MINIMAX METHODS FOR FINDING MULTIPLE SADDLE CRITICAL POINTS IN BANACH SPACES AND THEIR APPLICATIONS

A Dissertation

by

XUDONG YAO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2004

Major Subject: Mathematics

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ABSTRACT

Minimax Methods for Finding Multiple Saddle Critical Points in Banach Spaces and Their Applications. (August 2004)

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This dissertation was to study computational theory and methods for finding multiple saddle critical points in Banach spaces. Two local minimax methods were developed for this purpose. One was for unconstrained cases and the other was for constrained cases. First, two local minmax characterization of saddle critical points in Banach spaces were established. Based on these two local minmax characterizations, two local minimax algorithms were designed. Their flow charts were presented. Then convergence analysis of the algorithms were carried out. Under certain assumptions, a subsequence convergence and a point-to-set convergence were obtained. Furthermore, a relation between the convergence rates of the functional value sequence and corresponding gradient sequence was derived. Techniques to implement the algorithms were discussed. In numerical experiments, those techniques have been successfully implemented to solve for multiple solutions of several quasilinear elliptic boundary value problems and multiple eigenpairs of the well known nonlinear p-Laplacian operator. Numerical solutions were presented by their profiles for visualization. Several interesting phenomena of the solutions of quasilinear elliptic boundary value problems and the eigenpairs of the p-Laplacian operator have been observed and are open for further investigation. As a generalization of the above results, nonsmooth critical points were considered for locally Lipschitz continuous functionals. A local minmax characterization of nonsmooth saddle critical points was also established. To establish its version in Banach spaces, a new notion, pseudo-generalized-gradient has to be introduced. Based on the characterization, a local minimax algorithm for finding multiple nonsmooth saddle critical points was proposed for further study.

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CHAPTER I

INTRODUCTION

This dissertation is to study numerical methods and their related theory for computing multiple saddle critical points in Banach spaces. For a given Banach space B, let B^* be its topological dual, $\langle , \rangle$ the dual relation and $\|\cdot\|$ the norm on B. Let $J \in C^1(B, \mathbb{R})$. A point $u^* \in B$ is said to be a critical point of J iff u^* satisfies the Euler-Lagrange equation

$$\nabla J(u) = 0$$

where $\nabla J(u)$ is the gradient of J at u in the sense of the Fréchet derivative. Critical points of a C^1 functional are called smooth critical points (SCP). Let $J : B \to \mathbb{R}$ be locally Lipschitz continuous. Then the generalized-gradient of J in the sense of Clark [7] is defined as follows.

Definition I.1 *Let J be Lipschitz continuous near $u_0 \in B$. The generalized directional derivative $J^0(u_0; v)$ of J at u_0 in the direction of $v \in B$ is defined by*

$$J^0(x; v) = \limsup_{\substack{u \to u_0 \\ t \downarrow 0}} \frac{J(u + tv) - J(u)}{t}.$$

The generalized gradient $\partial J(u_0)$ of J at u_0 is a subset of B^* given by

$$\partial J(u_0) = \{\zeta \in B^* : J^0(u_0; v) \geq \langle \zeta, v \rangle, \forall v \in B\}.$$

According to Chang [3], $u^* \in B$ is a critical point of J iff u^* satisfies

$$0 \in \partial J(u^*).$$

This dissertation follows the style and format of *Mathematics of Computation.*

Critical points of a locally Lipschitz continuous functional are called nonsmooth critical points (NCP). If J is C^1, then $\partial J(u) = \{\nabla J(u)\}$, i.e., two definitions coincide. When u^* is a critical point (SCP or NCP), $c = J(u^*)$ is called the critical value of J at u^* and the set $J(c)^{-1} = \{u \in B : J(u) = J(u^*)\}$ is called a critical level. A problem is said to be *variational* if it can be converted to solving its Euler-Lagrange equation. Critical point theory is concerned with variational problems. The first candidates for critical points are the local maxima or minima to which the classical critical point theory was devoted in calculus of variation. Critical points u^* that are not local extrema are called *saddle (critical) points*, i.e., in any neighborhood $\mathcal{N}(u^*)$ of u^* there exist two points v, w such that $J(v) < J(u^*) < J(w)$. In physical systems, saddle points appear as *unstable* equilibria or transient excited states. Due to unstable nature, saddle critical points are very elusive to numerical approximation. Conventional numerical algorithms are designed to find stable (local extremum) solutions. New approaches and methods must be developed.

Variational methods have been proved to be powerful tools in solving nonlinear boundary value problems appearing in many disciplines where other methods may fail. The study of variational problems can be traced back to early as Fermat who proved in 1650 that the light follows the path that takes the least time to go from one point to an other. Newton and Leibnitz simultaneously and independently made the connection between calculus and derivatives with the variation of functions. Many great mathematicians, such as, Cauchy, Euler, Dirichlet, Lagrange, Poincare, etc., have made important contributions to critical point theory. Until beginning of the 20th century, mathematicians were looking for absolute minimizers of functions bounded from below. In 1905, in his thesis, Poincare treated a variational problem whose solution corresponded neither to a minimum nor to a maximum. This approach was revisited by Birkhoff in 1917 who succeeded to obtain a minimax principle in critical point theory and this principle was further generalized in late 1920s and early 1930s independently by Morse and Ljusternik-Schnirelman.

Since then, the minimax principle, which characterizes a saddle point as a solution to

$$\min_{A\in\mathcal{A}} \max_{v\in A} J(v) \tag{1.1}$$

for some collection $\mathcal{A}$ of subsets A in B, becomes one of the most popular approaches in critical point theory. As a typical example, the mountain pass lemma proved by Ambrosetti and Rabinowtz in 1973 [1] set a milestone for modern nonlinear analysis, since then many minimax theorems, such as various linking and saddle point theorems, have been successfully established to prove the existence of multiple critical points [4,10,19,21,1,25,26,27,2,24,3,13,etc.]. But most of them focus mainly on the existence issue and require one to solve a two-level *global* optimization problem, and therefore are not useful for algorithm implementation.

The first numerical minimax algorithm for finding smooth saddle critical points (SSCPs) basically with MI=1 was developed by Choi-McKenna [6] in 1993, where MI is the Morse index of a critical point u^* in a Hilbert space H which is defined as the maximum dimension of a subspace H^- of H on which $J''(u^*)$ is negative definite and MI is not defined in a Banach space. Ding-Costa-Chen [11] proposed a numerical minimax method in 1999 to capture SSCPs basically with MI=2. But no mathematical justification or convergence of the algorithms was established. A numerical local minimax algorithm together with its mathematical justification and convergence was successfully developed by Li-Zhou [17,18] in 2001, to find multiple SSCPs of MI=1,2,...n. All those three algorithms are formulated in Hilbert spaces, where the gradient and orthogonality played important roles. In fact, the gradient is used to find a search direction to update an approximation point and the orthogonality is used to prevent the search from degenerating to a lower critical level. In terms of minimax approach, in (1.1), at the first level, A is a 1D simplex in Choi-McKenna's method, a 2D simplex in Ding-Costa-Chen's method and an nD subspace in Li-Zhou's method.

However, many nonlinear problems in application, such as the wellknown nonlinear p-Laplacian equation in the study of non-Newtonian fluid flows [9, 15, 25], are formulated in Banach spaces and possess multiple solutions. How to find multiple SSCPs in Banach spaces? So far no such numerical methods are available in the literature. In this dissertation, a numerical local minimax method will be developed for this purpose. The key step in this development is to establish a mathematical justification, a local minmax characterization for SSCPs, in Banach spaces.

On the other hand, the popular hemivariational inequalities, which arise in mechanics when one wants to consider more realistic nonmonotone and multivalued stress-strain laws or bounded condition [22,23,14], require us to deal with NCPs. In fact, the local minmax characterization for SSCPs in a Banach space can be generalized to be a local minmax characterization for NCPs. The generalized local minmax characterization gives us a starting point to design a numerical local minimax method to find multiple nonsmooth saddle critical points (NSCPs).

When theory and methods for finding multiple SCPs are developed, it is quite natural to consider multiple constrained smooth critical point (CSCP) problems, which constitute an important part of critical point theory. An important class of multiple CSCP problems is nonlinear variational eigenpair problems. Linear eigenpair problems are a classical research topic both theoretically and numerically [28]. Huge literature is available. On nonlinear eigenpair problems, although many theoretical studies exist in the literature [28, 8], people's understanding is still limited. In particular, few numerical methods [16] can be found. In this dissertation, a numerical local minimax method will be developed to find multiple nonlinear eigenpairs.

In the sections of Chapter I, some related milestone results on the existence and computation of critical points and eigenpairs in contemporary critical point theory will be recalled. In Chapter II, a local minmax characterization for SSCPs will be established, a local minimax algorithm for finding multiple SSCPs will be designed, implementation techniques of the algorithm will be discussed and numerical experiment results on quasilinear elliptic PDEs will be presented by figures of their solution profiles for visualization. In Chapter III, some convergence results of the algorithm will be established and a relation between convergence rates of the functional values and their gradients will be presented. The smoothness of peak-selection will be discussed. As an application of our frame work, we give a proof to the existence of a nontrivial weak solution to a class of quasilinear elliptic PDEs. In Chapter IV, a local minmax characterization for a class of CSCP problems, i.e., iso-homogeneous nonlinear eigenpair problems will be established, a local minimax algorithm for finding multiple eigenpairs of this class eigenpair problems will be designed, numerical experiment results on eigenpairs of the wellknown nonlinear p-Laplacian operator, will be exhibited by figures of eigenfunction profiles with the corresponding eigenvalues for visualization. In Chapter V, several convergence results of the algorithm will be stated and the smoothness of peak-selection will be discussed. In Chapter VI, a local minmax characterization for NSCPs will be established. In order to establish such minmax characterization in Banach spaces, pseudo-generalized-gradient for locally Lipschitz continuous functionals has to be defined. A minimax algorithm for finding multiple NSCPs will be designed.

A. Existence of Multiple Saddle Critical Points

Many existence results for multiple SSCPs and NSCPs in various nonlinear problems are available in literatures. Some of them will be recalled in this section.

1. Existence of Multiple SSCPs

The following wellknown Palais-Smale condition [1] is frequently used in the study of the existence of SSCPs as a compactness assumption, which is, although not always, frequently satisfied by nonlinear PDE problems.

Definition I.2 *A functional $J \in C^1(B, \mathbb{R})$ is said to satisfy the Palais-Smale (PS) condition if any sequence $\{u_i\} \subseteq B$ such that $J(u_i)$ is bounded and $\nabla J(u_i) \to 0$ possesses a convergent subsequence.*

One of the simplest and most useful minimax theorems in the literature for saddle critical points, is the mountain pass lemma, established by Amhrosetti and Rabinowitz [1] in 1973.

Theorem I.1 *(*Mountain Pass Lemma*) Given a Banach space B and a functional $J \in C^1(B, \mathbb{R})$ satisfying the PS condition with $J(0) = 0$. Assume that*

(1) *there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$, and*

(2) *there is an $e \in B \setminus \partial B_\rho$ such that $J(e) \leq 0$.*

Then

$$c = \inf_{p \in C([0,1],B), p(0)=0, p(1)=e} \max_{t \in [0,1]} J(p(t))$$

is a critical value of J.

The mountain pass lemma sets a milestone for contemporary nonlinear analysis. It is used to prove the existence of the ground state. Since then, many linking theorems are also established to prove the existence of more saddle critical points in various nonlinear problems. The following linking theorem is due to Rabinowitz.

Theorem I.2 (Linking Theorem) *Given a Banach space B such that $B = L \oplus X$, where X, L are two closed subspaces of B and L has finite dimension. Assume that $J \in C^1(B, \mathbb{R})$ satisfies the PS condition and*

(1) *there are $\rho, \alpha > 0$ such that $J(v) \geq \alpha$, $\forall v \in \partial B_\rho \cap X$,*

(2) *there are $u \in X$ with $\|u\| = 1$ and a number $R > \rho$ such that $J(v) \leq 0$, $\forall v \in \partial Q$, where $Q = (\bar{B}_R \cap L) \oplus \{ru | r \in (0, R)\}$.*

Then

$$c = \inf_{\Gamma} \max_{v \in Q} J(h(u))$$

is a critical value, where

$$\Gamma = \{h \in C(\bar{Q}, B) | h = id \text{ on } \partial Q\}.$$

2. Existence of Multiple NSCPs

The nonsmooth version of the Palais-Smale condition is frequently used in the proof of the existence for NSCP and due to Chang [3].

Definition I.3 (Nonsmooth Palais-Smale Condition) *A locally Lipschitz continuous functional $J : B \to \mathbb{R}$ satisfies the nonsmooth Palais-Smale (PS) condition, if any sequence $\{J(u_n)\} \subset B$ such that $\{J(u_n)\}$ is bounded blow and $\{z_n\} \to 0$, where $z_n \in \partial J(u_n)$ with minimum norm, has a strongly convergent subsequence.*

By the nonsmooth version of the PS condition, several minimax theorems and linking theorems have been established and used to prove the existence of multiple NSSCPs. Similar to smooth cases, the following Theorem I.3 is for the existence of the ground state and Theorem I.4 is for the existence of more saddle critical points. These two minimax theorems are due to N. Kourogenis, P. Kandilakis and N. S. Papageorgiou.

Theorem I.3 *If*

(1) *B is a reflexive Banach space, $B = L \oplus M$ with dim $L < +\infty$,*

(2) *$J : B \to \mathbb{R}$ is a locally Lipschitz functional,*

(3) *there is $r > 0$ such that*

$$\max\{J(u) : u \in L, \|u\| = r\} < inf\{J(v) : v \in M\},$$

(4) *J satisfies the non-smooth Palais-Smale condition, and*

(5) *$c_0 \equiv \inf_{\gamma\in\Gamma} \max_{u\in D} J(\gamma(u))$ with $D \equiv \{u \in L : \|u\| \leq r\}$ and*

$$\Gamma \equiv \{\gamma \in C(D; X) : \gamma(u) = u \ for \ \|u\| = r\},$$

then $c_0 \geq inf_{v\in M} J(v)$ and c_0 is a critical value of J. Moreover, if $c_0 = inf_{v\in M} J(v)$, then there is a critical point $v_0 \in M$ with $c_0 = J(v_0)$.

Theorem I.4 *If*

(1) *B is a reflexive Banach space, $B = L \oplus M$ with dim $L < +\infty$,*

(2) *$J : B \to \mathbb{R}$ is a locally Lipschitz functional which is bounded below,*

(3) *J satisfies the non-smooth Palais-Smale condition,*

(4) *$J(0) = 0$ and $\inf_{v\in B} J(v) < 0$, and*

(5) *there is $r > 0$ such that*

$$\begin{aligned} J(u) &\leq 0 \ if \ u \in L \ and \ \|u\| \leq r, \\ J(u) &\geq 0 \ if \ u \in M \ and \ \|u\| \leq r, \end{aligned}$$

then, J has at least two non-trivial critical points.

B. Numerical Methods on Finding Multiple SSCPs

In this section, three numerical methods for finding SSCPs, which are related to the minimax algorithms in this dissertation, will be recalled. The first method is proposed by Choi and McKenna [6] in 1993. The flow chart of the algorithm in [6] is long. It is rewritten in [5]. The version in [5] reads basically as follows.

Algorithm I.1 *Modified Mountain Pass Method (Choi-Mckenna)*

Step 1. *Given an increasing direction v_0. Set $k = 0$.*

Step 2. *Solve $t_k = \arg\max_{t>0} J(tv_k)$.*

Step 3. *Find the steep descent direction d_k of J at $u_k = t_k v_k$. If $\|d_k\| \leq \epsilon$, stop the algorithm. Otherwise, do Step 4.*

Step 4. *Solve $s_k = \arg\max_{s>0}\{\max_{t>0} J(t(v_k + sd_k)) < J(u_k)\}$.*

Step 5. *Let $v_{k+1} = v_k + s_k d_k$. Update $k = k + 1$ and go to Step 2.*

The second method is designed by Ding, Costa and Chen [11] in 1999. The flow chart of the algorithm reads as follows.

Algorithm I.2 *High Linking Method (Ding-Costa-Chen)*

Step 1. *Find a point v such that $v_0 \neq 0$ and $J(v_0) \leq 0$.*

Step 2. *Apply the Modified Mountain Pass Method to find a mountain pass solution v_1 and u_1, u_2 satisfying*

$$J(v_1 + tu_1) < J(v_1), \quad J(v_1 + tu_2) < J(v_1) \quad for\ small\ t \neq 0.$$

Step 3. *Find* $t_1 > 0$ *and* $t_2 < 0$ *such that* $J(v_1 + t_1 u_1) \leq 0$ *and* $J(v_1 + t_2 u_1) \leq 0$, *and set* $g_1 = v_1 + t_1 u_1$ *and* $g_2 = v_1 + t_2 u_1$.

Step 4. *Find* $t_3 > 0$ *such that* $J(v_1 + t_3 u_2) \leq J(v_1)$, *and set* $g_3 = v_1 + t_3 u_2$.

Step 5. *Construct the triangle* $\triangle$ *by*

$$\triangle = \{\lambda_1 g_1 + \lambda_2 g_2 + (1 - \lambda_1 - \lambda_2) g_3 | \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 \leq 1\},$$

and find $v^* \in \triangle$ *such that* $J(v^*) = \max_{g \in \triangle} J(g)$.

Step 6. *If* v^* *is an interior point of* $\triangle$, *then go to next step. Otherwise, set* $u_2 = v^* - v_1$ *and go to Step 4.*

Step 7. *Set* $v_2 = v^*$, *compute* $w = \nabla J(v_2)$.

Step 8. *If* $\|v\| \leq \epsilon$, *then output* v_2 *and stop. Otherwise, set* $u_2 = (-v + v_2) - v_1$ *and go to next step.*

Step 9. *Repeat the same procedures as Step 4-6 to construct a new triangle* $\triangle$ *and find an interior point* $v^* \in \triangle$ *such that* $J(v^*) = \max_{g \in \triangle} J(g)$.

Step 10. *If* $J(v^*) < J(v_2)$, *go to Step 7. Otherwise, set* $w = \frac{1}{2} w$ *and* $u_2 = (-w + v_2) - v_1$, *then go to Step 9.*

The third method is established by Li and Zhou [17] in 2001. The flow chart of the algorithm reads as follows.

Algorithm I.3 *Local Minimax Method in the Hilbert Space (Li-Zhou)*

Assume that $u_1, ..., u_{n-1}$ *are* $n-1$ *found critical point of* J, $L = [u_1, ..., u_{n-1}]$ *and* λ, ϵ *are two positive numbers.*

Step 1. *Find an ascent direction* $v_n^1 \in L^\perp$ *at* u_{n-1}.

Step 2. *Solve for*

$$u_n^1 = \sum_{i=1}^{n-1} t_i^1 u_i + t_n^1 v_n^1 = \arg \max_{t_i \in R, i=1,...,n-1, t_n>0} J(\sum_{i=1}^{n-1} t_i u_i + t_n v_n^1)$$

with initial point $(0, ..., 0, 1)$ *and set* $k = 1$.

Step 3. *Compute the descent direction* w_n^k *of* J *at* u_n^k, $w_n^k = -\nabla J(u_n^k)$.

Step 4. *If* $\|w_n^k\| < \epsilon$, *then stop and output* u_n^k. *Otherwise, go to Step 5.*

Step 5. *Let* $v_n^k(s) = \frac{v_n^k + s w_n^k}{\|v_n^k + s w_n^k\|}$ *and solve for*

$$p(v_n^k(s)) = \sum_{i=1}^{n-1} t_i^k u_i + t_n^k v_n^k(s) = \arg \max_{t_i \in R, i=1,...,n} J(\sum_{i=1}^{n-1} t_i u_i + t_n v_n^k(s)).$$

with initial guess $(t_1^k, t_2^k, ..., t_n^k)$. *Set*

$$s_n^k = \max\{s | \lambda \geq s\|w_n^k\| \geq 0, J(p(v_n^k(s))) - J(p(v_n^k)) \leq -\frac{1}{2} s t_n^k \|w_n^k\|^2\}.$$

Let $v_n^{k+1} = v_n^k(s_n^k) = \frac{v_n^k + s_n^k w_n^k}{\|v_n^k + s_n^k w_n^k\|}$ *and* $u_n^{k+1} = p(v_n^{k+1}) = \sum_{i=1}^{n-1} t_i^{k+1} u_i + t_n^{k+1} v_n^{k+1}$. $k = k + 1$ *and go to Step 3.*

C. A Class of Quasilinear Elliptic PDEs

All three numerical methods in Section B are for finding critical points in Hilbert spaces. However, many problems in application have to be formulated as finding critical points in Banach spaces. For example, the weak solutions of the following class of quasilinear elliptic PDEs

$$\begin{cases} \Delta_p u + f(x, u) &= 0, \quad in \quad \Omega \\ u &= h, \quad on \quad \partial\Omega, \end{cases}$$

are the SCPs of some functional in Banach spaces where Δ_p denotes the p-Laplacian operator defined, for $p > 1$, by

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}),$$

$|\cdot|$ is the Euclidean norm, f satisfies some standard conditions [20], h is a constant and Ω is a domain in $\mathbb{R}^n$. When $p = 2$, Δ_p becomes the usual Laplacian operator Δ. This class of quasilinear elliptic PDEs ($p \neq 2$) appears in non-Newtonian fluids, some reaction-diffusion problems, flow through porous media, nonlinear elasticity, glaceology and petroleum extraction [9]. It has also geometrical interest for $p \geq 2$ [9].

D. Existence of Eigenpairs

Eigenpair problems constitute an important class of multiple CSCP problems. In this section, some important existence results will be recalled. The first wellknown result is of linear eigenpair problems.

Theorem I.5 *(*Courant Maximum-Minimum Principle*, [28]) Consider the linear eigenvalue problem*

$$Au = \lambda u, \quad u \in H, \ \lambda \in \mathbb{R}$$

with the aid of

$$\pm\frac{\lambda_n^{\pm}}{2} = \begin{cases} \sup_{S_k \in L_n^{\pm}} \inf_{u \in S_k} \pm F(u), \\ 0 \quad L_n^{\pm}. \end{cases} \tag{1.2}$$

for $n = 1, 2, \ldots$. In this connection, we assume

(H_1) *H is a real separable Hilbert space with inner product $\langle , \rangle$ and $dim(H) = \infty$. The operator $A : H \to H$ is nonzero, linear, symmetric and compact. Denote*

$$F(u) = 2^{-1}\langle Au, u\rangle, \quad G(u) = 2^{-1}\langle u, u\rangle.$$

(H_2) *$S = \{u \in H | \|u\| = 1\}$ and $S_k = S \cap H_k$, where H_k is a k-dimensional linear subspace of H.*

(H_3) *L_n is the set of all S_k with $k \geq n$ and*

$$L_n^{\pm} = \{S_k \in L_n | \pm F(u) > 0 \text{ on } S_k\}.$$

Let $\pm\lambda_n^{\pm} > 0$ for $+$ or $-$. Then the following four assertions hold:

(a) *$\lambda = \lambda_k^{\pm}$ is an eigenvalue of A. All eigenvalues $\lambda \neq 0$ of A can be obtained in this way with the aid of (1.2).*

(b) *The multiplicity of λ is equal to the number of indices k for which $\lambda_k^{\pm} = \lambda$.*

(c) *There exist eigenvectors $u_1, ..., u_n$ of A such that $\langle u_i, u_j \rangle = \delta_{ij}$ for $i, j = 1, ..., n$ and such that*

$$\pm\frac{\lambda_n^{\pm}}{2} = \min_{u \in S_n} \pm F(u),$$

where $S_n = S \cap span\{u_1, ..., u_n\} \in L_n^{\pm}$.

(d) *$\lambda_n^{\pm} \to 0$ as $n \to \infty$.*

The second theorem is for nonlinear eigenpair problems.

Theorem I.6 *([28]) For fixed $\alpha > 0$, consider the eigenvalue problem*

$$F'(u) = \lambda G'(u), \quad u \in N_\alpha \ \lambda \in \mathbb{R}, \tag{1.3}$$

where the level set

$$N_\alpha = \{u \in B | G(u) = \alpha\}.$$

with the aid of

$$\pm\frac{\lambda_n^{\pm}}{2} = \begin{cases} \sup_{S_k \in K_n^{\pm}} \inf_{u \in K} \pm F(u), \\ 0 \quad K_n^{\pm} \end{cases} \tag{1.4}$$

$n = 1, 2, ...,$ where $K_n^{\pm}$ denotes the class of all compact symmetric subsets K of N_α such that $gen(K) \geq n$ and $\pm F(u) > 0$ on K and

$$\pm\chi_{\pm} = \begin{cases} \textit{supremum over all } m \textit{ such that } \pm c_m > 0, \\ 0 \quad \textit{for } c_1^{\pm} = 0. \end{cases} \tag{1.5}$$

In this connection, we assume

(H_1) *B is a real reflexive separable Banach space with $dim(B) = \infty$ and $F, G : B \to \mathbb{R}$ are even function functionals such that $F, G \in C^1(B, \mathbb{R})$ and $F(0) = G(0) = 0$. In particular, it follows from this that F' and G' are odd potential operators.*

(H_2) *The operator F' is strongly continuous and $F(u) \neq 0$, $u \in \bar{co}(N_\alpha)$ implies that $F'(u) \neq 0$.*

(H_3) *The operator G' is uniformly continuous on bounded sets and satisfies*

$$u_n \rightharpoonup u, \; G'(u_n) \to v \quad \textit{implies} \quad u_n \to u \textit{ as } n \to \infty.$$

(H_4) *The level set N_α is bounded and*

$$u \neq 0 \quad \textit{implies} \quad \langle G'(u), u \rangle > 0, \;\; \lim_{t\to\infty} G(tu) = +\infty,$$

and

$$\inf_{u \in N_\alpha} \langle G'(u), u \rangle > 0.$$

Then the following five assertions hold:

(1) *Existence of an eigenvalue. If $\pm c_n^{\pm} > 0$ ($+$ or $-$), then (1.3) possesses a pair $(u_m^{\pm}, -u_m^{\pm})$ of eigenvectors with the eigenvalue $\lambda_m^{\pm} \neq 0$ and $F(u_m^{\pm}) = c_m^{\pm}$. If F' and G' are positive homogeneous, i.e., $F'(tu) = tF'(u)$ and $G'(tu) = tG'(u)$ for all $u \in B$ and $t > 0$, then $c_m^{\pm} = \alpha\lambda_m^{\pm}$.*

(2) *Multiplicity. (1.3) has at least* $\chi_+ + \chi_-$ *pairs* $(u, -u)$ *of eigenvectors with eigenvalues that are different from zero.*

If $\pm c_n^\pm = \pm c_{n+1}^\pm = \cdots = \pm c_{n+p}^\pm > 0$, $p \geq 1$ *(+ or −), then the set of all eigenvectors of (1.3) such that* $F(u) = c_n^\pm$ *has genus great than or equal to* $p+1$. *In particular, this set is infinite.*

(3) *Critical levels.* $\pm\infty \geq \pm c_1^\pm \geq \pm c_n^\pm \geq \cdots \geq 0$ *and* $c_n^\pm \to 0$ *as* $n \to \infty$.

(4) *Infinitely many Eigenvalues. If* $\chi_+ = \infty$ *or* $\chi_- = \infty$ *and* $F(u) = 0$, $u \in \bar{co}(N_\alpha)$ *implies* $\langle F'(u), u\rangle = 0$, *then there is a sequence* $\{\lambda_n\}$ *of infinitely many distinct eigenvalues for (1.3) such that* $\lambda_n \to 0$ *as* $n \to \infty$.

(5) *Weak convergence of eigenvectors. Assume that* $F(u) = 0$, $u \in \bar{co}(N_\alpha)$ *implies* $u = 0$. *Then* $\max(\chi_+, \chi_-) = \infty$ *and there is a sequence of eigenpairs* (λ_n, u_n) *of (1.3) such that* $u_n \rightharpoonup 0$, $\lambda_n \to 0$ *as* $n \to \infty$ *and* $\lambda_n \neq 0$ *for all* n.

Remark I.1 *The symmetry of a subset and the genus of a symmetric subset need an explanation.*

(1) A subset K *of a Banach space* B *is said to be symmetric iff* $u \in K$ *implies* $-u \in K$.

(2) The genus of a symmetric subset K *of a Banach space* B, *denoted as* $gen(K)$, *is defined as*

$\langle 1\rangle$ $gen(\phi) = 0$.

$\langle 2\rangle$ *If* $K \neq \phi$, $gen(K)$ *is the smallest natural number* $n \geq 1$ *for which a zero-free mapping* $f : K \to R^n - 0$, *where* f *is odd and continuous, exists.*

$\langle 3\rangle$ *If* $K \neq \phi$ *and no such* n *exists,* $gen(K) = +\infty$.

E. Nonlinear Eigenpair Models

Many models in physics and chemistry are related to nonlinear eigenpairs. As examples, three models are presented.

Example 1.(Non-Newtonian Fluids [9]) The quasilinear elliptic equation

$$\Delta_p u + \lambda u = 0, \quad p > 1,\ \lambda > 0, \tag{1.6}$$

appears in the study of non-Newtonian fluids. Indeed, when studying the laws of motion of fluid media, Newton fluids are usually considered to be those for which the relation between the shear stress τ and the velocity gradient $\frac{du}{dx}$ (for simplicity we will here restrict ourselves to the plane case) takes form

$$\tau = \mu \frac{du}{dx}. \tag{1.7}$$

However, this approximation is satisfactory only for a limited number of actual fluid media. Dispersive media treated according to a continuum model do not obey the law given by (1.7). The motions of such non-Newtonian fluids are studied in rheology. Usually (1.7) is substituted by the power rheological law

$$\tau = \mu \left|\frac{du}{dx}\right|^{p-2} \frac{du}{dx}, \quad p > 1. \tag{1.8}$$

The quantities μ and p are the rheological characteristics of the medium. Media with $p > 2$ are called dilatant fluids, and those with $p < 2$ are called pseudoplastics. When $p = 2$, they are Newtonian fluids.

Example 2.(Singular equations [9]) The study of some reaction-diffusion problems leads to formulations such as the following

$$\begin{cases} \Delta u + \lambda u^{-k} = 0 \ \ in \ \ \Omega \\ u = 1 \ \ on \ \ \partial\Omega \end{cases}$$

where $\lambda > 0$ and $0 < k < 1$. This eigenpair problem appears as the limiting case of some models in heterogeneous chemical catalyst kinetics (Langmuir-Hinshelwood model) where the equation is

$$\Delta u + \lambda u^m (\frac{\epsilon + 1}{\epsilon + u})^{m+k} = 0 \quad in\ \Omega, \tag{1.9}$$

with $k > 0$, $m \geq 1$, $\lambda > 0$ and $\epsilon > 0$ small, as well as in models in enzyme kinetics

$$\Delta u + \lambda \frac{u^m}{\epsilon + u^{m+k}} = 0 \quad in\ \Omega. \tag{1.10}$$

When $\epsilon \to 0$, the equations (1.9) and (1.10) become

$$\Delta u + \lambda u^{-k} = 0 \quad in \quad \Omega.$$

Example 3.(p-Laplacian Operator [15]) The p-Laplacian operator has various applications, for instance, in stellar dynamic structure and in flows through porous media when the D'Arcy's law does not remain valid. The weighted eigenpair problem of the p-Laplacian operator is defined as

$$\begin{cases} \Delta_p u + \lambda w |u|^{p-2} u &= 0, \ x \in \Omega, \\ u &= 0, \ \in \partial\Omega \end{cases}$$

where w is a weight function, Ω is a bounded region and $p > 1$. When $w \equiv 1$, the problem becomes the standard eigenpair problem of the p-Laplacian operator

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u &= 0, \ x \in \Omega, \\ u &= 0, \ \in \partial\Omega. \end{cases}$$

CHAPTER II

A MINIMAX METHOD FOR SSCPS IN BANACH SPACES

Assume that B is a Banach space and $J \in C^1(B, \mathbb{R})$. u^* is critical point of J iff u^* satisfies the Euler-Lagrange equation, i.e.,

$$\nabla J(u^*) = 0.$$

A. A Local Minmax Characterization for SSCPs

For a subspace $B' \subseteq B$, denote $S_{B'} = \{v | v \in B', \|v\| = 1\}$ as the unit sphere in B'. Assume that $B = L \oplus L'$, where L (called a support) and L' are closed subspaces of B, and $\mathcal{P} : B \to L'$ is the corresponding linear projection with a bound $M \geq 1$.

Definition II.1 *A set-valued mapping $P : S_{L'} \to 2^B$ is the* peak mapping *of J w.r.t. L if $\forall v \in S_{L'}$, $P(v)$ is the set of all local maximum points of J in the subspace $[L, v] = \{tv + w | w \in L,\ t \in R\}$. A single-valued mapping $p : S_{L'} \to B$ is a* peak selection *of J w.r.t. L if*

$$p(v) \in P(v), \quad \forall v \in S_{L'}.$$

For a given $v \in S_{L'}$, we say that J has a local peak selection *w.r.t. L at v if there is a neighborhood $\mathcal{N}(v)$ of v and a single-valued mapping $p : \mathcal{N}(v) \cap S_{L'} \to B$ satisfying*

$$p(u) \in P(u), \quad \forall u \in \mathcal{N}(v) \cap S_{L'}.$$

Definition II.2 *Let $u \in B$ be a point s.t. $\nabla J(u) \neq 0$. For given $\theta \in (0, 1]$, a point $\Psi(u) \in B$ is a pseudo-gradient of J at u w.r.t. θ if*

$$\|\Psi(u)\| \leq 1, \quad \langle \nabla J(u), \Psi(u) \rangle \geq \theta \|\nabla J(u)\|. \tag{2.1}$$

Denote $\hat{B} = \{u \in B : \nabla J(u) \neq 0\}$. A pseudo-gradient flow of J with a constant θ is *a continuous mapping* $\Psi : \hat{B} \to B$ *s.t.* $\forall u \in \hat{B}$, $\Psi(u)$ *satisfies (2.1).*

Remark II.1 Note that the number 1 in (2.1) can be replaced by any number $m \geq 1$, since it can be absorbed by the constant θ to become $0 < \frac{\theta}{m} \leq 1$.

Lemma II.1 *Let* $0 < \theta < 1$ *be given. For* $v_0 \in S_{L'}$, *if* p *is a local peak selection of* J *w.r.t.* L *at* v_0 *s.t.* $\nabla J(p(v_0)) \neq 0$ *and* $\Psi(p(v_0)) \in B$ *is a pseudo-gradient of* J *at* $p(v_0)$ *w.r.t. the constant* θ, *then there exists a (modified) pseudo-gradient* $G(p(v_0))$ *of* J *at* $p(v_0)$ *w.r.t. the constant* θ *s.t.*

(a) $G(p(v_0)) \in L'$, $0 < \|G(p(v_0))\| \leq M$ *where* $M \geq 1$ *is the bound of the linear projection* $\mathcal{P}$ *from* B *to* L'*;*

(b) $\langle \nabla J(p(v_0)), G(p(v_0)) \rangle = \langle \nabla J(p(v_0)), \Psi(p(v_0)) \rangle$*;*

(c) If $\Psi(p(v_0))$ *is the value of a pseudo-gradient flow* $\Psi(\cdot)$ *of* J *at* $p(v_0)$, *then* $G(\cdot)$ *is continuous and* $G(p(v_0))$ *is called the value of a modified pseudo-gradient flow of* J *at* $p(v_0)$.

Proof. Let $G(p(v_0)) = \mathcal{P}(\Psi(p(v_0))) \in L'$. Then $\|G(p(v_0))\| \leq M\|\Psi(p(v_0))\| \leq M$. Denote $\Psi(p(v_0)) = \Psi_L(p(v_0)) + G(p(v_0))$ for some vector $\Psi_L(p(v_0)) \in L$. By the definition of a peak selection p, we have $\langle \nabla J(p(v_0)), \Psi_L(p(v_0)) \rangle = 0$. Thus

$$\langle \nabla J(p(v_0)), G(p(v_0)) \rangle = \langle \nabla J(p(v_0)), \Psi(p(v_0)) \rangle \geq \theta \|\nabla J(p(v_0))\| > 0.$$

Therefore $G(p(v_0))) \neq 0$ is a pseudo-gradient of J at $p(v_0)$ w.r.t. θ. The results follow.

Lemma II.2 *For each* $v \in X$ *with* $\|v\| = 1$, *it holds*

$$\|v - \frac{v - w}{\|v - w\|}\| \leq \frac{2\|w\|}{\|v - w\|}, \quad \forall w \in B.$$

Proof. In fact,

$$\begin{aligned} \|v - \frac{v-w}{\|v-w\|}\| &= \frac{\|v(\|v-w\|-1)+w\|}{\|v-w\|} \le \frac{\|v\|\,|\,\|v-w\|-1| + \|w\|}{\|v-w\|} \\ &= \frac{|\,\|v-w\| - \|v\|\,| + \|w\|}{\|v-w\|} \le \frac{2\|w\|}{\|v-w\|}. \end{aligned}$$

The next lemma is crucial, which shows the relation between the gradient of J and the variation of a peak selection. It will be used to establish a local minmax characterization of saddle points and to design a stepsize rule in a local minimax algorithm.

Lemma II.3 *For $v_0 \in S_{L'}$, if there is a local peak selection p of J w.r.t. L at v_0 satisfying (1) p is continuous at v_0, (2) $d(p(v_0), L) > 0$ and (3) $\nabla J(p(v_0)) \neq 0$, then there exists $s_0 > 0$ s.t. as $0 < s < s_0$*

$$J(p(v_s)) - J(p(v_0)) < -\frac{s\theta|t_0|\|\nabla J(v_0)\|}{4} \tag{2.2}$$

where $p(v_0) = t_0 v_0 + w_0$ with $t_0 \neq 0$ and $w_0 \in L$,

$$v_s = \frac{v_0 - sign(t_0)sG(p(v_0))}{\|v_0 - sign(t_0)sG(p(v_0))\|}$$

and $G(p(v_0))$ is a modified pseudo-gradient of J at $p(v_0)$ as defined in Lemma II.1.

Proof. Since $J \in C^1(B, \mathbb{R})$, we have

$$J(p(v_s) = J(p(v_0)) + \langle \nabla J(p(v_0)), p(v_s) - p(v_0)\rangle + o(\|p(v_s) - p(v_0)\|). \tag{2.3}$$

Since p is a peak selection, we have $\langle \nabla J(p(v_0)), v_0\rangle = \langle \nabla J(p(v_0)), v\rangle = 0,\ \forall v \in L$. Thus

$$\langle \nabla J(p(v_0)), p(v_s) - p(v_0)\rangle = t_s\langle \nabla J(p(v_0)), v_s\rangle$$

$$= \quad -\frac{sign(t_0)t_s s\langle \nabla J(p(v_0)), G(p(v_0))\rangle}{\|v_0 - sign(t_0)sG(p(v_0))\|} = -\frac{sign(t_0)t_s s\langle \nabla J(p(v_0)), \Psi(p(v_0))\rangle}{\|v_0 - sign(t_0)sG(p(v_0))\|}$$

by Lemma II.1 where $p(v_s) = t_s v_s + w_s$ and $w_s \in L$. When p is continuous at v_0 and $B = L \oplus L'$, we have $t_s \to t_0$ and $w_s \to w_0$ as $s \to 0$. Then, by the definition of a pseudo-gradient, as $s > 0$ is small

$$\langle \nabla J(p(v_0)), p(v_s) - p(v_0)\rangle \leq -\frac{s\theta|t_0|}{\|v_0 - sign(t_0)sG(p(v_0))\|}\|\nabla J(p(v_0))\|. \qquad (2.4)$$

Hence, by (2.3) and (2.4), there is $s_0 > 0$ s.t. as $0 < s < s_0$,

$$J(p(v_s)) - J(p(v_0)) < -\frac{s\theta|t_0|\|\nabla J(p(v_0))\|}{2\|v_0 - sign(t_0)sG(p(v_0))\|}. \qquad (2.5)$$

Choose $s > 0$ small such that $\|v_0 - sign(t_0)sG(p(v_0))\| \leq 2$. Then

$$J(p(v_s)) - J(p(v_0)) < -\frac{s\theta|t_0|\|\nabla J(p(v_0))\|}{4}. \qquad (2.6)$$

The following theorem characterizes saddle points as local minimax solutions.

Theorem II.1 *Let $v_0 \in S_{L'}$. Suppose that J has a local peak selection p w.r.t. L at v_0 satisfying (1) p is continuous at v_0, (2) $d(p(v_0), L) > 0$ and (3) v_0 is a local minimum point of $J(p(\cdot))$. Then $p(v_0)$ is a critical point of J.*

Proof. Suppose $p(v_0)$ is not a critical point, then, by Lemma II.3, there is $s_0 > 0$ s.t.

$$J(p(v_s)) < J(p(v_0)) - \frac{s\theta|t_0|\|\nabla J(v_0)\|}{4}, \quad \forall s \in (0, s_0)$$

where $p(v_0) = t_0 v_0 + w_0$ ($t_0 \neq 0$ and $w_0 \in L$), $v_s = \dfrac{v_0 - sign(t_0)sG(p(v_0))}{\|v_0 - sign(t_0)sG(p(v_0))\|}$ and $G(p(v_0))$ is a modified pseudo-gradient of J with the constant θ at $p(v_0)$ as defined in Lemma II.1. This contradicts the assumption that v_0 is a local minimum point of $J(p(v))$.

The following Ekeland's variational principle will be used later.

Lemma II.4 (Ekeland's variational principle, *[27]) Let X be a complete metric space and $J : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous functional bounded from below. Then for any $\epsilon > 0$ and $x_0 \in X$ with $J(x_0) < +\infty$, there is $\bar{x} \in X$ such that*

$$J(\bar{x}) + \epsilon d(x_0, \bar{x}) \leq J(x_0) \ \ \textit{and} \ \ J(x) + \epsilon d(x, \bar{x}) > J(\bar{x}), \ \forall x \in X \textit{ and } x \neq \bar{x}.$$

By Ekeland's variational principle and the PS condition, we have the following existence theorem.

Theorem II.2 *Let $J \in C^1(B, \mathbb{R})$ satisfy the PS condition. If there is a peak selection p of J w.r.t. L satisfying (1) p is continuous, (2) $d(p(v), L) \geq \alpha, \forall v \in S_{L'}$ for some $\alpha > 0$ and (3) $\inf_{v \in S_{L'}} J(p(v)) > -\infty$, then there is $v_0 \in S_{L'}$ s.t. $p(v_0)$ is a critical point of J, and*

$$J(p(v_0)) = \min_{v \in S_{L'}} J(p(v)).$$

Proof. Since $S_{L'}$ is a closed subset and $J(p(\cdot))$ is a continuous function on $S_{L'}$, bounded from below, by Ekeland's variational principle, for any integer n, there is $v_n \in S_{L'}$ s.t.

$$J(p(v_n)) \leq \inf_{v \in S_{L'}} J(p(v)) + \frac{1}{n} \tag{2.7}$$

and

$$J(p(v)) - J(p(v_n)) \geq -\frac{1}{n}\|v - v_n\|, \ \forall v \in S_{L'}, v \neq v_n. \tag{2.8}$$

By Lemma II.3 and Lemma II.2, for some $v \in S_{L'}$ and close to v_n,

$$J(p(v)) - J(p(v_n)) < -\frac{\theta d(p(v_n), L)\|\nabla J(v_n)\|}{16M}\|v - v_n\|.$$

Thus

$$\|\nabla J(p(v_n))\| < \frac{16M}{n\theta d(p(v_n), L)} \leq \frac{16M}{n\theta\alpha}. \tag{2.9}$$

By the PS condition, $\{p(v_n)\}$ has a subsequence, denoted again by $\{p(v_n)\}$, converging to a point $u_0 \in B$. If denote $p(v_n) = t_n v_n + x_n$ where $t_n \in R$ and $x_n \in L$, then, $\{t_n v_n\}$ is convergent since $B = L \oplus L'$. Hence, $\{|t_n|\}$ is convergent. Assume $\{t_n\}$ is a convergent subsequence. Denote $t_0 = \lim_{n\to\infty} t_n$. Then, by our assumption (2), $|t_0| \geq \alpha > 0$. Thus, $v_n \to v_0 \in S_{L'}$. Since p is continuous, by (2.9), $p(v_0)$ is a critical point of J and by (2.7), $J(p(v_0)) = \min_{v\in S_{L'}} J(p(v))$.

B. A Local Minimax Algorithm for SSCPs in Banach Spaces

1. Flow Chart of the Algorithm

Let $u_1, u_2, ..., u_{n-1}$ be $n-1$ previously found critical points of J, $L = [u_1, u_2, ..., u_{n-1}]$, $B = L \oplus L'$. Given $\varepsilon, \lambda > 0$ and $\theta \in (0, 1)$. A flow chart of the algorithm reads:

Step 1: Let $v^1 \in S_{L'}$ be an increasing-decreasing direction at u_{n-1}.

Step 2: Set $k = 1$ and solve for

$$\begin{aligned} u^k &= p(v^k) = t_0^k v^k + t_1^k u_1 + \cdots + t_{n-1}^k u_{n-1} \\ &= \arg\max\{J(t_0 v^k + t_1 u_1 + \cdots + t_{n-1} u_{n-1}) | t_i \in R, i = 0, 1, ..., n-1\}. \end{aligned}$$

Step 3: Find a descent direction $w^k = -sign(t_0^k) G^k$ of J at u^k, where $G^k \in L'$ is a modified pseudo-gradient of J at $u^k = p(v^k)$ with the constant θ as defined in Lemma II.1.

Step 4: If $\|\nabla J(p(v^k))\| < \epsilon$, then output u^k, stop. Otherwise, do Step 5.

Step 5: For each $s > 0$, let $v^k(s) = \frac{v^k + s w^k}{\|v^k + s w^k\|}$ and use the initial point $(t_0^k, t_1^k, ..., t_{n-1}^k)$ to solve for

$$u^k(s) = p(v^k(s)) = \arg\max\Big\{J(t_0 v^k(s) + \sum_{i=1}^{n-1} t_i u_i) | t_i \in R, i = 0, 1, ..., n-1\Big\},$$

then set $u^{k+1} = p(v^{k+1}) = p(v^k(s^k))$ where

$$s^k = \max_{m \in N}\{s = \frac{\lambda}{2^m} | 2^m > \|w^k\|, J(u^k(s)) - J(u^k) \le -\frac{\theta}{4}|t_0^k| s \|\nabla J(u^k)\|\}.$$

Step 6: Update $k = k+1$ and go to Step 3.

Remark II.2 It is worthwhile making some remarks on the algorithm:

(a) If B is a Hilbert space, by taking $L' = L^\perp$ and $G^k = \nabla J(u^k)$, it becomes Li-Zhou's algorithm.

(b) Step 5 will not stop until $\|\nabla J(u^k)\| < \epsilon$ since by Lemma II.1, $\nabla J(u^k) \neq 0$ implies $G(u^k) \neq 0$.

(c) There are two key steps: (1) computation of a modified pseudo-gradient, (2) optimization. (2) can be done by some standard optimization method. The implementation of (1) will be addressed later.

(d) To implement Step 3, we can either follow a modified pseudo-gradient flow given by Lemma II.1, i.e., to keep the continuity of G^k in u^k or just find a modified pseudo-gradient.

(e) The following theorem indicates that the algorithm is stable.

Theorem II.3 *In the algorithm, if* $u^k = p(v^k) \notin L$, $\nabla J(u^k) \neq 0$ *and* p *is continuous at* $v^k \in S_{L'}$, *then* $s^k > 0$ *and* $u^{k+1} = p(v^k(s^k))$ *is well defined. Consequently* $J(u^{k+1}) < J(u^k)$.

Proof. By the setpsize rule and Lemma II.3.

2. Computation of Pseudo-Gradient

In this section, we present some formulas to compute a pseudo-gradient and a pseudo-gradient flow in $L^p(\Omega)$ $(p > 1)$. Their modified versions follow from a projection to a subspace. Assume that Ω is a measurable space with measure μ and $\|\cdot\|_p$ represents the norm in $L^p(\Omega)$. Let us recall some wellknown results.

Lemma II.5 *([10]) Let $f, \{f_n\}$ be in $L^p(\Omega), 1 \le p < \infty$,*

(a) if $f_n \to f$ in $L^p(\Omega)$, then $\{f_n\}$ has a subsequence that converges to f pointwise a.e.;

(b) if $f_n \overset{a.e.}{\to} f$ and $\|f_n\|_p \to \|f\|_p$, then $f_n \to f$ in $L^p(\Omega)$.

Lemma II.6 *Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $f, f_n \in L^q(\Omega)$ s.t. $f_n \to f$. Then $sign(f_n)|f_n|^{\frac{1}{p-1}} \to sign(f)|f|^{\frac{1}{p-1}}$ in $L^p(\Omega)$, where*

$$sign(g)(x) = \begin{cases} 1 & if \quad g(x) \ge 0, \\ -1 & if \quad g(x) < 0, \end{cases} \quad \forall g \in L^q(\Omega).$$

Proof. It suffices to show that any subsequence, denoted always by $\{sign(f_n)|f_n|^{\frac{1}{p-1}}\}$, has a subsequence that converges to $sign(f)|f|^{\frac{1}{p-1}}$ in $L^p(\Omega)$. Since $f_n \to f$ in $L^q(\Omega)$, by Lemma II.5, we have $|f_n|^{\frac{1}{p-1}} \overset{a.e.}{\to} |f|^{\frac{1}{p-1}}$. It follows,

$$sign(f_n)(x)|f_n(x)|^{\frac{1}{p-1}} \overset{a.e.}{\to} sign(f)(x)|f(x)|^{\frac{1}{p-1}}.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$ and $f_n \to f$ in $L^q(\Omega)$, it leads to

$$\|sign(f_n)|f_n|^{\frac{1}{p-1}}\|_p^p = \|f_n\|_q^q \to \|sign(f)|f|^{\frac{1}{p-1}}\|_p^p = \|f\|_q^q.$$

By Lemma II.5, the proof is complete.

Theorem II.4 *Let $p \geq 2$ and $\frac{1}{p}+\frac{1}{q}=1$. Assume that $J : L^p(\Omega) \to \mathbb{R}$ is Fréchet differentiable at $f \in L^p(\Omega)$ s.t. $\nabla J(f) \neq 0$. Let $G(f) = sign(\nabla J(f))|\nabla J(f)|^{\frac{1}{p-1}}$. Then*

$$\Psi(f) = \frac{G(f)}{\|\nabla J(f)\|_q^{q-1}}$$

is a pseudo-gradient of J at f with the constant 1. If in addition, J is C^1, then Ψ is a pseudo-gradient flow of J with constant 1.

Proof. $\|\Psi(f)\|_p = 1$ can be seen from

$$\|G(f)\|_p = (\int_\Omega |\nabla J(f)|^{\frac{p}{p-1}} d\mu)^{\frac{1}{p}} = \|\nabla J(f)\|_q^{\frac{q}{p}} = \|\nabla J(f)\|_q^{q-1}.$$

On the other hand,

$$\langle \nabla J(f), G(f)\rangle = \int_\Omega \nabla J(f)(x) G(f)(x) d\mu = \int_\Omega |\nabla J(f)(x)|^{\frac{p}{p-1}} d\mu = \|\nabla J(f)\|_q^q.$$

Hence $\langle \nabla J(f), \Psi(f)\rangle = \|\nabla J(f)\|_q$ and $\Psi(f)$ is a pseudo-gradient at f with the constant 1.

To show Ψ is continuous. Let $f_0 \in L^p(\Omega)$ with $\nabla J(f_0) \neq 0$ and $\{f_n\} \subseteq L^p(\Omega)$ s.t. $f_n \to f_0$. Since $J \in C^1(L^p(\Omega), R)$, we have $\nabla J(f_n) \to \nabla J(f_0)$ in $L^q(\Omega)$ and $\|\nabla J(f_n)\|_q \to \|\nabla J(f_0)\|_q$. Then Lemma II.6 leads to

$$G(f_n) \to G(f_0) \text{ in } L^p(\Omega), \text{ i.e. } \Psi(f_n) \to \Psi(f_0) \text{ in } L^p(\Omega).$$

Theorem II.5 *Let $1 < p < 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\mu(\Omega) < \infty$ and $\theta = \max(1, (\mu(\Omega))^{\frac{1}{p}-\frac{1}{2}})$. If $J : L^p(\Omega) \to \mathbb{R}$ is Fréchet differentiable at f with $\nabla J(f) \neq 0$. Then*

$$\Psi(f) = \frac{\nabla J(f)}{\theta \|\nabla J(f)\|_2}$$

is a pseudo-gradient of J *at* f *with the constant* θ^{-2}*. If in addition,* J *is* C^1*, then* Ψ *is a pseudo-gradient flow of* J *with the constant* θ^{-2}*.*

Proof. By the *Hölder* inequality, we have

$$\|\nabla J(f)\|_p \leq \|\nabla J(f)\|_2(\mu(\Omega))^{\frac{1}{p}-\frac{1}{2}} \quad \text{or} \quad \|\Psi(f)\|_p \leq 1. \tag{2.10}$$

It follows

$$\langle \nabla J(f), \Psi(f)\rangle = \int_\Omega \nabla J(f)(x)\frac{\nabla J(f)(x)}{\theta\|\nabla J(f)\|_2}d\mu = \frac{\|\nabla J(f)\|_2}{\theta} \geq \frac{\|\nabla J(f)\|_p}{\theta^2}.$$

Hence $\Psi(f) = \frac{\nabla J(f)}{\theta\|\nabla J(f)\|_2}$ is a pseudo-gradient of J at f with the constant θ^{-2}.

To show Ψ is continuous, let $\{f_n\} \subset L^p(\Omega)$ s.t. $f_n \to f$ in $L^p(\Omega)$. Since $J \in C^1(L^p(\Omega), R)$, $\nabla J(f_n) \to \nabla J(f)$ in $L^q(\Omega)$. It follows $\nabla J(f_n) \to \nabla J(f)$ in $L^p(\Omega)$ and $\|\nabla J(f_n)\|_2 \to \|\nabla J(f)\|_2$, since $\frac{1}{p}+\frac{1}{q} = 1$, $1 < p < 2 < q$ and $\mu(\Omega) < \infty$. Hence $\Psi(f_n) \to \Psi(f)$ in $L^p(\Omega)$, i.e., Ψ is a pseudo-gradient flow of J with the constant θ^{-2}.

In a general Banach space B, when $\nabla J(f) \neq 0$ is computed in B^* at some $f \in B$, a pseudo-gradient of J at f corresponding to a constant $0 < \theta < 1$ can be computed through

$$\sup_{\psi\in B, \|\psi\|_B=1} \langle \frac{\nabla J(f)}{\theta\|\nabla J(f)\|_{B^*}}, \psi\rangle,$$

which has an upper bound $\frac{1}{\theta}$. It seems to us that it is extremely difficult, in this case, to derive an explicit formula for computing a pseudo-gradient for a functional $J : W^{1,p}(\Omega) \to R$. Instead we develop some numerical techniques to do the job in the next section.

C. Numerical Experiment to Quasilinear Elliptic PDEs

Consider solving the following quasilinear elliptic BVP for multiple solutions:

$$\Delta_p u(x) + f(x,u(x)) = 0, x \in \Omega, \quad u \in B \equiv W_0^{1,p}(\Omega),\ p > 1, \tag{2.11}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ and $\Delta_p u(x) = \mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x))$ is the nonlinear p-Laplacian differential operator, which has a variety of applications in physical fields, such as in fluid dynamics when the shear stress $\vec{\tau}$ and the velocity gradient ∇u of the fluid are related in the manner $\vec{\tau}(x) = r(x)|\nabla u|^{p-2}\nabla u$, where $p = 2, p < 2, p > 2$ if the fluid is Newtonian, pseudoplastic, dilatant, respectively. The p-laplacian operator also appears in the study of flow in a porous media ($p = \frac{3}{2}$), nonlinear elasticity ($p > 2$) and glaciology ($p \in (1, \frac{4}{3})$) [9]. So far people's knowledge about solutions to (2.11) is still very limited. We hope to examine the qualitative behavior of solutions and find new phenomena through numerical investigation. We have $B^* \equiv W_0^{-1,q}(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Under certain standard conditions on f, weak solutions of (2.11) coincide with critical points of the functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega F(x,u(x))dx \quad \text{where} \quad F(x,t) = \int_0^t f(x,s)ds. \tag{2.12}$$

For $u \in B$, to find the gradient $d = \nabla J(u) \in B^*$, for each $v \in B$, we have

$$\begin{aligned}
\langle d, v\rangle &= \int_\Omega \nabla d(x)\nabla v(x)\,dx = \int_\Omega -\Delta d(x) v(x)\,dx = \frac{d}{dt}|_{t=0} J(u+tv) \\
&= \int_\Omega \Big(|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) - f(x,u(x))v(x)\Big)dx \\
&= \int_\Omega (-\Delta_p u(x) - f(x,u(x)))v(x)dx.
\end{aligned}$$

Thus $d = \nabla J(u)$ can be computed through solving the linear elliptic equation

$$\begin{cases} \Delta d(x) &= \Delta_p u(x) + f(x, u(x)), \quad x \in \Omega, \\ d(x) &= 0, \qquad\qquad\qquad\qquad\quad x \in \partial\Omega. \end{cases} \tag{2.13}$$

Where since $\Delta_p u(x) + f(x, u(x)) \in W_0^{-1,q}(\Omega)$, we have $d \in W_0^{1,q}(\Omega)$. When $u = p(v)$ for some $v \in S_{L'}$, by the definition of a peak selection, $d = \nabla J(u)$ satisfies

$$\langle d, w \rangle = \int_\Omega \nabla d(x) \nabla w(x)\, dx = 0, \quad \forall w \in L,$$

i.e., $d = \nabla J(u) \perp L$. In our numerical examples, we check the ratio

$$\gamma = \frac{\|d\|_2^2}{\|d\|_p \cdot \|d\|_q}, \tag{2.14}$$

where $\|\cdot\|_r$ is the norm in $W^{1,r}(\Omega)$. $\gamma \leq 1$ by the Hölder inequality. If $\gamma > \alpha > 0$, then $G(u) = \frac{d}{\|d\|_p} \in L'$ is a modified pseudo-gradient of J at u as in Lemma II.1. It is interesting to point out that although we have not been able to analytically prove $\gamma > \alpha > 0$, we can numerically check this ratio in each computation. All our numerical examples show that the ratio γ is a way above 0. For $p > 2$, since $B \subset B^*$, we define $L' = L^\perp = \{v \in B : \langle u, v \rangle = 0, \forall u \in L\}$. For $p < 2$, $\nabla J(u) \in B^* \subset B$, it can be used directly in the algorithm.

Next, we apply our numerical minimax algorithm to find multiple solutions for the p-Emden-Fowler Equation:

$$\Delta_p u(x) + |u(x)|^{q-1} u(x) = 0, \; x \in \Omega, \quad u \in W_0^{1,p}(\Omega) \tag{2.15}$$

and the p-Henon Equation:

$$\Delta_p u(x) + |x - \vec{1}|^r |u(x)|^{q-1} u(x) = 0, \; x \in \Omega, \quad u \in W_0^{1,p}(\Omega) \tag{2.16}$$

where $|\cdot|$ is the Euclidean norm, $\vec{1} = (1, ..., 1)$, $1 < p < q + 1 < p^*$ with $p^* = \frac{np}{n-p}$ for

$p < n$ and $p^* = \infty$ for $n \leq p$, and n is the dimension of the domain space. In our computation, $\Omega = [0, 2] \times [0, 2] \subset \mathbb{R}^2$.

Note that the right-hand-side of (2.13) involves an evaluation of a higher-order derivative of a numerical solution u, i.e., $\Delta_p u(x)$, which causes difficulty for using linear finite elements. To solve the problem, we utilize a weak form of (2.13)

$$\int_\Omega \Delta d(x) v(x)\, dx = \int_\Omega \left(\Delta_p u(x) + f(x, u(x))\right) v(x)\, dx \quad \forall v \in W_0^{1,p} \tag{2.17}$$

and the identity

$$\int_\Omega \Delta_p u(x) v(x)\, dx = -\int_\Omega |\nabla u(x)|^p \nabla v(x)\, dx \quad \forall v \in W_0^{1,p} \tag{2.18}$$

to replace the higher-order derivative term by a first-order derivative term. Thus linear finite elements can be applied. Here either 400×400 or 800×800 linear square elements are used. Since different values of p have different physical applications, we will use different values for p also for the parameter r to examine their solution profiles. We use $\varepsilon = \|\nabla J(u_k)\| < 10^{-3}$ to stop the iterations. The profiles of solutions are presented as follows, Fig.1-Fig.18.

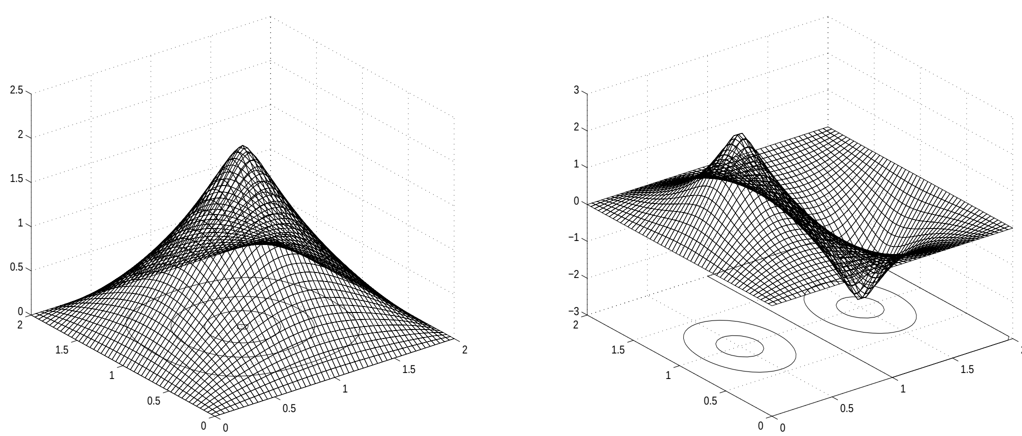

Fig. 1. Equation (2.15) with $p = 3.0$, $q = 7.0$. The ground state with $J = 4.4829$ (left) and a solution with $J = 40.9568$ (right).

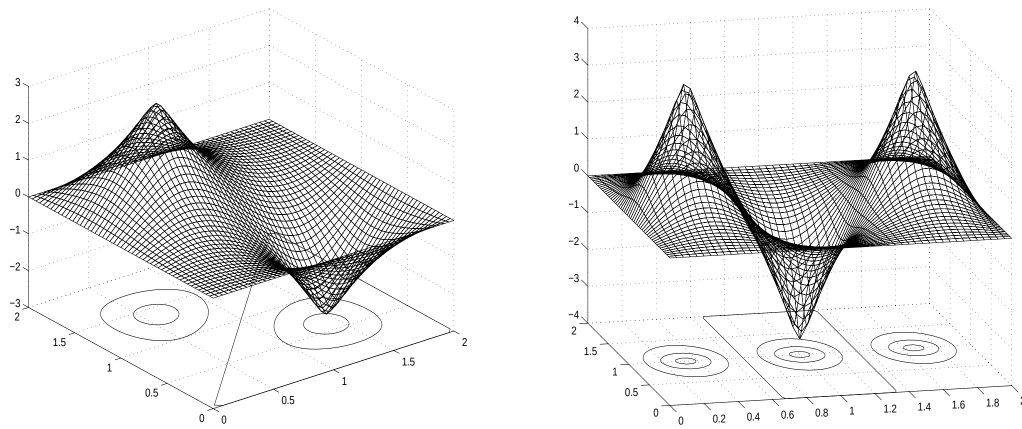

Fig. 2. Same equation as Fig. 1. Two solutions with $J = 34.4457$ (left) and $J = 181.7966$ (right).

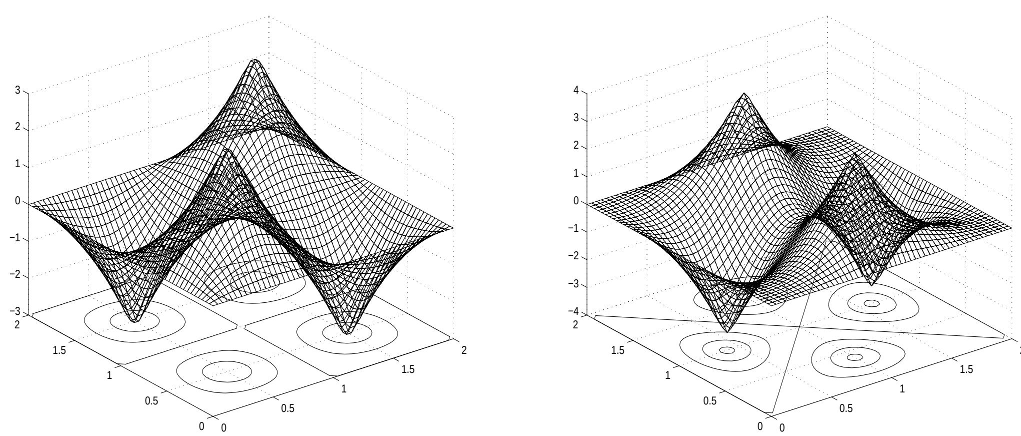

Fig. 3. Same equation as Fig. 1. Two solutions with $J = 1124.8750$ (left) and $J = 124.8750$ (right).

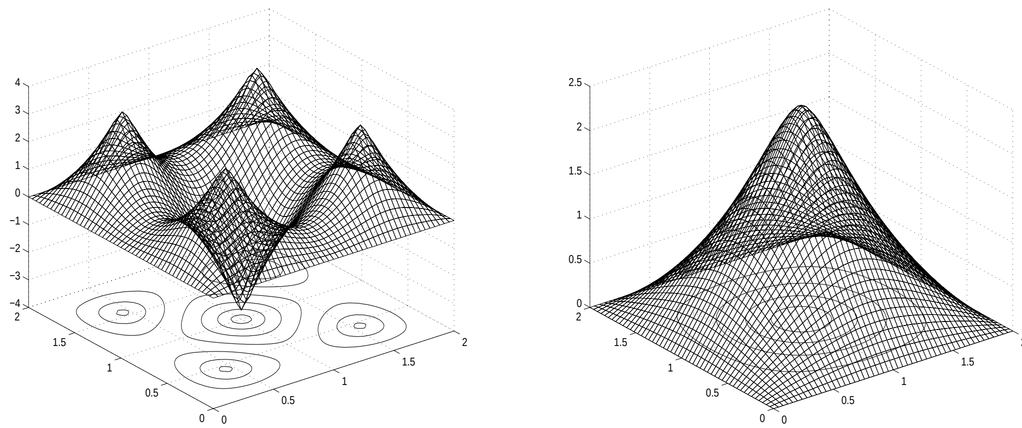

Fig. 4. Same equation as Fig. 1, a solution with $J = 228.2925$ (left). Equation (2.15) with $p = 2.5$, $q = 5.0$, the ground state with $J = 5.9398$ (right).

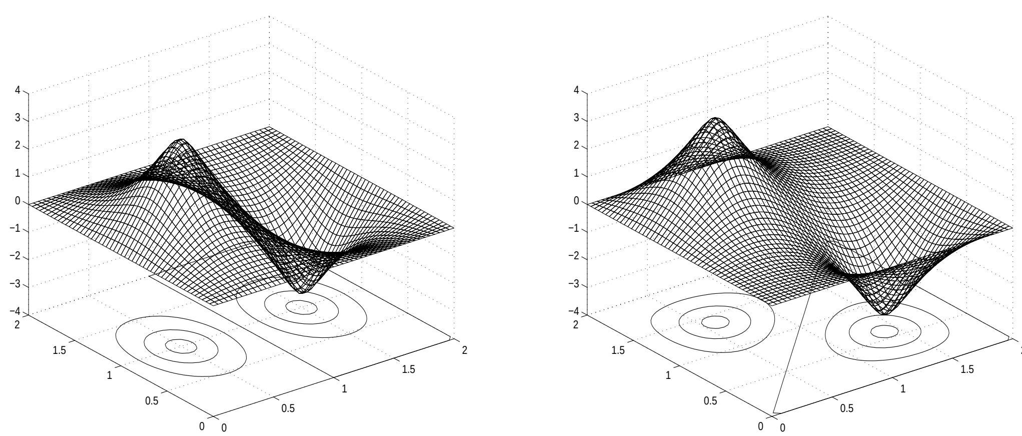

Fig. 5. Equation (2.15) with $p = 2.5$, $q = 5.0$. Two solutions with $J = 40.1451$ (left) and $J = 35.4001$ (right).

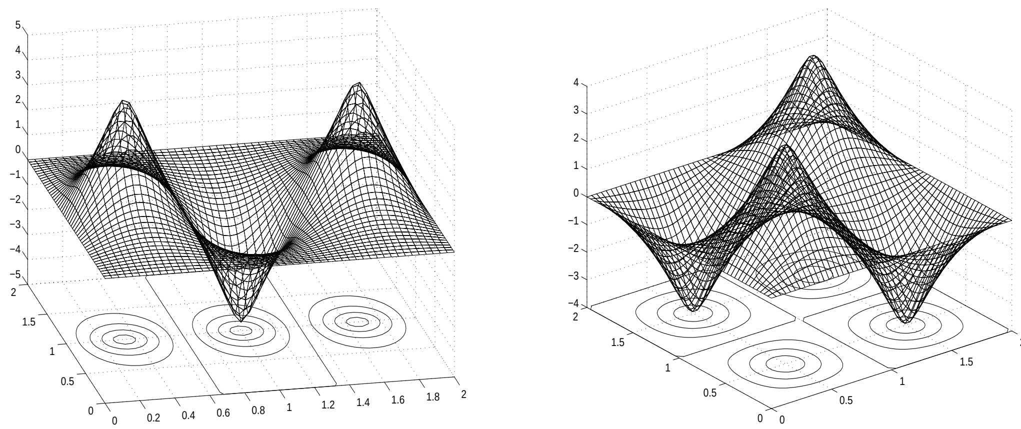

Fig. 6. Same equation as Fig. 5. Two solutions with $J = 149.7131$ (left) and $J = 115.84532$ (right).

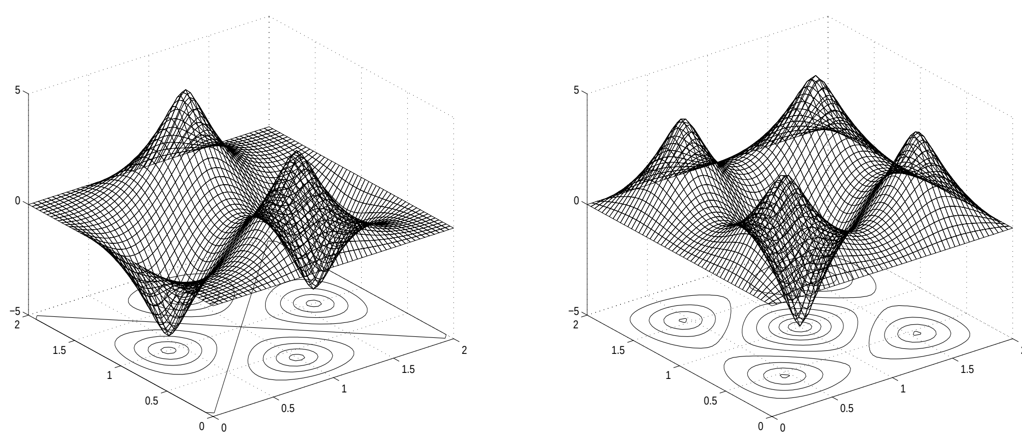

Fig. 7. Same equation as Fig. 5. Two solutions with $J = 156.3368$ (left) and $J = 193.5180$ (right).

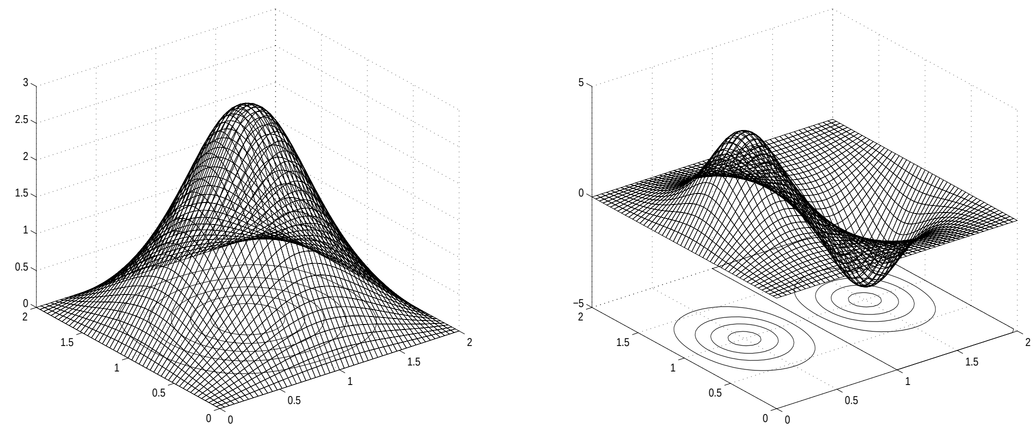

Fig. 8. Equation (2.15) with $p = 1.75$, $q = 3.0$. The ground state with $J = 7.0745$ (left) and a solution with $J = 25.4653$ (right).

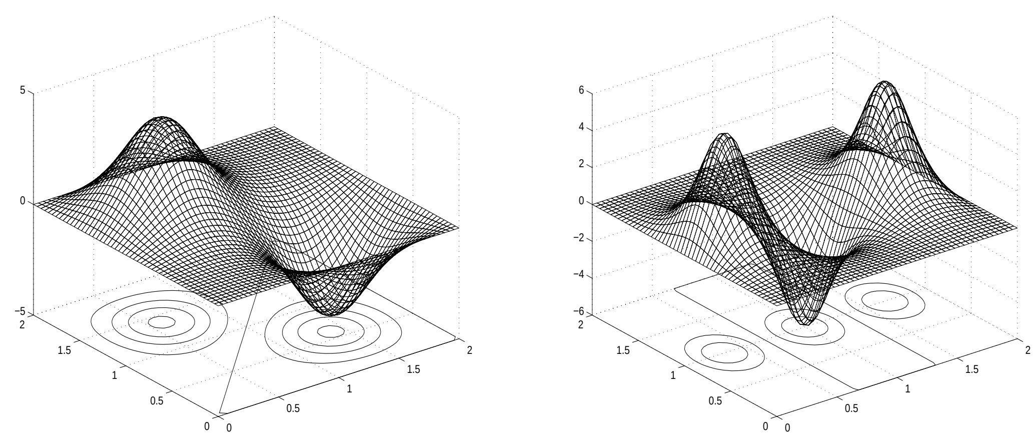

Fig. 9. Same equation as Fig. 8. Two solutions with $J = 24.0274$ (left) and $J = 59.4209$ (right).

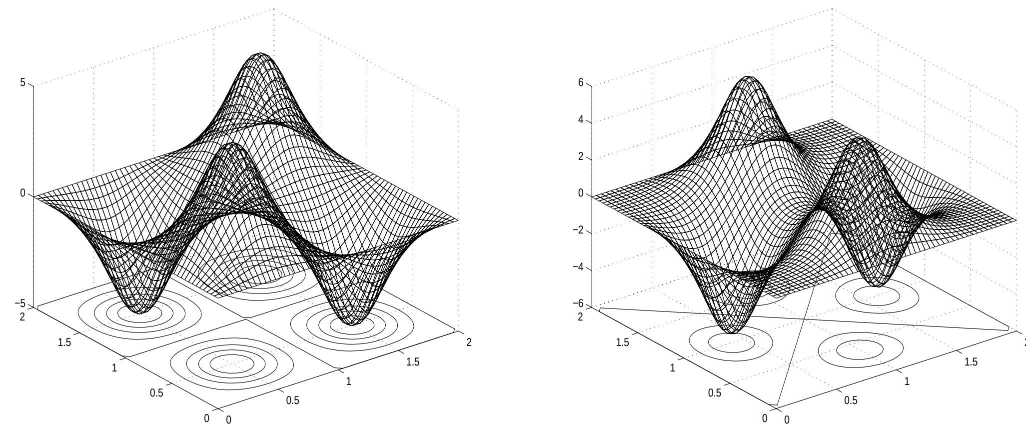

Fig. 10. Same equation as Fig. 8. Two solutions with $J = 61.1246$ (left) and $J = 70.6261$ (right).

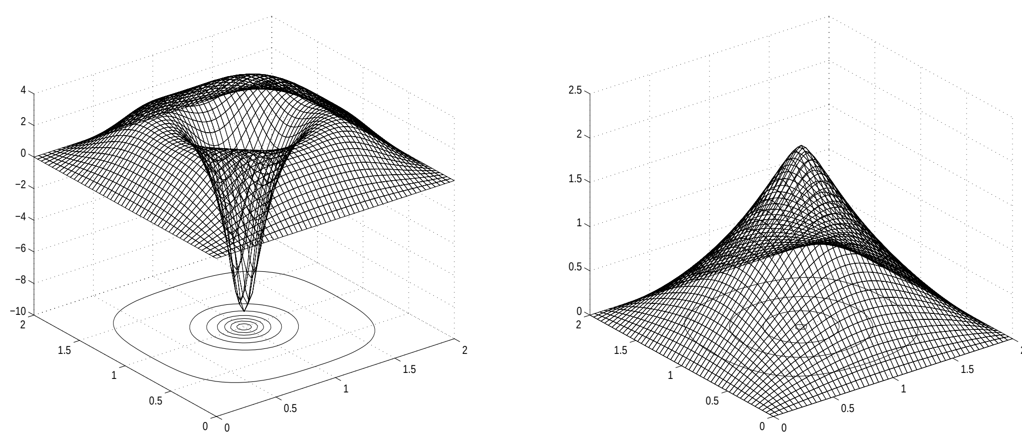

Fig. 11. Same equation as Fig. 8, a solutions with $J = 77.2337$ (left). The ground state of (2.16) with $J = 4.48854$, $p = 3.0$, $q = 7.0$, $r = 0.001$ (right).

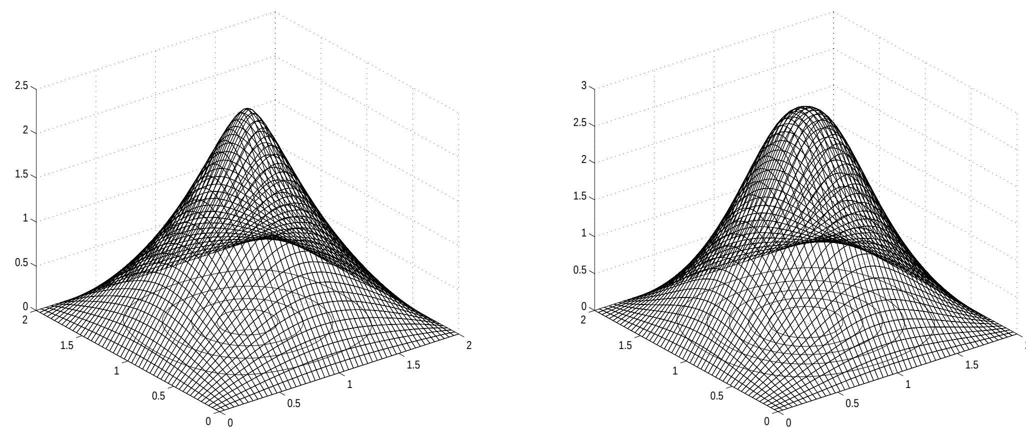

Fig. 12. Two ground states of (2.16) with $J = 5.947472$, $p = 2.5$, $q = 5.0$, $r = 0.001$ (left) and with $J = 7.082540$, $p = 1.75$, $q = 3.0$, $r = 0.001$ (right).

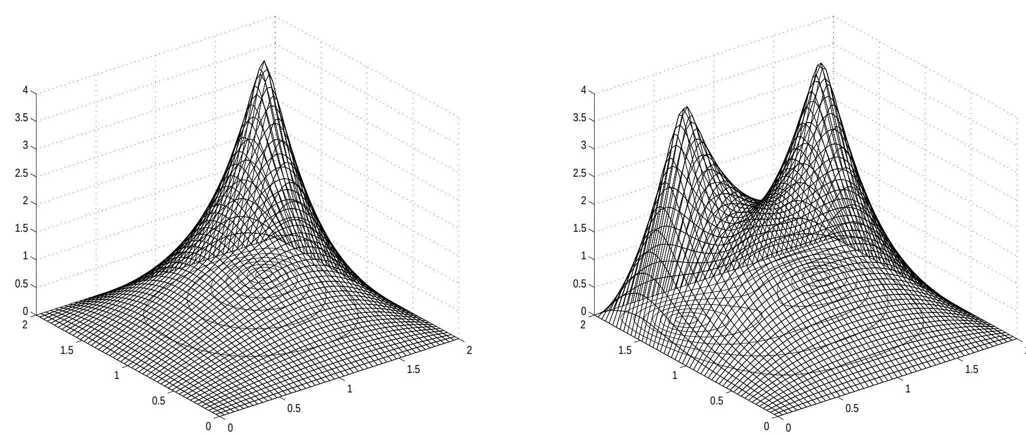

Fig. 13. Equation (2.16) with $p = 3.0$, $q = 7.0$, $r = 7.0$. A ground state with $J = 60.4600$ (left) and a solution with $J = 116.2310$ (right).

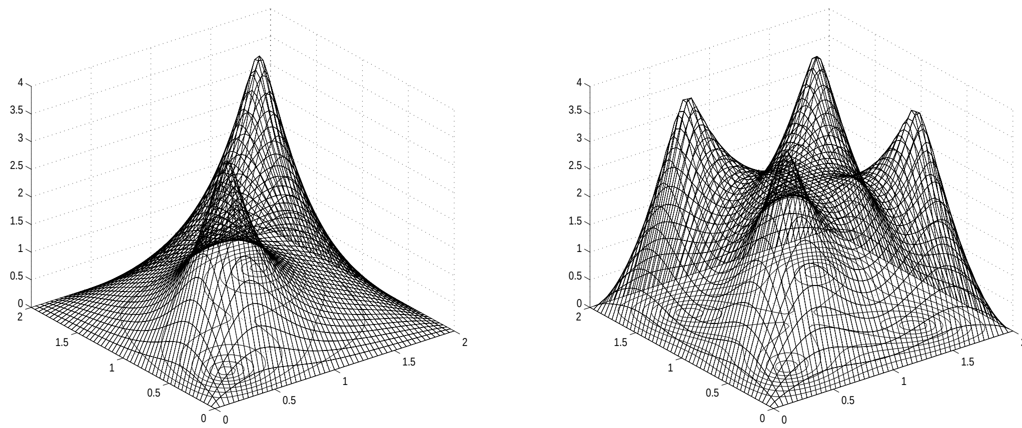

Fig. 14. Same equation as Fig. 13. Two solutions with $J = 118.9060$ (left) and $J = 219.8671$ (right).

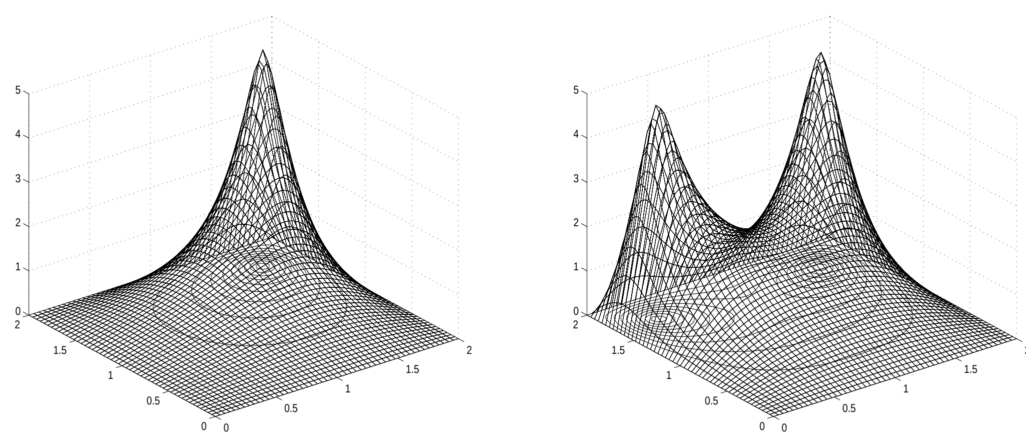

Fig. 15. Equation (2.16) with $p = 2.5$, $q = 5.0$, $r = 7.0$. A ground state with $J = 54.2139$ (left) and a solution with $J = 105.6687$ (right).

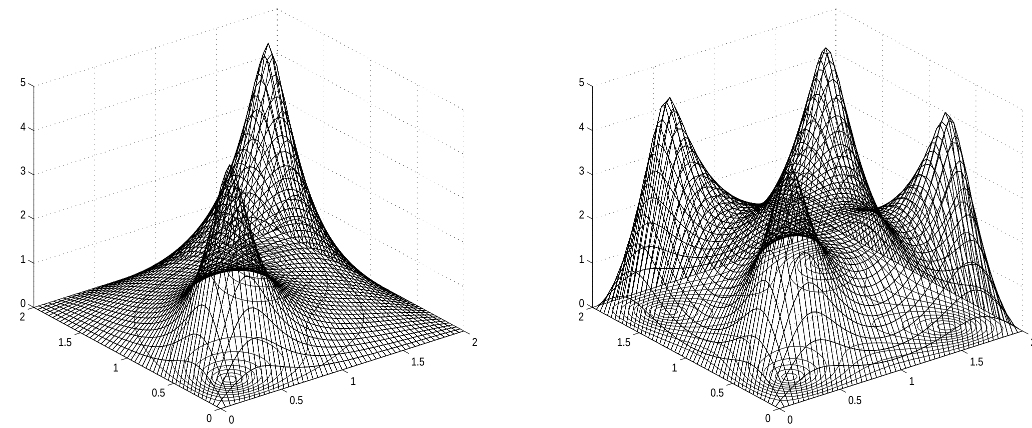

Fig. 16. Same equation as Fig. 15. Two solutions with $J = 107.1374$ (left) and $J = 203.5262$ (right).

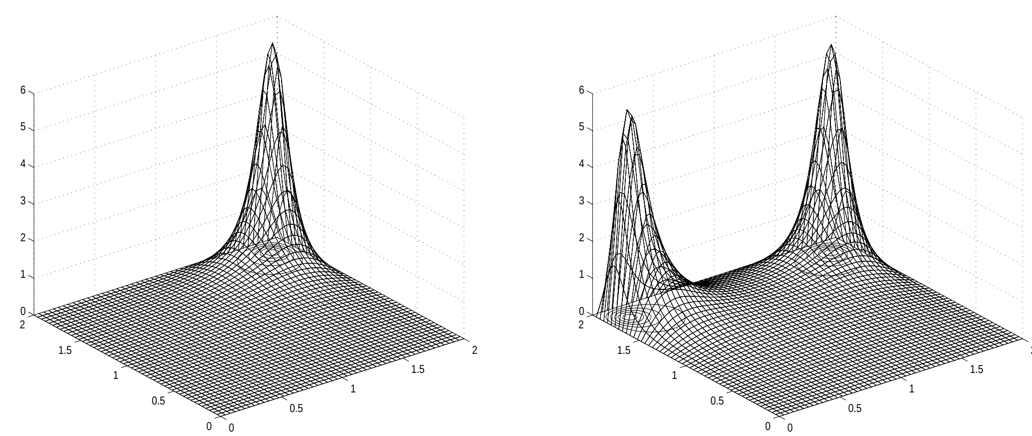

Fig. 17. Equation (2.16) with $p = 1.75$, $q = 3.0$, $r = 7.0$. A ground state with $J = 15.7588$ (left) and a solution with $J = 31.3832$ (right).

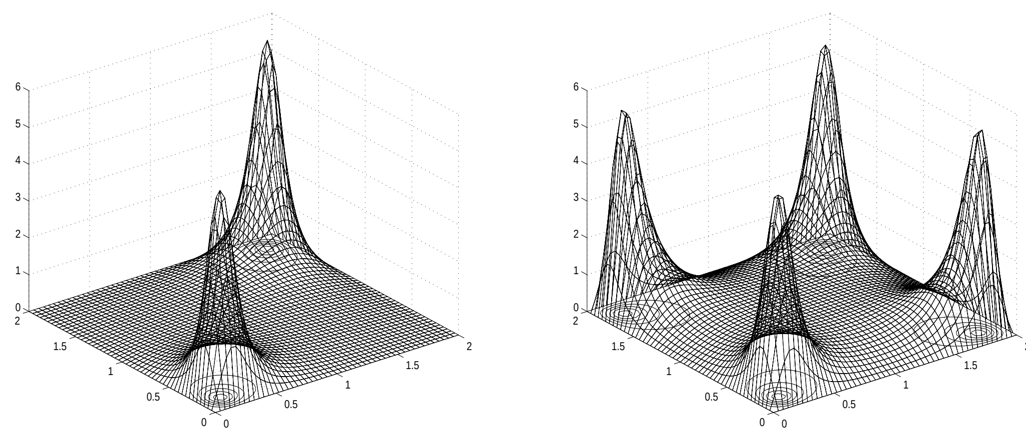

Fig. 18. Same as Fig. 17. Two solutions with $J = 31.4178$ (left) and $J = 62.2163$ (right).

Remark II.3 It is to the best of our knowledge that the above solutions are the first time to be computed and visualized. Several interesting phenomena have been observed, e.g., for fixed p and q and let r increase, the ground state breaks its symmetry, compare Fig. 11 with Fig 13, Fig. 12 with Figs 15 and 17. Once the symmetry is broken, it leads to four asymmetric ground states. Those phenomena are still open to be analytically verified.

Table.I and Table.II show some argument and symmetry used in the algorithm.

Table I. $\gamma_{\min}$ is the minimum among ratios defined by (2.14) in the last 10 iterations of the computation for each solution and L is the support in the computation.

Solution	γ_{min}	L	Solution	γ_{min}	L
left, Fig.1 (u_1)	0.88	$\{0\}$	right, Fig.6	0.93	$\{0\}$
right, Fig.1	0.93	$\{u_1\}$	left, Fig.7	0.92	$\{0\}$
left, Fig.2	0.65	$\{u_1\}$	right, Fig.7	0.91	$\{u_2\}$
right, Fig.2	0.73	$\{0\}$	left, Fig.8 (u_3)	0.90	$\{0\}$
left, Fig.3	0.83	$\{0\}$	right, Fig.8 (u_4)	0.96	$\{u_3\}$
right, Fig.3	0.88	$\{0\}$	left, Fig.9	0.91	$\{u_3\}$
left, Fig.4	0.74	$\{u_1\}$	right, Fig.9	0.93	$\{0\}$
right, Fig.4 (u_2)	0.95	$\{0\}$	left, Fig.10	0.98	$\{u_3, u_4(x,y), u_4(y,x)\}$
left, Fig.5	0.97	$\{u_2\}$	right, Fig.10	0.94	$\{0\}$
right, Fig.5	0.98	$\{u_2\}$	left, Fig.11	0.91	$\{u_3\}$
left, Fig.6	0.91	$\{0\}$	right, Fig.11		$\{0\}$
Fig.12$\sim$Fig.18		$\{0\}$			

Table II. Symmetry listed is used in the computation for each solution.

Solutions	Symmetry
right, Fig.2; left, Fig.6; right, Fig.9	locally odd about $x = \frac{2}{3}, \frac{4}{3}$
left, Fig.3; right, Fig.6	odd about $x = 1, y = 1$
right, Fig.3; left, Fig.7; right, Fig.10	odd about $y = x, x + y = 2$
left, Fig.4; right, Fig.7; left, Fig.11	4-rotation
right, Fig.13; right, Fig.15; right, Fig.17	even about $x = 1$
left, Fig.14; left, Fig.16; right, Fig.18	even about $x + y = 2$
right, Fig.14; right, Fig.16; right, Fig.18	4-rotation

CHAPTER III

CONVERGENCE OF THE 1ST MINIMAX ALGORITHM

For a subspace $B' \subseteq B$, denote $S_{B'} = \{v|v \in B', \|v\| = 1\}$. Assume that $B = L \oplus L'$ for some closed subspaces L and L' and $\mathbb{P}$ is the corresponding linear projection operator $B \to L'$ with bound $M \geq 1$. For each $v \in S_{L'}$, let $[L, v] = \{tv + w|w \in L, t \in R\}$.

Definition III.1 A set-valued mapping $P : S_{L'} \to 2^B$ is the L-$\perp$ *mapping* of J if $\forall v \in B_{L'}, P(v) = \{u \in [L, v] : \langle \nabla J(u), w\rangle = 0, \forall w \in [L, v]\}$. A single-valued mapping $p : S_{L'} \to B$ is an L-$\perp$ *selection* of J if $p(v) \in P(v), \forall v \in S_{L'}$. For a given $v \in S_{L'}$, we say that J has a *local* L-$\perp$ *selection* at v if an L-$\perp$ selection p is locally defined near v.

Lemma III.1 *If J is C^1, then the graph $G = \{(u, v) : v \in S_{L^\perp}, u \in P(v) \neq \emptyset\}$ is closed.*

Proof. Let $(u_n, v_n) \in G$ and $(u_n, v_n) \to (u_0, v_0)$. We have $u_n \in [L, v_n]$, $\nabla J(u_n) \perp [L, v_n]$ and $u_n = t_n v_n + v_n^L \to u_0$ for some scalar t_n and point $v_n^L \in L$. Denote $u_0 = u_0' + u_0^L$ for some $u_0' \in L'$ and $u_0^L \in L$. It follows $v_n^L - u_0^L = u_n - u_0 - \mathbb{P}(u_n - u_0) \to 0$ and $t_n v_n - u_0' = \mathbb{P}(u_n - u_0) \to 0$, i.e., $t_n v_n \to u_0' = t_0 v_0$ for some scalar t_0, because $v_n \to v_0$. Thus $u_n \to u_0 = t_0 v_0 + u_0^L \in [L, v_0]$ and $\nabla J(u_0) \perp [L, v_0]$ because J is C^1. Therefore $v_0 \in S_{L^\perp}$ and $u_0 \in P(v_0)$, i.e., $(u_0, v_0) \in G$.

It is clear that if P is the peak mapping of J w.r.t. L, then P is the L-$\perp$ mapping of J. This generalization exceeds the scope of a minimax principle, the most popular approach in critical point theory. It enables us to treat non-minimax type saddle points, such as the wellknown monkey saddle, or a problem without a mountain pass structure at all. See Example 2.1 in [29].

Lemma III.2 *Let $0 < \theta < 1$ be given. For $v_0 \in S_{L'}$, if p is a local L-$\perp$ selection of J at v_0 s.t. $\nabla J(p(v_0)) \neq 0$ and $\Psi(p(v_0)) \in B$ is a pseudo-gradient of J at $p(v_0)$ w.r.t. θ, then there exists a (modified) pseudo-gradient $G(p(v_0))$ of J at $p(v_0)$ w.r.t. θ s.t.*

(a) $G(p(v_0)) \in L'$, $0 < \|G(p(v_0))\| \leq M$ *where $M \geq 1$ is the bound of the linear projection $\mathbb{P}$ from B to L';*

(b) $\langle \nabla J(p(v_0)), G(p(v_0)) \rangle = \langle \nabla J(p(v_0)), \Psi(p(v_0)) \rangle$*;*

(c) If $\Psi(p(v_0))$ is the value of a pseudo-gradient flow $\Psi(\cdot)$ of J at $p(v_0)$, then $G(\cdot)$ is continuous and $G(p(v_0))$ is called the value of a modified pseudo-gradient flow of J at $p(v_0)$.

Lemma III.3 *For $v_0 \in S_{L'}$, if J has a local L-$\perp$ selection p at v_0 satisfying (1) p is continuous at v_0, (2) $d(p(v_0), L) > \alpha > 0$ and (3) $\nabla J(p(v_0)) \neq 0$. Then, there exists $s_0 > 0$ such that for $0 < s < s_0$*

$$J(p(v(s))) - J(p(v_0)) < -\frac{\theta s}{4}|t_0|\|\nabla J(p(v_0))\| \tag{3.1}$$

where $p(v_0) = t_0 v_0 + w_0$ for some $t_0 \in R, w_0 \in L$, $v(s) = \dfrac{v_0 - sign(t_0)sG(p(v_0))}{\|v_0 - sign(t_0)sG(p(v_0))\|}$ and $G(p(v_0))$ is a modified pseudo-gradient of J with θ at $p(v_0)$ as defined in Lemma III.2.

The proof of the above two lemmas can follow a similar argument of Lemma II.1 and II.3. The inequality in (3.1) will be used to define a stepsize rule for the algorithm. We have

Theorem III.1 *Let $v_0 \in S_{L'}$. Assume that J has a local L-$\perp$ selection p at v_0 such that (1) p is continuous at v_0, (2) $d(p(v_0), L) > 0$ and (3) v_0 is a local minimum point of $J(p(v))$. Then, $p(v_0)$ is a critical point of J.*

A. A Unified Convergence Result

In this section, we prove a unified and abstract convergence result which is independent of the algorithm. This result is designed to cover several different cases for the algorithm.

Denote

$$K = \{u \in B | \nabla J(u) = 0\} \text{ and } K_c = \{u \in B | \nabla J(u) = 0, J(u) = c\}.$$

If J satisfies the PS condition, K_c is a compact set. Now we are ready to prove an abstract convergence result.

Theorem III.2 *Let $V \subset B$ be open and $U = V \cap S_{L'} \neq \emptyset$. Assume that $J \in C^1(B, \mathbb{R})$ satisfies the PS condition,*

(1) *p is a continuous L-$\perp$ selection of J in $\bar{U}$, where $\bar{U}$ is the closure of U on $S_{L'}$,*

(2) $\inf_{v \in U} d(p(v), L) > \alpha > 0$,

(3) $\inf_{v \in \partial \bar{U}} J(p(v)) > c = \inf_{v \in U} J(p(v)) > -\infty$, *where $\partial \bar{U}$ is the boundary of $\bar{U}$ on $S_{L'}$.*

Then, $K_c^p = p(U) \cap K_c \neq \emptyset$ and for any $\{v_k\} \subset U$ with $J(u_k) \to c$ where $u_k = p(v_k)$,

(a) $\forall \varepsilon > 0$, *there is $\bar{k} > 0$ such that $d(K_c^p, u_k) < \varepsilon$, $\forall k > \bar{k}$;*

(b) *If in addition, $\nabla J(p(\cdot))$ is Lipschitz continuous in U, then there is a constant C such that $\|\nabla J(u_k)\| \leq C(J(u_k) - c)^{\frac{1}{2}}$.*

Proof. Define

$$\hat{J}(p(v)) = \begin{cases} J(p(v)) & v \in \bar{U}, \\ +\infty & v \notin \bar{U}. \end{cases}$$

Then, $\hat{J}(p(\cdot))$ is lower semicontinuous and bounded from below on the complete metric space $S_{L'}$. Let $\{v_k\} \subset U$ be any sequence such that $J(p(v_k)) \to c$. By our assumption (c), such sequence always exists. Denote $u_k = p(v_k)$. Applying Ekeland's variational principle to $\hat{J}(p(\cdot))$, for every $v_k \in U$ and $\delta_k = (J(u_k) - c)^{\frac{1}{2}}$, there is $\bar{v}_k \in S_{L'}$ such that

$$\hat{J}(p(\bar{v}_k)) - \hat{J}(p(v)) \leq \delta_k \|\bar{v}_k - v\|, \quad \forall v \in S_{L'} \tag{3.2}$$

$$\hat{J}(p(\bar{v}_k)) - \hat{J}(p(v_k)) \leq -\delta_k \|\bar{v}_k - v_k\|. \tag{3.3}$$

By the definition of $\hat{J}(p(\cdot))$ and assumptions on p, we have $\bar{v}_k \in \bar{U}$,

$$J(p(\bar{v}_k)) - J(p(v)) \leq \delta_k \|\bar{v}_k - v\|, \quad \forall v \in S_{L'} \tag{3.4}$$

$$J(p(\bar{v}_k)) - J(p(v_k)) \leq -\delta_k \|\bar{v}_k - v_k\|. \tag{3.5}$$

It follows $c \leq J(p(\bar{v}_k)) \leq J(u_k) - \delta_k \|\bar{v}_k - v_k\|$, or

$$\|\bar{v}_k - v_k\| \leq \delta_k^{\frac{1}{2}}. \tag{3.6}$$

and $d(L, p(\bar{v}_k)) > \alpha$ when k is large. Then $J(p(v_k)) \to c$ implies $J(p(\bar{v}_k)) \to c$. By condition (3), we have $\bar{v}_k \in U$ for large k. For those large k, if $\nabla J(p(\bar{v}_k)) \neq 0$, by Lemma II.3, when s is small,

$$J(p(\bar{v}_k(s))) - J(p(\bar{v}_k)) \leq -\frac{\alpha\theta}{8M} \|\nabla J(p(\bar{v}_k))\| \|\bar{v}_k(s) - \bar{v}_k\|$$

where $\bar{v}_k(s) = \frac{\bar{v}_k + s\bar{w}_k}{\|\bar{v}_k + s\bar{w}_k\|} \in U$, $\bar{w}_k = -sign(t_0^k) G(p(\bar{v}_k)))$, $p(\bar{v}_k) = t_0^k \bar{v}_k + u_L^k$ for some $u_L^k \in L$ and $G(p(\bar{v}_k))$ is a modified pseudo-gradient of J at $p(\bar{v}_k)$. Hence

$$\|\nabla J(p(\bar{v}_k))\| \leq \frac{16M}{\alpha\theta} \delta_k^{\frac{1}{2}} \tag{3.7}$$

which implies $\nabla J(p(\bar{v}_k)) \to 0$ and then $\nabla J(p(v_k)) \to 0$ by (3.6). $\{J(p(v_k))\}$ is already bounded. By the PS condition, $\{u_k\}$ has a subsequence that converges to a critical point u^*. It is clear that $J(u^*) = c$ and $u^* \in K_c^p \neq \emptyset$. Let β be any limit point of $\{d(K_c^p, u_k)\}$ and $u_{k_i} = p(v_{k_i}) \in \{u_k\}$ such that $\lim_{i\to\infty} d(K_c^p, u_{k_i}) = \beta$. By the PS condition, $\{p(\bar{v}_{k_i})\}$ has a subsequence that converges to a critical point $\bar{u}$. Again $J(\bar{u}) = c$ and $\bar{u} \in K_c^p$, i.e., $\beta = 0$. Thus conclusion (a) holds.

If in addition, $\nabla J(p(\cdot))$ is Lipschitz continuous in U with a Lipschitz constant ℓ_1, from (3.6) and (3.7), we have

$$\begin{aligned}\|\nabla J(p(v_k))\| &\leq \|\nabla J(p(\bar{v}_k))\| + \|\nabla J(p(v_k)) - \nabla J(p(\bar{v}_k))\| \\ &\leq \frac{16M}{\alpha\theta}\delta_k^{\frac{1}{2}} + \ell_1\|\bar{v}_k - v_k\| \;\leq\; (\frac{16M}{\alpha\theta} + \ell_1)(J(u_k) - c)^{\frac{1}{2}}.\end{aligned}$$

Corollary III.1 *Let $J \in C^1(B,\mathbb{R})$ satisfy the PS condition, V_1 and V_2 be open in L' with $\emptyset \neq U_2 \equiv V_2 \cap S_{L'} \subset V_1 \cap S_{L'} \equiv U_1$. If p is a continuous L-$\perp$ selection of J in U_1 with*

(1) $\inf_{v\in U_1} d(p(v), L) \geq \alpha > 0, c = \inf_{v\in U_1} J(p(v)) > -\infty$ *and* $K_c^p = p(U_1)\cap K \subset K_c$,

(2) *there is $d > 0$ with*

$$\inf\{J(p(v))|v \in U_1, d(v, \partial U_1) \leq d\} = a > b = sup\{J(p(v))|v \in U_2\},$$

(3) *given $\{v_k\}$ such that $v_1 \in U_2$, $\|v_{k+1} - v_k\| < d$, $J(u_{k+1}) < J(u_k)$ and $\{u_k\}$ has a subsequence that converges to a critical point u_0, where $u_k = p(v_k)$. Then*

(a) $\forall \varepsilon > 0$, *there is $\bar{k} > 0$ such that* $\quad d(K_c^p, u_k) < \varepsilon, \quad \forall k > \bar{k}$;

(b) *If in addition, $\nabla J(p(\cdot))$ is Lipschitz continuous in U_1, then there is a constant C such that $\|\nabla J(u_k)\| \leq C(J(u_k) - c)^{\frac{1}{2}}$.*

Proof. First, we prove that $v_k \in U_1$ and $d(v_k, \partial U_1) > d$, $k = 1, 2, ...$ In fact, if

$v_k \in U_1$, $d(v_k, \partial U_1) > d$ and $J(u_k) \le b$, then $v_{k+1} \in U_1$ and $J(u_{k+1}) < b$, i.e., $v_{k+1} \in U_1$ and $d(v_{k+1}, \partial U_1) > d$. Thus, for $v_1 \in U_2$, $v_k \in U_1$ and $d(v_k, \partial U_1) > d$, $k = 1, 2, ...$ Since $K_c^p = p(U_1) \cap K \subset K_c$ and $\{u_k\}$ has a subsequence that converges to a critical point u_0, we have $u_0 \in K_c^p \neq \emptyset$. Denote $U = \{v \in U_1 | d(v, \partial U_1) > d\}$. Then by the monotonicity of $\{J(u_k)\}$, we have $J(u_k) \to c = \inf_{v \in U} J(p(v))$ as $k \to \infty$, and

$$\inf_{v \in \partial \bar{U}} J(p(v)) \ge a > b \ge J(p(v_1)) \ge c = \inf_{v \in U} J(p(v)).$$

Thus all the assumptions of Theorem III.2 are satisfied and the conclusions follow.

B. A Min-Orthogonal Algorithm & Subsequence Convergence

Definition III.2 *Let $v_0 \in S_{L'}$ and p be a local L-$\perp$ selection of J at v_0 with $\nabla J(p(v_0)) \neq 0$. A point $w \in L'$ is a* descent direction *of $J(p(\cdot))$ at v_0 if there is $s_0 > 0$ such that*

$$J(p(v_0(s))) < J(p(v_0)), \quad \forall\, 0 < s < s_0 \quad \textit{where} \quad v_0(s) = \frac{v_0 + sw}{\|v_0 + sw\|}.$$

The local min-orthogonal characterization of a saddle point, Theorem III.1, suggests to devise the following local min-orthogonal algorithm.

Assume that $L = [u^1, u^2, ..., u^{n-1}]$ where $u^1, u^2, ..., u^{n-1}$ are $n-1$ previously found critical points of J. For given $\lambda, \varepsilon > 0$ and $\theta \in (0, 1)$. Let $B = L \oplus L'$.

Step 1: Let $v_1 \in S_{L'}$ be an ascent-descent direction at u^{n-1}.

Step 2: Set $k = 1$. Solve for $u_k \equiv p(v_k) \equiv t_0^k v_k + t_1^k u^1 + \cdots + t_{n-1}^k u^{n-1}$ such that $t_0^k \neq 0$,

$$\langle \nabla J(p(v_k)), v_k \rangle = 0 \quad \text{and} \quad \langle \nabla J(p(v_k)), u^i \rangle = 0,\ i = 1, 2, ..., n-1.$$

Step 3: Find a descent direction w_k of $J(p(\cdot))$ at v_k.

Step 4: If $\|\nabla J(u_k)\| \le \varepsilon$, then output $u_k = p(v_k)$, stop. Otherwise, do Step 5.

Step 5: For each $s > 0$, denote $v_k(s) = \dfrac{v_k + sw_k}{\|v_k + sw_k\|}$ and set $v_{k+1} = v_k(s_k)$ where

$$s_k = \max\{\frac{\lambda}{2^m}|m \in N, 2^m > \|w_k\|, J(p(v_k(\frac{\lambda}{2^m}))) - J(u_k) < -\frac{\theta|t_0^k|}{4}(\frac{\lambda}{2^m})\|\nabla J(u_k)\|\}.$$

Step 6: Update $k = k + 1$ and go to Step 3.

Remark III.1 *About the algorithm, we need point out the following facts.*

(1) *In Step 2, one way to solve the equations while satisfying the nondegenerate condition $t_0^k \neq 0$ is to find a local maximum point u_k of J in the subspace $[L, v_k]$, i.e., $u_k = p(v_k)$ and p becomes a peak selection of J w.r.t. L.*

(2) *In Step 3, there are many different ways to select a descent direction w_k. However, when a descent direction is selected, a corresponding stepsize rule in Step 5 has to be designed such that it can be achieved and leads to converge to a critical point. For example, when a negative modified pseudo-gradient flow $-G_k$, or a negative modified pseudo-gradient is used as a descent direction, a positive step size s_k for the current stepsize rule in Step 5 can always be obtained. In some case, when the negative gradient $-\nabla J(p(v_k))$ is used as a descent direction, the stepsize rule in Step 5 has to be modified as in Case 3 below.*

Now let us first assume that a negative modified pseudo-gradient (flow) is used as a descent direction.

Definition III.3 *For each $v \in S_{L'}$ with $\|\nabla J(p(v))\| \neq 0$, write $p(v) = t_0 v + v_L$ for some $v_L \in L$ and define the stepsize at v as*

$$s(v) = \max_{\lambda \geq s > 0}\{s|\lambda > s\|w\|, J(p(v(s))) - J(p(v)) \leq -\frac{1}{4}\theta|t_0|s\|\nabla J(p(v))\|\}$$

where

$$v(s) = \frac{v + sw}{\|v + sw\|}, \quad w = -sign(t_0)G$$

and G is either a modified pseudo-gradient of J with θ at $p(v)$ or the value of a modified pseudo-gradient flow of J with θ at $p(v)$.

Then it is easy to check that $\frac{1}{2}s(v_k) \le s_k \le s(v_k)$ and by Lemma 3.1 we have

Lemma III.4 *If p is a local L-$\perp$ selection of J at $v \in S_{L'}$ such that (1) p is continuous at v, (2) $d(p(v), L) > 0$ and (3) $\nabla J(p(v)) \neq 0$, then $s(v) > 0$.*

To verify the condition that $\{u_k\}$ has a subsequence that converges to a critical point in Corollary III.1, let us make the following uniform stepsize assumption for $\{u_k\}$ and then verify it for different cases.

(H) if $v_0 \in S_{L'}$ with $\nabla J(p(v_0)) \neq 0$ and $u_k \to p(v_0)$, then there is $s_0 > 0$ such that $s(v_k) \ge s_0$ when k is large.

Theorem III.3 *Let $J \in C^1(B, \mathbb{R})$ satisfy the PS condition and p be an L-$\perp$ selection of J such that (1) p is continuous on $S_{L'}$, (2) $\inf_{1 \le k < \infty} d(p(v_k), L) \ge \alpha > 0$, (3) $\inf_{1 \le k < \infty} J(p(v_k)) > -\infty$, (4) $\{p(v_k)\}$ satisfies Assumption (H), then*
(a) $\{v_k\}_{k=1}^{\infty}$ has a subsequence $\{v_{k_i}\}$ such that $u_{k_i} = p(v_{k_i})$ converges to a critical point of J;
(b) if a subsequence $u_{k_i} \to u_0$ as $i \to \infty$, then $u_0 = p(v_0)$ is a critical point of J.

Proof. (a) By the stepsize rule and Lemma II.2, for $k = 1, 2, ...,$ we have

$$J(u_{k+1}) - J(u_k) \le -\frac{1}{4}\theta\alpha s_k \|\nabla J(p(v_k))\| \le -\frac{1}{16M}\theta\alpha\|v_{k+1} - v_k\|\|\nabla J(p(v_k))\|. \quad (3.8)$$

Suppose that there is $\delta > 0$ such that $\|\nabla J(p(v_k))\| \ge \delta$ for any k. From (3.8), we have

$$J(u_{k+1}) - J(u_k) \le -\frac{1}{16M}\theta\alpha\delta\|v_{k+1} - v_k\|, \quad \forall k = 1, 2, ... \quad (3.9)$$

Adding up (3.9) gives

$$\lim_{k\to\infty} J(u_k) - J(u_1) = \sum_{k=1}^{\infty}[J(u_{k+1}) - J(u_k)] \le -\frac{1}{16M}\theta\alpha\delta \sum_{k=1}^{\infty} \|v_{k+1} - v_k\|, \quad (3.10)$$

i.e., $\{v_k\}$ is a Cauchy sequence. Thus $v_k \to \hat{v} \in S_{L'}$. By the continuity of p, $\|\nabla J(p(\hat{v}))\| \ge \delta > 0$. On the other hand, adding up (3.8) gives

$$\lim_{k\to\infty} J(u_k) - J(u_1) \le -\frac{1}{4}\theta\alpha \sum_{k=1}^{\infty} s_k \|\nabla J(p(v_k))\| \le -\frac{1}{4}\theta\alpha\delta \sum_{k=1}^{\infty} s_k,$$

or $s_k \to 0$ as $k \to \infty$. It leads to a contradiction to assumption (4). Therefore, there is a subsequence $\{v_{k_i}\}$ such that $\|\nabla J(p(v_{k_i}))\| \to 0$ as $i \to \infty$ and $\{J(p(v_{k_i}))\}$ is convergent. By the PS condition, $\{p(v_{k_i})\}$ has a subsequence that converges to a critical point u_0.

(b) Suppose $u_0 = p(v_0)$ is not a critical point. Then there is $\delta > 0$ such that $\|\nabla J(u_{k_i})\| > \delta, i = 1, 2, ...$ Similar to (3.8), we have

$$J(u_{k_i+1}) - J(u_{k_i}) \le -\frac{1}{4}\theta\alpha s_{k_i} \|\nabla J(u_{k_i})\| < -\frac{1}{4}\theta\alpha\delta s_{k_i}.$$

Since $\sum_{k=1}^{\infty}[J(u_{k+1}) - J(u_k)] = \lim_{k\to\infty} J(u_k) - J(u_1), \lim_{i\to\infty}(J(u_{k_i+1}) - J(u_{k_i})) = 0.$ Hence, $\lim\limits_{i\to\infty} s_{k_i} = 0$. It leads to a contradiction to Assumption (H). Thus u_0 is a critical point.

First, we discuss case 1, i.e., use a negative modified pseudo-gradient flow as a descent direction.

In Step 3 of the algorithm we choose $w_k = -sign(t_0^k)G(p(v_k))$ where $G(p(v_k))$ is the value of a modified pseudo-gradient flow of J at $p(v_k) = t_0^k v_k + v_k^L$ for some $v_k^L \in L$.

Lemma III.5 *If p is a local L-$\perp$ selection of J at $v_0 \in S_{L'}$ such that (1) p is continuous at v_0, (2) $d(p(v_0), L) > 0$ and (3) $\nabla J(p(v_0)) \neq 0$, then Assumption (H) is* *satisfied, or, there exist* $\epsilon, s_0 > 0$ *such that for each* $v \in S_{L'}$ *with* $\|v - v_0\| < \epsilon$,

$$J(p(v(s_0))) - J(p(v)) < -\frac{s_0\theta|t_v|}{4}\|\nabla J(p(v))\|$$

where $v(s_0) = \frac{v+sign(t_v)s_0G(p(v))}{\|v+sign(t_v)s_0G(p(v))\|}$, $p(v) = t_v v + w_v$ *for some* $w \in L$ *and* $G(p(v))$ *is the value of a modified pseudo-gradient flow of* J *at* $p(v)$ *with constant* θ.

Proof. By Lemma III.4, there is $\bar{s} > 0$ such that as $0 < s < \bar{s}$

$$J(p(v_0(s))) - J(p(v_0)) < -\frac{s\theta|t_0|}{4}\|\nabla J(p(v_0))\| \tag{3.11}$$

where $v_0(s) = \frac{v_0 - sign(t_0)sG(p(v_0))}{\|v_0 - sign(t_0)sG(p(v_0))\|}$ and $p(v_0) = t_0 v_0 + w_0$ for some $w_0 \in L$. Actually, for fixed s, the two sides of (3.11) are continuous in v_0. Thus, there are $\epsilon, s_0 > 0$ such that

$$J(p(v(s_0))) - J(p(v)) < -\frac{s_0\theta|t_v|}{4}\|\nabla J(p(v))\|, \quad \forall v \in S_{L'} \text{ with } \|v - v_0\| \leq \epsilon.$$

Second, we discuss case 2, i.e., use a negative modified pseudo-gradient as a descent direction.

In Step 3 of the algorithm we choose $w_k = -sign(t_0^k)G(p(v_k))$ where $G(p(v_k))$ is a modified pseudo-gradient of J at $p(v_k) = t_0^k v_k + v_k^L$ for some $v_k^L \in L$. Since pseudo-gradients may be chosen from different pseudo-gradient flows, we lost the continuity. To compensate the loss, we assume that an L-$\perp$ selection p of J is Lipschitz continuous.

Lemma III.6 *Let* p *be a local* L-$\perp$ *selection of* J *at* $v_0 \in S_{L'}$. *If (1)* p *is Lipschitz continuous in a neighborhood of* v_0, *(2)* $d(p(v_0), L) > 0$ *and (3)* $\nabla J(p(v_0)) \neq 0$, *then*

Assumption (H) is satisfied, or there are $\epsilon, s_0 > 0$ *such that*

$$J(p(v(s_0))) - J(p(v)) < -\frac{1}{4}s_0\theta|t_v|\|\nabla J(p(v))\|, \quad \forall v \in B_{L'} \;\; \textit{with} \;\; \|v - v_0\| < \epsilon$$

where

$$v(s_0) = \frac{v - sign(t_v)s_0G(p(v))}{\|v - sign(t_v)s_0G(p(v))\|}, \quad p(v) = t_v v + v^L \;\; \textit{for some}\; v^L \in L$$

and $G(p(v))$ *is a modified pseudo-gradient of* J *at* $p(v)$ *with constant* θ*.*

Proof. First, denote $p(v(s)) = t_v^s v(s) + w_v(s)$ for some $w_v(s) \in L$, we have

$$J(p(v(s))) - J(p(v)) = \langle \nabla J(p(v)) + (\nabla J(\zeta(v,s)) - \nabla J(p(v))), p(v(s)) - p(v)\rangle \quad (3.12)$$

where $\zeta(v,s) = (1-\lambda)p(v) + \lambda p(v(s))$ for some $\lambda \in [0,1]$. By assumption (1) and Lemma II.2,

$$\|p(v(s)) - p(v)\| \leq \ell\|v(s) - v\| \leq \frac{2\ell s\|G(p(v))\|}{\|v - sign(t_v)sG(p(v))\|} \leq 4\ell Ms. \quad (3.13)$$

On the other hand, by the definition of an L-$\perp$ selection of J, as $s > 0$ is small and for any v close to v_0, denote $v(s) = \dfrac{v - sign(t_v)sG(p(v))}{\|v - sign(t_v)sG(p(v))\|}$, we have

$$\begin{aligned} \langle \nabla J(p(v)), p(v(s)) - p(v)\rangle &= -\frac{sign(t_v)t_v^s s\langle \nabla J(p(v)), G(p(v))\rangle}{\|v - sign(t_v)sG(p(v))\|} \\ = -\frac{|t_v^s|s\langle \nabla J(p(v)), \Psi(p(v))\rangle}{\|v - sign(t_v)sG(p(v))\|} &\leq -\frac{s\theta|t_v|\|\nabla J(p(v))\|}{2} < 0, \end{aligned} \quad (3.14)$$

$$\begin{aligned} &|\langle \nabla J(\zeta(v,s)) - \nabla J(p(v)), p(v(s)) - p(v)\rangle| \\ \leq\;& \|\nabla J(\zeta(v,s)) - \nabla J(p(v))\| \; \|p(v(s)) - p(v)\| \leq \frac{s\theta|t_v|\|\nabla J(p(v))\|}{4} \end{aligned} \quad (3.15)$$

where in the last inequality, since J is C^1 and by assumptions (2) and (3), we have

$$\|\nabla J(\zeta(v,s)) - \nabla J(p(v))\| \leq \frac{\theta|t_v|\|\nabla J(p(v))\|}{16\ell M}. \quad (3.16)$$

By (3.12) there exist $s_0, \epsilon > 0$ such that

$$J(p(v(s_0))) - J(p(v)) \leq -\frac{s_0\theta|t_v|\|\nabla J(p(v))\|}{4}, \ \forall v \in S_{L'} \text{ with } \|v - v_0\| < \epsilon. \tag{3.17}$$

Finally, we discuss case 3, i.e., use a practical technique for a descent direction.

To solve a class of quasilinear elliptic PDEs, some very useful practical techniques are developed in Chapter II for numerical implementation to compute descent search directions. Let $B = W_0^{1,p}(\Omega) = L \oplus L'$ for some closed subspaces L, L' in B, $p > 1$ and $B^* = W_0^{-1,q}(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathcal{P}$ be an L-$\perp$ selection of J. For $v \in S_{L'}, u = \mathcal{P}(v)$, let $\delta J(u)$ be the gradient of J at u w.r.t. the usual (B, B^*) duality. By the definition of $\mathcal{P}, \delta J(u) \perp L$. But $\delta J(u) \in B^*$, thus cannot be used as a search direction in B. Our gradient $d = \nabla J(u)$ is a solution to

$$\Delta d(x) = -\delta J(u)(x), \ x \in \Omega, \quad d(x)|_{\partial\Omega} = 0.$$

We have $\nabla J(u) \in W_0^{1,q}(\Omega) \subset B^*$ and for any $w \in B$,

$$\begin{aligned} & \langle d, w \rangle_{W_0^{1,q} \times W_0^{1,p}} \equiv \langle \nabla d, \nabla w \rangle_{L^q \times L^p} \equiv \int_\Omega \nabla d(x) \cdot \nabla w(x)\, dx \\ = & \int_\Omega -\Delta d(x) w(x)\, dx = \int_\Omega \delta J(u)(x) w(x)\, dx \equiv \langle \delta J(u), w \rangle_{W_0^{-1,q} \times W_0^{1,p}}. \end{aligned}$$

In this sense, $d = \nabla J(u)$ can be used as a gradient of J at u and in particular

$$\langle \nabla J(u), w \rangle_{W_0^{1,q} \times W_0^{1,p}} = \langle \delta J(u), w \rangle_{W_0^{-1,q} \times W_0^{1,p}} = 0, \quad \forall w \in L. \tag{3.18}$$

Then we will discuss, a few paragraphs late, how to choose L' such that $\nabla J(u) \in L'$. Since

$$\|\delta J(u)\|_{W_0^{-1,q}} = \sup_{\|w\|_{W_0^{1,p}} = 1} |\langle \delta J(u), w \rangle_{W_0^{-1,q} \times W_0^{1,p}}|$$

$$= \sup_{\|w\|_{W_0^{1,p}}=1} |\langle d, w\rangle_{W_0^{1,q}\times W_0^{1,p}}| = \sup_{\|\nabla w\|_{L^p}=1} |\langle \nabla d, \nabla w\rangle_{L^q\times L^p}| \leq \|d\|_{W_0^{1,q}},$$

$\nabla J(u_k) \to 0 \implies \delta J(u_k) \to 0$, i.e., the PS condition of J in terms of δJ implies the PS condition of J in terms of ∇J. From now on, $\langle\cdot,\cdot\rangle_{(1,1)} = \langle\cdot,\cdot\rangle_{W_0^{1,q}\times W_0^{1,p}}$, $\langle\cdot,\cdot\rangle_{(-1,1)} = \langle\cdot,\cdot\rangle_{W_0^{-1,q}\times W_0^{1,p}}$ and $\langle\cdot,\cdot\rangle$ means $\langle\cdot,\cdot\rangle_{(1,1)}$ whenever ∇J is involved. Based on the understanding that when a nice smooth initial guess v_0 is used, we may expect that actually nice functions are used to approximate a critical point, i.e., all the points $v_k, u_k = \mathcal{P}(v_k)$ and $\nabla J(u_k)$ are nice. Motivated by pseudo-gradients, to find a descent search direction, we check the ratio

$$\frac{\|\nabla J(u_k)\|_2^2}{\|\nabla J(u_k)\|_q \|\nabla J(u_k)\|_p} \geq \theta > 0 \quad \forall k = 1, 2, ..., \tag{3.19}$$

where $\|\cdot\|_r$ is the $W_0^{1,r}(\Omega)$-norm. When (3.19) is satisfied, $\phi(u_k) = \frac{\nabla J(u_k)}{\|\nabla J(u_k)\|_p}$ is not only in $W_0^{1,r}(\Omega)$ with $r = q, 2, p$, but also a modified pseudo-gradient of J w.r.t. θ at u_k, i.e., $\phi(u_k) \in S_{L'}$, $\|\phi(u_k)\|_p = 1$ and $\langle \delta J(u_k), \phi(u_k)\rangle_{(-1,1)} \geq \theta \|\delta J(u_k)\|_{W_0^{-1,q}}$. However, we cannot assume that $\phi(u_k)$ is the value of a modified pseudo-gradient flow of J at $\mathcal{P}(v_k)$, simply because we do not have any information about the ratio at other points.

Thus, when $-\phi(u_k)$ is used as a descent search direction, this case can be covered by Case 2. But in implementation, the lower bound θ in (3.19) is usually not known beforehand. In particular, we do not know whether or not the ratio is satisfied at a limit point of the sequence. Hence, Step 3 in the algorithm is modified to be

Step 3: Find a descent direction w_k of J at $u_k = \mathcal{P}(v_k)$, $w_k = -sign(t_0^k)\nabla J(u_k)$. Compute the ratio $\theta_k = \dfrac{\|w_k\|_2^2}{\|w_k\|_p \|w_k\|_q} > 0$;

and the stepsize rule in Step 5 has to be changed to

$$s_k = \max\Big\{s = \frac{\lambda}{2^m} | m \in N, 2^m > \|w_k\|, J(\mathcal{P}(v_k(s))) - J(u_k) \le \frac{|t_0^k| s}{-4} \|\nabla J(u_k)\|_2^2\Big\}. \tag{3.20}$$

Next we show that if $0 < \|\nabla J(\mathcal{P}(v_0))\|_2 < +\infty$, a positive stepsize can always be attained.

Lemma III.7 *For $v_0 \in S_{L'}$, if J has a local L-$\perp$ selection $\mathcal{P}$ at v_0 satisfying (1) $\mathcal{P}$ is continuous at v_0, (2) $d(\mathcal{P}(v_0), L) > \alpha > 0$ and (3) $0 < \|\nabla J(\mathcal{P}(v_0))\|_2 < +\infty$. Then there exists $s_0 > 0$ such that as $0 < s < s_0$*

$$J(\mathcal{P}(v_0(s))) - J(\mathcal{P}(v_0)) < -\frac{|t_0| s}{4} \|\nabla J(\mathcal{P}(v_0))\|_2^2 \tag{3.21}$$

where $v_0(s) = \frac{v_0 - sign(t_0) s \nabla J(\mathcal{P}(v_0))}{\|v_0 - sign(t_0) s \nabla J(\mathcal{P}(v_0))\|}$ and $\mathcal{P}(v_0) = t_0 v_0 + w_L$ for some $t_0 \in R$, $w_L \in L$.

Proof. Since $\|\mathcal{P}(v_0(s)) - \mathcal{P}(v_0)\| \to 0$ as $s \to 0$, we have

$$\begin{aligned} & J(\mathcal{P}(v_0(s)) - J(\mathcal{P}(v_0)) \\ = \;& \langle \nabla J(\mathcal{P}(v_0)), \mathcal{P}(v_0(s)) - \mathcal{P}(v_0)\rangle + o(\|\mathcal{P}(v_0(s)) - \mathcal{P}(v_0)\|) \\ = \;& -\frac{|t_0^s| s \|\nabla J(\mathcal{P}(v_0))\|_2^2}{\|v_0 - s\nabla J(\mathcal{P}(v_0))\|} + o(\|\mathcal{P}(v_0(s)) - \mathcal{P}(v_0)\|) < -\frac{|t_0| s}{4} \|\nabla J(\mathcal{P}(v_0))\|_2^2 \end{aligned}$$

where $\mathcal{P}(v_0(s)) = t_0^s v_0(s) + w_L^s$ for some $t_0^s \in R$, $w_L^s \in L$ and the last inequality holds for $0 < s < s_0$ for some $s_0 > 0$.

Now we discuss how to choose L' such that $\nabla J(u_k) \in L'$.

For $p < 2$, $\nabla J(v) \in W_0^{1,q} \subset B$. Although when L is finite-dimensional, theoretically there is a closed subspace L' such that $B = L \oplus L'$, in general it is difficult to find an explicit formula for L'. Here we develop a different approach for convergence analysis. Denote $L^\perp = \{v \in W_0^{1,q} | \langle v, u\rangle_{(1,1)} = 0, \; \forall u \in L\}$ and L' to be the $\|\cdot\|_p$-norm closure of $L^\perp$ in B. It is clear that L' is closed in B. When $L = \{0\}$, $L' = B$ holds.

But when $\dim(L) > 0$, $B = L \oplus L'$ fails to hold. Thus this case has to be handled with extra care. We still use $S_{L'}$ as the domain to define an L-$\perp$ selection $\mathcal{P}$ as in Definition III.1. However if an initial guess v_1 is chosen in $S_{L^\perp}$, we have $v_k \in S_{L^\perp}$ for all $k = 2, 3, ...$

Lemma III.8 *Let $J \in C^1(B, \mathbb{R})$ and $v_0 \in S_{L'}$. Let $\mathcal{P}$ be a local L-$\perp$ selection of J at v_0 such that $\mathcal{P}$ is continuous at v_0 and $d(\mathcal{P}(v_0), L) > 0$. If $\nabla J(\mathcal{P}(v_0)) \neq 0$, then there exists $s_0 > 0$ and $\varepsilon > 0$ such that*

$$J(\mathcal{P}(v(s_0)) - J(\mathcal{P}(v)) < -\frac{|t_v| s_0}{4} \|\nabla J(\mathcal{P}(v))\|_2^2, \quad \forall v \in S_{L'}, \|v - v_0\| < \varepsilon,$$

where $\mathcal{P}(v) = t_v v + w_v$ and $w_v \in L$.

Proof. By Lemma III.7, we have

$$J(\mathcal{P}(v_0(s))) - J(\mathcal{P}(v_0)) < -\frac{|t_0| s}{4} \|\nabla J(\mathcal{P}(v_0))\|_2^2, \tag{3.22}$$

where $\mathcal{P}(v_0) = t_0 v_0 + w_0$ and $w_0 \in L$. When $p < 2$, we have $q > 2$. J is C^1 implies that ∇J is continuous in $\|\cdot\|_2$-norm. For fixed s, all the terms in (3.22) are continuous in v_0. Thus there exists $s_0 > 0$ and $\varepsilon > 0$ such that

$$J(\mathcal{P}(v(s_0))) - J(\mathcal{P}(v)) < -\frac{|t_v| s_0}{4} \|\nabla J(\mathcal{P}(v))\|_2^2, \quad \forall v \in S_{L'}, \|v - v_0\| < \varepsilon.$$

With the new stepsize rule and Lemma III.8, if $\theta_k > \theta > 0$ in Step 3 is satisfied, we can verify Theorem III.3. The proof is similar. We only need to replace (3.8) by

$$\begin{aligned} J(u_{k+1}) - J(u_k) &< -\frac{\alpha s_k}{4} \|\nabla J(u_k)\|_2^2 < -\frac{\alpha \theta s_k}{4} \|\nabla J(u_k)\|_p \|\nabla J(u_k)\|_q \\ &< -\frac{\alpha \theta}{8} \|v_{k+1} - v_k\| \|\nabla J(u_k)\| \end{aligned} \tag{3.23}$$

and then follow the proof.

Then the unified convergence result, Corollary III.1 holds for this practical technique.

In all our numerical examples carried out so far, (3.19) is satisfied. We also note that the ratio is stable for $1 < p \leq 2$ and gets worse as $p \to +\infty$. Thus for $p > 2$, instead of assuming (3.19) holds and using $-\phi(u_k)$ as a descent search direction, we only assume $\|\nabla J(u_k)\|_p \leq M$ for some $M > 0$ and directly verify that $-\nabla J(u_k)$ is a descent search direction in B.

For $p > 2$, $B \subset B^*$. Let $L' = L^\perp = \{u \in B | \langle u, v\rangle_{(1,1)} = 0, \ \forall v \in L\}$. Thus it can be verified that L' is closed in B and $B = L \oplus L'$ holds at least when L is finite-dimensional. If $\|\nabla J(\mathcal{P}(v_k))\|_p < +\infty$, then $\nabla J(\mathcal{P}(v_k)) \in L'$ by the definition of $\mathcal{P}$ at $v_k \in S_{L'}$ and (3.18). Since J is C^1 means that δJ is continuous in $W_0^{-1,q}(\Omega)$, but ∇J is not necessarily continuous in $\|\cdot\|_2$-norm or $\|\cdot\|_p$-norm, we need an L-$\perp$ selection $\mathcal{P}$ to be locally Lipschitz continuous.

Lemma III.9 *If $w_k \to w \neq 0$ in $W_0^{1,q}(\Omega)(q > 1)$ and $w_k \in W_0^{1,r}(\Omega)(r > 1), k = 1, 2, ...$ and Ω is bounded, then $\inf_k \|w_k\|_r > 0$.*

Proof. (1) The case $q \geq r$ is trivial, since $w_k \to w$ in $W_0^{1,q}(\Omega) \Rightarrow w_k \to w$ in $W_0^{1,p}(\Omega)$. (2) For $q < r$, if $\inf_k \|w_k\|_r = 0$, then there is $\{w_{k_n}\}$ such that $\lim_{n\to\infty} \|w_{k_n}\|_r = 0$, i.e., $\lim_{n\to\infty} w_{k_n} = 0$ in $W_0^{1,r}(\Omega)$. Then $\lim_{n\to\infty} w_{k_n} = 0$ in $W_0^{1,q}(\Omega)$, i.e., $w = 0$. It is a contradiction.

Lemma III.10 *Let $J \in C^1(B, \mathbb{R})$ and $v_0 \in S_{L'}$. Assume $\mathcal{P}$ is a local L-$\perp$ selection of J at v_0 such that (1) $\mathcal{P}$ is locally Lipschitz continuous (2) $d(\mathcal{P}(v_0), L) > 0$ and (3) $\nabla J(\mathcal{P}(v_0)) \neq 0$. Then for any $v_k \in S_{L'}$, $\lim_{k\to\infty} v_k = v_0$ and $\|\nabla J(\mathcal{P}(v_k))\| < M$ for some constant M, $u_k = \mathcal{P}(v_k)$ satisfies Assumption (H).*

Proof. Let $u_k = t_k v_k + v_k^L$ and $\mathcal{P}(v_k(s)) = t_k^s v_k(s) + v_k^L(s)$ for some $v_k^L, v_k^L(s) \in L$, we have

$$J(\mathcal{P}(v_k(s))) - J(u_k) = \langle \nabla J(u_k) + (\nabla J(\zeta(v_k, s)) - \nabla J(u_k)), \mathcal{P}(v_k(s)) - u_k \rangle \quad (3.24)$$

where $\zeta(v_k, s) = (1-\lambda_k)u_k + \lambda_k \mathcal{P}(v_k(s))$ for some $\lambda_k \in [0,1]$. By assumption (1) and Lemma II.2, it leads to

$$\|\mathcal{P}(v_k(s)) - u_k\| \leq \ell \|v_k(s) - v_k\| \leq \frac{2\ell s \|\nabla J(u_k)\|}{\|v_k - sign(t_k) s \nabla J(u_k)\|}.$$

On the other hand, by the definition of an L-$\perp$ selection of J, as $s > 0$ is small and k is large,

$$\langle \nabla J(u_k), \mathcal{P}(v_k(s)) - u_k \rangle = -\frac{sign(t_k) t_k^s s \|\nabla J(u_k)\|_2^2}{\|v_k - sign(t_k) s \nabla J(u_k)\|} \leq -\frac{s|t_k|}{2} \|\nabla J(u_k)\|_2^2 < 0$$

where $v_k(s) = \dfrac{v_k - sign(t_k) s \nabla J(u_k)}{\|v_k - sign(t_k) s \nabla J(u_k)\|}$. Since J is C^1 and by assumptions (2) and (3), and Lemma III.9, there exist $\delta > 0$ such that when $s > 0$ is small and k is large,

$$\frac{|t_k| \|v_k - sign(t_k) s \nabla J(u_k)\| \|\nabla J(u_k)\|_2^2}{8\ell \|\nabla J(u_k)\|} > \delta > 0.$$

Thus we can choose $s > 0$ small and k large such that

$$\|\nabla J(\zeta(v_k, s)) - \nabla J(u_k)\| \leq \frac{|t_k| \|v_k - sign(t_k) s \nabla J(u_k)\| \|\nabla J(u_k)\|_2^2}{8\ell \|\nabla J(u_k)\|}.$$

Hence

$$\begin{aligned} & |\langle \nabla J(\zeta(v_k, s)) - \nabla J(u_k), \mathcal{P}(v_k(s)) - u_k \rangle| \\ \leq \; & \|\nabla J(\zeta(v_k, s)) - \nabla J(u_k)\| \|\mathcal{P}(v_k(s)) - u_k\| \; \leq \; \frac{s|t_k| \|\nabla J(u_k)\|_2^2}{4}. \end{aligned}$$

By (3.24), there exist $\bar{k}, s_0 > 0$ such that when $0 < s < s_0$,

$$J(\mathcal{P}(v_k(s))) - J(u_k) \leq -\frac{s|t_k| \|\nabla J(u_k)\|_2^2}{4}, \; \forall k > \bar{k}.$$

With the new stepsize rule and the conditions $\|\nabla J(u_k)\|_p < M$, we can also verify Theorem III.3. The proof is similar. Note that when $\|\nabla J(u_k)\|_q > \delta_0$ for some $\delta_0 > 0$, $\|\nabla J(u_k)\|_2 > \delta$ for some $\delta > 0$ and there is always a $\beta > 0$ such that $\|v_k + s_k w_k\|_p \geq \beta$, $k = 1, 2....$ We only need to replace (3.8) and (3.9) by

$$\begin{aligned} J(u_{k+1}) - J(u_k) &< -\frac{\alpha s_k}{4}\|\nabla J(u_k)\|_2^2 \leq -\frac{\alpha s_k}{4}\delta^2 = -\frac{\alpha s_k \delta^2}{4M}M \\ &\leq -\frac{\alpha s_k \delta^2}{8M}\|\nabla J(u_k)\|_p \leq -\frac{\alpha\beta\delta^2}{16M}\|v_{k+1} - v_k\| \end{aligned}$$

where the last inequality follows from Lemma II.2 and then follow the proof. The unified convergence result, Corollary III.1 also follows.

C. An Application to Nonlinear p-Laplacian PDE

As an application, let us consider the following quasilinear elliptic boundary-value problem on a bounded smooth domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} \Delta_p u(x) + f(x, u(x)) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \qquad u \in B \equiv W^{1,p}(\Omega),\ p > 1, \tag{3.25}$$

where Δ_p defined by $\Delta_p u(x) = div(|\nabla u(x)|^{p-2}\nabla u(x))$ is the p-Laplacian operator which has a variety of applications in physical fields, such as in fluid dynamics when the shear stress and the velocity gradient are related in certain manner where $p = 2, p < 2, p > 2$ if the fluid is Newtonian, pseudoplastic, dilatant, respectively. The p-Laplacian operator also appears in the study of flow in a porous media ($p = \frac{3}{2}$), nonlinear elasticity ($p > 2$) and glaciology ($p \in (1, \frac{4}{3})$). Under certain standard conditions on f, it can be shown that a point $u^* \in W_0^{1,p}(\Omega)$ is a weak solution of (3.25) if and only if u^* is a critical point of the functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega F(x, u(x))dx \quad \text{where} \quad F(x,t) = \int_0^t f(x,s)ds. \tag{3.26}$$

Since conditions (1), (2) and (3) in Theorem III.2 are basic assumptions in our results and new in the literature, we verify them in this section. Let us assume some of the standard growth and regularity conditions in the literature. Set the Sobolev exponent $p^* = \frac{np}{n-p}$ for $p < n$ and $p^* = \infty$ for $p \geq n$. Assume

(a) $f \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R}), f(x,0) = 0, \frac{f(x,t\xi)}{|t\xi|^{p-2}t\xi}$ monotonically increases to $+\infty$ in t,

(b) For each $\varepsilon > 0$, there is $c_1 = c_1(\varepsilon) > 0$ such that $f(x,t)t < \varepsilon|t|^p + c_1|t|^{p^*}$, $\forall t \in \mathbb{R},\ x \in \Omega$.

It is clear that $u = 0$ is a critical point of least value J and $f(x,u) = |u|^{q-2}u$ for $q > p$ satisfies condition (a). For each $v \in B$ with $\|v\| = 1$ and $t > 0$, let $g(t) = J(tv)$. We have

$$\begin{aligned} g'(t) &= \langle \nabla J(tv), v\rangle = \int_\Omega \Big(t^{p-1}|\nabla v(x)|^p - f(x,tv(x))v(x)\Big)dx \\ &= t^{p-1}\Big(1 - \int_\Omega \frac{f(x,tv(x))|v(x)|^p}{|tv(x)|^{p-2}tv(x)}\Big)dx. \end{aligned}$$

Thus, by condition (a), there is a unique $t_v > 0$ such that $g'(t_v) = 0$, i.e., for $L = \{0\}$ and each $v \in S_B$, the L-$\perp$ selection (actually a peak selection) $\mathcal{P}(v) = t_v v$ is uniquely determined with $J(\mathcal{P}(v)) > 0$ and

$$\begin{aligned} g''(t) &= (p-1)t^{(p-2)} - \int_\Omega f'_\xi(x,tv(x))v^2(x)\,dx \\ &< (p-1)t^{(p-2)} - \int_\Omega \frac{(p-1)}{t} f(x,tv(x))v(x)\,dx = \frac{p-1}{t}g'(t). \end{aligned}$$

The last inequality follows from taking a derivative of condition (a) w.r.t. t. Thus condition (3) in Theorem III.2 is always satisfied for any L. Next let us recall that when $L = [u_1, u_2, ..., u_{n-1}]$, by the definition of an L-$\perp$ selection, $\mathcal{P}(v) = t_0 v + t_1 u_1 + \cdots + t_{n-1}u_{n-1}$ is solved from

$$\langle \nabla J(t_0 v + t_1 u_1 + \cdots + t_{n-1}u_{n-1}), v\rangle = 0, \tag{3.27}$$

$$\langle \nabla J(t_0 v + t_1 u_1 + \cdots + t_{n-1}u_{n-1}), u_i\rangle \;=\; 0, \quad i = 1, ..., n.$$

If $u = \mathcal{P}(v) = t_0 v + t_1 u_1 + \cdots + t_{n-1}u_{n-1}$ satisfies (3.27) and at u, the $n \times n$ matrix

$$Q = \begin{bmatrix} \langle J''(u)v, v\rangle_{(-1,1)} & \langle J''(u)u_1, v\rangle_{(-1,1)} & \cdots & \langle J''(u)u_{n-1}, v\rangle_{(-1,1)} \\ \langle J''(u)v, u_1\rangle_{(-1,1)} & \langle J''(u)u_1, u_1\rangle_{(-1,1)} & \cdots & \langle J''(u)u_{n-1}, u_{n-1}\rangle_{(-1,1)} \\ \cdots & & \cdots & \\ \langle J''(u)v, u_{n-1}\rangle_{(-1,1)} & \langle J''(u)u_1, u_{n-1}\rangle_{(-1,1)} & \cdots & \langle J''(u)u_{n-1}, u_{n-1}\rangle_{(-1,1)} \end{bmatrix}$$

is invertible, i.e., $|Q| \neq 0$, then by the implicit function theorem, around u, the L-$\perp$ selection $\mathcal{P}$ is well-defined and continuously differentiable. The condition $|Q| \neq 0$ can be easily and numerically checked. For the current case $L = \{0\}$, we have $Q = g''(t_v) < 0$. Thus the L-$\perp$ selection $\mathcal{P}$ is C^1. To show that $d(\mathcal{P}(v), L) \geq \alpha > 0$ for all $v \in S_B$, by (b), for any $\varepsilon > 0$, there is $c_1 = c_1(\varepsilon)$ such that $f(x, v(x))v(x) < \varepsilon |v(x)|^p + c_1 |v(x)|^{p^*}$. It follows

$$\begin{aligned} \int_\Omega f(x, v(x))v(x)\, dx \;&<\; \varepsilon \int_\Omega |v(x)|^p\, dx + c_1 \int_\Omega |v(x)|^{p^*}\, dx \\ &\qquad \text{(by the Poincare and Sobolev inequalities)} \\ &\leq\; \varepsilon c_0(\Omega) \int_\Omega |\nabla v(x)|^p\, dx + c_1 c_2(\Omega) \Big(\int_\Omega |\nabla v(x)|^p\, dx\Big)^{\frac{p^*}{p}} \\ &=\; \Big[\varepsilon c_0(\Omega) + c_1 c_2(\Omega) \Big(\int_\Omega |\nabla v(x)|^p\, dx\Big)^{\frac{p^*}{p}-1}\Big] \int_\Omega |\nabla v(x)|^p\, dx. \end{aligned}$$

Thus

$$\begin{aligned} \langle \nabla J(v), v\rangle \;&\geq\; \Big[1 - \varepsilon c_0(\Omega) - c_1 c_2(\Omega) \Big(\int_\Omega |\nabla v(x)|^p\, dx\Big)^{\frac{p^*}{p}-1}\Big] \int_\Omega |\nabla v(x)|^p\, dx \\ &=\; \Big[1 - \varepsilon c_0(\Omega) - c_1 c_2(\Omega) \|v\|^{p^*-p}\Big] \|v\|^p. \end{aligned}$$

It follows that for any small $\varepsilon > 0, c_1, c_0(\Omega)$ and $c_2(\Omega)$, there is $t_0 > 0$ such that when $0 < \|v\| = t < t_0$, we have $\langle \nabla J(v), v\rangle \geq \Big[1 - \varepsilon c_0(\Omega) - c_1 c_2(\Omega) t^{p^*-p}\Big] t^p > 0$. Therefore the L-$\perp$ selection $\mathcal{P}(v)$ satisfies $\|\mathcal{P}(v)\| > t_0$ or $d(\mathcal{P}(v), L) > t_0 > 0, \forall v \in S_B$ where

$L = \{0\}$.

To assure that the energy function J in (3.26) satisfies the PS condition, we need

(c) $|f(x,u)| \leq C(1+|u|^{q-1})$, $\forall u \in \mathbb{R}$, $x \in \Omega$ for some positive constant C and $1 \leq q \leq p^*$,

(d) there is $\theta > p$, $M > 0$ such that $|u| \geq M$ implies

$$0 < \theta F(x,u) \leq uf(x,u),$$

[20]. It is easy to check $f(x,u) = |u|^{q-2}u$, where $p < q \leq p^*$, satisfies (c) and (d).

By the above discussion and Theorem III.2, we have following existence theorem.

Theorem III.4 *If f in (3.25) and F in (3.26) satisfy the conditions (a), (b), (c) and (d), then the quasilinear elliptic boundary value problem (3.25) has a nontrivial weak solution.*

CHAPTER IV

A MINIMAX METHOD FOR NONLINEAR EIGENPAIRS

Let B be a Banach space, $\langle , \rangle$ the dual relation and $\|\cdot\|$ the norm in B. Consider the following eigenpair problem, for given $\alpha > 0$, find $(\lambda, u) \in \mathbb{R} \times (B \setminus \{0\})$ such that

$$\begin{cases} F'u = \lambda G'u \quad \text{or} \quad \langle F'u, v\rangle = \lambda \langle G'u, v\rangle, \ \forall v \in B \\ \text{subject to} \quad G(u) = \alpha \end{cases} \tag{4.1}$$

where F' and G' are the *Fréchet* derivatives of two functionals F and G in $C^1(B, \mathbb{R})$. Such (λ, u) is called an *eigenpair* where λ is an *eigenvalue* and u is an *eigenfunction* corresponding to λ. Since (4.1) is a constrained critical point problem, let us define the Lagrange functional

$$\mathcal{L}(u, \lambda) = F(u) - \lambda(G(u) - \alpha). \tag{4.2}$$

Then critical points (u, λ) of $\mathcal{L}(u, \lambda)$ are eigenpairs (λ, u) of (4.1) and vice versa. Under certain conditions, existence of countable critical points (u_m, λ_m) to (4.2) can be established (see Proposition 44.26 in [28]). We assume that the eigenpair problem (4.1) satisfies the following iso-homogeneous condition, i.e., there is $k \neq 0$ such that

$$F'(tu) = t^k F'(u) \text{ and } G'(tu) = t^k G'(u), \ \forall t > 0, u \in B. \tag{4.3}$$

Let $U = \{u \in B | G(u) = 0\}$. We assume that $U \cap S$ contains only isolated points on the unit sphere S of B and $F(u) \neq 0$, $\forall u \in U \setminus \{0\}$. Then the Rayleigh quotient J can be defined by

$$J(u) = \frac{F(u)}{G(u)}, \ \forall u \in B \setminus U. \tag{4.4}$$

It is easy to check that $J \in C^1(B \setminus U, \mathbb{R})$.

A. Characterization of Eigenpairs

Lemma IV.1 *Under the homogeneous condition, a pair* (λ, u) *is an eigenpair of (4.1), if and only if* u *is a critical point of* J *and* $\lambda = J(u)$ *is the corresponding critical value.*

Proof. The "if" part is always true. To see the "only if" part, let $u \in B \setminus U$, we have

$$\int_0^1 \langle F'(tu), u \rangle dt = \int_0^1 \frac{d}{dt} F(tu) dt = F(u).$$

Similarly, $\int_0^1 \langle G'(tu), u \rangle dt = G(u)$. Thus, if (λ, u) is an eigenpair of (4.1), i.e., $F'(u) = \lambda G'(u)$, then with the homogeneous condition we have $F(u) = \lambda G(u)$,

$$\lambda = \frac{F'(u)}{G'(u)} = \frac{F(u)}{G(u)} \equiv J(u) \quad \text{and} \quad J'(u) = \frac{F'(u)G(u) - F(u)G'(u)}{G^2(u)} = 0.$$

Remark IV.1 *Several points need to be remarked.*

(a) *Due to the homogeneous condition,* α *in (4.1) can be replaced by any nonzero number. For the Rayleigh quotient, we have* $J(tu) \equiv J(u)$ *for any* $u \in B \setminus U$*. Thus* $\langle \nabla J(u), u \rangle = 0$*. From now on we limit* J *on the unit sphere* S *of* $B \setminus U$*;*

(b) *Lemma IV.1 gives the equivalence between eigenpairs of (4.1) and critical points of (4.4);*

(c) *Another important consequence of Lemma IV.1 is that if critical points* u_k *are found in a way that their critical values are in a monotone (increasing) manner, then eigenvalues* λ_k *are obtained in the same monotone (increasing) manner. Thus it is easy for us to discuss whether or not we miss any eigenfunctions.*

1. A Local Minmax Characterization of Eigenpairs

In order to solve our eigenpair problems, we need to modify the local minimax method in Chapter II. Let us introduce the following definitions. Let $L = [u_1, u_2, ..., u_{n-1}]$ be the space spanned by given linearly independent $u_1, u_2, ..., u_{n-1} \in B$ and $B = L \oplus L'$. Let $\mathcal{P} : B \to L'$ be the corresponding linear projection operator. Let $S_{L'}$ be the unit sphere in $L' \setminus U$. For each $u \in S_{L'}$ denote $[L, u]_S = \{w = \sum_{k=1}^{n-1} t_k u_k + t_0 u | \sum_{k=1}^{n-1} t_k^2 + t_0^2 = 1\}$.

Definition IV.1 *A mapping $P : S_{L'} \to 2^B$ is the peak mapping of J w.r.t. L if for each $u \in S_{L'}$, $P(u)$ is the set of all local maximum points of J on $[L, u]_S$, i.e., $w \in [L, u]_S$ is in $P(u)$ if and only if there is a neighborhood $\mathcal{N}(w)$ of w such that $J(v) \leq J(w)$, $\forall v \in [L, u]_S \cap \mathcal{N}(w)$. A single-valued mapping $p : S_{L'} \to B$ is a peak selection of J w.r.t. L if $p(v) \in P(v)$, $\forall v \in S_{L'}$. For a given $u \in S_{L'}$, p is said to be a* local *peak selection of J at u if the peak mapping P is locally defined near u and $p(v) \in P(v)$ when v is near u.*

Remark IV.2 *Several points should be remarked for the definition IV.1.*

(a) *If $U = \{0\}$, J is a continuous function on the nonempty compact set $[L, u]_S$ for each $u \in S_{L'}$. Since any global maximum point of J on $[L, u]_S$ is indeed a local maximum point of J on $[L, u]_S$ as well, $P(u)$ is always nonempty;*

(b) *According to the definition, $P(u)$ contains no points of U except at points $v \in U \cap [L, u]_S$ where $\lim_{w \in [L,u]_S, w \to v} J(v) = +\infty$. Due to the monotone decreasing feature of our local minimax method, $J(p(\cdot))$ has a barrier at u. Thus the search of the algorithm will keep away from such points. We may simply exclude all those points and focus our discussion only on those $u \in S_{L'}$ with $P(u) \cap U = \emptyset$.*

Lemma IV.2 *For each $u_0 \in S_{L'}$, if p is a local peak selection of J at u_0 such that $p(u_0) \notin U$, then $\langle \nabla J(p(u_0)), u_i \rangle = 0,\ i = 0, 1, ..., n-1$.*

Proof. By the assumption, let $w = p(u_0) = \sum_{i=0}^{n-1} t_i u_i \in [L, u_0]_S \setminus U$ where $\sum_{i=0}^{n-1} t_i^2 = 1$. Thus $\nabla J(w)$ exists. For each $i = 0, 1, ..., n-1$, if $\langle \nabla J(w), u_i \rangle \neq 0$, we denote $w(s) = \frac{w + s u_i}{c(s)}$ where $c(s) = [(t_i + s)^2 + \sum_{k=0, k\neq i}^{n-1} t_k^2]^{\frac{1}{2}}$. Since $B \setminus U$ is open, $w \in B \setminus U$ and $w(s) \to w$ as $s \to 0$, there exists $s_0 > 0$ such that when $0 < |s| < s_0$, we have $w(s) \in [L, u_i]_S \setminus U$ and

$$J(w(s)) - J(w) = \langle \nabla J(w), \frac{s}{c(s)} u_i \rangle + o(\|w(s) - w\|),$$

where we have used the fact that $\langle \nabla J(w), w \rangle = 0$. Thus when $|s|$ is small, the term $\frac{s}{c(s)} \langle \nabla J(w), u_i \rangle$ dominates the difference of $J(w(s)) - J(w)$. Since this term can be made either positive or negative as we wish by properly selecting $s \neq 0$, it leads to a contradiction that w is a local maximum of J on $[L, u_0]_S$. Therefore $\langle \nabla J(w), u_i \rangle = 0$.

Lemma IV.3 *Let a local peak selection p of J be continuous at $\bar{u} \in S_{L'}$ with $\nabla J(p(\bar{u})) \neq 0$. When $s > 0$ is small and $\bar{u}(s) = \frac{\bar{u} + s w(\bar{u})}{\|\bar{u} + s w(\bar{u})\|}$, we have*

$$J(p(\bar{u}(s))) < J(p(\bar{u})) - \frac{1}{4} s\theta |t_n| \|\nabla J(p(\bar{u}))\|$$

where $w(\bar{u}) = -sign(t_n)\mathcal{P}(G(p(\bar{u})))$, $p(\bar{u}) = t_1 u_1 + \cdots + t_{n-1} u_{n-1} + t_n \bar{u}$ with $t_n \neq 0$ $\sum_{k=1}^{n} t_k^2 = 1$ and $G(p(\bar{u}))$ is a pseudo-gradient of J at $p(\bar{u})$ with constant $\theta \in (0, 1)$, i.e.,

$$\|G(p(\bar{u}))\| \leq 1, \quad \text{and} \quad \langle \nabla J(p(\bar{u})), G(p(\bar{u})) \rangle \geq \theta \|\nabla J(p(\bar{u}))\|.$$

Proof. Since $\langle \nabla J(w), w\rangle = 0,\ \forall w \in B \setminus U$ and p is continuous at $\bar{u}$ implies that $p(\bar{u}) \notin U$, when $s > 0$ is small, we have

$$J(p(\bar{u}(s))) = J(p(\bar{u})) + \langle \nabla J(p(\bar{u})), p(\bar{u}(s))\rangle + o(\|p(\bar{u}(s)) - p(\bar{u})\|).$$

On the other hand, by Lemma IV.2, as $s > 0$ is small,

$$\begin{aligned} \langle \nabla J(p(\bar{u})), p(\bar{u}(s))\rangle &= -\frac{sign(t_n)t_n(s)s}{\|\bar{u}+sw(\bar{u})\|}\langle \nabla J(p(\bar{u})), \mathcal{P}(G(p(\bar{u})))\rangle \\ &< -\frac{1}{2}s\theta|t_n|\|\nabla J(p(\bar{u}))\| \end{aligned}$$

where $p(\bar{u}(s)) \equiv p(\frac{\bar{u}+sw(\bar{u})}{\|\bar{u}+sw(\bar{u})\|}) = t_1(s)u_1 + \cdots + t_{n-1}(s)u_{n-1} + t_n(s)\frac{\bar{u}+sw(\bar{u})}{\|\bar{u}+sw(\bar{u})\|}$. Hence, when $s > 0$ is small,

$$J(p(\bar{u}(s))) < J(p(\bar{u})) - \frac{1}{4}s\theta|t_n|\|\nabla J(p(\bar{u}))\|$$

Remark IV.3 *Several points on Lemma IV.2 and IV.3 need to be remarked.*

(a) *The last inequality in the proof of Lemma IV.3 implies that if* $p(\bar{u}) \notin U$, *then* $p(\bar{u}(s)) \notin U$ *as well;*

(b) *From the last two lemmas, it is clear that the notion of a peak selection* $p(\bar{u})$ *can be generalized to satisfy* $\langle J(p(\bar{u})), \bar{u}\rangle = \langle J(p(\bar{u})), u_i\rangle = 0$, $i = 1, ..., n-1$.

As a direct consequence of Lemma IV.3, we have the following local minmax characterization of eigenpairs of (4.1).

Theorem IV.1 *Assume that a local peak selection* p *of* J *is continuous at* $\bar{u} \in S_{L'}$. *If* $J(p(\bar{u})) = \min_{u\in S_{L'}} J(p(u))$ *and* $d(p(\bar{u}), L) > 0$, *then* $p(\bar{u})$ *is a critical point of* J, *i.e.,* $p(\bar{u})$ *is an eigenfunction of (4.1) and* $\lambda = J(p(\bar{u}))$ *is the corresponding eigenvalue.*

2. Comparison with Other Characterizations

Theorem IV.1 serves as a local minmax characterization of eigenpairs to (4.1) under the iso-homogeneous condition. It states that when the first $n-1$ linearly independent eigenfunctions $u_1, u_2, ..., u_{n-1}$ are found this way, by setting $L = [u_1, ..., u_{n-1}]$ and $\mathcal{M} = \{p(u) | u \in S_{L'}\}$, the nth eigenfunction u_n can be found through finding a local minimum of J on $\mathcal{M}$ or solving a local minimax problem

$$\min_{u \in S_{L'}} \max_{v \in [u_1, ..., u_{n-1}, u]_S} J(v), \tag{4.5}$$

Theorem IV.2 *For the wellknown linear eigenpair problem, find* $(\lambda, u) \in \mathbb{R} \times (B - \{0\})$ *such that*

$$Fu = \lambda Gu, \tag{4.6}$$

where F *and* G *are two linear, self-adjoint operators and* G *is positive definite in a Hilbert space* B*, the local minimax method (4.5) is equivalent to the Rayleigh-Ritz method, i.e., by letting* $\langle u, v\rangle_G = \langle Gu, v\rangle$ *and* $\|u\|_G = (\langle u, v\rangle_G)^{\frac{1}{2}}$ *be the equivalent inner product and norm on* B*,* $L' = L^{\perp} = \{u \in B | \langle u, u_i\rangle_G = 0, i = 1, ..., n-1\}$ *and* $S_{L^{\perp}} = \{u \in L^{\perp} | \|u\|_G = 1\}$*, thus*

(1) *if* $u_n = \arg\min_{u \in S_{L^{\perp}}} \max_{v \in [u_1, ..., u_{n-1}, u]_S} J(v)$*, then* $u_n = \arg\min_{u \in S_{L^{\perp}}} J(u)$*;*

(2) *if* $u_n = \arg\min_{u \in S_{L^{\perp}}} J(u)$*, then* $u_n = \arg\min_{u \in S_{L^{\perp}}} \max_{v \in [u_1, ..., u_{n-1}, u]_S} J(v)$*.*

Proof. (1) It is known that $\langle u_i, u_j\rangle_G = 0$, $1 \le i < j \le n$. Thus we only have to find $v = p(u) \in L^{\perp}$. Then (4.5) reduces to the wellknown orthogonal method of Rayleigh-Ritz

$$\min_{u \in S_{L^{\perp}}} \max_{v \in [u]_S} J(v) = \min_{u \in S_{L^{\perp}}} J(u). \tag{4.7}$$

(2) if $u_n = \arg\min_{u \in S_{L^\perp}} J(u) = \arg\min_{u \in S_{L^\perp}} \langle Fu, u\rangle$. Then there is a neighborhood $\mathcal{N}(u_n)$ of u_n such that for all $u \in \mathcal{N}(u_n) \cap S_{L^\perp}$,

$$\max_{v \in [u_1,\ldots,u_{n-1},u]_S} \langle Fv, v\rangle \geq \langle Fu, u\rangle \geq \langle Fu_n, u_n\rangle.$$

On the other hand, $\forall u \in [u_1, \ldots, u_{n-1}, u_n]_S$, we have $u = \sum_{i=1}^n c_i u_i$ with $\sum_{i=1}^n c_i^2 = 1$. Then

$$\begin{aligned} J(u) &= \langle Fu, u\rangle = \sum_{i=1}^n \sum_{j=1}^n c_i c_j \langle Fu_i, u_j\rangle = \sum_{i=1}^n \sum_{j=1}^n c_i c_j \lambda_i \langle Gu_i, u_j\rangle \\ &= \sum_{i=1}^n c_i^2 \lambda_i \leq \lambda_n = J(u_n), \end{aligned}$$

where λ_i is the eigenvalue corresponding to u_i. Therefore

$$J(u_n) = \langle Fu_n, u_n\rangle = \min_{u \in S_{L^\perp}} \max_{v \in [u_1,\ldots,u_{n-1},u_n]_S} \langle Fv, v\rangle = \min_{u \in S_{L^\perp}} \max_{v \in [u_1,\ldots,u_{n-1},u_n]_S} J(v).$$

As for nonlinear eigenpair problems, the Courant-Fischer minimax principle states that u_n can be found through solving $\min_{W_n} \max_{v \in W_n \cap S} J(v)$ where the minimum is taken over *all* the subspaces W_n of dimension n in B and S is the unit sphere in B. The Courant-Fischer minimax principle is originally designed for linear eigenpair problems. People then found that it was also valid for nonlinear eigenpair problems where the homogeneous condition is satisfied. The Ljusternik-Schnirelman minimax principle which is commonly regarded as a generalization of the Courant-Fischer minimax principle, is used to characterize saddle points of a nonlinear functional J through solving $\inf_{K \in \mathcal{K}_n} \sup_{u \in K} J(u)$ where $\mathcal{K}_n$ is the class of all compact subsets K of B with $\text{ind}(K) \geq n$, sup is the global maximum of J on K and inf is the global minimum over $\mathcal{K}_n$. When the homogeneous condition is satisfied and J is the Rayleigh quotient, the Ljusternik-Schnirelman minimax principle coincides with the Courant-

Fischer minimax principle. But they are all two-level global minmax characterizations and do not provide much help in algorithm implementation. While our local minmax characterization in (4.5) can be implemented as the following numerical algorithm.

B. A Local Minimax Algorithm for Eigenpairs

Assume that $u_1, u_2, ..., u_{n-1}$ are previously found $n-1$ critical points of J with $\|u_i\| = 1,\ i = 1, 2, ..., n-1$. Let $L = [u_1, u_2, ..., u_{n-1}], B = L \oplus L'$ and $\mathcal{P} : B \to L'$ be the corresponding linear continuous projection operator. Given $\theta \in (0, 1)$ and $\lambda > 0$.

Step 1. Let $k = 1$. Choose $v^1 \in S_{L'}$ such that $p(v^1) \in [L, v^1]_S \setminus U$ where $u^1 = p(v^1) = \sum_{i=1}^{n-1} t_i^1 u_i + t_n^1 v^1$ is solved from

$$u^1 = \arg\max \Big\{ J(v) | v = \sum_{i=1}^{n-1} t_i u_i + t_n v^1, \sum_{i=1}^{n} t_i^2 = 1 \Big\}.$$

Step 2. Compute a descent direction $w^k = -sign(t_n^k)\mathcal{P}(G^k)$, where G^k is a pseudo-gradient of J at u^k with constant θ.

Step 3. If $\|\nabla J(p(v^k))\| \leq \epsilon$, then output $u^k = p(v^k)$, stop. Otherwise, do Step 4.

Step 4. Denote $v^k(s) = \dfrac{v^k + sw^k}{\|v^k + sw^k\|}$ and

$$u^k(s) = p(v^k(s)) = \arg\max \Big\{ J(v) | v = \sum_{i=1}^{n-1} t_i u_i + t_n v^k(s) \notin U, \sum_{i=1}^{n} t_i^2 = 1 \Big\},$$

where $(t_1^k, ..., t_n^k)$ is used as an initial point. Let

$$s^k = \max_{m \in N} \Big\{ s = \frac{\lambda}{2^m} | 2^m > \|w^k\|, J(u^k(\frac{\lambda}{2^m})) - J(u^k) \leq -\frac{s\theta |t_n^k|}{4} \|\nabla J(u^k)\| \Big\},$$

Step 5. Set $v^{k+1} = v^k(s^k), u^{k+1} = p(v^{k+1}) \equiv \sum_{i=1}^{n-1} t_i^{k+1} u_i + t_n^{k+1} v^{k+1}$ and $k = k + 1$. Go to Step 2.

Remark IV.4 *About the algorithm, we need point out the followings.*

(a) *If $p(v_n^1) \in [L, v_n^1] \setminus U$ is satisfied, then $p(v_n^k) \in [L, v_n^k] \setminus U$ for all $k = 1, 2, \ldots.$*

(b) *In Step 3, we can either following a pseudo-gradient flow or just find a pseudo-gradient at the current point. The projection is important to avoid the degeneracy. For computation of a pseudo-gradient or a pseudo-gradient flow in $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ spaces, see Chapter II and Chapter III.*

(c) *It is easy to check that $\frac{1}{2}s_k \leq s^k \leq s_k$ where s_k is the step-size defined by s_k*

$$= \max_{0<s\leq\lambda}\{s|J(u^k(s)) - J(u^k) \leq -\frac{1}{4}s\theta t_n^k\|\nabla J(u^k)\|, \lambda \geq s\|\nabla J(u^k)\| > 0\}. \quad (4.8)$$

(d) *A computational technique can be used to find a pseudo-gradient, Chapter II. In this case w^k in Step 2 should be $w^k = -sign(t_n^k)\nabla J(u^k)$ and the inequality to decide s^k in Step 4 should be*

$$J(p(v^k(\frac{\lambda}{2^m}))) - J(u^k) \leq -\frac{t_n^k}{4}\frac{\lambda}{2^m}\|\nabla J(p(v^k))\|_2^2.$$

In fact, the expression of $w(\bar{u})$ in Lemma IV.3 should be $w(\bar{u}) = -sign(t_n)\nabla J(\bar{u})$ and the inequality should be $J(p(\bar{u}(s))) < J(p(\bar{u})) - \frac{1}{4}s|t_n|\|\nabla J(p(\bar{u}))\|_2^2$ under the assumption $\|\nabla J(p(\bar{u}))\|_2 < +\infty$, where $\|\cdot\|_2$ is the norm of $W^{1,2}(\Omega)$.

(e) *The algorithm is stable in the sense $J(u^{k+1}) < J(u^k)$.*

C. Numerical Experiment to Eigenpairs of p-Laplacian

In this section we carry out several numerical experiments to find the (weighted) eigenpairs of the p-Laplacian operator on the domain $\Omega = [0, 2] \times [0, 2]$. The weight function is either $w(x) \equiv 1$ or $w(x) = |x - \bar{1}|^q$, where $\bar{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. In Section 2, the Rayleigh quotient J has been defined.

To compute the gradient $d = \nabla J(u)$ at u, for each $v \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\langle d, v\rangle &= -\int_\Omega \Delta d(x) v(x)\,dx \equiv \frac{d}{dt}|_{t=0} J(u+tv)\\ &= \frac{p}{b^2}\int_\Omega (-b\Delta_p u(x) - aw(x)|u(x)|^{p-2}u(x))v(x)\,dx\end{aligned}$$

which leads to solve a linear Poisson problem

$$\begin{cases}\Delta d(x) = \frac{p}{b^2}(b\Delta_p u(x) + aw(x)|u(x)|^{p-2}u(x)), & x \in \Omega\\ d|_{\partial\Omega} = 0\end{cases}$$

where $a = \int_\Omega |\nabla u(x)|^p dx$ and $b = \int_\Omega w(x)|u(x)|^p dx$. Then by using $\nabla J(u)$, we can follow the practical techniques developed in Chapter II and Chapter III to find a pseudo-gradient.

In our numerical computations, 800×800 or 1000×1000 linear square elements are used. Next for each case, the profiles of the first seven numerically computed eigenfunctions and their eigenvalues λ_i for $w(x) = 1$ and the first five for $w(x) = |x - \bar{1}|^q$ are displayed. The profiles of eigenfunctions are presented as follows, Fig.19-39.

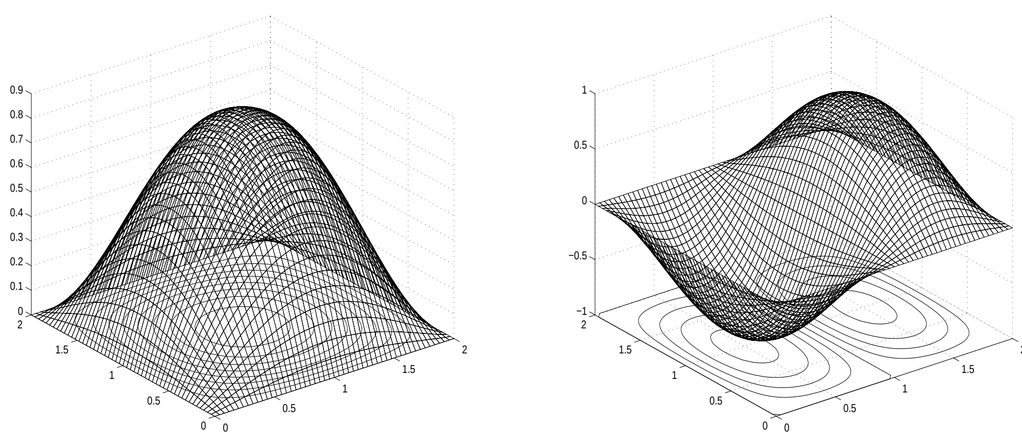

Fig. 19. Eigenfunctions of Δ_p, p=1.75. $\lambda_1 = 4.245837$ (left) and $\lambda_2 = 9.317313$ (right).

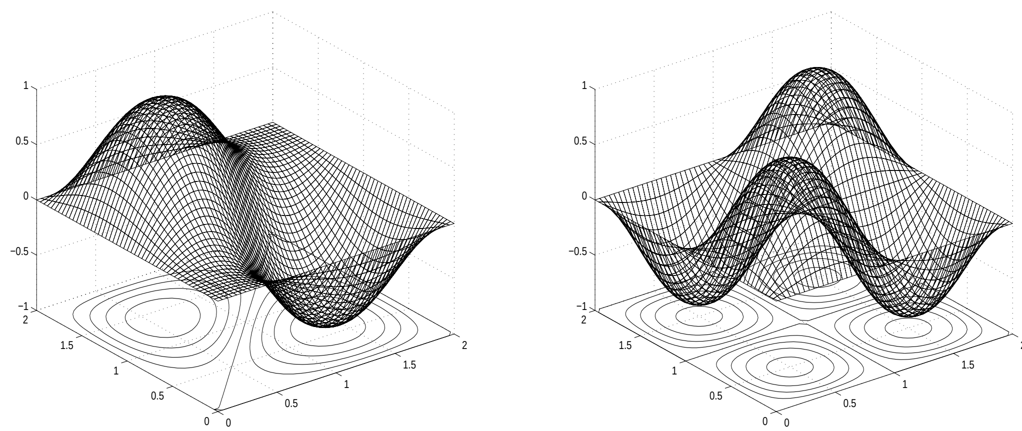

Fig. 20. Eigenfunctions of Δ_p, p=1.75. $\lambda_3 = 9.407816$ (left) and $\lambda_4 = 14.280496$ (right).

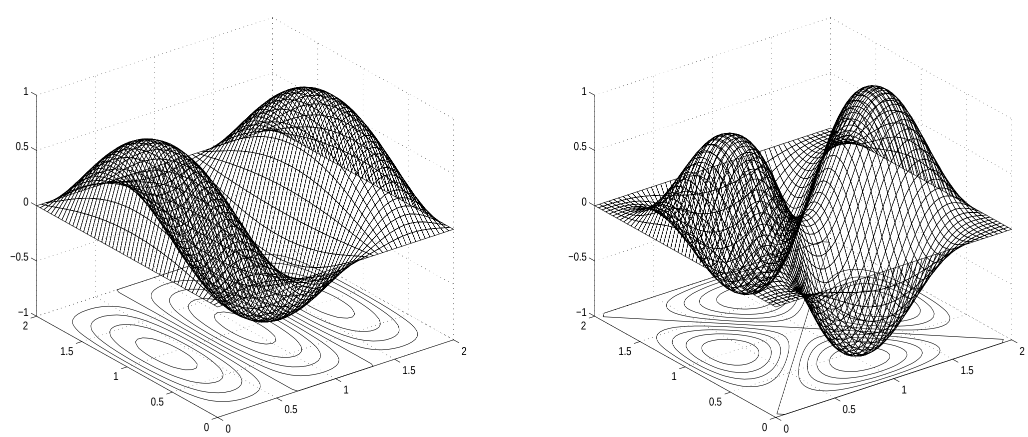

Fig. 21. Eigenfunctions of Δ_p, p=1.75. $\lambda_5 = 16.837822$ (left) and $\lambda_6 = 17.254568$ (right).

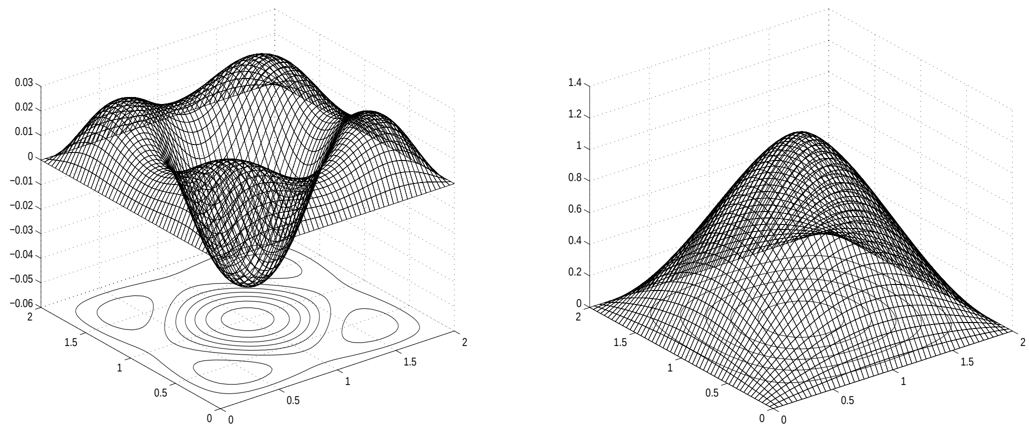

Fig. 22. Eigenfunction of Δ_p, p=1.75. $\lambda_7 = 23.366003$ (left) and eigenfunction of Δ_p, p=2.5. $\lambda_1 = 20.798476$ (right).

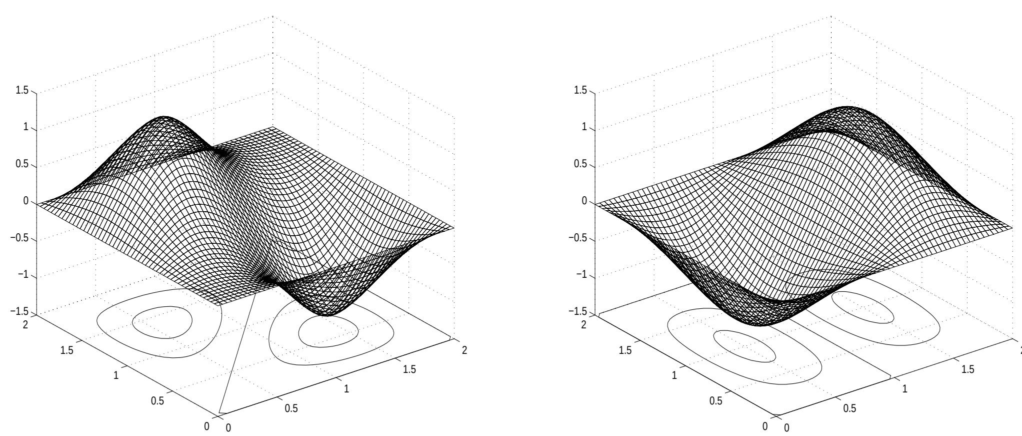

Fig. 23. Eigenfunctions of Δ_p, p=2.5. $\lambda_2 = 20.289627$ (left) and $\lambda_3 = 20.798476$ (right).

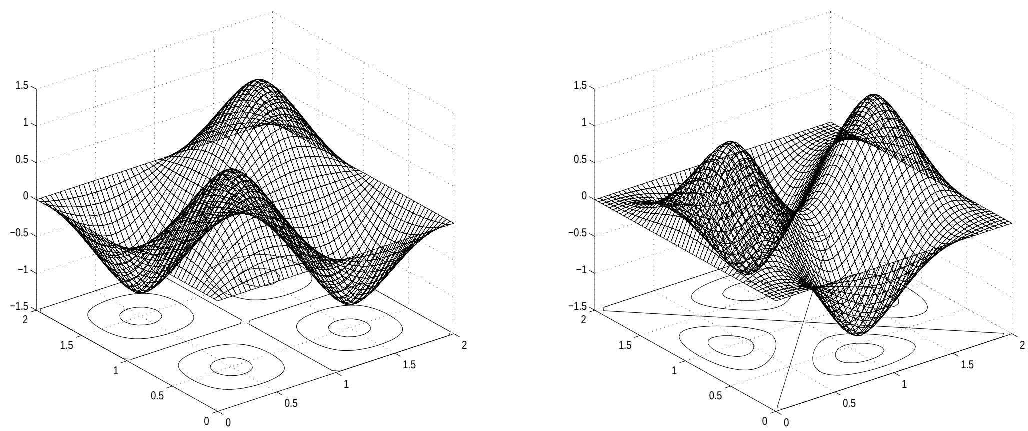

Fig. 24. Eigenfunctions of Δ_p, p=2.5. $\lambda_4 = 35.944786$ (left) and $\lambda_5 = 48.259806$ (right).

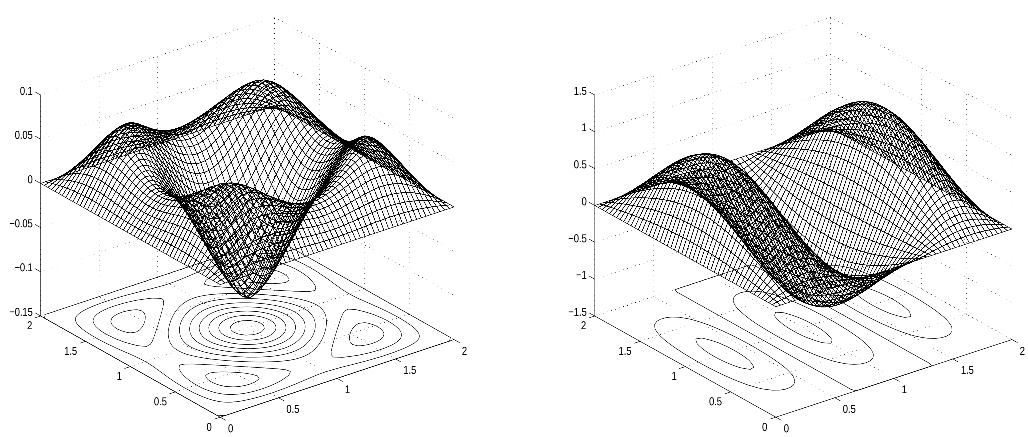

Fig. 25. Eigenfunctions of Δ_p, p=2.5. $\lambda_6 = 49.679394$ (left) and $\lambda_7 = 51.104811$ (right).

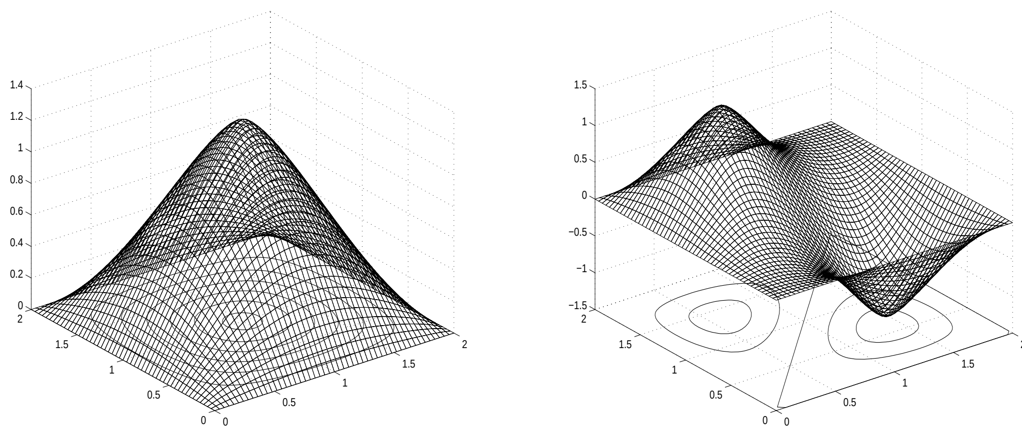

Fig. 26. Eigenfunctions of Δ_p, p=3.0. $\lambda_1 = 7.844420$ (left) and $\lambda_2 = 32.098661$ (right).

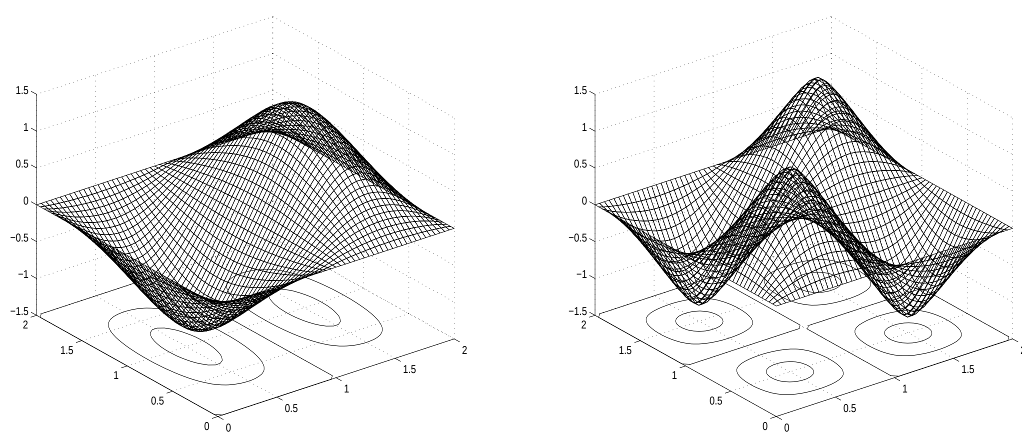

Fig. 27. Eigenfunctions of Δ_p, p=3.0. $\lambda_3 = 33.947805$ (left) and $\lambda_4 = 62.748593$ (right).

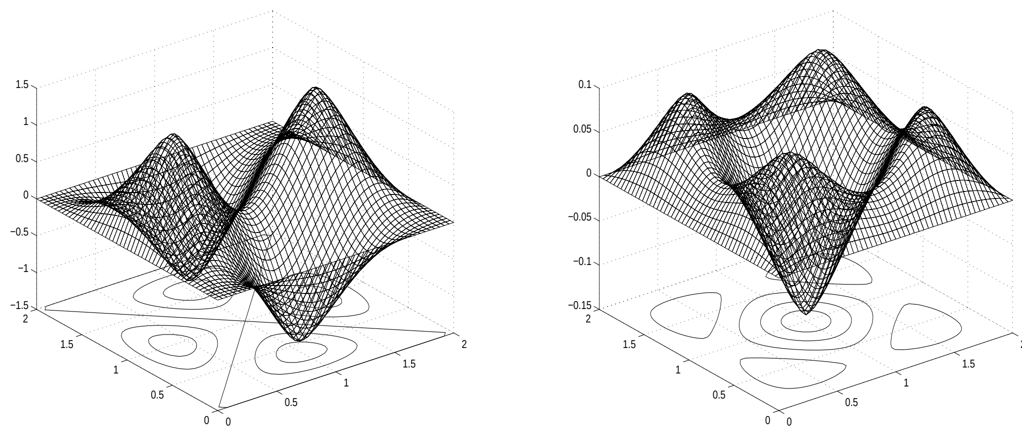

Fig. 28. Eigenfunction of Δ_p, p=3.0. $\lambda_5 = 90.795294$ (left) and $\lambda_6 = 94.932100$ (right).

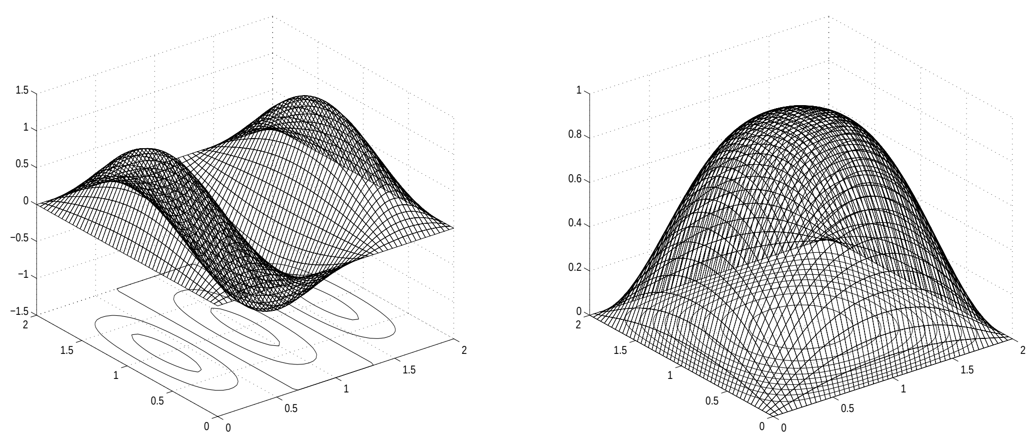

Fig. 29. Eigenfunction of Δ_p, p=3.0. $\lambda_7 = 102.660394$ (left) and weighted eigenfunction of Δ_p p=1.75, q=0.5. $\lambda_1 = 6.088006$ (right).

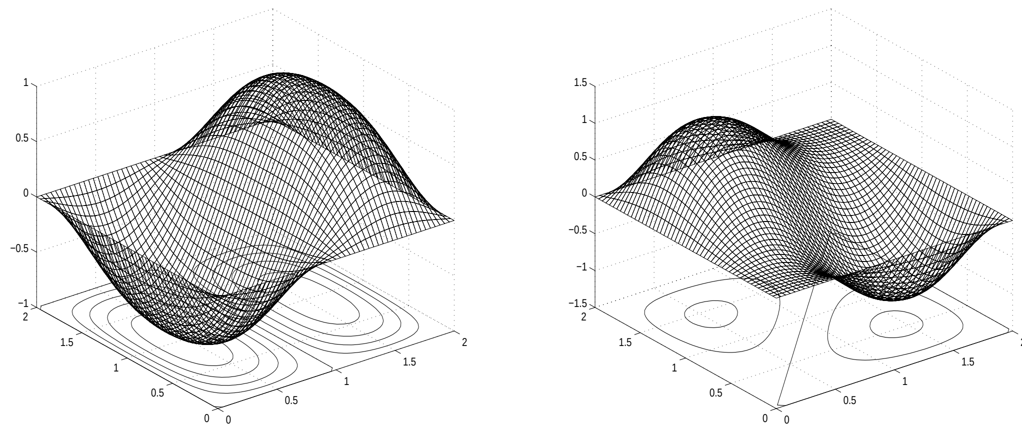

Fig. 30. Weighted eigenfunctions of Δ_p, p=1.75, q=0.5. $\lambda_2 = 11.775095$ (left) and $\lambda_3 = 11.938270$ (right).

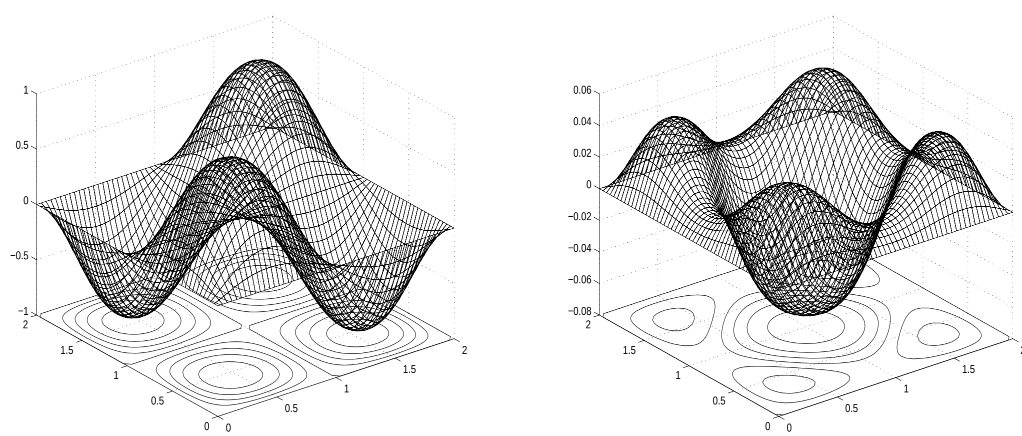

Fig. 31. Weighted eigenfunctions of Δ_p, p=1.75, q=0.5. $\lambda_4 = 16.633820$ (left) and $\lambda_5 = 23.366003$ (right).

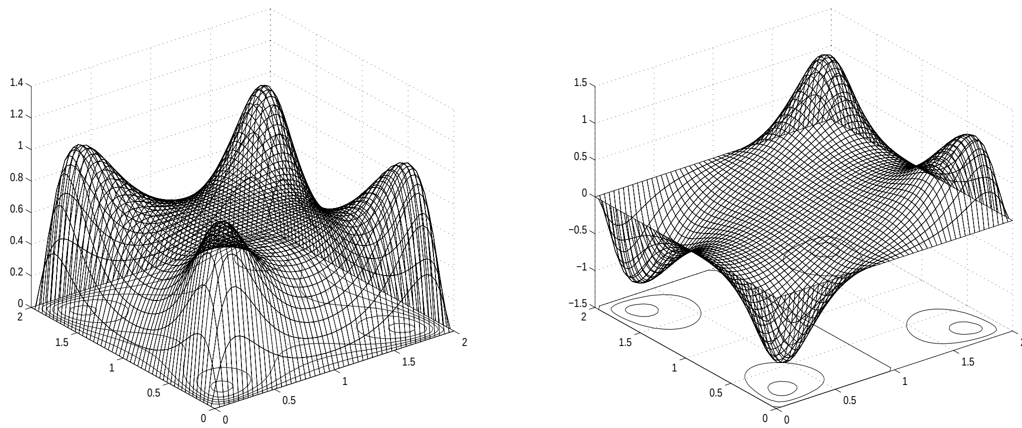

Fig. 32. Weighted eigenfunctions of Δ_p, p=1.75, q=6.0. $\lambda_1 = 18.714875$ (left) and $\lambda_2 = 20.312840$ (right).

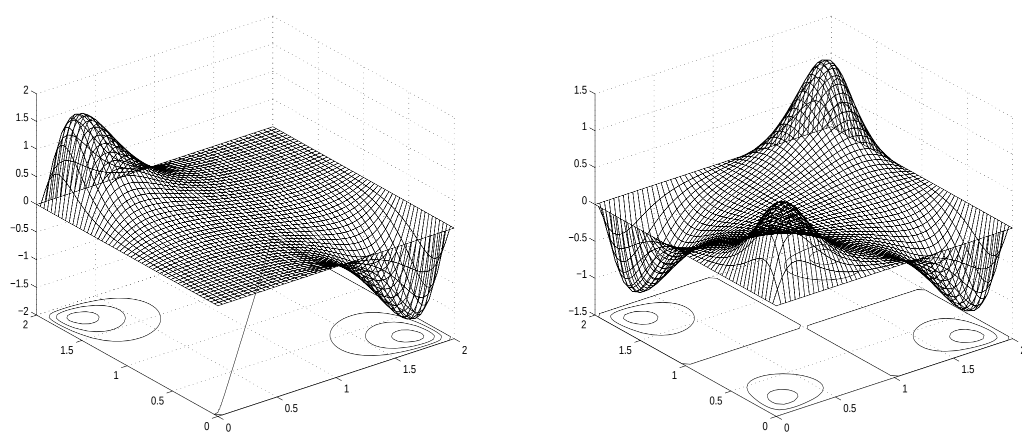

Fig. 33. Weighted eigenfunctions of Δ_p, p=1.75, q=6.0. $\lambda_3 = 20.425545$ (left) and $\lambda_4 = 20.738396$ (right).

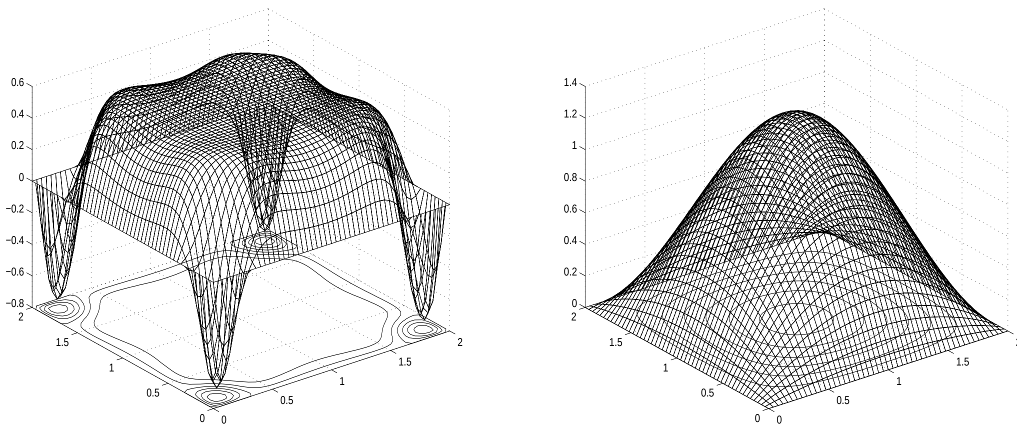

Fig. 34. Weighted eigenfunctions of Δ_p, p=1.75, q=6.0. $\lambda_5 = 34.801623$ (left) and p=2.5, q=0.5. $\lambda_1 = 10.185286$ (right).

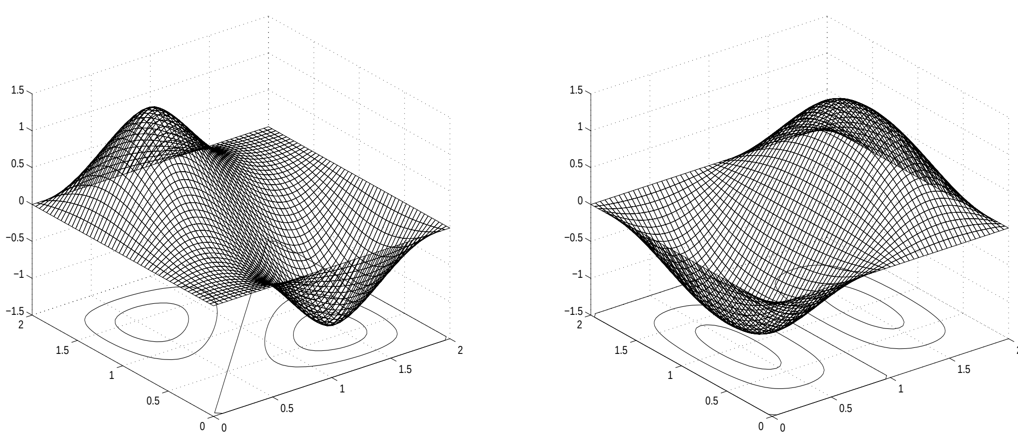

Fig. 35. Weighted eigenfunctions of Δ_p, p=2.5, q=0.5. $\lambda_2 = 26.174362$ (left) and $\lambda_3 = 26.991732$ (right).

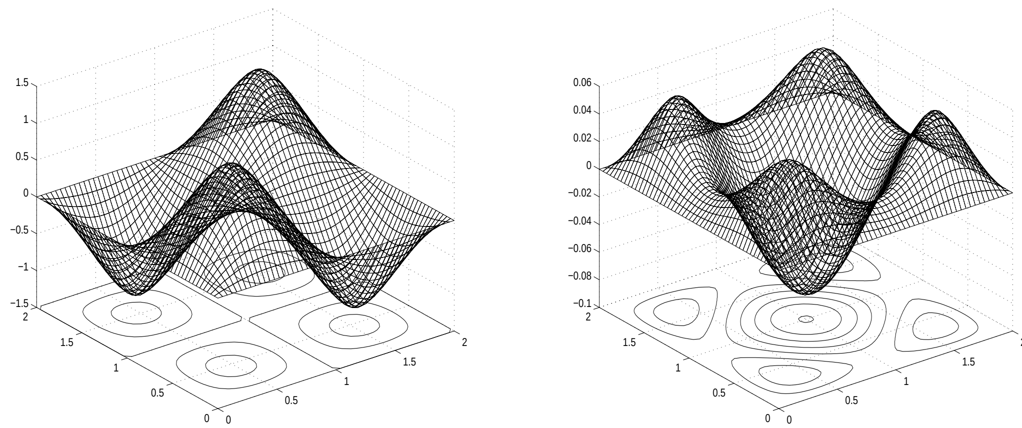

Fig. 36. Weighted eigenfunctions of Δ_p, p=2.5, q=0.5. $\lambda_4 = 42.140740$ (left) and $\lambda_5 = 69.931326$ (right).

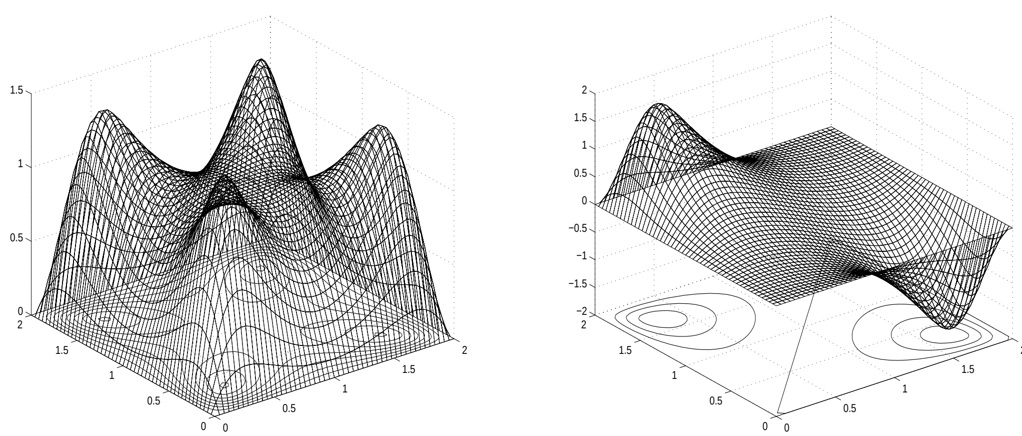

Fig. 37. Weighted eigenfunctions of Δ_p, p=2.5, q=6.0. $\lambda_1 = 65.223275$ (left) and $\lambda_2 = 70.878805$ (right).

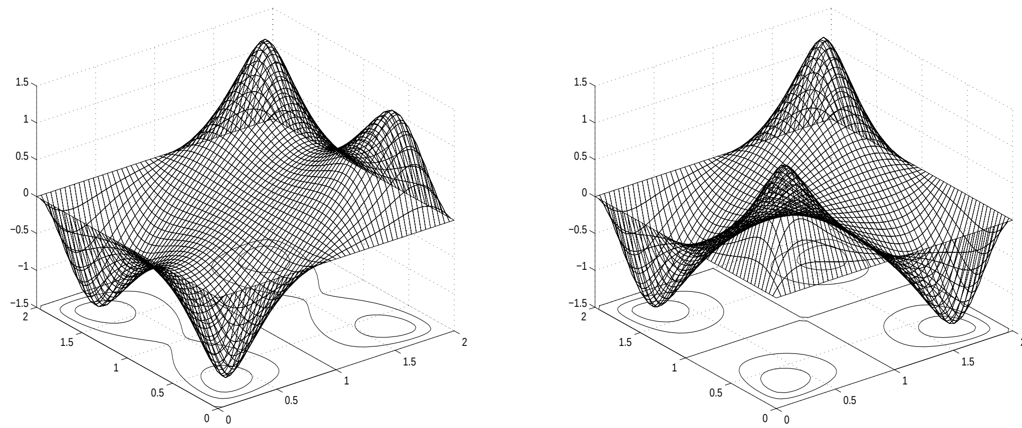

Fig. 38. Weighted eigenfunctions of Δ_p, p=2.5, q=6.0. $\lambda_3 = 71.815461$ (left) and $\lambda_4 = 74.271235$ (right).

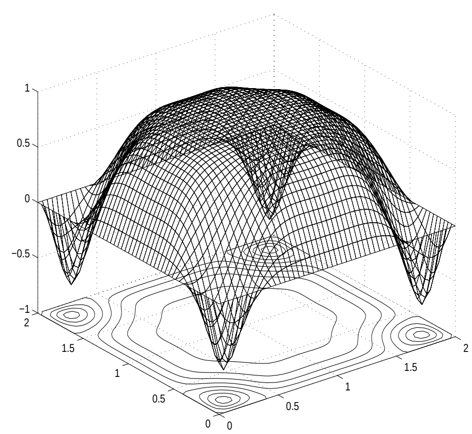

Fig. 39. Weighted eigenfunctions of Δ_p, p=2.5, q=6.0. $\lambda_5 = 161.729721$.

Several interesting phenomena related to the (weighted) eigenpairs of Δ_p on the square domains have been observed in our numerical experiments.

(a) By comparing Fig. 19 (left), Fig. 22 (right) and Fig. 26 (left), Fig. 19 (right), Fig. 23 (right) and Fig. 27 (left), Fig. 20 (left), Fig. 23 (left) and Fig. 26 (right), Fig. 21 (left), Fig. 25 (right) and Fig. 29 (left), Fig. 21 (right), Fig. 24 (right) and Fig. 28 (left), Fig. 20 (right), Fig. 24 (left) and Fig. 27 (right), Fig. 22 (left), Fig. 25 (left) and Fig. 28 (right), we observe that for different values of p, the eigenfunctions in the same group have the same number of peaks, their locations are also quite similar and peaks become sharper when p becomes larger.

(b) By comparing Fig. 19 (right), Fig. 23 (right) and Fig. 27 (left) (side-to-side peaks), Fig. 20 (left), Fig. 23 (left) and Fig. 26 (right) (corner-to-corner peaks), we found that when p crosses 2, the sequential order of the eigenvalues of the eigenfunctions with side-to-side peaks and the eigenfunctions with corner-to-corner peaks switches. Numerical computation shows that when $p = 2$, the eigenvalues of the eigenfunctions with side-to-side peaks and the eigenfunctions with corner-to-corner peaks are the same. By comparing Fig. 21 (left), Fig. 25 (right) and Fig. 29 (left) (3-peak), Fig. 21 (right), Fig. 24 (right) and Fig. 28 (left) (4-peak), Fig. 22 (left), Fig. 25 (left) and Fig. 28 (right) (5-peak), the sequential order of the eigenvalues of the eigenfunctions with 3-peak, 4-peak, 5-peak changes when p crosses 2. Numerical computation shows when $p = 2$, it seems their eigenvalues are same.

(c) If we pay attention to the peak locations and compare Fig. 29 (right) and Fig.32 (left), Fig. 30 (left) and Fig.32 (right), Fig. 31 (right) and Fig. 34 (left), Fig. 34 (right) and Fig. 37 (left), Fig. 35 (right) and Fig. 38 (left), Fig. 36 (right) and Fig. 39, we can see that the peaks prefer the corners when q increases and crosses some number.

(d) To the weighted eigenpair problem of the p-Laplacian, there is a corresponding

p-Henon equation

$$-\Delta_p u + |x - \vec{1}|^r |u|^q u = 0,\ x \in \Omega, \quad u|_{\partial\Omega} = 0.$$

Usually, $q > p-2$. By the numerical experiments in Chapter II, the symmetry of the ground state will be broken when the difference $r - q$ becomes large. This interesting case is called a symmetry breaking phenomenon. But by our numerical experiments in this chapter, it seems that for the weighted eigenfunction problems, the symmetry breaking phenomenon never took place.

CHAPTER V

CONVERGENCE OF THE 2ND MINIMAX ALGORITHM

Let $L = [u_1, u_2, ..., u_{n-1}]$ be the space spanned by given linearly independent $u_i \in B$, $i = 1, ..., n-1$, $B = L \oplus L'$ and $\mathcal{P} : B \to L'$ the corresponding linear projection operator. Let $S_{L'}$ be the unit sphere in $L' \setminus U$ where $U = \{u \in B | G(u) = 0\}$. For each $u \in S_{L'}$ denote $[L, u]_S = \{w = \sum_{k=1}^{n-1} t_k u_k + t_0 u | \sum_{k=1}^{n-1} t_k^2 + t_0^2 = 1\}$.

Definition V.1 *A set-valued mapping $P : S_{L'} \to 2^B$ is the L-$\perp$ mapping of J if $\forall v \in S_{L'}, P(v) = \{u \in [L, v]_S | \sum_{i=1}^{n} t_i^2 = 1, \langle \nabla J(u), u_i \rangle = 0, i = 1, .., n-1\}$. A single-valued mapping $p : S_{L'} \to B$ is an L-$\perp$ selection of J if $p(v) \in P(v)$, $\forall v \in S_{L'}$. For a given $v \in S_{L'}$, we say that J has a local L-$\perp$selection at v if there is a neighborhood $\mathcal{N}(v)$ of v and $p : \mathcal{N}(v) \cap S_{L'} \to B$ such that $p(u) \in P(u)$, $\forall u \in \mathcal{N}(v) \cap S_{L'}$.*

By L-$\perp$ selection, we have the following lemma and theorem which generalize Lemma IV.3 and Theorem IV.1.

Lemma V.1 *Given an L-$\perp$ peak selection p of J which is continuous at $\bar{u} \in S_{L'}$ with $\nabla J(p(\bar{u})) \neq 0$. When $s > 0$ is small and $\bar{u}(s) = \frac{\bar{u}+sw(\bar{u})}{\|\bar{u}+sw(\bar{u})\|}$, we have*

$$J(p(\bar{u}(s))) < J(p(\bar{u})) - \frac{1}{4} s\theta |t_n| \|\nabla J(p(\bar{u}))\|$$

where $w(\bar{u}) = -sign(t_n)\mathcal{P}(\Psi(p(\bar{u})))$, $p(\bar{u}) = t_1 u_1 + \cdots + t_{n-1} u_{n-1} + t_n \bar{u}$ with $t_n \neq 0$, $\sum_{k=1}^{n} t_k^2 = 1$ and $\Psi(p(\bar{u}))$ is a pseudo-gradient of J at $p(\bar{u})$ with constant $\theta \in (0, 1)$.

Theorem V.1 *Assume that an L-$\perp$ peak selection p of J is continuous at $\bar{u} \in S_{L'}$. If $J(p(\bar{u})) = \min_{u \in S_{L'}} J(p(u))$ and $d(p(\bar{u}), L) > 0$, then $p(\bar{u})$ is a critical point of J, i.e., $p(\bar{u})$ is an eigenfunction of (4.1) and $\lambda = J(p(\bar{u}))$ is the corresponding eigenvalue.*

A. A Min-Orthogonal Algorithm

By Lemma V.1 and Theorem V.1, a min-orthogonal algorithm can be designed through replacing a peak selection in the minimax algorithm by an L-$\perp$ selection. The flow chart reads as follows.

Assume that $L = [u^1, u^2, ..., u^{n-1}]$ where $u^1,\ u^2,\ ...,\ u^{n-1}$ are $n-1$ previously found eigenfunctions. For given $\lambda, \varepsilon > 0$ and $\theta \in (0,1)$. Let $B = L \oplus L'$.

Step 1: Let $v_1 \in S_{L'}$ be an ascent-descent direction at u^{n-1}.

Step 2: Set $k = 1$. Solve for $u_k \equiv p(v_k) \equiv t_1^k u^1 + \cdots + t_{n-1}^k u^{n-1} + t_n^k v_k$ such that $t_n^k \neq 0$,

$$\langle \nabla J(t_1^k u_1 + \cdots + t_{n-1}^k u_{n-1} + t_n^k v_k), u_i \rangle = 0,\ i = 1, .., n-1,\ \sum_{i=1}^{n} (t_i^k)^2 = 1.$$

Step 3: Find a descent direction w_k of $J(p(\cdot))$ at v_k.

Step 4: If $\|\nabla J(u_k)\| \leq \varepsilon$, then output $u_k = p(v_k)$, stop. Otherwise, do Step 5.

Step 5: For each $s > 0$, denote $v_k(s) = \dfrac{v_k + s w_k}{\|v_k + s w_k\|}$ and set $v_{k+1} = v_k(s_k)$ where

$$s_k = \max\{\frac{\lambda}{2^m} | m \in N, 2^m > \|w_k\|, J(p(v_k(\frac{\lambda}{2^m}))) - J(u_k) < -\frac{\theta |t_n^k|}{4} (\frac{\lambda}{2^m}) \|\nabla J(u_k)\|\}.$$

Step 6: Update $k = k + 1$ and go to Step 3.

To establish some convergence results of the algorithm, for simplicity we assume that $U = \{0\}$ and the following version of the PS condition is needed.

Definition V.2 *Given $u_i \in B$ with $\|u_i\| = 1$, $i = 1, ..., n-1$. A functional $J \in C^1(B, \mathbb{R})$ is said to satisfy the Palais-Smale (PS) condition if any sequence $\{v_i\} \subseteq B$ with $\|v_i\| = 1$ satisfies $J(w_i)$ is bounded and $\nabla J(w_i) \to 0$, where $w_i = \sum_{j=1}^{n-1} t_j(v_i) u_j + t_n(v_i) v_i$, $\sum_{j=1}^{n} (t_j(v_i))^2 = 1$, then $\{w_i\}$ possesses a convergent subsequence.*

B. Statement of Convergence Results

Similar convergence results as in Chapter III can be established through some modifications in the corresponding proofs. We only state our conclusions and omit all proofs.

1. A Unified Convergence Result

Theorem V.2 *Let $V \subset B$ be open and $W = V \cap S_{L'} \neq \emptyset$. Assume that $J \in C^1(B, \mathbb{R})$ satisfies the PS condition,*

(1) *p is a continuous L-$\perp$ selection of J in $\bar{W}$, where $\bar{W}$ is the closure of W on $S_{L'}$,*

(2) $\inf_{v \in W} d(p(v), L) > \alpha > 0$,

(3) $\inf_{v \in \partial\bar{W}} J(p(v)) > c = \inf_{v \in W} J(p(v)) > -\infty$, *where $\partial\bar{W}$ is the boundary of $\bar{W}$ on $S_{L'}$.*

Then, $K_c^p = p(W) \cap K_c \neq \emptyset$ and for any $\{v_k\} \subset W$ with $J(u_k) \to c$ where $u_k = p(v_k)$,

(a) $\forall \varepsilon > 0$, *there is $\bar{k} > 0$ such that* $d(K_c^p, u_k) < \varepsilon$, $\forall k > \bar{k}$;

(b) *If in addition, $\nabla J(p(\cdot))$ is Lipschitz continuous in W, then there is a constant C such that $\|\nabla J(u_k)\| \leq C(J(u_k) - c)^{\frac{1}{2}}$.*

Corollary V.1 *Let $J \in C^1(B, \mathbb{R})$ satisfy the PS condition, V_1 and V_2 be open in L' with $\emptyset \neq W_2 \equiv V_2 \cap S_{L'} \subset V_1 \cap S_{L'} \equiv W_1$. Assume p is a continuous L-$\perp$ selection of J in W_1 with*

(1) $\inf_{v \in W_1} d(p(v), L) > \alpha > 0, c = \inf_{v \in W_1} J(p(v)) > -\infty$ *and* $K_c^p = p(W_1) \cap K \subset K_c$

(2) *there is $d > 0$ with*

$$\inf\{J(p(v)) | v \in W_1, d(v, \partial W_1) \leq d\} = a > b = sup\{J(p(v)) | v \in W_2\},$$

(3) *given* $\{v_k\}$ *such that* $v_1 \in W_2$, $\|v_{k+1} - v_k\| < d$, $J(u_{k+1}) < J(u_k)$ *and* $\{u_k\}$ *has a subsequence that converges to a critical point* u_0, *where* $u_k = p(v_k)$. *Then*

(a) $\forall \varepsilon > 0$, *there is* $\bar{k} > 0$ *such that* $\quad d(K_c^p, u_k) < \varepsilon, \quad \forall k > \bar{k}$;

(b) *If in addition,* $\nabla J(p(\cdot))$ *is Lipschitz continuous in* W_1, *then there is a constant* C *such that* $\|\nabla J(u_k)\| \le C(J(u_k) - c)^{\frac{1}{2}}$.

2. Subsequence Convergence

Theorem V.3 *Let* $J \in C^1(B, \mathbb{R})$ *satisfy the PS condition and* p *be an* L-$\perp$ *selection of* J *such that (1)* p *is continuous on* $S_{L'}$, *(2)* $\inf_{1 \le k < \infty} d(p(v_k), L) \ge \alpha > 0$, *(3)* $\inf_{1 \le k < \infty} J(p(v_k)) > -\infty$ *and (4)* $w_k = -sign(t_n^k)\mathcal{P}(\Psi(u_k))$ *in Step 3 of the algorithm, where* $\Psi(\cdot)$ *is a pseudo-gradient flow with the constant* $\theta \in (0, 1]$ *and* $\mathcal{P} : B \to L'$ *is the linear projection operator, then*

(a) $\{v_k\}_{k=1}^{\infty}$ *has a subsequence* $\{v_{k_i}\}$ *such that* $u_{k_i} = p(v_{k_i})$ *converges to a critical point of* J;

(b) *if a subsequence* $u_{k_i} \to u_0$ *as* $i \to \infty$, *then* $u_0 = p(v_0)$ *is a critical point of* J.

Theorem V.4 *Let* $J \in C^1(B, \mathbb{R})$ *satisfy the PS condition and* p *be an* L-$\perp$ *selection of* J *such that (1)* p *is locally Lipschitz continuous on* $S_{L'}$, *(2)* $\inf_{1 \le k < \infty} d(p(v_k), L) \ge \alpha > 0$, *(3)* $\inf_{1 \le k < \infty} J(p(v_k)) > -\infty$ *and (4)* $w_k = -sign(t_n^k)\mathcal{P}(\Psi(u_k))$ *in Step 3 of the algorithm, where* $\Psi(u_k)$ *is a pseudo-gradient at* u_k *with constant* $\theta \in (0, 1]$ *and* $\mathcal{P} : B \to L'$ *is the linear projection operator, then*

(a) $\{v_k\}_{k=1}^{\infty}$ *has a subsequence* $\{v_{k_i}\}$ *such that* $u_{k_i} = p(v_{k_i})$ *converges to a critical point of* J;

(b) *if a subsequence* $u_{k_i} \to u_0$ *as* $i \to \infty$, *then* $u_0 = p(v_0)$ *is a critical point of* J.

When $B = W_0^{1,q}(\Omega)$ $(q > 1)$, a practical technique mentioned in Chapter III can be used in the min-orthogonal algorithm. Set $w_k = -sign(t_n^k)\nabla J(u_k)$ and compute

$$\gamma_k = \frac{\|\nabla J(u_k)\|_2^2}{\|\nabla J(u_k)\|_q \|\nabla J(u_k)\|_r}$$

in Step 3 of the algorithm where $\frac{1}{q} + \frac{1}{r} = 1$ and modify the stepsize rule in Step 5 as

$$s_k = \max\{\frac{\lambda}{2^m} | m \in N, 2^m > \|w_k\|, J(p(v_k(\frac{\lambda}{2^m}))) - J(u_k) < -\frac{|t_n^k|}{4}(\frac{\lambda}{2^m})\|\nabla J(u_k)\|_2^2\}.$$

For $1 < q < 2$, L' is the $\|\cdot\|_q$-norm closure of $L^\perp$ in B where $L^\perp = \{v \in W_0^{1,r}(\Omega) | \int_\Omega vudx = 0, \ \forall u \in L\}$ and for $q \geq 2$, $L' = L^\perp = \{u \in B | \int_\Omega uvdx = 0, \ \forall v \in L\}$. Then we have the following two subsequence convergence results.

Theorem V.5 *Assume that $J \in C^1(B, \mathbb{R})$ satisfies the PS condition, $q \in (0, 1]$ and p is an L-$\perp$ selection of J such that (1) p is continuous on $S_{L'}$, (2) $\inf_{1 \leq k < \infty} d(p(v_k), L) \geq \alpha > 0$, (3) $\inf_{1 \leq k < \infty} J(p(v_k)) > -\infty$ and (4) $\gamma_k \geq \gamma_{min} > 0$ where γ_{min} is a constant, then*

- **(a)** *$\{v_k\}_{k=1}^\infty$ has a subsequence $\{v_{k_i}\}$ such that $u_{k_i} = p(v_{k_i})$ converges to a critical point of J;*
- **(b)** *if a subsequence $u_{k_i} \to u_0$ as $i \to \infty$, then $u_0 = p(v_0)$ is a critical point of J.*

Theorem V.6 *Assume that $q > 2$, $J \in C^1(B, \mathbb{R})$ satisfies the PS condition and p is an L-$\perp$ selection of J such that (1) p is locally Lipschitz continuous on $S_{L'}$,(2) $\inf_{1 \leq k < \infty} d(p(v_k), L) \geq \alpha > 0$, (3) $\inf_{1 \leq k < \infty} J(p(v_k)) > -\infty$ and (4) $\|\nabla J(u_k)\|_q \leq M$, $\forall k$, where $M > 0$ is a constant or (4)' $\gamma_k \geq \gamma_{min} > 0$, $\forall k$, where γ_{min} is a constant, then*

- **(a)** *$\{v_k\}_{k=1}^\infty$ has a subsequence $\{v_{k_i}\}$ such that $u_{k_i} = p(v_{k_i})$ converges to a critical point of J;*

(b) *if a subsequence* $u_{k_i} \to u_0$ *as* $i \to \infty$*, then* $u_0 = p(v_0)$ *is a critical point of* J*.*

C. On the Smoothness of L-$\perp$ Selection

Since the continuity or smoothness of an L-$\perp$ selection is important for our algorithm design and convergence analysis, the following method can be used to check whether or not p is continuously differentiable. According to the definition of an L-$\perp$ selection, when $L = [u_1, u_2, ..., u_{n-1}]$, $p(v) = t_1 u_1 + \cdots + t_{n-1} u_{n-1} + t_n v$, where $\sum_{i=1}^n t_i^2 = 1$ and $v \in S_{L'}$, is solved from

$$\langle \nabla J(t_1 u_1 + \cdots + t_{n-1} u_{n-1} + t_n v), u_i \rangle \quad = \quad 0, \quad i = 1, ..., n \text{ and } \sum_{i=1}^n t_i^2 = 1. \quad (5.1)$$

To apply the implicit function theorem to (5.1), we need to resolve the problem caused by the normalization condition $\sum_{i=1}^n t_i^2 = 1$ which prevents us from taking derivative w.r.t. t_i. Since the right hand side of (5.1) contains all zeros, by the homogeneous condition, the normalized condition $\sum_{i=1}^n t_i^2 = 1$ can always be satisfied afterward through dividing each t_i by $(\sum_{i=1}^n t_i^2)^{\frac{1}{2}}$. Thus this condition can be released. Then for given $v \in S_{L'}$, the system (5.1) contains n unknown $t_1, ..., t_n$ but $n-1$ equations. To obtain a square Hessian matrix of (5.1) and keep in mind of the nondegeneracy condition $d(p(v), L) > 0$ in our local minimax characterization, Theorem IV.1, let us consider solving

$$\langle \nabla J(t_1 u_1 + \cdots + t_{n-1} u_{n-1} + v), u_i \rangle = 0, \quad i = 1, ..., n-1. \quad (5.2)$$

That is, we force $t_n = 1$ in (5.1). The implicit function theorem states that if $u = p(v) = t_1u_1 + \cdots + t_{n-1}u_{n-1} + v$ satisfies (5.2) and at u, the $(n-1)\times(n-1)$ matrix

$$Q = \begin{bmatrix} \langle J''(u)u_1, u_1\rangle & \cdots & \langle J''(u)u_{n-1}, u_{n-1}\rangle \\ \cdots & \cdots & \\ \langle J''(u)u_1, u_{n-1}\rangle & \cdots & \langle J''(u)u_{n-1}, u_{n-1}\rangle \end{bmatrix}$$

is invertible, i.e., $|Q| \neq 0$, then $(t_1(w), ..., t_{n-1}(w))$ can be solved from (5.2) around v and $(t_1(w), ..., t_{n-1}(w))$ is continuously differentiable around v, i.e.,

$$p(w) = \sum_{i=1}^{n-1} t_i(w)u_i + w$$

satisfies (5.2). Thus the L-$\perp$ selection p is well-defined and continuously differentiable near $v \in S_{L'}$. Then we can normalize $p(w)$ through a differentiable operation, i.e., multiplying each $t_i(w)$ and 1 by the number $t_n(w) = 1/(\sum_{i=1}^{n-1} t_i(w)^2 + 1)^{\frac{1}{2}}$ to get $p(w) = (\sum_{i=1}^{n-1} t_i(w)u_i + w)/t_n(w),\ w \in [L, w]_S$ for all w near v in $S_{L'}$. Such $p(w)$ satisfies (5.1). The condition $|Q| \neq 0$ can be easily checked in numerical computation.

CHAPTER VI

A NONSMOOTH MINMAX CHARACTERIZATION

Let B be a Banach space, B^* its dual space, $\langle , \rangle$ the dual relation and $\| \cdot \|$ its norm. Let $J : B \to \mathbb{R}$ be a locally Lipschitz continuous functional. Then according to Chang [3], a point $u^* \in B$ is a critical point of J iff

$$0 \in \partial J(u^*), \tag{6.1}$$

where $\partial J(u^*)$ is the generalized gradient of J at u^* in the sense of Clark [7]. If J is C^1, (6.1) reduces to $\nabla J(u^*) = 0$ where $\nabla J(u^*)$ is the gradient of J at u^*, i.e., (6.1) becomes the wellknown Euler-Lagrange equation.

Let us recall some basic lemmas for the generalized gradient of locally Lipschitz continuous functionals which will be used later for convenience.

Lemma VI.1 *([7]) Assume that J is Lipschitz continuous in a neighborhood $\mathcal{N}(u_0)$ of u_0 with Lipschitz constant K, i.e., $|J(u) - J(v)| \le K\|u - v\|$, $\forall u, v \in \mathcal{N}(u_0)$. Then*

(1) *For all $u \in \mathcal{N}(u_0)$, $\partial J(u)$ is a nonempty, convex, weak*-compact subset of B^* and $\|w\| \le K$, $\forall w \in \partial J(u)$.*

(2) *Let B be a Hilbert space. For each $u \in \mathcal{N}(u_0)$, if $z \in \partial J(u)$ such that $\|z\| = \min\{\|\zeta\| : \zeta \in \partial J(u)\}$, then we have*

$$\langle z, \zeta \rangle \ge \|z\|^2, \quad \forall \zeta \in \partial J(u).$$

Lemma VI.2 *(Lebourg, [7]) Let $u, v \in B$. Assume that J is Lipschitz continuous in an open set which contains the line segment $\{\lambda u + (1 - \lambda)v : \lambda \in [0, 1]\}$. Then there is $\lambda_0 \in (0, 1)$ such that*

$$J(u) - J(v) \in \langle \partial J(\lambda_0 u + (1 - \lambda_0)v), u - v \rangle.$$

To locally Lipschitz continuous functional, we can define peak mapping, peak selection and local peak selection in the same way as Definition II.1.

Definition VI.1 *A set-valued mapping $G : B \to 2^{B^*}$ is said to be weakly upper semicontinuous at $u \in B$, if for all $u_k \to u$ and $v_k \in G(u_k)$, there is $w_k \in G(u)$ such that $w_k - v_k \to 0$ weakly. G is said to be weakly upper semicontinuous if it is weakly upper semicontinuous at each point in B.*

A. A Local Minmax Characterization for NSCPs

1. A Characterization in Hilbert Spaces

First let us consider the case in a Hilbert space H. By using the generalized gradient, we are able to establish a local minmax characterization for multiple nonsmooth critical points in H which generalizes the corresponding results in [17, 18] for multiple smooth critical points in H. The following lemma plays an important role in the local minimax method.

Lemma VI.3 *Let H be a Hilbert space with $H = L \oplus L^\perp$ for a closed subspace $L \subset H$ and $J : H \to \mathbb{R}$. Assume that p is a local peak selection of J w.r.t. L at $v \in S_{L^\perp}$ and J is locally Lipschitz continuous in a neighborhood of $p(v)$ such that*

(1) *p is continuous at v and $dis(p(v), L) > 0$,*

(2) *$z \in \partial J(p(v))$ such that $\|z\| = \min\{\|w\| : w \in \partial J(p(v))\} > 0$, and*

(3) *the set-valued mapping $G : u \to \partial J(u)$, $\forall u \in \mathcal{N}(p(v))$ is weakly upper semicontinuous at $p(v)$, where $\mathcal{N}(p(v))$ is a neighborhood of $p(v)$.*

Then as $s > 0$ is sufficient small,

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4}|t_v|\|z\|^2, \tag{6.2}$$

where $v(s) = \dfrac{v - sign(t_v)sz_{L^\perp}}{\|v - sign(t_v)sz_{L^\perp}\|}$, $p(v) = t_v v + w_v$, $w_v \in L$ *and* $z = z_L + z_{L^\perp}$, $z_L \in L$, $z_{L^\perp} \in L^\perp$.

Proof. By Lemma VI.2, for t close to t_v, $w \in L$ close to w_v and $s > 0$ sufficient small,

$$J\Big(\alpha(s,t,w) - \frac{sign(t_v)stz}{\|v - sign(t_v)sz_{L^\perp}\|}\Big) - J(\alpha(s,t,w)) = -\frac{sign(t_v)st}{\|v - sign(t_v)sz_{L^\perp}\|}\langle z_{v,w}, z\rangle$$

where $\alpha(s,t,w) = \frac{tv}{\|v - sign(t_v)sz_{L^\perp}\|} + w$ and $z_{v,w} \in \partial J\Big(\alpha(s,t,w) - \lambda_{v,w}\frac{sign(t_v)stz}{\|v - sign(t_v)sz_{L^\perp}\|}\Big)$ for some $\lambda_{v,w} \in (0,1)$. Since p is a peak selection, for t close to t_v, w close to w_v and s sufficiently small, we have

$$J(p(v)) \geq J(\alpha(s,t,w)).$$

Hence

$$J\Big(\alpha(s,t,w) - \frac{sign(t_v)stz}{\|v - sign(t_v)sz_{L^\perp}\|}\Big) - J(p(v)) \leq -\frac{sign(t_v)st}{\|v - sign(t_v)sz_{L^\perp}\|}\langle z_{v,w}, z\rangle.$$

On the other hand, by assumption (3), there is $\zeta_{v,w} \in \partial J(p(v))$ such that

$$|\langle \zeta_{v,w} - z_{v,w}, z\rangle| \leq \frac{1}{2}\|z\|^2$$

for t close to t_v, w close to w_v and $s > 0$ sufficiently small. Thus, by Lemma VI.1,

$$\begin{aligned} & J\Big(\alpha(s,t,w) - \frac{sign(t_v)stz}{\|v - sign(t_v)sz_{L^\perp}\|}\Big) - J(p(v)) \\ \leq\ & -\frac{sign(t_v)st}{\|v - sign(t_v)sz_{L^\perp}\|}(-|\langle z_{v,w} - \zeta_{v,w}, z\rangle| + \langle \zeta_{v,w}, z\rangle) \leq -\frac{1}{4}s|t_v|\|z\|^2. \end{aligned}$$

Then

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4}s|t_v|\|z\|^2,$$

as $s > 0$ sufficiently small by letting $t = t(s)$, $w = w(s) + \dfrac{sign(t_v)st(s)z_L}{\|v - sign(t_v)sz_{L^\perp}\|}$, where $p(v(s)) = t(s)v(s) + w(s)$, $w(s) \in L$.

Remark VI.1 *Several points on this lemma need to be remarked.*

(a) $z_{L^\perp} \neq 0$, *since* $z \neq 0$ *and* p *is a peak selection, see Lemma VI.6.*

(b) The inequality (6.2) is an important result which can be used to not only derive a local minmax characterization of nonsmooth saddle points as presented in Theorem VI.1 but also design a stepsize rule for the local minimax algorithm, see Step 5 in the flow chart of the algorithm in Section 3.

(c) If $H = \mathbb{R}^n$, *to a locally Lipschitz function* J, *the set-valued mapping* $G : u \to \partial J(u)$, $\forall u \in H$, *is upper semicontinuous [7]. If* J *is* C^1, *then* $\partial J(u) = \{\nabla J(u)\}$, *i.e,* G *is upper semicontinuous.*

By Lemma VI.3, a minmax characterization for nonsmooth critical points in a Hilbert space can be immediately derived as follow.

Theorem VI.1 *Let* H *be a Hilbert space with* $H = L \oplus L^\perp$ *for a closed subspace* $L \subset H$ *and* $J : H \to \mathbb{R}$. *Assume that* p *is a local peak selection of* J *w.r.t.* L *at* $v \in S_{L^\perp}$ *and* J *is locally Lipschitz continuous in a neighborhood of* $p(v)$ *such that*

(1) p *is continuous at* v *and* $dis(p(v), L) > 0$,

(2) *the set-valued mapping* $G : u \to \partial J(u)$, $\forall u \in \mathcal{N}(p(v))$ *is weakly upper semicontinuous at* $p(v)$, *where* $\mathcal{N}(p(v))$ *is a neighborhood of* $p(v)$, *and*

(3) $J(p(v)) = local\text{-}min_{u \in S_{L^\perp}} J(p(u))$.

Then $p(v)$ *is a critical point of* J.

Proof. If $p(v)$ is not a critical point of J, let $z \in \partial J(p(v))$ satisfying $\|z\| = \min\{\|w\| : w \in \partial J(p(v))\} > 0$, then by Lemma VI.3, as $s > 0$ sufficiently small,

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4}s|t_v|\|z\|^2,$$

where $v(s) = \dfrac{v - sign(t_v)sz_{L^\perp}}{\|v - sign(t_v)sz_{L^\perp}\|}$, $p(v) = t_v v + w_v$, $w_v \in L$ and $z = z_L + z_{L^\perp}$, $z_L \in L$, $z_{L^\perp} \in L^\perp$. It is a contradiction to assumption (3).

2. A Characterization in Reflexive Banach Spaces

Now we start to establish a local minmax characterization for nonsmooth saddle points in Banach spaces. Since in this case, the generalized gradient $\partial J(u)$ is in B^* not B, a point in $\partial J(u)$ cannot be used to update an iteration point $u \in B$. Thus as long as numerical algorithms are concerned, a new notion has to be developed. Motivated by the notion of a pseudo-gradient for C^1 functional in Banach spaces, we introduce the following definition which is crucial for later development.

Definition VI.2 *Let B be a reflexive Banach space and $J : B \to \mathbb{R}$ be Lipschitz continuous near a point $u_0 \in B$. Let $\mu = \min\{\|z\|_{B^*} : z \in \partial J(u_0)\} > 0$. Then the pseudo-generalized-gradient (PGG) $\Psi J(u_0)$ of J at u_0 is defined by* $\Psi J(u_0)$

$$= \{z^* \in B : \|z^*\| = \mu, \langle w, z^*\rangle \geq \langle z, z^*\rangle = \mu^2, z \in \partial J(u_0), \|z\|_{B^*} = \mu, \forall w \in \partial J(u_0)\}.$$

Lemma VI.4 *Assume that B is a reflexive Banach space, $J : B \to \mathbb{R}$ is Lipschitz continuous near a point $u_0 \in B$ and u_0 is not a critical point. Then the PGG $\Psi J(u_0)$ of J at u_0 is a nonempty, convex. If in addition, B^* is locally uniformly convex and $\|\cdot\|_{B^*}$ is Fréchet differentiable on $B^* \setminus \{0\}$, then $\Psi J(u_0) = \{\|z\|_{B^*}\|z\|'_{B^*}\}$ where z is the unique point of minimum norm in $\partial J(u_0)$.*

Proof. Let $\mu = \min\{\|z\|_{B^*} : z \in \partial J(u_0)\}$ and $S(\mu) = \{u \in B^* : \|u\|_{B^*} \leq \mu\}$. If $0 \in \partial J(u_0)$, i.e., $\mu = 0$ and $z = 0$, then $\Psi J(u_0) = \{0\}$. If $0 \notin \partial J(u_0)$, then $\mu > 0$ and there is $z \in \partial J(u_0)$ such that $\|z\|_{B^*} = \mu > 0$ since $\partial J(u_0)$ is nonempty, convex and *weak**-compact. Note that $\text{int}S(\mu) \cap \partial J(u_0) = \emptyset$ and $z \in S(\mu) \cap \partial J(u_0)$, by Lemma VI.1 and the separation theorem [28], there is a $z^* \in B^{**} = B$ such that

(1) $\|z^*\| = \|z\|_{B^*} = \mu,$ and

(2) $\inf_{w\in\partial J(u_0)}\langle w, z^*\rangle = \langle z, z^*\rangle = \sup_{u\in S(\mu)}\langle u, z^*\rangle.$

On the other hand,

$$\sup_{u\in S(\mu)}\langle u, z^*\rangle = \sup_{\{u\in B^*:\|u\|=\|z\|\}}\langle u, z^*\rangle = \|z^*\|\|z\|_{B^*} = \|z\|^2_{B^*} = \mu^2.$$

Hence

$$\langle w, z^*\rangle \geq \langle z, z^*\rangle = \|z\|^2_{B^*} = \mu^2, \quad \forall w \in \partial J(u_0).$$

Thus $\Psi J(u_0)$ is nonempty. To show that $\Psi J(u_0)$ is convex, let $z_1^*, z_2^* \in \Psi J(u_0)$ and $0 < \alpha < 1$. There exist $z_1, z_2 \in \partial J(u_0)$ such that $\|z_1\|_{B^*} = \|z_2\|_{B^*} = \mu > 0$ and

$$\langle w, z_i^*\rangle \geq \langle z_i, z_i^*\rangle = \mu^2, \quad \forall w \in \partial J(u_0),\ i = 1, 2.$$

Since $\|z_1^*\| = \|z_2^*\| = \mu$, we have

$$\|\alpha z_1^* + (1-\alpha)z_2^*\| \leq \mu \text{ and } \|\alpha z_1 + (1-\alpha)z_2\| \leq \mu,$$

and for all $w \in \partial J(u_0)$,

$$\begin{aligned} &\langle w, \alpha z_1^* + (1-\alpha)z_2^*\rangle = \alpha\langle w, z_1^*\rangle + (1-\alpha)\langle w, z_2^*\rangle \\ \geq\ & \alpha\langle z_1, z_1^*\rangle + (1-\alpha)\langle z_2, z_2^*\rangle = \alpha\mu^2 + (1-\alpha)\mu^2 = \mu^2. \end{aligned}$$

In particular for $w = \alpha z_1 + (1-\alpha)z_2 \in \partial J(u_0)$, we have

$$\mu^2 \leq \langle \alpha z_1 + (1-\alpha)z_2, \alpha z_1^* + (1-\alpha)z_2^*\rangle \leq \|\alpha z_1 + (1-\alpha)z_2\|_{B^*}\|\alpha z_1^* + (1-\alpha)z_2^*\| \leq \mu^2.$$

Therefore we must have

$$\begin{aligned} \langle \alpha z_1 + (1-\alpha)z_2, \alpha z_1^* + (1-\alpha)z_2^*\rangle &= \mu^2, \\ \|\alpha z_1^* + (1-\alpha)z_2^*\| = \|\alpha z_1 + (1-\alpha)z_2\|_{B^*} &= \mu \end{aligned}$$

and for all $w \in \partial J(u_0)$,

$$\langle w, \alpha z_1^* + (1-\alpha)z_2^* \rangle \geq \langle \alpha z_1 + (1-\alpha)z_2, \alpha z_1^* + (1-\alpha)z_2^* \rangle = \mu^2,$$

i.e., $\alpha z_1^* + (1-\alpha)z_2^* \in \Psi J(u_0)$ and thus $\Psi J(u_0)$ is a convex set.

If in addition, B^* is locally uniformly convex and $\|\cdot\|_{B^*}$ is Frechet differentiable on $B^* \setminus \{0\}$, then there is only one $z \in \partial J(u_0)$ with $\|z\|_{B^*} = \mu$ and $S(\mu) \cap \partial J(u_0) = \{z\}$. The set $\{u \in B^* : \langle \|z\|'_{B^*}, u - z \rangle = 0\}$ is the tangent plane of the sphere $S(\mu)$ at z. On the other hand, B is reflexive and $\|\cdot\|'_{B^*}$ exists on $B^* \setminus \{0\}$ imply that B is locally uniformly convex. Since $\Psi J(u_0)$ is a convex set in B such that for any $z^* \in \Psi J(u_0)$, we have $\|z^*\| = \|z\|_{B^*} = \mu$, the set $\Psi J(u_0)$ can contain at most one point z^*. The hyperplane corresponding to z^* separates $\partial J(u_0)$ from $S(\mu)$ at z. Such a separating hyperplane must be a tangent plane of $S(\mu)$ at z. Since $g(v) = \|v\|_{B^*}$ is Frechet differentiable at z, such a tangent plane is unique. We have

$$\langle \|z\|'_{B^*}, w - z \rangle \geq 0 \geq \langle \|z\|'_{B^*}, u - z \rangle, \forall w \in \partial J(u_0), u \in S(\mu).$$

Since $\langle \|z\|'_{B^*}, z \rangle = \|z\|_{B^*} = \mu$, we have

$$\langle \|z\|_{B^*} \|z\|'_{B^*}, w \rangle \geq \langle \|z\|_{B^*} \|z\|'_{B^*}, z \rangle = \mu^2 = \|z\|^2_{B^*} \geq \langle \|z\|_{B^*} \|z\|'_{B^*}, u \rangle,$$

$\forall w \in \partial J(u_0), u \in S(\mu)$, which implies $\| \|z\|_{B^*} \|z\|'_{B^*} \| = \|z\|_{B^*} = \mu$ and then $\Psi J(u_0) = \{z^*\} = \{\|z\|_{B^*} \|z\|'_{B^*}\}$.

Remark VI.2 *Several points on this lemma need to be remarked.*

(a) When B is a Hilbert space, $z^* = z$.

(b) When J is a C^1 functional, z^* is a pseudo-gradient of J at u_0 with

$$\|z^*\| = \|\nabla J(u_0)\| \quad \textit{and} \quad \langle z^*, \nabla J(u_0) \rangle \geq \|\nabla J(u_0)\|^2.$$

(c) By the Kadec-Troyanski theorem (pp. 603-605, [28]), in every reflexive Banach space B, an equivalent norm $\|\cdot\|_B$ can be introduced so that B and B^* are locally uniformly convex and therefore $\|\cdot\|_B$ and $\|\cdot\|_{B^*}$ are Frechet differentiable on $B\setminus\{0\}$ and $B^*\setminus\{0\}$. Thus in this case, we may use the norm $\|\cdot\|_B$ as the default norm $\|\cdot\|$ on B.

Then replacing the generalized gradient by the PGG and with some modification, the following lemma can be verified in a similar way as in Lemma VI.3.

Lemma VI.5 *Assume that J is locally Lipschitz continuous in B and p is a local peak selection of J w.r.t L at $v \in S_{L^\perp}$ such that*

(1) *p is continuous at v and $dis(p(v), L) > 0$,*

(2) *$z^* \in B$ is the PGG of J at $p(v)$ with $\|z^*\| > 0$, and*

(3) *the set-valued mapping $G : u \to \partial J(u)$, $\forall u \in \mathcal{N}(p(v))$ is weakly upper semicontinuous at $p(v)$, where $\mathcal{N}(p(v))$ is a neighborhood of $p(v)$.*

Then

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4}s|t_v|\|z\|^2_{B^*},$$

where $v(s) = \dfrac{v - sign(t_v)sz^*_{L'}}{\|v - sign(t_v)sz^*_{L'}\|}$, $p(v) = t_v v + w_v$, $w_v \in L$, $z^* = z^*_L + z^*_{L'}$, $z^*_L \in L$, $z^*_{L'} \in L'$ and z is a point of minimum norm in $\partial J(p(v))$.

By Lemma VI.5, the minmax characterization for nonsmooth critical points in Banach spaces can be written as follow.

Theorem VI.2 *Assume that J is locally Lipschitz continuous in B and p is a local peak selection of J w.r.t L at $v \in S_{L'}$ such that*

(1) *p is continuous at v and $dis(p(v), L) > 0$,*

(2) *the set-valued mapping* $G : u \to \partial J(u)$, $\forall u \in \mathcal{N}(p(v))$ *is weakly upper semicontinuous at* $p(v)$, *where* $\mathcal{N}(p(v))$ *is a neighborhood of* $p(v)$, *and*

(3) $J(p(v)) = \text{local-min}_{u \in S_{L'}} J(p(u))$.

Then $p(v)$ *is a critical point of* J, *i.e.,* $0 \in \partial J(p(v))$.

B. A Local Minimax Algorithm

Before we present the algorithm, we need the following lemma to show that Step 3 in the algorithm can be carried out once a nonsmooth saddle critical point has not been reached.

Lemma VI.6 *Let* B *be a reflexive Banach space with* $B = L \oplus L'$ *for some closed subspaces* L, L' *in* B *and* $J : B \to \mathbb{R}$. *Assume* p *is a local peak selection of* J *w.r.t* L *at* $v_0 \in S_{L'}$, J *is locally Lipschitz continuous near* $u_0 = p(v_0)$ *and the set-valued mapping* $G : u \to \partial J(u)$ *is weakly upper semicontinuous at* u_0. *If* u_0 *is not a critical point, then* $\mathcal{P}(z^*) \neq 0, \forall z^* \in \Psi J(u_0)$ *where* $\mathcal{P} : B \to L'$ *is the projection operator.*

Proof. Since u_0 is not a critical point of J, we have $\mu = \min\{\|z\| : z \in \partial J(u_0)\} > 0$. If $\mathcal{P}(z^*) = 0$, then $z^* \in L$, $u_0 + tz^* \in [L, v_0]$. When $t > 0$ is sufficiently small, by Lemma VI.2, there exist $\lambda \in (0, 1)$, $\zeta_t \in \partial J(u_0 + \lambda t z^*)$ and $\zeta_0 \in \partial J(u_0)$ such that

$$\begin{aligned} J(u_0 + tz^*) - J(u_0) &= t\langle \zeta_t, z^* \rangle = t(\langle \zeta_t - \zeta_0, z^* \rangle + \langle \zeta_0, z^* \rangle) \\ &\geq t(-\frac{1}{2}\mu^2 + \mu^2) = \frac{t}{2}\mu^2 > 0, \end{aligned}$$

where the first inequality is due to the conditions that $G : u \to \partial J(u)$ is weakly upper semicontinuous and $z^* \in \Psi J(u_0)$. It leads to a contradiction to the assumption that $u_0 = p(v_0)$ is a local maximum point of J in $[L, v_0]$. Now we are ready to present the algorithm.

Assume that u_1,...,u_{n-1} are $n-1$ previously found nonsmooth critical points of a locally Lipschitz continuous functional J in a reflexive Banach space B. Let $L = \{u_1, ..., u_{n-1}\}$, $B = L \oplus L'$ and $\mathcal{P}$ be the corresponding projection operator from B to L'. Given $\varepsilon, \lambda > 0$.

A flow chart of the algorithm reads:

Step 1: Let $v^1 \in S_{L'}$ be an increasing-decreasing direction at u_{n-1}.

Step 2: Set $k = 1$ and solve for

$$\begin{aligned} u^k &= p(v^k) = t_0^k v^k + t_1^k u_1 + \cdots + t_{n-1}^k u_{n-1} \\ &= \arg\max\{J(t_0 v_n^k + t_1 u_1 + \cdots + t_{n-1} u_{n-1}) | t_i \in R, i = 0, 1, ..., n-1\}. \end{aligned}$$

Step 3: Find a descent direction $w^k = -\text{sign}(t_0^k)\mathcal{P}(z^k)$ at u^k, where $z^k \in \Psi J(u^k)$.

Step 4: If $\|u^k - u^{k-1}\| < \varepsilon$, then output u^k, stop. Otherwise, do Step 5.

Step 5: For each $s > 0$, use the initial point $(t_0^k, t_1^k, ..., t_{n-1}^k)$ to solve for

$$p(v^k(s)) = \arg\max\left\{J(t_0 v^k(s) + \sum_{i=1}^{n-1} t_i u_i) | t_i \in R, i = 0, 1, ..., n-1\right\},$$

where $v^k(s) = \frac{v^k + s w^k}{\|v^k + s w^k\|}$, then set $u^{k+1} = p(v^{k+1}) = p(v^k(s^k))$ where s^k satisfies

$$s^k = \max\{s = \frac{\lambda}{2^m} | m \in N, 2^m > \|w^k\|, J(p(v^k(s))) - J(p(v^k)) \le -\frac{1}{4}|t_0^k| s \|z^k\|^2\}.$$

Step 6: Update $k = k + 1$ and go to Step 3.

Remark VI.3 *Several points on the algorithm need to be remarked.*

(a) By Lemmas VI.5 and VI.6, a positive step size in Step 5 of the algorithm can always be obtained if a critical point has not been reached. Therefore the algorithm is

a strict descending method, i.e., $J(u^{k+1}) < J(u^k),\ \forall k = 1, 2, ...$

(b) When B *is a Hilbert space,* L' *will be chosen as* $L^\perp$ *and* z^k *is the point of minimum norm in* $\partial J(u^k)$.

(c) When J *is a* C^1 *functional, this algorithm will reduce to the local minimax algorithm in [17, 18] if* B *is a Hilbert space and the local minimax algorithm in Chapter II if* B *is a reflexive Banach space except Step 3 where for smooth saddle critical points [17, 18] and Chapter II,*

$$\|\nabla J(u^k)\| \le \varepsilon \quad or \quad \|G^k\| \le \varepsilon,$$

where G^k *is a modified pseudo-gradient of* J *at* u^k*, is naturally used as a criterion to stop iteration in the algorithm. For nonsmooth saddle critical points, one may think to use*

$$\|\mathcal{P}(z^k)\| \le \varepsilon \tag{6.3}$$

as a criterion to stop iteration in the algorithm. But it is easy to construct a Lipschitz continuous functional J*, e.g.,* $J(u) = |u|$, $u \in \mathbb{R}$ *such that* u_0 *is a nonsmooth critical point of* J *and* $u^k \to u_0 \in B$ *satisfies*

$$\|\mathcal{P}(z^k)\| > \beta > 0, \forall z^k \in \Psi J(u^k).$$

Hence in general (6.3) cannot be used as a criterion to stop iteration in the algorithm. Instead we may use $\|u^k - u^{k-1}\| < \varepsilon$ *or* $|J(u^k) - J(u^{k-1})| < \varepsilon$ *or* $\|v^k - v^{k-1}\| < \varepsilon$ *which is equivalent to* $\|s^k \mathcal{P}(z^k)\| < \varepsilon$*, as a criterion to stop the iteration of the algorithm. Those criteria are commonly used in numerical computation.*

(d) Other definitions of generalized gradient may also be used to derive local minmax characterization of nonsmooth saddle critical points. We are conducting further study and implementation of the algorithm.

CHAPTER VII

FURTHER TOPICS AND CONCLUSION

A. Further Topics

1. Elliptic Neumann Boundary Value Problem

Consider the following quasilinear elliptic Neumann boundary value problem

$$\begin{cases} \Delta_p u - l|u|^{p-2}u + f(x,u) &= 0, \quad x \in \Omega, \\ \frac{\partial u}{\partial n} &= 0, \quad x \in \partial\Omega, \end{cases}$$

where Δ_p is p-Laplacian operator with $p > 1$, Ω is a bounded domain and $l > 0$.

In the space $W_n^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) | \frac{\partial u}{\partial n} = 0 \ on \ \partial\Omega\}$, we define

$$\|u\|_{W_n^{1,p}(\Omega)} = \int_\Omega (|\nabla u|^p + l|u|^p)dx, \quad \forall u \in W_r^{1,p}(\Omega).$$

Then, the energy function to the above quasilinear elliptic Neumann boundary value problem is

$$J(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + l|u|^p)dx - \int_\Omega F(x,u)dx,$$

where $F(x,u) = \int_0^u f(x,s)ds$. Thus, the gradient $d = \nabla J(u)$ of J at u can be calculated by solving the following linear elliptic equation

$$\begin{cases} \Delta d - ld &= \Delta_p u - l|u|^{p-2}u + f(x,u), \quad x \in \Omega, \\ \frac{\partial d}{\partial n} &= 0, \quad x \in \partial\Omega, \end{cases}$$

So far, people's knowledge on the existence of solutions to the problem and their properties is still quite limited. From the theoretical analysis in this dissertation, our algorithm should work for this problem. So computational theory and methods developed in this dissertation can be used to provide some tools for further investigation on the problem.

2. Lagrange Multiplier Methods for Eigenpairs

In Chapter IV, we consider the following eigenpair problem, for given $\alpha > 0$, find $(\lambda, u) \in \mathbb{R} \times (B \setminus \{0\})$ such that

$$\begin{cases} F'u = \lambda G'u \quad \text{or} \quad \langle F'u, v\rangle = \lambda\langle G'u, v\rangle, \ \forall v \in B \\ \text{subject to} \quad G(u) = \alpha \end{cases} \tag{7.1}$$

where F' and G' are the *Fréchet* derivatives of two functionals F and G in $C^1(B, \mathbb{R})$ and B is a Banach space with the dual relation $\langle , \rangle$ and the norm $\| \cdot \|$. Such (λ, u) is called an *eigenpair* where λ is an *eigenvalue* and u is an *eigenfunction* corresponding to λ. As a special case, the iso-homogeneous eigenpair problem has been solved in Chapter IV. Here we consider more general cases.

Define the Lagrange functional

$$\mathcal{L}(\lambda, u) = F(u) - \lambda(G(u) - \alpha). \tag{7.2}$$

Then critical points (u, λ) of $\mathcal{L}(u, \lambda)$ are eigenpairs (λ, u) of (7.1) and vice versa. By this equivalence, we can define a peak-selection in $\mathbb{R} \times B$ and get a minmax characterization for the critical points of (7.2). Then, a minimax algorithm can be designed for finding multiple saddle critical points of (7.2), i.e., multiple eigenpairs of (7.1). As a matter of fact, a peak-selection in $\mathbb{R} \times B$ has already been defined, a minmax characterization for critical points of (7.2) has been established and a minimax algorithm for capturing multiple saddle critical points of (7.2) has been designed. Our numerical experiment on several models in [9] shows us that the algorithm is successful. This is an ongoing research.

3. Nonsmooth Saddle Critical Points

In Chapter VI, a minimax algorithm for capturing multiple nonsmooth saddle critical points has been proposed and needs to be implemented. Techniques for such implementation may need to be developed. Numerical experiment on some models needs to be done. It is another ongoing research.

B. Conclusion

Two local minimax methods together with their related theory have been developed in this dissertation for computing multiple saddle critical points in Banach spaces. The first is for unconstrained smooth cases and the second is for a class of constrained smooth cases, i.e., the iso-homogeneous nonlinear eigenpair problems in Banach spaces. They are two-level local optimization methods. The first level is a local maximization and the second is a local minimization. Hence they can be realized numerically. There are two key steps in devising these two minimax methods. The first is to define a peak-selection and the second is to establish a minmax characterization for multiple saddle critical points. Such an approach has been generalized to design a minimax algorithm for unconstrained nonsmooth saddle critical points in Banach spaces.

Based on the methods, two numerical minimax algorithms have been designed for finding multiple smooth saddle critical points in Banach spaces. Pseudo-gradient has been used to find a descent direction for the local minimization at the second level and projection is used to avoid the degeneracy. To implement the algorithms, techniques to compute a pseudo-gradient are proposed. In particular, the method to compute our gradient of $J \in C^1(W_0^{1,p}, \mathbb{R})$ $(p > 1)$ is noteworthy. A unified convergence and several subsequence convergence results have been established for the algorithms. A relation between the convergence rates of the functional value and its derivative has been derived. To get convergence results, peak-selections have been generalized to L-$\perp$ selections. By these L-$\perp$ selections, L-$\perp$ characterizations are established and min-L-$\perp$ algorithms can be designed. By this generalization, the smoothness of peak-selections can be numerically checked. Several numerical experiments to solve a class of quasilinear elliptic PDEs for multiple solutions and to find multiple eigenpairs of the p-Laplacian operator are carried out. Several interesting phenomena have been observed. As an application of our theory, we verify the existence of a nontrivial solution to a class of quasilinear elliptic PDEs.

A minimax algorithm has been designed for finding multiple nonsmooth saddle critical points. To do so, a pseudo-generalized-gradient has been introduced. Some interesting properties of a pseudo-generalized-gradient have been found.

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