alpha-1)a_0r^{\alpha} \\
&\quad\quad - \sum_{j=1}^{\infty} a_j(\alpha+j)(\alpha+j-1)r^{\alpha+j} - 2\alpha a_0 r^{\alpha} - 2\sum_{j=1}^{\infty} a_j(\alpha+j)r^{\alpha+j} \\
&= \alpha(\alpha-1)a_0r^{\alpha-2} + (\alpha+1)\alpha a_1 r^{\alpha-1} + (\alpha+2)(\alpha+1)a_2r^{\alpha} + (\alpha+3)(\alpha+2)a_3r^{\alpha+1} \\
&\quad + \sum_{j=2}^{\infty}(\alpha+j+2)(\alpha+j+1)a_{j+2}r^{\alpha+j} - \alpha(\alpha-1)a_0r^{\alpha} - (\alpha+1)\alpha a_1 r^{\alpha+1} - \\
&\quad \sum_{j=2}^{\infty} a_j(\alpha+j)(\alpha+j-1)r^{\alpha+j} - 2\alpha a_0r^{\alpha} - 2(\alpha+1)a_1r^{\alpha+1} - 2\sum_{j=2}^{\infty} a_j(\alpha+j)r^{\alpha+j}.
\end{aligned}$$

(III.64)

Also using (III.53) and (III.54) we have

$$\left(\frac{\frac{1}{4}-N^2}{r^2} - c^2r^2 + 2rc\frac{J_N'(cr)}{J_N(cr)} + \chi\right)\phi(r)$$
$$= \left(\frac{1}{4}-N^2\right)\left[a_0r^{\alpha-2} + a_1r^{\alpha-1} + a_2r^{\alpha} + a_3r^{\alpha+1}\right] + \left(\frac{1}{4}-N^2\right)\sum_{j=2}^{\infty} a_{j+2}r^{\alpha+j}$$
$$- c^2\sum_{j=2}^{\infty} a_{j-2}r^{\alpha+j} + \gamma_0r^{\alpha} + \gamma_1r^{\alpha+1} + \sum_{j=2}^{\infty}\gamma_jr^{\alpha+j} + \chi(a_0r^{\alpha} + a_1r^{\alpha+1}) + \chi\sum_{j=2}^{\infty}a_jr^{\alpha+j}. \tag{III.65}$$

Substituting (III.64) and (III.65) into (III.63) and equating different coefficients of $r^{\alpha+j}$ we have:

- *coefficient of* $r^{\alpha-2}$:

  $\alpha(\alpha-1)a_0 + \left(\frac{1}{4}-N^2\right)a_0 = 0$. We take

  $$a_0 \neq 0. \tag{III.66}$$

  Hence using the same argument as in $c = 0$ case (i.e., Case I), we have

  $$\alpha = \frac{1}{2} + N. \tag{III.67}$$

- *coefficient of* $r^{\alpha-1}$:

  $\alpha(\alpha+1)a_1 + \left(\frac{1}{4}-N^2\right)a_1 = 0$ which gives (using the value of α in (III.67)):

  $$(2N+1)a_1 = 0.$$

  So we must have

  $$a_1 = 0. \tag{III.68}$$

- *coefficient of* r^{α}:

$$a_2(\alpha+2)(\alpha+1) - \alpha(\alpha-1)a_0 - 2\alpha a_0 + \left(\frac{1}{4} - N^2\right) a_2 + \gamma_0 + \chi a_0 = 0. \quad \text{(III.69)}$$

But using (III.56): $\gamma_0 = a_0 b_0 = 2N a_0$. Simplifying (III.69) using (III.67) we have

$$4(N+1)a_2 = (N^2 + \frac{3}{4} - \chi)a_0, \quad \text{(III.70)}$$

which is equivalent to

$$4(N+1)a_2 = a_0[\left(N + \frac{1}{2}\right)\left(N + \frac{3}{2}\right) - 2N - \chi]. \quad \text{(III.71)}$$

- *coefficient of* $r^{\alpha+1}$:

$a_3(\alpha+3)(\alpha+2) + a_1(\alpha+1)\alpha - 2a_1(\alpha+1) + a_3\left(\frac{1}{4} - N^2\right) + a_1\chi + \gamma_1 = 0$. Therefore using (III.68) we have

$$a_3[(\alpha+3)(\alpha+2) + \left(\frac{1}{4} - N^2\right)] = -\gamma_1. \quad \text{(III.72)}$$

But $\gamma_1 = a_0 b_1 + a_1 b_0 = a_0 \cdot 0 + 0 \cdot b_1 = 0$. Therefore (III.72) gives (with the value of α from (III.67)

$$a_3(6N+9) = 0$$

i.e.,

$$a_3 = 0. \quad \text{(III.73)}$$

- *coefficient of* $r^{\alpha+j}$*, where* $j = 2, 3, 4, \ldots$: $(\alpha+j+2)(\alpha+j+1)a_{j+2} - a_j(\alpha+j)(\alpha+j-1) - 2a_j(\alpha+j) + a_{j+2}\left(\frac{1}{4} - N^2\right) - c^2 a_{j-2}$
$+ a_j\chi + \gamma_j = 0$. But $\gamma_j = \sum_{k=0}^{j} a_k b_{j-k}$, and hence we have (with the value of α from (III.67)): $(j+2)(2N+j+2)a_{j+2} - a_j[(N+j+\frac{1}{2})(N+j+\frac{3}{2}) - \chi] - c^2 a_{j-2} + \sum_{k=0}^{j} a_k b_{j-k} = 0$.

which gives

$$(j+2)(2N+j+2)a_{j+2} = a_j\left[\left(N+j+\frac{1}{2}\right)\left(N+j+\frac{3}{2}\right) - \chi - b_0\right] - a_{j-1}b_1 + (c^2 - b_2)a_{j-2} - \sum_{k=0}^{j-3} a_k b_{j-k}. \quad \text{(III.74)}$$

We have now two cases :

Case I: (j in (III.74) is odd)

We already have found from (III.68) and (III.73): $a_1 = 0$ and $a_3 = 0$. Now, let $j = 3$ in (III.74). Then we can find (using $b_{2k+1} = 0$, for $k = 0, 1, 2, 3, \dots$)

$$a_5 = 0$$

Proceeding in this way and using (III.74) we can show

$$a_{2k+1} = 0, \quad \text{(III.75)}$$

$k = 0, 1, 2, 3, \dots$

Case II: (j in (III.74) is even)

Let $j = 2m$ (Note: since in (III.74) $j = 2, 3, 4, \dots.$ we have $m = 1, 2, 3, \dots.$) Then using the values of b_0 and b_1 and (from Case I) $a_{2m-1} = 0$ in (III.74) we have:

$(2m+2)(2N+2m+2)a_{2m+2} = a_{2m}[\left(N+2m+\frac{1}{2}\right)\left(N+2m+\frac{3}{2}\right) - \chi - 2N] + (c^2 + \frac{4}{N+1}\left(\frac{c}{2}\right)^2)a_{2m-2} - (b_4 a_{2m-4} + b_6 a_{2m-6} + \cdots + b_{2m-4}a_4 + b_{2m-2}a_2 + b_{2m}a_0)$,

Which is the same as: $(2m+2)(2N+2m+2)a_{2m+2} = a_{2m}[\left(N+2m+\frac{1}{2}\right)\left(N+2m+\frac{3}{2}\right) - \chi - 2N] + c^2\left(\frac{N+2}{N+1}\right)a_{2m-2} - (b_4 a_{2m-4} + b_6 a_{2m-6} + \cdots + b_{2m-4}a_4 + b_{2m-2}a_2 + b_{2m}a_0)$.

(III.76)

Remark III.16. Though (III.76) is valid for $m = 1, 2, 3, \ldots$ when $m = 0$, (III.76) gives

$$4(N+1)a_2 = a_0[\left(N+\frac{1}{2}\right)\left(N+\frac{3}{2}\right) - 2N - \chi]. \tag{III.77}$$

But (III.77) is identical to (III.71) that we obtained earlier. So (III.76) is actually valid for $m = 0, 1, 2, 3, 4, \ldots$...

Remark III.17. Let

$$\chi' = \chi + 2N. \tag{III.78}$$

Then first few relations using (III.76):

- When $m = 0$:

$$4(N+1)a_2 - \left(N+\frac{1}{2}\right)\left(N+\frac{3}{2}\right)a_0 = -\chi' a_0.$$

- When $m = 1$:

$$4\cdot 2(N+2)a_4 - \left(N+\frac{5}{2}\right)\left(N+\frac{7}{2}\right)a_2 - c^2\left(\frac{N+2}{N+1}\right)a_0 = -\chi' a_2.$$

- When $m = 2$:

$$4\cdot 3(N+3)a_6 - \left(N+\frac{9}{2}\right)\left(N+\frac{11}{2}\right)a_4 - c^2\left(\frac{N+2}{N+1}\right)a_2 + b_4 a_0 = -\chi' a_4.$$

- When $m = 3$:

$$4\cdot 4(N+4)a_8 - \left(N+\frac{13}{2}\right)\left(N+\frac{15}{2}\right)a_6 - c^2\left(\frac{N+2}{N+1}\right)a_4 + b_4 a_2 + b_6 a_0 = -\chi' a_6.$$

- When $m = 4$:

$$4\cdot 5(N+5)a_{10} - \left(N+\frac{17}{2}\right)\left(N+\frac{19}{2}\right)a_8 - c^2\left(\frac{N+2}{N+1}\right)a_6 + b_4 a_4 + b_6 a_2 + b_8 a_0 = -\chi' a_8.$$

- When $m = 5$:

$$4 \cdot 6(N+6)a_{12} - \left(N + \frac{21}{2}\right)\left(N + \frac{23}{2}\right)a_{10} - c^2\left(\frac{N+2}{N+1}\right)a_8 + b_4 a_6 + b_6 a_4 + b_8 a_2 + b_{10} a_0 = -\chi' a_{10}.$$

. . . etc.

So we have the equation

$$\mathbf{A} \cdot \mathbf{X} = -\chi' \mathbf{X}, \tag{III.79}$$

where

$$\mathbf{X} = \begin{pmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \\ a_8 \\ a_{10} \\ . \\ . \\ . \end{pmatrix}, \tag{III.80}$$

and

$$\mathbf{A} = \begin{pmatrix} f_1 & g_1 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \\ -b' & f_2 & g_2 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \\ b_4 & -b' & f_3 & g_3 & 0 & 0 & 0 & \dots & \dots & \dots \\ b_6 & b_4 & -b' & f_4 & g_4 & 0 & 0 & \dots & \dots & \dots \\ b_8 & b_6 & b_4 & -b' & f_5 & g_5 & 0 & \dots & \dots & \dots \\ b_{10} & b_8 & b_6 & b_4 & -b' & f_6 & g_6 & \dots & \dots & \dots \\ . & . & . & . & . & . & . & \dots & \dots & \dots \\ . & . & . & . & . & . & . & \dots & \dots & \dots \\ . & . & . & . & . & . & . & \dots & \dots & \dots \end{pmatrix}, \tag{III.81}$$

where,

$$f_j = -\left(N + \frac{4j-3}{2}\right)\left(N + \frac{4j-1}{2}\right), \quad j = 1, 2, 3, 4, \dots \tag{III.82}$$

and

$$g_j = 4j(N + j), \quad j = 1, 2, 3, 4, \dots \tag{III.83}$$

and

$$b' = c^2\left(\frac{N+2}{N+1}\right), \tag{III.84}$$

and we can find b_j's from (III.54)-(III.61). *Eigenvalues* $-\chi'$ *of (III.79) should give bounded* $\phi(r)$ *as* $r \to 1$. *i.e.,* $\sum_{j=0}^{\infty} a_{2j} < \infty$.

Approximation of eigenvalues:

We will now approximate eigenvalues $-\chi'$ of (III.79) for large N and small c so that $|b_4| << 1$. In that case (III.79) approximately yields

$$-b'a_{2j-2} + (f_j + \chi')a_{2j} + g_j a_{2j+2} = 0, \qquad \text{(III.85)}$$

i.e.,

$$-c^2\left(\frac{N+2}{N+1}\right)a_{2j-2} - \left(\left(N+\frac{4j-3}{2}\right)\left(N+\frac{4j-1}{2}\right)-\chi'\right)a_{2j} + 4j(N+j)a_{2j+2} = 0, \qquad \text{(III.86)}$$

$j = 0, 1, 2, 3, \ldots$ with the notation $a_{-2} = 0$.

This recursion formula for the expansion coefficients constitutes a linear homogeneous difference equation of the second order. A second order difference equation corresponds to a differential equation of second order, so that there are two non-trivial independent solutions (cf. [28]). Examination of (III.86) reveals in fact that as r approaches infinity, either $\frac{a_{2j}}{a_{2j-2}}$ increases as $\frac{4j^2}{c^2}\left(\frac{N+1}{N+2}\right)$, or goes to zero as $-\frac{c^2}{4j^2}\left(\frac{N+2}{N+1}\right)$. Of these two solutions the former leads to a divergent series. Hence we choose the latter. The condition that the limit of $\frac{a_{2j}}{a_{2j-2}}$ be zero as j becomes infinity enables us to obtain a transcendental equation in χ'. We will develop this idea now.

Suppose

$$N_j = \frac{1}{c^2}\left(\frac{N+1}{N+2}\right)\frac{a_{2j}}{a_{2j-2}}, \quad j = 1, 2, 3, \ldots. \qquad \text{(III.87)}$$

Since we want a convergent series therefore we must have $N_j \to 0$ as $j \to \infty$. Suppose $\gamma_j = \left(N + \frac{4j-3}{2}\right)\left(N + \frac{4j-1}{2}\right)$ and $\beta_j = 4c^2\left(\frac{N+2}{N+1}\right)j(N+j)$. Hence (III.85) becomes

$$N_{j+1} = \beta_j(\gamma_j - \chi') - \frac{\beta_j}{N_j} \qquad \text{(III.88)}$$

and reciprocally

$$N_j = \frac{\beta_j}{\beta_j(\gamma_j - \chi') - N_{j+1}}, \quad j = 1, 2, 3, \ldots \qquad \text{(III.89)}$$

and

$$N_1 = \frac{1}{c^2}\left(\frac{N+2}{N+1}\right)\frac{a_2}{a_0} = \frac{1}{c^2}\left(\frac{N+2}{N+1}\right)\frac{\left(N+\frac{1}{2}\right)\left(N+\frac{3}{2}\right)-\chi'}{4(N+1)}.$$

By iteration of (III.89) we get from the condition that $\lim_{j\to\infty} N_j = 0$, the convergent infinite continued fraction

$$N_{j+1} = \cfrac{\beta_{j+1}}{\beta_{j+1}(\gamma_{j+1}-\chi') - \cfrac{\beta_{j+2}}{\beta_{j+2}(\gamma_{j+2}-\chi') - \cfrac{\beta_{j+3}}{\beta_{j+3}(\gamma_{j+3}-\chi') - \ldots}}}. \qquad \text{(III.90)}$$

On the other hand, by iteration of (III.88) we obtain, from the condition that $a_{2j} = 0$ for $j < 0$, the finite continued fraction

$$N_{j+1} = \beta_j\gamma_j - \beta_j\chi' - \cfrac{\beta_j}{\beta_{j-1}(\gamma_{j-1}-\chi') - \cfrac{\beta_{j-1}}{\beta_{j-2}(\gamma_{j-2}-\chi') - \ldots}}, \qquad \text{(III.91)}$$

where the last partial denominator is $\beta_1(\gamma_1 - \chi')$. Equating two continued fractions of (III.90) and (III.91) we obtain a transcendental equation, the roots of which give the required χ'. That is we have

$$\chi' = \gamma_j - \cfrac{1}{\beta_{j-1}(\gamma_{j-1}-\chi') - \cfrac{\beta_{j-1}}{\beta_{j-2}(\gamma_{j-2}-\chi') - \ldots}}$$

$$- \left(\frac{\beta_{j+1}}{\beta_j}\right) \cfrac{1}{\beta_{j+1}(\gamma_{j+1}-\chi') - \cfrac{\beta_{j+2}}{\beta_{j+2}(\gamma_{j+2}-\chi') - \cfrac{\beta_{j+3}}{\beta_{j+3}(\gamma_{j+3}-\chi') - \ldots}}}. \qquad \text{(III.92)}$$

Once we find χ' from (III.92) we can solve the system of equations (III.79) to get the coefficients a_{2j}, $j = 0, 1, 2, 3, \ldots$. Hence we can approximate the solution of (III.43) for arbitrary values of c by

$$\phi(r) = k \sum_{j=0}^{\infty} a_{2j} r^{N+\frac{1}{2}+2j}$$

(where k is a constant and we can set $k = 1$ for simplicity), where the coefficients a_{2j} 's are as described above. Also, once we find χ' from (III.79) we can compute the eigenvalues χ of (III.43) from the relation (III.78).

3. Comparison with other works

As mentioned in Svidzinsky, Chang and Scully's papers [42, 53] a solution of (III.31) is given by $\psi(\mathbf{x}) = j_n(cr)Y_{nm}(\xi)$. That is $\phi(r) = rj_n(cr)$ is a solution of (III.35). But

$$\phi(r) = rj_n(cr) = \frac{1}{c}\sqrt{\frac{\pi}{2}} K_{N+\frac{1}{2}}(cr).$$

Plugging this to the LHS of (III.42) we have

$$\frac{d}{dr}\left((1-r^2)\frac{d\phi(r)}{dr}\right) + \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right)\phi(r) + 2rc\frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)}\phi(r)$$
$$= (-c^2 - N^2 - N - 1)\frac{1}{c}\sqrt{\frac{\pi}{2}} K_{N+\frac{1}{2}}(cr)$$
$$= (-c^2 - N^2 - N - 1)\phi(r).$$

This shows $\phi(r) = rj_n(cr)$ actually satisfies (III.42) with $\chi = c^2 + N^2 + N + 1$.

Another way of verification is as follows:

We can write

$$\phi(r) = rj_N(cr) = \sqrt{\frac{\pi}{2}}\{\frac{1}{0!\Gamma(N+\frac{3}{2})}\left(\frac{c}{2}\right)^N r^{N+1} - \frac{1}{1!\Gamma(N+\frac{5}{2})}\left(\frac{c}{2}\right)^{N+2} r^{N+3} + \frac{1}{2!\Gamma(N+\frac{7}{2})}\left(\frac{c}{2}\right)^{N+4} r^{N+5} - \frac{1}{3!\Gamma(N+\frac{9}{2})}\left(\frac{c}{2}\right)^{N+6} r^{N+7} + \frac{1}{4!\Gamma(N+\frac{11}{2})}\left(\frac{c}{2}\right)^{N+8} r^{N+9} - \ldots\}.$$

Therefore $a_0 = \sqrt{\frac{\pi}{2}}\frac{1}{0!\Gamma(N+\frac{3}{2})}\left(\frac{c}{2}\right)^N$, $a_2 = -\sqrt{\frac{\pi}{2}}\frac{1}{1!\Gamma(N+\frac{5}{2})}\left(\frac{c}{2}\right)^{N+2}$, etc ...
Replacing N by $(N+\frac{1}{2})$ in the equations (III.53)-(III.84) we have $\chi' = \chi + 2\left(N+\frac{1}{2}\right) = \chi + 2N + 1$. Now with $\chi = c^2 + N^2 + N + 1$, we obtain $\chi' = (N+1)(N+2) + c^2$. We can also show that with this value of χ' the coefficients a_0, a_2, a_4 ...etc. satisfies (III.79) [with N replaced by $(N+\frac{1}{2})$]. Thus $\phi(r) = rj_n(cr)$ actually satisfies (III.42).

4. Alternative set of complete eigenfunctions for (III.35)

Theorem III.18. *$rj_N(cA_kr)$ is a complete orthogonal basis of eigenfunctions of (III.35) in $L^2(0,1)$, where A_k are the roots of the equation $A_k = \nu\frac{j_N(cA_k)}{j_{N-1}(cA_k)}$ and $\nu = \frac{N+\frac{1}{2}-\gamma}{c}$ where $\gamma > -\left(N+\frac{1}{2}\right)$.*

Proof. Using the relation

$$\int r^2 j_n(ar)j_n(r)dr = \frac{r^2}{1-a^2}\left[aj_n(r)j_{n-1}(ar) - j_{n-1}(r)j_n(ar)\right]$$

it is easy to prove the orthogonality of $rj_N(cA_kr)$. Therefore we will now show that $rj_N(cA_kr)$ are actually complete in $L^2(0,1)$.
We observe

$$rj_N(cr) = \sqrt{\frac{\pi}{2cr}}rJ_{N+\frac{1}{2}}(cr) = \lambda\sqrt{r}J_{N+\frac{1}{2}}(cr),$$

where $\lambda = \sqrt{\frac{\pi}{2c}}$.

Differentiating with respect to r we obtain

$$j_N(cr) + crj_N'(cr) = \lambda \left[\frac{1}{2\sqrt{r}} J_{N+\frac{1}{2}}(cr) + c\sqrt{r} J_{N+\frac{1}{2}}'(cr) \right].$$

Divide LHS by $rcj_N(cr)$ and RHS by its equivalent $c\lambda\sqrt{r}J_{N+\frac{1}{2}}(cr)$ to get

$$\frac{1}{2cr} + \frac{j_N'(cr)}{j_N(cr)} = \frac{J_{N+\frac{1}{2}}'(cr)}{J_{N+\frac{1}{2}}(cr)}. \tag{III.93}$$

We know the identity

$$\frac{N+1}{cr} j_N(cr) + j_N'(cr) = j_{N-1}(cr).$$

This gives

$$\frac{j_N'(cr)}{j_N(cr)} = \frac{j_{N-1}(cr)}{j_N(cr)} - \frac{N+1}{cr}. \tag{III.94}$$

Using (III.94) we obtain from (III.93)

$$\frac{j_{N-1}(cr)}{j_N(cr)} - \left(\frac{1}{2} + N\right)\frac{1}{cr} = \frac{J_{N+\frac{1}{2}}'(cr)}{J_{N+\frac{1}{2}}(cr)}. \tag{III.95}$$

Suppose

$$x = cr$$

Then we have

$$\frac{j_{N-1}(x)}{j_N(x)} - \left(\frac{1}{2} + N\right)\frac{1}{x} = \frac{J_{N+\frac{1}{2}}'(x)}{J_{N+\frac{1}{2}}(x)}. \tag{III.96}$$

Let us now denote the solution of

$$\gamma J_{N+\frac{1}{2}}(x) + xJ_{N+\frac{1}{2}}'(x) = 0 \tag{III.97}$$

by B_k, where we choose $\gamma > -\left(N + \frac{1}{2}\right)$.

Hence (III.97) gives

$$\frac{J_{N+\frac{1}{2}}'(B_k)}{J_{N+\frac{1}{2}}(B_k)} = -\frac{\gamma}{B_k}.$$

Then we must have from (III.96) that B_k satisfies

$$\frac{j_{N-1}(B_k)}{j_N(B_k)} - \left(N + \frac{1}{2}\right)\frac{1}{B_k} = -\frac{\gamma}{B_k},$$

i.e.,

$$\frac{j_{N-1}(B_k)}{j_N(B_k)} = \frac{N + \frac{1}{2} - \gamma}{B_k},$$

i.e,

$$B_k = \left(N + \frac{1}{2} - \gamma\right)\frac{j_N(B_k)}{j_{N-1}(B_k)}.$$

Suppose that when $x = B_k$ we have $r = A_k$. That is $cA_k = B_k$.

Then we have

$$A_k = \frac{N + \frac{1}{2} - \gamma}{c}\frac{j_N(cA_k)}{j_{N-1}(cA_k)},$$

i.e.,

$$A_k = \nu\frac{j_N(cA_k)}{j_{N-1}(cA_k)}, \tag{III.98}$$

where $\nu = \frac{N+\frac{1}{2}-\gamma}{c}$. Hence the zeros A_k of (III.98) are zeros cA_k of (III.97). But we know from [16] that with $cA_k > 0$ satisfying (III.97), $J_{N+\frac{1}{2}}(cA_k r)$ is an orthogonal basis for $L^2_w(0,1)$ where $w(r) = r$.

i.e., $\sqrt{r}J_{N+\frac{1}{2}}(cA_k r)$ is an orthogonal basis for $L^2(0,1)$.

i.e., $rj_N(cA_k r)$ is an orthogonal basis of $L^2(0,1)$. □

Now we will show that the set of eigenfunctions obtained in the previous theorem is different than the set of eigenfunctions obtained in our earlier work.

Theorem III.19. *The only $rj_N(cAr)$ that satisfies (III.42) is when $A = 1$.*

Proof. We have

$$rj_N(Acr) = \frac{1}{Ac}\sqrt{\frac{\pi}{2}}K_{N+\frac{1}{2}}(Acr).$$

Let $\phi(r) = \frac{1}{Ac}\sqrt{\frac{\pi}{2}}K_{N+\frac{1}{2}}(Acr)$ and we substitute this to

$$\frac{d}{dr}\left((1-r^2)\frac{d\phi(r)}{dr}\right)+\left(\frac{\frac{1}{4}-N^2}{r^2}-c^2r^2\right)\phi(r)+2rc\frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)}\phi(r) = -\chi\phi(r). \quad \text{(III.99)}$$

Then the LHS of (III.99) becomes (dropping the constant $\frac{1}{Ac}\sqrt{\frac{\pi}{2}}$)

$$\begin{aligned} LHS = A^2c^2r^2K_{N+\frac{1}{2}}(Acr) + \left(\frac{1}{4} - (N+\frac{1}{2})^2\right)K_{N+\frac{1}{2}}(Acr) \\ - A^2c^2K_{N+\frac{1}{2}}(Acr) - 2AcrK'_{N+\frac{1}{2}}(Acr) - c^2r^2K_{N+\frac{1}{2}}(Acr) \\ + 2rc\frac{K'_{N+\frac{1}{2}}(Acr)}{K_{N+\frac{1}{2}}(Acr)}K_{N+\frac{1}{2}}(Acr) - K_{N+\frac{1}{2}}(Acr). \end{aligned}$$

Thus in order to have (III.99) satisfied for some constant χ we must have

$$\begin{aligned} (A^2-1)c^2r^2K_{N+\frac{1}{2}}(Acr) - 2AcrK'_{N+\frac{1}{2}}(Acr) + 2rc\frac{K'_{N+\frac{1}{2}}(Acr)}{K_{N+\frac{1}{2}}(Acr)}K_{N+\frac{1}{2}}(Acr) \\ = \lambda_1 K_{N+\frac{1}{2}}(Acr), \end{aligned}$$

for some constant λ_1. Hence we must have

$$(A^2-1)c^2r^2 - 2Arc\frac{K'_{N+\frac{1}{2}}(Acr)}{K_{N+\frac{1}{2}}(Acr)} + 2rc\frac{K'_{N+\frac{1}{2}}(Acr)}{K_{N+\frac{1}{2}}(Acr)} = \lambda_1, \quad \text{(III.100)}$$

which is equivalent to

$$(A^2-1)c^2r^2 - 2Arc\frac{J'_{N+\frac{1}{2}}(Acr)}{J_{N+\frac{1}{2}}(Acr)} + 2rc\frac{J'_{N+\frac{1}{2}}(Acr)}{J_{N+\frac{1}{2}}(Acr)} = \lambda_1. \quad \text{(III.101)}$$

(III.101) must be identically satisfied for all $r \in (0,1)$. Comparing the coefficient of r^2 we have

$$(A^2-1)c^2 + \frac{4}{N+1}\left(\frac{Ac}{2}\right)^2 - \frac{4}{N+1}\left(\frac{c}{2}\right)^2 = 0.$$

This implies $A = \pm 1$. Discarding the negative solution we have

$$A = 1.$$

□

D. Dependency of $p(r)$ and $q(r)$ in the three-dimensional problem

1. Special case with $p(r) = 1 - r^2$

We want to find a self-adjoint differential operator M_r such that

$$M_r Ker(r, r') = M_{r'} Ker(r, r').$$

Take $M_r = \frac{d}{dr}\left(p(r)\frac{d}{dr}\right) + q(r)$ with $p(1) = 0$. Then,

$$M_r Ker(r, r') = p'(r)\frac{d}{dr}\left(K_{N+\frac{1}{2}}(cr)\right) K_{N+\frac{1}{2}}(cr') + p(r)\frac{d^2}{dr^2}\left(K_{N+\frac{1}{2}}(cr)\right) K_{N+\frac{1}{2}}(cr') + q(r) K_{N+\frac{1}{2}}(cr) K_{N+\frac{1}{2}}(cr'). \quad \text{(III.102)}$$

But it is known that

$$\frac{d^2}{dr^2}\left(K_{N+\frac{1}{2}}(cr)\right) = -\left(c^2 + \frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2}\right) K_{N+\frac{1}{2}}(cr). \quad \text{(III.103)}$$

Using (III.103) we obtain from (III.102)

$$M_r Ker(r, r') = p'(r)\frac{d}{dr}\left(K_{N+\frac{1}{2}}(cr)\right) K_{N+\frac{1}{2}}(cr') + \left(-p(r)\left(c^2 + \frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2}\right) + q(r)\right) K_{N+\frac{1}{2}}(cr) K_{N+\frac{1}{2}}(cr'). \quad \text{(III.104)}$$

We choose

$$p(r) = 1 - r^2.$$

Then,

$$M_r Ker(r, r') = -2rcK'_{N+\frac{1}{2}}(cr) K_{N+\frac{1}{2}}(cr') + \left(-(1-r^2)\left(c^2 + \frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2}\right) + q(r)\right) K_{N+\frac{1}{2}}(cr) K_{N+\frac{1}{2}}(cr'),$$

i.e.,

$$M_r Ker(r,r') = -2rcK'_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr') -$$
$$\left(\left(c^2 - \frac{1}{4} + (N+\frac{1}{2})^2\right) + \left(\frac{\frac{1}{4}-(N+\frac{1}{2})^2}{r^2} - c^2r^2\right)\right) K_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr') +$$
$$q(r)K_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr'). \quad \text{(III.105)}$$

We denote

$$k = \left(c^2 - \frac{1}{4} + (N+\frac{1}{2})^2\right)$$

and

$$A(r) = \left(\frac{\frac{1}{4}-(N+\frac{1}{2})^2}{r^2} - c^2r^2\right),$$

where k is a constant. Therefore (III.105) becomes

$$M_r Ker(r,r') = -2rcK'_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr') - (k + A(r))\, K_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr')$$
$$+ q(r)K_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr'), \quad \text{(III.106)}$$

i.e.,

$$M_r Ker(r,r') = \left(-2rcK'_{N+\frac{1}{2}}(cr) - (k + A(r))\, K_{N+\frac{1}{2}}(cr) + q(r)K_{N+\frac{1}{2}}(cr)\right) K_{N+\frac{1}{2}}(cr'). \quad \text{(III.107)}$$

But we want

$$M_r Ker(r,r') = M_{r'} Ker(r,r').$$

Hence (III.107) gives the necessary condition

$$-2rcK'_{N+\frac{1}{2}}(cr) - (k + A(r))\, K_{N+\frac{1}{2}}(cr) + q(r)K_{N+\frac{1}{2}}(cr) = \mu' K_{N+\frac{1}{2}}(cr), \quad \text{(III.108)}$$

where μ' is a constant independent of r. Let

$$\mu = \mu' + k.$$

Then

$$-2rcK'_{N+\frac{1}{2}}(cr) - A(r)K_{N+\frac{1}{2}}(cr) + q(r)K_{N+\frac{1}{2}}(cr) = \mu K_{N+\frac{1}{2}}(cr), \tag{III.109}$$

which gives

$$q(r) = \mu + A(r) + 2rc\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)}, \tag{III.110}$$

i.e., (by the expression for $A(r)$): the most general form of $q(r)$ is given by

$$q(r) = \mu + \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right) + 2rc\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)}, \tag{III.111}$$

where μ is a constant independent of r.

Particular Case: When $\mu = -1$, using Lemma III.11

$$2rc\frac{K'_N(cr)}{K_N(cr)} = 1 + 2rc\frac{J'_N(cr)}{J_N(cr)}$$

we have from (III.111):

$$q(r) = \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right) + 2rc\frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)}.$$

We used this $q(r)$ in our previous works.

2. Generalization

Let us now generalize the concept of the last subsection.

Let $M_r = \frac{d}{dr}\left(p(r)\frac{d}{dr}\right) + q(r)$, with $p(1) = 0$.

Then, as we have found in (III.104) we have

$$M_r Ker(r,r') = p'(r)\frac{d}{dr}\left(K_{N+\frac{1}{2}}(cr)\right)K_{N+\frac{1}{2}}(cr') + \left(-p(r)\left(c^2 + \frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2}\right) + q(r)\right)K_{N+\frac{1}{2}}(cr)K_{N+\frac{1}{2}}(cr'),$$

i.e.,

$$M_r Ker(r,r') = p'(r) c K'_{N+\frac{1}{2}}(cr) K_{N+\frac{1}{2}}(cr') + \left(-p(r)\left(c^2 + \frac{\frac{1}{4}-(N+\frac{1}{2})^2}{r^2}\right) + q(r)\right) K_{N+\frac{1}{2}}(cr) K_{N+\frac{1}{2}}(cr').$$

Let

$$B(r) = \left(\frac{\frac{1}{4}-(N+\frac{1}{2})^2}{r^2}\right),$$

i.e.,

$$M_r Ker(r,r') = \Big(p'(r) c K'_{N+\frac{1}{2}}(cr) + \left[-p(r)\left(c^2 + B(r)\right) + q(r)\right] K_{N+\frac{1}{2}}(cr)\Big) K_{N+\frac{1}{2}}(cr'). \tag{III.112}$$

Let

$$f(r) = \Big(p'(r) c K'_{N+\frac{1}{2}}(cr) + \left(-p(r)\left(c^2 + B(r)\right) + q(r)\right) K_{N+\frac{1}{2}}(cr)\Big).$$

Then (III.112) gives

$$M_r Ker(r,r') = f(r) K_{N+\frac{1}{2}}(cr').$$

But, we want

$$M_r Ker(r,r') = M_{r'} Ker(r,r').$$

Therefore

$$f(r) K_{N+\frac{1}{2}}(cr') = f(r') K_{N+\frac{1}{2}}(cr).$$

Which gives

$$\frac{f(r)}{K_{N+\frac{1}{2}}(cr)} = \frac{f(r')}{K_{N+\frac{1}{2}}(cr')}.$$

LHS is a function of r whereas RHS is a function of r'. So we must have :

$$\frac{f(r)}{K_{N+\frac{1}{2}}(cr)} = \frac{f(r')}{K_{N+\frac{1}{2}}(cr')} = \mu',$$

where μ' is a constant independent of both r and r'., i.e., we have

$$p'(r)cK'_{N+\frac{1}{2}}(cr) + \left[-p(r)\left(c^2 + B(r)\right) + q(r)\right] K_{N+\frac{1}{2}}(cr) = \mu' K_{N+\frac{1}{2}}(cr). \quad \text{(III.113)}$$

As a consequence

$$q(r) = -p'(r)c\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)} + p(r)c^2 + p(r)B(r) + \mu'. \quad \text{(III.114)}$$

Also the boundary value is given to be $p(1) = 0$. So given a function $q(r)$ there exist unique $p(r)$ that satisfies (III.114) with $p(1) = 0$.

Lemma III.20. *Choose* $\mu' = -\left(c^2 - \frac{1}{4} + (N + \frac{1}{2})^2\right) - 1 = -\left(c^2 + \frac{3}{4} + (N + \frac{1}{2})^2\right)$ *in (III.114). Then* $p(r) = 1 - r^2$ *if and only if* $q(r) = \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right) + 2rc\frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)}$.

Proof. We have already proved in last subsection that if $p(r) = 1 - r^2$ then we must have $q(r) = \mu + \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right) + 2rc\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)}$ for any constant μ. Choosing $\mu = -1$ we have shown previously that $q(r) = \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right) + 2rc\frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)}$. We also observe from the last subsection that $\mu = \mu' + k = \mu' + \left(c^2 - \frac{1}{4} + (N + \frac{1}{2})^2\right)$. Hence $\mu = -1$ actually gives $\mu' = -\left(c^2 + \frac{3}{4} + (N + \frac{1}{2})^2\right)$.

We will now prove the converse. Let $q(r) = \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right) + 2rc\frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)} = \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right) + 2rc\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)} - 1$.

Let $A(r) = \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right)$. Then $q(r) = A(r) + 2rc\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)} - 1$.

So, from (III.114) we can write

$$p'(r)c\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)} = p(r)(c^2 + B(r)) - A(r) - 2rc\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)} + 1 + \mu'. \quad \text{(III.115)}$$

But observe that with $A(r) = \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2\right)$ and $B(r) = \left(\frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2}\right)$ we have

$$(1 - r^2)(c^2 + B(r)) = k + A(r),$$

where $k = \left(c^2 - \frac{1}{4} + (N+\frac{1}{2})^2\right)$.

So, (III.115) can be written as

$$p'(r)c\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)} = p(r)(c^2+B(r)) - (1-r^2)(c^2+B(r)) + k - 2rc\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)} + 1 + \mu'. \tag{III.116}$$

But with $\mu' = -\left(c^2 + \frac{3}{4} + (N+\frac{1}{2})^2\right)$ we have $k+1+\mu' = 0$. Therefore (III.116) gives

$$p'(r)c\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)} = \left(p(r) - (1-r^2)\right)(c^2+B(r)) - 2rc\frac{K'_{N+\frac{1}{2}}(cr)}{K_{N+\frac{1}{2}}(cr)}. \tag{III.117}$$

But the differential equation of $p(r)$ given in (III.117) with the boundary condition $p(1) = 0$ is satisfied by $p(r) = 1 - r^2$. By the uniqueness of solution thus we have the the only possibility $p(r) = 1 - r^2$. □

E. Completeness of eigenfunctions

Notation: We define $\Omega = D(L) =$ domain of $L = \{\phi \in L^2(0,1) | L\phi \in L^2(0,1), \phi(0) = 0, \lim_{r\to 1} |\phi(r)| < \infty\}$. For short we speak the boundary conditions $\phi(0) = 0$, $\phi(1) = \lim_{r\to 1} |\phi(r)| < \infty$ of B_1, B_2 respectively.

Lemma III.21. *Consider self-adjoint operators L and K such that $LKf = KLf$, for all $f \in D(L)$. Assume for L with boundary conditions B_1 and B_2 the eigenspace corresponding to each eigenvalue is one-dimensional. Also, assume that the set of all eigenfunctions of L with B_1 and B_2 is a complete system in $L^2(0,1)$. Then the set of the eigenfunctions of L is a subset of the set of eigenfunctions of K, and hence there exist a complete set of eigenfunctions of K in $L^2(0,1)$. Moreover, if the eigenspace corresponding to each eigenvalue of K is one-dimensional then the eigenfunctions of L and K are identical.*

Proof. Let $L\phi = \lambda\phi$ with ϕ satisfying B_1 and B_2. Now, since $LK = KL$, $L(K\phi) =$

$K(L\phi) = \lambda(K\phi)$. But L has eigenspace of dimension one corresponding to λ. So $K\phi = \mu\phi$. Therefore the set of the eigenfunctions of L is a subset of the set of eigenfunctions of K. Given the eigenfunctions of L with B_1 and B_2 are complete in $L^2(0,1)$. Thus there exist a complete set of eigenfunctions of K in $L^2(0,1)$. Next, suppose that the eigenspace corresponding to each eigenvalue of K is one-dimensional. Let $S = \{\phi_1, \phi_2, \phi_3, \dots\}$ be the complete set (in $L^2(0,1)$) of eigenfunctions of L correspondind to the eigenvalues $\{\mu_1, \mu_2, \mu_3, \dots\}$ of K (we have already proved eigenfunctions of L are eigenfunctions of K). From the given criteria we must have all the μ_i's are distinct. We observe K has a set of eigenfunctions having subspace S. Let ξ be an eigenfunction of K corresponding to the eigenvalue μ. Let $\xi = \sum_i a_i\phi_i$ where the convergence of the series is in $L^2(0,1)$. Then $K\xi = \mu\xi$ gives $a_i(\mu - \mu_i) = 0$, for all i. If $\mu \neq \mu_i$, for all i, then $a_i = 0$, for all i and hence $\xi = 0$. If $\mu = \mu_i$ for some i then we can take $a_i \neq 0$ for that i. Hence $\xi = a_i\phi_i$. The lemma is proved. □

Let us reconsider the equation

$$\frac{d}{dr}\left((1-r^2)\frac{d\phi(r)}{dr}\right) + \left(\frac{\frac{1}{4}-(N+\frac{1}{2})^2}{r^2} - c^2r^2\right)\phi(r) + 2rc\frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)}\phi(r) = -\chi\phi(r). \tag{III.118}$$

We consider the eigenfunctions of (III.118) with $B_1(\phi)$ and $B_1(\phi)$. That is,

$$\phi(0) = 0, \tag{III.119}$$

and

$$\lim_{r\to 1}|\phi(r)| < \infty. \tag{III.120}$$

Remark III.22. We observe the boundary conditions given by (III.119) and (III.120) are self-adjoint with respect to the differential operator given by LHS of (III.118).

Lemma III.23. *For the singular Sturm-Liouville problem (III.118) with boundary* *conditions (III.119) and (III.120), all the eigenvalues are real and the eigenfunctions corresponding to distinct eigenvalues are orthogonal in* $L^2(0, 1)$.

Proof. Since the boundary conditions are self-adjoint the proof goes in the same way as in a regular Sturm-Liouville problem. (cf. [16]). □

Lemma III.24. *The eigenspace for any eigenvalue* χ *in (III.118) is 1-dimensional.*

Proof. The fundamental existence theorem for ordinary differential equations says that for any constants c_1 and c_2 there is a unique solution for (III.118) satisfying the initial conditions $\phi(0) = c_1$ and $\phi'(0) = c_2$. That is, a solution is specified by two arbitrary constants c_1 and c_2. Hence the space of all solutions of (III.118) is *at most* 2-dimensional. For our problem $\phi(0) = 0$, gives $c_1 = 0$. So the dimension of the solution space is reduced to one. Hence the eigenspace for any eigenvalue χ in (III.118) is 1-dimensional. □

Liouville Normal Form: The general second-order differential equation

$$a(X)\frac{d^2Y}{dX^2} + b(X)\frac{dY}{dX} + \{\lambda - c(X)\}Y = 0$$

can be reduced to the normal form as follows: Suppose $a(X) > 0$ almost everywhere, and let

$$x = \int \frac{dX}{\sqrt{a(X)}}.$$

Then

$$\frac{d^2Y}{dx^2} + \beta(x)\frac{dY}{dx} + \{\lambda - \gamma(x)\}Y = 0,$$

where

$$\beta(x) = \frac{b(X) - \frac{1}{2}a'(X)}{\sqrt{a(X)}},$$

$$\gamma(x) = c(X).$$

Putting

$$Y = f(x)y$$

where

$$f(x) = e^{-\frac{1}{2}\int \beta(x)dx}$$

we obtain:

$$\frac{d^2y}{dx^2} + \{\lambda - \frac{1}{4}\beta^2(x) - \frac{1}{2}\beta'(x) - \gamma(x)\}y = 0.$$

This is the standard form. The argument assumes that $b(X)$ is differentiable and $a(X)$ twice differentiable.

We will now transform (III.118) into Liouville's normal form. We rewrite (III.118) as

$$\frac{d}{dr}\left((1-r^2)\frac{d\phi(r)}{dr}\right) + \left(\chi + \frac{\frac{1}{4} - (N+\frac{1}{2})^2}{r^2} - c^2r^2 + 2rc\frac{J'_{N+\frac{1}{2}}(cr)}{J_{N+\frac{1}{2}}(cr)}\right)\phi(r) = 0, \tag{III.121}$$

where $r \in [0, 1]$. Let

$$r = \sin x$$

and

$$\phi = y\sqrt{\sec x}.$$

Then $x \in [0, \pi/2]$ and (III.121) becomes

$$\frac{d^2y}{dx^2} + (\chi + (\frac{1}{4} - (N+\frac{1}{2})^2)\csc^2 x - c^2\sin^2 x + 2c\sin x\frac{J'_{N+\frac{1}{2}}(c\sin x)}{J_{N+\frac{1}{2}}(c\sin x)} - \frac{1}{4}\tan^2 x + \frac{1}{2}\sec^2 x)y = 0,$$

which is equivalent to:

$$\frac{d^2y}{dx^2} + (\chi + (\frac{1}{4} - (N+\frac{1}{2})^2)\csc^2 x - c^2\sin^2 x + 2c\sin x\frac{J'_{N+\frac{1}{2}}(c\sin x)}{J_{N+\frac{1}{2}}(c\sin x)} + \frac{1}{4}\tan^2 x + \frac{1}{2})y = 0, \tag{III.122}$$

i.e.,

$$\frac{d^2y}{dx^2} + (\chi - q(x))\, y = 0, \tag{III.123}$$

where

$$q(x) = -\left((\frac{1}{4} - (N+\frac{1}{2})^2) \csc^2 x - c^2 \sin^2 x + 2c \sin x \frac{J'_{N+\frac{1}{2}}(c \sin x)}{J_{N+\frac{1}{2}}(c \sin x)} + \frac{1}{4}\tan^2 x + \frac{1}{2} \right), \tag{III.124}$$

where $x \in [0, \pi/2]$.

Remark III.25. Singularities of (III.118) at $r = 0$ and $r = 1$ are same as the singularities of (III.123) at $x = 0$ and $x = \pi/2$ respectively.

Endpoint Classifications: Define $L_{loc}(J, \mathbb{C})$ to be the set of functions f satisfying $f \in L([a, b], \mathbb{C})$ for every compact subinterval $[a, b]$ of J. Also define $L^2(J, w) = \{f : J \to \mathbb{C}, \int_J |f|^2 w < \infty\}$ the Hilbert space of square integrable functions with weight w if $w > 0$ a.e. on J.

Consider the equation

$$(py')' + (\lambda w - q)y = 0 \tag{III.125}$$

where $\lambda \in \mathbb{C}$, on J with

$$J = (a, b), \quad -\infty \le a < b \le \infty, \quad 1/p, q, w \in L_{loc}(J, \mathbb{C}). \tag{III.126}$$

We use the definitions from [61]. The (finite or infinite) endpoint a:

- is *regular* (R) if, in addition to (III.126)

  $$1/p, q, w \in L((a, d), \mathbb{C})$$

  holds for some (and hence any) $d \in J$;

- is *limit-circle* (LC) if all solution of the equation (III.125) are in $L^2((a, d), |w|)$

for some (and hence any) $d \in (a, b)$;

- is *limit-point* (LP) if it is *not* LC;

- is *oscillatory* (O) if $1/p, q, w, \lambda$ are all real-valued and there is a nontrivial real-valued solution with an infinite number of zeros in any right neighborhood of a;

- is *non-oscillatory* (NO) if $1/p, q, w, \lambda$ are all real-valued and is *not* O;

- is *limit-circle-oscillatory* (LCO) if it is both LC and O;

- and is *limit-circle-non-oscillatory* (LCNO) if it is both LC and NO.

Similar definitions are made at b. An endpoint is called *singular* if it is *not regular.*

We now state two lemmas. Proofs of the lemmas can be found in [61].

Lemma III.26. *LC and LP classification are independent of* $\lambda \in \mathbb{C}$*.*

Lemma III.27. *For an endpoint in the LC case and* $p > 0$ *a.e., the LCO and LCNO classifications are independent of* $\lambda \in \mathbb{R}$*.*

Next we proceed to prove the following lemmas.

Lemma III.28. *For (III.118) the endpoint* $r = 1$ *[or, for (III.123) the endpoint* $x = \pi/2$*] is of LC type. (More precisely it is of LCNO type).*

Proof. Since $\lim_{x \to \frac{\pi}{2}-} (x - \dfrac{\pi}{2})^2 \tan^2 x = 1$ and $\tan^2 x \le \frac{1}{(x - \frac{\pi}{2})^2}$ near $x = \frac{\pi}{2}-$, therefore $|q(x)| \le \frac{1}{4}(x - \frac{\pi}{2})^{-2} + A$, near $x = \frac{\pi}{2}$ (from left side) for some constant A. Thus it follows from [37] (or, [56], section 5.25, pp-127) that for (III.123) the endpoint $x = \frac{\pi}{2}$ (consequently the endpoint $r = 1$ for (III.118)) is of LC type. Also, with

$$\chi = c^2 + (N + 1)(N + 2) - 2N - 1$$

a solution of (III.118) is

$$\phi(r) = rj_N(r) = \frac{1}{c}\sqrt{\frac{\pi}{2}}K_{N+\frac{1}{2}}(cr)$$

while the second independent solution will also be non-oscillatory. Hence $r = 1$ is of LCNO type for $\chi = c^2 + (N+1)(N+2) - 2N - 1$. Moreover, by using Lemma III.27 we obtain that for any χ, the endpoint $r = 1$ of (III.118) is LCNO. Same is true for the endpoint $x = \pi/2$ of (III.123). This proves the Lemma. □

Lemma III.29. *For (III.118) when $N \geq 1$ the endpoint $r = 0$ [or, for (III.123) the endpoint $x = 0$] is of LP type. For (III.118) when $N = 0$ the endpoint $r = 0$ [or, for (III.123) the endpoint $x = 0$] is a regular type.*

Proof. From (III.124) we have

$$q(x) = -\left((\frac{1}{4} - (N + \frac{1}{2})^2) \csc^2 x - c^2 \sin^2 x + 2c \sin x \frac{J'_{N+\frac{1}{2}}(c \sin x)}{J_{N+\frac{1}{2}}(c \sin x)} + \frac{1}{4}\tan^2 x + \frac{1}{2} \right),$$

where $x \in [0, \pi/2]$.

When $N = 0$ there is no singularity of $q(x)$ at $x = 0$ and so the endpoint is regular. When $N \geq 1$ if x near 0 and positive then $\sin x \leq x$ gives $\frac{1}{x^2} \leq \csc^2 x$. Thus

$$\frac{1}{x^2} \leq -\left(\frac{1}{4} - (N+\frac{1}{2})^2\right)\frac{1}{x^2} \leq -\left(\frac{1}{4} - (N+\frac{1}{2})^2\right)\csc^2 x.$$

Hence we can choose constant A so that for sufficiently small x

$$q(x) \geq \frac{3}{4}x^{-2} + A$$

So by cf. [56], section 5.25, pp-127 we conclude $x = 0$ is a limit point for (III.123). That is, when $N \geq 1$, $r = 0$ is of LP for (III.118). □

Lemma III.30. *Given any Sturm-Liouville Problem (SLP) with endpoints which are either regular or LCNO there exists a regular SLP which has exactly same spectrum* *as in this singular problem and furthermore the eigenfunctions of the given singular problem* $\{y_n : n \in \mathbb{N}\}$ *are related to the eigenfunctions* $\{z_n : n \in \mathbb{N}\}$ *of the corresponding regular problem by the equation* $y_n(t) = v(t)z_n(t)$, $t \in (a,b)$, $n \in \mathbb{N}$ *for some function* v *in the maximal domain of the singular problem which satisfies* $v(t) > 0$ *for* $t \in (a,b)$.

Proof. cf. [31]. □

Corollary III.31. *For a singular SLP with endpoints which are either regular or LCNO the spectrum is discrete.* □

We write $x \in [0,\pi/2] = I_1 \cup I_2$ where $I_1 = \{x \in [0,\pi/4]\}$ and $I_2 = \{x \in [\pi/4,\pi/2]\}$ [or, we write $r \in [0,1] = I_1 \cup I_2$ where $I_1 = \{r \in [0,1/2]\}$ and $I_2 = \{r \in [1/2,1]\}$]. Let $\theta_1(x,\chi)$, $\theta_2(x,\chi)$ be solutions of (III.123) such that $\theta_1(\pi/4) = 0$, $\theta_1'(\pi/4) = -1$, $\theta_2(\pi/4) = 1$, $\theta_2'(\pi/4) = 0$ Then there are functions $m_1(\chi)$ and $m_2(\chi)$ such that for non-real χ

$$\xi_1(x,\chi) = \theta_2(x,\chi) + m_1(\chi)\theta_1(x,\chi)$$

is $L^2(0,\pi/4)$, and

$$\xi_2(x,\chi) = \theta_2(x,\chi) + m_2(\chi)\theta_1(x,\chi)$$

is $L^2(\pi/4,\pi/2)$.

Then the Wronskian:

$$W(\xi_1,\xi_2) = m_1(\chi) - m_2(\chi).$$

m_1 and m_2 are known as *Weyl-Titchmarsh functions* and *they are meromorphic if and only if the corresponding spectrum is discrete.*

Lemma III.32. $m_1(\chi)$ *is a meromorphic function.*

Proof. We use Lemma III.29 for (III.123) on I_1. Therefore we have either only singularity at $x = 0$ of LP type (for $N \geq 1$) or no singularity at $x = 0$ (for $N = 0$). The spectrum is clearly discrete for $N = 0$ and hence in that case $m_1(\chi)$ is meromorphic.

For $N \geq 1$ we have for some suitable constant A

$$q(x) \geq \frac{3}{4}x^{-2} + A \geq -\frac{1}{4}x^{-2} + A$$

Hence by cf. [56], section 5.25 pp-127, the spectrum is discrete for $N \geq 1$. Thus $m_1(\chi)$ is meromorphic. □

Lemma III.33. *$m_2(\chi)$ is a meromorphic function.*

Proof. For (III.123) on I_2 we have the only singularity at $x = \frac{\pi}{2}$ is of LCNO type (by Lemma III.28). Now on I_2 for (III.123) applying Lemma III.30 (and the corollary for Lemma III.30) we have (since the endpoint $x = \frac{\pi}{4}$ is regular) the spectrum is discrete. So, $m_2(\chi)$ is meromorphic.

□

$$\text{Let } G(x,y,\chi) = \begin{cases} \dfrac{\xi_2(x,\chi)\xi_1(y,\chi)}{m_2(\chi) - m_1(\chi)} & y \leq x \\ \\ \dfrac{\xi_1(x,\chi)\xi_2(y,\chi)}{m_2(\chi) - m_1(\chi)} & y > x \end{cases} \quad \text{and}$$

$$\Phi(x,\chi) = -\int_0^{\frac{\pi}{2}} G(x,y,\chi)f(y)dy$$

where f is the arbitrary function to be expanded. *We know from [56] that the expansion of f will reduce to a series if both $m_1(\chi)$ and $m_2(\chi)$ are meromorphic functions.* In the present case by Lemma III.32 and Lemma III.33 we have both $m_1(\chi)$ and $m_2(\chi)$ meromorphic functions. So, any function $f \in L^2(0, \pi/2)$ can be expanded in a series of eigenfunctions of (III.123). Also, by Lemma III.23 the eigenfunctions are orthogonal (since Liouville's reduction transforms orthogonal functions in $L^2(D)$ to orthogonal functions in $L^2(D)$). So we have the following theorems:

Theorem III.34. *Eigenfunctions of (III.123), with* $\phi(0) = 0$ *and* $\lim_{x \to \frac{\pi}{2}} |\phi(x)| < \infty$ *form a complete orthogonal set in* $L^2(0, \pi/2)$. □

Theorem III.35. *Eigenfunctions of (III.118), with* $\phi(0) = 0$ *and* $\lim_{r \to 1} |\phi(r)| < \infty$ *form a complete orthogonal set in* $L^2(0, 1)$. □

F. Uniqueness of operator for one-dimensional problem

We prove the uniqueness of differential operator for one-dimensional problem. For one-dimensional problem we have

$$Ker(x, y) = \frac{\sin c(x-y)}{c(x-y)}.$$

We want to find a self-adjoint differential operator M_x such that $M_x Ker(x, y) = M_y Ker(x, y)$.

Let, $M_x = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$ with $p(1) = p(-1) = 0$.

Then we can derive

$$M_x Ker(x, y) = \frac{1}{c}\left(\frac{\cos c(x-y)}{(x-y)^2}[-2cp(x) + cp'(x)(x-y)]\right) + \frac{1}{c}\left(\frac{\sin c(x-y)}{(x-y)^3}[-c^2 p(x)(x-y)^2 + 2p(x) - p'(x)(x-y) + q(x)(x-y)^2]\right).$$

So, to have

$$M_x \frac{\sin c(x-y)}{c(x-y)} = M_y \frac{\sin c(x-y)}{c(x-y)}$$

we must have

$$\frac{[-2cp(x) + cp'(x)(x-y)]}{(x-y)^2} = \frac{[-2cp(y) + cp'(y)(y-x)]}{(x-y)^2}, \qquad \text{(III.127)}$$

i.e.,

$$-2cp(x) + cp'(x)(x-y) = -2cp(y) + cp'(y)(y-x), \qquad \text{(III.128)}$$

and

$$\frac{[-c^2p(x)(x-y)^2 + 2p(x) - p'(x)(x-y) + q(x)(x-y)^2]}{(x-y)^3} = \frac{[-c^2p(y)(x-y)^2 + 2p(y) - p'(y)(y-x) + q(y)(x-y)^2]}{(x-y)^3}, \qquad \text{(III.129)}$$

i.e.,

$$-c^2p(x)(x-y)^2 + 2p(x) - p'(x)(x-y) + q(x)(x-y)^2 = -c^2p(y)(x-y)^2 + 2p(y) - p'(y)(y-x) + q(y)(x-y)^2. \qquad \text{(III.130)}$$

Using (III.128) we obtain from (III.130)

$$-c^2p(x) + q(x) = -c^2p(y) + q(y). \qquad \text{(III.131)}$$

Since in (III.131) LHS depends only on x and RHS depends only on y we must have

$$-c^2p(x) + q(x) = -c^2p(y) + q(y) = \lambda, \qquad \text{(III.132)}$$

where λ is a constant. Thus,

$$q(x) = c^2p(x) + \lambda. \qquad \text{(III.133)}$$

Now we proceed to find $p(x)$. We have from (III.128) :

$$p'(x) + p'(y) = 2\left(\frac{p(x) - p(y)}{x-y}\right). \qquad \text{(III.134)}$$

[**Note**: by the $'$ notation we mean the derivative with respect to the argument variable. Precisely $p'(x) = \frac{dp(x)}{dx}$ and $p'(y) = \frac{dp(y)}{dy}$.]

Differentiating (III.134) with respect to x we have (also keeping in mind that x and

y are independent variables):

$$p''(x) = 2\left(\frac{p'(x)(x-y) - (p(x)-p(y))}{(x-y)^2}\right), \tag{III.135}$$

and

$$p''(y) = 2\left(\frac{-p'(y)(x-y) + (p(x)-p(y))}{(x-y)^2}\right). \tag{III.136}$$

Subtracting (III.136) from (III.135) we have

$$p''(x) - p''(y) = \frac{2}{(x-y)^2}\left[(p'(x)+p'(y))(x-y) - 2(p(x)-p(y))\right].$$

Using (III.134) we then have

$$p''(x) - p''(y) = \frac{2}{(x-y)^2}\left[2(p(x)-p(y)) - 2(p(x)-p(y))\right] = 0.$$

So

$$p''(x) = p''(y)$$

Thus we must have

$$p''(x) = p''(y) = \mu_1.$$

where μ_1 is a constant. Now $p''(x) = \mu_1$ gives

$$p(x) = \mu_1 x^2 + \mu_2 x + \mu_3.$$

But we have the boundary condition $p(1) = p(-1) = 0$. The boundary conditions give $\mu_1 + \mu_3 = 0$ and $\mu_2 = 0$.

Therefore

$$p(x) = \mu(1-x^2), \tag{III.137}$$

where $\mu = \mu_3$ is a constant. So, (III.137) gives **the uniqueness** of $p(x)$ up to a multiplicative constant. Let us take $\mu = 1$ and $\lambda = -c^2$ in (III.133). Then, from

(III.133) we have:

$$q(x) = c^2(1 - x^2) - c^2,$$

i.e.,

$$q(x) = -c^2 x^2. \tag{III.138}$$

Note: This $q(x)$ is exactly what Slepian and Pollak used in [46]. □

Notation: Throughout the following work we will consider the ϕ square integrable and twice differentiable in $(-\sigma, \sigma)$ for some $\sigma > 0$. We define $\Lambda = \{\phi \in L^2(-\sigma, \sigma) \cap C^2([-\sigma, \sigma]) | \lim_{r \to \pm\sigma} |\phi(r)| < \infty\}$.
We proceed to prove the main theorem of this section.

Theorem III.36. *Suppose K is a self adjoint operator with simple spectrum. Suppose*

$$L_1 = \frac{d}{dx}\left(p_1(x)\frac{d}{dx}\right) + q_1(x)$$

and

$$L_2 = \frac{d}{dx}\left(p_2(x)\frac{d}{dx}\right) + q_2(x)$$

with $p_1(\pm\sigma) = p_2(\pm\sigma) = 0$ and $p_1, p_2 \in \Lambda$, are two operators on $[-\sigma, \sigma]$ that commute with K for the functions $f \in \Lambda$. Assume both L_1 and L_2 have a set of eigenfunctions in Λ which are complete orthogonal in $L^2(-\sigma, \sigma)$. Then

$$p_2(x) \equiv kp_1(x)$$

and

$$q_2(x) \equiv kq_1(x) + k'$$

for some $k, k' \in \mathbb{C}$ and $x \in (-\sigma, \sigma)$.

Proof. It is easy to show that the conclusion for Lemma III.21 is valid for functions f such that $f \in \Lambda$. Therefore from the condition of the above theorem using Lemma III.21 for the operators L_1 and K we have eigenfunctions of L_1 and K are same in $L^2(-\sigma, \sigma)$. Similarly using Lemma III.21 for the operators L_2 and K we have eigenfunctions of L_2 and K are same in $L^2(-\sigma, \sigma)$. Hence the eigenfunctions for L_1 and L_2 are identical in $L^2(-\sigma, \sigma)$.

Since for both L_1 and L_2 when considered on functions on Λ have one-dimensional eigenspace corresponding to each eigenvalue, therefore we must have

$$L_1 L_2 \phi = L_2 L_1 \phi, \tag{III.139}$$

for all $\phi \in \Lambda$. But

$$L_1 L_2 \phi = \Gamma_1(x) \frac{d^4\phi}{dx^4} + \Gamma_2(x) \frac{d^3\phi}{dx^3} + \Gamma_3(x) \frac{d^2\phi}{dx^2} + \Gamma_4(x) \frac{d\phi}{dx} + \Gamma_5(x)\phi, \tag{III.140}$$

where

$$\Gamma_1(x) = p_1(x) p_2(x), \tag{III.141}$$

$$\Gamma_2(x) = 3p_1(x) p_2'(x) + p_1'(x) p_2(x), \tag{III.142}$$

$$\Gamma_3(x) = 3p_1(x) p_2''(x) + 2p_1'(x) p_2'(x) + p_1(x) q_2(x) + q_1(x) p_2(x), \tag{III.143}$$

$$\Gamma_4(x) = p_1(x) p_2'''(x) + 2p_1(x) q_2'(x) + p_1'(x) p_2''(x) + q_2(x) p_1'(x) + q_1(x) p_2'(x), \tag{III.144}$$

$$\Gamma_5(x) = p_1(x) q_2''(x) + p_1'(x) q_2'(x) + q_1(x) q_2(x). \tag{III.145}$$

Similarly

$$L_2 L_1 \phi = \tilde{\Gamma_1}(x) \frac{d^4\phi}{dx^4} + \tilde{\Gamma_2}(x) \frac{d^3\phi}{dx^3} + \tilde{\Gamma_3}(x) \frac{d^2\phi}{dx^2} + \tilde{\Gamma_4}(x) \frac{d\phi}{dx} + \tilde{\Gamma_5}(x)\phi, \tag{III.146}$$

where

$$\tilde{\Gamma_1}(x) = p_1(x) p_2(x), \tag{III.147}$$

$$\tilde{\Gamma_2}(x) = 3p_2(x)p_1'(x) + p_2'(x)p_1(x), \tag{III.148}$$

$$\tilde{\Gamma_3}(x) = 3p_2(x)p_1''(x) + 2p_2'(x)p_1'(x) + p_2(x)q_1(x) + q_2(x)p_1(x), \tag{III.149}$$

$$\tilde{\Gamma_4}(x) = p_2(x)p_1'''(x) + 2p_2(x)q_1'(x) + p_2'(x)p_1''(x) + q_1(x)p_2'(x) + q_2(x)p_1'(x), \tag{III.150}$$

$$\tilde{\Gamma_5}(x) = p_2(x)q_1''(x) + p_2'(x)q_1'(x) + q_2(x)q_1(x). \tag{III.151}$$

Now, $L_1L_2\phi = L_2L_1\phi$, for all $\phi \in D(L)$ implies

$$\Gamma_i(x) = \tilde{\Gamma_i}(x),$$

for $i = 1, 2, 3, 4, 5$

- $\Gamma_1(x) = \tilde{\Gamma_1}(x)$, gives an identity.
- $\Gamma_2(x) = \tilde{\Gamma_2}(x)$, gives

  $$p_2(x) = kp_1(x), \tag{III.152}$$

  for some constant k.
- $\Gamma_3(x) = \tilde{\Gamma_3}(x)$, gives an identity when (III.152) is utilized.
- $\Gamma_4(x) = \tilde{\Gamma_4}(x)$, gives (using (III.152))

  $$q_2(x) = kq_1(x) + k', \tag{III.153}$$

  where k' is another constant (and k is the same constant appeared in (III.152)).
- $\Gamma_5(x) = \tilde{\Gamma_5}(x)$, gives an identity when (III.152) and (III.153) are utilized.

Clearly (III.152) and (III.153) gives the required results and thus the theorem is proved. □

Corollary III.37. *Suppose K is a symmetric, positive definite convolution operator which admits eigenfunctions in Λ which are either even or odd. Then if there exist* *operators of the form as in Theorem III.36 which commutes with K then that operator is unique.*

Proof. A similar proof as of Lemma II.7 shows that K has simple spectrum. Hence the result follows immediately from Theorem III.36. $\square$

G. Numerical results

NOTATION: $e \pm xxx = 10^{\pm xxx}$ We will consider the case for $N = 10$ and $c = .5$: Eigenvalues $-\chi'$ can be approximated from (III.92) and given by (with N replaced by $(N + \frac{1}{2})$ in (III.79)-(III.87) of our previous work):

$-\chi_1 = -132.250121139998e + 000$

$-\chi_2 = -182.218606730622e + 000$

$-\chi_3 = -240.198559950980e + 000$

$-\chi_4 = -306.185016923963e + 000$

$-\chi_5 = -380.175438302939e + 000$

$-\chi_6 = -462.168414139272e + 000$

$-\chi_7 = -552.163110205855e + 000$

$-\chi_8 = -650.159007151745e + 000$

$-\chi_9 = -756.155767878764e + 000$

$-\chi_{10} = -870.153165820800e + 000$

Corresponding approximated coefficients of 10 eigenvectors (v_1 through v_{10}) are given in Table I to Table V at the end of this chapter. Table I to Table V thus gives the coefficients $\{a_0, a_2, a_4, \ldots, a_{26}\}$ of the expansion of (N replaced by $(N + \frac{1}{2})$ in our previous work):

$$\phi(r) = \sum_{j=0}^{\infty} a_{2j} r^{N+1+2j}$$

In Fig. 1 to Fig. 10 we present the graphs of first 10 eigenfunctions (v_1 through v_{10}).

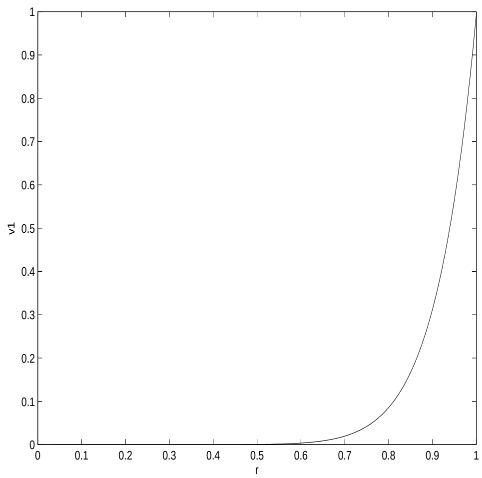

Fig. 1. Eigenfunction v_1 corresponding to eigenvalue χ_1.

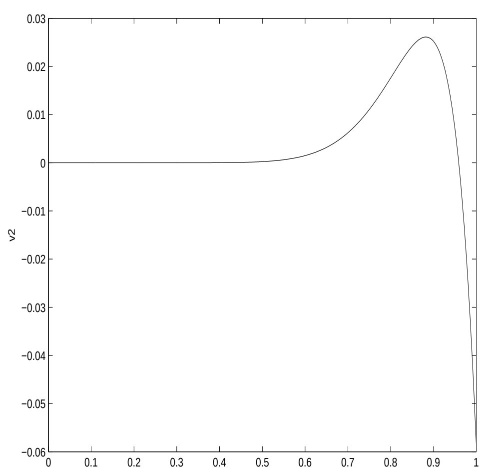

Fig. 2. Eigenfunction v_2 corresponding to eigenvalue χ_2.

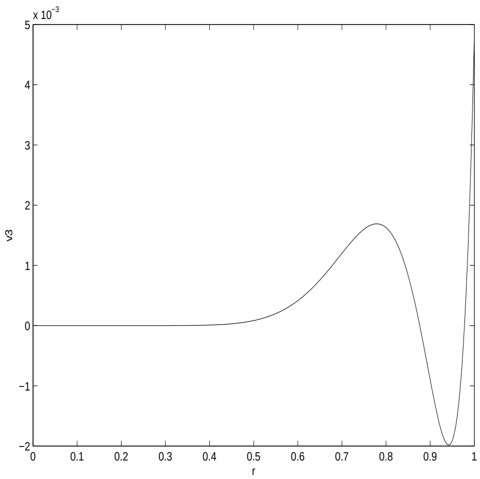

Fig. 3. Eigenfunction v_3 corresponding to eigenvalue χ_3.

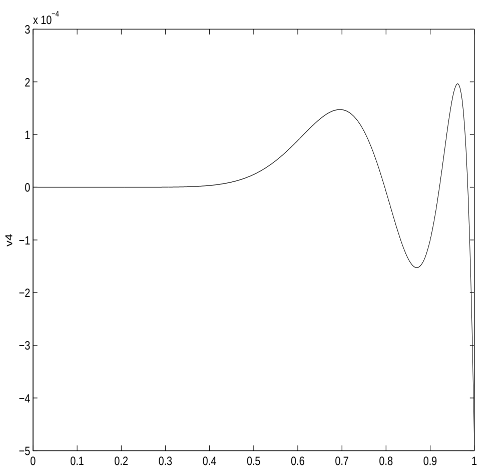

Fig. 4. Eigenfunction v_4 corresponding to eigenvalue χ_4.

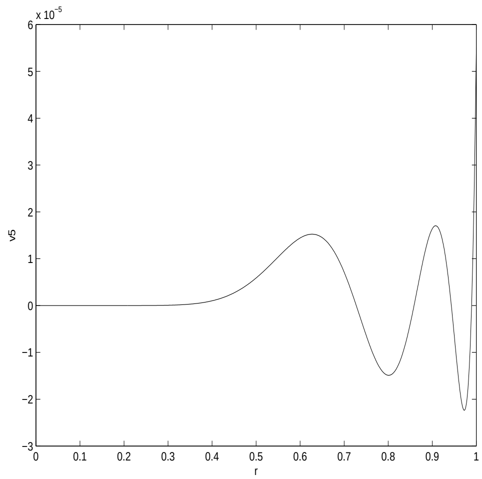

Fig. 5. Eigenfunction v_5 corresponding to eigenvalue χ_5.

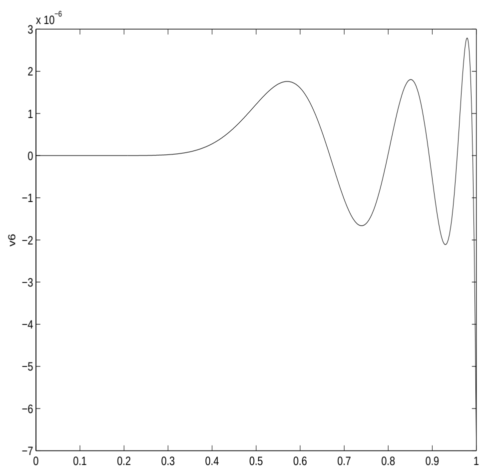

Fig. 6. Eigenfunction v_6 corresponding to eigenvalue χ_6.

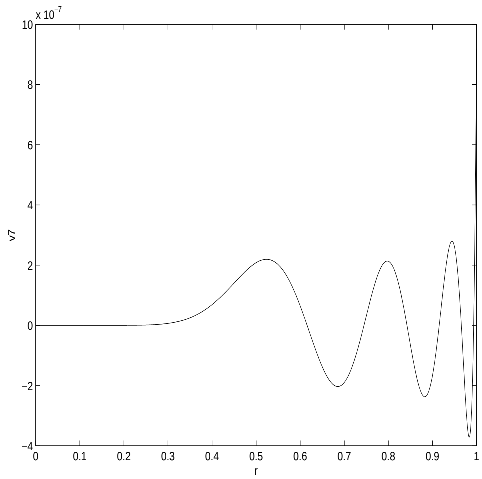

Fig. 7. Eigenfunction v_7 corresponding to eigenvalue χ_7.

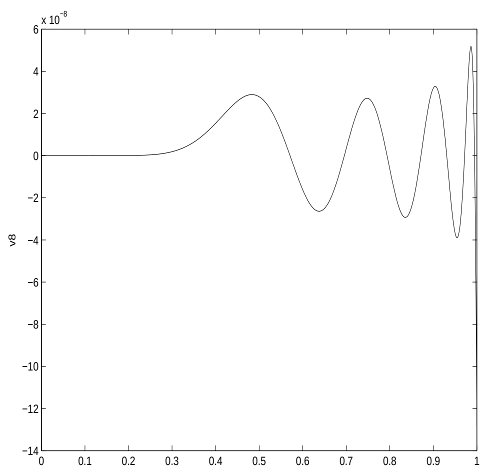

Fig. 8. Eigenfunction v_8 corresponding to eigenvalue χ_8.

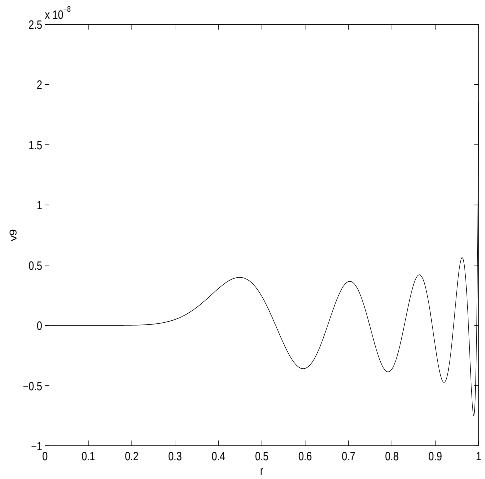

Fig. 9. Eigenfunction v_9 corresponding to eigenvalue χ_9.

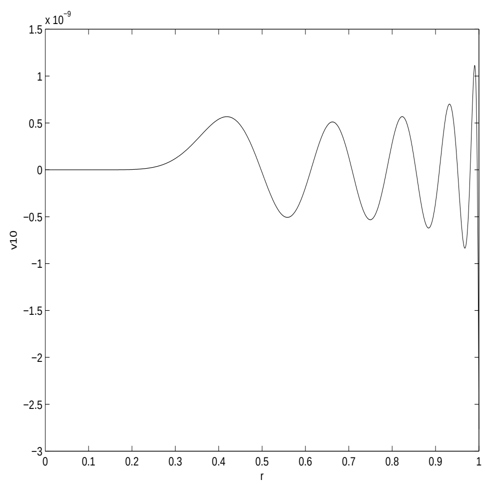

Fig. 10. Eigenfunction v_{10} corresponding to eigenvalue χ_{10}.

H. Tables

Table I. First few coefficients for the eigenvectors v_1 and v_2.

Coefficients	v_1	v_2
a_0	$999.985217505498e-003$	$675.451490685599e-003$
a_2	$-5.43733570832087e-003$	$-737.396364703309e-003$
a_4	$12.2832064245212e-006$	$3.44746409190612e-003$
a_6	$-17.2615745994078e-009$	$-6.65444926119112e-006$
a_8	$14.6198528806105e-012$	$8.30755720607264e-009$
a_{10}	$-14.6833939754400e-015$	$-5.59295427236881e-012$
a_{12}	$-8.28549877604746e-018$	$7.25300987182962e-015$
a_{14}	$-53.5167908775138e-021$	$9.87677880991563e-018$
a_{16}	$-189.652472186803e-024$	$43.6650178905477e-021$
a_{18}	$-720.767346429098e-027$	$160.106580916380e-024$
a_{20}	$-2.75926299573150e-027$	$607.789316300708e-027$
a_{22}	$-10.6772993307145e-030$	$2.33297946895398e-027$
a_{24}	$-41.6842918575617e-033$	$9.05060864842052e-030$
a_{26}	$-163.990074084501e-036$	$35.4238897739336e-033$

Table II. First few coefficients for the eigenvectors v_3 and v_4.

Coefficients	v_3	v_4
a_0	$344.746087154042e-003$	$155.800569267870e-003$
a_2	$-810.891960408735e-003$	$-589.959234667101e-003$
a_4	$472.864253735594e-003$	$733.064346527957e-003$
a_6	$-1.93944965262965e-003$	$-300.482516229658e-003$
a_8	$3.25292595825168e-006$	$1.09787705185421e-003$
a_{10}	$-3.69119792818096e-009$	$-1.62099460118373e-006$
a_{12}	$1.86984573044179e-012$	$1.70521201230316e-009$
a_{14}	$-3.65846582415599e-015$	$-581.638219805724e-015$
a_{16}	$-7.55488718552041e-018$	$2.02504177236703e-015$
a_{18}	$-29.9883009715888e-021$	$5.22023202345821e-018$
a_{20}	$-111.264133545678e-024$	$19.8898173991007e-021$
a_{22}	$-422.767245946025e-027$	$74.1747256422476e-024$
a_{24}	$-1.62631077397138e-027$	$282.210254943080e-027$
a_{26}	$-6.32297481792567e-030$	$1.08761601566888e-027$

Table III. First few coefficients for the eigenvectors v_5 and v_6.

Coefficients	v_5	v_6
a_0	$64.6677714010751e-003$	$25.0874901579561e-003$
a_2	$-348.890271989908e-003$	$-180.067322612757e-003$
a_4	$691.590553351989e-003$	$504.559934674806e-003$
a_6	$-599.004977171548e-003$	$-692.260543633547e-003$
a_8	$192.324725052537e-003$	$466.578934177618e-003$
a_{10}	$-633.579988524058e-006$	$-124.277706948462e-003$
a_{12}	$832.065943847455e-009$	$372.761193498578e-006$
a_{14}	$-825.610602969357e-012$	$-439.108057967154e-009$
a_{16}	$143.872275976080e-015$	$417.307708887440e-012$
a_{18}	$-1.19568775624226e-015$	$-2.34433371048981e-015$
a_{20}	$-3.48623437668741e-018$	$734.544959475480e-018$
a_{22}	$-13.0461452061164e-021$	$2.29887772538821e-018$
a_{24}	$-48.7851623572161e-024$	$8.52847800263557e-021$
a_{26}	$-185.850944617855e-027$	$31.9454509242405e-024$

Table IV. First few coefficients for the eigenvectors v_7 and v_8.

Coefficients	v_7	v_8
a_0	$9.19765050162771e-003$	$3.21219812264544e-003$
a_2	$-84.0111617684820e-003$	$-36.1832476087941e-003$
a_4	$311.003322937772e-003$	$169.403861559830e-003$
a_6	$-599.423409269805e-003$	$-428.965133231587e-003$
a_8	$636.381163052443e-003$	$636.544148893743e-003$
a_{10}	$-353.949032618667e-003$	$-555.111858982462e-003$
a_{12}	$81.0252141424255e-003$	$264.197488405026e-003$
a_{14}	$-223.065089704634e-006$	$-53.2328830867823e-003$
a_{16}	$237.344951893137e-009$	$135.428319681709e-006$
a_{18}	$-218.989961982175e-012$	$-130.918067577922e-009$
a_{20}	$-35.9147894831927e-015$	$118.691155485700e-012$
a_{22}	$-462.277048540968e-018$	$39.5310741042795e-015$
a_{24}	$-1.50914392564148e-018$	$295.379169415805e-018$
a_{26}	$-5.57355653972508e-021$	$989.684359031189e-021$

Table V. First few coefficients for the eigenvectors v_9 and v_{10}.

Coefficients	v_9	v_{10}
a_0	$1.07522343804556e-003$	$346.661990727125e-006$
a_2	$-14.5892806655314e-003$	$-5.56281838967078e-003$
a_4	$83.7681182359883e-003$	$38.2816528736560e-003$
a_6	$-266.921985849956e-003$	$-148.918619346875e-003$
a_8	$518.013932645790e-003$	$362.169449724903e-003$
a_{10}	$-628.794997076006e-003$	$-572.770850244477e-003$
a_{12}	$467.435352356885e-003$	$590.597547793815e-003$
a_{14}	$-195.102884267521e-003$	$-383.787896145751e-003$
a_{16}	$35.1996995454345e-003$	$142.994125609753e-003$
a_{18}	$-83.2331713482651e-006$	$-23.4009021143565e-003$
a_{20}	$73.4641541750645e-009$	$51.6872738576394e-006$
a_{22}	$-66.1406495808850e-012$	$-41.8299333983849e-009$
a_{24}	$-33.0527088686279e-015$	$37.7477344668050e-012$
a_{26}	$-190.629268922604e-018$	$24.9953548059809e-015$

CHAPTER IV

CONCENTRATION PROBLEM OVER DISJOINT INTERVALS

A. Introduction

It is important in *communication theory* to solve the *concentration problem*. There are many different problems which fall into this category. In this chapter we shall precisely study the problem to determine which bandlimited signals lose the smallest fraction of their energy when restricted in a given time interval. Slepian et. al. in their papers (see [46, 47, 51]) considered this problem for a *connected symmetric* time and frequency interval. In their paper they obtained that for this problem the solution actually correspond to the eigenfunction corresponding to the largest eigenvalue of the integral equation $\int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)}\psi(y)dy = \lambda\psi(x)$, where $|x| \leq 1$. A commuting differential operator can be found for this integral operator and thus it is easy to solve the problem for connected symmetric interval. However it turns out that for *disjoint interval* cases the problem is more involved.

In this chapter we shall study the *concentration problem* for disjoint intervals. We shall consider the problem of finding the bandlimited signals (for *disjoint frequency domains*) which lose the smallest fraction of their energy when restricted in a given connected and symmetric time interval.

Throughout this chapter we denote the Fourier transform pair as

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt$$

and

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega.$$

We use Parseval's Theorem to get $\langle g(t), g(t)\rangle = \frac{1}{2\pi}\langle G(\omega), G(\omega)\rangle$, that is

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega.$$

Suppose $g(t)$ is a bandlimited signal, *Fourier Transform of* $g(t) = G(\omega)$ and $|G(\omega)| = 0$ when ω is outside $E = [a_1, b_1] \cup [a_2, b_2]$. Then $Bg = g$ and we want to maximize

$$\lambda = \frac{||DB(g)||^2}{\int_{-\infty}^{\infty} |g(t)|^2 dt} = \frac{\int_{-c}^{c} |g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt}, \tag{IV.1}$$

where $g(t)$ is a bandlimited signal, *Fourier Transform of* $g(t) = G(\omega)$ and $|G(\omega)| = 0$ when ω is outside $E = [a_1, b_1] \cup [a_2, b_2]$. Then

$$\lambda = \frac{\frac{1}{4\pi^2}\int_{-c}^{c} dt \int_E e^{iyt} G(y) dy \int_E e^{-ixt} G(x)^* dx}{\frac{1}{2\pi}\int_E |G(x)|^2 dx} = \frac{1}{2\pi}\frac{\int_E dx \int_E \frac{2\sin c(x-y)}{(x-y)} G(x)^* G(y) dy}{\int_E G(x) G(x)^* dx}$$
$$= \frac{\int_E dx \int_E \frac{\sin c(x-y)}{\pi(x-y)} G(x)^* G(y) dy}{\int_E G(x) G(x)^* dx}$$

Here $G(x)$ is an arbitrary function in $L^2(E)$. We will maximize λ in the above expression. Observing $||G||^2 = ||G^*||^2 = \int_E G(x)G(x)^* dx$, we can write

$$\lambda = \int_E dx \int_E \frac{\sin c(x-y)}{\pi(x-y)} \frac{G(x)^*}{||G*||} \frac{G(y)}{||G||} dy \tag{IV.2}$$

If we denote $A(x) = \frac{G(x)^*}{||G*||}$ and $B(y) = \frac{G(y)}{||G||}$, then clearly $||A||^2 = ||B||^2 = 1$. We define

$$C_B(x) = \int_E \frac{\sin c(x-y)}{\pi(x-y)} B(y) dy. \tag{IV.3}$$

We shall consider the following problem:

Maximize

$$\lambda = \int_E A(x) C_B(x) dx \tag{IV.4}$$

under the condition $||A||^2 = 1$.

To solve this variational problem we define (noting $A(x)B(x) = A(x)A(x)^* = ||A||^2$)

$$\Omega(A) = \int_E A(x)C_B(x)dx - \nu(\int_E A(x)B(x)dx - 1)$$
$$= \int_E (C_B(x) - \nu B(x))\, A(x)dx + \nu$$

where ν is the *Lagrange multiplier*. The *Euler-Lagrange equation* for this problem will then yield

$$C_B(x) - \nu B(x) = 0,$$

which gives

$$\int_E \frac{\sin c(x-y)}{\pi(x-y)} B(y)dy = \nu B(x),$$

i.e.,

$$\int_{[a_1,b_1]\cup[a_2,b_2]} \frac{\sin c(x-y)}{\pi(x-y)} B(y)dy = \nu B(x), \tag{IV.5}$$

where $x \in E$. Therefore to maximize λ in (IV.1) we *must* find the maximum eigenvalue ν for (IV.5). This is the *Fourier Transform* characterization of our problem.

For *angular prolate spheroidal functions* $S_{0n}(c,x)$ and $\lambda_n(c) = \frac{2c}{\pi}[R^{(1)}_{0n}(c,1)]^2$, where $R^{(1)}_{0n}(c,x)$ are *radial prolate spheroidal functions* and $n = 0, 1, 2, \ldots$. Slepian and Pollak defined in [46]

$$\psi_n(c,x) = \frac{\sqrt{\lambda_n(c)}}{||S_{0n}(c,t)||} S_{0n}(c,t).$$

We shall refer to this function as Slepian's function. Slepian functions are studied extensively in [46, 47, 48, 49, 50].

Proposition IV.1. *Let* $\{\psi_n(c,x)\}_{n=0}^{\infty}$ *be the Slepian's functions. Then*

$$\frac{\sin c(x-y)}{\pi(x-y)} = \sum_{n=0}^{\infty} \psi_n(c,x)\psi_n(c,y),$$

in $L^2((-1,1) \times (-1,1))$.

Proof. Fix $x \in (-1,1)$. Then $\frac{\sin c(x-y)}{\pi(x-y)}$ is a square integrable function with respect to y. Therefore it can be represented as an expansion of $\{\psi_n(y)\}_{n=0}^{\infty}$ as

$$\frac{\sin c(x-y)}{\pi(x-y)} = \sum_{n=0}^{\infty} c_n(x)\psi_n(y), \tag{IV.6}$$

and the $c_n(x)$'s can be calculated as

$$c_n(x) = \left(\int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)}\psi_n(y)dy\right) / \left(\int_{-1}^{1} |\psi_n(y)|^2 dy\right).$$

On the other hand, Slepian's functions $\{\psi_n(x)\}_{n=0}^{\infty}$ are by definition normalized to satisfy

$$\int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)}\psi_n(y)dy = \lambda_n \psi_n(x),$$

and

$$\int_{-1}^{1} |\psi_n(y)|^2 dy = \lambda_n.$$

Hence

$$c_n(x) = \lambda_n \psi_n(x) / \int_{-1}^{1} |\psi_n(y)|^2 dy = \psi_n(x).$$

This completes the proof. □

B. Proof of existence and some properties

We consider *symmetric disjoint intervals.* For our convenience we consider $J = [a_1, b_1] \cup [a_2, b_2] = [-2,-1] \cup [1,2]$. We have the following problem

$$\int_{-2}^{-1} \frac{\sin c(x-y)}{\pi(x-y)} f(y)dy + \int_{1}^{2} \frac{\sin c(x-y)}{\pi(x-y)} f(y)dy = \mu f(x) \tag{IV.7}$$

where $x \in [-2,-1] \cup [1,2]$. The integration on the left hand side defines an operator from $L^2((-2,-1) \cup (1,2)) \to L^2((-2,-1) \cup (1,2))$ as follows

$$K(f)(x) = \left(\int_{-2}^{-1} + \int_{1}^{2}\right) \frac{\sin c(x-y)}{\pi(x-y)} f(y)dy,$$

with the kernel given by

$$Ker(x,y) = \frac{\sin c(x-y)}{\pi(x-y)}.$$

Clearly $Ker(x,y)$ is bounded (as $|Ker(x,y)| = |\frac{\sin c(x-y)}{\pi(x-y)}| \leq \frac{c}{\pi}$), and thus square integrable. Moreover it is symmetric with respect to x and y. It follows from a classical theorem in functional analysis that K is a compact symmetric operator (cf. [23, 29]). Therefore K is a *compact, symmetric operator* from $L^2((-2,-1)\cup(1,2))$ to $L^2((-2,-1)\cup(1,2))$. Furthermore, it possesses countably many eigenvalues which can be ordered as

$$\mu_0 \geq \mu_1 \geq \dots \mu_n \geq \mu_{n+1} \geq \dots \to 0$$

and its *orthonormal eigenfunctions* is *complete* in $L^2((-2,-1)\cup(1,2))$.

Proposition IV.2. *K is positive definite.*

Proof. Define an extension operator $\bar{E} : L^2((-2,-1)\cup(1,2)) \to L^2(-2,2)$ as follows:

$$\bar{E}(f)(x) = \begin{cases} f(x), & x \in (-2,-1)\cup(1,2) \\ \\ 0, & x \in (-1,1) \end{cases}$$

Then

$$K(f)(x) = \int_{-2}^{2} \frac{\sin c(x-y)}{\pi(x-y)} \bar{E}(f)(y)dy,$$

and

$$\langle K(f), f\rangle = \left(\int_{-2}^{-1} + \int_{1}^{2}\right) K(f)(x)f(x)dx = \int_{-2}^{2}\left(\int_{-2}^{2} \frac{\sin c(x-y)}{\pi(x-y)}\bar{E}(f)(y)dy\right)\bar{E}(f)(x)dx. \tag{IV.8}$$

It is well known that the kernel $\frac{\sin c(x-y)}{\pi(x-y)}$ is positive definite over any connected symmetric interval, i.e.,

$$\int_{-2}^{2}\left(\int_{-2}^{2} \frac{\sin c(x-y)}{\pi(x-y)}\bar{E}(f)(y)dy\right)\bar{E}(f)(x)dx \geq 0.$$

Hence we must have $\langle K(f), f\rangle \geq 0$, $\forall f \in L^2((-2,-1)\cup(1,2))$ and $\langle K(f), f\rangle = 0$ implies that $\bar{E}(f)(x) \equiv 0$ where $x \in [-2,2]$. That is, $f(x) \equiv 0$, $x \in (-2,-1)\cup(1,2)$. This completes the proof. □

Suppose we are considering the interval $E = [-b_2, -a_2] \cup [a_2, b_2]$, where $a_2, b_2 > 0$. Let $\phi_n(x)$ and $\phi_m(x)$ be eigenfunctions of (IV.7) corresponding to two distinct eigenvalues μ_n and μ_m. We define

$$\Phi_i(x) = \sqrt{\mu_i}\frac{\phi_i(x)}{||\phi_i(x)||_{L^2(E)}}, \quad i = 0, 1, 2, \cdots$$

Then the following proposition follows easily.

Proposition IV.3.

$$\int_{[-a_2,-b_2]\cup[a_2,b_2]} \Phi_n(x)\Phi_m(x)dx = \mu_n\delta_{mn}.$$

Next we shall prove that $\{\Phi_i(x)\}_{i=0}^{\infty}$ are orthonormal in $(-\infty,\infty)$. We extend $\Phi_i(x)$, $i = 0, 1, 2, \cdots$ from E to $(-\infty,\infty)$ analytically by using (IV.7) for the generalized case.

Lemma IV.4. *Suppose* $c_1 \geq c_2 > 0$. *Then*

$$\int_{-\infty}^{\infty} \rho_{c_1}(t-u)\rho_{c_2}(u-s)du = \rho_{c_2}(t-s)$$

where $\rho_c(\tau) = \frac{\sin c\tau}{\pi\tau}$.

Proof. It is clear that we can rewrite $\rho_c(\tau) = \frac{1}{2\pi}\int_{-c}^{c} e^{i\omega\tau}d\omega$. Since for any c, $\rho_c(\tau)$ is even, $\rho_c(t-u) = \rho_c(u-t) = \frac{1}{2\pi}\int_{-c}^{c} e^{i\omega(u-t)}d\omega$. If we take $f(u) = \rho_{c_1}(u-t)$ then the Fourier transform is given by $F(\omega) = e^{-i\omega t}\chi_{(-c_1,c_1)}$, where $\chi_{(a,b)}$ is the characteristic function of the interval (a,b). Then taking $g(u) = \rho_{c_2}(u-s)$ we have $G(\omega) = e^{-i\omega s}\chi_{(-c_2,c_2)}$. We use Parseval's Theorem to get $\langle f(u), g(u)\rangle = \frac{1}{2\pi}\langle F(\omega), G(\omega)\rangle$. Using $c_1 \geq c_2 > 0$, we thus obtain

$$\int_{-\infty}^{\infty} \rho_{c_1}(t-u)\rho_{c_2}(u-s)du = \frac{1}{2\pi}\int_{-c_2}^{c_2} e^{-i\omega t}e^{i\omega s}d\omega = \frac{1}{2\pi}\int_{-c_2}^{c_2} e^{i\omega(s-t)}d\omega = \rho_{c_2}(s-t) = \rho_{c_2}(t-s).$$

□

We shall now prove the following proposition.

Proposition IV.5.

$$\int_{-\infty}^{\infty} \Phi_n(t)\Phi_m(t)dx = \delta_{mn}.$$

Proof. We know that for all n and m, $\Phi_n(t)$, $\Phi_m(t)$ are solutions of

$$\mu f(t) = \int_{[-a_2,-b_2]\cup[a_2,b_2]} \rho_c(t-s)f(s)ds, \quad t \in [-a_2,-b_2]\cup[a_2,b_2], \tag{IV.9}$$

where $\rho_c(\tau) = \frac{\sin(c\tau)}{\pi\tau} = \frac{1}{2\pi}\int_{-c}^{c} e^{i\omega\tau}d\omega$. Now using (IV.9) we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Phi_n(t)\Phi_m(t)dt \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{\mu_n}\int_{[-a_2,-b_2]\cup[a_2,b_2]} \rho_c(t-s)\Phi_n(s)ds\right)\left(\frac{1}{\mu_m}\int_{[-a_2,-b_2]\cup[a_2,b_2]} \rho_c(t-u)\Phi_m(u)du\right)dt \\
&= \frac{1}{\mu_n\mu_m}\int_{[-a_2,-b_2]\cup[a_2,b_2]} du \int_{[-a_2,-b_2]\cup[a_2,b_2]} \Phi_n(s)\Phi_m(u)ds \int_{-\infty}^{\infty} \rho_c(u-t)\rho_c(t-s)dt
\end{aligned}$$

Use Lemma IV.4 to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \Phi_n(t)\Phi_m(t)dt &= \frac{1}{\mu_n\mu_m}\int_{[-a_2,-b_2]\cup[a_2,b_2]} du \int_{[-a_2,-b_2]\cup[a_2,b_2]} \Phi_n(s)\Phi_m(u)\rho_c(u-s)ds \\
&= \frac{1}{\mu_n\mu_m}\int_{[-a_2,-b_2]\cup[a_2,b_2]} \Phi_m(u)du \int_{[-a_2,-b_2]\cup[a_2,b_2]} \rho_c(u-s)\Phi_n(s)ds
\end{aligned}$$

Finally using (IV.9) and Proposition IV.3 we obtain

$$\int_{-\infty}^{\infty} \Phi_n(t)\Phi_m(t)dx = \delta_{mn}.$$

This completes the proof. □

C. Algorithm description

We consider disjoint interval problems. Let E be any bounded measurable set of positive measure in $[-1,1]$. Consider

$$\int_E \frac{\sin c(x-y)}{\pi(x-y)} f(y)\mathrm{d}y = \mu f(x). \tag{IV.10}$$

For E not in $[-1,1]$, we may transform the problem into $[-1,1]$ by changing variable. We may not find an equivalent differential operator, since the interval is irregular. Suppose a solution is represented as an expansion of the Slepian's eigenfunctions as

$$f(x) = \sum_{n=0}^{\infty} a_n \psi_n(x). \tag{IV.11}$$

Then using Proposition IV.1 and (IV.11) in (IV.10), we have

$$\int_E \sum_{n=0}^{\infty} \psi_n(x)\psi_n(y) \sum_{m=0}^{\infty} a_m \psi_m(y)\mathrm{d}y = \mu \sum_{n=0}^{\infty} a_n \psi_n(x).$$

or

$$\sum_n \psi_n(x)\{\sum_m \int_E \psi_n(y)\psi_m(y)\mathrm{d}y\} a_m = \mu \sum_n a_n \psi_n(x)$$

Comparing the coefficients of $\psi_n(x)$ on both sides, we get

$$\sum_m \{\int_E \psi_n(y)\psi_m(y)\mathrm{d}y\} a_m = \mu a_n, \quad n = 0, 1, 2, \cdots \tag{IV.12}$$

We define

$$\tilde{d}_{nm} = \int_E \psi_n(y)\psi_m(y)dy \tag{IV.13}$$

$$\tilde{D} = (\tilde{d}_{nm})_{\infty\times\infty},$$

and

$$\tilde{A} = (a_0, a_1, a_2, \cdots)^T.$$

Then (IV.12) can be represented as

$$\tilde{D}\tilde{A} = \mu\tilde{A}. \tag{IV.14}$$

It is easy to see that (IV.12) is a eigenvalue problem of linear algebraic equations system of infinite order. An approximate solution can be described as

$$f_N(x) = \sum_{n=0}^{N-1} a_n \psi_n(x).$$

Its corresponding linear algebraic equations system would be

$$\tilde{D}_N \tilde{A}_N = \mu \tilde{A}_N, \tag{IV.15}$$

where $\tilde{D}_N = (\tilde{d}_{nm})_{N\times N}$ and $\tilde{A}_N = [a_0, a_1, \cdots, a_{N-1}]^T$.

D. Convergence analysis

It is clear that

$$\begin{aligned}|\tilde{d}_{nm}| = |\int_E \psi_n(y)\psi_m(y)dy| &\leq \sqrt{\int_E |\psi_n(y)|^2 dy}\sqrt{\int_E |\psi_m(y)|^2 dy} \\ &\leq \sqrt{\int_{-1}^{1} |\psi_n(y)|^2 dy}\sqrt{\int_{-1}^{1} |\psi_m(y)|^2 dy} = \sqrt{\lambda_n \lambda_m}.\end{aligned}$$

Therefore

$$\tilde{d}_{nm}^2 \leq \lambda_n \lambda_m, \tag{IV.16}$$

where λ_i satisfies $\int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)} \psi_i(y)\mathrm{d}y = \lambda_i \psi_i(x),\ i = 0, 1, 2, \cdots$ In [48] it is proved that

$$\sum_{i=0}^{\infty} \lambda_i = \Delta < \infty \tag{IV.17}$$

and for $m \geq \frac{2c}{\pi}$,

$$\lambda_m \leq \frac{\frac{\log^+ c}{\pi^2} + 1}{m - \frac{2c}{\pi}}.$$

where $\log^+ c = \max\{\log c, 0\}$. Another important tool is the *Wielandt-Hoffman theorem* as follows (see [60]).

Proposition IV.6. *If $C = A + B$, where A, B and C are symmetric matrices having eigenvalues α_i, β_i and γ_i respectively arranged in non-increasing order, then*

$$\sum_i (\gamma_i - \alpha_i)^2 \leq \|B\|_E^2,$$

where $\|\cdots\|_E$ refers to the Frobenius norm of matrices. For matrix $A = (a_{ij})_{m\times n}$, Frobenius norm is defined by $||A||_{Frob} = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2}$.

Now we prove the following convergence theorem.

Theorem IV.7. *Let $\mu_0^{(N)}$, $\mu_1^{(N)}$, $\cdots$, $\mu_n^{(N)}$ be the first $n+1$ eigenvalues of (IV.15). Then they converge as $N \to \infty$. That is to say*

$$\lim_{N\to\infty} \mu_i^{(N)} = \mu_i, \quad i = 0, 1, \cdots, n.$$

Proof. Let N and M be two non-negative integer, $\mu_i^{(N)}$ and $\mu_i^{(N+M)}$, $i = 0, 1, \cdots, N-1$ be the first N eigenvalues of $\tilde{D}_N$ and $\tilde{D}_{N+M}$ respectively. For convenience, we extend $\tilde{D}_N$ to $N + M$ rows and columns by zero. Denote the extension by $\tilde{D}^{(N,M)}$, i.e.,

$$\tilde{D}^{(N,M)} = (\tilde{d}_{ij}^{(N,M)})_{(N+M)\times(N+M)},$$

where

$$\tilde{d}_{ij}^{(N,M)} = \begin{cases} \tilde{d}_{ij}, & 0 \leq i, j \leq N-1 \\ 0, & N \leq i \text{ or } j \leq N+M-1. \end{cases}$$

Of course, $\mu_i^{(N)}$, $i = 0, 1, \cdots, N-1$ are also the first N eigenvalues of $D^{(N,M)}$, and the remainders are zero. It follows from Proposition IV.6, (IV.16) and (IV.17) that

$$\begin{aligned}\sum_{i=0}^{N-1} |\mu_i^{(N+M)} - \mu_i^{(N)}|^2 &\le \|\tilde{D}_{N+M} - \tilde{D}^{(N,M)}\|_E^2 \\ &= \sum_{m=0}^{N+M-1} \sum_{n=N}^{N+M-1} \tilde{d}_{nm}^2 + \sum_{n=0}^{N-1} \sum_{m=N}^{N+M-1} \tilde{d}_{nm}^2 \\ &< \sum_{m=0}^{\infty} \sum_{n=N}^{N+M-1} \tilde{d}_{nm}^2 + \sum_{n=0}^{\infty} \sum_{m=N}^{N+M-1} \tilde{d}_{nm}^2. \\ &\le \sum_{m=0}^{\infty} \sum_{n=N}^{N+M-1} \lambda_n \lambda_m + \sum_{n=0}^{\infty} \sum_{m=N}^{N+M-1} \lambda_n \lambda_m \\ &= 2\Delta \sum_{n=N}^{N+M-1} \lambda_n\end{aligned}$$

Fix $\epsilon > 0$. Choose N in such a way that $\sum_{n=N}^{\infty} \lambda_n < \frac{\epsilon}{4\Delta}$. The choice of such N is possible as $\sum_{n=0}^{\infty} \lambda_n < \infty$. Therefore we have $\forall \epsilon > 0$, $\exists N$ such that for all $M > 0$

$$\sum_{i=0}^{N-1} |\mu_i^{(N+M)} - \mu_i^{(N)}|^2 < \frac{\epsilon}{2}.$$

Then for any $N_1, N_2 > N$, $|\mu_i^{(N_1)} - \mu_i^{(N_2)}|^2 < \epsilon$ and hence $\{\mu_i^{(N)}\}_{N=0}^{\infty}$ is a Cauchy sequence for $i = 0, 1, 2, \cdots$. This completes the proof. □

Remark IV.8. The above deduction did not take into account the result that for any fixed c, the $\lambda_n(c)$, $n = 0, 1, 2, \cdots$, form a positive sequence bounded away from 1 and approaching 0 exponentially with n. This fact has been stated or used in a number of papers (see [5, 6, 24, 25, 30, 36, 51, 52, 58]). So for sufficiently large N, $\mu_i^{(N)}$ is actually rapidly converging to μ_i.

Remark IV.9. Letting $M \to \infty$ we obtain from the proof of Theorem IV.7

$$\sum_{i=0}^{N-1} |\mu_i^{(N)} - \mu_i|^2 \le 2\Delta \sum_{n=N}^{\infty} \lambda_n.$$

From [58] it follows that for a fixed c, and sufficiently large n, $\lambda_n \sim \frac{1}{e^3}\left(\frac{4}{ec}(n+\frac{1}{2})\right)^{-2n-1}$. Therefore for large n $\lambda_n < \frac{1}{e^3}\left(\frac{4n}{ec}\right)^{-2n-1}$. Choose n large enough so that $\frac{4n}{ec} > 2$. Hence for sufficiently large n, $\lambda_n < \frac{1}{e^3}\frac{1}{2^{2n+1}}$. Hence $\sum_{n=N}^{\infty}\lambda_n < \frac{1}{3e^3 2^{2N-1}}$, for sufficiently large N. Therefore for sufficiently large N,

$$\sum_{i=0}^{N-1}|\mu_i^{(N)} - \mu_i|^2 < 2\Delta\frac{1}{3e^3 2^{2N-1}}.$$

This implies that the error of numerical eigenvalues decays at an exponential rate.

Remark IV.10. The convergence of the eigenvectors follows from the rapid decay of the $\tilde{d}$'s and the perturbation theory of eigenvectors (see section 24 in Chapter II of [60]). Let $\tilde{A}_N^{(i)}$ and $\tilde{A}_{N+1}^{(i)}$ be two eigenvectors related to $\mu_i^{(N)}$ and $\mu_i^{(N+1)}$ respectively. Then it follows from arguments as in [60] that

$$\|\tilde{A}_{N+1}^{(i)} - \tilde{A}_N^{(i)}\|_2 \le (N+1)^{3/2}\sqrt{2\Delta\lambda_N\sum_{n=N}^{\infty}\lambda_n}.$$

So it follows from Remark IV.9, that for sufficiently large N

$$\|\tilde{A}_{N+1}^{(i)} - \tilde{A}_N^{(i)}\|_2 < (N+1)^{3/2}\frac{1}{e^3 2^{2N}}\sqrt{\frac{2\Delta}{3}}.$$

Therefore the error of numerical eigenvectors decays at an exponential rate.

E. Numerical results

We took $E = [-1,-0.5]\cup[0.5,1]$, $N = 8$ and solved (IV.15) with Matlab7. Our maximum eigenvalue is $\mu_0 = 0.2638$, its corresponding eigenfunction is $\phi_0 = 0.9991\psi_0 + 0.042\psi_2$; the second largest eigenvalue is $\mu_1 = 0.0542$, its corresponding eigenfunction is ψ_1; the other eigenvalues are of below 10^{-4} degree. We computed $\mu_0\phi_0(x)$ at

$$x = -0.9, -0.8, -0.7, -0.6, -0.5, 0.5, 0.6, 0.7, 0.8, 0.9,$$

and got data as follows.

$$0.1305, 0.1341, 0.1374, 0.1402, 0.1426, 0.1426, 0.1402, 0.1374, 0.1341, 0.1305.$$

Then we computed $[\int_{-1}^{-0.5} + \int_{0.5}^{1}] \frac{\sin(x-y)}{\pi(x-y)} \phi_0(y) \mathrm{d}y$ at

$$x = -0.9, -0.8, -0.7, -0.6, -0.5, 0.5, 0.6, 0.7, 0.8, 0.9,$$

and got data as follows.

$$0.1305, 0.1341, 0.1373, 0.1402, 0.1426, 0.1426, 0.1402, 0.1373, 0.1341, 0.1305.$$

The values of $\mu_1 \phi_1$ at those points are

$$-0.0146, -0.0132, -0.0118, -0.0102, -0.0086, 0.0086, 0.0102, 0.0118, 0.0132, 0.0146.$$

The values of $[\int_{-1}^{-0.5} + \int_{0.5}^{1}] \frac{\sin(x-y)}{\pi(x-y)} \phi_1(y) \mathrm{d}y$ at those points are

$$-0.0147, -0.0132, -0.0118, -0.0102, -0.0086, 0.0086, 0.0102, 0.0118, 0.0132, 0.0147.$$

The numerical experiments show that our algorithm converges rapidly at high accuracy.

CHAPTER V

CONCLUSIONS

In our work we have found a complete set of eigenfunctions for the superradiance problem in three dimensions. The commuting differential operator obtained for the radial part for three-dimensional problem is significant as this can be generalized easily for higher dimensions. We have also studied many interesting properties of this differential operator. As we have already seen the three dimensional superradiance problem actually reduces to solving some integral equation of the radial part which has separable kernel. Thus only one (up to multiplicity) eigenfunction corresponds to non-zero eigenvalue. All the rest of the eigenfunctions correspond to the zero-eigenvalue of the problem. We extracted a complete set of eigenfunctions from this null set of the radial integral operator by finding a differential operator that commutes with the integral operator. However it is clear that this differential operator is non-unique. In our case we took $p(r) = 1 - r^2$ and chose $q(r)$ accordingly. The reason for this is twofold. First it keeps our calculations as simple as possible, and second, our differential operator is surprisingly similar (though not identical) with the differential operator obtained by Slepian [49] with a different kernel integral equation in two dimensions. But we could as well have chosen a different $p(r)$ as long as it is sufficiently smooth and $p(1) = 0$. Correspondingly it is essential to modify $q(r)$ as well. A detailed study of the dependence of $q(r)$ over $p(r)$ has been given in Chapter III, section D. However the uniqueness of the commuting differential operator for the superradiance kernel restricted in one dimension [46] is established in Chapter III, section F. Indeed without doing any computation it follows from the main result of that section.

In our problem we considered the integral over the unit ball with center at the origin. This can be generalized to any ball in 3-dimension without much difficulty.

However, there are still some questions which are to be addressed. For instance the generalization that we have provided is a natural generalization when we view the series expansion of the kernel for the three-dimensional superradiance problem

$$\frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_n(cr) j_n(cr') Y_{nm}(\xi) Y_{nm}^{*}(\eta),$$

where $\mathbf{x} = (r, \xi)$ and $\mathbf{y} = (r', \eta)$ are in $\mathbb{R}^3$. That is in our work we generalized the series expansion for the kernel. In near future we want to study the superradiance problem with kernel

$$\frac{\sin(c|\mathbf{x}-\mathbf{y}|)}{c|\mathbf{x}-\mathbf{y}|}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ for any dimension n and find a complete set of eigenfunctions for that problem. Also it might be useful to find a self adjoint operator in higher dimension that commutes directly with the original integral operator (i.e., we do not need to derive an equation for the radial part of the eigenfunctions). In that case we can carry out an analysis similar to [18] or [57] to obtain a complete set of solutions for the original problem.

The eigenfunctions that appeared as the solution of the differential equation have many interesting properties. They are complete in $L^2(0,1)$ and with the increasing eigenvalue the number of zeros in $(0,1)$ for the corresponding eigenfunction also increases. The difference between the number of zeros in $(0,1)$ of any two successive eigenfunction is exactly 1. Other properties of the solution worth further investigation. Another interesting further research may be carried out on the properties of the eigenvalues of the differential equation related to the *three-dimensional superradiance problem*. More research is needed for better approximation of these eigenvalues as the truncation of continued fraction is not always very effective for computational purpose. It will be useful if a convergence rate formula for the eigenvalues can be derived.

It will be useful to consider the three dimensional problem over different domains. The domains can be disconnected set, topological groups etc. Even the domain may be non-symmetric. It will be a challenging problem to extend our results for such generalized domains.

In Chapter IV, we mostly worked over the union of disjoint intervals E. In future we want to consider any set $E \subseteq [-1, 1]$, having positive measure. Numerical methods may not be useful in that case and we need to develop a technique of finding analytic eigenfunctions. A problem similar to the higher dimensional generalization of Slepian's original concentration problem has many physical applications. For example, they are used in the *superradiance problem* in quantum optics (see [38, 39]). In future we want to generalize the concentration problem over higher dimensions. This generalization will be useful for physical applications.

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