LOCAL COHOMOLOGY: COMBINATORICS AND D-MODULES

A Thesis

by

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ABSTRACT

In this dissertation, we study combinatorial and D-module theoretic aspects of local cohomology.

Viewing local cohomology from the point of view of A-hypergeometric systems, the quasidegree set of the non-top local cohomology modules supported at the maximal ideal $\mathfrak{m}$ of a toric ideal determine parameters β where the rank of the corresponding hypergeometric system is higher than expected. We discuss the *Macaulay2* package, *Quasidegrees*, and its main functions. The main purpose of *Quasidegrees* is to compute where the rank of the solution space of an A-hypergeometric system is higher than expected.

Local duality gives a vector space isomorphism between local cohomology and Ext-modules. However, the proof of local duality is nonconstructive. We recall a combinatorial construction by Irena Peeva and Bernd Sturmfels to minimally resolve codimension 2 lattice ideals. With motivation coming from A-hypergeometric systems, we use the construction by Peeva and Sturmfels to construct an explicit local duality isomorphism for codimension 2 lattice ideals.

In general, local cohomology modules of a ring may not be finitely generated. However, they still may possess other finiteness properties. In 1990, Craig Huneke asked if the number of associated prime ideals of a local cohomology module is finite. Using characteristic free D-module techniques inspired by Glennady Lyubeznik, we answer Huneke's question in the affirmative for local cohomology modules over Stanley-Reisner rings.

DEDICATION

This dissertation is dedicated to my family for their continued love and support in this endeavor.

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Contributors

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1. INTRODUCTION

Local cohomology was introduced by Alexander Grothendieck to prove Lefschetz-type theorems. Since its introduction, local cohomology has found numerous connections with other areas of mathematics.

Algebraically, local cohomology modules can be used to measure the dimension and depth of a module on an ideal. As a consequence, local cohomology can be used to test if a ring is Cohen-Macaulay or Gorenstein. Local cohomology can also assist in answering the question of how many generators an ideal has up to radical. The feature of local cohomology that allows us to answer this question is the fact that local cohomology of a module supported at an ideal is the same as the local cohomology of the module supported at the radical of the ideal, that is $H^i_I(M) = H^i_{\sqrt{I}}(M)$ for an R-module M and ideal $I \subset R$. Furthermore, if I is generated by n elements, there is no local cohomology in cohomological degree larger than n.

From the D-module point of view, local cohomology is used to determine the parameters at which the rank of an associated system of partial differential equations, called the A-hypergeometric system with parameter β, is higher than expected. A-hypergeometric systems have an associated toric ideal in the polynomial ring. Toric ideals have a rich combinatorics that is used to gain an understanding of their local cohomology. Conversely, D-modules help us gain a better understanding of the structure of local cohomology modules. For example, D-modules are used to determine if the number of associated prime ideals of a local cohomology module is finite.

For the geometric interpretation of local cohomology, let X be a topological space. A subset $V \subset X$ is *locally closed* if it is the intersection of an open subset and a closed subset of X. Let $\mathscr{F}$ be an abelian sheaf on X, V be a locally closed in X, and

U is an open subset of X in which V is closed in. Set $\Gamma_V(X, \mathscr{F})$ to be the subset of sections $s \in \Gamma(U, \mathscr{F})$ such that s restricted to the complement $U \setminus V$ is zero. One can show that this definition is independent of U and $\Gamma_V(X, -)$ is left exact. The i-th right derived functor of $\Gamma_V(X, -)$ evaluated at $\mathscr{F}$ is the i-th local cohomology group $H^i_V(X, \mathscr{F})$. Since $\Gamma_V(X, -)$ is left exact, $H^0_V(X, -) \cong \Gamma_V(X, -)$.

In this dissertation, we study computational, combinatorial, and D-module aspects of local cohomology. This dissertation is organized as follows.

In Chapter 2, we give the definition and properties of local cohomology. We also recall basic notions from commutative algebra and homological algebra that will be used throughout this dissertation.

Chapter 3 begins by defining the quasidegree set of a module. When the module is presented by a matrix with monomial entries, the quasidegree set can be computed combinatorially using standard pairs of monomial ideals coming from the entries of the presentation matrix. We discuss the main functions of the *Macaulay2* package *Quasidegrees*. The chapter concludes with a discussion on the motivation for writing *Quasidegrees*, namely A-hypergeometric systems. We defined A-hypergeometric systems and recall results by Matusevich, Miller, and Walther that give a connection between quasidegrees of the non-top local cohomology modules of toric ideals and the parameters at which the dimension of the corresponding A-hypergeometric system is higher than expected.

The goal of chapter 4 is to describe an isomorphism for local duality for the non-top local cohomology modules of codimension 2 lattice ideals. The chapter begins by describing the combinatorial construction for the minimal free resolution of codimension 2 lattice ideals by Irena Peeva and Bernd Sturmfels, found in [12]. We then show combinatorial properties of the Ext-modules of codimension 2 lattice ideals. Using the combinatorial data, we construct an isomorphism for local duality of codimension 2 lattice ideals.

Craig Huneke asked whether the set of associated primes of a local cohomology module is finite. The goal of Chapter 5 is to answer Huneke's question for local cohomology modules of Stanley-Reisner rings. The chapter begins by recalling the necessary definitions and properties for simplicial complexes and Stanley-Reisner rings as well as the needed background from the theory of D-modules. Using D-module techniques inspired by Gennady Lyubeznik, chapter 5 concludes by answering Huneke's question question in the affirmative for Stanley-Reisner rings whose associated simplicial complex is a T-space.

2. PRELIMINARIES

This chapter provides the necessary definitions and properties in commutative and homological algebra that will be used throughout this dissertation. We follow definitions and conventions in [5].

For the remainder, let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\Bbbk$ be a field of characteristic 0 unless otherwise specified. Let $S = \Bbbk[x_1, \ldots, x_n] = \Bbbk[x]$ be a polynomial ring in n variables over the field $\Bbbk$. We use multi-index notation throughout.

Notation 2.0.1. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. Then $x^{\mathbf{a}}$ denotes the monomial $x_1^{a_1} \cdots x_n^{a_n}$. A polynomial in S may be written as $\sum_{\mathbf{a}} c_{\mathbf{a}} x^{\mathbf{a}}$ for some $c_{\mathbf{a}} \in \Bbbk$. If $\mathbf{a} \in \mathbb{Z}^n$, then $\sum_{\mathbf{a}} c_{\mathbf{a}} x^{\mathbf{a}} \in \Bbbk[x^{\pm}]$ is a Laurent polynomial.

2.1 Commutative algebra

Let R be a commutative ring, M be an R-module, and $U \subset R$ be a multiplicatively closed subset. The *localization of* M *at* U, written $M[U^{-1}]$ or $U^{-1}M$, is the set of equivalence classes of pairs (m, u) with $m \in M$ and $u \in U$ with the equivalence relation $(m, u) \sim (m', u')$ if there is an element $v \in U$ such that $v(mu' - m'u) = 0$ in M. Denote the equivalence class (m, u) by m/u. The localization $M[U^{-1}]$ is an R-module by defining

$$m/u + m'/u' = (mu' + m'u)/(uu') \text{ and } r(m/u) = (rm)/u$$

for $m, m' \in M$, $u, u' \in U$, and $r \in R$. If $M = R$, then the localization is a ring with multiplication defined as $(r/u)(r'/u') = (rr')/(uu')$. There is a natural inclusion $M \to M[U^{-1}]$ defined by $m \mapsto m/1$. If $U = R \setminus P$ for some prime ideal $P \subset R$, then we write the *localization at* P as M_P. If $f \in R$ and $U = \{f^i : i \geq 0\}$, then we write the *localization at* f as M_f.

A commutative ring R is *graded* if we can write R as a decomposition as an additive group $R = \bigoplus_{i \in G} R_i$ such that $R_i R_j \subset R_{i+j}$ for an abelian monoid G. We may call R a G-graded ring and say R is G-graded to emphasize which grading on R we are using. Similarly, an R-module M is graded by G if R is G-graded and M has a decomposition $M = \bigoplus_{i \in G}$ such that $R_i M_j \subset M_{i+j}$. An element $m \in M$ is *homogeneous* if $m \in M_i$ for some $i \in G$ and the element m is said to have degree i. It may be convenient to shift the degrees of the module. If M is a G-graded module, define $M(g)$ to be the G-graded module where $M(g)_i = M_{g+i}$. Let $f : M \to N$ be a homomorphism of G-graded R-modules. Then f is said to be graded, or homogeneous, of degree d if $f(M_i) \subset N_{i+d}$ for all i.

Example 2.1.1. Let $S = \Bbbk[x_1, \ldots, x_n]$. Then S has an $\mathbb{N}$-grading by setting $S = \bigoplus_{i \in \mathbb{N}} S_i$ where $S_i = \Bbbk \cdot \{x^{\mathbf{a}} : a_1 + \cdots + a_n = i\}$ is the $\Bbbk$-vector space generated by monomials of degree i. This is the *standard grading* on S.

Example 2.1.2. Let $S = \Bbbk[x_1, \ldots, x_n]$. Then S has a $\mathbb{N}^n$-grading by setting $S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_{\mathbf{a}}$ where $S_{\mathbf{a}} = \Bbbk \cdot \{x^{\mathbf{a}}\}$ is the $\Bbbk$-vector space with basis $x^{\mathbf{a}}$.

A proper ideal $I \subset R$ is *prime* if $xy \in R$ and $x \notin I$ implies $y \in I$. If M is an R-module, then a prime ideal $P \subset R$ is said to be *associated* to M if there is an $m \in M$ such that $P = \operatorname{ann}(m) := \{r \in R : rm = 0\}$. We call P an associated prime ideal of M, or simply an associated prime if the context is understood. The set of associated primes of M is denoted $\operatorname{Ass}(M)$. A module M is called P-*primary* if P is the only associated prime of M. If $I \subset R$ is an ideal and $M = R/I$, we say that P is an associated prime of I if P is an associated prime of R/I. Similarly, we say that I is P-primary when R/I is P-primary.

A prime ideal P is *minimal* over an ideal I if P contains I and does not contain any other prime ideal that contains I. The minimal primes of a module M are the minimal primes over the annihilator $\operatorname{ann}(M) := \{r \in R : rM = 0\}$ of M. The primes in $\operatorname{Ass} M$ that are not minimal are called *embedded primes of M*.

A submodule $N \subset M$ is *irreducible* if it cannot be written as the intersection of two distinct submodules of M. If $N \subset M$ is a submodule of M, a *decomposition* of N is an expression of the form $N = N_1 \cap \cdots \cap N_l$ where the N_i are submodules of M. If each N_i is irreducible, such a decomposition is called an *irreducible decomposition*. If each N_i is primary, the decomposition is called a *primary decomposition*. If we cannot omit any N_i in the decomposition, then we call such a decomposition *irredundant*.

Theorem 2.1.3 (Primary decomposition)**.** *Let R be a Noetherian ring and M be an R-module.*

1. *If $N \subset M$ is an irreducible submodule, then N is primary.*

2. *If $N = N_1 \cap N_2 \cap \cdots \cap N_l$ with $\operatorname{Ass}(M/N_i) = \{P_i\}$ is an irredundant primary decomposition of a proper submodule $N \subset M$, then $\operatorname{Ass}(M/N) = \{P_1, \ldots, P_l\}$.*

3. *Every proper submodule of M has a primary decomposition.*

The *dimension* of a ring R, denoted $\dim(R)$ is the supremum of the lengths of chains of distinct prime ideals in R where the length of the chain $P_0 \subset P_1 \subset \cdots \subset P_r$ of prime ideals is r. The dimension of a module M is taken to be $\dim(R/\operatorname{ann}_R(M))$. The *height* of a prime ideal $P \subset R$, denoted $\operatorname{height}(P)$, is the supremum of the lengths of strictly decreasing $P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_l$ chains of prime ideals starting from P. We remark that $\operatorname{height}(P) = \dim(R_P)$. The height of an ideal $I \subset R$ is the minimum of the heights of primes that contain I.

Let R be a ring and M be an R-module. A *nonzero divisor* in R is an element $x \in R$ such that for every $r \neq 0$, $xr \neq 0$ A sequence $x_1, \ldots, x_n$ of elements of R is a *regular sequence* if $x_1, \ldots, x_n$ generate a proper ideal of R and x_i is a nonzero divisor in $R/(x_1, \ldots, x_{i-1})$. There is an analogous definition for modules. An element $x \in R$ is a *nonzero divisor on M*, or *M-regular*, if $xm \neq 0$ for all $0 \neq m \in M$. A sequence $x_1, \ldots, x_n$ is an *M-sequence*, or an *M-regular sequence*, if $(x_1, \ldots, x_n)M \neq M$ and x_i is a nonzero divisor on $M/(x_1, \ldots, x_{i-1})M$. An ideal that is generated by a regular sequence is called a *complete intersection*.

Let I be an ideal of a ring R and let M be a finitely generated R-module such that $IM \neq M$. The *depth* of I on M, denoted $\operatorname{depth}(I, M)$, is the length of the longest regular sequence on M contained in I. If R is local or homogeneous, then $\operatorname{depth}(M) := \operatorname{depth}(\mathfrak{m}, M)$ where $\mathfrak{m}$ is the maximal homogeneous ideal in R.

2.2 Homological algebra

Let R be a commutative ring with 1 and M be a left R-module. A sequence

$$\mathcal{F}_\bullet : 0 \leftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \leftarrow \cdots$$

of maps of R-modules is a *chain complex*, or simply a *complex*, if $\phi_{i+1} \circ \phi_i = 0$ for all i. The module F_i is said to be in *homological degree i*. The *length* of the chain complex $\mathcal{F}_\bullet$ is the largest homological degree l in which F_l is a nonzero module in $\mathcal{F}_\bullet$. We call a chain complex *exact* if $\ker \phi_i = \operatorname{im} \phi_{i+1}$. If the module M is graded, we require the homomorphisms ϕ_i to be graded of degree 0.

A *projective resolution* of M is a chain complex $\mathcal{P}_\bullet$ of projective R-modules

$$\mathcal{P}_\bullet : 0 \leftarrow P_0 \xleftarrow{\phi_1} P_1 \xleftarrow{\phi_2} P_2 \leftarrow \cdots$$

that is exact everywhere except in homological degree 0, where we require $M = P_0/\operatorname{im}\phi_1$. The *projective dimension* of M is the minimum length of a projective resolution of M. We often *augment* the projective resolution $\mathcal{P}_\bullet$ by placing $0 \leftarrow M \leftarrow F_0$ at the left end to make $\mathcal{P}_\bullet$ exact everywhere.

Dually, an *injective resolution* of M is a chain complex $\mathcal{J}^\bullet$ of injective R-modules

$$\mathcal{J}^\bullet : 0 \to J^0 \xrightarrow{\phi_0} J^1 \xrightarrow{\phi_1} J^2 \xrightarrow{\phi_2} \cdots .$$

The *injective dimension* of M is the minimum length of an injective module of M.

The functor $\operatorname{Hom}_R(-, N)$ is a contravariant left-exact functor. Let M be an R-module and $\mathcal{P} : P_0 \leftarrow P^1 \leftarrow \cdots$ a projective resolution of M. The i-th homology module of the complex $\operatorname{Hom}(M, N) = \operatorname{Hom}(\mathcal{P}, N) : 0 \to \operatorname{Hom}_R(P_0, N) \to \operatorname{Hom}_R(P^1, N) \to \cdots$ is denoted $\operatorname{Ext}^i_R(M, N)$. Alternatively, $\operatorname{Ext}^i_R(M, N)$ is the i-th right derived functor $R^i \operatorname{Hom}(M, -)(N)$. The functor $\operatorname{Hom}_R(M, -)$ is a covariant left-exact functor. If $\mathcal{J} : J_0 \to J_1 \to \cdots$ is an injective resolution of N, we may also define $\operatorname{Ext}^i_R(M, N)$ to be the i-th homology of the complex $\operatorname{Hom}(M, N) = \operatorname{Hom}(M, \mathcal{J}^\bullet) : 0 \to \operatorname{Hom}_R(J_0, N) \to \operatorname{Hom}_R(J^1, N) \to \cdots$ and these two notions agree.

2.3 Local cohomology

This section introduces the main algebraic object of our study. Let $\mathbf{f} = f_1, \ldots, f_l$ be elements in R. The *Čech complex* of R on $\mathbf{f}$ is the complex

$$\check{C}^\bullet(\mathbf{f}; R) : 0 \to R \to \bigoplus_{i=1}^{l} R_{f_i} \to \bigoplus_{1 \le i < j \le l} R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_l}$$

with the homomorphism being the canonical inclusions and signs coming from the Koszul resolution on $\mathbf{f}$. The Čech complex of an R-module M over $\mathbf{f}$ is the complex

$$\check{C}^\bullet(\mathbf{f};M) = \check{C}^\bullet(\mathbf{f};R) \otimes_R M.$$

The i-th Čech cohomology of $\mathbf{f}$ on M is

$$\check{H}^i(\mathbf{f};M) = H^i(\check{C}^\bullet(\mathbf{f};M)).$$

Let $\mathbf{f} = f_1, \ldots, f_l$ be a set of generators for the ideal I. The *i-th local cohomology module* of M supported at the ideal I is

$$H^i_I(M) := \check{H}^i(\mathbf{f};M).$$

We can also define local cohomology of M supported at the ideal I as follows. Let $\Gamma_I(M) = \{m \in M : I^t m = 0 \text{ for some } t \in \mathbb{N}\}$. We call $\Gamma_I(-)$ the I-torsion functor. The I-torsion functor is left-exact. The i-th right derived functor of $\Gamma_I(M)$ is the i-th local cohomology module supported at I and denoted $H^i_I(-)$, that is,

$$H^i_I(M) = H^i(\Gamma_I(\mathcal{J}^\bullet)$$

where $\mathcal{J}^\bullet$ is an injective resolution of M. Viewing the local cohomology modules as the right derived functors of the I-torsion functor yields the following.

Proposition 2.3.1. *For any $m \in H^i_I(M)$, there exists some $t \in \mathbb{N}$ such that $I^t m = 0$.*

In general, local cohomology is difficult to compute. However, when M is a finitely generated module over the polynomial ring S, we have a vector space isomorphism between local cohomology and Ext-modules.

Theorem 2.3.2 (Local duality)**.** *Let $S = \Bbbk[x_1, \ldots, x_n]$ and $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ be the homogeneous maximal ideal. If M is a finitely generated graded S-module under the standard grading, then there is a graded $\Bbbk$-vector space isomorphism*

$$H^i_{\mathfrak{m}}(M) \cong Ext^{n-i}(M, S(-n)).$$

The following theorem tells us that the vanishing of local cohomology depends on the depth and dimension of our module. Conversely, the non-vanishing of local cohomology gives us information about the depth and dimension of the module.

Theorem 2.3.3 (Vanishing of local cohomology)**.** *Let M be a finitely generated S-module.*

1. *If $i < \operatorname{depth} M$ or $i > \dim M$ then $H^i_{\mathfrak{m}}(M) = 0$.*
2. *If $i = \operatorname{depth} M$ or $i = \dim M$ then $H^i_{\mathfrak{m}}(M) \neq 0$.*

Theorem 2.3.3 is also called depth sensitivity. The following is an immediate result of Theorem 2.3.3.

Corollary 2.3.4. *A finitely generated S-module M is Cohen-Macaulay if and only if M has one non-zero local cohomology module.*

3. COMPUTING QUASIDEGREES AND A-HYPERGEOMETRIC SYSTEMS

Let $S = \Bbbk[x_1, \dots, x_n]$ be a $\mathbb{Z}^d$-graded polynomial ring over a field $\Bbbk$ and let $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ denote the homogeneous maximal ideal in S. Let $M = \bigoplus_{\beta \in \mathbb{Z}^d} M_\beta$ be a $\mathbb{Z}^d$-graded S-module. The *true degree set* of M is

$$\operatorname{tdeg}(M) = \{\beta \in \mathbb{Z}^d \mid M_\beta \neq 0\}.$$

The *quasidegree set* of M, denoted $\operatorname{qdeg}(M)$, is the Zariski closure in $\mathbb{C}^d$ of $\operatorname{tdeg}(M)$.

In this chapter, we discuss computing the quasidegree set of a finitely generated $\mathbb{Z}^d$-graded S-module presented as the cokernel of a monomial matrix. We then discuss the *Macaulay2* [7] package, *Quasidegrees* [1], whose purpose is to compute the quasidegree set of such a finitely generated $\mathbb{Z}^d$-graded module. Lastly, we apply *Quasidegrees* to compute where rank jumps of A-hypergeometric systems occur.

By a monomial matrix, we mean a homogeneous map of degree zero whose matrix representation has entries that are either zero or a monomial in S. More precisely, a *monomial matrix* is a map

$$\bigoplus S(-\alpha_i) \xleftarrow{\begin{array}{cc} & \begin{matrix} \cdots & \alpha_j & \cdots \end{matrix} \\ \begin{matrix} \vdots \\ \alpha_i \\ \vdots \end{matrix} & \begin{bmatrix} & & \\ & \lambda_{ij}x^{\alpha_{ij}} & \\ & & \end{bmatrix} \end{array}} \bigoplus S(-\alpha_j) : \phi$$

where the columns of ϕ are labeled by *source degrees* and the rows of ϕ are labeled by *target degrees*.

3.1 Combinatorics of quasidegree sets

Suppose M is $\mathbb{Z}^d$-graded module that has a presentation by the monomial matrix $\phi : \bigoplus_{\alpha_j} S(-\alpha_j) \to \bigoplus_{\alpha_i} S(-\alpha_i)$, that is,

$$M = \frac{\bigoplus_{\alpha_i} S(-\alpha_i)}{\operatorname{im} \phi}.$$

The basis vector of $S(-\alpha_j)$ maps to $\bigoplus_i \lambda_{ij} x^{\alpha_{ij}}$. Thus the true degree set and quasidegree set of M are determined by the monomials outside of the image of ϕ. Since ϕ is assumed to be a monomial matrix, the image of ϕ is a monomial module.

We use the idea of standard pairs of monomial ideals to index the monomials outside a monomial ideal. Monomials outside of a monomial ideal are called *standard monomials* and are a $\Bbbk$-basis for S/I. Given a monomial x^α and a subset $Z \subset \{x_1, \ldots, x_n\}$, the pair (x^α, Z) indexes the monomials $x^\alpha \cdot x^\beta$ where $\operatorname{supp}(x^\beta) \subset Z$. A *standard pair* of a monomial ideal $I \subset R$ is a pair (x^α, Z) satisfying:

1. $\operatorname{supp}(x^\alpha) \cap Z = \varnothing$,
2. all of the monomials indexed by (x^α, Z) are outside of I,
3. (x^α, Z) is maximal in the sense that $(x^\alpha, Z) \not\subseteq (x^\beta, Y)$ for any other pair (x^β, Y) satisfying the first two conditions.

If our polynomial ring is in two or three variables, we can represent monomial ideals geometrically in the positive quadrant or orthant using a staircase diagram. Nonzero monomials correspond to their exponent vectors. We demonstrate this in two variables. Let $S = \Bbbk[x, y]$ and let $I = \langle x^{\alpha_1} y^{\beta_1}, x^{\alpha_2} y^{\beta_2}, \ldots, x^{\alpha_t} y^{\beta_t} \rangle$ be a monomial ideal where $\alpha_1 > \alpha_2 > \cdots > \alpha_t \geq 0$ and $\beta_t > \beta_{t-1} > \cdots > \beta_1 \geq 0$. The *staircase diagram* of I is the region of the plane that contains the exponent vectors of the monomials in I. The lattice points in the nonnegative first quadrant that are not in I represent the standard monomials. The staircase diagram of I is illustrated in Figure 3.1.

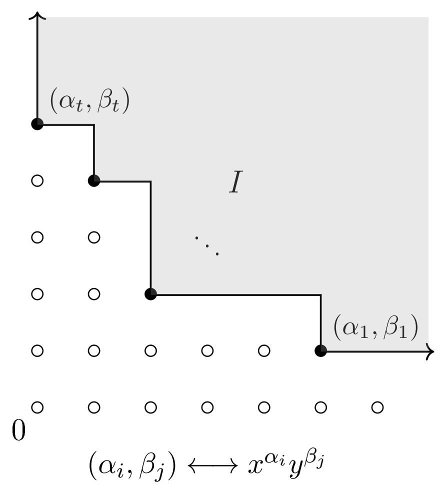


Figure 3.1: A general staircase diagram

Example 3.1.1. Let $S = \Bbbk[x, y]$ with the $\mathbb{N}^2$-grading by letting $\deg(x) = \binom{1}{0}$ and $\deg(y) = \binom{0}{1}$. Let $I = \langle y^3, xy^2, x^3y \rangle$. In Figure 3.2, we represent the ideal I geometrically by its staircase diagram.

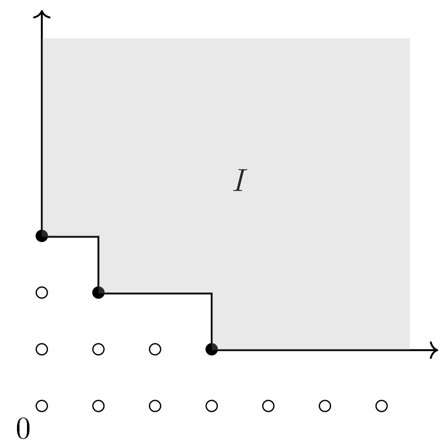


Figure 3.2: The staircase diagram for Example 3.1.1

The standard pairs are $(1, \{x\})$, $(y, \varnothing)$, $(xy, \varnothing)$, $(x^2y, \varnothing)$, and $(y^2, \varnothing)$. The quasidegrees can be read from the standard pairs. In this example, the quasidegree set is

$$\mathrm{qdeg}(S/I) = \{(\xi, 0) : \xi \in \mathbb{C}\} \cup (0, 1) \cup (1, 1) \cup (2, 1) \cup (0, 2).$$

The following routine is implemented in *Quasidegrees* to compute the quasidegree set of M. We first find a monomial presentation of M so that M is the cokernel of a monomial matrix ϕ. We then compute the standard pairs of the ideals generated by the rows of ϕ and to each standard pair we associate the degrees of the corresponding variables.

3.2 Quasidegrees

In this section, we discuss some of the functions in the *Macaulay2* package, *Quasidegrees*. The main function of *Quasidegrees* is the method `quasidegrees`. The input of `quasidegrees` is either a module M that can be presented as the cokernel of a monomial matrix or an ideal and the output is the quasidegree set of the module. If the input is an ideal I with base ring S, then `quasidegrees` computes the quasidegree set of the S-module $M = S/I$. *Quasidegrees* represents the quasidegree set as a list of pairs (α, Z) with $\alpha \in \mathbb{Q}^d$ and $Z \subset \mathbb{Q}^d$ where the pair (α, Z) represents the plane

$$\alpha + \sum_{\beta \in Z} \mathbb{C} \cdot \beta.$$

The union of all planes over all such pairs in the output is the quasidegree set of M.

The following is an example of *Quasidegrees* computing the quasidegree set of an S-module with the standard grading:

```
i1 : S=QQ[x,y,z]

o1 = S

o1 : PolynomialRing

i2 : I=ideal(x*y,y*z)

o2 = ideal (x*y, y*z)

o2 : Ideal of S

i3 : M=S^1/I

o3 = cokernel | xy yz |

                                    1

o3 : S-module, quotient of S

i4 : Q = quasidegrees M

o4 = {{0, {| 1 |}}, {0, {| 1 |, | 1 |}}}

o4 : List
```

The above example displays a caveat of *Quasidegrees* in that there may be some redundancies in the output. By a redundancy, we mean when one plane in the output is contained in another. The redundancy above is clear:

$$\mathrm{qdeg}(\mathbb{Q}[x,y,z]/\langle xy,yz\rangle) = \mathbb{C} = \{\xi_1+\xi_2 \in \mathbb{C} \mid \xi_1,\xi_2 \in \mathbb{C}\}.$$

The function `removeRedundancy` gets rid of redundancies in the list of planes:

```
i5 : removeRedundancy Q

o5 = {{0, {| 1 |, | 1 |}}}

o5 : List
```

3.3 Quasidegrees and hypergeometric systems

The motivation for writing the package *Quasidegrees* is to study the rank jumps of A-hypergeometric systems of partial differential equations. Let $A = [a_1\ a_2\ \cdots\ a_n]$ be an integer $(d\times n)$-matrix such that $\mathbb{Z}A = \mathbb{Z}^d$, that is, the columns of A span $\mathbb{Z}^d$ as a lattice. We also assume that the cone over the columns of A is pointed. There is a natural $\mathbb{Z}^d$-grading on $S = \Bbbk[x_1,\ldots,x_n]$ by the columns of A by letting $\deg(x_j) = a_j$, the j-th column of A. A module that is homogeneous with respect to this grading is said to be *A-graded* and have an *A-grading*. By the assumptions on A, S is positively graded by A, that is, the only polynomials of degree 0 are the constants. Given a matrix A and a polynomial ring S in n variables, the method `toAgradedRing` makes S an A-graded module. For example, let $A = \begin{pmatrix} 1&1&1&1&1\\ 0&0&1&1&0\\ 0&1&1&0&-2 \end{pmatrix}$. We make the polynomial ring $\mathbb{Q}[x_1,x_2,x_3,x_4,x_5]$ A-graded :

```
i6 : A=matrix{{1,1,1,1,1},{0,0,1,1,0},{0,1,1,0,-2}}

o6 = | 1 1 1 1 1  |

     | 0 0 1 1 0  |

     | 0 1 1 0 -2 |

              3        5
```

```
o6 : Matrix ZZ  <--- ZZ

i7 : S=QQ[x_1..x_5]

o7 = S

o7 : PolynomialRing

i8 : S=toAgradedRing(A,S)

o8 = S

o8 : PolynomialRing

i9 : describe S

o9 = QQ[x , x , x , x , x , Degrees => {{1}, {1}, {1}, {1},
         1   2   3   4   5                {0}  {0}  {1}  {1}
                                          {0}  {1}  {1}  {0}
     ------------------------------------------------------
     {1 }}, Heft => {1, 2:0}, MonomialOrder =>
     {0 }
     {-2}
     ------------------------------------------------------
     {MonomialSize => 32}, DegreeRank => 3]
     {GRevLex => {5:1}  }
     {Position => Up    }
```

The *toric ideal* of A in S is the binomial ideal

$$I_A = \langle x^u - x^v : Au = Av,\ u, v \in \mathbb{Z}^n \rangle.$$

The method `toricIdeal` computes the toric ideal of A in the ring S. We continue with the A and S from the above example and compute the toric ideal I_A in S:

```
i10 : I=toricIdeal(A,S)

                            2    2     2             3
o10 = ideal (x x  - x x , x x  - x x , x x  - x x x , x  -
              1 3    2 4   1 4    3 5   1 4    2 3 5   1
      ------------------------------------------------------
       2
      x x )
       2 5

o10 : Ideal of S
```

We now introduce A-hypergeometric systems. Given a matrix $A \in \mathbb{Z}^{d \times n}$ as above and a parameter $\beta \in \mathbb{C}^d$, the *A-hypergeometric system with parameter* $\beta \in \mathbb{C}^d$ [13], denoted $H_A(\beta)$, is the system of partial differential equations:

$$\frac{\partial^{|v|}}{\partial x^v}\phi(x) = \frac{\partial^{|u|}}{\partial x^u}\phi(x) \text{ for all } u, v,\ Au = Av$$
$$\sum_{j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j}\phi(x) = \beta_i \phi(x),\ \text{ for } i = 1, \ldots, d.$$

Such systems are sometimes called *GKZ-hypergeometric systems*. The function `gkz` in the *Macaulay2* package *Dmodules* computes this system as an ideal in the Weyl algebra. The *holonomic rank* of $H_A(\beta)$ is

$$\operatorname{rank}(H_A(\beta)) = \dim_{\mathbb{C}} \left\{ \begin{array}{l} \text{germs of holomorphic solutions of } H_A(\beta) \\ \text{near a generic nonsingular point} \end{array} \right\}.$$

In general, the rank of a hypergeometric system is an upper semi-continuous function of β [10, Theorem 2.6]. Set $\mathrm{vol}(A)$ as $d!$ times the Euclidean volume of $\mathrm{conv}(A \cup \{0\})$ the convex hull of the columns of A and the origin in $\mathbb{R}^d$. The following theorem gives the parameters β for which $\mathrm{rank}(H_A(\beta))$ is higher than expected:

Theorem 3.3.1. [10] *Let $H_A(\beta)$ be an A-hypergeometric system with parameter β. If $\beta \in \mathrm{qdeg}(\bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(S/I_A))$ then $\mathrm{rank}(H_A(\beta)) > \mathrm{vol}(A)$. Otherwise, $\mathrm{rank}(H_A(\beta)) = \mathrm{vol}(A)$.*

Since Theorem 3.3.1 was the initial motivation for *Quasidegrees*, the package has a method `quasidegreesLocalCohomology` (abbreviated `QLC`) to compute the quasidegree set of the local cohomology modules $H^i_{\mathfrak{m}}(S/I_A)$. If the input is an integer i and the S-module S/I_A, then the method computes $\mathrm{qdeg}(H^i_{\mathfrak{m}}(S/I_A))$. If the input is only the module S/I_A, the method computes the quasidegree set in Theorem 3.3.1.

We use graded local duality to compute the local cohomology modules supported at $\mathfrak{m}$ of a finitely generated A-graded S-module. The algorithm implemented for `QLC` is essentially the algorithm for `quasidegrees` applied to the Ext-modules of M with the additional twist of ε_A coming from local duality. For our purposes, we exploit the fact that the higher syzygies of S/I_A are generated by vectors whose entires are monomials or 0 (see [11], Chapter 9).

Continuing our running example, we use `quasidegreesLocalCohomology` to compute the quasidegree set of $\bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(S/I_A)$:

```
i11 : M=S^1/I

o11 = cokernel | x_1x_3-x_2x_4 x_1x_4^2-x_3^2x_5

x_1^2x_4-x_2x_3x_5 x_1^3-x_2^2x_5 |

                              1

o11 : S-module, quotient of S

i12 : quasidegreesLocalCohomology M
```

```
o12 = {{| 0 |, {| 1  |}}}
        | 0 |   | 0  |
        | 1 |   | -2 |

o12 : List
```

Thus

$$\operatorname{qdeg}\left(\bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(S/I_A)\right) = \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right] + \mathbb{C}\cdot\left[\begin{smallmatrix}1\\0\\-2\end{smallmatrix}\right]. \tag{3.1}$$

To confirm our computations, we use the methods `gkz` and `holonomicRank` from the package *Dmodules* to compute $\operatorname{rank}(H_A(0))$ and $\operatorname{rank}(H_A(\beta))$ for two different choices of β in (3.1) and demonstrate a rank jump:

```
i13 : holonomicRank gkz(A,{0,0,0}) -- vol A in this case

o13 = 4

i14 : holonomicRank gkz(A,{0,0,1})

o14 = 5

i15 : holonomicRank gkz(A,{3/2,0,-2})

o15 = 5
```

The source code for `quasidegrees` and `quasidegreeslocalcohomology` can be found in the appendix.

4. AN ISOMORPHISM OF LOCAL DUALITY FOR CODIMENSION 2 TORIC IDEALS

4.1 Introduction

Let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring over $\Bbbk$ in n variables and let $A = (\mathbf{a}_{i,j}) \in \mathbb{Z}^{d\times n}$ be an integer matrix of full rank. We denote the j-th column of A by $\mathbf{a}_j$. Assume that the first row of A is all 1's and the cone over the columns of A is pointed. The toric ideal of A in S is the binomial ideal

$$I_A := \langle x^{\mathbf{u}} - x^{\mathbf{v}} \in S : A\mathbf{u} = A\mathbf{v}\rangle \subset S.$$

The *codimension* of I_A, denoted $\operatorname{codim}(I_A)$, is $n-d$. For the remainder of this chapter, we assume that $\operatorname{codim} I_A = 2$. Then the integer kernel of A $\ker_{\mathbb{Z}}(A)$ is a 2-dimensional sublattice of $\mathbb{Z}^n$. Let $B \in \mathbb{Z}^{n\times 2}$ be an integer matrix whose columns generate $\ker_{\mathbb{Z}}(A)$ and let $\mathbf{b}_i = (b_{i1}\ b_{i2})$ be the i-th row of B. The vector arrangement of the rows of B in the Euclidean plane is called the *Gale diagram* of A, or of I_A. The Gale diagram of A is unique up to an $SL_2(\mathbb{Z})$ action on $\mathbb{Z}^2$. Gale diagrams of codimension 2 toric ideals characterize Cohen-Macaulayness [12].

Theorem 4.1.1. *A toric ideal I_A is not Cohen-Macaulay if and only if A has a Gale diagram that intersects each of the four open quadrants of $\mathbb{Z}^2$.*

We do not consider Cohen-Macaulay codimension 2 toric ideals because their local cohomology is well understood. By interchanging the rows of B and corresponding columns of A, we may assume for the remainder that the first four columns of B

have the following signs:

$$B = \begin{pmatrix} + & + \\ - & + \\ - & - \\ + & - \\ \vdots & \vdots \end{pmatrix}.$$

Notation 4.1.2. The support of a vector $\mathbf{b} \in \mathbb{Z}^d$ is the set $\operatorname{supp}(\mathbf{b}) := \{i : b_i \neq 0\}$. A vector $\mathbf{b} \in \mathbb{Z}^d$ can be written as $\mathbf{b} = \mathbf{b}_+ - \mathbf{b}_-$ where the entries of $\mathbf{b}_+$ and $\mathbf{b}_-$ are non-negative and $\operatorname{supp}(\mathbf{b}_+) \cap \operatorname{supp}(\mathbf{b}_-) = \varnothing$. If B is a matrix, write B_i to be the i-th column of B. The *positive support* of a monomial $x^{\mathbf{u}}$ is $\operatorname{supp}(x^{\mathbf{u}}) = \{x_i : u_i > 0\}$ and the *negative support* of a monomial $x^{\mathbf{u}}$ to be $\operatorname{nsupp}(x^{\mathbf{u}}) = \{x_i : u_i < 0\}$.

Since we assume the codimension of I_A is 2, the vanishing of local cohomology means that the only non-top local cohomology occurs in cohomological degree $n-3$. By local duality, we have the following isomorphism.

Theorem 4.1.3 (Graded local duality)**.** *There is a vector space isomorphism*

$$Ext^3_S(S/I_A, S)_\alpha \cong \operatorname{Hom}_{\Bbbk}(H^{n-3}_{\mathfrak{m}}(S/I_A)_{-\alpha+\epsilon_A}, \Bbbk)$$

where $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ *and* ϵ_A *is the sum of the columns of* A.

The goal of this chapter is to produce an explicit isomorphism for Theorem 4.1.3.

4.2 Combinatorics of resolving codimension 2 toric ideals

Irena Peeva and Bernd Sturmfels give a combinatorial construction for the minimal free resolution of a codimension 2 toric ideal in [12]. In this section, we recall their construction.

Let G_A be the Gale diagram of A. Rotating the G_A by $\pi/2$ gives the *Gale* diagram* $G_A^* = \{\mathbf{b}_1^*, \ldots, \mathbf{b}_n^*\}$ where $\mathbf{b}_i^* = (-b_{i,2}, b_{i,1})$. After cyclically relabeling the vectors $\mathbf{b}_i^*$, we may assume that no element of G_A^* lies in the interior of a cone $\operatorname{pos}(\mathbf{b}_i, \mathbf{b}_{i+1}) = \{\lambda \mathbf{b}_i + \mu \mathbf{b}_{i+1} : \lambda, \mu \in \mathbb{R}_{\geq 0}\}$. Let $\mathcal{H}_i$ be the minimal generating set of the semigroup $\mathbb{Z}^2 \cap \text{pos}(\mathbf{b}_i^*, \mathbf{b}_{i+1}^*)$. The *Hilbert basis of* G_A^* is defined to be the set

$$\mathcal{H}_A = \{\mathbf{u} \in \mathbb{Z}^2 : \mathbf{u}, -\mathbf{u} \in \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_n\},$$

where the antipodal points of $\mathcal{H}_A$ are identified. The elements of the Hilbert basis $\mathcal{H}_A$ correspond to the generators of the lattice ideal I_A.

Theorem 4.2.1. *If I_A is not a complete intersection, the elements $\mathbf{u} \in \mathcal{H}_A$ correspond to a unique set of A graded binomial generators $x^{(B\mathbf{u})_+} - x^{(B\mathbf{u})_-}$ of I_A.*

A lattice polytope is the convex hull of a finite set of integer lattice points. A lattice polytope is said to be *primitive* if has no lattice points in its interior. A primitive parallelogram is a *syzygy quadrangle* if and only if each vertex of the parallelogram is *supported* by at least one vector in the Gale diagram G_A, that is, if v is a vertex of the parallelogram, then there is some $\mathbf{b}_i$ such that $\mathbf{b}_i \cdot v \geq \mathbf{b}_i \cdot p$ for every point p in the parallelogram. Furthermore, a primitive parallelogram is a syzygy quadrangle if and only if its diagonal represents a minimal generator of I_A. Let the generators of I_A corresponding to diagonals of a syzygy quadrangle be

$$\alpha = x^{\mathbf{u}_+}x^{\mathbf{t}}x^{\mathbf{p}} - x^{\mathbf{u}_-}x^{\mathbf{s}}x^{\mathbf{r}} \quad \text{and} \quad \beta = x^{\mathbf{v}_+}x^{\mathbf{s}}x^{\mathbf{p}} - x^{\mathbf{v}_-}x^{\mathbf{t}}x^{\mathbf{r}}$$

where $x^{\mathbf{p}}$ is the greatest common factor between the first term of α and the first term of β, $x^{\mathbf{t}}$ is the greatest common factor between the first term of α and the second term of β, $x^{\mathbf{s}}$ is the greatest common factor between the second term of α and the first term of β, and $x^{\mathbf{r}}$ is the greatest common factor between the second term of α and the second term of β. The edges of the syzygy quadrangle are represented by the binomials

$$\gamma = x^{\mathbf{u}_+}x^{\mathbf{v}_+}x^{2\mathbf{p}} - x^{\mathbf{u}_-}x^{\mathbf{v}_-}x^{2\mathbf{r}} \quad \text{and} \quad \delta = x^{\mathbf{u}_+}x^{\mathbf{v}_-}x^{2\mathbf{t}} - x^{\mathbf{u}_-}x^{\mathbf{v}_+}x^{2\mathbf{s}}.$$

There is a chain complex associated to the syzygy quadrangle $P_{\mathbf{u}}$ called the *quadrangle complex* of $P_{\mathbf{u}}$

$$0 \to S \xrightarrow{\begin{pmatrix} -x^{\mathbf{s}} \\ x^{\mathbf{t}} \\ x^{\mathbf{r}} \\ -x^{\mathbf{p}} \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} x^{\mathbf{v}_+}x^{\mathbf{p}} & x^{\mathbf{v}_-}x^{\mathbf{r}} & -x^{\mathbf{v}_-}x^{\mathbf{t}} & -x^{\mathbf{v}_+}x^{\mathbf{s}} \\ x^{\mathbf{u}_-}x^{\mathbf{r}} & x^{\mathbf{u}_+}x^{\mathbf{p}} & x^{\mathbf{u}_-}x^{\mathbf{s}} & x^{\mathbf{u}_+}x^{\mathbf{t}} \\ -x^{\mathbf{t}} & -x^{\mathbf{s}} & 0 & 0 \\ 0 & 0 & x^{\mathbf{p}} & x^{\mathbf{r}} \end{pmatrix}} S^4 \xrightarrow{(\alpha\ \beta\ \gamma\ \delta)} S. \tag{4.1}$$

The twists of S^4 in homological degree one are

$$S(-A\cdot(\mathbf{u}_+ + \mathbf{t} + \mathbf{p})) \oplus S(-A\cdot(\mathbf{v}_+ + \mathbf{s} + \mathbf{p}))\oplus$$
$$S(-A\cdot(\mathbf{u}_+ + \mathbf{v}_+ + 2\mathbf{p})) \oplus S(-A\cdot(\mathbf{u}_+ + \mathbf{v}_- + 2\mathbf{t})),$$

the twists of S^4 in homological degree two are

$$S(-A\cdot(\mathbf{u}_+ + \mathbf{t} + 2\mathbf{p} + \mathbf{v}_+)) \oplus S(-A\cdot(\mathbf{v}_+ + \mathbf{s} + \mathbf{p} + \mathbf{v}_- + \mathbf{r}))\oplus$$
$$S(-A\cdot(\mathbf{u}_+ + \mathbf{v}_+ + 2\mathbf{p} + \mathbf{v}_- + \mathbf{t})) \oplus S(-A\cdot(\mathbf{u}_+ + \mathbf{v}_- + 2\mathbf{t} + \mathbf{v}_+ + \mathbf{s})),$$

and the twist of S in homological degree three is

$$S(-A\cdot(\mathbf{u}_+ + \mathbf{t} + 2\mathbf{p} + \mathbf{v}_+ + \mathbf{s})) = S(-A\cdot(B_{1+} + B_{2+})).$$

We now present the combinatorial construction found in [12] for the minimal free resolution of I_A. For $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$, denote the convex hull of $(0,0)$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$) by

$$[\mathbf{u}, \mathbf{v}] = \mathrm{conv}((0,0), \mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}).$$

We construct a directed tree of primitive quadrangles as follows. Let the unit square $[(0,1),(1,0)]$ be the root of this tree. For a quadrangle that is not the unit square, there are two outgoing edges to quadrangles $[\mathbf{u}+\mathbf{v},\mathbf{v}]$ and $[\mathbf{u},\mathbf{u}+\mathbf{v}]$. Since the unit square has two representations, $[(0,1),(1,0)] = [(0,1),(-1,0)]$, the root has four outgoing edges. The resulting tree is called the *master tree* and is depicted in Figure 4.1, as found in [12].

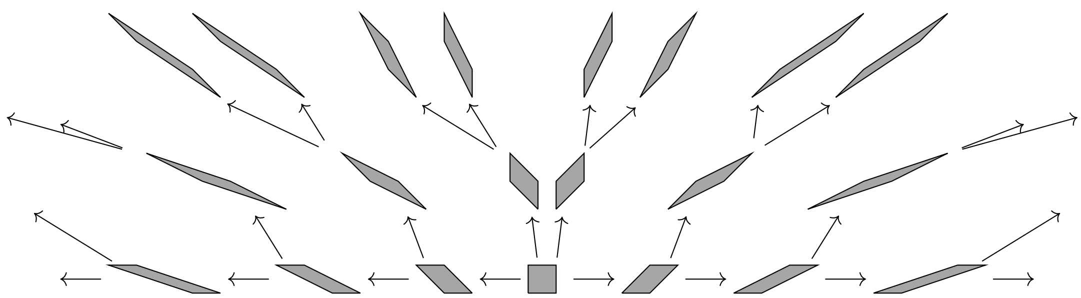


Figure 4.1: Master tree

The subgraph of the master tree consisting of all syzygy quadrangles is called the homology tree of I_A, denoted $\mathcal{T}_A$. Moving along the tree corresponds to applying the generators of $SL(2,\mathbb{Z})$ and their inverses to the parallelograms where the generators of $SL(2,\mathbb{Z})$ are $\mathfrak{s} = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $\mathfrak{t} = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.

We now construct the quadrangle complex combinatorially for the unit square. Syzygy quadrangles corresponds to the minimal third syzygies. The minimal second syzygies are the triangles, called syzygy triangles, obtained by deleting a vertex from a syzygy quadrangle. The generators of I_A are the edges and diagonals of the syzygy quadrangle obtained by deleting a vertex from the syzygy triangles. The quadrangle complex for a syzygy quadrangle is

$$\mathbf{F} : 0 \to S \to S^4 \to S^4 \to S.$$

The differentials are given schematically in Figure 4.2.

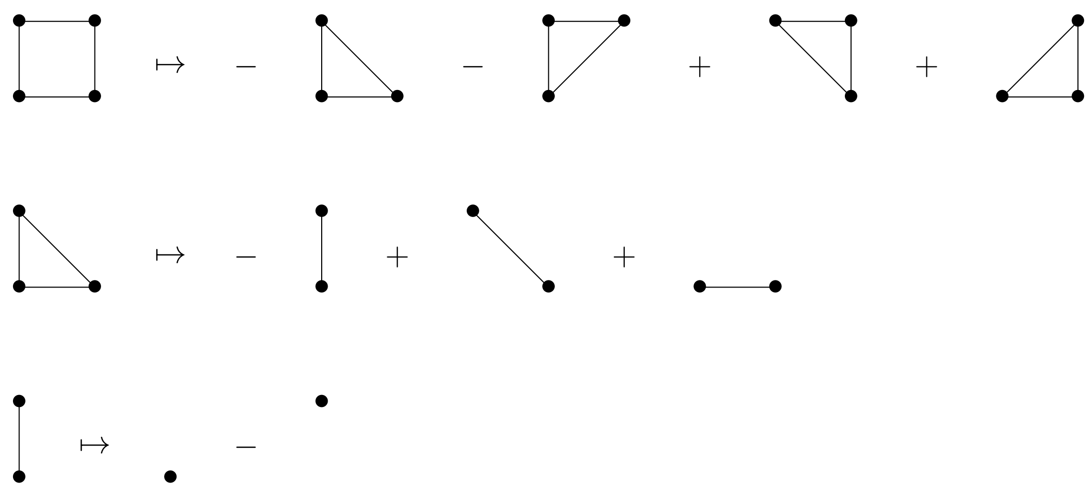


Figure 4.2: Schematic of the differentials in a quadrangle complex

The algebraic information for the quadrangle complex $\mathbf{F}$ is found as follows. Let

$$\mathbf{m}_3 = B_{1+} + B_{2+}, \qquad \mathbf{m}_4 = \mathbf{m}_3 - B_1.$$
$$\mathbf{m}_2 = \mathbf{m}_3 - B_2, \qquad \mathbf{m}_1 = \mathbf{m}_3 - B_1 - B_2.$$

Label a vertex of the unit square by the monomial with exponent vector being the corresponding supporting vector. Note that $\gcd(x^{\mathbf{m}_1}, x^{\mathbf{m}_2}, x^{\mathbf{m}_3}, x^{\mathbf{m}_4}) = 1$. Let D be an ordered subset of the ordered monomials $[x^{\mathbf{m}_1}, x^{\mathbf{m}_2}, x^{\mathbf{m}_3}, x^{\mathbf{m}_4}]$. Then D is a generator that corresponds to a syzygy quadrangle, triangle, edge, or vertex so that the quadrangle complex for the unit square is $\mathbf{F} : S \to S^4 \to S^4 \to S$ has basis elements being the symbols $[D]$. The differential of the quadrangle complex is defined to be

$$D \mapsto \sum_{x^{\mathbf{m}} \in D} \operatorname{sign}(x^{\mathbf{m}}, D) \cdot \gcd(D \setminus x^{\mathbf{m}}) \cdot [D \setminus x^{\mathbf{m}}]$$

where $\operatorname{sign}(x^{\mathbf{m}}, D) = (-1)^{t+1}$ if $x^{\mathbf{m}}$ is the t-th monomial in the ordered list $[D]$.

If the syzygy quadrangle P is not the unit square, there is an element $\mathfrak{u} \in SL(2, \mathbb{Z})$ that acts on the unit square to give P. To get the syzygy quadrangle complex for P, let $\mathbf{m}_3 = (B \cdot \mathfrak{u})_{1+} + (B \cdot \mathfrak{u})_{2+}$ and then proceed as before. The minimal free resolution of I_A is

$$\mathbf{F}_{I_A} = \bigoplus_{P \in \mathcal{T}_A} \mathbf{F}_P$$

where the sum runs over the homology tree of I_A.

4.3 Combinatorics of local cohomology for codimension 2 toric ideals

Without loss of generality we assume that S/I_A has exactly one syzygy quadrangle. From Theorem 4.1.3, we are interested in $\operatorname{Ext}^3_S(S/I_A, S)$. Applying $\operatorname{Hom}(-, S)$ to the syzygy complex (4.1) and taking homology gives

$$\operatorname{Ext}^3(S/I_A, S) \cong S(A \cdot (B_{1+} + B_{2+}))/\langle x^{\mathbf{s}}, x^{\mathbf{t}}, x^{\mathbf{r}}, x^{\mathbf{p}} \rangle.$$

Thus the standard monomials of $\langle x^{\mathbf{s}}, x^{\mathbf{t}}, x^{\mathbf{r}}, x^{\mathbf{p}} \rangle$ are a $\Bbbk$-basis for $\operatorname{Ext}^3_S(S/I_A, S)$ and are indexed by the standard pairs of $\langle x^{\mathbf{s}}, x^{\mathbf{t}}, x^{\mathbf{r}}, x^{\mathbf{p}} \rangle$. We will see in this section that if $(x^{\mathbf{u}}, Z)$ is a standard pair of $\langle x^{\mathbf{s}}, x^{\mathbf{t}}, x^{\mathbf{r}}, x^{\mathbf{p}} \rangle$, then Z is a codimension 2 face of $\operatorname{conv}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, the convex hull of the columns of A. We will use these codimension 2 faces to make a map between the vector spaces in local duality.

The local cohomology supported at the maximal ideal of a toric ring can be computed using the Ishida complex. Let $P = \operatorname{conv}(A)$ be the convex hull of the columns of A and label the vertices of P by the corresponding indeterminates $x_1, \dots, x_n$. We allow x_i to possibly lie in the interior of a face. Note that $\dim(P) = d - 1$ since the first row of A is assumed to be all 1's. Denote the set of t-dimensional faces of P by $\mathscr{F}_P^t$. In general, if the dimension of the largest linear subspace of $\operatorname{pos}(A)$ has dimension l, then P is taken to be a codimension $l+1$ transverse linear section of $\operatorname{pos}(A)$. The largest linear subspace of a polyhedron P is the *lineality space* of P.

The *Ishida complex* of I_A (or of A) is the cocomplex

$$\mho_A^\bullet : S/I_A \to \bigoplus_{v \in \mathscr{F}_P^0} (S/I_A)_v \to \cdots \to \bigoplus_{F \in \mathscr{F}_P^{d-2}} (S/I_A)_F \to (S/I_A)_{x_1 \cdots x_n}.$$

The differentials ∂ consist of natural localization maps $(S/I_A)_{F \in \mathscr{F}^{t-1}} \to (S/I_A)_{G \in \mathscr{F}^t}$ with signs as in the algebraic cochain complex of P [4, Theorem6.2.5]. The terms S/I_A and $(S/I_A)_{x_1, \dots, x_n}$ are in cohomological degree 0 and d respectively. We remark that the Ishida complex is a subcomplex of the Čech complex.

Theorem 4.3.1. *Let $\mathfrak{m}$ be the multigraded maximal ideal of S/I_A and M be an S/I_A-module. Then*

$$H^i_{\mathfrak{m}}(M) \cong H^i(M \otimes \mho_A^\bullet).$$

For our purposes, we are interested in $H^{n-3}_{\mathfrak{m}}(S/I_A)$. By Theorem 4.3.1, this is the cohomology of the sequence

$$\bigoplus_{F \in \mathscr{F}_P^{d-3}} (S/I_A)_F \to \bigoplus_{F \in \mathscr{F}_P^{d-2}} (S/I_A)_F \to (S/I_A)_{x_1 \cdots x_n}.$$

We now analyze the combinatorics of the polytope P that will allow us to compute local cohomology in later sections. For the remainder, let $x^{\mathbf{p}}, x^{\mathbf{r}}, x^{\mathbf{s}}, x^{\mathbf{t}}$ be as in equation 4.1. Let P be a convex set in $\mathbb{R}^d$. The *relative interior of* P is the interior of P in the smallest affine space that contains P. We will frequently rely on the following theorem.

Theorem 4.3.2. *[17, Theorem 5.6] Let* $P = \operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ *and let* B *be as above. Then* F *is a face of* P *if and only if either* $F = P$ *or* $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_k : x_k \notin F))$.

Proposition 4.3.3. *Let* $(x^{\mathbf{u}}, Z)$ *be a standard pair of* $\langle x^{\mathbf{s}}, x^{\mathbf{t}}, x^{\mathbf{r}}, x^{\mathbf{p}} \rangle$. *Then* Z *is a codimension 2 face of* P.

Proof. Note that $Z = \{x_1, \dots, x_n\} \setminus \{x_i, x_j, x_k, x_l\}$ where $i \in \operatorname{supp}(\mathbf{x}^{\mathbf{p}})$, $j \in \operatorname{supp}(\mathbf{x}^{\mathbf{t}})$, $k \in \operatorname{supp}(\mathbf{x}^{\mathbf{r}})$, and $l \in \operatorname{supp}(\mathbf{x}^{\mathbf{s}})$. Then the corresponding row vectors of B, $\mathbf{b}_i$, $\mathbf{b}_j$, $\mathbf{b}_k$, and $\mathbf{b}_l$, intersect each open quadrant of $\mathbb{R}^2$. Thus Z is a face of the polytope P since $0 \in \operatorname{relint}(\operatorname{conv}\{\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k, \mathbf{b}_l\})$.

To show Z is a codimension 2 face of P, we use Theorem 4.3.2 to construct part of the face lattice of P. There are three cases to consider for the arrangement of $\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k, \mathbf{b}_l$.

Case 1. Suppose $\{\mathbf{b}_i, \mathbf{b}_k\}$ and $\{\mathbf{b}_k, \mathbf{b}_l\}$ are two sets of linearly dependent vectors. Figure 4.3 is an example of such a vector configuration. Then removing any one vector gives a line segment through the origin and another vector emanating from the origin. In such a vector arrangement, $0 \notin \operatorname{relint}(\operatorname{conv}\{\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k\})$. Removing two linearly dependent vectors, say $\mathbf{b}_j$ and $\mathbf{b}_l$, gives a line segment through 0 and clearly 0 is in the relative interior of such a vector arrangement. Thus, $Z \cup \{x_j, x_l\}$ is a face of P. Finally, removing either of $\mathbf{b}_i$ or $\mathbf{b}_k$ leaves a vector, which has empty interior in the Euclidean topology. We conclude that $Z \cup \{x_j, x_l\}$ is a codimension 1 face of P and Z is a codimension 2 face of P.

Case 2. Suppose two vectors, say $\mathbf{b}_i$ and $\mathbf{b}_k$, are on a line through the origin and the other two vectors, $\mathbf{b}_j$ and $\mathbf{b}_l$, are not on a line through the origin. Figure 4.4 is

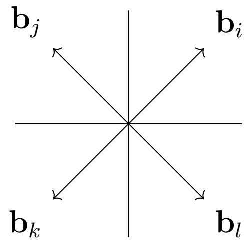


Figure 4.3: The vector arrangement in case 1 of Proposition 4.3.3

an example of such a vector configuration. Then removing either $\mathbf{b}_j$ or $\mathbf{b}_l$ from the configuration leaves a line segment through the origin and a vector emanating from the origin. The relative interior of the convex hull of such a configuration does not contain 0. If we remove $\mathbf{b}_j$ and $\mathbf{b}_l$ from the configuration, then $\operatorname{conv}\{\mathbf{b}_i, \mathbf{b}_k\}$ is a line segment through the origin and so $0 \in \operatorname{relint}(\operatorname{conv}\{\mathbf{b}_i, \mathbf{b}_k\})$. Thus $Z \cup \{x_j, x_l\}$ is a codimension 1 face of P and Z is a codimension 2 face of P.

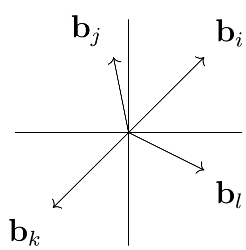


Figure 4.4: The vector arrangement in case 2 of Proposition 4.3.3

Case 3. Suppose that no two of $\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k, \mathbf{b}_l$ lie on a line through the origin and let α, β, γ, and δ be the angles between $\mathbf{b}_i$ and $\mathbf{b}_j$, $\mathbf{b}_j$ and $\mathbf{b}_k$, and so on respectively. Figure 4.5 is an example of such a vector configuration. Then $\alpha+\beta+\gamma+\delta = 2\pi$. If the sum of any two consecutive angles is greater than π, then 0 is in the relative interior of the convex hull of the corresponding vector arrangement. Suppose $\alpha + \beta < \pi$ and $\beta + \gamma < \pi$. Then $\gamma + \delta > \pi$ and $0 \in \mathrm{relint}(\mathrm{conv}\{\mathbf{b}_k, \mathbf{b}_l, \mathbf{b}_i\})$. If we remove any two of $\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k, \mathbf{b}_l$, then the relative interior of the remaining two vectors is a line segment that misses the origin. If we remove any three of $\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k, \mathbf{b}_l$, then the relative interior of the remaining vector is empty. Thus Z is a codimension 2 face of P.

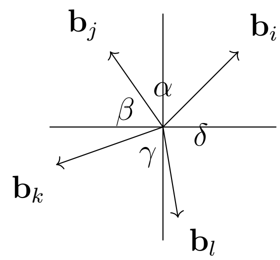


Figure 4.5: The vector arrangement in case 3 of Proposition 4.3.3

$\square$

It is well known that a codimension 2 face of a polytope is the intersection of exactly two facets. Under our assumptions, we can say more.

Proposition 4.3.4. *Let $(x^{\mathbf{u}}, Z)$ be as above and suppose Z is the intersection of two facets $Z = F_1 \cap F_2$. Then there exists $x_{j_1} \in F_1$ and $x_{j_2} \in F_2$ such that $j_1 \neq j_2$ and $1 \leq j_1, j_2 \leq 4$.*

Proof. By Proposition 4.3.3, Z is a codimension 2 face. There are three cases to consider.

Case 1. If there are $x_{j_1}, x_{j_2} \in Z$, where $1 \leq j_1, j_2 \leq 4$, then we are done since Z is the intersection of two facets.

Case 2. Suppose only one of x_1, x_2, x_3, and x_4 are in Z. Without loss of generality, suppose $x_4 \in Z$. There are three subcases to consider.

(a) If $\mathbf{b}_1$ and $\mathbf{b}_3$ are linearly dependent, then $Z \cup \{x_2\}$ is a facet.

(b) Suppose the angle between $\mathbf{b}_1$ and $\mathbf{b}_3$ is less than π. Since Z is a codimension 2 face of P and $0 \notin \operatorname{relint}(\operatorname{conv}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3))$, there must be some $j > 4$ such that $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_j))$. Then $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_j))$ and so $Z \cup \{x_2\}$ is a facet.

(c) Suppose the angle between $\mathbf{b}_1$ and $\mathbf{b}_3$ is greater than π. Since Z is a codimension 2 face and $0 \notin \operatorname{relint}(\operatorname{conv}(\mathbf{b}_{i_1}, \mathbf{b}_{i_2}))$ for $i_1, i_2 \in \{1, 2, 3\}$, there must be some $j > 4$ such that $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_j))$Then $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_j, \mathbf{b}_{i_1}, \mathbf{b}_{i_2}))$ for some $i_1, i_2 \in \{1, 2, 3\}$. Thus, the union of Z with the remaining vector is a facet.

Case 3. Suppose $x_1, x_2, x_3, x_4 \notin Z$. There are three subcases to consider.

(a) If $\mathbf{b}_1$, $\mathbf{b}_3$, and 0 are collinear and $\mathbf{b}_2$, $\mathbf{b}_4$, and 0 are collinear, then $Z \cup \{x_1, x_3\}$ and $Z \cup \{x_2, x_4\}$ are facets whose intersection is Z.

(b) Without loss of generality, suppose that $\mathbf{b}_1$, $\mathbf{b}_3$, and 0 are collinear and $\mathbf{b}_2$ and $\mathbf{b}_4$ are linearly independent. If the angle between $\mathbf{b}_2$ and $\mathbf{b}_4$ is greater than π, then $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4))$ and otherwise, $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4))$. So either $Z \cup \{x_1\}$ or $Z \cup \{x_3\}$ is a facet of P. Since $\mathbf{b}_1$, $\mathbf{b}_3$, and 0 are collinear, $Z \cup \{x_2, x_4\}$ is a facet of P.

(c) Suppose that no two of $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, and $\mathbf{b}_4$ lie on a line through the origin. Then there is an i, $1 \leq i \leq 4$ such that the angle between $\mathbf{b}_i$ and $\mathbf{b}_{i+2}$ is greater than π. Then $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_i, \mathbf{b}_{i+1}, \mathbf{b}_{i+2}))$ and $Z \cup \{x_{i+3}\}$ is a facet of P.

For the other facet, consider the angle θ between $\mathbf{b}_{i+1}$ and $\mathbf{b}_{i+3}$. If $\theta > \pi$, then $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_{i+1}, \mathbf{b}_{i+2}, \mathbf{b}_{i+3}))$ and $Z \cup \{x_i\}$ is a facet. If $\theta < \pi$, then we have that $0 \in \operatorname{relint}(\operatorname{conv}(\mathbf{b}_i, \mathbf{b}_{i+1}, \mathbf{b}_{i+3}))$ and $Z \cup \{x_{i+2}\}$ is a facet. This concludes case 3.

□

4.4 A local duality map for codimension 2 toric ideals

In this section, we describe an isomorphism for Theorem 4.1.3. Let

$$\mathbf{v}_3 = B_{1+} + B_{2+} - \sum_{i=1}^{n} \mathbf{e}_i, \qquad \mathbf{v}_4 = \mathbf{v}_3 - B_1.$$

$$\mathbf{v}_2 = \mathbf{v}_3 - B_2, \qquad \mathbf{v}_1 = \mathbf{v}_3 - B_1 - B_2$$

where $\mathbf{e}_i$ is the i-th standard basis vector of $\mathbb{R}^n$.Let $f \in \operatorname{Ext}^3_S(S/I_A, S)$. The standard monomials of $\langle x^{\mathbf{p}}, x^{\mathbf{r}}, x^{\mathbf{s}}, x^{\mathbf{t}}\rangle$ form a $\Bbbk$-basis for $\operatorname{Ext}^3_S(S/I_A, S)$. Since the standard monomials of $\langle x^{\mathbf{p}}, x^{\mathbf{r}}, x^{\mathbf{s}}, x^{\mathbf{t}}\rangle$ are indexed by standard pairs, we let $f x^{\mathbf{u}} x^{\mathbf{z}} \in \operatorname{Ext}^3_S(S/I_A, S)$ where the monomial $x^{\mathbf{u}}x^{\mathbf{z}}$ is indexed by the standard pair $(x^{\mathbf{u}}, Z)$. By Theorem 4.3.3 and Theorem 4.3.4, Z is the intersection of 2 facets, F_1 and F_2, such that there exists $x_{j_1} \in F_1$ and $x_{j_2} \in F_2$ and $1 \leq j_1 \neq j_2 \leq 4$. We define the map $\phi : \operatorname{Ext}^3_S(S/I_A, S) \to H^{n-3}_{\mathfrak{m}}(S/I_A)$ by

$$x^{\mathbf{u}}x^{\mathbf{z}} \mapsto \frac{1}{x^{\mathbf{u}}x^{\mathbf{z}}}\left(x^{\mathbf{v}_{j_1}} \oplus x^{\mathbf{v}_{j_2}}\right). \tag{4.2}$$

Proposition 4.4.1. $\frac{1}{x^{\mathbf{u}}x^{\mathbf{z}}}\left(x^{\mathbf{v}_{j_1}} \oplus -x^{\mathbf{v}_{j_2}}\right) \in H^{n-3}_{\mathfrak{m}}(S/I_A)$.

Proof. Since $x^{\mathbf{v}_{j_1}}$ and $x^{\mathbf{v}_{j_2}}$ have the same degree, then $\partial(x^{\mathbf{v}_{j_1}} \oplus -x^{\mathbf{v}_{j_2}}) = x^{\mathbf{v}_{j_1}} - x^{\mathbf{v}_{j_2}} \in I_A$ and so $\partial(x^{\mathbf{v}_{j_1}} \oplus -x^{\mathbf{v}_{j_2}})$ is 0 in $(S/I_A)_{x_1 \cdots x_n}$. Thus $x^{\mathbf{v}_{j_1}} \oplus -x^{\mathbf{v}_{j_2}}$ is in the kernel of ∂.

We now show that $x^{\mathbf{v}_{j_1}} \oplus -x^{\mathbf{v}_{j_2}}$ is not in the image of ∂. To do this, we show that there is no solution $\mathbf{z} \in \mathbb{Z}^2$ to the inequality $B\mathbf{z} \leq \mathbf{v}_i - \mathbf{u} - \mathbf{z}$ for $i = 1, \ldots, 4$. It is enough to show that there is no solution $\mathbf{z} \in \mathbb{Z}^2$ to the inequality $B\mathbf{z} \leq \mathbf{v}_i - \mathbf{u} - \mathbf{l}$ for $i = 1, \ldots, 4$.

Case 1. $i = 1$: When $i = 1$, we are considering the inequality

$$B\mathbf{z} \leq B_{1+} + B_{2+} - B_1 - B_2 - \sum_{i=1}^{n} \mathbf{e}_i.$$

If $z_1, z_2 > 0$, the third inequality fails. If $z_1 \leq 0, z_2 > 0$, the fourth inequality fails. If $z_1, z_2 \leq 0$, the first inequality fails. If $z_1 > 0, z_2 \leq 0$, then the second inequality fails.

Case 2. $i = 2$: When $i = 2$, we are considering the inequality

$$B\mathbf{z} \leq B_{1+} + B_{2+} - B_2 - \sum_{i=1}^{n} \mathbf{e}_i.$$

If $z_1, z_2 > 0$, the fourth inequality fails. If $z_1 \leq 0, z_2 > 0$, the third inequality fails. If $z_1, z_2 \leq 0$, the second inequality fails. If $z_1 > 0, z_2 \leq 0$, then the first inequality fails.

Case 3. $i = 3$: When $i = 3$, we are considering the inequality

$$B\mathbf{z} \leq B_{1+} + B_{2+} - \sum_{i=1}^{n} \mathbf{e}_i.$$

If $z_1, z_2 > 0$, the first inequality fails. If $z_1 \leq 0, z_2 > 0$, the second inequality fails. If $z_1, z_2 \leq 0$, the third inequality fails. If $z_1 > 0, z_2 \leq 0$, then the fourth inequality fails.

Case 4. $i = 4$: When $i = 4$, we are considering the inequality

$$B\mathbf{z} \leq B_{1+} + B_{2+} - B_1 - \sum_{i=1}^{n} \mathbf{e}_i.$$

If $z_1, z_2 > 0$, the second inequality fails. If $z_1 \leq 0, z_2 > 0$, the first inequality fails. If $z_1, z_2 \leq 0$, the fourth inequality fails. If $z_1 > 0, z_2 \leq 0$, then the third inequality fails.

$\square$

Proposition 4.4.2. *The map ϕ is well-defined.*

Proof. Let $x^{\mathbf{u}_1}x^{\mathbf{z}_1} = x^{\mathbf{u}_2}x^{\mathbf{z}_2} \in \mathrm{Ext}^3_S(S/I_A, S)$. Suppose $x^{\mathbf{u}_1}x^{\mathbf{z}_1}$ and $x^{\mathbf{u}_2}x^{\mathbf{z}_2}$ are indexed by the standard pair $(x^{\mathbf{u}}, Z)$. Since Z is a codimension 2 face, $Z = F_1 \cap F_2$ for facets F_1 and F_2. Fix the coordinates corresponding to F_1 and F_2. By construction, we choose x_{j_1} and x_{j_2} such that $1 \leq j_1, j_2 \leq 4$ and $j_1 \neq j_2$. Suppose we choose possibly different indexes for $x^{\mathbf{u}_1}x^{\mathbf{z}_1}$ and $x^{\mathbf{u}_2}x^{\mathbf{z}_2}$, say j_1, j_2 and j_1', j_2' respectively so that

$$\begin{aligned} x^{\mathbf{u}_1}x^{\mathbf{z}_1} &\mapsto \frac{1}{x^{\mathbf{u}_1}x^{\mathbf{z}_1}}(x^{\mathbf{v}_{j_1}} \oplus -x^{\mathbf{v}_{j_2}}) \\ x^{\mathbf{u}_2}x^{\mathbf{z}_2} &\mapsto \frac{1}{x^{\mathbf{u}_2}x^{\mathbf{z}_2}}(x^{\mathbf{v}_{j_1'}} \oplus -x^{\mathbf{v}_{j_2'}}). \end{aligned}$$

Since each component has the same degree, the difference of the images is zero in local cohomology and so the map is well-defined. $\square$

Proposition 4.4.3. *Let $(x^{\mathbf{u}}, Z)$ be a standard pair of $\langle x^{\mathbf{p}}, x^{\mathbf{r}}, x^{\mathbf{s}}, x^{\mathbf{t}}\rangle$ and suppose that $x^{\mathbf{u}}x^{\mathbf{z}} \mapsto \frac{1}{x^{\mathbf{u}}x^{\mathbf{z}}}(x^{\mathbf{v}_{j_1}} \oplus -x^{\mathbf{v}_{j_2}})$ under the map 4.2. Fix a facet F of P such that $Z \subset F$, that is, fix a coordinate in the image of ϕ. Then $\mathrm{nsupp}(\frac{x^{\mathbf{v}_i}}{x^{\mathbf{u}}x^{\mathbf{z}}})$ properly contains a codimension 2 face of P and is contained in a facet of P.*

Proof. Clearly $\text{supp}(x^{\mathbf{z}}) \subset F$. Since $(x^{\mathbf{u}}, Z)$ is a standard pair of $\langle x^{\mathbf{p}}, x^{\mathbf{r}}, x^{\mathbf{s}}, x^{\mathbf{t}} \rangle$, $Z = \{x_1 \ldots, x_n\} \setminus \{x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}\}$ where $x_{j_1} \in \text{supp}(x^{\mathbf{p}})$, $x_{j_2} \in \text{supp}(x^{\mathbf{r}})$, and so on. By Proposition 4.3.3, Z is a codimension 2 face. Let Figure 4.6 represent the the Gale diagram of F.

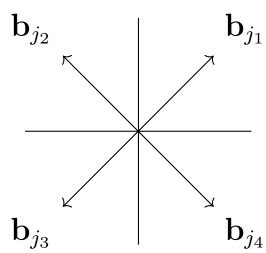


Figure 4.6: Gale diagram in the proof of Proposition 4.4.3

Fix a $1 \leq j \leq 4$ such that x_j is in a facet that contains Z. Observe that $\text{nsupp}(x^{\mathbf{v}_j}) = \{x_i : \mathbf{b}_i \text{ and } \mathbf{b}_j \text{ are in the same quadrant}\}$. Then if we look at the vector arrangement of the corresponding variables in $\text{nsupp}(\frac{x^{\mathbf{v}_j}}{x^{\mathbf{z}}})$, the vector arrangement Figure 4.7, with possibly additional vectors and no vectors in the j-th quadrant. Since we fixed our coordinate in the image of ϕ, this means that $\text{nsupp}(\frac{x^{\mathbf{v}_j}}{x^{\mathbf{z}}})$ does not contain the codimension 2 face Z. Furthermore, from the vector arrangement, $\text{nsupp}(\frac{x^{\mathbf{v}_j}}{x^{\mathbf{z}}})$ is a subset of one of the facets that contain Z.

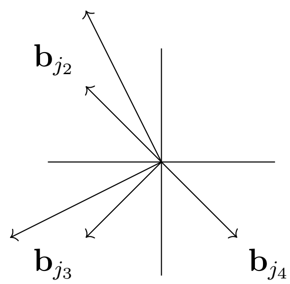


Figure 4.7: Gale diagram of the complement of nsupp$(\frac{x_j^{\mathbf{v}}}{x^{\mathbf{u}}})$

We now show that dividing by $x^{\mathbf{u}}$ keeps this inclusion. If $x_k \in \text{supp}(x^{\mathbf{u}}) \cap F$, then the inclusion holds. Suppose $x_k \in \text{supp}(x^{\mathbf{u}}) \setminus F$. We claim that x_k gets canceled out by $x^{\mathbf{v}_j}$. Now, x_k has a different sign convention than x_j. Note that x_k is in the support of exactly one of $x^{\mathbf{p}}$, $x^{\mathbf{r}}$, $x^{\mathbf{s}}$, and $x^{\mathbf{t}}$. The exponent of x_k is strictly less than the exponent of the monomial that x_k appears in since $x \in \text{supp}(x^{\mathbf{u}})$ and $(x^{\mathbf{u}}, Z)$ is a standard pair. But the monomials $x^{\mathbf{p}}$, $x^{\mathbf{r}}$, $x^{\mathbf{s}}$, and $x^{\mathbf{t}}$ are made up of the greatest common divisors of the $x^{\mathbf{v}_j}$'s. Thus, the exponent of x_k in $x^{\mathbf{v}_j}$ is at least the exponent of x_k in $x^{\mathbf{u}}$. Thus x_k in $x^{\mathbf{u}}$ cancels with $x^{\mathbf{v}}$. $\square$

Proposition 4.4.4. *The map ϕ in is injective.*

Proof. Let $f \in \text{Ext}_S^3(S/I_A, S)$ and $f \mapsto 0 \in H_{\mathfrak{m}}^{n-3}(S/I_A)$. Then

$$f = \sum \lambda \cdot x^{\mathbf{u}} x^{\mathbf{z}} \mapsto \sum \lambda \frac{1}{x^{\mathbf{u}} x^{\mathbf{z}}} \left(x^{\mathbf{v}} \oplus -x^{\mathbf{v}}\right)$$

If the image of f under the map is 0, then the image is 0 at every coordinate $F \in \mathfrak{u}_P^{d-2}$. We now restrict to the coordinate at $F \in \mathscr{F}_P^{d-2}$. Suppose that $\phi(f)\big|_F = 0$ and

$$\phi(f)\big|_F = \sum \frac{1}{x^{\mathbf{u}} x^{\mathbf{z}}} x^{\mathbf{v}_j}.$$

Then $\operatorname{nsupp}\left(\frac{1}{x^{\mathbf{u}}x^{\mathbf{z}}}x^{\mathbf{v}_j}\right)$ is contained in F and properly contains a codimension 2 face of P by Theorem 4.4.3. In particular, each summand is not 0. Furthermore, since $\deg(x^{\mathbf{v}_1}) = \cdots = \deg(x^{\mathbf{v}_4})$, it remains to show that $\deg(x^{\mathbf{u}}x^{\mathbf{z}})$ are distinct. This follows from the fact that the $x^{\mathbf{u}}x^{\mathbf{z}}$ form a $\Bbbk$-basis for $\operatorname{Ext}^3_S(S/I_A, S)$. Thus, if $\phi(f) = 0$, then the coefficients are all 0 and so $f = 0$. Injectivity of ϕ follows. $\square$

Since ϕ is injective, we have the following consequence by Theorem 4.1.3.

Corollary 4.4.5. *The map ϕ in equation 4.2 is also surjective. Thus ϕ is an isomorphism.*

4.5 Examples

In this last section, we illustrate the isomorphism in 4.2. In the first three examples, $\mathbb{N}A$ has one syzygy quadrangle. In the last example, $\mathbb{N}A$ has two syzygy quadrangles.

Example 4.5.1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & 11 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 2 \\ -2 & 3 \\ -1 & -4 \\ 1 & -1 \end{pmatrix}.$$

Then

$$\mathbf{v}_3 = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 0 \end{pmatrix} \qquad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} \qquad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 4 \\ 0 \end{pmatrix}$$

and

$$x^{\mathbf{v}_3} = \frac{x_1^3 x_2^2}{x_3} \qquad x^{\mathbf{v}_4} = \frac{x_1 x_2^4}{x_4} \qquad x^{\mathbf{v}^2} = \frac{x_1 x_3^3 x_4}{x_2} \qquad x^{\mathbf{v}_1} = \frac{x_2 x_3^4}{x_1}.$$

Figure 4.8 shows the convex hull of the columns of A and the Gale diagram of B.

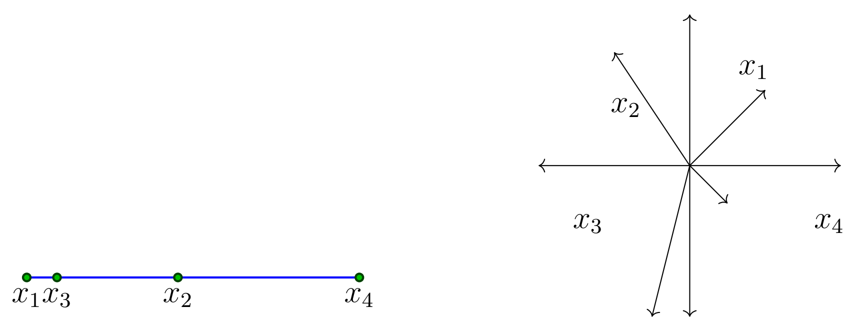


Figure 4.8: conv(A) and the Gale diagram of B in Example 4.5.1

The toric ideal of A in S is

$$I_A = \langle x_2^2 x_3 - x_1^2 x_4, x_2^5 - x_3^3 x_4^2, x_1^2 x_2^3 - x_3^4 x_4, x_1^4 x_2 - x_3^5 \rangle.$$

The minimal free resolution of S/I_A is

$$\mathcal{F} : S \xrightarrow{\begin{pmatrix} x_2^2 \\ -x_1^2 \\ -x_4 \\ x_3 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} -x_1^2 x_2 & -x_2^3 & -x_3^4 & x_3^3 x_4 \\ -x_4 & 0 & -x_2^2 & 0 \\ x_3 & -x_4 & x_1^2 & x_2^2 \\ 0 & x_3 & 0 & x_1^2 \end{pmatrix}} S^4 \to S$$

so $\mathrm{Ext}_S^1(S/I_A, S) \cong S/\langle x_1^2, x_2^2, x_3, x_4 \rangle$. The standard pairs of $\langle x_1^2, x_2^2, x_3, x_4 \rangle$ are

$(1, \varnothing), (x_1, \varnothing), (x_1x_2, \varnothing)$ and $(x_2, \varnothing)$. Then

$$1 \mapsto x^{\mathbf{v}_1} \oplus x^{\mathbf{v}_4} \qquad\qquad x_1 \mapsto \frac{1}{x_1}(x^{\mathbf{v}_1} \oplus x^{\mathbf{v}_4})$$

$$x_1x_2 \mapsto \frac{1}{x_1x_2}(x^{\mathbf{v}_1} \oplus x^{\mathbf{v}_4}) \qquad\qquad x_2 \mapsto \frac{1}{x_2}(x^{\mathbf{v}_1} \oplus x^{\mathbf{v}_4}).$$

Example 4.5.2. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & -2 & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ -2 & -1 \\ 3 & -1 \\ -1 & -1 \end{pmatrix}.$$

Then

$$\mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 2 \\ -1 \end{pmatrix} \qquad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 3 \\ 0 \end{pmatrix} \qquad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Figure 4.9 shows the convex hull of the columns of A and the Gale diagram of B.

The minimal free resolution of S/I_A is

$$\mathcal{F} : S \xrightarrow{\begin{pmatrix} -x_3x_5 \\ -x_4 \\ x_2 \\ x_1 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} -x_2x_3 & -x_1x_4^2 & -x_3^2x_5 & -x_4^3 \\ x_1 & 0 & 0 & x_3x_5 \\ x_4 & -x_3x_5 & -x_1^2 & x_1x_2 \\ 0 & x_2 & x_4 & 0 \end{pmatrix}} S^4 \to S.$$

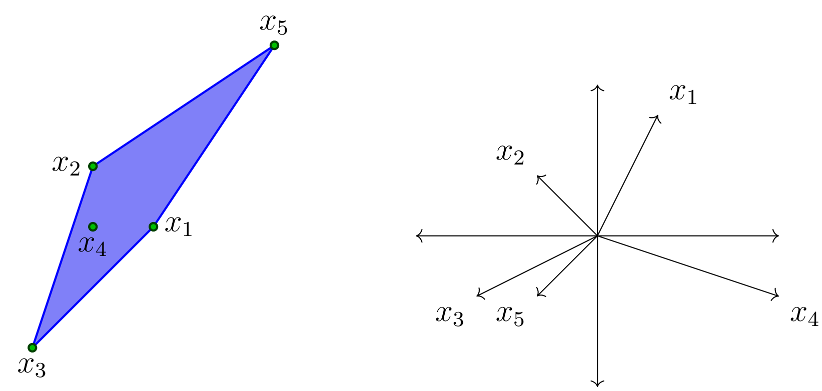


Figure 4.9: conv(A) and the Gale diagram of B in Example 4.5.2

The standard pairs of $\langle x_3x_5, x_4, x_2, x_1\rangle$ are $(1, \{x_3\})$ and $(1, \{x_5\})$. The codimension 2 face $\{x_5\}$ is the intersection of facets $\{x_1, x_5\}$ and $\{x_2, x_5\}$. For each facet, we only have one x_i such that $1 \leq i \leq 4$ and these variables are distinct. The monomials indexed by the standard pair $(1, \{x_5\})$ are of the form x_5^e for $e \geq 0$ and so

$$x_5^e \mapsto \frac{1}{x_5^e}\left(x^{\mathbf{v}_1} \oplus -x^{\mathbf{v}_2}\right).$$

The monomials indexed by $(1, \{x_3\})$ are of the form x_3^f. The codimension 2 face $\{x_3\}$ is the intersection of facets $\{x_1, x_3\}$ and $\{x_2, x_3\}$. For this example, we choose

$$x_3^f \mapsto \frac{1}{x_3^f}\left(x^{\mathbf{v}_1} \oplus -x^{\mathbf{v}_2}\right).$$

We could have also sent x_3^f to $\frac{1}{x_3^f}\left(x^{\mathbf{v}_3} \oplus -x^{\mathbf{v}_2}\right)$ instead, along with other choices, but any choice is equal in local cohomology.

Example 4.5.3. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -2 \\ 1 & -1 \\ -2 & -1 \\ 2 & 2 \end{pmatrix}.$$

Then

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 3 \end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}.$$

Figure 4.10 shows the convex hull of the columns of A and the Gale diagram of B.

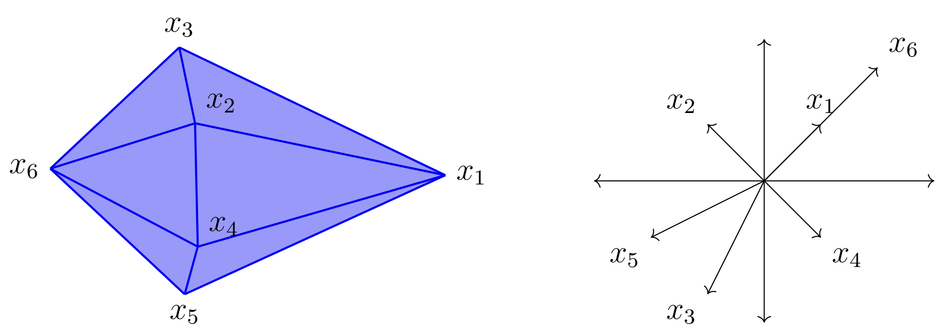


Figure 4.10: conv(A) and the Gale diagram of B in Example 4.5.3

The toric ideal of A in S is

$$I_A = \langle x_3x_4^2 - x_2^2x_5, x_3^2x_4x_5 - x_1x_2x_6^2, x_2x_3x_5^2 - x_1x_4x_6^2, x_3^3x_5^3 - x_1^2x_6^4 \rangle.$$

The minimal free resolution of S/I_A is

$$\mathcal{F} : S \xrightarrow{\begin{pmatrix} -x_1x_6^2 \\ -x_3x_5 \\ -x_4 \\ x_2 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} -x_3x_5 & x_1x_6^2 & 0 & 0 \\ x_4 & -x_2x_5 & -x_1x_6^2 & -x_3x_5^2 \\ -x_2 & x_3x_4 & -x_3^2x_5 & -x_1x_6^2 \\ 0 & 0 & x_2 & x_4 \end{pmatrix}} S^4 \to S$$

so $\mathrm{Ext}^3_S(S/I_A, S) \cong S/\langle x_1x_6^2, x_3x_5, x_4, x_2 \rangle$. The standard pairs of $\langle x_1x_6^2, x_3x_5, x_4, x_2 \rangle$ are $(1, \{x_5, x_6\})$, $(1, \{x_3, x_6\})$, $(1, \{x_1, x_5\})$, $(x_6, \{x_1, x_5\})$, $(1, \{x_3, x_1\})$, and $(x_6, \{x_3, x_1\})$.

Let $x_1^e x_5^f$ be a monomial indexed by $(1, \{x_1, x_5\})$. The two facets with x_1 and x_5 as vertices are the facets $x_1x_4x_5$ and $x_1x_3x_6x_5$. We make the choice

$$x_1^e x_5^f \mapsto \frac{1}{x_1^e x_5^f}(x^{\mathbf{v}_1} \oplus x^{\mathbf{v}_3}).$$

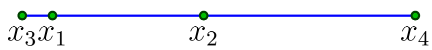
If $x_1^e x_5^f x_6$ is a monomial indexed by $(x_6, \{x_1, x_5\})$. Then

$$x_1^3 x_5^f x_6 \mapsto \frac{1}{x_1^e x_5^f x_6}(x^{\mathbf{v}_1} \oplus x^{\mathbf{v}_3}).$$

Example 4.5.4. This examples demonstrates the case when I_A has two syzygy quadrangles. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 6 & 0 & 13 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 1 \\ -3 & 2 \\ -3 & -2 \\ 1 & -1 \end{pmatrix}.$$

Figure 4.11 shows the convex hull of the columns of A and the Gale diagram of B.



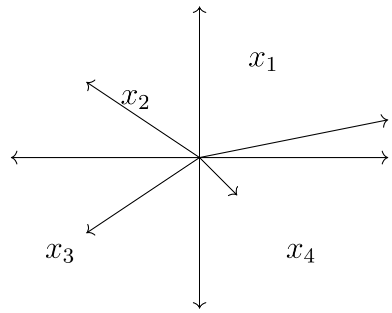


Figure 4.11: conv(A) and the Gale diagram of B in Example 4.5.4

The two syzygy quadrangles of I_A are the unit square Q and the quadrangle $Q \cdot \mathfrak{s}^{-1}$ as shown in Figure 4.12.

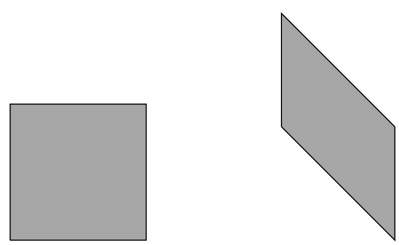


Figure 4.12: The two syzygy quadrangles of I_A.

The matrix $B \cdot \mathfrak{s}^{-1}$ is

$$B \cdot \mathfrak{s}^{-1} = \begin{pmatrix} 4 & 1 \\ -5 & 2 \\ -1 & -2 \\ 2 & -1 \end{pmatrix}.$$

The toric ideal of A in S is

$$I_A = \langle x_1x_2^2 - x_3^2x_4, x_2^3x_3^3 - x_1^5x_4, x_2^5x_3 - x_1^4x_4^2, x_1^6 - x_2x_3^5, x_2^7 - x_1^3x_3x_4^3 \rangle.$$

The minimal free resolution of S/I_A is

$$\mathcal{F} : S^2 \xrightarrow{\begin{pmatrix} x_2^2 & x_3^2 \\ 0 & -x_2^2 \\ 0 & x_4 \\ -x_3 & 0 \\ x_1 & 0 \\ x_4 & x_1 \end{pmatrix}} S^6 \longrightarrow S^5 \longrightarrow S.$$

The copy of S in homological degree 3 corresponding the unit square is the map given by the second column and is twisted by $\left(\begin{smallmatrix} 9 \\ 31 \end{smallmatrix}\right)$. The copy of S in homological degree 3 that corresponds to the rhomboid is the map given by the first column and is twisted by $\left(\begin{smallmatrix} 9 \\ 43 \end{smallmatrix}\right)$. The twist can be checked by computing $(B \cdot \mathfrak{s}^{-1})_{1+} + (B \cdot \mathfrak{s}^{-1})_{2+}$. Then $\operatorname{Ext}_S^1(S/I_A, S) \cong S\left(\begin{smallmatrix} 9 \\ 43 \end{smallmatrix}\right)/\langle x_1, x_2^2, x_3, x_4\rangle \oplus S\left(\begin{smallmatrix} 9 \\ 31 \end{smallmatrix}\right)/\langle x_1, x_2^2, x_3^2, x_4\rangle$. The standard pairs of $\langle x_1, x_2^2, x_3, x_4\rangle$ are $(1, \varnothing)$ and $(x_2, \varnothing)$. The standard pairs of $\langle x_1, x_2^2, x_3^2, x_4\rangle$ are $(1, \varnothing)$, $(x_2, \varnothing)$, $(x_2x_3, \varnothing)$, and $(x_3, \varnothing)$. The map corresponding the the unit square is done as in Example 4.5.1. Then

$$\mathbf{v}_3 = \begin{pmatrix} 4 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 6 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 2 \end{pmatrix} \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 4 \\ 2 \\ 0 \end{pmatrix}.$$

The copy of S corresponding to the rhomboid,

$$1 \mapsto x^{\mathbf{v}_3} \oplus x^{\mathbf{v}_4} \qquad\qquad x_2 \mapsto \frac{1}{x_2}(x^{\mathbf{v}_3} \oplus x^{\mathbf{v}_4}).$$

5. ASSOCIATED PRIME IDEALS OF LOCAL COHOMOLOGY MODULES OVER STANLEY-REISNER RINGS

While local cohomology modules are not finitely generated in general, they may possess other finiteness properties that enable us to gain an understanding of their structure. One such property is the number of associated prime ideals of the local cohomology modules. Let R be a $\Bbbk$-algebra where $\Bbbk$ is a field of arbitrary characteristic. Huneke asked if the number of associated prime ideals of a local cohomology module $H^i_I(R)$ is finite [6]. Huneke's question has been answered in the affirmative for smooth $\mathbb{Z}$-algebras [3], Gorenstein rings of finite F-representation type [16], among other rings. There do exists examples by Katzman [8], Singh [14], and Singh and Swanson [15] where local cohomology modules do not have finitely many associated prime ideals. One such example due to Singh [14] is if we let $R = \mathbb{Z}[u, v, w, x, y, z]/\langle ux + vy + wz\rangle$ and $I = \langle x, y, z\rangle R$. Then $H^3_I(R)$ has infinitely many associated prime ideals.

Madsen, Wheeler, and I answered Huneke's question in the affirmative when R is a Stanley-Reisner ring over a field of arbitrary characteristic whose associated simplicial complex is a T-space [2]. Our approach was inspired by Lyubeznik's characteristic-free approach in [9].

5.1 Simplicial complexes and Stanley-Reisner rings

An (abstract) *simplicial complex* on the vertex set $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of V that is closed under inclusion, that is, if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$. We assume $\{x_i\} \in \Delta$ for all i. The elements of Δ are called *faces*. The maximal faces of Δ under inclusion are called *facets*. The *dimension* of a face $\sigma \in \Delta$ is $\dim \sigma = |\sigma| - 1$. The *$f$-vector* of Δ is the vector $(f_{-1}, f_0, \ldots, f_{\dim\Delta-1})$ where f_i is the number of i-dimensional faces of Δ. The *dimension* of Δ is the maximum of the dimensions of the faces of Δ.

If $V' \subset V$ is a subset of the vertices of Δ, define $\Delta_{V'} := \{\sigma \subset V' : \sigma \in \Delta\}$. The *link* of a face $\sigma \in \Delta$ is

$$\operatorname{link}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cup \tau \in \Delta \text{ and } \sigma \cap \tau = \varnothing\}.$$

The *star* of a vertex $x \in V$ is

$$\operatorname{st}(x) = \{\sigma \in \Delta : \sigma \cup \{x\} \in \Delta\}$$

and the *core* of the vertex set V of Δ is

$$\operatorname{core}_\Delta(V) = \{v \in V : \operatorname{st}(x) \neq V\}.$$

Define $\operatorname{core}\Delta := \Delta_{\operatorname{core} V}$.

A face $\sigma \in \Delta$ can be *separated from* a face $\tau \in \Delta$ if there exists a facet containing σ but not containing τ.

Definition 5.1.1. A simplicial complex Δ is a *T-space* if for every face $\sigma \in \Delta$, if $\tau \not\subset \sigma$, then σ can be separated from τ.

The following equivalent criterion for a T-space is found in [19].

Proposition 5.1.2. *Let Δ be a simplicial complex on vertex set V. Then*

1. *Δ is a T-space if and only if σ may be separated from $\{v\}$ for any face $\sigma \in \Delta$ and vertex $v \in V \setminus \sigma$.*

2. *Given a face $\sigma \in \Delta$ and a vertex $v \in V \setminus \sigma$, then σ can be separated from $\{v\}$ if and only if $\sigma \cap \operatorname{core} V$ can be separated from $\{v\}$ in $\operatorname{core}\Delta$.*

The T-space property on a simplicial complex should be thought of as a simplicial analogue to the separation axioms in topology. Figure 5.1 and Figure 5.2 show examples of simplicial complexes that are not T-spaces. Figure 5.3 and Figure 5.4 show examples of T-spaces.

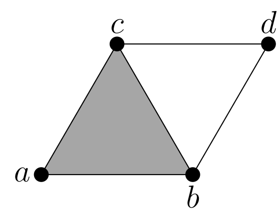


Figure 5.1: A simplicial complex that is not a T-space

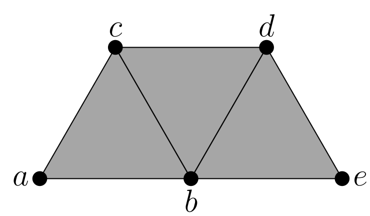


Figure 5.2: A pure simplicial complex that is not a T-space

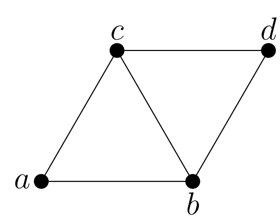


Figure 5.3: A pure T-space

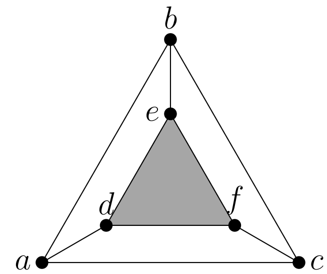


Figure 5.4: A non-pure T-space

Let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\Bbbk$ and let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$. The *Stanley-Reisner ideal*, or *face ideal*, of Δ in S is the ideal generated by the non-faces of Δ

$$I_\Delta := \langle x_{i_1} \cdots x_{i_t} : \{x_{i_1}, \ldots, x_{i_t}\} \notin \Delta \rangle \subset S.$$

The *Stanley-Reisner ring*, or *face ring*, of Δ over $\Bbbk$ is the quotient ring

$$\Bbbk[\Delta] := \frac{S}{I_\Delta}.$$

The Krull dimension of $\Bbbk[\Delta]$ is $\dim \Bbbk[\Delta] = \dim \Delta + 1$.

Proposition 5.1.3. *[4, Theorem 5.1.7] The Hilbert function of a Stanley-Reisner ring $\Bbbk[\Delta]$ with f-vector $f = (f_0, f_1, \ldots, f_{d-1})$ is*

$$H(\Bbbk[\Delta], t) = \begin{cases} 1, & t = 0 \\ \sum_{i=0}^{d-1} f_i \binom{d-1}{i}, & t > 0 \end{cases}.$$

The Hilbert series of $\Bbbk[\Delta]$ *under the fine grading is*

$$F(\Bbbk[\Delta], \lambda) = \sum_{\sigma \in \Delta} \prod_{x_i \in \sigma} \frac{\lambda_i}{1 - \lambda_i}.$$

5.2 Differential operators on Stanley-Reisner rings

Let R be a commutative $\Bbbk$-algebra. We define the ring of $\Bbbk$-linear differential operators on R as follows. Set $D_0(R; \Bbbk) = R$ viewed as a subring of the $\Bbbk$-algebra $\mathrm{Hom}_{\Bbbk}(R, R)$ by the multiplication map $r \mapsto (s \mapsto rs)$. For each $i \geq 0$, let

$$D_{i+1}(R; \Bbbk) = \{P \in \mathrm{Hom}_k(R, R) : [P, r] \in D_i(R; \Bbbk) \text{ for each } r \in R\},$$

where the bracket $[P, r] = P \cdot r - r \cdot P$ is the commutator and the product is composition. The $\Bbbk$-subalgebra of $\mathrm{Hom}_{\Bbbk}(R, R)$,

$$D(R; \Bbbk) = \bigcup_{i \geq 0} D_i(R; , \Bbbk),$$

is the *ring of* $\Bbbk$*-linear differential operators* on R. The elements of $D_i(R; \Bbbk)$ are the differential operators of *order* i on R. This gives R the structure of a left $D(R; \Bbbk)$-module. When we reference R with its $D(R; \Bbbk)$-module structure, we call R a D-*module*.

Example 5.2.1. Let $\Bbbk$ be a field of characteristic 0 and $R = \Bbbk[x]$ be the polynomial ring over $\Bbbk$. Then $D(R; \Bbbk)$ is

$$D(R; \Bbbk) = \Bbbk\langle x, \tfrac{\partial}{\partial x} : [\tfrac{\partial}{\partial x}, x] = 1\rangle,$$

where $\frac{\partial}{\partial x}$ is the usual partial differential operator for x.

If the characteristic of $\Bbbk$ is $p > 0$, then $D(R;\Bbbk)$ is generated by R and all the divided powers $\frac{1}{t!}\frac{\partial^t}{\partial x^t}$ for $t > 0$.

Example 5.2.2. More generally, let $\Bbbk$ be a field of characteristic 0 and $R = \Bbbk[x_1, \ldots, x_n]$ be the n-variate polynomial ring over $\Bbbk$. Then $D(R;\Bbbk)$ is

$$D(R;\Bbbk) = \Bbbk\langle x_1, \ldots, x_n, \tfrac{\partial}{\partial x_1}, \ldots, \tfrac{\partial}{\partial x_n} : [x_i, x_j] = [\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}] = 0, [\tfrac{\partial}{\partial x_i}, x_j] = \delta_{ij}\rangle.$$

If the characteristic of $\Bbbk$ is $p > 0$, then $D(R;\Bbbk)$ is generated by R and all the divided powers $\frac{1}{t!}\frac{\partial^t}{\partial x_i^t}$ for $i = 1, \ldots, n$ and $t > 0$.

For the remainder, we use the following notation for partial differential operators.

Notation 5.2.3. Let $\partial_i^t = \frac{\partial^t}{\partial x_i^t}$ be the t-th divided power for x_i. If $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, then $\partial^{\mathbf{a}} = \partial_1^{a_1}\partial_2^{a_2}\cdots\partial_n^{a_n}$. We write $\frac{\partial f}{\partial x_i}$ if the partial differential operator for x_i is acting of f. When ∂_i and f are being multiplied in as elements in a D-module, we write $\partial_i \cdot f$ or ∂f.

For the remainder of this section, we examine the D-module structure of Stanley-Reisner rings.

Theorem 5.2.4. *[4] Let $\Bbbk[\Delta]$ be a Stanley-Reisner ring over a field $\Bbbk$. Then the minimal prime ideals of I_Δ are in bijection with the facets of Δ.*

The minimal prime $P = \langle x_{i_1}, \ldots, x_{i_t}\rangle$ of I_Δ corresponds to the face of complementary vertices $\{x_j : i \neq i_1, \ldots, i_t\}$. We use the above characterization of minimal primes of I_Δ along with the following theorem of Traves [18] to determine $D(\Bbbk[\Delta];\Bbbk)$.

Theorem 5.2.5 (Theorem 3.5)**.** *[18] Given a monomial ring $R = \Bbbk[x]/J$ with no nonzero nilpotent elements, an element of the Weyl algebra $x^{\mathbf{a}}\partial^{\mathbf{b}}$ is in $D(R)$ if and only if for each minimal prime P of R, we have either $x^{\mathbf{a}} \in P$ or $x^{\mathbf{b}} \notin P$.*

In particular, $D(R;\Bbbk)$ *is generated as a* $\Bbbk$*-algebra by* $\{x^{\mathbf{a}}\partial^{\mathbf{b}} : x^{\mathbf{a}} \in P \text{ or } x^{\mathbf{b}} \notin P$ *for each minimal prime* P *of* $R\}$*, and these are a free basis for* $D(R;\Bbbk)$ *as a left* $\Bbbk$*-module.*

Tripp [19, Theorem 5.7] showed that for a Stanley-Reisner ring $\Bbbk[\Delta]$ over a field of characteristic 0, $D(\Bbbk[\Delta];\Bbbk)$ is left Noetherian if and only if $\operatorname{core}\Delta$ is a T-space.

Theorem 5.2.6. *Let* $R = \Bbbk[\Delta]$ *be a Stanley-Reisner ring where* $\operatorname{core}\Delta$ *is a* T*-space. Then*

$$D = D(R;\Bbbk) = R\langle x_i\partial_i^t : 1 \leq i \leq n, t \geq 0\rangle.$$

Proof. Theorem 5.2.5 gives that D is generated as a $\Bbbk$-module by $x^{\mathbf{a}}\partial^{\mathbf{b}}$ such that for every minimal prime P of $\Bbbk[\Delta]$, either $x^{\mathbf{a}} \in P$ or $x^{\mathbf{b}} \notin P$. The operators $x\partial^t$ satisfy this condition so $R\langle x_i\partial_i^t : 1 \leq i \leq n, t \geq 0\rangle \subseteq D$.

For the reverse inclusion, suppose that $x^{\mathbf{a}}\partial^{\mathbf{b}} \in D$. We assume $\operatorname{supp}(x^{\mathbf{a}}) \in \Delta$ since otherwise $x^{\mathbf{a}} \in I_\Delta$. The facets of Δ correspond to the minimal prime ideals of I_Δ. Choose a facet σ containing $\operatorname{supp}(x^{\mathbf{a}})$. Then $x^{\mathbf{a}} \notin P_\sigma$ and by [18] $\operatorname{supp}(x^{\mathbf{b}}) \subset \sigma$. Hence, $\operatorname{supp}(x^{\mathbf{b}}) \in \Delta$. Furthermore, $\operatorname{supp}(x^{\mathbf{a}})$ cannot be separated from $\operatorname{supp}(x^{\mathbf{b}})$ and so $\operatorname{supp}(x^{\mathbf{b}}) \subset \operatorname{supp}(x^{\mathbf{a}})$ since Δ is a T-space. It follows that for all $i = 1, \ldots, n$ we have $a_i \geq b_i$ and so $x_i\partial_i^{b_i}$ divides $x^{\mathbf{a}}\partial^{\mathbf{b}}$. □

Following the convention in [18], a D-submodule is called a *D-stable ideal*.

Theorem 5.2.7. *Let* $\Bbbk[\Delta]$ *be a Stanley-Reisner ring such that* $\operatorname{core}\Delta$ *is a* T*-space. Then*

1. *An ideal* $J \subset \Bbbk[\Delta]$ *is* D*-stable if and only if it is a squarefree monomial ideal.*

2. *The length of* $\Bbbk[\Delta]$ *as a* D*-module is* $\lambda_D(\Bbbk[\Delta]) = |\Delta|$.

Proof. 1. By [18, Lemma 4.4], every D-stable ideal is a square free monomial ideal. For the converse, let $J \subset \Bbbk[\Delta]$ be a squarefree monomial ideal. It suffices to show that J is stable under $x_i\partial_i^t$. Since J is a monomial ideal, it is enough to show stability for monomials. Let $x^{\mathbf{a}} \in J$. If $a_i < t$, then $x_i\partial_i^t = 0 \in J$. Otherwise, $x_i\frac{\partial^t x^{\mathbf{a}}}{t!\partial x_i^t} = \binom{a_i}{t}x^{\mathbf{a}-\mathbf{b}}$, where $\mathbf{b} = (0, \ldots, t-1, \ldots, 0)$ with $t-1$ in position i. Since $a_i \geq t$, $|\mathbf{a}-\mathbf{b}| = |\mathbf{a}|$

2. We induct on the number of facets of Δ. Let Δ' be a obtained by deleting one facet of Δ. Then $\Bbbk[\Delta']$ is a D-submodule of $\Bbbk[\Delta]$. Moreover, any D-submodule of $I_{\Delta'}$ must also be a squarefree monomial ideal and so the submodule corresponds to a simplicial complex between Δ' and Δ. Since Δ' is obtained by deleting one facet of Δ, this is not possible. Hence $\lambda_D(I_{\Delta'}) = 1$. By induction, $\lambda_D(\Bbbk[\Delta]/I_{\Delta'}) = |\Delta'| = |\Delta| - 1$. Therefore $\lambda_D(\Bbbk[\Delta]) = 1 + (|\Delta| - 1) = |\Delta|$.

□

Lemma 5.2.8. *The partial differential operators ∂_i on $\Bbbk[x_1, \ldots, x_n]$ satisfy the following relation*

$$x_i\partial_i^t x_i = x^2\partial_i^t + tx\partial_i^{t-1} \tag{5.1}$$

where we let $\partial_i^0 = 1$.

Proof. We induct on t. For $t = 1$, multiplying the Leibniz rule on the left by x gives

$$x\partial x = x^2\partial + x.$$

Assume equation 5.1 for $t-1$. Then

$$\begin{aligned}
\partial^t x &= \partial(\partial^{t-1}x) \\
&= \partial(x\partial^{t-1} + (t-1)\partial^{t-2}) \\
&= \partial x\partial^{t-1} + (t-1)\partial^{t-1} \\
&= (x\partial + 1)\partial^{t-1} + (t-1)\partial^{t-1} \\
&= x\partial^t + t\partial^{t-1}.
\end{aligned}$$

Multiplying on the left by x gives the desired result. $\square$

Lemma 5.2.9. *For any $f, g \in R$ and $j \geq 0$,*

$$x\partial^t\left(\frac{g}{f^j}\right) = \frac{1}{f^j}x\partial^t(g) - \sum_{i=1}^{t}\frac{1}{f^j}\frac{\partial^i f^j}{\partial x^i}x\partial^{t-i}\left(\frac{g}{f^j}\right)$$

with $x\partial^0 = x$.

Proof. Multiplying the product rule on the left by x gives

$$x\partial^t(fg) = \sum_{i=0}^{t}\frac{\partial f}{\partial x}x\partial^{t-i}(g).$$

Rewriting the above equation as an operator on g gives

$$x\partial^t f = \sum_{i=0}^{t}\frac{\partial^i f}{\partial x^i}x\partial^{t-i}.$$

Thus

$$x\partial^t f - \sum_{i=1}^{t} \frac{\partial^i f}{\partial x^i} x\partial^{t-i} = \partial^0(f) \cdot x\partial^t$$
$$= f \cdot x\partial^t.$$

Replacing f with f^j and then multiplying on the left by $\frac{1}{f^j}$ gives

$$x\partial^t = \frac{1}{f^j} x\partial^t f^j - \sum_{i=1}^{t} \frac{1}{f^j} \frac{\partial^i f^j}{\partial x^i} x\partial^{t-i}.$$

Applying the above operator to $\frac{g}{f^j} \in R_f$ gives the result. $\square$

We remark that the proof of Lemma 5.2.9 describes how R_f inherits a D-module structure from R. Furthermore, Lemma 5.2.9 holds if we replace g with an element from D-module so we emphasize that M_f inherits a D-module structure from M.

Proposition 5.2.10. *Let $\mathfrak{m} = (x_1 - c_1, \ldots, x_n - c_n)$ be a $\Bbbk$-rational maximal ideal of $\Bbbk[\Delta]$, that is, the natural map $\Bbbk \hookrightarrow \Bbbk[\Delta]/\mathfrak{m}$ is a bijection.*

1. *A $\Bbbk$-basis for $D/D\mathfrak{m}$ is given by $\{x_{a_1} \cdots x_{a_l} \partial_{a_1}^{t_1} \cdots \partial_{a_l}^{t_l} : \{x_{a_1}, \ldots, x_{a_l}\} \in \Delta\}$.*

2. *For any $w \in D/D\mathfrak{m}$ that is not in the field $\Bbbk$, there exists $f \in \mathfrak{m}$ such that $fw = 1$.*

Proof. 1. Note that $\{x^{\mathbf{a}}\partial^{\mathbf{t}} : \mathrm{supp}(x^{\mathbf{a}}) \cup \mathrm{supp}(x^{\mathbf{t}}) \in \Delta\}$ is a $\Bbbk$ vector space basis for D. We now determine the relations among these generators in $D/D\mathfrak{m}$. Denote $x_{a_1} \cdots x_{a_l} \partial_{a_1}^{t_1} \cdots \partial_{a_l}^{t_l}$ such that $\{x_{a_1}, \ldots, x_{a_l}\} \in \Delta$ by $\mathbf{x}\partial^{\mathbf{t}}$. Suppose $\mathbf{a} \in \mathbb{N}^n$ is a vector with $a_i \geq 1$ for some fixed i and consider $x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}}(x_i - c_i)$ where $\mathbf{e}_i$ is the i-th standard basis vector with a 1 in the i-th coordinate and 0 in all other coordinates. Note that these elements are a $\Bbbk$-vector space basis for $D\mathfrak{m}$.

If $t_i \geq 2$, then by Lemma 5.2.8

$$\begin{aligned} x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}}(x_i - c_i) &= x^{\mathbf{a}-\mathbf{e}_i}(\mathbf{x}\partial^{\mathbf{t}} x_i) - c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}} \\ &= x^{\mathbf{a}}\mathbf{x}\partial^{\mathbf{t}} + t_i x^{\mathbf{a}-\mathbf{e}_i} - c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}}. \end{aligned}$$

Since $x^{\mathbf{a}_i-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}}(x_i - c_i) \in D\mathfrak{m}$, we have the following congruence in $D/D\mathfrak{m}$

$$x^{\mathbf{a}}\mathbf{x}\partial^{\mathbf{t}} \equiv -t_i x^{\mathbf{a}_i-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}-\mathbf{e}_i} + c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}}. \tag{5.2}$$

If $t_i = 0$, then

$$x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}}(x_i - c_i) = x^{\mathbf{a}}\mathbf{x}\partial^{\mathbf{t}} - c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}},$$

which implies we have the congruence

$$x^{\mathbf{a}}\mathbf{x}\partial^{\mathbf{t}}(x_i - c_i) \equiv c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}}. \tag{5.3}$$

Lastly, when $t_i = 1$, Lemma 5.2.8 gives

$$\begin{aligned} x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}}(x_i - c_i) &= x^{\mathbf{a}-\mathbf{e}_i}(\mathbf{x}\partial^{\mathbf{t}} x_i) - c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}} \\ &= x^{\mathbf{a}}\mathbf{x}\partial^{\mathbf{t}} + x^{\mathbf{a}}\mathbf{x}\partial^{\mathbf{t}-\mathbf{e}_i} - c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}}. \end{aligned}$$

Thus, in $D/D\mathfrak{m}$,

$$\begin{aligned} x^{\mathbf{a}}\mathbf{x}\partial^{\mathbf{t}} &\equiv -x^{\mathbf{a}}\mathbf{x}\partial^{\mathbf{t}-\mathbf{e}_i} + c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}} \\ &\equiv -c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}-\mathbf{e}_i} + c_i x^{\mathbf{a}-\mathbf{e}_i}\mathbf{x}\partial^{\mathbf{t}} \end{aligned} \tag{5.4}$$

where the second congruence follows from equation 5.3 applied to $x^{\mathbf{a}}\mathbf{x}\partial^{\mathbf{t}-\mathbf{e}_i}$.

Observe that in each of equations 5.2, 5.3, and 5.4, the exponent vectors $\mathbf{a} - \mathbf{e}_i$

occurring on the right side of the equations are smaller coordinatewise than the exponent vectors $\mathbf{a}$ occurring on the left side of the equations. We then use these congruences to reduce any element of $D/D\mathfrak{m}$ to terms with exponent vector $\mathbf{0}$. Hence we reduce to linear combinations of elements in $\{x_{a_1}\cdots x_{a_l}\partial_{a_1}^{t_1}\cdots\partial_{a_l}^{t_l} : \{x_{a_1},\dots,x_{a_l}\}\in\Delta\}$ and so this set is a basis for $D/D\mathfrak{m}$ as a $\Bbbk$-vector space.

2. Applying the congruences in 5.2, 5.3, and 5.4 in the case $\mathbf{a}=\mathbf{e}_i$ and rearranging gives

$$(x_i-c_i)\mathbf{x}\partial^{\mathbf{t}} \equiv \begin{cases} -t_i\mathbf{x}\partial^{\mathbf{t}-\mathbf{e}_i}, & t_i>1 \\ -c_i\mathbf{x}\partial^{\mathbf{t}-\mathbf{e}_i}, & t=1 \\ 0, & t_i=0. \end{cases}$$

In particular, we see that $(\mathbf{x}-\mathbf{c})^{\mathbf{a}}$ with $\mathbf{c}=(c_1,\dots,c_n)$ annihilates $\mathbf{x}\partial^{\mathbf{t}}$ if $a_i>t_i$ for any i while $(\mathbf{x}-\mathbf{c})^{\mathbf{t}}\mathbf{x}\partial^{\mathbf{t}}\in\Bbbk^*$.

Write $w=\sum_{j=1}^{s} b_j\mathbf{x}\partial^{\mathbf{t}_j}$ where $b_j\in\Bbbk^*$. Pick any l such that t_l is the maximum among all $t_j's$ and let $\mathbf{a}=t_l$. Let $f'=(\mathbf{x}-\mathbf{c})^{\mathbf{a}}$. By our choice of l, for any $j\neq l$, there is some index i for which $a_i>t_i$, so that f' annihilates $\mathbf{x}\partial^{\mathbf{t}_j}$. Thus $f'w=b_l(f'\mathbf{x}\partial^{\mathbf{t}_l})$, and $f'\mathbf{x}\partial^{\mathbf{t}_l}$ is a unit by our observations above. Put $f=(f'w)^{-1}f'$ to get $fw=1$.

□

Proposition 5.2.11. *Let $\mathfrak{m}\subset\Bbbk[\Delta]$ be a maximal ideal, let M be a D-module, and let $z\in M$ be an element such that $\operatorname{ann}_R(z)=\mathfrak{m}$. The set $\{x_{a_1}\cdots x_{a_l}\partial_{a_1}^{t_1}\cdots\partial_{a_l}^{t_l}\cdot z : \{x_{a_1},\dots,x_{a_l}\}\in\Delta\}$ is linearly independent over $\Bbbk$*

Proof. Let $\tilde{\Bbbk}$ denote the algebraic closure of $\Bbbk$ and let $\tilde{R}=\tilde{\Bbbk}\otimes_{\Bbbk}\Bbbk[\Delta]=\tilde{\Bbbk}[\Delta]$, $\tilde{\mathfrak{m}}=\mathfrak{m}\tilde{R}$, $\tilde{D}=\tilde{k}\otimes_{\Bbbk}D=D(\tilde{R};\tilde{\Bbbk})$, and $\tilde{M}=\tilde{\Bbbk}\otimes_{\Bbbk}M$. Then $\tilde{M}$ is a $\tilde{D}$-module and $M\subset\tilde{M}$. It suffices to show that $\{x_{a_1}\cdots x_{a_l}\partial_{a_1}^{t_1}\cdots\partial_{a_l}^{t_l}\cdot z : \{x_{a_1},\dots,x_{a_l}\}\in\Delta\}$ is linearly independent in $\tilde{M}$ over $\tilde{\Bbbk}$. Since $\tilde{\Bbbk}/\Bbbk$ is separable, $\tilde{\Bbbk} \otimes_{\Bbbk} (R/\mathfrak{m})$ is reduced. Thus there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ such that $\mathfrak{m} = \bigcap_{i=1}^{s} \mathfrak{m}_i$. Since $\tilde{\Bbbk}$ is algebraically closed, each $\mathfrak{m}_i$ is $\Bbbk$-rational. By the Chinese remainder theorem,

$$\tilde{D}/\tilde{D}\tilde{\mathfrak{m}} \cong \tilde{D} \otimes_{\tilde{R}} (\tilde{R}/\tilde{\mathfrak{m}}) \cong \tilde{D} \otimes_{\tilde{R}} \left(\bigoplus_{i=1}^{s} (\tilde{R}/\mathfrak{m}_i) \right) \cong \bigoplus_{i=1}^{s} (\tilde{D}/\tilde{D}\mathfrak{m}_i). \tag{5.5}$$

Since $\tilde{\mathfrak{m}} = \operatorname{ann}_{\tilde{R}}(z)$, there is a D-module map

$$\begin{aligned} \phi : &\tilde{D}/\tilde{D}\tilde{m} \to \tilde{M} \\ &w \mapsto w \cdot z. \end{aligned}$$

We claim that ϕ is injective. Indeed, suppose there is some $w \in \tilde{D}/\tilde{D}\tilde{\mathfrak{m}}$ such that $w \cdot z = 0$ but $w \neq 0$. For $1 \leq i \leq s$, let w_i be the image of w in $\tilde{D}/\tilde{D}\mathfrak{m}_i$. By Proposition 5.2.10, there is some $f_i \in \mathfrak{m}_i$ such that $f_i w_i = 1$. It follows that there is some $f \in R$ such that $f w_i \in \tilde{\Bbbk}$ for all i and at least one $f w_i \neq 0$. Then under the correspondence above in 5.5, the tuple $(f w_1, \ldots, f w_s)$ corresponds to a nonzero element $f w \in \tilde{R}/\tilde{\mathfrak{m}}$. Then $(fw) \cdot z = f(w \cdot z) = 0$ and so $fw \in \operatorname{ann}_{\tilde{R}/\tilde{\mathfrak{m}}}(z) = 0$, a contradiction. Thus ϕ is injective.

By Proposition 5.2.10, the set $\{x_{a_1} \cdots x_{a_l} \partial_{a_1}^{t_1} \cdots \partial_{a_l}^{t_l} : \{x_{a_1}, \ldots, x_{a_l}\} \in \Delta\}$ is linearly independent in $\tilde{D}/\tilde{D}\tilde{\mathfrak{m}}$ since it is independent in each $\tilde{D}/\tilde{D}\mathfrak{m}_i$. Therefore, its image $\{x_{a_1} \cdots x_{a_l} \partial_{a_1}^{t_1} \cdots \partial_{a_l}^{t_l} \cdot z : \{x_{a_1}, \ldots, x_{a_l}\} \in \Delta\}$ is linearly independent in $\tilde{M}$. $\square$

5.3 Associated primes of local cohomology modules of Stanley-Reisner rings

Let R be a homomorphic image of the polynomial ring S over a field $\Bbbk$. The ring of differential operators $D(R; \Bbbk)$ has a natural filtration by total degree $\mathcal{B} = B_0 \subset$

$B_1 \subset \cdots$, where

$$B_t = \Bbbk \cdot \{x^{\mathbf{a}}\partial^{\mathbf{b}} : \sum_{j=1}^{n}(a_j + b_j) \le t\} \cap D(R;\Bbbk),$$

called the *Bernstein filtration*. For the remainder, $\mathcal{B}$ will denote the Bernstein filtration on $D(R;\Bbbk)$. A $\Bbbk$-*filtration* on a D-module M is an ascending chain of $\Bbbk$-vector spaces $F_0 \subset F_1 \subset \cdots$ such that $\bigcup_i F_i = M$ and $B_i F_j \subset F_{i+j}$. A D-module M is *holonomic* if it has a $\Bbbk$-filtration $F_0 \subset F_1 \subset \cdots$ with $\dim_{\Bbbk} F_i \le C i^{\dim R}$ where C is a constant independent of i.

Proposition 5.3.1. *Let M be a D-module and let $z \in M$ such that $P = \operatorname{ann}_R(z)$ is a prime ideal in R. Then for all d, $\dim_{\Bbbk}(\mathcal{B}_d z) \ge \sum_{i=0}^{d} H(R,i)$.*

Proof. Let $h = \operatorname{height} P$, $\mathcal{K}$ be the field of fractions of R/P, and let $Q \subset \Bbbk[x]$ be a prime ideal such that $P = QR$. After permuting the variables of S, we may assume that $x_{h+1}, \ldots, x_n$ are algebraically independent over $\Bbbk$ in $\mathcal{K}$, and $\mathcal{K}$ is finite over $\mathcal{L} = \Bbbk(x_{h+1}, \ldots, x_n)$. Set $\mathcal{R} = \mathcal{L} \otimes_T R$, $\mathcal{M} = \mathcal{L} \otimes_T M$.

Note that $\sigma = \{x_{h+1}, \ldots, x_n\}$ is a face of Δ, and $\mathcal{R} = \mathcal{L}[\Gamma]$ where $\Gamma = \operatorname{link}_{\Delta} \sigma$. The ideal $P\mathcal{R} = \operatorname{ann}_{\mathcal{R}}(z)$ is a maximal ideal of $\mathcal{R}$ so by Proposition 5.2.11, the set

$$\{x_{a_1} \cdots x_{a_l} \partial_{a_1}^{t_1} \cdots \partial_{a_l}^{t_l} : \{x_{a_1}, \ldots, x_{a_l}\} \in \Gamma\}$$

is linearly independent in $\mathcal{M}$. Hence the set

$$Y := \{(x_{l+1}^{a_l+1} \cdots x_n^{a_n}) x_{a_1} \cdots x_{a_l} \partial_{a_1}^{t_1} \cdots \partial_{a_l}^{t_l} : \{x_{a_1}, \ldots, x_{a_l}\} \in \Gamma\}$$

is linearly independent in M. Let $Y_d = Y \cap \mathcal{B}_d z$. Then $Y_d \subseteq \mathcal{B}_d z$ and since the elements of Y_d are linearly independent, we have the inequality $\dim_{\Bbbk} \mathcal{B}_d z \ge |Y_d|$.

Under the correspondence $x_i\partial_i^t \mapsto x_i^{t+1}$, the set Y_d is in bijection with the set of monomials not in the Stanley-Reisner ideal I_Δ of degree at most d. Hence $|Y_d| = \sum_{j=0}^{d} H(R,j)$. □

Theorem 5.3.2. *Every holonomic D-module M has finite length in the category of D-modules and $l(M) < C \cdot n!$ for some constant C.*

Proof. Let $\mathcal{F} = F_0 \subset F_1 \subset \cdots \subset F_l = M$ be a D-module filtration of M and let $\mathcal{G}$ be a $\Bbbk$-filtration on M. Then there is a constant C such that $\dim_\Bbbk G_i \leq Ci^d$ for all i. For each $1 \leq s \leq l$, $\mathcal{G}$ induces a filtration $\mathcal{G}^{(s)}$ on F_s/F_{s-1} where $G_j^{(s)} = (G_j \cap F_s)/(G_i \cap F_{s-1})$.

Choose a prime $P_s \in \operatorname{Ass}_R(F_s/F_{s-1})$. Then there exists $z_s \in F_s/F_{s-1}$ such that $P_s = \operatorname{ann}_R(z_s)$. Assume that $z_s \in \mathcal{G}_{j_s}^{(s)}$. Then for all $i \geq j_s$, $\mathcal{B}_{i-j_s}z_s \subset \mathcal{G}_i^s$. By Proposition 5.3.1, $\dim_\Bbbk \mathcal{G}_i^{(s)} \geq \sum_{j=0}^{i-j_s} H(R,j)$. Therefore

$$Ci^r \geq \dim_\Bbbk \mathcal{G}_i = \sum_{s=1}^{l} \dim_\Bbbk \mathcal{G}_i^{(s)} \geq \sum_{s=1}^{l}\sum_{j=0}^{i-j_s} H(R,J).$$

Since the Hilbert function eventually equals a polynomial of degree $r-1$ and the leading coefficient is at least $\frac{1}{(r-1)!}$, the function $\sum_{j=0}^{i-j_s} H(R,j)$ is eventually a polynomial of degree r with leading coefficient at least $\frac{1}{r!}$. Thus $C \geq \frac{l}{r!}$ and so $l \leq Cr!$. Finally we get $\lambda_D(M) \leq Cr! \leq \infty$. □

Before continuing, we recall Faulhaber's formula on the sum of p-th powers of the first m positive integers.

Proposition 5.3.3 (Faulhaber's formula)**.** *The sum of pth powers of the first m positive integers is*

$$\sum_{a=1}^{m} a^p = \frac{1}{p+1}\sum_{i=0}^{p}(-1)^i\binom{p+1}{i}B_i m^{p+1-i}$$

where B_i is the i-th Bernoulli number.

Theorem 5.3.4. *Let $R = \mathbb{k}[x_1, \ldots, x_n]/I_\Delta$ be a Stanley-Reisner ring whose associated simplicial complex is a T-space. Then R is a holonomic D-module.*

Proof. Let $r = \dim R$ and $\mathcal{F} = F_0 \subset F_1 \subset \cdots$ be a filtration on R where F_t consists of the elements of R with degree at most t under the standard grading. Then $\mathcal{F}$ is a $\mathbb{k}$-filtration.

We show that there is a constant C such that for all i, $\dim_{\mathbb{k}} F_i \leq Ci^d$. By Proposition5.1.3,

$$\begin{aligned}\dim_{\mathbb{k}} F_t &= \sum_{i=0}^{t} H(R,i) = 1 + \sum_{i=1}^{t} H(R,i) \\ &= 1 + \sum_{i=0}^{t} \left(\sum_{j=0}^{r-1} (-1)^{r-1-j} e_{r-1-j} \binom{i+j}{j} \right).\end{aligned}$$

Let e be the maximum multiplicity $e = \max_{0 \leq j \leq r-1} e_{r-1-j}$. Then

$$\dim_{\mathbb{k}} F_t \leq 1 + \sum_{i=0}^{t} \sum_{j=0}^{r-1} \frac{e}{j!} (i+j)(i+j-1) \cdots (i+1).$$

For each l, the a-th coefficient of the polynomial $(i+j)(i+j-1)\cdots(i+1)$ is the sum of all products consisting of a factors chosen among $1, \ldots, j$. There are $\binom{j}{a}$ choices with the largest such product equal to $j(j-1)\cdots(j-a+1) = \frac{j!}{(j-a)!}$ so we get

$$\begin{aligned}\dim_{\mathbb{k}} F_t &\leq 1 + \sum_{i=1}^{t} \sum_{j=0}^{r-1} \frac{e}{j!} \sum_{a=0}^{j} a! \binom{j}{a}^2 i^a \\ &= 1 + e \sum_{i=1}^{t} \sum_{j=0}^{r-1} \frac{e}{j!} \sum_{a=0}^{j} \frac{a!}{j!} \binom{j}{a}^2 i^a\end{aligned}$$

Since $j \leq r-1$, the binomial coefficients satisfy $\binom{j}{a} \leq \binom{r-1}{a}$ for any a and so we get the inequality

$$\frac{a!}{j!}\binom{j}{a}^2 = \frac{j!}{(j-a)!^2} \leq \frac{(r-1)!}{(r-1-a)!} \cdot \frac{1}{(j-1)!} \leq \frac{(r-1)!}{(r-1-a)!}.$$

We now obtain a bound independent of j as follows.

$$\begin{aligned}
\dim_{\Bbbk} F_t &\leq 1 + e \sum_{i=1}^{t} \sum_{j=0}^{r-1} \sum_{a=0}^{j} \frac{(r-1)!}{(r-1-a)!} i^a \\
&\leq 1 + e \sum_{i=1}^{t} \sum_{j=0}^{r-1} \sum_{a=0}^{r-1} \frac{(r-1)!}{(r-1-a)!} i^a \\
&= 1 + e \sum_{i=1}^{t} r \sum_{a=0}^{j} \frac{(r-1)!}{(r-1-a)!} i^a \\
&= 1 + er \sum_{a=0}^{r-1} \frac{(r-1)!}{(r-1-a)!} \sum_{i=1}^{t} i^a \\
&= 1 + er \sum_{a=0}^{r-1} \frac{(r-1)!}{(r-1-a)!} \left(\sum_{m=0}^{a} \frac{(-1)^m}{a+1} \binom{a+1}{m} B_m t^{a+1-m} \right).
\end{aligned}$$

The last line is a degree r polynomial in t. To get an explicit bound on all the coefficients, it remains to eliminate an index. Let $B = \max_{0 \leq m \leq r-1} |B_m|$. For any a,

$$\frac{(r-1)!}{(r-1-a)!} = a!\binom{r-1}{a} \leq a!\binom{r-1}{\lfloor \frac{r-1}{2} \rfloor} \leq (r-1)!\binom{r-1}{\lfloor \frac{r-1}{2} \rfloor}. \tag{5.6}$$

By inequality 5.6,

$$\dim_{\Bbbk} F_t \leq 1 + er \left(r(r-1)!\binom{r-1}{\lfloor \frac{r-1}{2} \rfloor} \right) \sum_{m=0}^{r-1} \binom{r}{m} B t^{r-m}.$$

We let C be the largest coefficient that appears

$$C = eBr \left(r!\binom{r-1}{\lfloor \frac{r-1}{2} \rfloor} \right) \binom{r}{\lfloor \frac{r}{2} \rfloor}$$

and conclude that R is holonomic. $\square$

Theorem 5.3.5. *Let $R = \Bbbk[\Delta]$ be a Stanley-Reisner ring whose associated simplicial complex is a T-space. Let $D = D(R;\Bbbk)$ be the ring of $\Bbbk$-linear differential operators on R and suppose that M is a D-module. Then for any $f \in R$, if M is holonomic, then so is M_f. In particular, R_f is a holonomic D-module.*

Proof. Let d be the degree of the polynomial f, $r = \dim R$, and let $\mathcal{B} : B_0 \subset B_1 \subset \cdots$ be the Bernstein filtration. Since M is holonomic, there is a $\Bbbk$-filtration on M, $\mathcal{F} : F_0 \subset F_1 \cdots$ such that $\dim_{\Bbbk} F_i \leq Ci^r$ for some constant C. Define the filtration $\mathcal{F}' : F_0' \subset F_1' \subset \cdots$ by

$$F_i' = \Bbbk \cdot \left\{ \frac{u}{f^i} : u \in F_{i(d+1)} \right\}.$$

If $\mathcal{F}'$ is a $\Bbbk$-filtration, then

$$\dim_{\Bbbk}(F_i') \leq \dim_{\Bbbk} F_{i(d+1)} \leq C(i(d+1))^r.$$

Setting $C' = C(d+1)^r$ gives that M_f is holonomic.

It remains to show that $\mathcal{F}'$ is a $\Bbbk$-filtration. First note that $\bigcup_i F_i' \subseteq M_f$. For the reverse inclusion, choose $\frac{u}{f^w} \in M_f$ where $u \in F_i$ for some i. If $i \leq w(d+1)$, then $u \in F_i \subseteq F_{w(d+1)}$ and hence, $\frac{u}{f^w} \in M_w'$ by definition. On the other hand, if $i > w(d+1)$ the put $j = i - w(d+1)$. Since $f^j \in B_{jd}$, it follows that $f^j u \in F_{jd+i}$. Rewriting

$$jd + i = jd + (j + w(d+1)) = (j+w)(d+1)$$

gives $\frac{u}{f^w} = \frac{f^j u}{f^{j+w}} \in F_{j+w}' \subseteq \bigcup_i F_i'$ and so $\bigcup_i F_i' = F_f$.

We now show that for all i, j, $B_i F_j' \subseteq F_{i+j}'$. Let $\frac{u}{f^j} \in F_j'$ with $u \in F_{j(d+1)}$. By Lemma 5.2.6, we show that for $t \geq 0$, we have the containment $x_i \partial_i^t F_j' \subseteq F_{t+1+j}'$. We induct on t. For the base case, we have $x_i \partial_i = x_i$. Since $x_i f \in B_{d+1}$, it follows that

$(x_i f)(u) \in F_{j(d+1)+(d+1)}$. Then $x\frac{u}{f^j} = \frac{xfu}{f^{j+1}} \in F'_{j+1}$ as desired.

Assume for $1 \leq l \leq t$ that $(x\partial^{t-l} \cdot \frac{u}{f^j} \in F'_{(t-s+1)+j}$. By Lemma 5.2.9, it suffices to show that each of

$$\frac{1}{f^j} x\partial^t \cdot u \qquad \text{and} \qquad \frac{1}{f^j}\frac{\partial^l f^j}{\partial x^l} \cdot x\partial^{t-l} \cdot \frac{u}{f^j}$$

are in $F'_{(t+1)+j}$. Since $f \in B_d$, $x\partial^t \in B_{t+1}$, and $u \in F_{j(d+1)}$, we have

$$f^{t+1} \cdot x\partial^t \in B_{d(t+1)+t+1} = B_{(t+1)(d+1)}$$

and hence

$$f^{t+1} \cdot x\partial^t(u) \in F_{(t+1)(d+1)+j(d+1)} = F_{(t+1+j)(d+1)}.$$

Thus

$$\frac{1}{f^j} x\partial^t(u) = \frac{f^{t+1} x\partial^t(u)}{f^{t+1+j}} \in F'_{t+1+j}.$$

For the other term, there exists an element $u_s \in F_{(t-s+1+j)(d+1)}$ such that

$$x\partial^{t-s}\left(\frac{u}{f^j}\right) = \frac{u_s}{f^{t-s+1+j}}$$

by the induction hypothesis.

We claim by induction on j that f^{j-s} divides $\frac{\partial^s f^j}{\partial x^s}$. But this is just the usual power rule in $D(S; \Bbbk)$. When $j = 0$, we have $\frac{\partial^s(1)}{\partial x^s} = 0$. Assume the claim as our inductive hypothesis. Write $\frac{\partial^s f^{j-1}}{\partial x^s}$ as a quotient $\frac{v_{s,j-1}}{f^{s-j+1}}$. Using the usual product rule for higher order derivatives gives

$$\begin{aligned}\frac{\partial^t(f^{j-1}\cdot f)}{\partial x^t} &= \sum_{s=0}^{t}\frac{\partial^s f^{j-1}}{\partial x^s}\cdot\frac{\partial^{t-s}f}{\partial x^{t-s}}\\ &= \sum_{s=0}^{t}\frac{v_{s,j-1}}{f^{s-j+1}}\cdot\frac{v_{t-s,1}}{f^{(t-s)-1}}\\ &= \sum_{s=0}^{t}\frac{v_{s,j-1}v_{t-s,1}}{f^{s-j+1+t-s+1}}.\end{aligned}$$

We conclude that $\frac{\partial^t f^j}{\partial x^t}$ divides f^{j-t} and since t was arbitrary, we replace it with s.

Thus we have

$$\frac{1}{f^j}\frac{\partial^s f^j}{\partial x^s}\cdot\partial^{t-s}\left(\frac{u}{f^j}\right)=\frac{v_{s,j}u_s}{f^{t+j}}. \tag{5.7}$$

The polynomial $\frac{\partial^s f^j}{\partial x^s}$ has degree $dj-s$ so the polynomial $v_{s,j}$ has degree $dj-s-d(j-s)=ds-s$. Hence

$$v_{s,j}u_s\in F_{ds-s+(t-s+1+j)(d+1)}\subset F_{(t+1+j)(d+1)}$$

because $(t-s+1+j)(d+1)+s(d-1)<(t-s+1+j)(d+1)+s(d+1)=(t+1+j)(d+1)$. Thus $\frac{v_{s,j}u_s}{f^j}\in F'_{t+j}$ and by equation 5.7, we are done. $\square$

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APPENDIX A

MACAULAY2 SCRIPTS FOR QUASIDEGREES

This appendix contains summaries and the source code for the main functions of the *Macaulay2* package *Quasidegrees*.

The main function of the package *Quasidegrees* is the method `quasidegrees`. The input is a finitely generated module over the polynomial ring that can be presented as a monomial matrix. The output is a list Q that represents the quasidegree set of M. The list Q consists of pairs $(\mathbf{u}, F)$ where $\mathbf{u} \in \mathbb{Z}^d$ and F is a list of vectors in $\mathbb{Z}^d$. The pair $(\mathbf{u}, F)$ represents $\mathbf{u} + \sum_{\mathbf{b} \in F} \mathbb{C} \cdot \mathbf{b}$. If the user inputs an ideal $I \subset R$ instead of a module, `quasidegrees` is executed on the module R/I.

```
quasidegrees=method()
quasidegrees(Module) := (M) -> (
    R := ring M;
    P := presentation M;
    if not isMonomialMatrix P then error
        "Expected module to be presented by a matrix with monomial entries";
    if not isPositivelyGraded M then error
        "Module is not positively graded";
    if not isHomogeneous M then error
        "Module is not homogeneous with respect to ambient grading";
    E := entries P;
    D := degrees target P;
--  We make the following list of pairs S.
--  The first entry is a degree twist in the presentation of M.
--  The second entry are the standard pairs of the corresponding row.
    S := apply(#D, i -> (
        {vector(D_i), standardPairs monomialIdeal E_i}
        )
    );
--  Next we make a list T representing the quasidegree set of M as follows.
--  The variables in S get assigned their degrees and then
```

```
--  shifted by the corresponding degree twist in the
--  presentation of M.
    T := flatten(apply(S, s-> (
        apply((s_1), w -> (
            if w_0==1 then (
                if w_1 =={} then (
                    {s_0,{}}
                    )
                else {s_0, apply(w_1, x -> vector degree x)}
                )
            else{s_0 + (vector(degree(w_0))), apply(w_1, x -> vector degree x)}
            )
        )
    )
    )
    );
    toList set T
)


quasidegrees(Ideal) := (I) -> (
    R := ring I;
    M := R^1/I;
    quasidegrees M
    )
```

The other main function of *Quasidegrees*, and the motivation for writing the package *Quasidegrees*, is the function `quasidegreesLocalCohomology`, abbreviated `QLC`. The input for `QLC` is an integer i and a finitely generated module M over a polynomial ring that is presented by a matrix with monomial entries. `QLC` returns a list Q representing the quasidegree set of the i-th local cohomology module of M supported at $\mathfrak{m}$, $H^i_{\mathfrak{m}}(M)$. Local duality is used to compute the local cohomology module. As in `quasidegrees`, the list Q consists of pairs $(\mathbf{u}, F)$ where $\mathbf{u} \in \mathbb{Z}^d$ and F is a list of vectors in $\mathbb{Z}^d$. The pair $(\mathbf{u}, F)$ represents $\mathbf{u} + \sum_{\mathbf{b} \in F} \mathbb{C} \cdot \mathbf{b}$. If the user inputs an ideal $I \subset R$ instead of a module, `QLC` is executed on the module R/I.

```
quasidegreesLocalCohomology = method()
quasidegreesLocalCohomology(ZZ, Module) := (i,M) -> (
    R := ring M;
    n := numgens R;
    v := gens R;
    e := vector sum apply(v,x -> degree x);
--  use Local Duality
    N := Ext^(n-i)(M,R);
    P := presentation N;
    if not isMonomialMatrix P then error
        "Expected module to be presented by a matrix with monomial entries";
    if not isPositivelyGraded N then error
        "Module is not positively graded";
    if not isHomogeneous N then error
        "Module is not homogeneous with respect to ambient grading";
    E := entries P;
    D := degrees target P;
--  We make the following list of pairs S.
--  The first entry is a degree twist in the presentation of M.
--  The second entry is a list of the standard pairs of the monomial ideal
--  generated by the entries of the corresponding row.
    S := apply(#D, i -> (
        {vector(-D_i), standardPairs monomialIdeal E_i}
        )
    );
--  We next make a list T representing the quasidegree set of M.
--  The variables in S are assigned their degrees and then
--  shifted by the corresponding degree twist in the
--  presentation of M.
    T := flatten(apply(S, s-> (
        apply((s_1), w -> (
            if w_0==1 then (
                if w_1 =={} then (
                    {s_0,{}}
                    )
                else {s_0, apply(w_1, x -> vector degree x)}
                )
            else{s_0 - (vector(degree(w_0))), apply(w_1, x -> vector degree x)}
            )
```

```
        )
    )
    )
    );
    Q := toList set T;
    apply( #Q, j -> {((Q_j)_0)-e,(Q_j)_1})
)


quasidegreesLocalCohomology(ZZ, Ideal) := (i,I) -> (
    R := ring I;
    M := R^1/I;
    quasidegreesLocalCohomology(i,M)
    )
```