ME617 - Handout 7

(Undamped) Modal Analysis of MDOF Systems

The governing equations of motion for a *n*-DOF linear mechanical system with viscous damping are:

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{D}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U}_{(t)} = \mathbf{F}_{(t)} \qquad (1)$$

where $\mathbf{U}, \dot{\mathbf{U}},$ and $\ddot{\mathbf{U}}$ are the vectors of generalized displacement, velocity and acceleration, respectively; and $\mathbf{F}_{(t)}$ is the vector of generalized (external forces) acting on the system.
$\mathbf{M}, \mathbf{D}, \mathbf{K}$ represent the matrices of inertia, viscous damping and stiffness coefficients, respectively[1].

The solution of Eq. (1) is uniquely determined once initial conditions are specified. That is,

$$\text{at } t = 0 \;\rightarrow \mathbf{U}_{(0)} = \mathbf{U}_o, \;\; \dot{\mathbf{U}}_{(0)} = \dot{\mathbf{U}}_o \qquad (2)$$

In most cases, i.e. conservative systems, the inertia and stiffness matrices are **SYMMETRIC**, i.e. $\mathbf{M} = \mathbf{M}^T \;,\; \mathbf{K} = \mathbf{K}^T$. The kinetic energy (*T)* and potential energy (*V*) in a conservative system are

$$T = \frac{1}{2}\dot{\mathbf{U}}^T\,\mathbf{M}\,\dot{\mathbf{U}}, \quad V = \frac{1}{2}\mathbf{U}^T\,\mathbf{K}\,\mathbf{U} \qquad (3)$$

[1] The matrices are square with *n*-rows = *n* columns, while the vectors are n-rows.

In addition, since $T > 0$, then **M** is a positive definite matrix[2]. If $V > 0$, then **K** is a positive definite matrix. $V=0$ denotes the existence of a rigid body mode, and makes **K** a semi-positive matrix.

In MDOF systems, a **natural state** implies a certain configuration of shape taken by the system during motion. Moreover a MDOF system does not possess only ONE natural state but a *finite number* of states known as **natural modes of vibration.** Depending on the initial conditions or external forcing excitation, the system can vibrate in any of these modes or a combination of them. To each mode corresponds a unique frequency knows as **a natural frequency.** There are as many natural frequencies as natural modes.

The modeling of a *n*-DOF mechanical system leads to a set of *n*-coupled 2nd order ODEs, Hence the motion in the direction of one DOF, say *k,* depends on or it is coupled to the motion in the other degrees of freedom, $j=1,2\ldots n$.

In the analysis below, for a proper choice of generalized coordinates, known as **principal or natural coordinates,** the system of *n*-ODE describing the system motion is independent of each other, i.e. uncoupled. The natural coordinates are linear combinations of the (actual) physical coordinates, and conversely. Hence, the motion in physical coordinates can be construed or interpreted as the superposition or combination of the motions in each natural coordinate.

[2] **Positive definite** means that the determinant of the matrix is greater than zero. More importantly, it also means that all the matrix eigenvalues will be positive. A **semi-positive** matrix has a zero determinant, with at least an eigenvalues equaling zero.

For simplicity, begin the analysis of the system by neglecting damping, **D=0**. Hence, Eq.(1) reduces to

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\,\mathbf{U}_{(t)} = \mathbf{F}_{(t)} \qquad (4)$$

and at $t = 0 \rightarrow \mathbf{U}_{(0)} = \mathbf{U}_o, \;\; \dot{\mathbf{U}}_{(0)} = \dot{\mathbf{U}}_o$

Presently, set the external force **F=0,** and let's find the **free vibrations response** of the system.

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{0} \qquad (5)$$

The solution to the homogenous Eq. (5) is simply

$$\mathbf{U} = \boldsymbol{\varphi}\cos(\omega t - \theta) \qquad (6)$$

which denotes a periodic response with a typical frequency ω. From Eq. (6),

$$\ddot{\mathbf{U}} = -\boldsymbol{\varphi}\,\omega^2\cos(\omega t - \theta) \qquad (7)$$

Note that Eq. (6) is a simplification of the more general solution

$$\mathbf{U} = \boldsymbol{\varphi}\,e^{st} \text{ with } s = i\,\omega \text{ and where } i = \sqrt{-1} \qquad (8)$$

Substitution of Eqs. (6) and (7) into the EOM (5) gives:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{0} \;\; \rightarrow$$

$$\rightarrow -\mathbf{M}\boldsymbol{\varphi}\,\omega^2\cos(\omega t - \theta) + \mathbf{K}\boldsymbol{\varphi}\cos(\omega t - \theta) = \mathbf{0}$$

$$\rightarrow \left[-\mathbf{M}\,\omega^2 + \mathbf{K}\right]\boldsymbol{\varphi}\cos(\omega t - \theta) = \mathbf{0}$$

and since $\cos(\omega t - \theta) \neq 0$ for most times, then

$$\left[-\mathbf{M}\,\omega^2 + \mathbf{K}\right]\boldsymbol{\varphi} = \mathbf{0} \tag{9}$$

or

$$\omega^2 \mathbf{M}\boldsymbol{\varphi} = \mathbf{K}\boldsymbol{\varphi} \tag{10}$$

Eq. (10) is usually referred as the **standard eigenvalue problem** (mathematical jargon):

$$\mathbf{A}\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}$$
$$\text{where } \mathbf{A} = \mathbf{M}^{-1}\mathbf{K} \;\; \text{and } \lambda = \omega^2 \tag{11}$$

Eq.(9) is a set of *n*-homogenous algebraic equations. A nontrivial solution, $\boldsymbol{\varphi} \neq \mathbf{0}$ exists <u>if and only if</u> the determinant Δ of the system of equations is zero, i.e.

$$\Delta = \left|-\mathbf{M}\,\omega^2 + \mathbf{K}\right| = 0 \tag{12}$$

Eq. (12) is known as the **characteristic equation** of the system. It is a polynomial in $\omega^2 = \lambda$, i.e.

$$\Delta = 0 = a_0 + a_1\,\omega^2 + a_2\,\omega^4 + a_3\,\omega^6 + \ldots.a_n\,\omega^n$$
$$\Delta = 0 = a_0 + \sum_{i=1}^{n}\left(a_i\,\lambda^i\right) \tag{13}$$

This polynomial or characteristic equations has *n*-roots, i.e. the set $\{\lambda_k\}_{k=1,2,\ldots.n}$ or $\{\pm\omega_k\}_{k=1,2,\ldots.n}$ since $\omega = \pm\sqrt{\lambda}$.

The ω's are known as the **natural frequencies** of the system. In the MATH jargon, the λ's are known as the **eigenvalues (**of matrix **A)**

Knowledge summary

a) A n-DOF system has n-natural frequencies.

b) If **M** and **K** are positive definite, then
$0<\omega_1 \leq \omega_2 ....\omega_{n-1} \leq \omega_n$.

c) If **K** is semi-positive definite, then
$0=\omega_1 \leq \omega_2 ....\omega_{n-1} \leq \omega_n$, i.e. at least one natural frequency is zero, i.e. motion with infinite period. This is known as **rigid body mode**.

Note that each of the natural frequencies satisfies Eq. (9). Hence, associated to each **natural frequency** (or **eigenvalues**) there is a corresponding **natural mode vector** (**eigenvector**) such that

$$\left[-\mathbf{M}\,\lambda_i + \mathbf{K}\right]\boldsymbol{\varphi}_{(i)} = \mathbf{0}, \quad {}_{i=1,\ldots n} \qquad (14)$$

The n-elements of an eigenvector are real numbers (for undamped system), with all entries defined except for a constant. The eigenvectors are unique in the sense that the ratio between two elements is constant, i.e.

$$\left(\varphi_{(k)_j} \big/ \varphi_{(k)_i} \right) = \text{constant for any } j,i = 1,\ldots.n$$

The actual value of the elements in the vector is entirely arbitrary. Since Eq. (14) is homogenous, if $\boldsymbol{\varphi}$ is a solution, so it is $\alpha\boldsymbol{\varphi}$ for any arbitrary constant α. Hence, one can say that the **SHAPE** of a **natural mode** is **UNIQUE** but not its amplitude.

For MDOF systems with a large number of degrees of freedom, n>>3, the eigenvalue problem, Eq. (11), is solved numerically.

Nowadays, PCs and mathematical computation software allow, with a single (simple) command, the evaluation of all (or some) eigenvalues and its corresponding eigenvectors in real time, even for systems with thousands of DOFs.

Long gone are the days when the graduate student or practicing engineer had to develop his/her own efficient computational routines to calculate eigenvalues. Handout # 9 discusses briefly some of the most popular numerical methods to solve the eigenvalue problem.

A this time, however, let's assume the **set of eigenpairs** $\left\{\omega_i, \boldsymbol{\varphi}_{(i)}\right\}_{i=1,2...n}$ is known.

Properties of natural modes

The natural modes (or eigenvectors) satisfy important **orthogonality properties**. Recall that each eigenpair $\left\{\omega_i, \boldsymbol{\varphi}_{(i)}\right\}_{i=1,2...n}$ satisfies the equation

$$\left[-\mathbf{M}\,\omega_i^2 + \mathbf{K}\right]\boldsymbol{\varphi}_{(i)} = \mathbf{0},\ _{i=1,...n}. \qquad (15)$$

Consider two different modes, say mode-*j* and mode-*k,* each satisfying

$$\omega_j^2\,\mathbf{M}\,\boldsymbol{\varphi}_{(j)} = \mathbf{K}\,\boldsymbol{\varphi}_{(j)} \quad \text{and} \quad \omega_k^2\,\mathbf{M}\,\boldsymbol{\varphi}_{(k)} = \mathbf{K}\,\boldsymbol{\varphi}_{(k)} \qquad (16)$$

Pre-multiply the equations above by $\boldsymbol{\varphi}_{(k)}^T$ and $\boldsymbol{\varphi}_{(j)}^T$ to obtain

$$\omega_j^2\,\boldsymbol{\varphi}_{(k)}^T\mathbf{M}\,\boldsymbol{\varphi}_{(j)} = \boldsymbol{\varphi}_{(k)}^T\mathbf{K}\,\boldsymbol{\varphi}_{(j)}$$

and (17)

$$\omega_k^2\,\boldsymbol{\varphi}_{(j)}^T\mathbf{M}\,\boldsymbol{\varphi}_{(k)} = \boldsymbol{\varphi}_{(j)}^T\mathbf{K}\,\boldsymbol{\varphi}_{(k)}$$

Now, perform some matrix manipulations. The products $\boldsymbol{\varphi}^{\mathbf{T}}\mathbf{M}\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^{\mathbf{T}}\mathbf{K}\boldsymbol{\varphi}$ are scalars, i.e. not a matrix nor a vector. The transpose of a scalar is the number itself. Hence,

$$\left(\boldsymbol{\varphi}_{(j)}^T\mathbf{K}\,\boldsymbol{\varphi}_{(k)}\right)^T = \left(\mathbf{K}\,\boldsymbol{\varphi}_{(k)}\right)^T\left(\boldsymbol{\varphi}_{(j)}^T\right)^T$$

$$= \boldsymbol{\varphi}_{(k)}^T\mathbf{K}^T\boldsymbol{\varphi}_{(j)}$$

$$= \boldsymbol{\varphi}_{(k)}^T\mathbf{K}\,\boldsymbol{\varphi}_{(j)} \quad \text{since} \quad \mathbf{K} = \mathbf{K}^T$$

and

$$\left(\boldsymbol{\varphi}_{(j)}^T\mathbf{M}\,\boldsymbol{\varphi}_{(k)}\right)^T = \boldsymbol{\varphi}_{(k)}^T\mathbf{M}\,\boldsymbol{\varphi}_{(j)} \quad \text{since} \quad \mathbf{M} = \mathbf{M}^T$$

for symmetric systems. Thus, Eqs. (17) are rewritten as

$$\omega_j^2\,\boldsymbol{\varphi}_{(j)}^T\mathbf{M}\,\boldsymbol{\varphi}_{(k)} = \boldsymbol{\varphi}_{(j)}^T\mathbf{K}\,\boldsymbol{\varphi}_{(k)} \quad (a)$$

and (18)

$$\omega_k^2\,\boldsymbol{\varphi}_{(j)}^T\mathbf{M}\,\boldsymbol{\varphi}_{(k)} = \boldsymbol{\varphi}_{(j)}^T\mathbf{K}\,\boldsymbol{\varphi}_{(k)} \quad (b)$$

Subtract (b) from (a) above to obtain

$$\left(\omega_j^2 - \omega_k^2\right)\boldsymbol{\varphi}_{(j)}^T\mathbf{M}\,\boldsymbol{\varphi}_{(k)} = 0 \qquad (19)$$

if $\omega_j \neq \omega_k$, i.e. for TWO different natural frequencies; then it follows that for $j \neq k$ $\quad \boldsymbol{\varphi}_{(j)}^T \mathbf{M} \boldsymbol{\varphi}_{(k)} = 0$ and $\boldsymbol{\varphi}_{(j)}^T \mathbf{K} \boldsymbol{\varphi}_{(k)} = 0$ (20)

for $j = k$ $\quad \boldsymbol{\varphi}_{(j)}^T \mathbf{M} \boldsymbol{\varphi}_{(j)} = M_j$ and $\boldsymbol{\varphi}_{(j)}^T \mathbf{K} \boldsymbol{\varphi}_{(j)} = K_j = \omega_j^2 M_j$ (20)

where K_j and M_j are known as the j-modal stiffness and j-modal mass, respectively.

Define a modal matrix $\boldsymbol{\Phi}$ has as its columns each of the eigenvectors, i.e.

$$\boldsymbol{\Phi} = \left[\boldsymbol{\varphi}_1 \quad \boldsymbol{\varphi}_2 \quad .. \quad \boldsymbol{\varphi}_n \right] \tag{21}$$

and the modal properties are written as

$$\boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} = \left[M \right]; \quad \boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi} = \left[K \right] \tag{22}$$

where $[M]$ and $[K]$ are **diagonal matrices** containing the modal mass and stiffnesses, respectively.

The eigenvector set $\boldsymbol{\varphi}_{\mathbf{k=1,\ldots n}}$ is linearly independent. Hence, any vector $(\mathbf{v})$ in n-dimensional space can be described as a linear combination of the natural modes, i.e.

$$\mathbf{v} = \sum_{j=1}^{n} a_j \boldsymbol{\varphi}_{(j)} = \boldsymbol{\Phi}\,\mathbf{a} \tag{23}$$

$$\mathbf{v} = \boldsymbol{\varphi}_1 a_1 + \boldsymbol{\varphi}_2 a_2 + .. + \boldsymbol{\varphi}_n a_n = \left[\boldsymbol{\varphi}_1 \quad \boldsymbol{\varphi}_2 \quad .. \quad \boldsymbol{\varphi}_n \right] \begin{bmatrix} a_1 \\ a_2 \\ .. \\ a_n \end{bmatrix} = \boldsymbol{\Phi}\,\mathbf{a}$$

System Response in Modal Coordinates

The orthogonality property of the natural modes (eigenvectors) permits the simplification of the analysis for prediction of system response. Recall that the equations of motion for the undamped system are

$$\mathbf{M}\ddot{\mathbf{U}}+\mathbf{K}\mathbf{U}_{(t)}=\mathbf{F}_{(t)} \qquad (4)$$

and at $t=0 \rightarrow \mathbf{U}_{(0)}=\mathbf{U}_0,\ \dot{\mathbf{U}}_{(0)}=\dot{\mathbf{U}}_0$

Consider the **modal transformation** $\mathbf{U}_{(t)}=\mathbf{\Phi}\,\mathbf{q}_{(t)}$ (24)[3]

And with $\ddot{\mathbf{U}}_{(t)}=\mathbf{\Phi}\,\ddot{\mathbf{q}}_{(t)}$, then EOM (4) becomes:

$$\mathbf{M}\mathbf{\Phi}\ddot{\mathbf{q}}+\mathbf{K}\mathbf{\Phi}\mathbf{q}=\mathbf{F}_{(t)}$$

which offers no advantage in the analysis. However, premultiply the equation above by $\mathbf{\Phi}^T$ to obtain

$$\left(\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}\right)\ddot{\mathbf{q}}+\left(\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}\right)\mathbf{q}=\mathbf{\Phi}^T\mathbf{F}_{(t)} \qquad (25)$$

and using the properties of the natural modes, $\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}=\left[M\right];\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}=\left[K\right]$, then Eq. (25) becomes

$$\left[M\right]\ddot{\mathbf{q}}+\left[K\right]\mathbf{q}=\mathbf{Q}=\mathbf{\Phi}^T\mathbf{F}_{(t)} \qquad (26)$$

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[3] Eq. (24) sets the physical displacements **U** as a function of the modal coordinates **q**. This transformation merely uses the property of linear independence of the natural modes.

And since [*M*] and [*K*] are diagonal matrices. Eq. (26) is just a set of *n*-uncoupled ODEs. That is,

$$
\begin{aligned}
M_1\,\ddot{q}_1 + K_1\,q_1 &= Q_1 \\
M_2\,\ddot{q}_2 + K_2\,q_2 &= Q_2 \\
&\ldots\ldots \\
M_n\,\ddot{q}_n + K_n\,q_n &= Q_n
\end{aligned}
\qquad (27)
$$

Or $$M_j\,\ddot{q}_j + K_j\,q_j = Q_j \quad \text{with } \omega_{n_j} = \sqrt{K_j / M_j}, \quad j=1,2\ldots n \qquad (28)$$

The set of *q's* are known as **modal or natural coordinates** (**canonical or principal**, too). The vector $\mathbf{Q} = \mathbf{\Phi}^T \mathbf{F}_{(t)}$ is known as the **modal force vector.**

Thus, the **major advantage of the modal transformation (24)** is that in **modal space the EOMS are uncoupled**. Each equation describes a mode as a SDOF system.

The unique solution of Eqs. (28) needs of **initial conditions** specified **in modal space**, i.e. $\{\mathbf{q}_o, \dot{\mathbf{q}}_o\}$.

Using the modal transformation, $\mathbf{U}_o = \mathbf{\Phi}\,\mathbf{q}_o;\ \dot{\mathbf{U}}_o = \mathbf{\Phi}\,\dot{\mathbf{q}}_o$, it follows

$$\mathbf{q}_o = \mathbf{\Phi}^{-1}\mathbf{U}_o \quad ; \quad \dot{\mathbf{q}}_o = \mathbf{\Phi}^{-1}\,\dot{\mathbf{U}}_o \qquad (28)$$

However, Eq. (28) requires of the inverse of modal matrix $\mathbf{\Phi}$, i.e. $\mathbf{\Phi}^{-1}\mathbf{\Phi} = \mathbf{I}$. For systems with a large number of DOF, $n >> 1$, finding the matrix $\mathbf{\Phi}^{-1}$ is computationally expensive.

A more efficient to determine the initial state $\{\mathbf{q}_o, \dot{\mathbf{q}}_o\}$ in modal coordinates follows. Start with the fundamental transformation, $\mathbf{U}_o = \mathbf{\Phi}\,\mathbf{q}_o$, and premultiply this relationship by $\mathbf{\Phi}^T\mathbf{M}$ to obtain,

$$\mathbf{\Phi}^T\mathbf{M}\,\mathbf{U}_o = \mathbf{\Phi}^T\mathbf{M}\,\mathbf{\Phi}\,\mathbf{q}_o$$
$$= \left[M\right]\mathbf{q}_o,$$

since $\left[M\right] = \mathbf{\Phi}^T\mathbf{M}\,\mathbf{\Phi}$, hence

$$\mathbf{q}_o = \left[M\right]^{-1}\mathbf{\Phi}^T\mathbf{M}\,\mathbf{U}_o\,, \qquad \dot{\mathbf{q}}_o = \left[M\right]^{-1}\mathbf{\Phi}^T\mathbf{M}\,\dot{\mathbf{U}}_o \tag{29a}$$

or

$$q_{o_k} = \frac{1}{M_k}\boldsymbol{\varphi}_{(k)}^T\left(\mathbf{M}\,\mathbf{U}_o\right),\ \dot{q}_{o_k} = \frac{1}{M_k}\boldsymbol{\varphi}_{(k)}^T\left(\mathbf{M}\,\dot{\mathbf{U}}_o\right) \tag{29b}$$

Eqs. (29) are much easier to calculate efficiently when *n*-DOF is large. Note that finding the inverse of the modal mass matrix **[*M*]**$^{-1}$ is trivial, since this matrix is diagonal.

Comparing eqs. (28) and (29a) it follows that

$$\mathbf{\Phi}^{-1} = \left[M\right]^{-1}\mathbf{\Phi}^T\mathbf{M} \tag{30}$$

The solution of ODEs $M_j\,\ddot{q}_j + K_j\,q_j = Q_j$ with initial conditions $\left\{q_{o_j}, \dot{q}_{o_j}\right\}$ follows an identical procedure as in the solution of the SDOF response. That is, each modal response adds the homogeneous solution and the particular solution. The particular solution clearly depends on the time form of the modal force $Q(t)$, *i.e* step-load, ramp-load, pulse-load, periodic load, or arbitrary time form.

Free response in modal coordinates

Without modal forces, $Q=0$, the modal equations are

$$M_j \ddot{q}_{Hj} + K_j q_{Hj} = 0 = Q_j \qquad (31a)$$

with solutions, **for an elastic mode**

$$q_{Hj} = q_{o_j} \cos\left(\omega_{n_j} t\right) + \frac{\dot{q}_{o_j}}{\omega_{n_j}} \sin\left(\omega_{n_j} t\right) \quad \text{if } \omega_{n_j} \neq 0 \qquad (31b)$$

; and for a **rigid body mode**

$$q_{Hj} = q_{o_j} + \dot{q}_{o_j} t \quad \text{if } \omega_{n_j} = 0 \qquad (31c)$$

$j=1,2,\ldots n$

Forced response in modal coordinates

For step-loads, Q_{Sj}, the modal equations are

$$M_j \ddot{q}_j + K_j q_j = Q_{Sj} \qquad (32a)$$

and; for **an elastic mode,** $\omega_{n_j} \neq 0$,

$$q_j = q_{o_j} \cos\left(\omega_{n_j} t\right) + \frac{\dot{q}_{o_j}}{\omega_{n_j}} \sin\left(\omega_{n_j} t\right) + \frac{Q_{S_j}}{K_j}\left[1 - \cos\left(\omega_{n_j} t\right)\right] \qquad (32a)$$

; and for a **rigid body mode,** $\omega_{n_j} = 0$,

$$q_j = q_{o_j} + \dot{q}_{o_j} t + \frac{1}{2} \frac{Q_{S_j}}{M_j} t^2 \tag{32c}$$

j=1,2,….*n*

For periodic loads,, the modal equations are

$$M_j \ddot{q}_j + K_j q_j = Q_{P_j} \cos(\Omega t) \tag{33a}$$

with solutions

for **an elastic mode,** $\omega_{n_j} \neq 0$, and $\Omega \neq \omega_{n_j}$

$$q_j = C_j \cos\left(\omega_{n_j} t\right) + S_j \sin\left(\omega_{n_j} t\right) + \frac{Q_{P_j}}{K_j} \left[\frac{1}{1 - \left(\Omega / \omega_{n_j}\right)^2} \right] \cos\left(\Omega t\right)$$

(33b)

Note that if $\Omega = \omega_{n_j}$, a resonance appears that will lead to system destruction.

For a **rigid body mode,** $\omega_{n_j} = 0$,

$$q_j = q_{o_j} + \dot{q}_{o_j} t - \frac{Q_{P_j}}{M_j \Omega^2} \cos(\Omega t) \tag{33c}$$

For arbitrary-loads Q_j, the modal response is

$$q_j = q_{j_o} \cos(\omega_{n_j} t) + \frac{\dot{q}_{j_o}}{\omega_{n_j}} \sin(\omega_{n_j} t) + \frac{1}{M_j \omega_{n_j}} \int_0^t Q_{j(\tau)} \sin\left[\omega_{n_j} (t - \tau)\right] d\tau$$

(34)

for **an elastic mode,** $\omega_{n_j} \neq 0$.

System Response in Physical Coordinates

Once the response in modal coordinates is fully determined, the system response in physical coordinates follows using the modal transformation

$$\mathbf{U}_{(t)} = \mathbf{\Phi}\, \mathbf{q}_{(t)} =$$

$$\mathbf{U}_{(t)} = \begin{bmatrix} \boldsymbol{\varphi}_1 & \boldsymbol{\varphi}_2 & .. & \boldsymbol{\varphi}_n \end{bmatrix} \begin{bmatrix} q_{1_{(t)}} \\ q_2 \\ .. \\ q_n \end{bmatrix} = \boldsymbol{\varphi}_1 q_1 + \boldsymbol{\varphi}_2 q_2 + .. + \boldsymbol{\varphi}_n q_n$$

$$\mathbf{U}_{(t)} = \sum_{j=1}^{n} \boldsymbol{\varphi}_j\, q_{j_{(t)}} \qquad (35)$$

One important question follows: **are all the modal responses important and need be accounted for to obtain the response in physical coordinates? If not**, savings in computation time are evident. Hence, the physical response becomes

$$\mathbf{U}_{(t)} \approx \sum_{j=1}^{m} \boldsymbol{\varphi}_j\, q_{j_{(t)}}, \quad m < n \qquad (36)$$

If $m<n$, then how many modes are to be included to ensure the physical response is accurate? That is, which modes are important and which others are not?

Example: Consider the case of force excitation with frequency $\Omega \neq \omega_{n_j}$ and acting for very long times. The EOMs in physical space are

$$\mathbf{M\ddot{U}} + \mathbf{K\,U} = \mathbf{F_P}\cos(\Omega t)$$

Let's assume there is a little damping; hence, the steady state periodic response in modal coordinates is (see eq. (33b)):

$$q_j \approx \frac{Q_{P_j}}{K_j}\left[\frac{1}{1-\left(\Omega/\omega_{n_j}\right)^2}\right]\cos(\Omega t) \qquad (37a)$$

And thus,

$$\mathbf{U} = \mathbf{U_P}\cos(\Omega t) = \mathbf{\Phi\, q} = \sum_{j=1}^{n}\left(\boldsymbol{\varphi}_j \frac{Q_{P_j}}{K_j}\left[\frac{1}{1-\left(\Omega/\omega_{n_j}\right)^2}\right]\right)\cos(\Omega t)$$

(38)

The physical response is also periodic with same frequency as the force excitation.

Recall that $K_j = \omega_{n_j}^2\, M_j = \boldsymbol{\varphi}_{(j)}^T \mathbf{K}\,\boldsymbol{\varphi}_{(j)}$ and $Q_{P_j} = \boldsymbol{\varphi}_{(j)}^T \mathbf{F_P}$

However, nowadays the engineer in a hurry prefers to dump the problem into a super computer; and for $\mathbf{U} = \mathbf{U_P}\cos(\Omega t)$, finds the solution

$$\mathbf{U}_P = \left[\mathbf{K} - \Omega^2\,\mathbf{M}\right]^{-1}\mathbf{F_P} \qquad (39)$$

at **a fixed** excitation frequency Ω. Brute force substitutes beauty and elegance, time savings in lieu of understanding!

Example: Find natural frequencies and natural mode shapes of UNDAMPED system.

$ORIGIN := 1$

Given EOMs for a 2DOF - undamped- system:

$$\begin{pmatrix} M_2 & 0 \\ 0 & M_1 \end{pmatrix} \cdot \frac{d^2}{dt^2}\begin{pmatrix} X_2 \\ X_1 \end{pmatrix} + \begin{pmatrix} 2 \cdot K_2 & -2K_2 \\ -2 \cdot K_2 & 2 \cdot K_2 + K_1 \end{pmatrix} \cdot \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} = \begin{pmatrix} 0 \\ K_1 \cdot Z \end{pmatrix} \qquad \textbf{(1)}$$

where $M_2 = m_o$, $M_1 = 5\ m_o$, $K_2 = k_o$; $K_1 = 5\ k_o$

$$\begin{pmatrix} m_o & 0 \\ 0 & 5 \cdot m_o \end{pmatrix} \cdot \frac{d^2}{dt^2}\begin{pmatrix} X_2 \\ X_1 \end{pmatrix} + \begin{pmatrix} 2 \cdot k_o & -2k_o \\ -2 \cdot k_o & 2 \cdot k_o + 5 \cdot k_o \end{pmatrix} \cdot \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} = \begin{pmatrix} 0 \\ K_1 \cdot Z \end{pmatrix}$$

(a) PROCEDURE TO FIND NATURAL FREQUENCIES AND NATURAL MODES: Assume the motions are periodic with frequency ω, ie

$$X2 = a_1 \cdot \cos(\omega \cdot t) \qquad X1 = a_2 \cdot \cos(\omega \cdot t) \qquad \textbf{(2)}$$

Set the RHS of Eq. (1) equal to 0. Substitution of (2) into (1) gives

$$\begin{pmatrix} 2 \cdot k_o - m_o \cdot \omega^2 & -2k_o \\ -2 \cdot k_o & 7 \cdot k_o - 5 \cdot m_o \cdot \omega^2 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \cos(\omega \cdot t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

cancel cos(ωt) since it is NOT zero for all times

The homogeneous system of eqns

$$\begin{pmatrix} 2 \cdot k_o - m_o \cdot \omega^2 & -2k_o \\ -2 \cdot k_o & 7 \cdot k_o - 5 \cdot m_o \cdot \omega^2 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \textbf{(3)}$$

has a non-trivial solution if the determinant of the system of equations equals zero, i.e. if

$$\Delta(\omega) = \left(7 \cdot k_o - 5 \cdot m_o \cdot \omega^2\right) \cdot \left(2 \cdot k_o - m_o \cdot \omega^2\right) - 4 \cdot k_o^2 = 0$$

Let $\lambda = \omega^2$, and expanding the products in the determinant

$$0 = \lambda^2 \cdot 5m_o^2 - \lambda \cdot \left(7 \cdot k_o \cdot m_o + 10 \cdot k_o \cdot m_o\right) + 14 \cdot k_o^2 - 4 \cdot k_o^2$$

Let $\bar{\lambda} = \lambda \cdot \left(\frac{m_o}{k_o}\right)$ Leads to: $0 = \left[a \cdot \left(\bar{\lambda}\right)^2 + b \cdot \bar{\lambda} + c\right]$ **(4)**

with: $a := 5$ $\quad b := -17$ $\quad c := 10$

The roots (eigenvalues) of the characteristic equation are

$$\lambda_1 := \frac{-b - (b^2 - 4\cdot a\cdot c)^{0.5}}{2\cdot a} \qquad \lambda_2 := \frac{-b + (b^2 - 4\cdot a\cdot c)^{0.5}}{2\cdot a} \qquad \lambda = \begin{pmatrix} 0.757 \\ 2.643 \end{pmatrix} \left(\frac{k_o}{m_o}\right)$$

and the **natural frequencies** are:

$$\omega_1 := (\lambda_1)^{0.5} \qquad \omega_2 := (\lambda_2)^{0.5} \qquad \omega = \begin{pmatrix} 0.87 \\ 1.626 \end{pmatrix} \left(\frac{k_o}{m_o}\right)^{0.5}$$

Find the eigenvectors:

The two equations in (3) are **linearly dependent**. Thus, one cannot solve for a1 and a2. S $\phi_1 := 1$ arbitrarily; and from the first equation

for ω

$$\phi_2 = \frac{(2\cdot k_o - m_o\cdot \omega_1{}^2)}{2\cdot k_o} = \frac{(2\cdot k_o - 0.757\cdot k_o)}{2\cdot k_o} \qquad \phi_2 := \frac{(2 - 0.757)}{2}$$

$$\phi_2 = 0.621$$

$$\phi_1 := \phi$$

$$\phi_1 = \begin{pmatrix} 1 \\ 0.621 \end{pmatrix}$$

is the first eigenvector (natural mode)

(b) Explanation: DOF1 (X2) and DOF2 (X1) move in phase, with X2>X1

for ω_2

$$\phi_1 := 1$$

$$\phi_2 = \frac{(2\cdot k_o - m_o\cdot \omega_2{}^2)}{2\cdot k_o} = \frac{(2\cdot k_o - 2.643\cdot k_o)}{2\cdot k_o} \qquad \phi_2 := \frac{(2 - 2.643)}{2}$$

$$\phi_2 := \phi \qquad \phi_2 = \begin{pmatrix} 1 \\ -0.321 \end{pmatrix}$$

is the 2nd eigenvector (natural mode)

(b) Explanation: DOF1 (X2) and DOF2 (X1) move 180 deg OUT of phase, with |X2|>|X1|

(c) find the numerical value for each natural frequency:

Since

$$\omega := \begin{pmatrix} 0.87 \\ 1.626 \end{pmatrix} \cdot \left(\frac{k_o}{m_o}\right)^{0.5} \qquad m_o := \frac{1000 lb}{g} \qquad k_o := 10^5 \cdot \frac{lb}{in}$$

0

Note that mass must be expressed in physical units consistent with the problem, i.e.

$$\omega = \begin{pmatrix} 170.947 \\ 319.495 \end{pmatrix} \frac{rad}{sec}$$

$$f_n := \frac{\omega}{2\cdot \pi} \qquad f_n = \begin{pmatrix} 27.207 \\ 50.849 \end{pmatrix} Hz$$

$$m_o = 2.59 \frac{lb\cdot sec^2}{in}$$

Perform same work using a calculator

Use BUILT IN functions
-
Not much learning

$$M := \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \cdot m_o \qquad K := \begin{pmatrix} 2 & -2 \\ -2 & 7 \end{pmatrix} \cdot k_o$$

Let $Z := M^{-1} \cdot K$

$$\lambda := \mathrm{sort}(\mathrm{eigenvals}(Z)) \qquad \lambda = \begin{pmatrix} 2.921 \times 10^4 \\ 1.021 \times 10^5 \end{pmatrix} \frac{1}{\mathrm{sec}^2}$$

$$\omega_1 := (\lambda_1)^{.5}$$

$$\omega_2 := (\lambda_2)^{.5}$$

$$\omega = \begin{pmatrix} 170.914 \\ 319.495 \end{pmatrix} \frac{\mathrm{rad}}{\mathrm{sec}} \qquad \frac{\omega}{2 \cdot \pi} = \begin{pmatrix} 27.202 \\ 50.849 \end{pmatrix} \mathrm{Hz}$$

natural modes:

$$\phi_1 := \mathrm{eigenvec}(Z, \lambda_1) \qquad \phi_1 = \begin{pmatrix} 0.849 \\ 0.528 \end{pmatrix} \qquad \frac{(\phi_1)_2}{(\phi_1)_1} = 0.622$$

$$\phi_2 := \mathrm{eigenvec}(Z, \lambda_2) \qquad \phi_2 = \begin{pmatrix} 0.952 \\ -0.306 \end{pmatrix} \qquad \frac{(\phi_2)_2}{(\phi_2)_1} = -0.322$$

which are the same ratios as for the vectors found earlier

Example: Undamped Modal Analysis

$m_o := \frac{1000lb}{g}$ $\quad k_o := 10^5 \cdot \frac{lb}{in}$ $\quad g = 32.174 \frac{ft}{sec^2}$ $\quad$ ORIGIN := 1

Equations of motion:

$$\begin{pmatrix} m_o & 0 \\ 0 & 5 \cdot m_o \end{pmatrix} \cdot \frac{d^2}{dt^2} \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} + \begin{pmatrix} 2 \cdot k_o & -2k_o \\ -2 \cdot k_o & 7 \cdot k_o \end{pmatrix} \cdot \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} = \begin{pmatrix} 0 \\ k_o \cdot Z \end{pmatrix}$$

natural frequencies, modal matrix (eigenvectors)

$$\omega_n := \begin{pmatrix} 170.95 \\ 319.5 \end{pmatrix} \cdot \frac{rad}{sec} \qquad \Phi := \begin{pmatrix} 1 & 1 \\ 0.621 & -0.321 \end{pmatrix}$$

given: $Z_o := 0.01 \cdot in$ provides a $F_o := k_o \cdot Z_o$ constant force

Define matrices:

$$M := \begin{pmatrix} m_o & 0 \\ 0 & 5 \cdot m_o \end{pmatrix} \quad K := \begin{pmatrix} 2 \cdot k_o & -2k_o \\ -2 \cdot k_o & 7 \cdot k_o \end{pmatrix} \quad F := \begin{pmatrix} 0 \cdot lb \\ F_o \end{pmatrix}$$

at t=0s, Initial conditions: **system is at REST**

(a) FIND modal masses and stiffnesses

$M_M := \Phi^T \cdot M \cdot \Phi$ $\qquad K_M := \Phi^T \cdot K \cdot \Phi$

$$M_M = \begin{pmatrix} 7.584 & 8.534 \times 10^{-3} \\ 8.534 \times 10^{-3} & 3.925 \end{pmatrix} \frac{lb \cdot sec^2}{in}$$

non-diagonal elements are very small= non zero b/c of roundoff in numerical calculator

modal masses and stiffnesses:

Mode 1 $\quad M_{m_1} := M_{M_{1,1}}$ $\qquad K_{m_1} := (\omega_{n_1})^2 \cdot M_{M_{1,1}}$

Mode 2 $\quad M_{m_2} := M_{M_{2,2}}$ $\qquad K_{m_2} := (\omega_{n_2})^2 \cdot M_{M_{2,2}}$

$$M_m = \begin{pmatrix} 7.584 \\ 3.925 \end{pmatrix} \frac{lb \cdot sec^2}{in} \qquad K_m = \begin{pmatrix} 2.216 \times 10^5 \\ 4.006 \times 10^5 \end{pmatrix} \frac{lb}{in}$$

(b) Find initial moddal displacements and velocities and modal force vector (Q)

At time t=0s, the system is at REST at its static equilibrium position, hence the initial conditions are null displacements and null velocities. Of course, **the same applies to modal space, i.e. null initial displacements and velocities**

for generality, define: $X_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot ft$ $\begin{matrix} X1 \\ X2 \end{matrix}$ $\quad V_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \frac{ft}{sec}$ $\quad$ Calculate inverse of A matrix $\quad \Phi_{inv} := \Phi^{-1}$

and in modal coordinates (disp & velocities) $\quad q_o := \Phi_{inv} \cdot X_o$ $\qquad q_{o\_dot} := \Phi_{inv} \cdot V_o$ velocity

$$q_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ft \qquad q_{o\_dot} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \frac{ft}{sec}$$

No need for actual calculation
- a knowledge statement suffices

Define **modal force** $\quad Q := \Phi^T \cdot F$

$$Q = \begin{pmatrix} 621 \\ -321 \end{pmatrix} lb$$

Both natural modes will be excited

(c) Modal EOMs and modal responses

The EOMs in modal space are **uncoupled** and equal to

$$M_{m_i} \left(\frac{d^2}{dt^2} q_i \right) + K_{m_i} \cdot q_i = Q_i \qquad i = 1, 2$$

Using the cheat sheet, and since the Initial conditions are null, the response in modal coordinates are

$$\delta_{m_1} := \frac{Q_1}{K_{m_1}} \qquad \delta_{m_2} := \frac{Q_2}{K_{m_2}}$$

$$q_1(t) := \delta_{m_1} \cdot \left(1 - \cos\left(\omega_{n_1} \cdot t\right)\right) \qquad q_2(t) := \delta_{m_2} \cdot \left(1 - \cos\left(\omega_{n_2} \cdot t\right)\right)$$

where: $\omega_n = \begin{pmatrix} 170.95 \\ 319.5 \end{pmatrix} \frac{rad}{sec}$

where $\quad \delta_m = \begin{pmatrix} 2.802 \times 10^{-3} \\ -8.013 \times 10^{-4} \end{pmatrix} in$

are the "static" deflections in modal space. δ2 << δ1, **thus first modal response is MORE important**

(d) The response in physical coordinates, X1 and X2, equals (from transformation x=Aq)

$X_1(t) := q_1(t) + q_2(t)$

$$X_1(t) = \delta_{m_1}\cdot\left(1 - \cos\left(\omega_{n_1}\cdot t\right)\right) + \delta_{m_2}\cdot\left(1 - \cos\left(\omega_{n_2}\cdot t\right)\right)$$

with $\Phi = \begin{pmatrix} 1 & 1 \\ 0.621 & -0.321 \end{pmatrix}$

$X_2(t) := 0.621\cdot q_1(t) - 0.321\cdot q_2(t)$

for graph below:

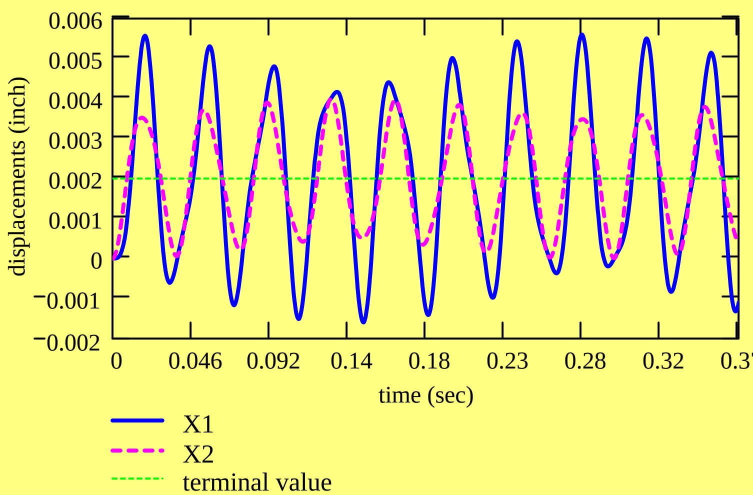
$$X_2(t) = \delta_{m_1}\cdot 0.621\cdot\left(1 - \cos\left(\omega_{n_1}\cdot t\right)\right) + \delta_{m_2}\cdot(-0.321)\cdot\left(1 - \cos\left(\omega_{n_2}\cdot t\right)\right)$$

$T_{large} := 10\cdot\left(\frac{2\cdot\pi}{\omega_{n_1}}\right)^1$

$\delta_{m_1}\cdot 0.621 = 1.74\times 10^{-3}\,in$ $\qquad \delta_{m_2}\cdot(-0.321) = 2.572\times 10^{-4}\,in$

Explanation: Since q1 and q2 are non-zero, then physical motion, X1 &X2, shows excitation of the TWO fundamental modes of vibration - **BUT response for second mode is much less**

GRAPHs not needed for exam:



Note that there is no damping or attenuation of motions.

Not too complicated physical response. It shows dominance of first mode (lowest natural freq or largest period)

$\frac{2\cdot\pi}{\omega_{n_1}} = 0.037\,sec$

$\frac{2\cdot\pi}{\omega_{n_2}} = 0.02\,sec$

Terminal condition:
If damping is present and since the applied force is a constant, the system will achieve a new steady state condition.

In the limit as t approaches very, very large values $\frac{d^2}{dt^2}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; hence ===> $K\cdot\begin{pmatrix} X_{1end} \\ X_{2end} \end{pmatrix} = F$

And equations of motion reduce to:

$$\begin{pmatrix} 2\cdot k_o & -2k_o \\ -2\cdot k_o & 7\cdot k_o \end{pmatrix}\cdot\begin{pmatrix} X_{1end} \\ X_{2end} \end{pmatrix} = \begin{pmatrix} 0 \\ F_o \end{pmatrix}$$

$F_o = 1\times 10^3\,lb$

And solving this system of equations using **Cramer's rule** $\Delta := 14\cdot k_o^2 - 4\cdot k_o^2$ determinant of system of eqns.

$X_{1end} := \frac{F_o\cdot 2\cdot k_o}{\Delta}$ $\qquad X_{2end} := \frac{2\cdot k_o\cdot F_o}{\Delta}$

$X_{1end} = 2\times 10^{-3}\,in$ $\qquad X_{2end} = 2\times 10^{-3}\,in$

Note that the graph of undamped periodic motions Z(t) and X(t) shows oscillatory motions abut these terminal or end values.

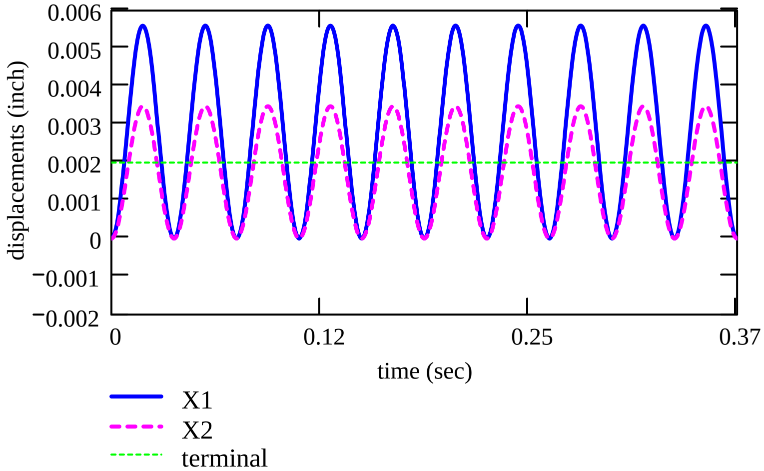
OR $K^{-1}\cdot F = \begin{pmatrix} 2\times 10^{-3} \\ 2\times 10^{-3} \end{pmatrix}in$

recall

$Z_o = 0.01\,in$

$\frac{Z_o}{X_{1end}} = 5$

COMPARE actual response with a response neglecting q2.
Indeed mode 2 does not afffect the physical response, except for motion X2 sligthly



Normalization of eigenvectors (natural modes)

Recall that the components of an eigenvector $\boldsymbol{\varphi}_j$ are ARBITRARY but for a multiplicative constant. If one of the elements of the eigenvector is assigned a certain value, then this vector becomes unique, since then *n-1* remaining elements are automatically adjusted to keep constant the ratio between any two elements in the vector.

In practice, the eigenvectors are normalized. The resulting vectors are called **NORMAL MODES**.

Some typical NORMS are

L_1 norm: $\left\|\mathbf{q}_{(j)}\right\| = 1 = \max\left(q_{j_k}\right)$ (39a)

L_2 norm: $\left\|\mathbf{q}_{(j)}\right\| = 1 = \sqrt{q_{j_1}^2 + q_{j_2}^2 + ... + q_{j_n}^2}$ (39b)

Or making the mass modal matrix equal to the identity matrix, **[*M*]=I**, i.e.

$$\boldsymbol{\varphi}_{(j)}^T \mathbf{M}\, \boldsymbol{\varphi}_{(j)} = M_j = 1 \quad (39c)$$

hence

$$\boldsymbol{\varphi}_{(j)}^T \mathbf{K}\, \boldsymbol{\varphi}_{(j)} = K_j = \omega_j^2 M_j = \omega_{nj}^2 \quad (39d)$$

This normalization has obvious advantages since it will reduce the number of operations when conducting the modal analysis. However, the physical significance of the modal equations is lost. Note that the modal Eqs. (26) become:

$$\ddot{q}_j + \omega_{n_j}^2 q_j = Q_j$$

Your lecturer recommends this normalization procedure be conducted only for systems with large number of degrees of freedom, *n>>>1*.

Note that the normalization process is a mere convenience, devoid of any physical significance.

Rayleigh's Energy Method

The method is a procedure to determine an approximate value (from above) for the fundamental natural frequency of a MDOF system. At times, the full solution of the eigenvalue problem is of NO particular interest and an estimate of the system lowest natural frequency suffices.

Recall that the pairs $\left\{\omega_i, \boldsymbol{\varphi}_{(i)}\right\}_{i=1,2...n}$ satisfy $\mathbf{K}\boldsymbol{\varphi}_{(i)} = \omega_i^2 \mathbf{M}\ \boldsymbol{\varphi}_{(i)}$

with properties $\boldsymbol{\Phi}^T \mathbf{M}\,\boldsymbol{\Phi} = \left[M\right];\quad \boldsymbol{\Phi}^T \mathbf{K}\,\boldsymbol{\Phi} = \left[K\right]$

i.e. with modal stiffness and masses calculated from:

$$K_i = \boldsymbol{\varphi}_{(i)}^T \mathbf{K}\boldsymbol{\varphi}_{(i)};\ M_i = \boldsymbol{\varphi}_{(i)}^T \mathbf{M}\boldsymbol{\varphi}_{(i)},\ \text{and}\ \omega_i^2 = K_i / M_i \quad (41)$$

That is,

$$\omega_i^2 = K_i / M_i = \frac{\tfrac{1}{2}\boldsymbol{\varphi}_{(i)}^T \mathbf{K}\boldsymbol{\varphi}_{(i)}}{\tfrac{1}{2}\boldsymbol{\varphi}_{(i)}^T \mathbf{M}\boldsymbol{\varphi}_{(i)}} \quad (42)$$

Above, the numerator relates to the **potential or strain energy** of the system for the *i-mode,* and the denominator to the **kinetic energy** for the same mode.

Consider an arbitrary vector **u** and define **Rayleigh's quotient** ***R*(u)** as

$$R(\mathbf{u}) = \frac{\frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}}{\frac{1}{2}\mathbf{u}^T\mathbf{M}\mathbf{u}} \tag{43}$$

$R(\mathbf{u})$ is a scalar whose value depends not only on the matrices **M** & **K,** but also on the choice of the vector **u.**

Clearly, if the arbitrary vector **u** coincides with (or is a multiple of) one of the natural mode vectors, then Rayleigh's quotient will deliver the exact natural frequency for that particular mode. It can also be shown that the quotient has a **stationary value**, i.e. a minimum, in the neighborhood of the system natural modes (eigenvectors). To show this, since **u** is an arbitrary vector and the natural modes are a set of linearly independent vectors, then one can represent

$$\mathbf{u} = \sum_{j=1}^{n} \boldsymbol{\varphi}_j \, c_j = \boldsymbol{\Phi}\,\mathbf{c} \tag{44}$$

Where $\mathbf{c}^T = \{c_1 \quad c_2 \quad .. \quad c_n\}$ is the vector of coefficients in the expansion. Substitution of the expression above into Rayleigh's quotient gives

$$R(\mathbf{u}) = \frac{\frac{1}{2}(\boldsymbol{\Phi}\mathbf{c})^T \mathbf{K}(\boldsymbol{\Phi}\mathbf{c})}{\frac{1}{2}(\boldsymbol{\Phi}\mathbf{c})^T \mathbf{M}(\boldsymbol{\Phi}\mathbf{c})} = \frac{\mathbf{c}^T\left(\boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi}\right)\mathbf{c}}{\mathbf{c}^T\left(\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi}\right)\mathbf{c}}$$

$$R(\mathbf{u}) = \frac{\mathbf{c}^T\left[K\right]\mathbf{c}}{\mathbf{c}^T\left[M\right]\mathbf{c}} \tag{45}$$

Assume the modes have been normalized with respect to the mass matrix, i.e.

$$R(\mathbf{u}) = \frac{\mathbf{c}^T \left[\omega_n^2 \right] \mathbf{c}}{\mathbf{c}^T \mathbf{I} \mathbf{c}} = \frac{\sum_{i=1}^{n} c_i^2 \omega_{n_i}^2}{\sum_{i=1}^{n} c_i^2} \qquad (46a)$$

Next, consider that the arbitrary vector **u** (which at this time can be regarded as an **assumed mode vector**) differs very little from the natural mode (eigenvector) $\boldsymbol{\varphi}_{(r)}$. This means that in the expansion of vector **u,** the coefficients $c_i << c_r$; for $i = 1, 2, \ldots n$ and $i \neq r$

Or

$c_i = \varsigma_i \, c_r$; $\varsigma_i << 1$ for $i = 1, 2, \ldots n$ and $i \neq r$

Then, Rayleigh's quotient is expressed as

$$R(\mathbf{u}) = \frac{c_r^2 \omega_{n_r}^2 + c_r^2 \sum_{i=1, i \neq r}^{n} \varsigma_i^2 \omega_{n_i}^2}{c_r^2 + c_r^2 \sum_{i=1, i \neq r}^{n} \varsigma_i^2}$$

$$R(\mathbf{u}) = \frac{\omega_{n_r}^2 + \sum_{i=1, i \neq r}^{n} \varsigma_i^2 \omega_{n_i}^2}{1 + \sum_{i=1, i \neq r}^{n} \varsigma_i^2} = \omega_{n_r}^2 \frac{1 + \sum_{i=1, i \neq r}^{n} \left(\varsigma_i \omega_{n_i} / \omega_{n_r} \right)^2}{1 + \sum_{i=1, i \neq r}^{n} \varsigma_i^2} \qquad (46b)$$

The quantities $\left\{ \varsigma_i^2 \right\}$ are small, of second order, hence $R(\mathbf{u})$ differs from the natural frequency by a small quantity of second order.

This implies that $R(\mathbf{u})$ has a **stationary value** in the vicinity of the modal vector $\boldsymbol{\varphi}_{(r)}$.

The most important property of **Rayleigh's quotient** is that it shows a **minimum value in the neighborhood of the fundamental mode,** i.e. when *r=1*.

$$R(\mathbf{u}) = \omega^2 = \omega_{n_1}^2 \frac{1 + \sum_{i=2}^{n} \left(\varsigma_i \omega_{n_i} / \omega_{n_1} \right)^2}{1 + \sum_{i=2}^{n} \varsigma_i^2}, \text{ since} \left(\omega_{n_i} / \omega_{n_1} \right) > 1 \quad (47)$$

Then each term in the numerator is greater than the corresponding one in the denominator. Hence, it follows that

$$R(\mathbf{u}) = \omega^2 \geq \omega_{n_1}^2 \quad (48)$$

i.e., Rayleigh's quotient provides an **upper bound to the first (lowest) natural frequency** of the undamped MDOF system. Clearly, the equality holds above if one selects $\mathbf{u} = c_1 \boldsymbol{\varphi}_{(1)}; c_1 \neq 0$.

Closure

Rayleigh's energy method is generally used when one is interested in a quick (but particularly accurate) estimate of the **fundamental** natural frequency of a continuous system, and for which a solution to the whole eigenvalue problem cannot be readily obtained. The method is based on the fact that the natural frequencies have stationary values in the neighborhood of the natural modes.

In addition, Rayleigh's quotient provides an upper bound to the first (lowest) natural frequency. **The engineering value of this approximation can hardly be overstated.** Rayleigh's energy method is the basis for the numerical computing of eigenvectors and eigenvalues as will be seen later.

Mode Acceleration Method

Recall that the response in physical coordinates is

$$\mathbf{U}_{(t)} \approx \sum_{j=1}^{m} \boldsymbol{\varphi}_j \, q_{j_{(t)}}, \;\; m < n \tag{36}$$

where $m<n$. The procedure is known as the mode displacement method.

This method, however, fails to give an accurate solution even when a static load is applied (See Structural Dynamics, by R. Craig, J. Wiley Pubs, NY, 1981.).

The difficulty is overcome by using the procedure detailed below. Recall that the system motion is governed by the set of equations

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{K}\,\mathbf{U}_{(t)} = \mathbf{F}_{(t)} \tag{4}$$

And, if there are no rigid body modes, i.e. all natural frequencies are greater than zero, then

$$\mathbf{U}_{(t)} = \mathbf{K}^{-1}\left(\mathbf{F}_{(t)} - \mathbf{M}\,\ddot{\mathbf{U}}\right) \tag{51}$$

where $\mathbf{K}^{-1}$ is a flexibility matrix. From Eq. (36),

$$\ddot{\mathbf{U}} \approx \sum_{j=1}^{m} \boldsymbol{\varphi}_{(j)} \, \ddot{q}_{j_{(t)}}, \;\; m < n \tag{52}$$

Hence, Eq, (51) can be written as

$$\mathbf{U}_{(t)} \approx \mathbf{K}^{-1}\mathbf{F}_{(t)} - \mathbf{K}^{-1}\mathbf{M}\sum_{j=1}^{m} \boldsymbol{\varphi}_{(j)}\, \ddot{q}_{j_{(t)}} \tag{53}$$

Using the **fundamental identity**,

$$\mathbf{K}\boldsymbol{\varphi}_{(i)} = \omega_i^2\, \mathbf{M}\boldsymbol{\varphi}_{(i)} \Rightarrow \frac{1}{\omega_i^2}\boldsymbol{\varphi}_{(i)} = \mathbf{K}^{-1}\mathbf{M}\boldsymbol{\varphi}_{(i)}$$

Write Eq. (53) as

$$\mathbf{U}_{(t)} \approx \mathbf{K}^{-1}\mathbf{F}_{(t)} - \sum_{j=1}^{m}\left(\frac{\boldsymbol{\varphi}_{(j)}}{\omega_j^2}\right)\ddot{q}_{j_{(t)}} \tag{54}$$

Note that

$$\mathbf{U}_S = \mathbf{K}^{-1}\mathbf{F}_{(t)} \tag{55}$$

is the displacement response vector due to a **"pseudo-static" force** **F**(t), i.e. without the system inertia accounted for. Hence write Eq. (54), as

$$\mathbf{U}_{(t)} \approx \mathbf{U}_{s(t)} - \sum_{j=1}^{m}\left(\frac{\boldsymbol{\varphi}_{(j)}}{\omega_j^2}\right)\ddot{q}_{j_{(t)}} \quad ; m<n \tag{56}$$

The second term above can be thought as the "**inertia induced response.**"

Example: Consider the case of force excitation with frequency $\Omega \neq \omega_{n_j}$ and acting for very long times. The EOMs in physical space are:

$$\mathbf{M}\,\ddot{\mathbf{U}}+\mathbf{K}\,\mathbf{U}=\mathbf{F_P}\cos(\Omega t)$$

With a little damping, the **steady state periodic response** in modal coordinates is

$$q_j \approx \frac{Q_{P_j}}{K_j}\left[\frac{1}{1-\left(\Omega/\omega_{n_j}\right)^2}\right]\cos(\Omega t) \tag{37a}$$

Recall that, using the **mode displacement method**, the response in physical coordinates is:

$$\mathbf{U} \approx \sum_{j=1}^{m}\left(\boldsymbol{\varphi}_j \frac{Q_{P_j}}{K_j}\left[\frac{1}{1-\left(\Omega/\omega_{n_j}\right)^2}\right]\right)\cos(\Omega t) \tag{38}$$

From each of the modal responses,

$$\ddot{q}_j \approx \frac{Q_{P_j}}{K_j}\left(-\Omega^2\right)\left[\frac{1}{1-\left(\Omega/\omega_j\right)^2}\right]\cos(\Omega t);$$

$$\frac{-\ddot{q}_j}{\omega_j^2} \approx \frac{Q_{P_j}}{K_j}\left(\frac{\Omega^2}{\omega_j^2}\right)\left[\frac{1}{1-\left(\Omega/\omega_j\right)^2}\right]\cos(\Omega t) \tag{57}$$

Since $K_j = \omega_j^2\, M_j$; then using the mode acceleration method, the response is

$$\mathbf{U} \approx \left\{\mathbf{U_{SP}} + \sum_{j=1}^{m}\boldsymbol{\varphi}_j \frac{Q_{P_j}}{K_j}\left(\frac{\Omega^2}{\omega_j^2}\right)\left[\frac{1}{1-\left(\Omega/\omega_j\right)^2}\right]\right\}\cos(\Omega t) \quad (58)$$

where the *pseudo-static* response is $\mathbf{U_{SP}} = \mathbf{K}^{-1}\,\mathbf{F_P}$. Now, in the limit, as the excitation frequency decreases, i.e., as $\Omega \rightarrow 0$, the second term in Eq. (58) above disappears, and hence the physical response becomes:

$$\mathbf{U} = \mathbf{U_{SP}} = \mathbf{K}^{-1}\,\mathbf{F_P} \tag{59}$$

which is the exact response, regardless of the number of modes chosen. Hence, the **mode acceleration method** is more accurate than the **mode displacement method**. Known disadvantages include more operations.

Finding the flexibility matrix is, in actuality, desirable. In particular, if derived from measurements, the flexibility matrix is easier to determine than the stiffness matrix.

STEP FORCED RESPONSE of Undamped 2-DOF mechanical system

ORIGIN := 1

Dr. Luis San Andres (c) MEEN 363, 617 February 2008

The undamped equations of motion are:

$$M\cdot\frac{d^2}{dt^2}X + K\cdot X = F_o \qquad (1)$$

where **M,K** are matrices of inertia and stiffness coefficients, and **X, V**=d**X**/dt, d^2**X**/dt^2 are the vectors of physical displacement, velocity and acceleration, respectively.
The FORCED undamped response to the initial conditions, at t=0, **Xo,Vo=**d**X/**dt, follows:

==

The equations of motion are:

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\cdot\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_{1o} \\ F_{2o} \end{pmatrix} \qquad (2)$$

1. Set elements of inertia and stiffness matrices

DATA FOR problem

$$M := \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix}\cdot kg \qquad K := \begin{pmatrix} 2\cdot 10^6 & -1\cdot 10^6 \\ -1\cdot 10^6 & 2\cdot 10^6 \end{pmatrix}\cdot\frac{N}{m} \qquad n := 2 \text{ \# of DOF}$$

Note M and K are symmetric matrices

initial conditions

$$X_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix}\cdot m \qquad V_o := \begin{pmatrix} 0.0 \\ 0 \end{pmatrix}\cdot\frac{m}{sec}$$

Applied force vector:

$$F_o := \begin{pmatrix} 10000 \\ -5000 \end{pmatrix}\cdot N$$

2. Find eigenvalues (undamped natural frequencies) and eigenvectors

Set determinant of system of eqns = 0

$$\Delta = \left[\left(K_{11} - M_{11}\cdot\omega^2\right)\cdot\left(K_{22} - M_{22}\cdot\omega^2\right) - \left(K_{12} - M_{12}\cdot\omega^2\right)\cdot\left(K_{21} - M_{21}\cdot\omega^2\right)\right] = 0 \qquad (2a)$$

$$\Delta = a\cdot\omega^4 + b\cdot\omega^2 + c = \left(a\cdot\lambda^2 + b\cdot\lambda + c\right) = 0 \quad \text{with } \lambda = \omega^2 \qquad (2b)$$

where the coefficients are:

$$a := M_{1,1}\cdot M_{2,2} - M_{1,2}\cdot M_{2,1}$$
$$b := K_{1,2}\cdot M_{2,1} - K_{1,1}\cdot M_{2,2} - K_{2,2}\cdot M_{1,1} + K_{2,1}\cdot M_{1,2} \qquad (2c)$$
$$c := K_{1,1}\cdot K_{2,2} - K_{1,2}\cdot K_{2,1}$$

The roots of equation (2b) are:

$$\lambda_1 := \frac{\left[-b-\left(b^2-4\cdot a\cdot c\right)^{.5}\right]\left[-b+\left(b^2-4\cdot a\cdot c\right)^{.5}\right]}{2\cdot a \quad\quad 2\cdot a} \qquad (3)$$

also known as eigenvalues. The **natural frequencies** follow as:

$$j := 1..n \qquad \omega_j := \left(\lambda_j\right)^{.5} \qquad f := \frac{\omega}{2\cdot\pi}$$

$$\omega = \begin{pmatrix} 112.6 \\ 217.53 \end{pmatrix}\frac{rad}{sec} \qquad (4)$$

$$f = \begin{pmatrix} 17.92 \\ 34.62 \end{pmatrix} Hz$$

Note that: $\Delta(\omega_1) = \Delta(\omega_2) = 0$

For each eigenvalue, the eigenvectors (**natural modes)** are

$$j := 1..n$$

$$a_j := \begin{bmatrix} 1 \\ \dfrac{K_{1,1} - M_{1,1}\cdot\lambda_j}{-\left(K_{1,2} - M_{1,2}\cdot\lambda_j\right)} \end{bmatrix}$$

Set arbitrarily first element of vector = 1

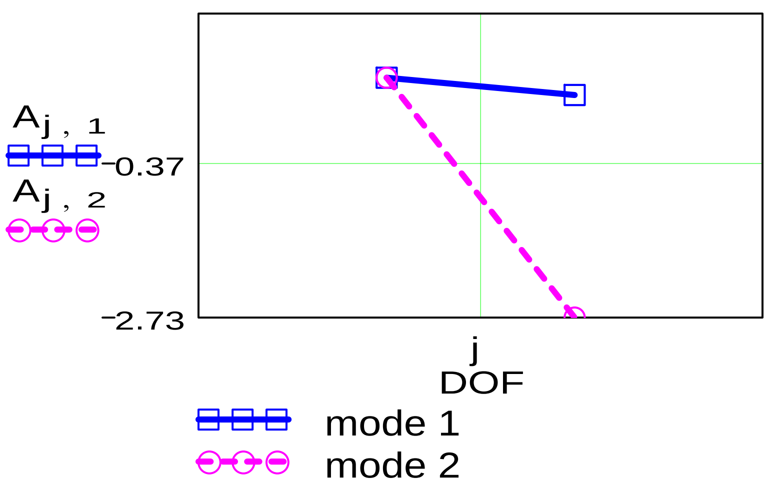
$$a_1 = \begin{pmatrix} 1 \\ 0.73 \end{pmatrix} \quad a_2 = \begin{pmatrix} 1 \\ -2.73 \end{pmatrix} \qquad (5)$$

MODAL matrix $\quad A^{\langle j \rangle} := a_j$

A is the matrix of eigenvectors (undamped modal matrix): each column corresponds to an eigenvector

$$A = \begin{pmatrix} 1 & 1 \\ 0.73 & -2.73 \end{pmatrix}$$

Plot the mode shapes:



3. Modal transformation of physical equations to (natural) modal coordinates

Using transformation: $X = A \cdot q$ (6)

EOMs (1) become uncoupled in modal space:

$$M_m \cdot \frac{d^2}{dt^2} q + K_m \cdot q = Q_m \quad (7)$$

with modal force vector: $Q_m = A^T \cdot F_o$ (8)

and initial conditions (modal displacement=q and modal velocity dq/dt=s)

$$q_o = M_m^{-1} \cdot \left(A^T \cdot M \cdot X_o\right) \qquad s_o = M_m^{-1} \cdot \left(A^T \cdot M \cdot V_o\right) \quad (9)$$

The modal responses are of the form: k=1....n

$$q_k = q_{o_k} \cdot \cos\left(\omega_k \cdot t\right) + \frac{s_{o_k}}{\omega_k} \cdot \sin\left(\omega_k \cdot t\right) + \frac{Q_{m_k}}{K_{m_{k,k}}} \cdot \left(1 - \cos\left(\omega_k \cdot t\right)\right) \quad \omega_k \neq 0 \quad (10a)$$

for an elastic mode

OR

$$q_k = q_{o_k} + s_{o_k} \cdot t + \frac{1}{2} \cdot \frac{Q_{m_k}}{M_{m_{k,k}}} \cdot t^2 \quad \text{for } \omega_k = 0 \quad (10b)$$

for a rigid body mode

And, the response in the physical coordinates is given by the superposition of the modal responses, i.e. $X(t) = A \cdot q(t)$ (5)

=== CHECK ==

Verify the orthogonality properties of the natural mode shapes

$M_m := A^T \cdot M \cdot A$

$$M_m = \begin{pmatrix} 126.79 & -2.24 \times 10^{-14} \\ -1.58 \times 10^{-14} & 473.21 \end{pmatrix} kg$$

$K_m := A^T \cdot K \cdot A$

$$K_m = \begin{pmatrix} 1.61 \times 10^{6} & 3.18 \times 10^{-10} \\ 3.51 \times 10^{-10} & 2.24 \times 10^{7} \end{pmatrix} \frac{N}{m}$$

$$\omega = \begin{pmatrix} 112.6 \\ 217.53 \end{pmatrix} s^{-1}$$

==

4. Find Modal and Physical Response for given initial condition and Constant Force vector

Recall the **vectors of initial conditions**

$$X_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m \qquad V_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \frac{m}{s}$$

and **Constant forces:**

$$F_o = \begin{pmatrix} 1 \times 10^4 \\ -5 \times 10^3 \end{pmatrix} m \frac{N}{m}$$

DATA FOR problem being analyzed:

4.a Find initial conditions in modal coordinates (displacement = q, velocity = s)

Set inverse of modal mass matrix

$$A_{inv} := M_m^{-1} \cdot \left(A^T \cdot M\right)$$

$$q_o := A_{inv} \cdot X_o \qquad s_o := A_{inv} \cdot V_o$$

$$q_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m \qquad s_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m\, s^{-1}$$

4.b Find Modal forces:

$$Q_m := A^T \cdot F_o \qquad Q_m = \begin{pmatrix} 6.34 \times 10^3 \\ 2.37 \times 10^4 \end{pmatrix} N$$

4.c Build Modal responses:

$$q_1(t) := q_{o_1} \cdot \cos(\omega_1 \cdot t) + \frac{s_{o_1}}{\omega_1} \cdot \sin(\omega_1 \cdot t) + \frac{Q_{m_1}}{K_{m_{1,1}}} \cdot \left(1 - \cos(\omega_1 \cdot t)\right)$$

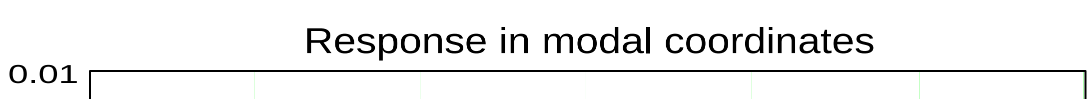
$$q_2(t) := q_{o_2} \cdot \cos(\omega_2 \cdot t) + \frac{s_{o_2}}{\omega_2} \cdot \sin(\omega_2 \cdot t) + \frac{Q_{m_2}}{K_{m_{2,2}}} \cdot \left(1 - \cos(\omega_2 \cdot t)\right)$$

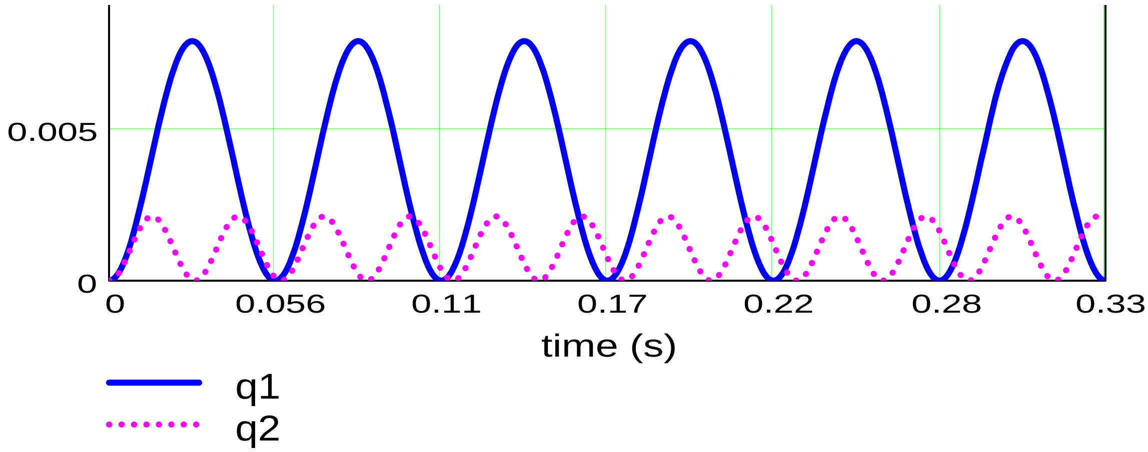
for plots:

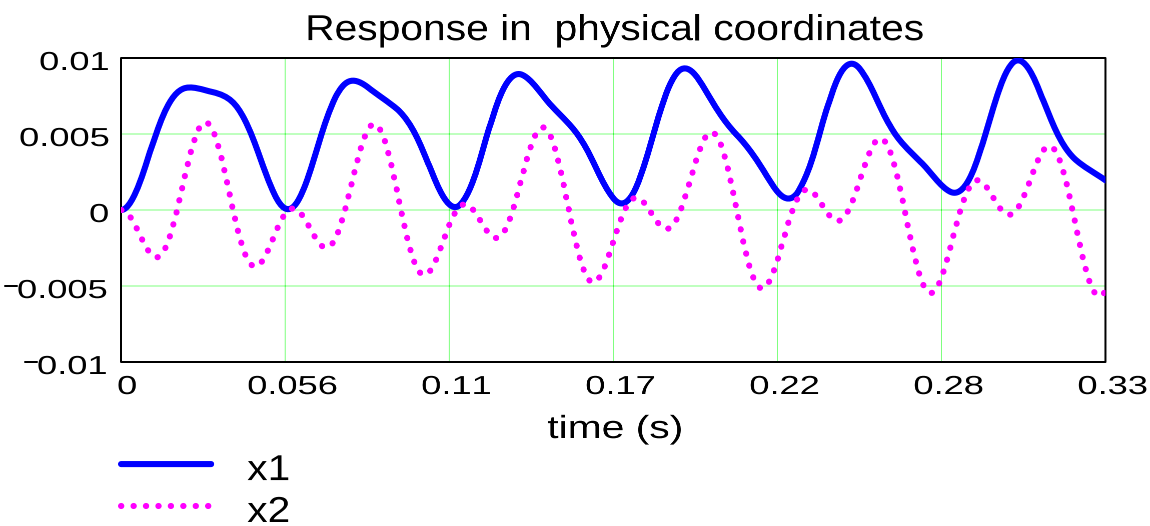
4.d Build Physical responses:

$$X(t) := a_1 \cdot q_1(t) + a_2 \cdot q_2(t) \qquad T_{plot} := \frac{6}{f_1}$$

4.e Graphs of Modal and Physical responses:







5. Interpret response: analyze results, provide recommendations

S-S displacement

$$K^{-1} \cdot F_o = \begin{pmatrix} 5 \times 10^{-3} \\ 0 \end{pmatrix} m$$

Recall natural frequencies & periods

$$f = \begin{pmatrix} 17.92 \\ 34.62 \end{pmatrix} Hz \qquad \frac{1}{f} = \begin{pmatrix} 0.056 \\ 0.029 \end{pmatrix} s$$

$$A = \begin{pmatrix} 1 & 1 \\ 0.73 & -2.73 \end{pmatrix}$$

$$\omega = \begin{pmatrix} 112.6 \\ 217.53 \end{pmatrix} s^{-1}$$

STEP FORCED RESPONSE of Undamped 2-DOF mechanical system

ORIGIN := 1

Dr. Luis San Andres (c) MEEN 363, 617 February 2008

The undamped equations of motion are:

$$M\cdot\frac{d^2}{dt^2}X + K\cdot X = F_o \qquad (1)$$

where **M,K** are matrices of inertia and stiffness coefficients, and **X, V**=d**X**/dt, d²**X**/dt² are the vectors of physical displacement, velocity and acceleration, respectively.
The FORCED undamped response to the initial conditions, at t=0, **Xo,Vo=**d**X/**dt, follows:

==

The equations of motion are:

WITH RIGID BODY MODE

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\cdot\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_{1o} \\ F_{2o} \end{pmatrix} \qquad (2)$$

1. Set elements of inertia and stiffness matrices

DATA FOR problem

$$M := \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix}\cdot kg \qquad K := \begin{pmatrix} 1\cdot 10^6 & -1\cdot 10^6 \\ -1\cdot 10^6 & 1\cdot 10^6 \end{pmatrix}\cdot\frac{N}{m} \qquad n := 2 \text{ \# of DOF}$$

Note M and K are symmetric matrices

initial conditions

$$X_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix}\cdot m \qquad V_o := \begin{pmatrix} 0.0 \\ 0 \end{pmatrix}\cdot\frac{m}{sec}$$

Applied force vector:

$$F_o := \begin{pmatrix} 1000 \\ -980 \end{pmatrix}\cdot N$$

---

2. Find eigenvalues (undamped natural frequencies) and eigenvectors

Set determinant of system of eqns = 0

$$\Delta = \left[\left(K_{11} - M_{11}\cdot\omega^2\right)\cdot\left(K_{22} - M_{22}\cdot\omega^2\right) - \left(K_{12} - M_{12}\cdot\omega^2\right)\cdot\left(K_{21} - M_{21}\cdot\omega^2\right)\right] = 0 \qquad (2a)$$

$$\Delta = a\cdot\omega^4 + b\cdot\omega^2 + c = \left(a\cdot\lambda^2 + b\cdot\lambda + c\right) = (\text{with } \lambda = \omega^2 \qquad (2b)$$

where the coefficients are:

$$a := M_{1,1}\cdot M_{2,2} - M_{1,2}\cdot M_{2,1}$$

$$b := K_{1,2}\cdot M_{2,1} - K_{1,1}\cdot M_{2,2} - K_{2,2}\cdot M_{1,1} + K_{2,1}\cdot M_{1,2} \qquad (2c)$$

$$c := K_{1,1}\cdot K_{2,2} - K_{1,2}\cdot K_{2,1}$$

The roots of equation (2b) are:

$$\lambda_1 := \frac{\left[-b-\left(b^2-4\cdot a\cdot c\right)^{.5}\right]}{2\cdot a}\;\frac{\left[-b+\left(b^2-4\cdot a\cdot c\right)^{.5}\right]}{2\cdot a} \qquad (3)$$

also known as eigenvalues. The **natural frequencies** follow as:

$j := 1 .. n \qquad \omega_j := \left(\lambda_j\right)^{.5} \qquad f := \frac{\omega}{2\cdot\pi}$

$$\omega = \begin{pmatrix} 0 \\ 173.21 \end{pmatrix} \frac{rad}{sec} \qquad (4)$$

$$f = \begin{pmatrix} 0 \\ 27.57 \end{pmatrix} Hz$$

Note that: $\Delta\left(\omega_1\right) = \Delta\left(\omega_2\right) = 0$

For each eigenvalue, the eigenvectors (**natural modes)** are

$j := 1 .. n$

$$a_j := \begin{bmatrix} 1 \\ \dfrac{K_{1,1}-M_{1,1}\cdot\lambda_j}{-\left(K_{1,2}-M_{1,2}\cdot\lambda_j\right)} \end{bmatrix}$$

Set arbitrarily first element of vector = 1

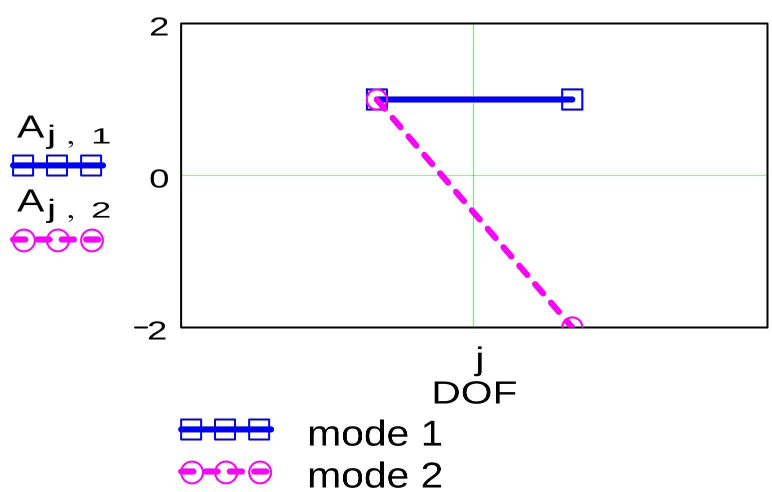
$$a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad (5)$$

MODAL matrix $\quad A^{\langle j\rangle} := a_j$

A is the matrix of eigenvectors (undamped modal matrix): each column corresponds to an eigenvector

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

Plot the mode shapes:



3. Modal transformation of physical equations to (natural) modal coordinates

Using transformation: (6)

Using transformation: $X = A \cdot q$ (6)

EOMs (1) become uncoupled in modal space:

$$M_m \cdot \frac{d^2}{dt^2} q + K_m \cdot q = Q_m \qquad (7)$$

with modal force vector:

$$Q_m = A^T \cdot F_o \qquad (8)$$

and initial conditions (modal displacement=q and modal velocity dq/dt=s)

$$q_o = M_m^{-1} \cdot \left(A^T \cdot M \cdot X_o\right) \qquad s_o = M_m^{-1} \cdot \left(A^T \cdot M \cdot V_o\right) \qquad (9)$$

The <u>modal responses</u> are of the form: k=1....n

$$q_k = q_{o_k} \cdot \cos\left(\omega_k \cdot t\right) + \frac{s_{o_k}}{\omega_k} \cdot \sin\left(\omega_k \cdot t\right) + \frac{Q_{m_k}}{K_{m_{k,k}}} \cdot \left(1 - \cos\left(\omega_k \cdot t\right)\right) \quad \omega_k \neq 0 \qquad (10a)$$

for an elastic mode

OR

$$q_k = q_{o_k} + s_{o_k} \cdot t + \frac{1}{2} \cdot \frac{Q_{m_k}}{M_{m_{k,k}}} \cdot t^2 \qquad \text{for } \omega_k = 0 \qquad (10b)$$

for a rigid body mode

And, the response in the physical coordinates is given by the superposition of the modal responses, i.e.

$$X(t) = A \cdot q(t) \qquad (5)$$

=== CHECK ===

Verify the orthogonality properties of the natural mode shapes

$$M_m := A^T \cdot M \cdot A \qquad M_m = \begin{pmatrix} 150 & 0 \\ 0 & 300 \end{pmatrix} kg$$

$$K_m := A^T \cdot K \cdot A \qquad K_m = \begin{pmatrix} 0 & 0 \\ 0 & 9 \times 10^6 \end{pmatrix} \frac{N}{m} \qquad \omega = \begin{pmatrix} 0 \\ 173.21 \end{pmatrix} s^{-1}$$

===

4. Find Modal and Physical Response for given initial condition and Constant Force vector

Recall the **vectors of initial conditions**

$$X_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m \qquad V_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \frac{m}{s}$$

and **Constant forces:**

$$F_o = \begin{pmatrix} 1 \times 10^3 \\ -980 \end{pmatrix} m \frac{N}{m}$$

DATA FOR problem being analyzed:

4.a Find initial conditions in modal coordinates (displacement = q, velocity = s)

Set inverse of modal mass matrix $A_{inv} := M_m^{-1} \cdot \left(A^T \cdot M\right)$

$$A_{inv} = \begin{pmatrix} 0.67 & 0.33 \\ 0.33 & -0.33 \end{pmatrix}$$

$q_o := A_{inv} \cdot X_o$ $\qquad$ $s_o := A_{inv} \cdot V_o$

$q_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m$ $\qquad$ $s_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m\, s^{-1}$

4.b Find Modal forces:

$Q_m := A^T \cdot F_o$ $\qquad$ $Q_m = \begin{pmatrix} 20 \\ 2.96 \times 10^3 \end{pmatrix} N$

4.c Build Modal responses:

$$q_1(t) := q_{o_1} + s_{o_1} \cdot t + \frac{Q_{m_1}}{M_{m_{1,1}}} \cdot \frac{t^2}{2}$$

response for rigid body mode

$$q_2(t) := q_{o_2} \cdot \cos\left(\omega_2 \cdot t\right) + \frac{s_{o_2}}{\omega_2} \cdot \sin\left(\omega_2 \cdot t\right) + \frac{Q_{m_2}}{K_{m_{2,2}}} \cdot \left(1 - \cos\left(\omega_2 \cdot t\right)\right)$$

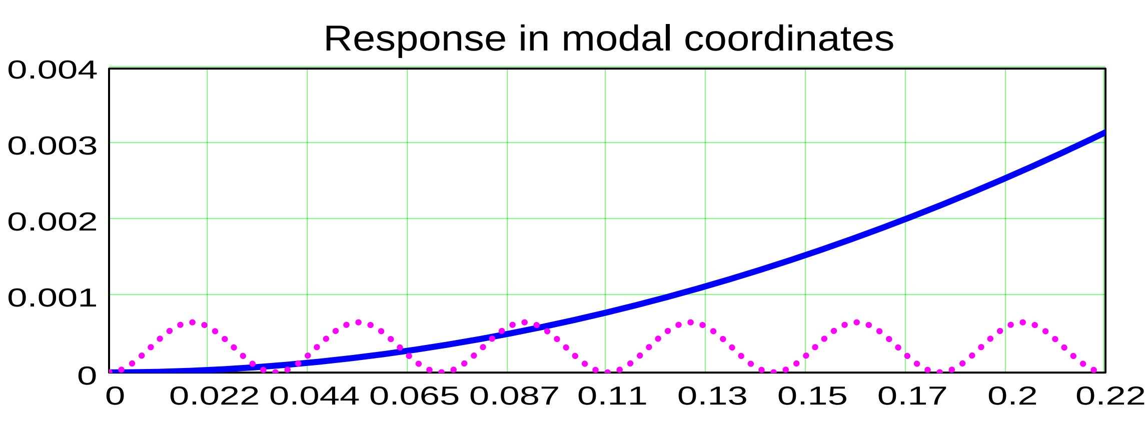
4.d Build Physical responses:

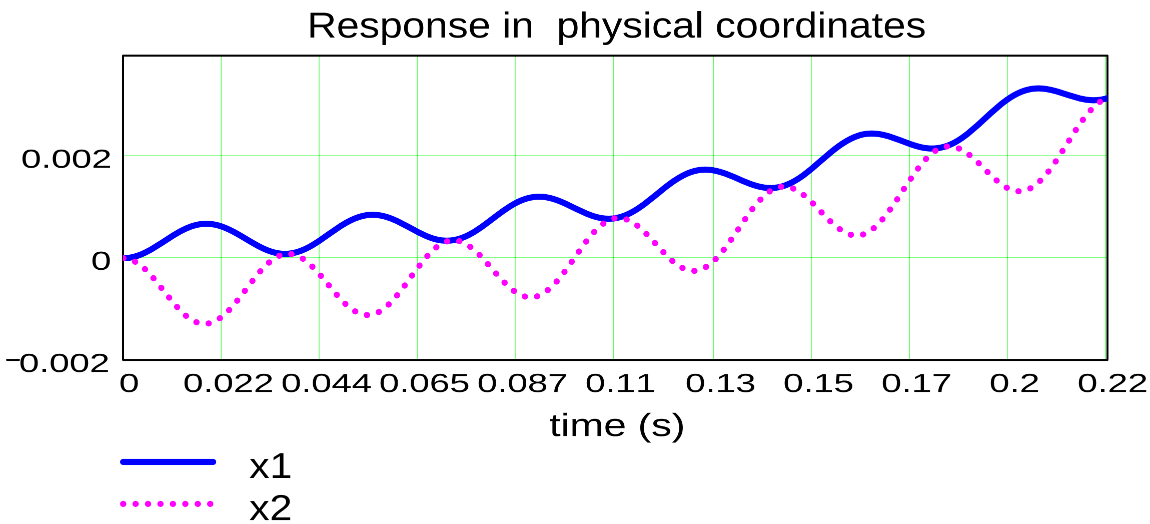
$$X(t) := a_1 \cdot q_1(t) + a_2 \cdot q_2(t)$$

for plots:

$$T_{plot} := \frac{6}{f_2}$$

4.e Graphs of Modal and Physical responses:





5. Interpret response: analyze results, provide recommendations

S-S displacement - NONE

Recall natural frequencies & periods

$$f = \begin{pmatrix} 0 \\ 27.57 \end{pmatrix} \text{Hz} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \qquad \omega = \begin{pmatrix} 0 \\ 173.21 \end{pmatrix} s^{-1}$$