ON THE COHOMOLOGY OF JOINS OF OPERATOR ALGEBRAS

A Dissertation

by

ALI-AMIR HUSAIN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2004

Major Subject: Mathematics

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ABSTRACT

On the Cohomology of Joins of Operator Algebras. (May 2004)

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Let $\mathfrak{A}$ be an abelian von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Then $M_n(\mathfrak{A})$ is a Hilbert C^*-module over $\mathfrak{A} \otimes \mathbb{C}1_n$. C^*-modules were originally defined and studied by Kaplansky and we outline the foundations of the theory and particular properties of $M_n(\mathfrak{A})$. Furthermore, we prove a structure theorem for ultraweakly closed submodules of $M_n(\mathfrak{A})$, using techniques from the theory of type I finite von Neumann algebras.

By analogy with the classical join in topology, the join $\mathcal{A} * \mathcal{B}$ for operator algebras $\mathcal{A}$ and $\mathcal{B}$ acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, was defined by Gilfeather and Smith. Assuming that $\mathcal{K}$ is finite dimensional, Gilfeather and Smith calculated the Hochschild cohomology groups for $\mathcal{A} * \mathcal{B}$ with coefficients in $\mathcal{L}(\mathbb{C}^n \oplus \mathcal{K})$.

We assume that $\mathfrak{A}$ is a maximal abelian von Neumann algebra acting on $\mathcal{H}$, $\mathcal{A}$ is a subalgebra of $\mathfrak{A} \mathbin{\overline{\otimes}} \mathcal{L}(\mathcal{K})$, and $\mathcal{B}$ is an ultraweakly closed subalgebra of $M_n(\mathfrak{A})$ containing $\mathfrak{A} \otimes \mathbb{C}1_n$. In this new context, we redefine the join $\mathcal{A} * \mathcal{B}$ and generalize the calculations of Gilfeather and Smith to multilinear maps on $\mathcal{A} * \mathcal{B}$ with values in $\mathfrak{A} \mathbin{\overline{\otimes}} \mathcal{L}(\mathbb{C}^n \oplus \mathcal{K})$. We then calculate $H^m(\mathcal{A} * \mathcal{B}, \mathfrak{A} \mathbin{\overline{\otimes}} \mathcal{L}(\mathbb{C}^n \oplus \mathcal{K}))$, for all $m \geq 0$.

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CHAPTER I

INTRODUCTION

Homology theory has its origins in the study of topological spaces. A sequence of modules $\{C_n\}_{n=0}^{\infty}$ and module homomorphisms $\partial : C_{n+1} \to C_n$ such that $\partial^2 = 0$, called a *chain complex*, is assigned to a topological space in such a way that if two topological spaces are isomorphic, then the homologies of their corresponding chain complexes are as well. In the 1940s, Cartan, Eilenberg, Mac Lane, et al. began to study properties of chain complexes independently of any underlying topological space. The resulting theory can be applied to a variety of mathematical objects in the same way that it is used to study topological spaces.

Hochschild [16, 17, 18] applied homological techniques to the study of an associative algebra $\mathcal{A}$ over a field. He constructed a cochain complex whose constituent modules are the sets of multilinear maps from $\mathcal{A}$ to a bimodule M over $\mathcal{A}$. Its cohomology groups are called the Hochschild cohomology groups of $\mathcal{A}$ with coefficients in M, denoted $H^n(\mathcal{A}, M)$.

Hochschild's theory was adapted by Johnson [21], Kadison, and Ringrose [22, 23], in the early 1970s, to examine a Banach algebra $\mathcal{A}$ acting continuously on a Banach space M. However, to accommodate the topological structure of $\mathcal{A}$ and M, the new theory was based on the subcomplex of continuous or, equivalently, bounded multilinear maps.

Given topological spaces X and Y, their join, $X * Y$, is the space obtained from $X \times I \times Y$, where I is the unit interval in $\mathbb{R}$, by making the identifications $(x, 0, y) \sim (x', 0, y)$ and $(x, 1, y) \sim (x, 1, y')$. Gilfeather and Smith [14] investigated

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an analogue for operator algebras and calculated its cohomology. Given operator algebras $\mathcal{A}$ and $\mathcal{B}$ acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, their join $\mathcal{A}*\mathcal{B}$ was defined as

$$\mathcal{A}*\mathcal{B} = \left\{ \begin{pmatrix} B & 0 \\ U & A \end{pmatrix} : A \in \mathcal{A}, U \in \mathcal{L}(\mathcal{K},\mathcal{H}), B \in \mathcal{B} \right\}$$

in $\mathcal{L}(\mathcal{K}\oplus\mathcal{H})$. When $\mathcal{K}$ is finite dimensional, they were able to express the cohomology of the join in terms of the cohomologies of $\mathcal{A}$ and $\mathcal{B}$ through the formula

$$H^m(\mathcal{A}*\mathcal{B},\mathcal{L}(\mathcal{K}\oplus\mathcal{H})) \cong \bigoplus_{k=0}^{m-1} H^k(\mathcal{A},\mathcal{L}(\mathcal{H})) \otimes H^{m-k-1}(\mathcal{B},\mathcal{L}(\mathcal{K})).$$

Although Gilfeather and Smith demonstrated that a formula for the cohomology of the join of two arbitrary operator algebras $\mathcal{A}$ and $\mathcal{B}$ in terms of the cohomologies of $\mathcal{A}$ and $\mathcal{B}$ alone is not possible, we extend their formula to a new class of operator algebras.

Let $\mathfrak{A}$ be an abelian von Neumann algebra acting on $\mathcal{H}$ and let $M_n(\mathfrak{A})$ be the algebra of matrices with entries from $\mathfrak{A}$. Suppose $\mathcal{A}$ is a norm closed subalgebra of $\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathcal{K})$ and $\mathcal{B}$ is an norm closed subalgebra of $M_n(\mathfrak{A})$. We define the join of $\mathcal{A}$ and $\mathcal{B}$ to be subalgebra $\mathcal{A}*\mathcal{B}$ of $\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathbb{C}^n\oplus\mathcal{K})$ given by

$$\mathcal{A}*\mathcal{B} = \left\{ \begin{pmatrix} B & 0 \\ U & A \end{pmatrix} : A \in \mathcal{A}, U \in \mathfrak{A}\otimes_* \mathcal{L}(\mathbb{C}^n,\mathcal{K}), B \in \mathcal{B} \right\}.$$

Note that the definitions coincide when $\mathfrak{A} = \mathbb{C}$.

Matrix subalgebras of $M_n(\mathfrak{A})$ are a natural infinite-dimensional analogue to the subalgebras of $M_n(\mathbb{C})$ studied by Gilfeather and Smith and we calculate the cohomology groups of the join in this new setting.

CHAPTER II

NOTATION AND PRELIMINARIES

Before proving the main results of this dissertation, we will define the basic terminology, establish notation, and recall some of the fundamental elements of functional analysis and the theory of operator algebras that will be used throughout. Unless otherwise stated, all vector spaces will be assumed to be over the complex numbers.

Since duality will play an important role in the sequel, we begin by reviewing some basic definitions and theorems regarding the weak* topology.

A. The Weak* Topology

Let $\mathfrak{X}$ be a normed space. For every $x \in \mathfrak{X}$, let $\hat{x} \in \mathfrak{X}^{**}$ be defined by $\hat{x} : f \mapsto f(x)$. Recall that the map $x \mapsto \hat{x}$ is an isometric linear embedding of $\mathfrak{X}$ into $\mathfrak{X}^{**}$, by the Hahn-Banach theorem, called the *canonical embedding*. We say that $\mathfrak{X}$ *reflexive*, if this embedding is surjective. Note that $\mathfrak{X}$ may be isometrically isomorphic to $\mathfrak{X}^{**}$ *without* being reflexive [19].

Every subspace $\mathfrak{Y}^*$ of $\mathfrak{X}^*$ induces a topology on $\mathfrak{X}$ which we will denote $\sigma(\mathfrak{X}, \mathfrak{Y}^*)$. In particular, the topology induced on $\mathfrak{X}^*$ by the image of $\mathfrak{X}$ in $\mathfrak{X}^{**}$ under the canonical embedding is called the *weak* topology* and $\mathfrak{X}^*$ with its weak* topology is denoted $(\mathfrak{X}^*, w^*)$. The property of the weak* topology that is most frequently useful is a consequence of Tychonov's theorem in topology.

Theorem 2.1 (Alaoglu). *Let $\mathfrak{X}$ be a normed space. Then the unit ball of $\mathfrak{X}^*$ is weak* compact.*

Although weak* convergent nets need not be bounded in norm, the next theorem [5, Theorem 5.12.1] often allows us to assume that our nets are bounded.

Theorem 2.2 (Krein-Šmulian). *If $\mathfrak{X}$ is a Banach space and A is a convex subset of $\mathfrak{X}^*$ such that $A \cap \{ f \in \mathfrak{X}^* : \|f\| \leq r \}$ is weak* closed for every $r > 0$, then A is weak* closed.*

The weak* topology is neither metrizable nor first countable, in general. However, when $\mathfrak{X}$ is a *separable* normed space, the unit ball $\mathfrak{X}_1^*$ of $\mathfrak{X}^*$ is metrizable and, furthermore, we have the following theorem [4, Theorem 2.3].

Theorem 2.3. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and let $\mathfrak{X}$ be separable. A linear mapping $S : (\mathfrak{X}^*, w^*) \to (\mathfrak{Y}^*, w^*)$ is continuous if and only if whenever a sequence $\{\varphi_n\}_{n=1}^{\infty}$ converges to 0 in $(\mathfrak{X}^*, w^*)$, then $\{T\varphi_n\}_{n=1}^{\infty}$ converges to 0 in $(\mathfrak{Y}^*, w^*)$.*

B. Algebras and Involutions

An *algebra* $\mathcal{A}$ is a vector space with an associative bilinear multiplication. We will assume that all algebras have a multiplicative unit $1_{\mathcal{A}}$ such that $\|1_{\mathcal{A}}\| = 1$. If an algebra $\mathcal{B}$ is contained in $\mathcal{A}$ and has the same unit as $\mathcal{A}$, we call $\mathcal{B}$ a *subalgebra* of $\mathcal{A}$. A linear subspace I of $\mathcal{A}$ is called a *left ideal* of $\mathcal{A}$, if $AI \subseteq I$, for every $A \in \mathcal{A}$. We define *right ideals* similarly and if I is both a left and right ideal of $\mathcal{A}$, it is called a *two-sided ideal* or simply an *ideal* of $\mathcal{A}$. A proper left ideal M is called *maximal*, if whenever it is contained in another proper left ideal I, then $M = I$. By Zorn's lemma, every proper left ideal is contained in a maximal left ideal.

An *involution* on an algebra $\mathcal{A}$ is a conjugate linear map $* : \mathcal{A} \to \mathcal{A}$ such that $a^{**} = a$, for all $a \in \mathcal{A}$ and $(ab)^* = b^* a^*$, for all $a, b \in \mathcal{A}$. An algebra $\mathcal{A}$ with an involution is called a $*$-algebra, but we shall encounter other classes of algebras in the sequel. An algebra $\mathcal{A}$ with a complete, submultiplicative norm such that $\|1\| = 1$ is called a *Banach algebra*. If there exists an isometric involution on $\mathcal{A}$, then we call $\mathcal{A}$ a *Banach $*$-algebra* and if, additionally, its norm satisfies $\|aa^*\| = \|a\|^2$, for every

$a \in \mathcal{A}$, $\mathcal{A}$ is called a *C^*-algebra*.

In a Banach algebra $\mathcal{A}$, we define the *spectrum* of $a \in \mathcal{A}$, denoted $\sigma(a)$, to be the set of all $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not invertible. The spectrum is always a non-empty compact set and when $\mathcal{A}$ is finite dimensional, $\sigma(a)$ is the set of eigenvalues of a. The spectral radius $r(a)$ is defined by $r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$. Note that $(\lambda - a)^{-1} = \sum_{n=0}^{\infty} a^n/\lambda^{n+1}$, for $|\lambda| > \|a\|$, so that $r(a) \leq \|a\|$. Furthermore, the spectral radius formula $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$ [28, Theorem 1.2.7] implies that if $\mathcal{A}$ is a C^*-algebra and $\|\cdot\|'$ is another C^*-norm on $\mathcal{A}$, then $\|\cdot\|' = \|\cdot\|$. We say that C^*-algebras have uniqueness of norm.

We distinguish several subsets of a C^*-algebra $\mathcal{A}$ related to its involution. We call $a \in \mathcal{A}$ *normal*, if $aa^* = a^*a$, an *isometry*, if $a^*a = 1$, a *co-isometry*, if $aa^* = 1$, and an element a that is both an isometry and a co-isometry is called *unitary*. We say that a is *self-adjoint*, if $a = a^*$ and, similarly, a subset $\mathcal{S}$ of $\mathcal{A}$ is called self-adjoint, if $\mathcal{S}^* = \{\, a^* : a \in \mathcal{S} \,\} = \mathcal{S}$. Finally, $a \in \mathcal{A}$ is called positive if $a = x^*x$, for some $x \in \mathcal{A}$.

Every positive element $a \in \mathcal{A}$ has a unique positive square root denoted $a^{1/2}$. Given an arbitrary $a \in \mathcal{A}$, however, we let $|a|$ denote the square root of a^*a. The set of positive elements of $\mathcal{A}_+$ is closed and forms a *positive cone* in $\mathcal{A}$, which means that $a + b \in \mathcal{A}_+$ and $\lambda a \in \mathcal{A}_+$, for all $a, b \in \mathcal{A}_+$ and $\lambda \geq 0$. We use $\mathcal{A}_+$ to define a partial order $\leq$ on the set of all self-adjoint elements $\mathcal{A}_{sa}$ by writing $a \leq b$, if $a, b \in \mathcal{A}_{sa}$ and $b - a \geq 0$.

A bounded linear map Φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called a *homomorphism* if $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathcal{A}$ and $\Phi(1) = 1$. If $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras and $\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathcal{A}$, we say that Φ is a $*$-homomorphism. We call a bijective homomorphism an *isomorphism*. If $\Phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism, then $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic*, denoted $\mathcal{A} \cong \mathcal{B}$. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, then injective $*$-homomorphisms are isometric [28, Theorem 3.1.5].

Let $\mathcal{A}$ be an abelian Banach algebra. A linear functional $f \in \mathcal{A}^*$ satisfying $f(ab) = f(a)f(b)$ for all $a, b \in \mathcal{A}$ is called a *multiplicative functional.* The set of multiplicative functionals Ω forms a weak* compact subset of $\mathcal{A}^*$. Since $f \mapsto \ker f$ is a bijection from Ω to the set of maximal ideals in $\mathcal{A}$, Ω is also called the *maximal ideal space* of $\mathcal{A}$. We denote the C^*-algebra of continuous, complex-valued functions on Ω by $\mathcal{C}(\Omega)$. The *Gelfand transformation* $\Gamma : \mathcal{A} \to \mathcal{C}(\Omega)$ defined by $\Gamma(a) = \hat{a} : f \mapsto f(a)$ is a contractive homomorphism. When $\mathcal{A}$ is a C^*-algebra, the Gelfand-Naimark theorem [11, Theorem 4.29]asserts that the Gelfand transformation is an isometric $*$-isomorphism.

C. Bounded Operators on Banach Spaces

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. The space of bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$ will be denoted $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$. When $\mathfrak{X} = \mathfrak{Y}$, we abbreviate $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ by $\mathcal{L}(\mathfrak{X})$ and we will abbreviate the notation for other spaces of operators similarly. Unless otherwise specified, $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ will be endowed with its usual norm topology.

The closed subspace of compact operators and the subspace of finite rank operators in $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ will be denoted $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ and $\mathcal{F}(\mathfrak{X}, \mathfrak{Y})$, respectively. The finite rank operators are linearly spanned by the operators of rank one and $\mathcal{F}(\mathfrak{X}, \mathfrak{Y}) \cong \mathfrak{Y} \otimes \mathfrak{X}^*$, where $\otimes$ denotes the algebraic tensor product. Hence, we let $y \otimes \phi$ denote the rank one operator $x \mapsto \phi(x)y$, where $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$, and $\phi \in \mathfrak{X}^*$. In particular, $\mathcal{C}(\mathfrak{X})$ and $\mathcal{F}(\mathfrak{X})$ are ideals in $\mathcal{L}(\mathfrak{X})$, the algebra of bounded linear operators on $\mathfrak{X}$, and $\mathcal{F}(\mathfrak{X})$ is contained in *every* non-zero ideal of $\mathcal{L}(\mathfrak{X})$.

The space of $m \times n$ matrices with entries in $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ is denoted $M_{m,n}(\mathcal{L}(\mathfrak{X}, \mathfrak{Y}))$. We consider matrices in $M_{m,n}(\mathcal{L}(\mathfrak{X}, \mathfrak{Y}))$ to be linear operators from n copies of $\mathfrak{X}$ to m copies of $\mathfrak{Y}$ and endow $M_{m,n}(\mathcal{L}(\mathfrak{X}, \mathfrak{Y}))$ with the corresponding operator norm.

Note that $M_{m,n}(\mathcal{L}(\mathfrak{X},\mathfrak{Y})) \cong \mathcal{L}(\mathfrak{X},\mathfrak{Y}) \otimes M_{m,n}(\mathbb{C})$. We let $\{E_{ij}\}_{i,j=1}^{n}$ denote the set of canonical matrix units in $M_{m,n}(\mathbb{C})$ and identify $A = (A_{ij})_{i,j=1}^{n}$ with $\sum_{i,j=1}^{n} A_{ij} \otimes E_{ij}$.

D. Tensor Products of Banach Spaces

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be Banach spaces. There are several norms on their algebraic tensor product $\mathfrak{X} \otimes \mathfrak{Y}$ such that $\mathfrak{X} \otimes \mathfrak{Y}$ is a dense linear submanifold of a Banach space $\mathfrak{Z}$. A norm $\|\cdot\|_{\alpha}$ on $\mathfrak{X} \otimes \mathfrak{Y}$ that satisfies $\|x \otimes y\|_{\alpha} = \|x\|_{\mathfrak{X}}\|y\|_{\mathfrak{Y}}$, for all $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$, is called a *cross norm*. The largest of the cross norms [32, Proposition 2.1] on $\mathfrak{X} \otimes \mathfrak{Y}$ is defined by

$$\|z\|_{\pi} = \inf\left\{ \sum_{i=1}^{n} \|x_i\|_{\mathfrak{X}}\|y_i\|_{\mathfrak{Y}} : x_1, \ldots, x_n \in \mathfrak{X},\ y_1, \ldots, y_n \in \mathfrak{Y},\ z = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

and called the *projective tensor norm* on $\mathfrak{X} \otimes \mathfrak{Y}$. The completion of $(\mathfrak{X} \otimes \mathfrak{Y}, \|\cdot\|_{\pi})$ is denoted $\mathfrak{X} \hat{\otimes}_{\pi} \mathfrak{Y}$ and called the *projective tensor product* of $\mathfrak{X}$ and $\mathfrak{Y}$. The projective tensor product has several important properties. We begin by describing the property from which its name is derived.

A bounded operator $T : \mathfrak{X} \to \mathfrak{Z}$ between Banach spaces is called a *quotient map*, if T is surjective and $\|z\|_{\mathfrak{Z}} = \inf\{\, \|x\|_{\mathfrak{X}} : x \in \mathfrak{X}, Tx = z\}$, for all $z \in \mathfrak{Z}$. In this case, T factors through the canonical projection $\pi : \mathfrak{X} \to \mathfrak{X}/\ker(T)$ and $\mathfrak{X}/\ker(T) \cong \mathfrak{Z}$ isometrically. Let $S : \mathfrak{W} \to \mathfrak{Y}$ and $T : \mathfrak{X} \to \mathfrak{Z}$ be quotient maps. Then there exists a unique bounded operator $S \hat{\otimes}_{\pi} T \in \mathcal{L}(\mathfrak{W} \hat{\otimes}_{\pi} \mathfrak{X}, \mathfrak{Y} \hat{\otimes}_{\pi} \mathfrak{Z})$ such that $(S \hat{\otimes}_{\pi} T)(w \otimes x) = (Sw) \otimes (Tx)$, for all $w \in \mathfrak{W}$ and $x \in \mathfrak{X}$. Furthermore, $S \hat{\otimes}_{\pi} T$ is also a quotient map [32, Proposition 2.5] — that is, the projective tensor product of quotient maps is a quotient map.

A bilinear operator $S : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z}$ is said to be bounded, if there exists $M > 0$ such that $\|S(x,y)\|_{\mathfrak{Z}} \leq M\|x\|_{\mathfrak{X}}\|y\|_{\mathfrak{Y}}$, for all $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$. The set of all bounded bilinear operators $S : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z}$ is denoted $\mathcal{B}(\mathfrak{X} \times \mathfrak{Y}, \mathfrak{Z})$ and, with the norm defined by $\|S\| = \inf\{\, \|S(x,y)\|_{\mathfrak{Z}} : x \in \mathfrak{X}_1,\, y \in \mathfrak{Y}_1 \,\}$, it is a Banach space. If $\pi : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X} \hat{\otimes}_\pi \mathfrak{Y}$ is the canonical bilinear map $(x, y) \mapsto x \otimes y$, then there is an isometric isomorphism $\varphi : \mathcal{B}(\mathfrak{X} \times \mathfrak{Y}, \mathfrak{Z}) \to \mathcal{L}(\mathfrak{X} \hat{\otimes}_\pi \mathfrak{Y}, \mathfrak{Z})$ making the following diagram commute [32, Theorem 2.9].

$$\begin{tikzcd}
\mathfrak{X} \times \mathfrak{Y} \arrow[r, "\pi"] \arrow[d, "S"'] & \mathfrak{X} \hat{\otimes}_\pi \mathfrak{Y} \arrow[dl, "\varphi(S)"] \\
\mathfrak{Z} &
\end{tikzcd}$$

An element of $\mathfrak{X} \otimes \mathfrak{Y}$ may also be identified with a bounded bilinear form on $\mathfrak{X}^* \times \mathfrak{Y}^*$ [32, Proposition 1.2] by letting $(x \otimes y)(f, g) = f(x)g(y)$, for all $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$. From this perspective, there is a canonical cross norm on $\mathfrak{X} \otimes \mathfrak{Y}$ defined by

$$\|z\|_\epsilon = \sup\left\{ \left| \sum_{i=1}^{n} f(x_i) g(y_i) \right| : f \in \mathfrak{X}_1^*,\, g \in \mathfrak{Y}_1^* \right\},$$

for all $z = \sum_{i=1}^{n} x_i \otimes y_i \in \mathfrak{X} \otimes \mathfrak{Y}$, called the *injective tensor norm* on $\mathfrak{X} \otimes \mathfrak{Y}$. The completion of $(\mathfrak{X} \otimes \mathfrak{Y}, \|\cdot\|_\epsilon)$ is denoted $\mathfrak{X} \hat{\otimes}_\epsilon \mathfrak{Y}$ and called the *injective tensor product* of $\mathfrak{X}$ and $\mathfrak{Y}$. Its name arises from the fact that if $\mathfrak{E}$ and $\mathfrak{F}$ are arbitrary closed subspaces of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, then $\mathfrak{E} \hat{\otimes}_\epsilon \mathfrak{F}$ embeds isometrically into $\mathfrak{X} \hat{\otimes}_\epsilon \mathfrak{Y}$ as a closed subspace.

E. Bounded Operators on Hilbert Spaces

We denote Hilbert spaces by $\mathcal{H}$ and $\mathcal{K}$ and define an inner product on their direct sum $\mathcal{H} \oplus \mathcal{K}$ by $\langle (h, k), (h', k') \rangle = \langle h, h' \rangle + \langle k, k' \rangle$, for all $h, h' \in \mathcal{H}$ and $k, k' \in \mathcal{K}$. Similarly, we define an inner product on the algebraic tensor product $\mathcal{H}_0$ of $\mathcal{H}$ and $\mathcal{K}$ by $\langle h \otimes k, h' \otimes k' \rangle = \langle h, h' \rangle \langle k, k' \rangle$, for all $h, h' \in \mathcal{H}$ and $k, k' \in \mathcal{K}$. The completion of $\mathcal{H}_0$ with respect to this inner product will be denoted $\mathcal{H} \otimes \mathcal{K}$.

For every operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, there is a unique operator $T^* \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, called the *adjoint* of T, that is defined by the equation $\langle Tx, y \rangle = \langle x, T^*y \rangle$, for every $x \in \mathcal{H}$ and $y \in \mathcal{K}$. Observe that the adjoint in $\mathcal{L}(\mathcal{H})$ is an involution that makes $\mathcal{L}(\mathcal{H})$ a C^*-algebra.

The positive and unitary elements of a C^*-algebra provide analogues to positive real numbers and complex numbers of modulus one, respectively. In particular, a unitary operator is an isometry and we define $T \in \mathcal{L}(\mathcal{H})$ to be a *partial isometry*, if T is an isometry when restricted to the orthogonal complement of its kernel, $\ker(T)$. Using this weaker notion of an isometry, we obtain a decomposition of bounded linear operators [28, Theorem 2.3.4] that is analogous to the *polar decomposition* of a complex number and is frequently useful.

Theorem 2.4 (Polar Decomposition). *Let $T \in \mathcal{L}(\mathcal{H})$. There exists a unique partial isometry U such that $T = U|T|$ and* $\ker(U) = \ker(T)$*. Furthermore, $U^*T = |T|$.*

In addition to the norm topology on $\mathcal{L}(\mathcal{H})$, there are other locally convex topologies on $\mathcal{L}(\mathcal{H})$ that are important. A net of operators $\{T_\alpha\}_{\alpha \in I}$ in $\mathcal{L}(\mathcal{H})$ is said to converge to T in the *strong operator topology* (SOT) if $\|T_\alpha x - Tx\| \to 0$ for every $x \in \mathcal{H}$ and $\{T_\alpha\}_{\alpha \in I}$ is said to converge to T in the *weak operator topology* (WOT) if $\langle T_\alpha x, y \rangle \to \langle Tx, y \rangle$ for all $x, y \in \mathcal{H}$. It is clear that every norm convergent net or sequence is SOT convergent and every SOT convergent net, by the Cauchy-Schwartz inequality, is WOT convergent. Although the strong and weak operator topologies do not coincide, in general, a convex subset of $\mathcal{L}(\mathcal{H})$ is SOT closed if and only if it is WOT closed [28, Theorem 4.2.7].

For every set of operators $\mathcal{S}$ in $\mathcal{L}(\mathcal{H})$, we define the *commutant* of $\mathcal{S}$, denoted $\mathcal{S}'$, to be the set of all operators commuting with $\mathcal{S}$. The commutant of $\mathcal{S}$ is a WOT closed subalgebra of $\mathcal{L}(\mathcal{H})$ and $\mathcal{S}$ is contained in $\mathcal{S}'' = (\mathcal{S}')'$.

A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ is said to have a *separating vector* $x \in \mathcal{H}$ if $Ax = 0$ implies that $A = 0$, for all $A \in \mathcal{A}$, and $x \in \mathcal{H}$ is called a *cyclic vector* for $\mathcal{A}$ if $\mathcal{A}x = \{ Ax : A \in \mathcal{A} \}$ is norm dense in $\mathcal{H}$. A cyclic vector $x \in \mathcal{H}$ for $\mathcal{A}$ is a separating vector for $\mathcal{A}'$. Indeed, if $B \in \mathcal{A}'$ and $Bx = 0$, then $BAx = ABx = 0$, for all $A \in \mathcal{A}$ and, hence, $B = 0$.

We call a WOT closed $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ a *von Neumann algebra*. There are two fundamental theorems in the theory of von Neumann algebras that we state for reference. The first is called *the double commutant theorem* and was discovered by von Neumann [40].

Theorem 2.5 (The Double Commutant Theorem). *Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{L}(\mathcal{H})$. Then $\mathcal{A}''$ is the WOT closure of $\mathcal{A}$. In particular, $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.*

The WOT closure of a C^*-subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ is apparently a von Neumann algebra. The next theorem, called the *Kaplansky density theorem* [25, Theorem 1], provides additional information about the way that $\mathcal{A}$ is embedded in its WOT closure.

Theorem 2.6 (The Kaplansky Density Theorem). *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ and $\mathcal{B}$ its WOT closure. Then the unit ball $\mathcal{A}_1$ of $\mathcal{A}$ is WOT dense in the unit ball $\mathcal{B}_1$ of $\mathcal{B}$ and the self-adjoint elements of $\mathcal{A}_1$ are WOT dense in the self-adjoint elements of $\mathcal{B}_1$.*

A bounded linear map Φ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is called positive if $\Phi(a) \geq 0$, for all $a \geq 0$. In this case, Φ is called normal if $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras and for every increasing net of positive elements $\{x_\alpha\}_{\alpha \in A}$ in $\mathcal{A}$ with supremum x, $\Phi(x) = \sup_{\alpha \in A} \Phi(x_\alpha)$. The image of a normal $*$-homomorphism is a von Neumann algebra and every normal map is continuous with respect to the ultraweak topology [10], which we will describe presently.

F. The Ultraweak Topology on $\mathcal{L}(\mathcal{H})$

Let $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal basis for $\mathcal{H}$ and $T \in \mathcal{L}(\mathcal{H})$. We define the *trace* of T, denoted $\operatorname{tr}(T)$, to be

$$\operatorname{tr}(T) = \sum_{\alpha\in I} \langle Te_\alpha, e_\alpha\rangle.$$

The trace of T need not be finite, but is independent of the choice of orthonormal basis and allows us to define, for $1 \le p \le \infty$, a collection of ideals

$$\mathcal{C}_p(\mathcal{H}) = \{\, T \in \mathcal{L}(\mathcal{H}) : \operatorname{tr}(|T|^p) < \infty\},$$

called the *Schatten p-classes*, all of which are contained in $\mathcal{C}(\mathcal{H})$. When endowed with the norm $\|\cdot\|_p : T \mapsto \big(\operatorname{tr}(|T|^p)\big)^{1/p}$, $\mathcal{C}_p(\mathcal{H})$ is a Banach $*$-algebra and if $T \in \mathcal{C}_1(\mathcal{H})$ and $1 \le p < q < \infty$, then $\|T\| \le \|T\|_q \le \|T\|_p$ [2, Proposition 1.1].

Of particular importance is $\mathcal{C}_1(\mathcal{H})$, called the algebra of *trace-class operators*. When restricted to $\mathcal{C}_1(\mathcal{H})$, the trace is linear and if $T \in \mathcal{C}_1(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$, then $\operatorname{tr}(ST) = \operatorname{tr}(TS)$ and $|\operatorname{tr}(ST)| \le \|S\|\|T\|_1$. We regard $\mathcal{C}_1(\mathcal{H})$ as a subspace of $\mathcal{C}(\mathcal{H})^*$ by mapping $T \in \mathcal{C}_1(\mathcal{H})$ to the bounded linear functional $S \mapsto \operatorname{tr}(ST)$ and, similarly, $\mathcal{L}(\mathcal{H})$ can be viewed as a subspace of $\mathcal{C}_1(\mathcal{H})^*$ by mapping $S \in \mathcal{L}(\mathcal{H})$ to $T \mapsto \operatorname{tr}(ST)$. Both maps are isometric isomorphisms [28, Theorems 4.2.1 and 4.2.3]of Banach spaces and we identify $\mathcal{C}_1(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$ with the dual spaces of $\mathcal{C}(\mathcal{H})$ and $\mathcal{C}_1(\mathcal{H})$, respectively.

The weak* topology induced by $\mathcal{C}_1(\mathcal{H})$ on $\mathcal{L}(\mathcal{H})$ is often called the *σ-weak* or *ultraweak topology*. A net $\{S_\alpha\}_{\alpha\in J}$ in $\mathcal{L}(\mathcal{H})$ converges ultraweakly to an operator $S \in \mathcal{L}(\mathcal{H})$ precisely when $\operatorname{tr}(S_\alpha T) \to \operatorname{tr}(ST)$, for every $T \in \mathcal{C}_1(\mathcal{H})$. Since $\operatorname{tr}((h \otimes k)T) = \langle Th, k\rangle$, for all $h, k \in \mathcal{H}$, every ultraweakly convergent net is WOT convergent.

Furthermore, the WOT is Hausdorff, and, by Alaoglu's theorem, the unit ball of $\mathcal{L}(\mathcal{H})$, denoted $\mathcal{L}(\mathcal{H})_1$, is ultraweakly compact. Hence, the identity map from $\mathcal{L}(\mathcal{H})_1$ with the ultraweak topology to $\mathcal{L}(\mathcal{H})_1$ with the WOT is a homeomorphism and the relative ultraweak and weak operator topologies coincide on bounded sets.

The Schatten p-classes are seen as non-commutative analogues of the ℓ_p spaces. If $T \in \mathcal{C}_p(\mathcal{H})$ and $\{\lambda_n\}_{n=0}^{\infty}$ is the set of eigenvalues of $|T|$, then $\|T\|_p = \left(\sum_{n=0}^{\infty} |\lambda_n|^p\right)^{1/p}$. Furthermore, in light of the relationships between dual spaces already discussed and the inclusions $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{C}_p(\mathcal{H}) \subseteq \mathcal{C}(\mathcal{H})$, $1 \leq p < \infty$, we may regard $\mathcal{F}(\mathcal{H})$, $\mathcal{C}(\mathcal{H})$, and $\mathcal{L}(\mathcal{H})$ as analogues of c_{00}, c_0, and ℓ_∞, respectively.

G. States and Representations

A *representation* of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$. An injective representation π is called *faithful* and if $\pi(\mathcal{A})' = \mathbb{C}$, π is called *irreducible*. A representation π is *cyclic* if $\pi(\mathcal{A})$ has a cyclic vector and π is *non-degenerate* if $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$. All irreducible and cyclic representations are clearly non-degenerate.

We call a linear functional $\rho \in \mathcal{A}^*$ *positive*, if $\rho(a) \geq 0$, for all $a \in \mathcal{A}_+$, or, equivalently, $\rho(1) = \|\rho\|$. A positive functional of norm one is called a *state* and the set of states in $\mathcal{A}^*$ is called the *state space* of $\mathcal{A}$, denoted $\mathcal{S}(\mathcal{A})$. If π is a cyclic representation of $\mathcal{A}$ on $\mathcal{H}$ and $x \in \mathcal{H}$ is a unit cyclic vector, then $\rho : a \mapsto \langle \pi(a)x, x\rangle$ is a state on $\mathcal{A}$. Conversely, for every state $\tau \in \mathcal{S}(\mathcal{A})$, there is an associated cyclic representation π_τ that is produced by the following method attributed to Gelfand, Naimark, and Segal [13, 34] and called the *GNS construction*.

Following the presentation of Murphy [28], we begin by defining a positive sesquilinear form on $\mathcal{A}$ by $\langle a, b\rangle = \tau(b^*a)$ and let $N_\tau = \{\, a \in \mathcal{A} : \tau(a^*a) = 0 \,\}$.

Since $\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$ [28, Theorem 3.3.7], N_τ is a closed left ideal of $\mathcal{A}$ and we may define an inner product on $\mathcal{A}/N_\tau$ by $\langle a+N_\tau, b+N_\tau\rangle = \tau(b^*a)$. Let $\mathcal{H}_\tau$ denote the completion of $\mathcal{A}/N_\tau$.

For all $a \in \mathcal{A}$, let $\varphi(a)$ be the linear map on $\mathcal{A}/N_\tau$ defined by $\varphi(a)(b + N_\tau) = ab + N_\tau$. Then

$$\|\varphi(a)(b + N_\tau)\|^2 = \tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b) = \|a\|^2\|b + N_\tau\|^2$$

and we may extend $\varphi(a)$ continuously to $\pi_\tau(a)$ on $\mathcal{H}_\tau$. The resulting map $\pi_\tau : \mathcal{A} \to \mathcal{L}(\mathcal{H}_\tau)$ is a representation of $\mathcal{A}$ on $\mathcal{H}_\tau$. Observe that $x_\tau = 1 + N_\tau$ is a cyclic vector for π_τ and $\tau(a) = \langle\pi_\tau(a)x_\tau, x_\tau\rangle$, for all $a \in \mathcal{A}$. We call π_τ the *GNS representation* associated with τ.

Now consider the category $\mathcal{C}$ of pairs $(\pi, \mathcal{M}(\pi))$, where $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a representation and $\mathcal{M}(\pi)$ is the WOT closure of $\pi(\mathcal{A})$ in $\mathcal{L}(\mathcal{H})$. A morphism between objects $(\pi, \mathcal{M}(\pi))$ and $(\rho, \mathcal{M}(\rho))$ of $\mathcal{C}$ is an ultraweakly continuous $*$-homomorphism $\tilde{\rho}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\pi"] \arrow[d, "\rho"'] & \mathcal{M}(\pi) \arrow[dl, "\tilde{\rho}"] \\
\mathcal{M}(\rho) &
\end{tikzcd}$$

Let $\pi = \bigoplus_{\tau\in\mathcal{S}(\mathcal{A})} \pi_\tau$ be the direct sum of the GNS representations of $\mathcal{A}$. Then $(\pi, \mathcal{M}(\pi))$ is a universally repelling object in $\mathcal{C}$ [39, Theorem 3.2.4]. Since universal objects are uniquely determined, we call π *the universal representation* of $\mathcal{A}$ and call $\mathcal{M}(\pi)$ *the universal enveloping von Neumann algebra* of $\mathcal{A}$. For every $0 \neq a \in \mathcal{A}$, there exists $\tau \in \mathcal{S}(\mathcal{A})$ such that $\tau(a^*a) = \|a^*a\|$ [28, Theorem 3.3.6], so $\pi_\tau(a) \neq 0$ and, consequently, the universal representation of every C^*-algebra is faithful. Furthermore, there is a correspondence between the states on $\pi(\mathcal{A})$ and $\mathcal{S}(\mathcal{A})$. Given a state ρ on $\pi(\mathcal{A})$, $\tau = \rho\pi$ is a state on $\mathcal{A}$ and $\tau(a) = \langle \pi_\tau(a)1 + N_\tau, 1 + N_\tau \rangle = \rho\pi(a)$. Consequently, every state on $\pi(\mathcal{A})$ is a vector state.

The ultraweak closure of a $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ is equal to its WOT closure [10]. In particular, if π is the universal representation of a C^*-algebra $\mathcal{A}$, then $\pi(\mathcal{A})$ is ultraweakly dense in its universal enveloping von Neumann algebra $\mathcal{M}(\pi)$ and every representation of $\mathcal{A}$ can be extended to a unique normal representation of $\mathcal{M}(\pi)$ [9]. More precisely, we have the following theorem.

Proposition 2.7. *Let $\mathcal{A}$ be a C^*-algebra, let π be its universal representation, and let $\mathcal{M}(\pi)$ be the universal enveloping von Neumann algebra of $\mathcal{A}$. Then for every representation $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ of $\mathcal{A}$ there is a unique normal representation $\tilde{\rho} : \mathcal{M}(\pi) \to \mathcal{L}(\mathcal{H})$ such that $\tilde{\rho}(\pi(x)) = \rho(x)$, for all $x \in \mathcal{A}$. Furthermore, $\tilde{\rho}(\mathcal{M}(\pi))$ is the ultraweak closure of $\rho(\mathcal{A})$.*

H. Continuous Hochschild Cohomology

Let $\mathcal{A}$ be a Banach algebra, and let M be a Banach space that is a bimodule over $\mathcal{A}$. If the left module action $(A, m) \mapsto Am$ and right module action $(m, A) \mapsto mA$ are bounded, then M is called a *Banach bimodule over* $\mathcal{A}$. In this case, we define $\mathcal{L}^0(\mathcal{A}, M)$ to be M and let $\mathcal{L}^n(\mathcal{A}, M)$ denote the space of all bounded n-linear maps, $f : \mathcal{A} \times \cdots \times \mathcal{A} \to M$, for $n > 0$. Elements of $\mathcal{L}^n(\mathcal{A}, M)$ are called *n-cochains.* The *coboundary maps* $\partial^n : \mathcal{L}^n(\mathcal{A}, M) \to \mathcal{L}^{n+1}(\mathcal{A}, M)$, often abbreviated ∂, are defined by

$$\begin{aligned}(\partial^n f)(a_1, \dots, a_{n+1}) = &a_1 f(a_2, \dots, a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_{n+1}) \\ &+ (-1)^{n+1} f(a_1, \dots, a_n) a_{n+1}.\end{aligned}$$

We define the subspace of *n-coboundaries* $\mathcal{B}^n(\mathcal{A}, M)$ to be the image of ∂^{n-1} and the subspace of *n-cocycles* $\mathcal{Z}^n(\mathcal{A}, M)$ to be the kernel of ∂^n. Since $\partial^2 = 0$, every coboundary is also a cocycle and we define the *Hochschild cohomology groups* of $\mathcal{A}$ with coefficients in M, denoted $H^n(\mathcal{A}, M)$, to be the quotient spaces $\mathcal{Z}^n(\mathcal{A}, M)/\mathcal{B}^n(\mathcal{A}, M)$.

A Banach bimodule M over $\mathcal{A}$ is said to be a *dual bimodule* over $\mathcal{A}$, if M is isometrically isomorphic to the dual space of a Banach space M_* and the maps $m \mapsto Am$ and $m \mapsto mA$ on M are weak* continuous for every $A \in \mathcal{A}$. If, additionally, $\mathcal{A}$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ and the maps $A \mapsto Am$ and $A \mapsto mA$ from $\mathcal{A}$ to M are ultraweak to weak* continuous, then M is called a *dual normal bimodule* over $\mathcal{A}$. We then say that $\rho \in \mathcal{L}^n(\mathcal{A}, M)$ is *normal*, if it is ultraweak to weak* continuous in each variable and denote the subspace of normal cochains by $\mathcal{L}^n_w(\mathcal{A}, M)$. Normal cocycles $\mathcal{Z}^n_w(\mathcal{A}, M)$ and normal coboundaries $\mathcal{B}^n_w(\mathcal{A}, M)$ are defined as above and because the boundary of a normal cochain is a normal cocycle, we may similarly define the *normal cohomology groups*, $H^n_w(\mathcal{A}, M)$.

Let M be a dual Banach bimodule over a Banach algebra $\mathcal{A}$. Johnson [21] observed that $\mathcal{L}^n(\mathcal{A}, M)$ is isometrically isomorphic to the dual space of $\mathcal{A} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A} \hat{\otimes}_\pi M_*$, where M_* is the predual of M, there are n copies of $\mathcal{A}$, and the duality is defined by

$$\langle a_1 \otimes \cdots \otimes a_n \otimes m_*, \xi \rangle = \langle m_*, \xi(a_1, \ldots, a_n) \rangle$$

for all $a_1, \ldots a_n \in \mathcal{A}$, $m_* \in M_*$, and $\xi \in \mathcal{L}^n(\mathcal{A}, M)$. There are two important dual bimodule actions [20] of $\mathcal{A}$ on $\mathcal{L}^n(\mathcal{A}, M)$. The first is given by

$$\begin{aligned} (a_0\xi)(a_1, \ldots, a_n) =& a_0\xi(a_1, \ldots, a_n) \\ (\xi a_0)(a_1, \ldots, a_n) =& \sum_{i=0}^{n-1} (-1)^j \xi(a_0, a_1, \ldots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \ldots, a_n) \\ &+ (-1)^n \xi(a_0, a_1, \ldots, a_{n-1}) a_n \end{aligned} \tag{2.1}$$

for all $a_0, \ldots, a_n \in \mathcal{A}$ and $\xi \in \mathcal{L}^n(\mathcal{A}, M)$, and there is a canonical isometric linear isomorphism $i_n : \mathcal{L}^n(\mathcal{A}, \mathcal{L}^p(\mathcal{A}, M)) \to \mathcal{L}^{n+p}(\mathcal{A}, M)$ defined by $(i_n\xi)(a_1, \ldots, a_{n+p}) = \xi(a_1, \ldots, a_n)(a_{n+1}, \ldots, a_{n+p})$, for all $a_1, \ldots, a_{n+p} \in \mathcal{A}$ and $n, p \geq 0$. Note that $i_{n+1}(\partial\xi) = \partial(i_n\xi)$ and, hence, $H^n(\mathcal{A}, \mathcal{L}^p(\mathcal{A}, M)) \cong H^{n+p}(\mathcal{A}, M)$ [16, Theorem 3.1].

Another dual bimodule action [20] of $\mathcal{A}$ on $\mathcal{L}^n(\mathcal{A}, M)$ may be defined by

$$(a_0\xi)(a_1, \ldots, a_n) = \xi(a_1, \ldots, a_n a_0)$$
$$(\xi a_0)(a_1, \ldots, a_n) = \xi(a_1, \ldots, a_n)a_0$$

for all $a_0, \ldots, a_n \in \mathcal{A}$. Then M may be replaced by $\mathcal{L}^p(\mathcal{A}, M)$, where $p > 0$, with the action defined in (2.1). Since i_n is weak* bicontinuous, for all $n \geq 0$, this new dual bimodule structure of $\mathcal{L}^n(\mathcal{A}, \mathcal{L}^p(\mathcal{A}, M))$ may be transferred onto $\mathcal{L}^{n+p}(\mathcal{A}, M)$ and, in this case, the bimodule operations are

$$(a_0\xi)(a_1, \ldots, a_{n+p}) = \xi(a_1, \ldots, a_n a_0, a_{n+1}, \ldots, a_{n+p})$$
$$\begin{aligned}(\xi a_0)(a_1, \ldots, a_{n+p})& \\ =&\xi(a_1, \ldots, a_n, a_0 a_{n+1}, \ldots, a_{n+p}) \\ &+ \sum_{i=1}^{p-1}(-1)^j \xi(a_1, \ldots, a_n, a_0, a_{n+1}, \ldots, a_{n+j-1}, a_{n+j}a_{n+j+1}, a_{n+j+2}, \ldots, a_{n+p}) \\ &+ (-1)^p \xi(a_1, \ldots, a_n, a_0, a_{n+1}, \ldots, a_{n+p-1})a_{n+p}\end{aligned}$$

for all $a_0, \ldots, a_{n+p} \in \mathcal{A}$ and $\xi \in \mathcal{L}^{n+p}(\mathcal{A}, M)$.

Each of the dual bimodule structures on $\mathcal{L}^n(\mathcal{A}, M)$ defined above plays a valuable role in the proofs of the averaging theorems contained in the sequel.

CHAPTER III

SUBMODULES OF MATRIX ALGEBRAS

We begin by proving some important facts about submodules of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$, where $\mathfrak{A}$ is an abelian von Neumann algebra, that will be useful in our cohomology calculations.

A. Orthogonal Complements in Matrix Algebras

Within the class of Banach spaces, Lindenstrauss and Tzafriri [27] characterized Hilbert spaces by the property that every closed subspace has a closed complement. A generalization of a Hilbert space called a C^*-module can be obtained by replacing the scalar inner product with an inner product having values in an abelian C^*-algebra. C^*-modules were first studied by Kaplansky [26] who defined them in the following manner.

Definition 3.1. Let $\mathcal{A}$ be an abelian C^*-algebra and let M be a left module over $\mathcal{A}$. We call a function $\langle \cdot\,, \cdot \rangle : M \times M \to \mathcal{A}$ an *inner product* on M, if it satisfies

(i) $\langle x, x \rangle \geq 0$ for all $x \in M$ and $\langle x, x \rangle = 0$ implies $x = 0$,

(ii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in M$,

(iii) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in M$.

An inner product defines a norm $|||\cdot|||$ on M by $|||x|||^2 = \|\langle x, x\rangle\|$. When $|||\cdot|||$ is complete, we call M a *C^*-module* over $\mathcal{A}$.

Let $\mathfrak{A}$ be an abelian von Neumann algebra acting on $\mathcal{H}$ and let Ω be the maximal ideal space of $\mathfrak{A}$. We then consider $M_n(\mathfrak{A}) \cong \mathfrak{A} \otimes M_n(\mathbb{C})$ to be a von Neumann algebra acting on $\mathcal{H} \otimes \mathbb{C}^n$ and let $\Phi : M_n(\mathfrak{A}) \to \mathfrak{A}$ be the sum of the diagonal entries of a matrix in $M_n(\mathfrak{A})$. We define an $\mathfrak{A}$-valued inner product on $M_n(\mathfrak{A}) \times M_n(\mathfrak{A})$ by $\langle A, B\rangle = \Phi(B^*A)$. Our most important examples of C^*-modules will be submodules of $M_n(\mathfrak{A})$. We now establish some facts about $M_n(\mathfrak{A})$ and the inner product we have defined.

Lemma 3.1. *Let $\mathcal{C}(\Omega, M_n(\mathbb{C}))$ be the algebra of continuous matrix valued functions on Ω with the supremum norm $\|A\|_\infty = \sup_{\omega\in\Omega} \|A(\omega)\|$ and involution $A^* = (a_{ji}^*)_{i,j=1}^n$, where $A = (a_{ij})_{i,j=1}^n \in \mathcal{C}(\Omega, M_n(\mathbb{C}))$ and $\|\cdot\|$ denotes the operator norm on $M_n(\mathbb{C})$. Then $M_n(\mathfrak{A})$ is $*$-isomorphic to $\mathcal{C}(\Omega, M_n(\mathbb{C}))$ as a C^*-algebra.*

Proof. First note that $\mathcal{C}(\Omega, M_n(\mathbb{C})) \cong \mathcal{C}(\Omega) \otimes M_n(\mathbb{C})$ algebraically, by identifying $A = (a_{ij})_{i,j=1}^n \in \mathcal{C}(\Omega, M_n(\mathbb{C}))$ with $\sum_{i,j=1}^n a_{ij} \otimes E_{ij}$, where $\{E_{ij}\}_{i,j=1}^n$ are the canonical matrix units in $M_n(\mathbb{C})$. Then, by choosing appropriate unit vectors in $\mathbb{C}^n$, we obtain

$$|a_{kl}(\omega)| = \|a_{kl}(\omega) \otimes E_{kl}\| \leq \|(a_{ij}(\omega))_{i,j=1}^n\| \leq \sum_{i,j=1}^n \|a_{ij}(\omega) \otimes E_{ij}\| = \sum_{i,j=1}^n |a_{ij}(\omega)|,$$

for all $1 \leq k, l \leq n$ and $\omega \in \Omega$, so the completeness of $\mathcal{C}(\Omega)$ with respect to the supremum norm implies the same for $\mathcal{C}(\Omega, M_n(\mathbb{C}))$. Furthermore, because the operator norm is a C^*-norm on $M_n(\mathbb{C})$,

$$\|A^*A\|_\infty = \sup_{\omega\in\Omega} \|A^*(\omega)A(\omega)\| = \sup_{\omega\in\Omega} \|A(\omega)\|^2 = \left(\sup_{\omega\in\Omega} \|A(\omega)\|\right)^2 = \|A\|_\infty^2,$$

for all $A \in \mathcal{C}(\Omega, M_n(\mathbb{C}))$, and we conclude that $\mathcal{C}(\Omega, M_n(\mathbb{C}))$ is a C^*-algebra.

Recall that the Gelfand transformation $\Gamma : \mathfrak{A} \to \mathcal{C}(\Omega)$ is a $*$-isomorphism. Consequently, $\Gamma \otimes 1_n : M_n(\mathfrak{A}) \to \mathcal{C}(\Omega, M_n(\mathbb{C}))$ defined by $(\Gamma \otimes 1_n)(A) = (\Gamma a_{ij})_{i,j=1}^n$, for all $A = (a_{ij})_{i,j=1}^n \in M_n(\mathfrak{A})$, is bijective,

$$\begin{aligned}(\Gamma \otimes 1_n)(AB) &= (\Gamma \otimes 1_n)\left(\sum_{k=1}^n a_{ik}b_{kj}\right)_{i,j=1}^n \\ &= \left(\sum_{k=1}^n \Gamma a_{ik}\Gamma b_{kj}\right)_{i,j=1}^n \\ &= (\Gamma \otimes 1_n)(A)(\Gamma \otimes 1_n)(B)\end{aligned}$$

for all $A = (a_{ij})_{i,j=1}^n, B = (b_{ij})_{i,j=1}^n \in M_n(\mathfrak{A})$, and $(\Gamma \otimes 1_n)(A^*) = (\Gamma a_{ji}^*)_{i,j=1}^n = ((\Gamma a_{ji})^*)_{i,j=1}^n = ((\Gamma \otimes 1_n)(A))^*$, for all $A = (a_{ij})_{i,j=1}^n \in M_n(\mathfrak{A})$. Therefore, $\Gamma \otimes 1_n$ is a $*$-isomorphism between C^*-algebras. $\square$

The identification made in Lemma 3.1 allows us to relate the operator norm on $M_n(\mathfrak{A})$ to the norm $|||\cdot|||$ induced by the inner product we have defined.

Lemma 3.2. *The operator norm and* $|||\cdot|||$ *are equivalent on* $M_n(\mathfrak{A})$.

Proof. Since the operator norm and $|||\cdot|||$ are equivalent on $M_n(\mathbb{C})$ [5, Theorem 3.3.1], there are constants $\alpha, \beta > 0$ such that $\alpha\|A\| \le |||A||| \le \beta\|A\|$, for all $A \in M_n(\mathbb{C})$. If $\Gamma : \mathfrak{A} \to \mathcal{C}(\Omega)$ is the Gelfand transformation, then $\Gamma \otimes 1_n : M_n(\mathfrak{A}) \to \mathcal{C}(\Omega, M_n(\mathbb{C}))$ is a $*$-isomorphism between C^*-algebras. In particular, Γ and $\Gamma \otimes 1_n$ are isometric. For all $A = (a_{ij})_{i,j=1}^n \in M_n(\mathfrak{A})$,

$$\begin{aligned}
\alpha\|A\| &= \alpha\|(\Gamma \otimes 1_n)(a_{ij})_{i,j=1}^n\|_\infty \\
&= \sup_{\omega\in\Omega} \alpha\|(\Gamma a_{ij}(\omega))_{i,j=1}^n\| \\
&\leq \sup_{\omega\in\Omega} |||(\Gamma a_{ij}(\omega))_{i,j=1}^n||| \\
&= \sup_{\omega\in\Omega} \left(\sum_{i,j=1}^n (\Gamma a_{ij}^*)(\omega)(\Gamma a_{ij})(\omega)\right)^{1/2} \\
&= \sup_{\omega\in\Omega} \left(\Gamma\left(\sum_{i,j=1}^n a_{ij}^* a_{ij}\right)(\omega)\right)^{1/2} \\
&= \left(\sup_{\omega\in\Omega} \Gamma\left(\sum_{i,j=1}^n a_{ij}^* a_{ij}\right)(\omega)\right)^{1/2} \\
&= \left\|\Gamma\left(\sum_{i,j=1}^n a_{ij}^* a_{ij}\right)\right\|_\infty^{1/2} \\
&= \left\|\sum_{i,j=1}^n a_{ij}^* a_{ij}\right\|^{1/2} \\
&= |||A|||
\end{aligned}$$

and, similarly, $|||A||| \leq \beta\|A\|$. ☐

Suppose M is a norm closed submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$. Since M is complete with respect to the operator norm on $M_n(\mathfrak{A})$, it is also complete with respect to $|||\cdot|||$ and, consequently, M is a C^*-module over $\mathfrak{A} \otimes 1_n$. Kaplansky realized, however, that the structure of a C^*-module was insufficient to mimic all of the main characteristics of a Hilbert space. He studied a class of C^*-modules having properties analogous to the SOT.

Definition 3.2. Let $\mathfrak{A}$ be a commutative von Neumann algebra. We say that M is a *W^*-module over* $\mathfrak{A}$ if it is a C^*-module over $\mathfrak{A}$ and has the following two properties:

(i) If $\{e_\alpha\}_{\alpha\in I}$ is a family of pairwise orthogonal projections in $\mathfrak{A}$ with supremum e and x is an element of M such that $e_\alpha x = 0$, for all $\alpha \in I$, then $ex = 0$.

(ii) If $\{e_\alpha\}_{\alpha\in I}$ is a family of pairwise orthogonal projections in $\mathfrak{A}$ with supremum 1 and $\{x_\alpha\}$ is a bounded subset of M, then there exists an element x in M such that $e_\alpha x_\alpha = e_\alpha x$, for all $\alpha \in I$.

A W^*-*submodule* of M is a norm closed submodule N of M which is also a W^*-module over $\mathfrak{A}$.

Lemma 3.3. *Let M be an ultraweakly closed submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A}\otimes 1_n$. Then M is a W^*-module over $\mathfrak{A}\otimes 1_n$.*

Proof. Suppose that $x \in M$ and $\{e_\alpha\}_{\alpha\in I}$ is a family of pairwise orthogonal projections in $\mathfrak{A}\otimes 1_n$ with supremum e such that $e_\alpha x = 0$, for all $\alpha \in I$. Let $\mathcal{F}$ be the collection of all finite subsets of I, partially ordered by inclusion, and let $S_F = \sum_{\alpha\in F} e_\alpha \le e$, for all $F \in \mathcal{F}$. Then $\{S_F\}_{F\in\mathcal{F}}$ is a bounded increasing net in $\mathfrak{A}\otimes 1_n$ and converges in the SOT to its least upper bound e [28, Theorem 4.1.1]. Since $S_F x = 0$, for all $F \in \mathcal{F}$, and $\{S_F x\}_{F\in\mathcal{F}}$ converges to ex in the SOT, $ex = 0$.

Now let $\{f_\beta\}_{\beta\in J}$ be a family of pairwise orthogonal projections in $\mathfrak{A}\otimes 1_n$ with supremum $1 = 1_{\mathfrak{A}}\otimes 1_n$ and let $\{y_\beta\}_{\beta\in J}$ be a bounded set in M. First assume that $y_\beta \ge 0$, for all $\beta \in J$, and B is a uniform bound for $\{y_\beta\}_{\beta\in J}$. Let $\mathcal{G}$ be the collection of all finite subsets of J, partially ordered by inclusion, and let

$$T_G = \sum_{\beta\in G} f_\beta y_\beta = \sum_{\beta\in G} f_\beta y_\beta f_\beta \le B1,$$

for all $G \in \mathcal{G}$. Then $\{T_G\}_{G\in\mathcal{G}}$ is a bounded increasing net in M and converges in the SOT to its least upper bound y. Since the ultraweak topology and the WOT coincide on bounded sets and the SOT is stronger than the WOT, $\{T_G\}_{G\in\mathcal{G}}$ also converges ultraweakly to y. Because M is ultraweakly closed, $y \in M$. Furthermore, given

$\beta_0 \in J$, then for all $G \in \mathcal{G}$ containing $\{\beta_0\}$, $f_{\beta_0} T_G = f_{\beta_0} y_{\beta_0}$ and, hence, $f_{\beta_0} y = f_{\beta_0} y_{\beta_0}$ as required.

For an arbitrary bounded collection of elements $\mathcal{S} = \{y_\beta\}_{\beta \in J} \subseteq M$, we consider the collections of the real and imaginary parts of each element in $\mathcal{S}$, denoted $\mathrm{Re}(\mathcal{S})$ and $\mathrm{Im}(\mathcal{S})$, respectively. Since $\mathcal{S}$ is uniformly bounded, we may assume that every element of $\mathrm{Re}(\mathcal{S})$ and $\mathrm{Im}(\mathcal{S})$ is positive, by adding a multiple of the identity, if necessary. Then there are self-adjoint elements $\mathrm{Re}(y), \mathrm{Im}(y) \in M_n(\mathfrak{A})$ such that $f_\beta \mathrm{Re}(y) = f_\beta \mathrm{Re}(y_\beta)$ and $f_\beta \mathrm{Im}(y) = f_\beta \mathrm{Im}(y_\beta)$, for all $\beta \in J$. Because $y = \mathrm{Re}(y) + i\mathrm{Im}(y)$ is a limit of elements of M in the ultraweak topology, we conclude that $y \in M$. $\square$

Definition 3.3. Let M be a W^*-module over $\mathfrak{A}$. We say that x and y are *orthogonal*, if $\langle x, y \rangle = 0$. The *orthogonal complement* $R^\perp$ of a subset R of M is the set of all $x \in M$ such that $\langle x, R \rangle = 0$.

A sequence or net of matrices in $M_n(\mathcal{L}(\mathcal{H}))$ is norm convergent entrywise if and only if it norm convergent in $M_n(\mathcal{L}(\mathcal{H}))$. Convergence is equivalent to entrywise convergence in all of the weaker topologies on $M_n(\mathcal{L}(\mathcal{H}))$, as well, and a simple consequence is stated as the next lemma.

Lemma 3.4. *Let $\mathcal{S}$ be a subset of $M_n(\mathfrak{A})$. Then $\mathcal{S}^\perp$ is an ultraweakly closed submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$.*

Proof. From the definition of an inner product, it is clear that $\mathcal{S}^\perp$ is a submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$. Suppose that $A = (a_{ij})_{i,j=1}^n \in \mathcal{S}$ and $\{B^\alpha\}_{\alpha \in I}$ is a net of matrices in $\mathcal{S}^\perp$ converging ultraweakly to $B = (b_{ij})_{i,j=1}^n \in M_n(\mathfrak{A})$. Let $B^\alpha = (b_{ij}^\alpha)_{i,j=1}^n$, for all $\alpha \in I$. Then $b_{ij}^\alpha \to b_{ij}$ ultraweakly, for all $1 \le i, j \le n$, and

$$\langle B, A \rangle = \sum_{i,j=1}^n a_{ji}^* b_{ji} = \sum_{i,j=1}^n a_{ji}^* \lim_{\alpha \in I} b_{ji}^\alpha = \lim_{\alpha \in I} \sum_{i,j=1}^n a_{ji}^* b_{ji}^\alpha = \lim_{\alpha \in I} \langle B^\alpha, A \rangle = 0,$$

so that $B \in \mathcal{S}^\perp$. $\square$

Kaplansky [26, Theorem 3] proved that, as in a Hilbert space, a W^*-module can be decomposed into a sum of any W^*-submodule and its orthogonal complement.

Theorem 3.5. *Let M be a W^*-module over $\mathfrak{A}$, let N be a W^*-submodule of M, and let $N^\perp$ be the orthogonal complement of N. Then $M = N \oplus N^\perp$.*

A mapping $F : M \to \mathfrak{A}$ is called a *linear functional*, if F is linear and homogeneous with respect to $\mathfrak{A}$. In this case, we say that F is *bounded*, if F is continuous with respect to the norm $|||\cdot|||$ induced by the inner product on M. Continuing the analogy with Hilbert space, Kaplansky [26, Theorem 5] showed that bounded linear functionals on M can be identified with elements of M.

Theorem 3.6. *Let $F : M \to \mathfrak{A}$ be a bounded linear functional. Then there exists a unique element $y \in M$ such that $F(x) = \langle x, y\rangle$, for all $x \in M$.*

By combining Lemma 3.3 and Theorem 3.6, we show that the projection of a W^*-submodule M of $M_n(\mathfrak{A})$ onto M is continuous and M is homeomorphic to the quotient $M_n(\mathfrak{A})/M^\perp$. Both facts, however, are consequences of the next proposition.

Proposition 3.7. *Let M be an ultraweakly closed submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$ and let F be an ultraweakly closed subset of M. Then $F + M^\perp$ is ultraweakly closed.*

Proof. Let $F_{kl} : M_n(\mathfrak{A}) \to \mathfrak{A}$ be the linear functional defined by $a = (a_{ij})_{i,j=1}^n \mapsto a_{kl}$, for all $1 \leq k, l \leq n$. Then F_{kl} is ultraweakly continuous and, by Lemma 3.2, F_{kl} is bounded. By Theorem 3.6, there exists $b_{kl} \in M$ such that $F_{kl}(a) = \langle a, b_{kl}\rangle$, for all $a \in M$ and $1 \leq k, l \leq n$, since M is a W^*-module over $\mathfrak{A} \otimes 1_n$.

Suppose that $\{f_\alpha + m_\alpha^\perp\}_{\alpha\in I}$ is a net in $F + M^\perp$, where $f_\alpha \in F$ and $m_\alpha^\perp \in M^\perp$, for all $\alpha \in I$, and suppose $f_\alpha + m_\alpha^\perp \to x \in M_n(\mathfrak{A})$. By Theorem 3.5, there exist $m \in M$ and $m^\perp \in M^\perp$ such that $x = m + m^\perp$. Then

$$F_{kl}(f_\alpha) = \langle f_\alpha, b_{kl}\rangle = \langle f_\alpha + m_\alpha^\perp, b_{kl}\rangle \to \langle x, b_{kl}\rangle = \langle m + m^\perp, b_{kl}\rangle = \langle m, b_{kl}\rangle = F_{kl}(m),$$

for all $1 \leq k, l \leq n$, so $f_\alpha \to m$ ultraweakly. Since F is ultraweakly closed, $m \in F$ and $x \in F + M^\perp$. $\square$

Corollary 3.8. *Let M be an ultraweakly closed submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$. Then the projection $E : M_n(\mathfrak{A}) \to M$ is continuous with respect to the ultraweak topology.*

Proof. Let F be ultraweakly closed in M. Then $E^{-1}(F) = F + M^\perp$ is ultraweakly closed. $\square$

Corollary 3.9. *Let M be an ultraweakly closed submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$. Then the restriction of the quotient map $\pi : M_n(\mathfrak{A}) \to M_n(\mathfrak{A})/M^\perp$ to M is an homeomorphism with respect to the ultraweak topology.*

Proof. Let β denote the restriction of π to M. By Theorem 3.5, β is bijective and if F is ultraweakly closed in M, then $\pi^{-1}(\beta(F)) = F + M^\perp$ is ultraweakly closed. We conclude that β is a closed map and a homeomorphism. $\square$

B. Ultraweakly Closed Submodules

The maximal ideal space Ω of an abelian von Neumann algebra $\mathfrak{A}$ is compact and extremally disconnected [39, Theorem 3.1.18], or *Stonian*. In particular, the closure of every open subset G of Ω is compact and open. This additional structure allows us to establish some properties of submodules of $M_n(\mathfrak{A})$ that are analogous to those of scalar matrices.

Stone [36, Theorem 17] proved that the algebra of real valued continuous functions $\mathcal{C}_\mathbb{R}(\Omega)$ on Ω is a boundedly complete lattice — that is to say, every uniformly bounded subset of $\mathcal{C}_\mathbb{R}(\Omega)$ has a least upper bound in $\mathcal{C}_\mathbb{R}(\Omega)$. The following lemma is a consequence that was first noted by Deckard and Pearcy [7, Lemma 2.1] for complex valued functions on Ω. By Lemma 3.1, the Gelfand transformation $\Gamma : \mathfrak{A} \to \mathcal{C}(\Omega)$ allows us to identify $M_n(\mathfrak{A})$ with $\mathcal{C}(\Omega, M_n(\mathbb{C}))$, the algebra of continuous matrix valued functions on Ω, and, henceforth, we tacitly use this fact.

Lemma 3.10. *Let M be an ultraweakly closed submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$. Suppose that $\mathcal{G} = \{G_i\}_{i \in I}$ is a collection of pairwise disjoint compact open subsets of Ω, and $\mathcal{S} = \{A_i\}_{i \in I} \subseteq M$ is a uniformly bounded collection of functions on Ω. Then there is a function $A \in M$ such that $A(\omega) = A_i(\omega)$, for all $\omega \in G_i$ and $i \in I$.*

Proof. Let χ_i be the characteristic function of G_i and let $p_i = (\Gamma \otimes 1_n)^{-1}(\chi_i \otimes 1_n)$, for all $i \in I$. Then $\{p_i\}_{i \in I}$ is a pairwise orthogonal family of projections in $\mathfrak{A} \otimes 1_n$. We may assume, without loss of generality, that the supremum p of $\{p_i\}_{i \in I}$ is 1. Otherwise, we add $1 - p$ to the collection and let the corresponding function be identically zero. By Lemma 3.3, there exists $A \in M$ such that $p_i A = p_i A_i$ and, hence, $A(\omega) = A_i(\omega)$, for all $i \in I$ and $\omega \in G_i$. $\square$

Using the notation in the proof of Lemma 3.10, let p be the least upper bound of $\{p_i\}_{i \in I}$ and let $A \in M_n(\mathfrak{A})$ and $B \in M_n(\mathfrak{A})$ satisfy the conclusion of the lemma. Then $pA = pB$ and we let $\sum_{i \in I} p_i A_i$ denote pA. In particular, for the matrix A constructed in the proof of Lemma 3.10, $A = pA$.

Although an arbitrary submodule of $M_n(\mathfrak{A})$ is not free over $\mathfrak{A} \otimes \mathbb{C}1_n$, we now show that, given an ultraweakly closed submodule M of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$, we can decompose M into a finite direct sum of free modules. We say that a free module is of *finite type* over $\mathfrak{A} \otimes \mathbb{C}1_n$, if it has a finite basis over $\mathfrak{A} \otimes \mathbb{C}1_n$.

Theorem 3.11. *Let M be an ultraweakly closed submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$. Then there are a finite number of pairwise disjoint open subsets $\{\mathcal{O}_k\}_{k=0}^{t}$ of Ω such that $\Omega = \bigcup_{k=0}^{t} \mathcal{O}_k^-$ and if χ_k is the characteristic function of $\mathcal{O}_k^-$ and $p_k = \chi_k \otimes 1_n$,* *then* $M_k = p_k M$ *is a free module of finite type over* $\mathcal{C}(\mathcal{O}_k^-)$*, for all* $0 \le k \le t$*. Furthermore,* $M \cong \bigoplus_{k=0}^t M_k$.

Proof. If $M = \{0\}$, the statement is trivial, so we assume that $M \neq \{0\}$. Since $M(\omega) = \{\, A(\omega) : A \in M \,\}$ is a subspace of $M_n(\mathbb{C})$, for all $\omega \in \Omega$, $M(\omega)$ is finite dimensional. We let $d(\omega)$ denote the dimension of $M(\omega)$ and let $d_0 = \sup_{\omega\in\Omega} d(\omega)$. Observe that $0 < d_0 \le n^2$.

Now define $\mathcal{O}_0 = \{\, \omega \in \Omega : d(\omega) = d_0 \,\}$. Given $\omega_0 \in \mathcal{O}_0$, there is a set of functions $\{A_k\}_{i=1}^{d_0} \subseteq M$ such that $\{A_k(\omega_0)\}_{k=1}^{d_0}$ is a basis for $M(\omega_0)$ over $\mathbb{C}$ and $\|A_k(\omega_0)\| < 1$, for all $1 \le k \le d_0$. By appending rows together, for example, we may also consider $\{A_k\}_{k=1}^{d_0}$ to be continuous functions taking values in $\mathbb{C}^{n^2}$. Then the set of functions $\{A_k\}_{k=1}^{d_0}$ forms a $n^2 \times d_0$ matrix C having a $d_0 \times d_0$ submatrix D such that $|D(\omega_0)| \neq 0$. Since the determinant of D is continuous, there exists a compact open neighbourhood U_0 of ω_0 such that $|D(\omega)| \neq 0$ and $\|A_k(\omega)\| \le 1$, for all $\omega \in U_0$ and $1 \le k \le d_0$. Consequently, $\{A_k(\omega)\}_{k=1}^{d_0}$ is a linearly independent set in $M_n(\mathbb{C})$, for all $\omega \in U_0$, and, for every $B \in M$, there are unique scalar valued functions $\{f_k\}_{k=1}^{d_0}$ on U_0 such that $B(\omega) = \sum_{k=1}^{d_0} f_k(\omega) A_k(\omega)$, for all $\omega \in U_0$. Cramer's rule implies that $\{f_k\}_{k=1}^{d_0} \subseteq \mathcal{C}(U_0)$ and we conclude, in particular, that $\mathcal{O}_0$ is an open subset of Ω.

Let $\mathcal{F}$ be the collection of all families of pairwise disjoint compact open subsets of $\mathcal{O}_0$, such that for all $\mathcal{G}_\alpha = \{\, G_i\}_{i\in I_\alpha} \in \mathcal{F}$, there is a set of matrix valued functions $\{A_k^\alpha\}_{k=1}^{d_0} \subseteq M$ supported on $G_\alpha = \left(\bigcup_{i\in I_\alpha} G_i\right)^-$ such that $\|A_k^\alpha\| \le 1$, for all $1 \le k \le d_0$, and, for all $B \in M$, there are unique functions $\{f_k^\alpha\}_{k=1}^{d_0} \subseteq \mathcal{C}(G_\alpha)$ such that $B(\omega) = \sum_{k=1}^{d_0} f_k^\alpha(\omega) A_k^\alpha(\omega)$, for all $\omega \in G_\alpha$. Then $\mathcal{F}$ is not empty and we define a partial order on $\mathcal{F}$ by writing $\mathcal{G}_\alpha \le \mathcal{G}_{\alpha'}$ if $\mathcal{G}_\alpha \subseteq \mathcal{G}_{\alpha'}$ and $A_k^\alpha(\omega) = A_k^{\alpha'}(\omega)$, for all $\omega \in G_\alpha \cap G_{\alpha'}$ and $1 \le k \le d_0$. If $\mathcal{C} = \{\mathcal{G}_\gamma\}_{\gamma\in\Gamma}$ is a chain in $\mathcal{F}$, then, by Lemma 3.10, $\mathcal{G} = \bigcup_{\gamma\in\Gamma} \mathcal{G}_\gamma$ is an upper bound for $\mathcal{C}$ in $\mathcal{F}$. By Zorn's lemma, there exists a maximal element $\mathcal{G}_0 = \{\, G_i \}_{i \in I_0} \in \mathcal{F}$.

Let $G_0 = \left(\bigcup_{i\in I_0} G_i\right)^-$ and assume, to obtain a contradiction, that $\Sigma = \mathcal{O}_0 \setminus G_0$ is not empty. Since Σ is an open subset of Ω, given $\sigma_0 \in \Sigma$, there is a compact open neighbourhood $V_0 \subseteq \Sigma$ of σ_0 and a set of matrix valued functions $\{E_k\}_{k=1}^{d_0} \subseteq M$ supported on V_0 such that $\|E_k\| \leq 1$, for all $1 \leq k \leq d_0$, and, for all $B \in M$, there exist unique functions $\{g_k\}_{k=1}^{d_0} \subseteq \mathcal{C}(V_0)$ such that $B(\omega) = \sum_{k=1}^{d_0} g_k(\omega)E_k(\omega)$, for all $\omega \in V_0$. This contradicts the maximality of $\mathcal{G}_0$. Therefore, $\mathcal{O}_0 \subseteq G_0$ and since the inclusion $\mathcal{O}_0^- \supseteq G_0$ is obvious, $\mathcal{O}_0^- = G_0$.

Now replace Ω with $\Omega' = \Omega \setminus \mathcal{O}_0^-$. Then Ω' is a compact open subset of Ω and let $d_1 = \sup_{\omega\in\Omega'} d(\omega) < d_0$. If $d_1 = 0$, then we are done. Otherwise, let $\mathcal{O}_1 = \{\, \omega \in \Omega' : d(\omega) = d_1 \,\}$ and we construct a set of matrix valued functions $\{A_k^1\}_{k=1}^{d_1} \subseteq M$ supported on $\mathcal{O}_1^-$ such that $\|A_k^1\| \leq 1$, for $1 \leq k \leq d_1$, and, for all $B \in M$, there are unique functions $\{f_k^1\}_{k=1}^{d_1} \subseteq \mathcal{C}(\mathcal{O}_1^-)$ such that $B(\omega) = \sum_{k=1}^{d_1} f_k^1(\omega)A_k^1(\omega)$, for all $\omega \in \mathcal{O}_1^-$. We continue in this manner until we have pairwise disjoint open subsets $\{\mathcal{O}_k\}_{k=0}^{t}$ of Ω such that $\Omega = \bigcup_{k=0}^{t} \mathcal{O}_k^- = \left(\bigcup_{k=0}^{t} \mathcal{O}_k\right)^-$, $d(\omega) = d_k$ when $\omega \in \mathcal{O}_k$, for all $0 \leq k \leq t$, and $0 \leq d_t < \cdots < d_1 < d_0 \leq n^2$. Furthermore, for all $0 \leq k \leq t$, we construct a set of d_k matrix valued functions $\{A_m^k\}_{m=1}^{d_k}$ supported on $\mathcal{O}_k^-$ such that $\|A_m^k\| \leq 1$, for all $1 \leq m \leq d_k$, and, for all $B \in M$, there are unique functions $\{f_m^k\}_{m=1}^{d_k} \in \mathcal{C}(\mathcal{O}_k^-)$ such that $B(\omega) = \sum_{m=1}^{d_k} f_m^k(\omega)A_m^k(\omega)$, for all $\omega \in \mathcal{O}_k^-$. Hence, if χ_k is the characteristic function of $\mathcal{O}_k^-$, $p_k = \chi_k \otimes 1_n$, and $M_k = p_k M$, for all $0 \leq k \leq t$, then $M \cong \bigoplus_{k=0}^{t} M_k$ and M_k is a free module of finite type over $\mathcal{C}(\mathcal{O}_k^-)$, for all $0 \leq k \leq t$. $\square$

Suppose that M and N are ultraweakly closed submodules of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes 1_n$ and $N \subseteq M$. The algorithm in the proof of Theorem 3.11 may be iterated to construct open sets that decompose Ω for *both* M and N. More precisely, we have the following corollary.

Corollary 3.12. *Let M and N be ultraweakly closed submodules of $M_n(\mathfrak{A})$ over $\mathfrak{A}\otimes 1_n$ such that $N \subseteq M$. Then there are a finite number of pairwise disjoint open subsets $\{\mathcal{O}_k\}_{k=0}^t$ of Ω such that $\Omega = \bigcup_{k=0}^t \mathcal{O}_k^-$ and if χ_k is the characteristic function of $\mathcal{O}_k^-$ and $p_k = \chi_k \otimes 1_n$, then $M_k = p_k M$ and $N_k = p_k N$ are free modules of finite type over $\mathcal{C}(\mathcal{O}_k^-)$. Furthermore, there is a finite basis for M_k over $\mathcal{C}(\mathcal{O}_k^-)$ containing a basis for N_k, for all $0 \leq k \leq t$.*

Proof. By using the decomposition of Theorem 3.11 for N, we may assume that N is a free module of finite type over $\mathfrak{A} \otimes 1_n$ and there exists an open set $\mathcal{O}$ that is dense in Ω such that $\dim N(\omega) = m > 0$, for all $\omega \in \mathcal{O}$. We fix a basis $\{A_k\}_{k=1}^m$ for N over $\mathfrak{A} \otimes 1_n$ and, using the technique in the proof of Theorem 3.11, we obtain a finite number of pairwise disjoint open subsets $\{\mathcal{O}_k\}_{k=0}^t$ of $\mathcal{O}$ such that $\Omega = \bigcup_{k=0}^t \mathcal{O}_k^-$ and if χ_k is the characteristic function of $\mathcal{O}_k^-$ and $p_k = \chi_k \otimes 1_n$, for all $0 \leq k \leq t$, then $M_k = p_k M$ is a free module of finite type over $\mathcal{C}(\mathcal{O}_k^-)$, for all $0 \leq k \leq t$. Since $\{A_k(\omega)\}_{k=1}^m$ is linearly independent, for all $\omega \in \mathcal{O}_k$, we may extend $\{A_k\}_{k=1}^m$ to a basis for M_k over $\mathcal{C}(\mathcal{O}_k^-)$, for every $0 \leq k < t$. $\square$

Remark 3.13. *Note that N need not be ultraweakly closed for the proof of Corollary 3.12 to be valid. It suffices that there exist a set of pairwise orthogonal projections $\{\chi_i\}_{i=1}^k$ in $\mathcal{C}(\Omega)$ such that $\sum_{i=1}^k \chi_i = 1$ and, for all $1 \leq i \leq k$, $(\chi_i \otimes 1_n)N$ has a basis $\{A_{ij}\}_{j=1}^{\ell_i}$ over $\mathcal{C}(\Omega_i^-)$, where Ω_i is open, Ω_i^- is the range of χ_i, and $\{A_{ij}(\omega)\}_{i=1}^{\ell_i}$ is linearly independent over $\mathbb{C}$, for all $\omega \in \Omega_i$. Of course, when N is ultraweakly closed, the proof of Theorem 3.11 shows that these conditions are satisfied.*

In later calculations it will not be possible to satisfy the conditions of Corollary 3.12, but we will be able to satisfy the weaker conditions of Remark 3.13. Consequently, we will need to know what linear independence over $\mathfrak{A} \otimes \mathbb{C}1_n$ implies about pointwise linear independence. We use a technique developed by Deckard and Pearcy [8] to solve systems of linear equations in Stonian spaces.

Lemma 3.14. *Let Λ be a Stonian space and let $A = (a_{ij}) \in M_{m,n}(\mathcal{C}(\Lambda))$. Suppose that $\mathcal{D}$ is dense in Λ and, for all $\lambda \in \mathcal{D}$, there is a non-trivial solution to the system of linear equations*

$$\begin{pmatrix} a_{11}(\lambda) & \cdots & a_{1n}(\lambda) \\ \vdots & \ddots & \vdots \\ a_{m1}(\lambda) & \cdots & a_{mn}(\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

Then there exists a set of functions $\{f_i\}_{i=1}^n$ in $\mathcal{C}(\Lambda)$, not all of which are identically zero, such that $\sum_{j=1}^n f_i a_{ij} = 0$, for all $1 \leq i \leq m$.

Proof. Let $r(\lambda)$ be the rank of $A(\lambda)$, for all $\lambda \in \Lambda$, and let $d_0 = \sup_{\lambda \in \Lambda} r(\lambda)$. Assume, without loss of generality, that $d_0 > 0$ and choose $\lambda_0 \in \Lambda$ such that $r(\lambda_0) = d_0$. Then there exists a $d_0 \times d_0$ submatrix $S = (s_{ij})$ of A such that $\det(S(\lambda_0)) \neq 0$ and, for notational convenience, we assume that $s_{ij} = a_{ij}$, for all $1 \leq i, j \leq d_0$. Since the determinant is continuous, there exists an open neighbourhood U_0 of λ_0 such that $\det(S(\lambda)) \neq 0$, for all $\lambda \in U_0$.

Now let $\lambda_1 \in \mathcal{D} \cap U_0$ and let U_1 be a compact open neighbourhood of λ_1 in U_0 with characteristic function χ_1. By assumption, there exist scalars $\{\mu_i\}_{i=1}^n$, not all of which are zero, such that $\sum_{j=1}^n \mu_j a_{ij}(\lambda_1) = 0$, for all $1 \leq i \leq m$. In particular, $d_0 < n$ and we define $f_i = \mu_i \chi_1$, for all $d_0 + 1 \leq i \leq n$. Then, by Cramer's rule, there exists a unique set of functions $\{f_i\}_{i=1}^{d_0}$ in $\mathcal{C}(\Lambda)$ such that $f_i = f_i \chi_1$ and $\sum_{j=1}^{d_0} f_j s_{ij} + \sum_{j=d_0+1}^n f_j a_{ij} = 0$, for all $1 \leq i \leq d_0$. Because the rank of $S(\lambda)$ is maximal, for all $\lambda \in U_1$, $\sum_{j=1}^n f_j a_{ij} = 0$, for all $1 \leq i \leq m$. Finally, as $f_i(\lambda_1) = \mu_i$, for all $1 \leq i \leq n$, the proof is complete. $\square$

As stated, Lemma 3.14 is applicable in a variety of situations, but we are only concerned with the following one.

Theorem 3.15. *Let* $\{A_i\}_{i=1}^k$ *be a linearly independent subset of* $M_n(\mathfrak{A})$ *over* $\mathfrak{A} \otimes \mathbb{C}1_n$ *and let* $\mathcal{O}$ *be the set of all points* $\omega \in \Omega$ *such that* $\{A_i(\omega)\}_{i=1}^k$ *is linearly independent over* $\mathbb{C}$. *Then* $\mathcal{O}$ *is open and dense in* Ω.

Proof. We may consider $\{A_i\}_{i=1}^k$ to be the columns of a $n^2 \times k$ matrix A with entries in $\mathcal{C}(\Omega)$. For every $\omega_0 \in \mathcal{O}$, there exists a $k \times k$ submatrix S of A such that the rank of $S(\omega_0) = k$. Then $\det(S(\omega)) \neq 0$ in an open neighbourhood of ω_0. Hence, $\mathcal{O}$ is open and it remains to show that $\mathcal{O}$ is dense in Ω.

Let $\Lambda = \Omega \setminus \mathcal{O}^-$ and assume, to obtain a contradiction, that Λ is not empty. For all $\omega \in \Lambda$, observe that the columns of $A(\omega)$ are linearly dependent and, therefore, $\ker(A(\omega)) \neq \{0\}$. Then, by Lemma 3.14, there exists a set of functions $\{f_i\}_{i=1}^k$ in $\mathcal{C}(\Omega)$, not all of which are identically zero, such that $\sum_{i=1}^k (f_i \otimes 1_n)A_i = 0$, a contradiction. $\square$

CHAPTER IV

AVERAGING TECHNIQUES IN COHOMOLOGY

In calculations involving continuous cocycles, it is often useful to replace a given cocycle $\rho \in \mathcal{Z}^n(\mathcal{A}, M)$ with an equivalent one that vanishes whenever one of its arguments is in a closed subalgebra $\mathcal{B}$ of $\mathcal{A}$. This is referred to as *averaging* in the theory of continuous cohomology.

A. Averaging over Amenable Algebras

Let G be a locally compact group. There exists a unique left-invariant regular Borel measure μ on G called its *Haar measure.* When G is compact and $\mu(G) = 1$, we call $\phi(f) = \int_G f\, d\mu$, where $f \in L^\infty(G)$, the *average* of f over G. In the absence of compactness, however, there is a weaker notion of averaging attributed to M. Day [6].

Definition 4.1. Let G be a locally compact group. A state $\phi \in L^\infty(G)^*$, also known as a *mean* on $L^\infty(G)$, is called *left-invariant* if $\phi(\delta_s * f) = \phi(f)$, where $f \in L^\infty(G)$, $s \in G$, and $(\delta_s * f)(t) = f(s^{-1}t)$ for all $t \in G$. We say that G is *amenable* if there exists a left-invariant mean on $L^\infty(G)$.

Amenable locally compact groups were characterized by Johnson and Ringrose [21, Theorem 2.5] in the following theorem.

Theorem 4.1. *Let G be a locally compact group. Then G is amenable if and only if $H^1(L^1(G), M) = 0$, whenever M is a dual Banach bimodule over $L^1(G)$.*

Johnson used Theorem 4.1 to extend the notion of amenability to Banach algebras. A Banach algebra $\mathcal{A}$ is called amenable, if $H^1(\mathcal{A}, M) = 0$ whenever M is a dual Banach bimodule over $\mathcal{A}$. Johnson, Kadison, and Ringrose [20, Theorem 4.1] proved that a continuous cocycle $\rho \in \mathcal{Z}^n(\mathcal{A}, M)$ can be averaged over an amenable subalgebra.

Theorem 4.2. *Let $\mathcal{A}$ be a Banach algebra, let M be a dual Banach bimodule over $\mathcal{A}$, and let $\mathcal{B}$ be a closed amenable subalgebra of $\mathcal{A}$. Suppose $\rho \in \mathcal{L}^n(\mathcal{A}, M)$ and $\partial\rho$ vanishes whenever any of its entries is in $\mathcal{B}$. Then there exists $\xi \in \mathcal{L}^{n-1}(\mathcal{A}, M)$ such that $\rho + \partial\xi$ vanishes whenever any of its entries lies in $\mathcal{B}$ and, in particular, $\partial\rho = \partial(\rho + \partial\xi)$.*

A locally compact group G is amenable if and only if there is a left-invariant mean on $\mathcal{C}_b(G)$, the algebra of bounded continuous complex-valued functions on G [31, Theorem 1.1.9]. We can also extend the definition of amenability to arbitrary topological groups by calling G amenable, if there is a left-invariant mean on $\mathcal{C}_b(G)$. When a C^*-algebra $\mathcal{B}$ is the norm closed linear span of a group of unitary operators G in $\mathcal{L}(\mathcal{H})$ that is amenable with respect to the *norm* topology, Kadison and Ringrose [23, Theorem 3.3] showed that $\mathcal{B}$ is an amenable Banach algebra.

As noted above, all compact groups are amenable, but the unitary group of a C^*-algebra $\mathcal{B}$ is not compact in the norm topology unless $\mathcal{B}$ is finite dimensional. The abelian groups form another important class of amenable groups. In applications, it is always possible to average over an abelian C^*-algebra because abelian groups are amenable with respect to the discrete topology [31, Examples 1.1.5].

The next proposition [29, Lemma 4.1] is a consequence of the definition of the coboundary map ∂ for Hochschild cohomology and is valid in a broad context.

Proposition 4.3. *Let $\mathcal{A}$ be an algebra, let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$, and let M be a bimodule over $\mathcal{A}$. If $\rho \in \mathcal{Z}^n(\mathcal{A}, M)$ vanishes whenever one its arguments lies in $\mathcal{B}$, then, for all $a_1, \ldots, a_n \in \mathcal{A}$ and $b \in \mathcal{B}$,*

(i) $\rho(ba_1, a_2, \ldots, a_n) = b\rho(a_1, \ldots, a_n)$,

(ii) $\rho(a_1, \ldots, a_{k-1}, a_k b, a_{k+1}, \ldots, a_n) = \rho(a_1, \ldots, a_k, b a_{k+1}, a_{k+2} \ldots, a_n)$, $1 \leq k < n$,

(iii) $\rho(a_1, \ldots, a_{n-1}, a_n b) = \rho(a_1, , \ldots, a_n) b$.

Suppose $\rho \in \mathcal{Z}^n(\mathcal{A}, M)$ vanishes whenever any of its first $1 \leq \ell < n$ arguments lies in a subalgebra $\mathcal{B}$ of $\mathcal{A}$. Then ρ satisfies conditions *(i)* and *(ii)*, for all $1 \leq k < \ell$, of Proposition 4.3 and is called *ℓ-multimodular* with respect to $\mathcal{B}$. If $\ell = n$, then ρ is simply called *multimodular* with respect to $\mathcal{B}$. Evidently, when $\mathcal{A}$ is a Banach algebra and M is a dual Banach bimodule over $\mathcal{A}$, Theorem 4.2 provides a sufficient condition for $\rho \in \mathcal{Z}^n(\mathcal{A}, M)$ to be replaced with $\zeta \in \mathcal{Z}^n(\mathcal{A}, M)$ within the same equivalence class of $H^n(\mathcal{A}, M)$ that is multimodular with respect to a closed amenable subalgebra $\mathcal{B}$ of $\mathcal{A}$.

Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(\mathcal{H})$, let M be a dual Banach bimodule over $\mathcal{A}$, and let $\mathcal{B}$ be an abelian C^*-subalgebra of the center of $\mathcal{A}$. Apparently, if $\xi \in \mathcal{L}^n(\mathcal{A}, M)$ is multimodular with respect to $\mathcal{B}$, then $\partial\xi$ is also multimodular. We let $\{\mathcal{L}^n(\mathcal{A}, M : \mathcal{B}), \partial^n\}_{n \geq 0}$ denote the subcomplex of $\{\mathcal{L}^n(\mathcal{A}, M), \partial^n\}_{n \geq 0}$ consisting of multimodular maps and use similar notation for coboundaries, cocycles and homology groups. In this case, the scalar field $\mathbb{C}$ may be replaced by $\mathcal{B}$ in our cohomology calculations. More generally, Sinclair and Smith [35, Theorem 3.2.7] showed that if $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ with an amenable unitary group, then it suffices to consider the multimodular complex.

Theorem 4.4. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(\mathcal{H})$, let M be a dual Banach bimodule over $\mathcal{A}$, and let $\mathcal{B}$ be C^*-subalgebra of $\mathcal{A}$ with an amenable unitary group. Then $H^n(\mathcal{A}, M : \mathcal{B}) \cong H^n(\mathcal{A}, M)$, for all $n \geq 0$.*

Every finite group of unitary operators G is amenable and, although Theorem 4.2 does not have a direct analogue for normal cocycles, the technique used in its proof is applicable when G is finite. More precisely, the following proposition is true [20, Lemma 5.3].

Proposition 4.5. *Let $\mathcal{A}$ be an operator algebra, let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$, and let G be a finite group of unitary operators in the center of $\mathcal{A}$ which linearly generate a C^*-algebra $\mathcal{D}$. Suppose M is a dual normal module over $\mathcal{A}$ and $\rho \in \mathcal{Z}_w^n(\mathcal{A}, M)$ vanishes whenever any of its arguments lie in $\mathcal{B}$. Then there exists $\xi \in \mathcal{L}_w^{n-1}(\mathcal{A}, M)$ such that $\rho + \partial\xi$ vanishes whenever any of its arguments is in $\mathcal{B}$ or $\mathcal{D}$.*

B. Averaging in Normal Cohomology

Averaging a normal cocycle over an amenable algebra proves to be a far greater challenge than averaging for continuous cocycles. Theorem 4.2 applies to normal cocycles, but the new cocycle may not be *normal*. However, for von Neumann algebras, Johnson, Kadison, and Ringrose [20] proved an analogue of Theorem 4.2 for normal cocycles. An essential element in its proof, is an extension theorem for normal multilinear maps.

A Banach space $\mathfrak{X}$ is said to be *weakly sequentially complete*, if every weakly Cauchy sequence converges. Grothendieck [15, Théorème 6] showed that any bounded linear map on $\mathcal{C}(K)$, where K is a compact Hausdorff space, to a weakly sequentially complete Banach space is weakly compact. Akemann [1, Theorem 2.8] applied Grothendieck's result in proving the following theorem.

Theorem 4.6. *Let $\mathfrak{X}$ be a Banach space such that $\mathfrak{X}^*$ is weakly sequentially complete and let $\mathcal{A}_*$ be the predual of a von Neumann algebra $\mathcal{A}$. Then every bounded linear map $T : \mathfrak{X} \to \mathcal{A}_*$ is weakly compact.*

Examples of operator algebras having a weakly sequentially complete dual space abound. As an illustration, following the presentation of Brown, Chevreau, and

Pearcy [4], we describe a large class of ultraweakly closed operator algebras, called *dual algebras*, all of which are generated by a single operator and have a weakly sequentially complete dual space.

Example 4.2. An operator $T \in \mathcal{L}(\mathcal{H})$ is called a *contraction*, if $\|T\| \leq 1$ and, in this case, T is called *completely non-unitary*, if its restriction to any non-zero reducing subspace is not unitary. For every contraction T, there exists a unique closed subspace $\mathcal{S}$ of $\mathcal{H}$ such that $\mathcal{S}$ is reducing for T, the restriction of T to $\mathcal{S}$ is unitary, and the restriction of T to $\mathcal{S}^\perp$ is completely non-unitary [37, Theorem 1.3.2].

Assume that T is a completely non-unitary contraction acting on a separable Hilbert space and let $\mathcal{A}_T$ be the dual algebra generated by T. By the Sz.-Nagy-Foiaş functional calculus [4, Theorem 3.2], there is a homomorphism $\varphi : H^\infty(\mathbb{T}) \to \mathcal{A}_T$, where $\mathbb{T}$ is the unit circle and $H^\infty(\mathbb{T})$ is the algebra of essentially bounded functions f in $L^2(\mathbb{T})$ such that $\langle f(t), e^{-int} \rangle = 0$, for all $n \in \mathbb{N}$. For all $f \in H^\infty(\mathbb{T})$, $\|\varphi(f)\| \leq \|f\|_\infty$ and we write $\varphi(f) = f(T)$.

Let $A^\infty(\mathbb{D})$ be the algebra of bounded holomorphic functions on the open unit disc $\mathbb{D}$. By taking pointwise radial limits, there is an isometric algebra isomorphism of $A^\infty(\mathbb{D})$ onto $H^\infty(\mathbb{T})$ [30, Theorem 17.10] and we denote the radial limit of $h \in A^\infty(\mathbb{D})$ by $\tilde{h}$. Suppose that $\sigma(T)$ is sufficiently large so that $\|h\|_\infty = \sup_{\lambda \in \sigma(T) \cap \mathbb{D}} |h(\lambda)|$, for all $h \in A^\infty(\mathbb{D})$. Because $h(\lambda) \in \sigma(\tilde{h}(T))$, for all $\lambda \in \sigma(T) \cap \mathbb{D}$ [12, Corollary 3.1], we have

$$\|\tilde{h}\|_\infty = \|h\|_\infty = \sup_{\lambda \in \sigma(T) \cap \mathbb{D}} |h(\lambda)| \leq \|\tilde{h}(T)\| \leq \|\tilde{h}\|_\infty$$

and $\mathcal{A}_T$ is isometrically isomorphic to $H^\infty(\mathbb{T})$. Since $H^\infty(\mathbb{T})^*$ is weakly sequentially complete [3, Corollary 5.4], $\mathcal{A}_T^*$ is also weakly sequentially complete.

When $\mathcal{A}$ is a C^*-algebra, $\mathcal{A}^{**}$ is isomorphic, as a Banach space, to its universal enveloping von Neumann algebra [38, Theorem 1]. Since the predual of a von Neumann algebra is weakly sequentially complete [33, Proposition 1], $\mathcal{A}^*$ is weakly sequentially complete. Johnson, Kadison, and Ringrose [20, Theorem 2.3] used Theorem 4.6 to extend multilinear maps on C^*-algebras.

Theorem 4.7. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C^*-algebras acting on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$, respectively, let M be the dual space of a Banach space M_*, and let $\rho : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to M$ be a bounded multilinear map that is separately ultraweak-weak* continuous. Then ρ extends uniquely, without increase of norm, to a separately ultraweak-weak* continuous map $\bar{\rho} : \mathcal{A}_1^- \times \cdots \times \mathcal{A}_n^- \to M$ on the product of the closures of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ in the ultraweak topology.*

Another key element required for averaging in normal cohomology is a theorem [24, Theorem 10.1.12] that relates an arbitrary representation of a C^*-algebra to its universal representation.

Theorem 4.8. *Let Φ be a representation of a C^*-algebra $\mathcal{A}$ and π be its universal representation. Then there is a projection P in the center of $\mathcal{M}(\pi)$ and an ultraweakly continuous $*$-isomorphism $\alpha : \mathcal{M}(\pi)P \to \mathcal{M}(\Phi)$ such that $\Phi(a) = \alpha(\pi(a)P)$, for all $a \in \mathcal{A}$.*

The conclusions of Theorem 4.8 are best summarized by the following commutative diagram.

$$\begin{tikzcd}
\pi(\mathcal{A}) \arrow[r] & \pi(\mathcal{A})P \arrow[r] \arrow[d, "\alpha"'] & \mathcal{M}(\pi)P \arrow[d, "\alpha"'] \\
\mathcal{A} \arrow[u, "\pi"] \arrow[r, "\Phi"] & \Phi(\mathcal{A}) \arrow[r] & \mathcal{M}(\Phi)P
\end{tikzcd}$$

Suppose $\mathcal{M}$ is a von Neumann algebra, $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{M}$ generated by an amenable unitary group, and N is a dual normal module over $\mathcal{M}$. Using Proposition 4.5, Theorem 4.7, and Theorem 4.8, Johnson, Kadison, and Ringrose [20, Lemma 5.4] proved that if $\rho \in \mathcal{Z}^n(\mathcal{M}, N)$ vanishes whenever any of its arguments lies in $\mathcal{B}$, then there exists $\xi \in \mathcal{L}^{n-1}(\mathcal{M}, N)$ such that $\rho + \partial\xi$ is a *normal* cocycle that vanishes whenever any of its arguments is in $\mathcal{B}$. Additionally [20, Lemma 5.5], they proved that if $\partial\xi \in \mathcal{B}^n(\mathcal{M}, N)$ is a normal cocycle, then there exists $\eta \in \mathcal{L}_w^{n-1}(\mathcal{M}, N)$ such that $\partial\xi = \partial\eta$. The combination of these results with Theorem 4.2 yields an averaging theorem for normal cohomology.

Theorem 4.9. *Let $\mathcal{M}$ be a von Neumann algebra, let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{M}$ generated by an amenable unitary group, and let N be a dual normal module over $\mathcal{M}$. Then, for all $\rho \in \mathcal{Z}_w^n(\mathcal{M}, N)$, there exists $\xi \in \mathcal{L}_w^{n-1}(\mathcal{M}, N)$ such that $\rho + \partial\xi$ vanishes whenever any of its arguments lies in $\mathcal{B}$.*

CHAPTER V

MULTILINEAR MAPS ON JOINS

Having established some preliminary results, we follow Gilfeather and Smith [14] and begin by investigating the structure of the multilinear maps we shall encounter in the sequel.

A. The Structure of Multilinear Maps

For every pair of operators $S \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{L}(\mathcal{K})$, there is a unique bounded operator $S \otimes T \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ such that $(S \otimes T)(h \otimes k) = Sh \otimes Tk$, for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$, and $\|S \otimes T\| = \|S\|\|T\|$ [28, Lemma 6.3.2]. Given subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{K})$, respectively, we regard their algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ as a subalgebra of $\mathcal{L}(\mathcal{H} \otimes \mathcal{K})$. Its norm closure is denoted $\mathcal{A} \otimes_* \mathcal{B}$ and called the *spatial tensor product* of $\mathcal{A}$ and $\mathcal{B}$. Furthermore, when $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, the von Neumann algebra generated by $\mathcal{A} \otimes \mathcal{B}$ is denoted $\mathcal{A} \overline{\otimes} \mathcal{B}$ and called *the von Neumann algebra tensor product* of $\mathcal{A}$ and $\mathcal{B}$. In this case, *the commutation theorem* for tensor products [39, Theorem 4.5.9] states that $(\mathcal{A} \overline{\otimes} \mathcal{B})' = \mathcal{A}' \overline{\otimes} \mathcal{B}'$ and the double commutant theorem furnishes a description of $\mathcal{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$ in terms of matrices.

Lemma 5.1. *Let $\mathcal{A}$ be a von Neumann algebra acting on $\mathcal{H}$ and let $S \in \mathcal{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$. If $\{f_\tau\}_{\tau \in T}$ is an orthonormal basis for $\mathcal{K}$ and $S = (s_{\tau\mu})_{\tau\mu \in T}$ is the matrix of S with respect to $\{f_\tau\}_{\tau \in T}$, then $s_{\mu\tau} \in \mathcal{A}$, for all $\tau, \mu \in T$.*

Proof. Since $\mathcal{A}' \otimes \mathbb{C}1_{\mathcal{K}} \subseteq (\mathcal{A} \overline{\otimes} \mathcal{L}(\mathcal{K}))'$, $xs_{\tau\mu} = s_{\tau\mu}x$, for all $x \in \mathcal{A}'$ and $\tau, \mu \in T$. Then, by the double commutant theorem, $s_{\tau\mu} \in \mathcal{A}'' = \mathcal{A}$. □

Definition 5.1. Let $\mathfrak{A}$ be an abelian von Neumann algebra acting on $\mathcal{H}$, let $\mathcal{A}$ be a norm closed subalgebra of $\mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$, and let $\mathcal{B}$ be a norm closed subalgebra of

$M_n(\mathfrak{A})$. The *join* of $\mathcal{A}$ and $\mathcal{B}$ is the subalgebra of $\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathbb{C}^n \oplus \mathcal{K})$ defined by

$$\mathcal{A} * \mathcal{B} = \left\{ \begin{pmatrix} B & 0 \\ U & A \end{pmatrix} : A \in \mathcal{A},\ U \in \mathfrak{A} \otimes_* \mathcal{L}(\mathbb{C}^n, \mathcal{K}),\ B \in \mathcal{B} \right\}.$$

Note that our definition of the join differs from the definition used by Gilfeather and Smith [14]. However, the two definitions coincide when $\mathfrak{A} = \mathbb{C}$.

Example 5.2. Let $\mathfrak{A}$ be the abelian von Neumann algebra whose maximal ideal space consists of two points and let $\mathcal{A}$ be a proper subalgebra of $M_2(\mathbb{C})$. Then $\mathfrak{A} \otimes \mathcal{A} \cong \mathcal{A} \oplus \mathcal{A}$ is always an algebra whose linear dimension is even. On the other hand, if $\mathcal{D}_2$ and $\mathcal{T}_2$ denote the subalgebras of diagonal and upper triangular matrices in $M_2(\mathbb{C})$, respectively, then $\mathcal{D}_2 \oplus \mathcal{T}_2$ is a proper subalgebra of $\mathfrak{A} \otimes M_2(\mathbb{C}) \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ of odd linear dimension. Evidently, there are subalgebras of $\mathfrak{A} \otimes M_2(\mathbb{C})$ that are *not* unitarily equivalent to $\mathfrak{A} \otimes \mathcal{A}$, for some proper subalgebra $\mathcal{A}$ of $M_2(\mathbb{C})$.

Notation 5.3. It will become necessary to distinguish between elements of the tensor product of Hilbert spaces and rank one operators between Hilbert spaces. Suppose that $h_0 \in \mathcal{H}$ and $k_0 \in \mathcal{K}$. Then $h_0 \otimes k_0$ denotes a vector in $\mathcal{H} \otimes \mathcal{K}$, while $k_0 \check{\otimes} h_0$ will denote the rank one operator defined by $h \mapsto \langle h, h_0 \rangle k_0$, for all $h \in \mathcal{H}$.

For the remainder of this chapter, $\mathfrak{A}$ will denote an maximal abelian von Neumann algebra acting on $\mathcal{H}$, $\mathcal{A}$ will denote a norm closed subalgebra of $\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathcal{K})$, and $\mathcal{B}$ will denote an ultraweakly closed subalgebra of $M_n(\mathfrak{A})$ containing $\mathfrak{A} \otimes \mathbb{C}1_n$. Furthermore, suppose that $B_0 \in \mathcal{B}$, $U_0 \in \mathfrak{A} \otimes_* \mathcal{L}(\mathbb{C}^n, \mathcal{K})$, and $A_0 \in \mathcal{A}$. Then X_0 will denote the fixed element of $\mathcal{A} * \mathcal{B}$ defined by $X_0 = \left(\begin{smallmatrix} B_0 & 0 \\ U_0 & A_0 \end{smallmatrix}\right)$ and when $U_0 = 0$, we shall also write $X_0 = B_0 \oplus A_0$.

Since $\mathcal{A} * \mathcal{B}$ contains an abelian subalgebra $(\mathfrak{A} \otimes \mathbb{C}1_n) \oplus \mathbb{C}1_{\mathcal{A}}$, if $m \geq 1$ and $\rho \in \mathcal{Z}^m(\mathcal{A} * \mathcal{B}, \mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathbb{C}^n \oplus \mathcal{K}))$, we may apply Theorem 4.2 to obtain an equivalent cocycle ζ that vanishes whenever any of its arguments belongs to $(\mathfrak{A} \otimes \mathbb{C}1_n) \oplus \mathbb{C}1_{\mathcal{A}}$. In particular, ζ is multimodular with respect to $(\mathfrak{A} \otimes \mathbb{C}1_n) \oplus \mathbb{C}1_{\mathcal{A}}$. From now on, we assume that every cocycle on $\mathcal{A} * \mathcal{B}$ with coefficients in $\mathfrak{A} \overline{\otimes} \mathcal{L}(\mathbb{C}^n \oplus \mathcal{K})$ vanishes whenever any of its entries is in $(\mathfrak{A} \otimes \mathbb{C}1_n) \oplus \mathbb{C}1_{\mathcal{A}}$ and the following decomposition, due to Gilfeather and Smith [14, Proposition 2.4], is a direct consequence.

Proposition 5.2. *Let $\rho \in \mathcal{Z}^m(\mathcal{A} * \mathcal{B}, \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathbb{C}^n \oplus \mathcal{K}))$. Then ρ is of the form*

$$\rho(X_1, \ldots, X_m) = \begin{pmatrix} \beta(B_1, \ldots B_m) & 0 \\ \sum_{j=1}^m \sigma_j(A_1, \ldots, A_{m-j}, U_{m-j+1}, B_{m-j+2}, \ldots, B_m) & \alpha(A_1, \ldots, A_m) \end{pmatrix}, \tag{5.1}$$

where $X_j \in \mathcal{A} * \mathcal{B}$ and σ_j is a bounded m-linear mapping with values in the ultraweak closure $(\mathfrak{A} \otimes \mathcal{L}(\mathbb{C}^n, \mathcal{K}))^-$ of $\mathfrak{A} \otimes \mathcal{L}(\mathbb{C}^n, \mathcal{K})$, for all $1 \leq j \leq m$. Furthermore, $\alpha \in \mathcal{Z}^m(\mathcal{A}, \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K}))$ and $\beta \in \mathcal{Z}^m(\mathcal{B}, M_n(\mathfrak{A}))$.

Note that the multilinear maps appearing in (5.1) inherit the multimodularity of ρ. In particular, for all $a \in \mathfrak{A}$ and $1 \leq j \leq m$, we have

$$\begin{aligned} \sigma_j(A_1, \ldots, A_{m-j}, (a \otimes 1_{\mathcal{K}})U_{m-j+1}, B_{m-j+2}, \ldots, B_m) \\ &= \sigma_j(A_1, \ldots, A_{m-j}, U_{m-j+1}(a \otimes 1_n), B_{m-j+2}, \ldots, B_m) \\ &= \sigma_j(A_1, \ldots, A_{m-j}, U_{m-j+1}, (a \otimes 1_n)B_{m-j+2}, \ldots, B_m) \\ &= \sigma_j(A_1, \ldots, A_{m-j}, U_{m-j+1}, B_{m-j+2}(a \otimes 1_n), \ldots, B_m) \\ &= \sigma_j(A_1, \ldots, A_{m-j}, U_{m-j+1}, B_{m-j+2}, \ldots, B_m)(a \otimes 1_n) \\ &= (a \otimes 1_{\mathcal{K}})\sigma_j(A_1, \ldots, A_{m-j}, U_{m-j+1}, B_{m-j+2}, \ldots, B_m) \end{aligned} \tag{5.2}$$

and β is homogeneous with respect to $\mathfrak{A} \otimes \mathbb{C}1_n$.

The multilinear maps appearing in the $(2,1)$ entry of (5.1) may be further decomposed. Our decomposition is similar to that obtained by Gilfeather and Smith

[14, Lemma 2.1] and, although the proof is substantially the same, it is included for completeness.

Lemma 5.3. *Let* $\gamma : \mathcal{A}\times\cdots\times\mathcal{A}\times\mathfrak{A}\otimes_*\mathcal{L}(\mathbb{C}^n,\mathcal{K})\times\mathcal{B}\times\cdots\times\mathcal{B}\to(\mathfrak{A}\otimes\mathcal{L}(\mathbb{C}^n,\mathcal{K}))^-$ *be a bounded* m*-linear function satisfying (5.2), where* $\mathcal{A}$ *occurs* $m-r-1$ *times and* $\mathcal{B}$ *occurs* r *times. Then* γ *is equal to a finite sum of* m*-linear functions of the following forms:*

(i) $(A_1,\ldots,A_{m-1},U_m)\mapsto\phi(A_1,\ldots,A_{m-1})U_mT$*, where* $\phi:\mathcal{A}\times\cdots\times\mathcal{A}\to\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathcal{K})$ *is a bounded* $(m-1)$*-linear map and* $T\in M_n(\mathfrak{A})$*, for* $r=0$*.*

(ii) $(A_1,\ldots,A_{m-r-1},U_{m-r},B_{m-r+1},\ldots,B_m)\mapsto\phi U_{m-r}\psi$*, where* $\phi:\mathcal{A}\times\cdots\times\mathcal{A}\to\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathcal{K})$ *is a bounded* $(m-r-1)$*-linear map and* $\psi:\mathcal{B}\times\cdots\times\mathcal{B}\to M_n(\mathfrak{A})$ *is a bounded* r*-linear map that is homogeneous with respect to* $\mathfrak{A}\otimes\mathbb{C}1_n$*, for* $0<r<m-1$*.*

(iii) $(U_1,B_2,\ldots,B_m)\mapsto SU_1\psi(B_2,\ldots,B_m)$*, where* $S\in\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathcal{K})$ *and* $\psi:\mathcal{B}\times\cdots\times\mathcal{B}\to M_n(\mathfrak{A})$ *is a bounded* $(m-1)$*-linear map that is homogeneous with respect to* $\mathfrak{A}\otimes\mathbb{C}1_n$*, for* $r=m-1$*.*

Proof. The proof of (ii) contains all of the essential elements of the argument and we omit the others.

By Theorem 3.5, $\mathcal{B}$ is complemented in $M_n(\mathfrak{A})$ and we may, therefore, assume that $\mathcal{B}=M_n(\mathfrak{A})$. Let $\{E_{ij}\}_{i,j=1}^n$ denote the canonical matrix units for $M_n(\mathbb{C})$ and $\{e_j\}_{j=1}^n$ be the canonical basis for $\mathbb{C}^n$. We use boldface to denote multi-indices $\mathbf{i}=(i_1,\ldots i_r)$ and $\mathbf{j}=(j_1,\ldots j_r)$. Define multilinear functions $\psi_{\mathbf{ij}pq}:M_n(\mathfrak{A})\times\cdots\times M_n(\mathfrak{A})\to M_n(\mathfrak{A})$ by

$$\psi_{\mathbf{ij}pq}(1_{\mathcal{H}}\otimes E_{s_1t_1},\ldots,1_{\mathcal{H}}\otimes E_{s_rt_r})=\begin{cases}1_{\mathcal{H}}\otimes E_{pq} & \text{if } \mathbf{s}=\mathbf{i} \text{ and } \mathbf{t}=\mathbf{j}\\ 0 & \text{otherwise}\end{cases}$$

and extend $\psi_{\mathbf{ij}pq}$ linearly to $M_n(\mathfrak{A})$. Now define $\phi_{\mathbf{ij}pq} : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ by

$$\phi_{\mathbf{ij}pq}(A_1, \ldots, A_{m-r-1})(h \otimes k) \\ = \gamma(A_1, \ldots, A_{m-r-1}, 1_{\mathcal{H}} \otimes (k \check{\otimes} e_p), E_{i_1j_1}, \ldots, E_{i_rj_r})(h \otimes e_q).$$

Since $\mathfrak{A}$ is maximal, $\mathfrak{A} = \mathfrak{A}'$ [11, Proposition 4.62] and γ takes values in $(\mathfrak{A} \otimes \mathcal{L}(\mathbb{C}^n, \mathcal{K}))^-$. By the commutation theorem, $\phi_{\mathbf{ij}pq}(A_1, \ldots, A_{m-r-1}) \in (\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}})' = \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$. Then, for all $a \in \mathfrak{A}$, $h \in \mathcal{H}$, $k \in \mathcal{K}$, and $1 \leq s, t \leq n$, (5.2) implies

(5.3)

$$\sum_{\mathbf{ij}pq} \phi_{\mathbf{ij}pq}(A_1, \ldots, A_{m-r-1})(a \otimes (k \check{\otimes} e_s))\psi_{\mathbf{ij}pq}(1_{\mathcal{H}} \otimes E_{g_1l_1}, \ldots, 1_{\mathcal{H}} \otimes E_{g_rl_r})(h \otimes e_t)$$
$$= \sum_{pq} \phi_{\mathbf{gl}pq}(A_1, \ldots, A_{m-r-1})(a \otimes (k \check{\otimes} e_s))(1_{\mathcal{H}} \otimes E_{pq})(h \otimes e_t)$$
$$= \sum_{p} \phi_{\mathbf{gl}pt}(A_1, \ldots, A_{m-r-1})(a \otimes (k \check{\otimes} e_s))(h \otimes e_p)$$
$$= \phi_{\mathbf{gl}st}(A_1, \ldots, A_{m-r-1})(a \otimes 1_{\mathcal{K}})(h \otimes k)$$
$$= \gamma(A_1, \ldots, A_{m-r-1}, 1_{\mathcal{H}} \otimes (k \check{\otimes} e_s), E_{g_1l_1}, \ldots, E_{g_rl_r})(a \otimes 1_n)(h \otimes e_t)$$
$$= \gamma(A_1, \ldots, A_{m-r-1}, a \otimes (k \check{\otimes} e_s), E_{g_1l_1}, \ldots, E_{g_rl_r})(h \otimes e_t).$$

By linearity, (5.3) holds for all $U_{m-r} \in \mathfrak{A} \otimes \mathcal{L}(\mathbb{C}^n, \mathcal{K})$, $B_1, \ldots, B_r \in \mathcal{B}$, and $x \in \mathcal{H} \otimes \mathbb{C}^n$. Finally, since γ is bounded, (5.3) must also be true for $U_{m-r} \in \mathfrak{A} \otimes_* \mathcal{L}(\mathbb{C}^n, \mathcal{K})$. $\square$

In our subsequent calculations, it is valuable to know when the sums appearing in Lemma 5.3 are equal to zero. The statement of the next lemma, while resembling [14, Lemma 2.2], is adapted to the present situation. However, its proof requires additional work to accomodate the case where both $\mathcal{H}$ and $\mathcal{K}$ are infinite dimensional.

Lemma 5.4. *Let $0 \leq r \leq m$ and $p \in \mathbb{N}$. Suppose $\phi_i : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$ is a $(m-r-1)$-linear map and $\psi_i : \mathcal{B} \times \cdots \times \mathcal{B} \to M_n(\mathfrak{A})$ is a r-linear map, for all*

$1 \leq i \leq p$. *If* $\{\psi_i\}_{i=1}^p$ *is linearly independent with respect to* $\mathfrak{A} \otimes \mathbb{C}1_n$ *and*

$$\sum_{i=1}^{p} \phi_i(A_1, \ldots, A_{m-r-1}) U \psi_i(B_1, \ldots, B_r) = 0,$$

for all $A_1, \ldots, A_{m-r-1} \in \mathcal{A}$, $U \in \mathfrak{A} \otimes_* \mathcal{L}(\mathbb{C}^n, \mathcal{K})$, *and* $B_1, \ldots, B_r \in \mathcal{B}$, *then* $\phi_i = 0$, *for all* $1 \leq i \leq p$. *A similar statement is true if* $\{\phi_i\}_{i=1}^p$ *is linearly independent with respect to* $\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}$.

Proof. We begin with the case where $m = 1$ and $r = 0$. Recall that a 0-linear map taking values in a Banach bimodule M is a fixed element of M. Let $\phi_i = S_i \in \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$ and $\psi_i = T_i \in M_n(\mathfrak{A})$, for all $1 \leq i \leq p$, and fix an orthonormal basis $\{f_\tau\}_{\tau \in T}$ for $\mathcal{K}$.

Every $A \in \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$ has a matrix $(a_{\tau\mu})_{\tau,\mu \in T}$ with respect to $\{f_\tau\}_{\tau \in T}$ and, by Lemma 5.1, $a_{\tau\mu} \in \mathfrak{A}$, for all $\tau, \mu \in T$. In particular, we let $S_i = (s^i_{\tau\mu})_{\tau,\mu \in T}$, for all $1 \leq i \leq p$. Then, for all $h_1, h_2 \in \mathcal{H}$, $\tau, \mu \in T$, and $1 \leq s, t \leq n$, we have

$$\begin{aligned}
(1_{\mathcal{H}} \otimes (e_t \check{\otimes} f_\mu)) S_i^*(h_2 \otimes f_\tau) &= \sum_{\nu \in T} (1_{\mathcal{H}} \otimes (e_t \check{\otimes} f_\mu))((s^i_{\tau\nu})^* h_2 \otimes f_\nu) \\
&= (s^i_{\tau\mu})^* h_2 \otimes e_t \\
&= ((s^i_{\tau\mu})^* \otimes 1_n)(h_2 \otimes e_t),
\end{aligned}$$

for all $1 \leq i \leq p$, and, consequently

$$\begin{aligned}
\sum_{i=1}^{p} \langle (s^i_{\tau\mu} \otimes 1_n) T_i(h_1 \otimes e_s), h_2 \otimes e_t \rangle &= \sum_{i=1}^{p} \langle T_i(h_1 \otimes e_s), ((s^i_{\tau\mu})^* \otimes 1_n)(h_2 \otimes e_t) \rangle \\
&= \sum_{i=1}^{p} \langle T_i(h_1 \otimes e_s), (1_{\mathcal{H}} \otimes (e_t \check{\otimes} f_\mu)) S_i^*(h_2 \otimes f_\tau) \rangle \\
&= \sum_{i=1}^{p} \langle S_i(1_{\mathcal{H}} \otimes (f_\mu \check{\otimes} e_t)) T_i(h_1 \otimes e_s), h_2 \otimes f_\tau \rangle \\
&= 0.
\end{aligned}$$

Since $\{T_i\}_{i=1}^p$ is linearly independent with respect to $\mathfrak{A}\otimes\mathbb{C}1_n$, $s^i_{\tau\mu} = 0$, for all $1 \le i \le p$ and $\tau, \mu \in T$, and, therefore, $S_i = 0$, for all $1 \le i \le p$.

If $m > 1$ and $0 \le r < m$, we fix $A_1, \dots, A_{m-r-1} \in \mathcal{A}$ and let the matrix of $\phi_i(A_1, \dots, A_{m-r-1})$ with respect to $\{f_\tau\}_{\tau\in T}$ be $(s^i_{\tau\mu})_{\tau,\mu\in T}$, for all $1 \le i \le p$. The preceding calculation shows that

$$\sum_{i=1}^{p} \langle (s^i_{\tau\mu} \otimes 1_n)\psi_i(B_1, \dots, B_r)(h_1 \otimes e_s), h_2 \otimes e_t\rangle = 0,$$

for all $\tau, \mu \in T$, $h_1, h_2 \in \mathcal{H}$, $1 \le s, t \le n$, and $B_1, \dots, B_r \in \mathcal{B}$. Because $\{\psi_i\}_{i=1}^p$ are linearly independent over $\mathfrak{A} \otimes \mathbb{C}1_n$, we conclude that $\phi_i(A_1, \dots, A_{m-r-1}) = 0$, for all $1 \le i \le p$. Since $A_1, \dots, A_{m-r-1} \in \mathcal{A}$ were arbitrary, $\phi_i = 0$, for all $1 \le i \le p$, and the proof is complete. $\square$

Note that the proof of Lemma 5.4 does not require that $\mathfrak{A}$ be maximal. In certain calculations, we shall replace $\mathfrak{A}$ by a von Neumann subalgebra of $\mathfrak{A}$. More precisely, the following lemma will be applicable.

Lemma 5.5. *Let $0 \le r \le m$, let $t \in \mathbb{N}$, and let $p \in \mathfrak{A}$ be a projection. Suppose $\phi_i : \mathcal{A}\times\cdots\times\mathcal{A} \to \mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathcal{K})$ is a $(m-r-1)$-linear map and $\psi_i : \mathcal{B}\times\cdots\times\mathcal{B} \to M_n(\mathfrak{A})$ is a r-linear map, for all $1 \le i \le t$. If $\{\psi_i\}_{i=1}^t$ is linearly independent with respect to $(p \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$ and*

$$\sum_{i=1}^{t} \phi_i(A_1, \dots, A_{m-r-1})U\psi_i(B_1, \dots, B_r) = 0,$$

for all $A_1, \dots, A_{m-r-1} \in \mathcal{A}$, $U \in \mathfrak{A}\otimes_*\mathcal{L}(\mathbb{C}^n, \mathcal{K})$, and $B_1, \dots, B_r \in \mathcal{B}$, then $(p\otimes 1_{\mathcal{K}})\phi_i = 0$, for all $1 \le i \le t$. A similar statement is true if $\{\phi_i\}_{i=1}^t$ is linearly independent with respect to $(p \otimes 1_{\mathcal{K}})(\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}})$.

By Proposition 5.2 and Lemma 5.3, every $\rho \in \mathcal{Z}^m(\mathcal{A} * \mathcal{B}, \mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathbb{C}^n \oplus \mathcal{K}))$ is a linear combination of maps of the form

$$\rho(X_1,\dots,X_m) = \begin{pmatrix} \beta(B_1,\dots,B_m) & 0 \\ \phi(A_1,\dots A_{m-r-1})U_{m-r}\psi(B_{m-r+1},\dots,B_m) & \alpha(A_1,\dots,A_m) \end{pmatrix}, \tag{5.4}$$

where $0 \le r < m$ and $X_1,\dots,X_m \in \mathcal{A} * \mathcal{B}$. The coboundaries of the constituent elements of ρ were calculated separately by Gilfeather and Smith [14] and recorded in a table that we reproduce for the sake of reference. Each coboundary in Table I is evaluated at $X_1,\dots,X_{m+1} \in \mathcal{A} * \mathcal{B}$.

Table I. Cochains and Their Coboundaries

ρ	$\partial\rho$
$\begin{pmatrix} \beta(B_1,\dots,B_m) & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} (\partial\beta)(B_1,\dots,B_{m+1}) & 0 \\ U_1\beta(B_2,\dots,B_{m+1}) & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ \phi U_{m-r}\psi & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ (\partial\phi)U_{m-r+1}\psi + (-1)^{m-r+1}\phi U_{m-r}(\partial\psi) & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 0 & \alpha(A_1,\dots,A_m) \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ (-1)^{m+1}\alpha(A_1,\dots,A_m)U_{m+1} & \partial\alpha \end{pmatrix}$

We now demonstrate that every cocycle in $\mathcal{Z}^n(\mathcal{A}*\mathcal{B}, \mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathbb{C}^n \oplus \mathcal{K}))$ is equivalent to a cocycle with *zeros* on the diagonal.

Proposition 5.6. *Let* $\rho \in \mathcal{Z}^m(\mathcal{A} * \mathcal{B}, \mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathbb{C}^n \oplus \mathcal{K}))$*. Then there is an equivalent cocycle* $\zeta \in \mathcal{Z}^m(\mathcal{A} * \mathcal{B}, \mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathbb{C}^n \oplus \mathcal{K}))$ *of the form*

$$\zeta(X_1,\dots,X_m) = \begin{pmatrix} 0 & 0 \\ \sum_{j=1}^m \gamma_j(A_1,\dots,A_{m-j},U_{m-j+1},B_{m-j+2},\dots,B_m) & 0 \end{pmatrix}, \tag{5.5}$$

where $X_1, \ldots, X_m \in \mathcal{A} * \mathcal{B}$. Furthermore, γ_j satisfies (5.2), for all $1 \leq j \leq m$.

Proof. We begin by applying Proposition 5.2 to ρ and, using the notation in (5.1), we evaluate $(\partial\rho)(X_1, \ldots, X_{m+1})$, where $X_j \in \mathcal{A}*\mathcal{B}$ and $B_j = 0$, for all $1 \leq j \leq m+1$. By Lemma 5.3, we may assume that $\sigma_1(A_1, \ldots, A_{m-1}, U_m) = \sum_{i=1}^{\ell} \phi_i(A_1, \ldots, A_{m-1})U_m\psi_i$, where $\phi_i : \mathcal{A}\times\cdots\times\mathcal{A} \to \mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathcal{K})$ and $\psi_i \in M_n(\mathfrak{A})$, for all $1 \leq i \leq \ell$. Then, applying Table I, the (2,1) entry of $(\partial\rho)(X_1, \ldots, X_{m+1})$ is

$$\sum_{i=1}^{\ell}(\partial\phi_i)(A_1, \ldots, A_m)U_{m+1}\psi_i + (-1)^{m+1}\alpha(A_1, \ldots, A_m)U_{m+1}(1_{\mathcal{H}} \otimes 1_n) = 0. \tag{5.6}$$

Now let N be the ultraweakly closed submodule of $M_n(\mathfrak{A})$ over $\mathfrak{A}\otimes\mathbb{C}1_n$ generated by $1_{\mathcal{H}}\otimes 1_n$ and $\{\psi_i\}_{i=1}^{\ell}$. By Theorem 3.11, there exist pairwise orthogonal projections $\{p_j\}_{j=1}^{t}$ such that $\sum_{j=1}^{t} p_j = 1_{\mathcal{H}}$ and $(p_j \otimes 1_n)N$ is a free module of finite type over $(p_j \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$, for all $1 \leq j \leq t$.

Choose j_0 such that $1 \leq j_0 \leq t$ and multiply (5.6) on the right by $(p_{j_0} \otimes 1_n)$. We may assume, by redefining $\{\phi_i\}_{i=1}^{\ell}$, if necessary, that $\{(p_{j_0} \otimes 1_n)\psi_i\}_{i=1}^{\ell}$ is a basis for $(p_{j_0} \otimes 1_n)N$ over $(p_{j_0} \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$ and $\psi_1 = p_{j_0} \otimes 1_n$. Then, by Lemma 5.5, $(p_{j_0} \otimes 1_{\mathcal{K}})(\partial\phi_1 + (-1)^{m+1}\alpha) = 0$. Since j_0 was arbitrary, we conclude that $\partial\phi_1 + (-1)^{m+1}\alpha = 0$.

Hence, if we let $\xi = \left(\begin{smallmatrix} 0 & 0 \\ 0 & (-1)^{m+1}\phi_1 \end{smallmatrix}\right)$ and replace ρ with $\eta = \rho + \partial\xi$, then η is an equivalent cocycle to ρ for which $\alpha = 0$. By Table I, η retains the form of (5.1) and the maps in the (2,1) entry of η satisfy (5.2). A similar calculation allows us to replace η with an equivalent cocycle ζ having the same form as (5.1) and such that $\alpha = \beta = 0$. □

B. The First Cohomology Groups of $\mathcal{A} * \mathcal{B}$

Notation 5.4. Having chosen $\mathcal{B}$ to be a algebra of matrices with entries in $\mathfrak{A}$ rather than $\mathbb{C}$ necessitates corresponding changes to the various coefficient spaces involved in our calculations. In particular, all multilinear maps on $\mathcal{A} * \mathcal{B}$ will take values in $\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathbb{C}^n \oplus \mathcal{K})$, all multilinear maps on $\mathcal{A}$ will take values in $\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathcal{K})$, and all multilinear maps on $\mathcal{B}$ will take values in $M_n(\mathfrak{A})$. The coefficient spaces will be omitted from future notation, for brevity.

These new coefficient spaces are all bimodules over $\mathfrak{A}$ and type I von Neumann algebras whose respective centers are $*$-isomorphic to $\mathfrak{A}$. For example, $M_n(\mathfrak{A})$ is a bimodule over $\mathfrak{A}$, if the module action is defined as

$$a \cdot A = (a \otimes 1_n)A = A(a \otimes 1_n) = A \cdot a,$$

for all $a \in \mathfrak{A}$ and $A \in M_n(\mathfrak{A})$. Similarly, the spaces of m-linear maps $\mathcal{L}^m(\mathcal{B})$ and cohomology groups $H^m(\mathcal{B})$ become bimodules over $\mathfrak{A}$, if we let

$$\begin{aligned}
(a \cdot \rho)(B_1, \ldots, B_m) &= (a \otimes 1_n)(\rho(B_1, \ldots, B_m)) \\
&= (\rho(B_1, \ldots, B_m))(a \otimes 1_n) \\
&= (\rho \cdot a)(B_1, \ldots, B_m),
\end{aligned}$$

for all $a \in \mathfrak{A}$, $\rho \in \mathcal{L}^m(\mathcal{B})$, and $B_1, \ldots, B_m \in \mathcal{B}$. This action is well defined on $H^m(\mathcal{B})$, because $a \cdot (\partial\rho) = \partial(a \cdot \rho)$, for all $a \in \mathfrak{A}$ and $\rho \in \mathcal{L}^m(\mathcal{B})$.

We shall express the cohomology groups of $\mathcal{A} * \mathcal{B}$ as the tensor product of $\mathfrak{A}$-bimodules which will be denoted $\otimes_{\mathfrak{A}}$ in contrast to the tensor product of complex vector spaces which we continue to denote by $\otimes$.

The results of the previous section demonstrate that cocycles on $\mathcal{A} * \mathcal{B}$ taking values in $\mathfrak{A}\,\overline{\otimes}\,\mathcal{L}(\mathbb{C}^n \oplus \mathcal{K})$ have a particularly simple form. We use it to determine the cohomology groups of $\mathcal{A} * \mathcal{B}$. While our calculations are based upon those of Gilfeather and Smith [14], in general, additional work is required.

Theorem 5.7. *$H^0(\mathcal{A} * \mathcal{B}) = \mathfrak{A} \otimes (1_n \oplus 1_{\mathcal{K}}) \cong \mathfrak{A}$.*

Proof. Since $\mathcal{A}*\mathcal{B}$ contains the abelian algebra $(\mathfrak{A}\otimes\mathbb{C}1_n)\oplus\mathbb{C}1_{\mathcal{A}}$, every $Y \in H^0(\mathcal{A}*\mathcal{B})$ must have the form $Y = T \oplus S$, where $T \in M_n(\mathfrak{A})$ and $S \in \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$. Let $\{e_j\}_{j=1}^n$ be the canonical basis for $\mathbb{C}^n$, let $\{f_\mu\}_{\mu\in M}$ be an orthonormal basis for $\mathcal{K}$, and let $X \in \mathcal{A} * \mathcal{B}$, where $A = B = 0$, $k \in \mathcal{K}$, $x \in \mathbb{C}^n$, and $U = 1_{\mathcal{H}} \otimes (k \check{\otimes} x)$. Then

$$XY - YX = \begin{pmatrix} 0 & 0 \\ (1_{\mathcal{H}} \otimes (k \check{\otimes} x))T - S(1_{\mathcal{H}} \otimes (k \check{\otimes} x)) & 0 \end{pmatrix} = 0.$$

Let $T = (t_{ij})_{i,j=1}^n$ be the matrix of T with respect to $\{e_j\}_{j=1}^n$ and let $S = (s_{\tau\mu})_{\tau,\mu\in M}$ be the matrix of S with respect to $\{f_\mu\}_{\mu\in M}$. In component form, the (2,1) entry of XY becomes

$$\begin{aligned} \langle (1_{\mathcal{H}} \otimes (f_\mu \check{\otimes} e_i))T(h_1 \otimes e_j), h_2 \otimes f_\tau\rangle &= \langle t_{ij}h_1 \otimes f_\mu, h_2 \otimes f_\tau\rangle \\ &= \delta_{\tau\mu}\langle t_{ij}h_1, h_2\rangle, \end{aligned}$$

for all $h_1, h_2 \in \mathcal{H}$, $\tau, \mu \in M$, and $1 \le i, j \le n$, while the (2,1) entry of YX is

$$\begin{aligned} \langle S(1_{\mathcal{H}} \otimes (f_\mu \check{\otimes} e_i))(h_1 \otimes e_j), h_2 \otimes f_\tau\rangle &= \delta_{ij} \sum_{\nu\in M} \langle s_{\nu\mu}h_1 \otimes f_\nu, h_2 \otimes f_\tau\rangle \\ &= \delta_{ij}\langle s_{\tau\mu}h_1, h_2\rangle. \end{aligned}$$

Apparently, if $i \ne j$ (respectively, $\tau \ne \mu$), then $t_{ij} = 0$ (respectively, $s_{\mu\tau} = 0$). On the other hand, when $i = j$ and $\tau = \mu$, $t_{ii} = s_{\mu\mu} = a \in \mathfrak{A}$, for all $1 \le i \le n$ and $\mu \in M$. Thus, $T = a \otimes 1_n$, $S = a \otimes 1_{\mathcal{K}}$, and $Y = a \otimes (1_n \oplus 1_{\mathcal{K}})$. □

A cocycle $\rho \in \mathcal{Z}^1(\mathcal{A} * \mathcal{B})$ is known as a *derivation*, because the cocycle equation reads $\rho(X_1X_2) = X_1\rho(X_2) + \rho(X_1)X_2$, for all $X_1, X_2 \in \mathcal{A} * \mathcal{B}$. In every equivalence class of $H^1(\mathcal{A} * \mathcal{B})$, we now show that there is a derivation defined by an element of $\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})$.

Lemma 5.8. *Every derivation $\rho \in \mathcal{Z}^1(\mathcal{A} * \mathcal{B})$ is equivalent to one of the form*

$$\rho(X) = \begin{pmatrix} 0 & 0 \\ \sum_{i=1}^p S_i U T_i & 0 \end{pmatrix}, \tag{5.7}$$

where $X \in \mathcal{A} * \mathcal{B}$, $S_1, \ldots, S_p \in \mathcal{Z}^0(\mathcal{A})$, and $T_1, \ldots, T_p \in \mathcal{Z}^0(\mathcal{B})$.

Proof. By Proposition 5.6 and Lemma 5.3, we may assume, for all $X \in \mathcal{A} * \mathcal{B}$, that

$$\rho(X) = \begin{pmatrix} 0 & 0 \\ \sum_{i=1}^p S_i U T_i & 0 \end{pmatrix},$$

where $S_1, \ldots, S_p \in \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$ and $T_1, \ldots, T_p \in M_n(\mathfrak{A})$. It only remains to show that $S_1, \ldots, S_p \in \mathcal{A}'$, and $T_1, \ldots, T_p \in \mathcal{B}'$.

Assume that both $\mathcal{B}'$ and $(\mathcal{B}')^\perp$ are free modules of finite type over $\mathfrak{A} \otimes \mathbb{C}1_n$ and assume there exists $1 \leq s < p$ such that $\{T_i\}_{i=1}^s$ (respectively $\{T_i\}_{i=s+1}^p$) is a basis for $\mathcal{B}'$ (respectively $(\mathcal{B}')^\perp$) over $\mathfrak{A} \otimes \mathbb{C}1_n$. Clearly, $\partial T_i = 0$, for all $1 \leq i \leq s$. Suppose that $\sum_{i=s+1}^p (a_i \otimes 1_n) \partial T_i = 0$, where $a_i \otimes 1_n \in \mathfrak{A} \otimes \mathbb{C}1_n$, for all $s+1 \leq i \leq p$. Since $\mathfrak{A} \otimes \mathbb{C}1_n \subseteq M_n(\mathfrak{A})'$, we have

$$\sum_{i=s+1}^p (a_i \otimes 1_n) \partial T_i = \partial \left(\sum_{i=s+1}^p (a_i \otimes 1_n) T_i \right) = 0,$$

so $\sum_{i=s+1}^p (a_i \otimes 1_n) T_i \in \mathcal{B}' \cap (\mathcal{B}')^\perp$. Then $\sum_{i=s+1}^p (a_i \otimes 1_n) T_i = 0$ and the linear independence of $\{T_i\}_{i=s+1}^p$ implies that $a_i \otimes 1_n = 0$, for all $s+1 \leq i \leq p$. Hence, $\{\partial T_i\}_{i=s+1}^p$ is a linearly independent set over $\mathfrak{A} \otimes \mathbb{C}1_n$.

We now calculate the coboundary equation for ρ. For all $X_1, X_2 \in \mathcal{A} * \mathcal{B}$, the

(2,1) entry of $(\partial\rho)(X_1, X_2)$ is

$$\sum_{i=1}^{s}(\partial S_i)(A_1)U_2T_i + \sum_{i=s+1}^{p}(\partial S_i)(A_1)U_2T_i + \sum_{i=s+1}^{p} S_iU_1(\partial T_i)(B_2) = 0, \tag{5.8}$$

by Table I. In particular, if $U_2 = 0$, then $\sum_{i=s+1}^{p} S_iU_1(\partial T_i)(B_2) = 0$ and Lemma 5.4 implies that $S_i = 0$, for all $s+1 \le i \le p$. Since (5.8) now reduces to $\sum_{i=1}^{s}(\partial S_i)(A_1)U_2T_i = 0$, another application of Lemma 5.4 demonstrates that $\partial S_i = 0$ — that is, $S_i \in \mathcal{A}'$, for all $1 \le i \le s$.

In general, both $\mathcal{B}'$ and $(\mathcal{B}')^{\perp}$ are ultraweakly closed submodules of $M_n(\mathfrak{A})$ over $\mathfrak{A}\otimes\mathbb{C}1_n$. By Corollary 3.12, there is the set of pairwise orthogonal projections $\{p_i\}_{i=0}^{t}$ in $\mathfrak{A}$ such that $\sum_{i=0}^{t} p_i = 1_{\mathcal{H}}$ and both $(p_i\otimes 1_n)\mathcal{B}'$ and $(p_i\otimes 1_n)(\mathcal{B}')^{\perp}$ are free modules of finite type over $(p_i \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$, for all $0 \le i \le t$. Hence, we may assume that $\{T_i\}_{i=1}^{p}$ generates $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes \mathbb{C}1_n$ and, for all $0 \le j \le t$, there exist $1 \le r(j) \le s(j) \le p$ such that $\{(p_j \otimes 1_n)T_i\}_{i=1}^{r(j)}$ (respectively $\{(p_j \otimes 1_n)T_i\}_{i=s(j)+1}^{p}$) is a basis for $(p_j \otimes 1_n)\mathcal{B}'$ (respectively $(p_j \otimes 1_n)(\mathcal{B}')^{\perp}$) over $(p_j \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$ and $(p_j \otimes 1_n)T_i = 0$, for all $r(j) + 1 \le i \le s(j)$.

Now choose j_0 such that $0 \le j_0 \le t$. Multiply the $(2, 1)$ entry of the coboundary equation on the right by $p_{j_0} \otimes 1_n$ to obtain

$$\left(\sum_{i=1}^{r(j_0)}(\partial S_i)(A_1)U_2T_i + \sum_{i=s(j_0)+1}^{p}(\partial S_i)(A_1)U_2T_i + \sum_{i=s(j_0)+1}^{p} S_iU_1(\partial T_i)(B_2)\right)(p_{j_0}\otimes 1_n) = 0.$$

Then, by Lemma 5.5, the preceding calculations prove that $(p_{j_0} \otimes 1_{\mathcal{K}})S_i \in \mathcal{A}'$, for all $1 \le i \le r(j_0)$, and, moreover, $(p_{j_0} \otimes 1_{\mathcal{K}})S_i = 0$, for all $s(j_0) + 1 \le i \le p$. Since

$$\sum_{i=1}^{p} S_iU(p_{j_0} \otimes 1_n)T_i = \sum_{i=1}^{r(j_0)} S_iU(p_{j_0} \otimes 1_n)T_i$$

and $U = \sum_{i=0}^{t} U(p_i \otimes 1_n)$, for all $U \in \mathfrak{A} \otimes_* \mathcal{L}(\mathbb{C}^n, \mathcal{K})$, we may replace S_i with $S_i - (p_{j_0} \otimes 1_{\mathcal{K}})S_i$, for all $r(j_0) + 1 \le i \le s(j_0)$, without changing $\rho(X)$. Because j_0

was arbitrary, we conclude that $S_i \in \mathcal{A}'$ and $T_i \in \mathcal{B}'$, for all $1 \leq i \leq p$. $\square$

Lemma 5.8 defines a surjective map from $\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})$ onto $H^1(\mathcal{A} * \mathcal{B})$. The majority of the work in the proof of the next theorem is in calculating the kernel of this map.

Theorem 5.9. $H^1(\mathcal{A} * \mathcal{B}) \cong H^0(\mathcal{A})/(\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} H^0(\mathcal{B})/(\mathfrak{A} \otimes \mathbb{C}1_n)$.

Proof. It is clear, by Table I, that any linear map $\rho \in \mathcal{L}^1(\mathcal{A} * \mathcal{B})$ of the form (5.7) is a derivation on $\mathcal{A} * \mathcal{B}$. We define a $\mathfrak{A}$-bilinear map $\phi : \mathcal{Z}^0(\mathcal{A}) \oplus \mathcal{Z}^0(\mathcal{B}) \to \mathcal{Z}^1(\mathcal{A} * \mathcal{B})$ by

$$\phi(S,T)(X) = \begin{pmatrix} 0 & 0 \\ SUT & 0 \end{pmatrix},$$

for all $(S,T) \in \mathcal{Z}^0(\mathcal{A}) \oplus \mathcal{Z}^0(\mathcal{B})$ and $X \in \mathcal{A} * \mathcal{B}$. If $\pi : \mathcal{Z}^0(\mathcal{A}) \oplus \mathcal{Z}^0(\mathcal{B}) \to \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})$ is the canonical map, then, by the universal property of the tensor product, there exists a unique $\mathfrak{A}$-linear map $\tilde{\phi} : \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}) \to \mathcal{Z}^1(\mathcal{A} * \mathcal{B})$ making the following diagram commute.

$$\begin{tikzcd} \mathcal{Z}^0(\mathcal{A}) \oplus \mathcal{Z}^0(\mathcal{B}) \arrow[r, "\pi"] \arrow[d, "\phi"'] & \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}) \arrow[dl, "\tilde{\phi}"] \\ \mathcal{Z}^1(\mathcal{A} * \mathcal{B}) & \end{tikzcd}$$

Let $\tilde{\pi} : \mathcal{Z}^1(\mathcal{A} * \mathcal{B}) \to H^1(\mathcal{A} * \mathcal{B})$ be the canonical projection and let $\psi = \tilde{\pi} \circ \tilde{\phi}$. By Lemma 5.8, ψ is surjective and we now calculate its kernel.

Recall that every $\xi \in \mathcal{B}^1(\mathcal{A} * \mathcal{B})$ is spacially implemented by an operator $Y \in \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathbb{C}^n \oplus \mathcal{K})$ — that is, $\xi(X) = XY - YX$, for all $X \in \mathcal{A} * \mathcal{B}$. If, additionally, $\xi \in \tilde{\phi}(\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}))$, then ξ vanishes on $\mathcal{B} \oplus \mathcal{A}$, so Y must be of the form $T \oplus S$, where $T \in \mathcal{Z}^0(\mathcal{B})$ and $S \in \mathcal{Z}^0(\mathcal{A})$. Then

$$\xi(X) = \begin{pmatrix} 0 & 0 \\ UT - SU & 0 \end{pmatrix},$$

for all $X \in \mathcal{A}*\mathcal{B}$, and $\xi = \tilde{\phi}(1_{\mathcal{A}} \otimes_{\mathfrak{A}} T - S \otimes_{\mathfrak{A}} 1_{\mathcal{B}})$. Thus, $\tilde{\phi}(\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})) \cap \mathcal{B}^1(\mathcal{A}*\mathcal{B}) \subseteq \tilde{\phi}(\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}))$. Since Table I implies the other inclusion, we conclude that $\tilde{\phi}(\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})) \cap \mathcal{B}^1(\mathcal{A} * \mathcal{B}) = \tilde{\phi}(\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}))$.

Now suppose that $D \in \ker(\tilde{\phi})$ and $D = \sum_{i=1}^{\ell} S_i \otimes_{\mathfrak{A}} T_i$, where $S_i \in \mathcal{Z}^0(\mathcal{A})$ and $T_i \in \mathcal{Z}^0(\mathcal{B})$, for all $1 \leq i \leq \ell$. By Theorem 3.11, there exist pairwise orthogonal projections $\{p_j\}_{j=0}^{t}$ in $\mathfrak{A}$ such that $\sum_{j=0}^{t} p_j = 1$ and $(p_j \otimes 1_n)\mathcal{Z}^0(\mathcal{B})$ is a free module of finite type over $(p_j \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$, for all $0 \leq j \leq t$. Consequently, we may assume that $\{T_i\}_{i=1}^{\ell}$ generates $\mathcal{Z}^0(\mathcal{B})$ over $\mathfrak{A} \otimes \mathbb{C}1_n$ and there exist $\{k_j\}_{j=0}^{t+1}$ such that $0 = k_0 \leq \cdots \leq k_{t+1} = \ell$ and $\{T_i\}_{i=k_j+1}^{k_{j+1}}$ is a basis for $(p_j \otimes 1_n)\mathcal{Z}^0(\mathcal{B})$ over $(p_j \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$, for all $0 \leq j \leq t$. Since

$$\tilde{\phi}(D) \begin{pmatrix} 0 & 0 \\ U(p_j \otimes 1_n) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \sum_{i=k_j+1}^{k_{j+1}} S_i U T_i & 0 \end{pmatrix} = 0,$$

for all $0 \leq j \leq t$ and $U \in \mathfrak{A} \otimes_* \mathcal{L}(\mathbb{C}^n, \mathcal{K})$, Lemma 5.5 implies that $(p_j \otimes 1_{\mathcal{K}})S_i = 0$, for all $k_j + 1 \leq i \leq k_{j+1}$ and $0 \leq j \leq t$. Thus, $\tilde{\phi}$ is injective and because $\tilde{\phi}(\ker(\psi)) = \tilde{\phi}(\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})) \cap \mathcal{B}^1(\mathcal{A} * \mathcal{B})$, $\ker(\psi) = \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})$.

We let $\hat{\pi} : \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}) \to \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}) / \ker(\psi)$ denote the canonical projection. By the first isomorphism theorem in algebra, there is an isomorphism $\hat{\psi} : (\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})) / \ker(\psi) \to H^1(\mathcal{A} * \mathcal{B})$ making the following diagram commute.

$$\begin{tikzcd} \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}) \arrow[r, "\hat{\pi}"] \arrow[d, "\psi"'] & \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}) / \ker(\psi) \arrow[dl, "\hat{\psi}"] \\ H^1(\mathcal{A} * \mathcal{B}) & \end{tikzcd}$$

We use the universal property of the tensor product to define linear maps $\omega : \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}) \to \mathcal{Z}^0(\mathcal{A})/(\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})/(\mathfrak{A} \otimes \mathbb{C}1_n)$ and $\sigma : \mathcal{Z}^0(\mathcal{A})/(\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})/(\mathfrak{A} \otimes \mathbb{C}1_n) \to (\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}))/\ker(\psi)$ by $a \otimes_{\mathfrak{A}} b \mapsto (a + \mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} (b + \mathfrak{A} \otimes \mathbb{C}1_n)$ and $(a + \mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} (b + \mathfrak{A} \otimes \mathbb{C}1_n) \mapsto (a \otimes_{\mathfrak{A}} b) + \ker(\psi)$, respectively. Note that $\sigma \circ \omega = \hat{\pi}$ and our diagram then reads

$$\begin{tikzcd}
\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}) \arrow[r, "\omega"] \arrow[d, "\psi"'] & (\mathcal{Z}^0(\mathcal{A})/\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} (\mathcal{Z}^0(\mathcal{B})/\mathfrak{A} \otimes \mathbb{C}1_n) \arrow[d, "\sigma"'] \\
H^1(\mathcal{A} * \mathcal{B}) & \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})/\ker(\psi) \arrow[l, "\hat{\psi}"]
\end{tikzcd}$$

We complete the proof by showing σ is an isomorphism. Since σ is clearly linear and surjective, it only remains to show that σ is injective. Suppose $\sigma y = 0$ for some $y \in \mathcal{Z}^0(\mathcal{A})/(\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})/(\mathfrak{A} \otimes \mathbb{C}1_n)$. Because ω is surjective, there exists $x \in \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B})$ such that $\omega x = y$. Now $\hat{\pi}x = (\sigma \circ \omega)x = 0$, so $x \in \ker(\psi) = \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes_{\mathfrak{A}} \mathcal{Z}^0(\mathcal{B}) \subseteq \ker(\omega)$. Then $y = \omega x = 0$. $\square$

There are several key elements in the calculation of $H^1(\mathcal{A}*\mathcal{B})$ that are important to note. Observe that $\mathcal{Z}^0(\mathcal{B}) = \mathcal{B}'$ is complemented in $\mathcal{L}^0(\mathcal{B}) = M_n(\mathfrak{A})$. Additionally, there is a set of pairwise orthogonal projections $\{p_i\}_{i=1}^t$ in $\mathfrak{A} \otimes \mathbb{C}1_n$ such that $\sum_{i=1}^t p_i = 1_{\mathcal{H}}$ and both $(p_i \otimes 1_n)\mathcal{B}'$ and $(p_i \otimes 1_n)(\mathcal{B}')^{\perp}$ are free modules of finite type over $(p_i \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$, for all $1 \leq i \leq t$. Before proceeding with the calculations of the higher cohomology groups, we must establish that $\mathcal{Z}^m(\mathcal{B})$ is complemented in $\mathcal{L}^m(\mathcal{B})$, for $m \geq 1$.

C. Multilinear Maps on $\mathcal{B}$

By Theorem 3.11, there exists a set of pairwise orthogonal projections $\{p_i\}_{i=1}^t$ in $\mathfrak{A}$ such that $\sum_{i=1}^t p_i = 1_{\mathcal{H}}$ and $\mathcal{B}_i = (p_i \otimes 1_n)\mathcal{B}$ is a free module of finite type over

$(p_i \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$, for all $1 \leq i \leq t$. We fix a basis $\{B_j^i\}_{j=1}^{\ell_i}$ for $\mathcal{B}_i$ over $(p_i \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$ that is annihilated by $1 - p_i \otimes 1_n$ and commence our study of multilinear maps on $\mathcal{B}$.

We may only consider maps in $\mathcal{L}^m(\mathcal{B}, M_n(\mathfrak{A}) : \mathfrak{A})$, by Theorem 4.4, and we will identify $\mathcal{L}^m(\mathcal{B}, M_n(\mathfrak{A}) : \mathfrak{A})$ with sums of multilinear arrays with entries from $M_n(\mathfrak{A})$. For all $\ell, m \geq 1$, the space of m-dimensional arrays $A = (A_{i_1,\ldots,i_m})_{i_1,\ldots i_m=1}^{\ell}$ with entries from $M_n(\mathfrak{A})$ will be denoted $A_{\ell,m}(M_n(\mathfrak{A}))$ and we define $A_{\ell,0}(M_n(\mathfrak{A}))$ to be $M_n(\mathfrak{A})$. We identify $A_{\ell,m}(M_n(\mathfrak{A}))$ with the von Neumann algebra $\bigoplus_{i=1}^{\ell^m} M_n(\mathfrak{A})$ of ℓ^m copies of $M_n(\mathfrak{A})$. Then there is a canonical norm topology on $A_{\ell,m}(M_n(\mathfrak{A}))$ and convergence is entrywise in the operator norm on $M_n(\mathfrak{A})$. Furthermore, the weak* topology on $M_n(\mathfrak{A})$ imposes a weak* topology on $A_{\ell,m}(M_n(\mathfrak{A}))$ and weak* convergence is also entrywise.

Recall that $\mathcal{L}^m(\mathcal{B})$ is the dual space of $\mathcal{B} \,\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B} \,\hat{\otimes}_\pi M_n(\mathfrak{A})_*$, where there are m copies of $\mathcal{B}$ and $M_n(\mathfrak{A})_*$ is the predual of $M_n(\mathfrak{A})$. The duality is defined by

$$\langle B_1 \otimes \cdots \otimes B_m \otimes A, \rho \rangle = \langle A, \rho(B_1, \ldots, B_m) \rangle,$$

for all $B_1, \ldots, B_m \in \mathcal{B}$ and $A \in M_n(\mathfrak{A})_*$, and $\mathcal{L}^m(\mathcal{B}, M_n(\mathfrak{A}) : \mathfrak{A})$ is weak* closed. We shall revisit the theory of C^*-modules where the weak* topology plays an important role.

Lemma 5.10. *There is a correspondence $\mathcal{L}^m(\mathcal{B}_i, M_n(p_i\mathfrak{A}) : \mathfrak{A}) \cong A_{\ell_i,m}(M_n(p_i\mathfrak{A}))$, for all $m > 0$ and $1 \leq i \leq t$, that is homogeneous with respect to $\mathfrak{A} \otimes \mathbb{C}1_n$ and a weak* homeomorphism.*

Proof. Let $1 \leq i_0 \leq t$ be fixed and let $\rho \in \mathcal{L}^m(\mathcal{B}_{i_0}, M_n(p_{i_0}\mathfrak{A}) : \mathfrak{A})$. Define $A^\rho_{i_1,\ldots,i_m} = \rho(B^{i_0}_{i_1}, \ldots, B^{i_0}_{i_m})$, for all $1 \leq i_1, \ldots, i_m \leq \ell_{i_0}$, and let $A^\rho = (A^\rho_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^{\ell_{i_0}}$. The mapping $\varphi_{i_0} : \mathcal{L}^m(\mathcal{B}_{i_0}, M_n(p_{i_0}\mathfrak{A}) : \mathfrak{A}) \to A_{\ell_{i_0},m}(M_n(p_{i_0}\mathfrak{A}))$ defined by $\rho \mapsto A^\rho$ is an isomorphism of Banach spaces that is homogeneous with respect to $\mathfrak{A}\otimes\mathbb{C}1_n$. Because

$$\langle B_{i_1}^{i_0}\otimes\cdots\otimes B_{i_m}^{i_0}\otimes A_*,\rho\rangle = \langle A_*,\rho(B_{i_1}^{i_0},\ldots,B_{i_m}^{i_0})\rangle = \langle A_*,A^{\rho}_{i_1,\ldots,i_m}\rangle,$$

for all $1\le i_1,\ldots,i_m\le \ell_{i_0}$ and $A_*\in M_n(\mathfrak{A})_*$, it is evident that φ_{i_0} is weak* continuous. Then φ_{i_0} is a weak* homeomorphism [4, Theorem 2.7]. □

As with the Hilbert spaces on which they are modelled, the direct sum of C^*-modules $\{M_i\}_{i=1}^p$ over $\mathfrak{A}$ is also a C^*-module over $\mathfrak{A}$. The inner product is defined as the sum of the inner products of the components and, hence, norm convergence in $\bigoplus_{i=1}^p M_i$ is equivalent to convergence in each component. In particular, $A_{\ell,m}(M_n(\mathfrak{A}))$ is a direct sum of C^*-modules and, by Lemma 3.2, the operator norm on $A_{\ell,m}(M_n(\mathfrak{A}))$ is equivalent to the norm induced by its inner product. The theorems of Chapter III remain valid for submodules of $A_{\ell,m}(M_n(\mathfrak{A}))$ and we use these theorems to calculate the remaining cohomology groups of the join.

D. The Higher Cohomology Groups of $\mathcal{A}*\mathcal{B}$

As in the previous sections, it will often suffice to consider the case where p is a projection in $\mathfrak{A}$ and $(p\otimes 1_n)\mathcal{B}$ is a free module of finite type over $(p\otimes 1_n)(\mathfrak{A}\otimes\mathbb{C}1_n)$. We may also assume that various weak* closed modules of multilinear maps on $(p\otimes 1_n)\mathcal{B}$ are free over $(p\otimes 1_n)(\mathfrak{A}\otimes\mathbb{C}1_n)$ of finite type. This is possible because only a finite number of steps are involved in the calculation of any particular cohomology group and we may refine a given partition of $1_{\mathcal{H}}$ at each step in such a way that every module involved in our calculation is a free module of finite type.

Notation 5.5. By Theorem 4.4, the cohomology of $\mathcal{B}$ is determined by the cochains that are homogeneous with respect to $\mathfrak{A}\otimes\mathbb{C}1_n$. Furthermore, by Proposition 5.3, the multilinear maps on $\mathcal{B}$ that will appear in the calculation of $H^m(\mathcal{A}*\mathcal{B})$ are also homogeneous with respect to $\mathfrak{A}\otimes\mathbb{C}1_n$. It will suffice, therefore, to consider cochains in $\mathcal{L}^m(\mathcal{B}, M_n(\mathfrak{A}) : \mathfrak{A})$ and we let $\mathcal{L}^m(\mathcal{B}) = \mathcal{L}^m(\mathcal{B}, M_n(\mathfrak{A}) : \mathfrak{A})$. Analogous notation will be used for cocycles, coboundaries, and cohomology groups.

Apply Theorem 3.11 to $\mathcal{B}$ to obtain a set of pairwise orthogonal projections $\{p_i\}_{i=1}^t$ in $\mathfrak{A}$ such that $\sum_{i=1}^t p_i = 1_{\mathcal{H}}$ and $\mathcal{B}_i = (p_i \otimes 1_n)\mathcal{B}$ is a free module of finite type over $(p_i\otimes 1_n)(\mathfrak{A}\otimes\mathbb{C}1_n)$, for all $1\leq i\leq t$. We let $\mathcal{L}^m(\mathcal{B}_i) = \mathcal{L}^m(\mathcal{B}_i, M_n(p_i\mathfrak{A}) : \mathfrak{A})$ and we identify $\mathcal{L}^m(\mathcal{B}_i)$ with $(p_i\otimes 1_n)\mathcal{L}^m(\mathcal{B})$, for all $1\leq i\leq t$. Observe that $\mathcal{L}^m(\mathcal{B})\cong \bigoplus_{i=1}^t \mathcal{L}^m(\mathcal{B}_i)$.

Now let $1\leq i_0\leq t$ be fixed. Following the procedure of Gilfeather and Smith [14], we define a sequence of bases for $\mathcal{L}^m(\mathcal{B}_{i_0})$, for all $m\geq 0$.

It is clear from the definition of the coboundary map that $\mathcal{Z}^m(\mathcal{B}_{i_0})$ is a weak* closed submodule of $\mathcal{L}^m(\mathcal{B}_{i_0})$, for all $m\geq 0$. By Lemma 5.10, we identify $\mathcal{L}^m(\mathcal{B}_{i_0})$ with a W^*-module of arrays with entries in $p_{i_0}\mathfrak{A}$. Then, by Theorem 3.5, $\mathcal{Z}^m(\mathcal{B}_{i_0})$ has a weak* closed complement $\mathcal{Z}^m(\mathcal{B}_{i_0})^\perp$ in $\mathcal{L}^m(\mathcal{B}_{i_0})$. We assume that both $\mathcal{Z}^m(\mathcal{B}_{i_0})$ and $\mathcal{Z}^m(\mathcal{B}_{i_0})^\perp$ are free modules of finite type over $(p_{i_0}\otimes 1_n)(\mathfrak{A}\otimes\mathbb{C}1_n)$, for all $m\geq 0$, as discussed above. For $m=0$, let $\{\psi_{0,2,j}^{i_0}\}$ be a basis for $\mathcal{Z}^0(\mathcal{B}_{i_0})$ such that $\psi_{0,2,1}^{i_0} = p_{i_0}\otimes 1_n$ and let $\{\psi_{0,3,j}^{i_0}\}$ be a basis for $\mathcal{Z}^0(\mathcal{B}_{i_0})^\perp$. If $\psi_{1,1,j}^{i_0} = \partial\psi_{0,3,j}^{i_0}$, for all j, then $\{\psi_{1,1,j}^{i_0}\}$ is a basis for $\mathcal{B}^1(\mathcal{B}_{i_0})$ over $(p_{i_0}\otimes 1_n)(\mathfrak{A}\otimes\mathbb{C}1_n)$. By Remark 3.13 and Theorem 3.15, there is a linearly independent set $\{\psi_{1,2,j}^{i_0}\}$ in $\mathcal{Z}^1(\mathcal{B}_{i_0})$ such that $\{\psi_{1,1,j}^{i_0}\}\cup\{\psi_{1,2,j}^{i_0}\}$ is a basis for $\mathcal{Z}^1(\mathcal{B}_{i_0})$. Similarly, for all $m\geq 1$, we construct bases for $\mathcal{Z}^m(\mathcal{B}_{i_0})^\perp$ and $\mathcal{Z}^{m+1}(\mathcal{B}_{i_0})$.

Having obtained bases for $\mathcal{B}^m(\mathcal{B}_i)$, $\mathcal{Z}^m(\mathcal{B}_i)$, and $\mathcal{Z}^m(\mathcal{B}_i)^\perp$, for all $m\geq 0$ and $1\leq i\leq t$, we combine them to form generating sets for $\mathcal{B}^m(\mathcal{B})$, $\mathcal{Z}^m(\mathcal{B})$, and $\mathcal{Z}^m(\mathcal{B})^\perp$. For all $1\leq \ell\leq 3$, $m\geq 0$, and for all j, let $\psi_{m,\ell,j}^i = 0$ when $\psi_{m,\ell,j}^i$ has not been defined already and let $\psi_{m,\ell,j} = \sum_{i=1}^t \psi_{m,\ell,j}^i$. Note, in particular, that $\psi_{0,2,1} = 1_{\mathcal{B}}$.

With generating sets of this form in hand, we may further simplify the cocycle in Proposition 5.6. Our decomposition is the same as that of Gilfeather and Smith [14, Proposition 4.2] and is the analogue of Lemma 5.8, for $m \geq 2$.

Lemma 5.11. *Let $\rho \in \mathcal{Z}^m(\mathcal{A} * \mathcal{B})$ and $m \geq 1$. Then there is an equivalent cocycle of the form*

$$\zeta(X_1, \ldots, X_m) = \begin{pmatrix} 0 & 0 \\ \sum_{i=0}^{m-1} \sum_j \phi_{i,2,j} U_{m-i} \psi_{i,2,j} & 0 \end{pmatrix}, \tag{5.9}$$

where $\phi_{i,2,j} \in \mathcal{Z}^{m-i-1}(\mathcal{A})$, for all $0 \leq i \leq m-1$ and all j. Moreover, $\phi_{0,2,1} = 0$ and $(p_s \otimes 1_{\mathcal{K}})\phi_{i,2,j} = 0$, whenever $(p_s \otimes 1_n)\psi_{i,2,j} = 0$, for all $1 \leq s \leq t$.

Proof. By Proposition 5.3 and Proposition 5.6, every cocycle in $\mathcal{Z}^m(\mathcal{A} * \mathcal{B})$ is equivalent to a cocycle of the form

$$\rho(X_1, \ldots, X_m) = \begin{pmatrix} 0 & 0 \\ \sum_{i=0}^{m-1} \sum_{k=1}^{3} \sum_j \phi_{i,k,j} U_{m-i} \psi_{i,k,j} & 0 \end{pmatrix},$$

where $\phi_{i,k,j} \in \mathcal{L}^{m-i-1}(\mathcal{A})$, for all i, j, k. We may assume, without loss of generality, that $(p_s \otimes 1_{\mathcal{K}})\phi_{i,k,j} = 0$, whenever $(p_s \otimes 1_n)\psi_{i,k,j} = 0$, for all $1 \leq s \leq t$. By Table I, the $(2,1)$ entry of $(\partial\rho)(X_1, \ldots, X_{m+1})$ is

$$\sum_{i,k,j} \partial\phi_{i,k,j} U_{m-i+1} \psi_{i,k,j} + \sum_{i,j} (-1)^{m-i+1} \phi_{i,3,j} U_{m-i} \psi_{i+1,1,j} = 0. \tag{5.10}$$

First let $1 \leq s_0 \leq t$, multiply (5.10) by $(p_{s_0} \otimes 1_n)$ on the right, and let $U_{m-i+1} = 0$, for all $0 \leq i \leq m-1$. Then we have

$$\sum_j \phi_{m-1,3,j} U_1 \psi_{m,1,j} (p_{s_0} \otimes 1_n) = 0$$

and, by Lemma 5.5, $(p_{s_0} \otimes 1_{\mathcal{K}})\phi_{m-1,3,j} = 0$, for all j. Similarly, if $1 \leq i_0 \leq m-1$ and

$U_{m-i+1} = 0$, for all $i \neq i_0$, then

$$\sum_{k,j} \partial\phi_{i_0,k,j} U_{m-i_0+1} \psi^{s_0}_{i_0,k,j} + \sum_j (-1)^{m-i_0} \phi_{i_0-1,3,j} U_{m-i_0+1} \psi^{s_0}_{i_0,1,j} = 0.$$

By Lemma 5.5, $(p_{s_0} \otimes 1_{\mathcal{K}})\partial\phi_{i_0,2,j} = 0$, $(p_{s_0} \otimes 1_{\mathcal{K}})\partial\phi_{i_0,3,j} = 0$ and $(p_{s_0} \otimes 1_{\mathcal{K}})(\partial\phi_{i_0,1,j} + (-1)^{m-i_0}\phi_{i_0-1,3,j}) = 0$, for all j. Finally, if $U_{m-i} = 0$, for all $0 \leq i \leq m-1$, then (5.10) becomes

$$\sum_{j,k} \partial\phi_{0,k,j} U_{m+1} \psi_{0,k,j}(p_{s_0} \otimes 1_n) = 0.$$

and, by Lemma 5.5, $(p_{s_0} \otimes 1_{\mathcal{K}})\partial\phi_{0,k,j} = 0$, for all j, $k = 2$, and $k = 3$. Since s_0 was arbitrary, we have the following relations.

(i) $\phi_{i,2,j} \in \mathcal{Z}^{m-i-1}(\mathcal{A})$, for all $0 \leq i \leq m-1$ and all j.

(ii) $\phi_{i,3,j} \in \mathcal{Z}^{m-i-1}(\mathcal{A})$, for all $0 \leq i \leq m-1$ and all j.

(iii) $\partial\phi_{i,1,j} = (-1)^{m-i+1}\phi_{i-1,3,j}$, for all $1 \leq i \leq m-1$ and all j.

(iv) $\phi_{m-1,3,j} = 0$, for all j.

The non-zero terms involving $\phi_{i,1,j}$ and $\phi_{i,3,j}$ may be subtracted from ρ by adding a coboundary, because (iii) and Table I imply that

$$\begin{aligned}
&\partial \begin{pmatrix} 0 & 0 \\ (-1)^{m-i+1}\phi_{i,1,j} U_{m-i} \psi_{i-1,3,j} & 0 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 0 & 0 \\ (-1)^{m-i+1}\partial\phi_{i,1,j} U_{m-i+1} \psi_{i-1,3,j} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \phi_{i,1,j} U_{m-i} \partial\psi_{i-1,3,j} & 0 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 0 & 0 \\ \phi_{i-1,3,j} U_{m-i+1} \psi_{i-1,3,j} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \phi_{i,1,j} U_{m-i} \psi_{i,1,j} & 0 \end{pmatrix},
\end{aligned}$$

for all $1 \leq i \leq m-1$ and all j. Hence, if we let $\xi_{i,j} = (-1)^{m-i} \left(\begin{smallmatrix} 0 & 0 \\ \phi_{i,1,j} U_{m-i} \psi_{i-1,3,j} & 0 \end{smallmatrix} \right)$, then $\zeta = \rho + \sum_{i,j} \partial\xi_{i,j}$ has the required form.

Now suppose that $\phi_{0,2,1} \neq 0$ and let $\eta = (-1)^{m+1}\left(\begin{smallmatrix} 0 & 0 \\ 0 & \phi_{0,2,1} \end{smallmatrix}\right)$. Then, since $\partial\phi_{0,2,1} = 0$ and $\psi_{0,2,1} = 1_{\mathcal{B}}$, Table I shows that $\zeta + \partial\eta$ satisfies all of the conditions in the statement of the lemma. $\square$

Having defined, in essence, a surjective map from $\bigoplus_{i=0}^{m-1} \mathcal{Z}^i(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-i-1}(\mathcal{B})$ onto $H^m(\mathcal{A} * \mathcal{B})$, we now calculate its kernel.

Lemma 5.12. *Let $\rho \in \mathcal{Z}^{m+1}(\mathcal{A} * \mathcal{B})$, let $m \geq 1$, and suppose that ρ has the form specified in Lemma 5.11. Then $\rho \in \mathcal{B}^{m+1}(\mathcal{A} * \mathcal{B})$ if and only if $\phi_{m,2,j} \in \mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}$ and $\phi_{i,2,j} \in \mathcal{B}^{m-i}(\mathcal{A})$, for all $0 \leq i \leq m-1$ and all j.*

Proof. Suppose, for all $0 \leq i \leq m-1$ and all j, $\phi_{m,2,j} = a_j \otimes 1_{\mathcal{K}}$ and $\phi_{i,2,j} = \partial\xi_{i,j}$, where $a_j \in \mathfrak{A}$ and $\xi_{i,j} \in \mathcal{L}^{m-i-1}(\mathcal{A})$. Then $\rho = \sum_j \partial\left(\begin{smallmatrix} (a_j\otimes 1_n)\psi_{m,2,j} & 0 \\ 0 & 0 \end{smallmatrix}\right) + \sum_{i,j} \partial\left(\begin{smallmatrix} 0 & 0 \\ \xi_{i,j}U_{m-i}\psi_{i,2,j} & 0 \end{smallmatrix}\right)$.

Conversely, suppose that $\rho = \partial\xi$, where $\xi \in \mathcal{L}^m(\mathcal{A} * \mathcal{B})$. By Theorem 4.2, we may assume that ξ vanishes whenever any of its entries is in $(\mathfrak{A} \otimes \mathbb{C}1_n) \oplus \mathbb{C}1_{\mathcal{A}}$. Although it is stated for cocycles, Proposition 5.2 applies to ξ in a weaker form. Combining its decomposition with Lemma 5.3, we assume that

$$\xi(X_1, \ldots X_m) = \begin{pmatrix} \beta(B_1, \ldots, B_m) & 0 \\ \sum_{i=0}^{m-1} \sum_{k=1}^{3} \sum_j \xi_{i,k,j} U_{m-i} \psi_{i,k,j} & \alpha(A_1, \ldots, A_m) \end{pmatrix},$$

where $\alpha \in \mathcal{L}^m(\mathcal{A})$, $\beta \in \mathcal{L}^m(\mathcal{B})$, and $\xi_{i,k,j} \in \mathcal{L}^{m-i-1}(\mathcal{A})$, for all i, j, k. We also assume that $(p_s \otimes 1_{\mathcal{K}})\xi_{i,k,j} = 0$, whenever $(p_s \otimes 1_n)\psi_{i,k,j} = 0$, for all $1 \leq s \leq t$. Since $\partial\xi = \rho$, Table I implies that $\alpha \in \mathcal{Z}^m(\mathcal{A})$, $\beta \in \mathcal{Z}^m(\mathcal{B})$, and

$$(5.11) \quad \sum_{i,k,j} \partial\xi_{i,k,j} U_{m-i+1} \psi_{i,k,j} + (-1)^{m+1} \alpha U_{m+1} 1_{\mathcal{B}} + 1_{\mathcal{A}} U_1 \beta = \sum_{i,j} \phi_{i,2,j} U_{m-i+1} \psi_{i,2,j}.$$

First let $\beta = \sum_j (\beta_j \otimes 1_n)\psi_{m,2,j}$, where $\beta_j \in \mathfrak{A}$, for all j. The coefficients $\{\beta_j\}$ are unique, if we insist that $p_s\beta_j = 0$, whenever $(p_s \otimes 1_n)\psi_{m,2,j} = 0$, for all $1 \leq s \leq t$. We repeat the procedure in the proof of Lemma 5.11. Let $1 \leq s_0 \leq t$, multiply (5.11) on the right by $p_{s_0}\otimes 1_n$, and let $U_i=0$, for all $2\le i\le m+1$. Then (5.11) reads

$$\sum_j(\beta_j\otimes 1_{\mathcal{K}})U_1\psi_{m,2,j}(p_{s_0}\otimes 1_n)=\sum_j\phi_{m,2,j}U_1\psi_{m,2,j}(p_{s_0}\otimes 1_n).$$

By Lemma 5.5, $(p_{s_0}\otimes 1_{\mathcal{K}})\phi_{m,2,j}=(p_{s_0}\otimes 1_{\mathcal{K}})(\beta_j\otimes 1_{\mathcal{K}})$, for all j. Next let $1\le i_0\le m-1$ and let $U_{m-i+1}=0$, for all $i\neq i_0$. We obtain

$$\sum_{k,j}\partial\xi_{i_0,k,j}U_{m-i_0+1}\psi_{i_0,k,j}(p_{s_0}\otimes 1_n)=\sum_j\phi_{i_0,2,j}U_{m-i_0+1}\psi_{i_0,2,j}(p_{s_0}\otimes 1_n)$$

and Lemma 5.5 implies that, for all j, $(p_{s_0}\otimes 1_{\mathcal{K}})\partial\xi_{i_0,1,j}=0$, $(p_{s_0}\otimes 1_{\mathcal{K}})\partial\xi_{i_0,3,j}=0$, and $(p_{s_0}\otimes 1_{\mathcal{K}})\phi_{i_0,2,j}=(p_{s_0}\otimes 1_{\mathcal{K}})\partial\xi_{i_0,2,j}$. Finally, let $U_i=0$, for all $1\le i\le m$, and then (5.11) becomes

$$\sum_{k,j}\partial\xi_{0,k,j}U_{m+1}\psi^{s_0}_{0,k,j}+(-1)^{m+1}\alpha U_{m+1}(p_{s_0}\otimes 1_n)=\sum_j\phi_{0,2,j}U_{m+1}\psi^{s_0}_{0,2,j}.$$

Recall that $\psi^{s_0}_{0,2,1}=p_{s_0}\otimes 1_n$ and $\phi_{0,2,1}=0$. Hence, by Lemma 5.5, $(p_{s_0}\otimes 1_{\mathcal{K}})(\partial\xi_{0,2,1}+(-1)^{m+1}\alpha)=0$, $(p_{s_0}\otimes 1_{\mathcal{K}})\phi_{0,2,j}=(p_{s_0}\otimes 1_{\mathcal{K}})\partial\xi_{0,2,j}$, for all $j\ge 2$, and $(p_{s_0}\otimes 1_{\mathcal{K}})\partial\xi_{0,3,j}=0$, for all j. Because s_0 was arbitrary, the following relations hold.

(i) $\phi_{m,2,j}\in\mathfrak{A}\otimes\mathbb{C}1_{\mathcal{K}}$, for all j.

(ii) $\partial\xi_{i,1,j}=0$, for all $1\le i\le m-1$ and all j.

(iii) $\partial\xi_{i,3,j}=0$, for all $0\le i\le m-1$ and all j.

(iv) $\partial\xi_{0,2,1}+(-1)^{m+1}\alpha=0$ and $\phi_{0,2,j}=\partial\xi_{0,2,j}$, for all $j\ge 2$.

(v) $\phi_{i,2,j}=\partial\xi_{i,2,j}$, for all $1\le i\le m-1$ and all j.

Since (i), (iv), and (v) are precisely the conditions in the statement of the lemma, the proof is complete. □

The calculation of $H^m(\mathcal{A}*\mathcal{B})$ is now a formality, as a large majority of the work is contained in Lemma 5.11 and Lemma 5.12.

Theorem 5.13. *For all $m \geq 2$,*

$$H^m(\mathcal{A} * \mathcal{B}) \cong H^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} H^0(\mathcal{B})/(\mathfrak{A} \otimes \mathbb{C}1_n) \oplus H^0(\mathcal{A})/(\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} H^{m-1}(\mathcal{B})$$
$$\bigoplus_{i=1}^{m-2} H^{m-i-1}(\mathcal{A}) \otimes_{\mathfrak{A}} H^i(\mathcal{B}).$$

Proof. Observe that $\mathcal{B}' = \mathcal{Z}^0(\mathcal{B})$ is a W^*-module over $\mathfrak{A} \otimes \mathbb{C}1_n$ and $\mathfrak{A} \otimes \mathbb{C}1_n$ is a W^*-submodule of $\mathcal{B}'$. By Theorem 3.5, $\mathcal{B}'/(\mathfrak{A} \otimes \mathbb{C}1_n) \cong (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp$ and we may choose $\{\psi_{0,2,j}\}_{j\geq 2}$ such that $\{\psi_{0,2,j}\}_{j\geq 2}$ generates $(\mathfrak{A} \otimes \mathbb{C}1_n)^\perp$ linearly over $\mathfrak{A} \otimes \mathbb{C}1_n$.

Define a linear mapping $\gamma_0 : \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp \to H^m(\mathcal{A} * \mathcal{B})$ by $\gamma_0(\phi_0 \otimes_{\mathfrak{A}} \psi_0)(X_1, \ldots, X_m) = \left(\begin{smallmatrix} 0 & 0 \\ \phi_0 U_m \psi_0 & 0 \end{smallmatrix}\right)$ and let $\pi_0 : \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp \to \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp / \ker(\gamma_0)$ be the canonical projection. Then, by the first isomorphism theorem, there is a unique injective map $\tilde{\gamma}_0$ making the following diagram commute.

$$\begin{tikzcd} \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp \arrow[r, "\pi_0"] \arrow[d, "\gamma_0"'] & \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp / \ker(\gamma_0) \arrow[dl, "\tilde{\gamma}_0"] \\ H^m(\mathcal{A} * \mathcal{B}) & \end{tikzcd} \tag{5.12}$$

Since, by Lemma 5.12, $\ker(\gamma_0) = \mathcal{B}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp$, we may define $\sigma_0 : H^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp \to \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp / \ker(\gamma_0)$ by $\sigma_0(\phi_0 \otimes_{\mathfrak{A}} \psi_0) = (\phi_0 \otimes_{\mathfrak{A}} \psi_0) + \ker(\gamma_0)$. If $\tau_0 : \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp \to H^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp$ is the canonical map, then $\sigma_0 \circ \tau_0 = \pi_0$ and (5.12) becomes

$$\begin{tikzcd} \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp \arrow[r, "\tau_0"] \arrow[d, "\gamma_0"'] & H^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp \arrow[d, "\sigma_0"'] \\ H^m(\mathcal{A} * \mathcal{B}) & \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp / \ker(\gamma_0) \arrow[l, "\tilde{\gamma}_0"] \end{tikzcd}$$

We now show that σ_0 is an isomorphism. Since σ_0 is obviously linear and surjective, it remains to show that σ_0 is injective. Suppose $\sigma_0 y = 0$. Because τ_0 is surjective, there exists $x \in \mathcal{Z}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^\perp$ such that $\tau_0 x = y$.

Then $\gamma_0 x = (\tilde{\gamma}_0 \circ \sigma_0)y = 0$, so $x \in \ker(\gamma_0) = \mathcal{B}^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes \mathbb{C}1_n)^{\perp}$. Hence, $y = \tau_0 x = 0$ and we note, in particular, the existence of an injective mapping $\Gamma_0 : H^{m-1}(\mathcal{A}) \otimes_{\mathfrak{A}} (\mathcal{Z}^0(\mathcal{B})/\mathfrak{A} \otimes \mathbb{C}1_n) \to H^m(\mathcal{A} * \mathcal{B})$ having the same image as γ_0.

Suppose that $1 \le i_0 \le m-2$. We define a linear mapping $\gamma_{i_0} : \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{i_0}(\mathcal{B}) \to H^m(\mathcal{A} * \mathcal{B})$ by $\gamma_{i_0}(\phi_{i_0} \otimes_{\mathfrak{A}} \psi_{i_0})(X_1, \ldots, X_m) = \left(\begin{smallmatrix} 0 & 0 \\ \phi_{i_0} U_{m-i_0} \psi_{i_0} & 0 \end{smallmatrix}\right)$ and let $\pi_{i_0} : \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{i_0}(\mathcal{B}) \to \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{i_0}(\mathcal{B})/\ker(\gamma_{i_0})$ be the canonical projection. Then there exists a unique injective map $\tilde{\gamma}_{i_0}$ making the following diagram commute, by the first isomorphism theorem.

(5.13)
$$\begin{tikzcd} \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{i_0}(\mathcal{B}) \arrow[r, "\pi_{i_0}"] \arrow[d, "\gamma_{i_0}"'] & \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{i_0}(\mathcal{B})/\ker(\gamma_{i_0}) \arrow[dl, "\tilde{\gamma}_{i_0}"] \\ H^m(\mathcal{A} * \mathcal{B}) & \end{tikzcd}$$

Observe that $\mathcal{Z}^{i_0}(\mathcal{B}) \cong \mathcal{B}^{i_0}(\mathcal{B}) \oplus M_{i_0}$, where M_{i_0} is the linear span of $\{\psi_{i_0,2,j}\}$ over $\mathfrak{A} \otimes \mathbb{C}1_n$. By Table I, $\mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{B}^{i_0}(\mathcal{B}) \subseteq \ker(\gamma_{i_0})$ and, by Lemma 5.12, $(\mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} M_{i_0}) \cap \ker(\gamma_{i_0}) = \mathcal{B}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} M_{i_0}$. Therefore, $\ker(\gamma_{i_0}) = \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{B}^{i_0}(\mathcal{B}) \oplus \mathcal{B}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} M_{i_0}$.

Let $\tau_{i_0} : \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{i_0}(\mathcal{B}) \to H^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} H^{i_0}(\mathcal{B})$ be the quotient map and let $\sigma_{i_0} : H^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} H^{i_0}(\mathcal{B}) \to \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{i_0}(\mathcal{B})/\ker(\gamma_{i_0})$ be defined by $\sigma_{i_0} : (\phi_{i_0} \otimes_{\mathfrak{A}} \psi_{i_0}) = (\phi_{i_0} \otimes_{\mathfrak{A}} \psi_{i_0}) + \ker(\gamma_{i_0})$. Then $\sigma_{i_0} \circ \tau_{i_0} = \pi_{i_0}$ and (5.13) now reads

$$\begin{tikzcd} \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{i_0}(\mathcal{B}) \arrow[r, "\tau_{i_0}"] \arrow[d, "\gamma_{i_0}"'] & H^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} H^{i_0}(\mathcal{B}) \arrow[d, "\sigma_{i_0}"'] \\ H^m(\mathcal{A} * \mathcal{B}) & \mathcal{Z}^{m-i_0-1}(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{i_0}(\mathcal{B})/\ker(\gamma_{i_0}) \arrow[l, "\tilde{\gamma}_{i_0}"] \end{tikzcd}$$

The same diagram chase used for σ_0 shows that σ_{i_0} is an isomorphism and, hence, $\Gamma_{i_0} = \tilde{\gamma}_{i_0} \circ \sigma_{i_0}$ is an injective map with the same image as γ_{i_0}.

Now let $\gamma_{m-1} : \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-1}(\mathcal{B}) \to H^m(\mathcal{A} * \mathcal{B})$ be defined by $\gamma_{m-1}(\phi_{m-1} \otimes_{\mathfrak{A}} \psi_{m-1})(X_1, \ldots, X_m) = \left(\begin{smallmatrix} 0 & 0 \\ \phi_{m-1} U_1 \psi_{m-1} & 0 \end{smallmatrix}\right)$. If M_{m-1} is the linear span of $\{\psi_{m-1,2,j}\}$ over

$\mathfrak{A} \otimes \mathbb{C}1_n$, then $\mathcal{Z}^{m-1}(\mathcal{B}) \cong \mathcal{B}^{m-1}(\mathcal{B}) \oplus M_{m-1}$. By Table I, $\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{B}^{m-1}(\mathcal{B}) \subseteq \ker(\gamma_{m-1})$ and, by Lemma 5.12, $(\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} M_{m-1}) \cap \ker(\gamma_{m-1}) = (\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} M_{m-1}$. Hence, $\ker(\gamma_{m-1}) = \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{B}^{m-1}(\mathcal{B}) \oplus (\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} M_{m-1}$.

Let $\pi_{m-1} : \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-1}(\mathcal{B}) \to \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-1}(\mathcal{B}) / \ker(\gamma_{m-1})$ be the canonical projection. Then, by the first isomorphism theorem, there exists a unique injective linear map $\tilde{\gamma}_{m-1}$ making the following diagram commute.

$$
\begin{tikzcd}
\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-1}(\mathcal{B}) \arrow[r, "\pi_{m-1}"] \arrow[d, "\gamma_{m-1}"'] & \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-1}(\mathcal{B}) / \ker(\gamma_{m-1}) \arrow[dl, "\tilde{\gamma}_{m-1}"] \\
H^m(\mathcal{A} * \mathcal{B}) &
\end{tikzcd} \tag{5.14}
$$

We factor π_{m-1} through $(\mathcal{Z}^0(\mathcal{A})/\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} H^{m-1}(\mathcal{B})$ by defining $\tau_{m-1} : \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-1}(\mathcal{B}) \to (\mathcal{Z}^0(\mathcal{A})/\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} H^{m-1}(\mathcal{B})$ as the quotient map and $\sigma_{m-1} : (\mathcal{Z}^0(\mathcal{A})/\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} H^{m-1}(\mathcal{B}) \to \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-1}(\mathcal{B}) / \ker(\gamma_{m-1})$ by $\sigma(\phi_{m-1} \otimes_{\mathfrak{A}} \psi_{m-1}) = (\phi_{m-1} \otimes_{\mathfrak{A}} \psi_{m-1}) + \ker(\gamma_{m-1})$. Then (5.14) becomes

$$
\begin{tikzcd}
\mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-1}(\mathcal{B}) \arrow[r, "\tau_{m-1}"] \arrow[d, "\gamma_{m-1}"'] & (\mathcal{Z}^0(\mathcal{A})/\mathfrak{A} \otimes \mathbb{C}1_{\mathcal{K}}) \otimes_{\mathfrak{A}} H^{m-1}(\mathcal{B}) \arrow[d, "\sigma_{m-1}"'] \\
H^m(\mathcal{A} * \mathcal{B}) & \mathcal{Z}^0(\mathcal{A}) \otimes_{\mathfrak{A}} \mathcal{Z}^{m-1}(\mathcal{B}) / \ker(\gamma_{m-1}) \arrow[l, "\tilde{\gamma}_0"]
\end{tikzcd}
$$

The same argument as for σ_0 shows that σ_{m-1} is an isomorphism and we have an injective map $\Gamma_{m-1} = \tilde{\gamma}_{m-1} \circ \sigma_{m-1}$ with the same image as γ_{m-1}.

To complete the proof, let $\Gamma = \bigoplus_{i=0}^{m-1} \Gamma_i$. By Lemma 5.11, Γ is surjective and, since Γ is clearly injective, Γ is an isomorphism. $\square$

CHAPTER VI

CONCLUSION

Let $\mathfrak{A}$ be an abelian von Neumann algebra acting on a Hilbert space $\mathcal{H}$. An ultraweakly closed submodule M of $M_n(\mathfrak{A})$ over $\mathfrak{A} \otimes \mathbb{C}1_n$ exhibits properties that are similar to a subspace of $M_n(\mathbb{C})$. Kaplansky [26] showed that M is complemented in $M_n(\mathfrak{A})$ and that every bounded linear functional on M is defined by an element of M. We demonstrate in Theorem 3.11 that there are a finite number of pairwise orthogonal projections $\{p_i\}_{i=0}^s$ in $\mathfrak{A}$ such that $\sum_{i=0}^s p_i = 1_{\mathcal{H}}$ and $(p_i \otimes 1_n)M$ is a free module of finite type over $(p_i \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$, for all $0 \leq i \leq t$. Furthermore, if N is an ultraweakly closed submodule of M, we prove that there exists a finite number of pairwise orthogonal projections $\{q_i\}_{i=0}^t$ such that $\sum_{i=0}^t q_i = 1_{\mathcal{H}}$ and both $(q_i \otimes 1_n)M$ and $(q_i \otimes 1_n)N$ are free modules of finite type over $(q_i \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$, for all $0 \leq i \leq t$. In particular, there is a finite basis for $(q_i \otimes 1_n)M$ containing a basis for $(q_i \otimes 1_n)N$ over $(q_i \otimes 1_n)(\mathfrak{A} \otimes \mathbb{C}1_n)$, for all $0 \leq i \leq t$.

We let $\mathfrak{A}$ be a maximal abelian subalgebra of $\mathcal{L}(\mathcal{H})$, let $\mathcal{A}$ be a norm closed subalgebra of $\mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K})$, and let $\mathcal{B}$ be an ultraweakly closed subalgebra of $M_n(\mathfrak{A})$ containing $\mathfrak{A} \otimes \mathbb{C}1_n$. We defined the join of $\mathcal{A}$ and $\mathcal{B}$ as

$$\mathcal{A} * \mathcal{B} = \left\{ \begin{pmatrix} B & 0 \\ U & A \end{pmatrix} : A \in \mathcal{A}, U \in \mathfrak{A} \otimes_* \mathcal{L}(\mathbb{C}^n, \mathcal{K}), B \in \mathcal{B} \right\}.$$

Using techniques developed by Gilfeather and Smith [14] and our results on submodules of $M_n(\mathfrak{A})$, we were able to decompose $\rho \in \mathcal{Z}^m(\mathcal{A} * \mathcal{B}, \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathbb{C}^n \oplus \mathcal{K}))$ into products of linear maps on $\mathcal{A}$, operators in $\mathfrak{A} \otimes_* \mathcal{L}(\mathbb{C}^n, \mathcal{K})$, and linear maps on $\mathcal{B}$. This decomposition was successively refined until a particularly simple form for ρ was attained. We then established necessary and sufficient conditions for a cocycle of this form to be a coboundary. Finally, we calculated $H^m(\mathcal{A} * \mathcal{B}, \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathbb{C}^n \oplus \mathcal{K}))$, for all $m \geq 0$, in terms of $H^k(\mathcal{A}, \mathfrak{A} \overline{\otimes} \mathcal{L}(\mathcal{K}))$ and $H^k(\mathcal{B}, M_n(\mathfrak{A}))$ and thereby generalized the theorem of Gilfeather and Smith [14, Theorem 4.1] to infinite dimensional matrix algebras.

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VITA

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