LOCALIZATION AND HIGHER SPIN/CFT DUALITIES

A Dissertation

by

YAODONG ZHU

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Chair of Committee,	Ergin Sezgin
Committee Members,	Christopher N. Pope
	Bhaskar Dutta
	Stephen A. Fulling
Head of Department,	Peter McIntyre

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ABSTRACT

Localization is a powerful tool to compute physical quantities such as partition functions, free energies and expectation values of certain operators exactly at any coupling in many supersymmetric theories. Due to this merit, the technique is able to provide highly nontrivial tests of AdS/CFT correspondence. We apply localization procedure to the most general three-dimensional $\mathcal{N} = 1$ Chern-Simons matter theories, which are not studied in the previous localization literature, and show that they can also be formally localized. The other focus in this body of work is the study of an important aspect of high energy physics, the higher spin theories, and their conjectured CFT duals. Higher spin theory is a remarkable extension of Einstein gravity in which mass particles of all spin are described by self-consistent and fully nonlinear field equations. We perform tests of the duality between supersymmetric higher spin theories in AdS_4 and the corresponding CFTs, by comparisons of the one loop free energies on both sides. We show that the mismatch between the free energies in the duality between Type-B higher spin theory/fermionic vector model cannot be solved by the introduction of supersymmetry. We then turn to another test of the HS/CFT correspondence, by comparing the tree-level three-point functions on both sides. We produce the full structures of three-point Witten diagrams for both parity-preserving and parity-violating bosonic HS theories, and show that they match perfectly with the corresponding ones on CFT side.

DEDICATION

To Ni Pi, and this world in 2217.

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1. INTRODUCTION

The AdS/CFT correspondence, also known as holography, represents a major advance in understanding non-perturbative aspects of quantum gravity and string theory, as well as quantum field theories by facilitating their studies in the corresponding weakly coupled theory [1]. Remarkable examples are several newly discovered superconformal Chern-Simons (CS) matter theories [2–4], which are conjectured to be dual to M-theory on AdS_7 orbifold backgrounds. Among the possible checks of these dualities are the computations of partition functions, free energies and expectation values of certain operators on both sides of the dual theories and making comparisons. These are usually difficult to do perturbatively, as the perturbative region of one theory is the strongly coupled region of its dual. However, the situation becomes much easier for theories with supersymmetry, thanks to the technique called localization. It was demonstrated, using localization, that finding the partition functions and expectation values of certain observables on CFT side reduces to calculations in matrix models. In particular, this process is coupling constant independent. In this spirit, localization has stimulated great amounts of studies in recent years [5–14], since it is a powerful tool in obtaining exact results in certain supersymmetric theories at any coupling and thus provides highly non-trivial tests of the conjectured dualities.

Under the theme of holography, another fascinating subject is the higher spin (HS)/CFT duality. HS theories are expected to emerge from the tensionless limit of string theory, and contain scalars, spin-1/2 fermions and an infinite tower of fields with spin $s \geq 1$ [15]. Their conjectured duals are conformal field theories with matter in vector representation of the gauge groups [16–19]. Notably, there are two types of parity invariant Vasiliev HS gravities, known as Type-A and B [18]. In their simplest forms, they both contain an infinite tower of massless even spin fields, each occurring once. They differ from each other in the parity of the spin-0 field, which is parity even (odd) in Type-A (B) theory. It has been conjectured that Type-A theory with $\Delta = 1$ boundary condition imposed on the scalar is dual to the $O(N)$ singlet sector of N free real scalars [16], while Type-B theory with $\Delta = 2$ boundary condition imposed on the pseudoscalar is dual to the $O(N)$ singlet sector of N free Majorana fermions [18] (for earlier work in which HS holography involving CFTs with matrix valued free fields, see [17]). These are HS symmetry preserving boundary conditions, with standard boundary conditions imposed on all other fields understood. The dual CFT can be altered by changing the boundary conditions imposed on the spin-0 field in such a way that they break HS symmetry. For instance, Type-A model with $\Delta = 2$ boundary condition on the scalar is conjectured to be dual to the critical $O(N)$ vector model [16], while Type-B model with $\Delta = 1$ boundary condition imposed on the pseudoscalar is conjectured to be dual to $O(N)$ Gross-Neveu model [18].

It has been noted that the HS/CFT duality is expected to arise in weakly coupled regimes of both bulk and boundary field theories. Therefore, one expects that higher spin AdS/CFT correspondence should be amenable to test order by order in perturbation theory. A highly non-trivial approach to test the conjectured duality is to match the one loop free energies on both sides of the duality [20,21]. Indeed, the one loop test supports the duality between Type-A model with the scalar field obeying ordinary boundary condition and the free bosonic vector model well. However, for Type-B model in Euclidean AdS_4 with the alternate boundary condition imposed on the pseudoscalar, it is observed that there exists a mismatch of its one loop free energy with the one of the conjectured dual theory, which is a fermionic vector model living on the boundary S^3.

As far as the checks of the HS/CFT duality are concerned, another important test complementary to the one stated above is the comparison of the three-point functions on both sides. Due to the non-localities in Vasiliev equations which describe the dynamics of HS theories, the computation of three-point function can be done so far at tree-level with one scalar leg on the bulk side. In this context the first three-point functions were computed in [22] in the parity even cases with spins $s_1 - s_2 - 0$, and they were found to match with those of the dual theories. In particular in [22] it was assumed that the results would give the correct CFT structures and it is possible to compare the leading coefficients only by choosing certain special kinematics.

This dissertation consists of three parts as follows. *(i)* We first apply localization to general $\mathcal{N} = 1$ superconformal theories in three-dimension and compute the partition function [23]. *(ii)* We then move on to the study of HS/CFT dualities and extend the previous one loop tests to a wider class of HS theories constructed by Konstein and Vasiliev [24], which contains supersymmetric HS theories as special cases, and their conjectured duals [25]. We will show that the mentioned mismatch of one loop free energies is also present in supersymmetric cases, and the problem remains to be an open one. The connection between *(i)* and *(ii)* is that in the higher spin limit, the $\mathcal{N} = 1$ superconformal theories we considered in the first part should be dual to $\mathcal{N} = 1$ HS theories with Chan-Paton factors in AdS_4. We left the verification of this duality to future works. *(iii)* We go beyond the previous one loop free energy computation, and compute the tree-level three-point Witten diagrams of HS theories in AdS_4, following [22]. In particular, we provide simplifications in the computation on bulk side and produce the full structures of the three-point functions for both parity-preserving and parity-violating HS theories, and compare them with the three-point correlation functions from the dual CFTs [26]. This will provide a novel approach to check the HS/CFT duality in addition to the one loop free energy test.

The rest of the dissertation is organized as follows. In chapter 2, we review the localization technique and describe our published results on the application of localization to general three-dimensional $\mathcal{N} = 1$ superconformal theories [23]. In chapter 3, we present our work on the one loop tests to supersymmetric HS/CFT dualities [25]. In chapter 4 we carry out the computation of three-point functions of both parity-preserving and parity-violating HS theories in AdS_4, and show that they match with the corresponding CFT results exactly [26].

2. $\mathcal{N} = 1$ CHERN-SIMONS MATTER THEORY AND LOCALIZATION*

The idea of localization was first introduced in [27] to great effect in computing observables in certain topologically twisted theories. More recently, initiated by the work of Pestun [5], supersymmetric localization is shown to be a powerful tool for computing exact results in supersymmetric quantum field theories at any coupling. Since then there has been a wave of exact results for non-topological observables on compact manifolds for theories in various dimensions and with various amounts of supersymmetry [6–14]. In particular the localization method has been used to great extent to attack the problems of various conjectured AdS/CFT dualities, as the perturbative region of one theory is dual to the strongly coupled of the dual, and it is hard to perform the check using ordinary perturbative methods.

This chapter is based on the work [23] in collaboration with Dr. Dimitrios Tsimpis.

2.1 The main idea

Consider a generic partition function with a deformation

$$\int d\varphi e^{iS-t\Delta V} \ , \tag{2.1}$$

where $\int d\varphi$ stands for integrations over all fields in consideration, S is the original action, t is a constant parameter and ΔV is the deformation. If the action is invariant under some symmetry transformation α, and ΔV is α-exact, i.e. $\Delta V = \delta_\alpha \widetilde{V}$, then a derivative of the

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partition function with respect to t would read

$$\begin{aligned}
&-t\int d\varphi\Delta V e^{iS-t\Delta V}\\
=&-t\int d\varphi\delta_\alpha\widetilde{V}e^{iS-t\Delta V}\\
=&-t\delta_\alpha\int d\varphi\widetilde{V}e^{iS-t\Delta V}+t\delta_\alpha(\int d\varphi)\widetilde{V}e^{iS-t\Delta V}+t\int d\varphi\widetilde{V}\delta_\alpha e^{iS-t\delta_\alpha\widetilde{V}}\\
=&t\int d\varphi\widetilde{V}e^{iS}\delta_\alpha e^{-t\delta_\alpha\widetilde{V}}\ ,
\end{aligned}\tag{2.2}$$

where we restrict the theory to the one with δ_α-invariant measure $\int d\varphi$, and we have used the fact $\delta_\alpha S = 0$. Interestingly, a closer observation of the last line indicates that, if $\delta_\alpha\widetilde{V}$ is also δ_α-closed, then this t-derivative would be zero. As such, the partition function (2.1) is independent of the deformation. An important consequence is that we can choose deformation ΔV to have a positive definite bosonic sector and tune t to be large, then the exponent would be dominated by the deformation and the integration would receive contribution only from the saddle points of the deformation. This technique is so called localization.

The crucial point in the above argument is that one needs to find a proper symmetry α of the theory which is nilpotent, $(\delta_\alpha)^2 = 0$, then $\delta_\alpha\widetilde{V}$ would automatically be closed. This gets facilitated in supersymmetric theories, where one can choose a proper supersymmetry transformation to be the α and have $(\delta_\alpha)^2 = 0$ satisfied. For examply, in 3D $\mathcal{N} = 2$ supersymmetric field theories, it is shown in [6] that if one keeps only one of the two supersymmetry parameters and set the other one to zero, then the supersymmetry variation corresponding to this choice parameters would be nilpotent and thus can be used in localization. However, it is not always true that there exists a nilpotent variation in all supersymmetric theories, and in these cases one may need to opt to the variation which squares to a transformation in the isometry group of the manifold, which in turn leads to

$\delta^2_\alpha \widetilde{V} = 0$ upon volume integration.

2.2 3D $\mathcal{N} = 1$ Chern-Simons matter theory

Starting from this section we introduce a specific model, the general $\mathcal{N} = 1$ Chern-Simons matter theory, then carry out localization procedure to study its partition function.

2.2.1 On-Shell

The general component form of the on-shell $\mathcal{N} = 1$ classically-superconformal CS Lagrangian with $Spin(5) \simeq Sp(2)$ global symmetry and gauge group $U(N) \times U(N)$ is given in [28]

$$\mathcal{L} = \mathcal{L}_{CS} + \mathcal{L}_{kin} + \mathcal{L}_4 + \mathcal{L}_6 \ , \tag{2.3}$$

where $\mathcal{L}_{CS}$ is the pure CS Lagrangian, $\mathcal{L}_{kin}$ is the matter kinetic term, $\mathcal{L}_4$ is the quartic interaction and $\mathcal{L}_6$ is the sextic potential. More specifically,†

$$\mathcal{L}_{CS} = \frac{k_1}{2\pi}\varepsilon^{\mu\nu\rho}\mathrm{tr}\left\{\frac{1}{2}A_\mu\partial_\nu A_\rho + \frac{i}{3}A_\mu A_\nu A_\rho\right\} - \frac{k_2}{2\pi}\varepsilon^{\mu\nu\rho}\mathrm{tr}\left\{\frac{1}{2}\hat{A}_\mu\partial_\nu \hat{A}_\rho + \frac{i}{3}\hat{A}_\mu \hat{A}_\nu \hat{A}_\rho\right\} \ , \tag{2.4}$$

where the normalization above was chosen to facilitate the derivation of the superconformal invariance; A_μ, $\hat{A}_\mu$ are gauge fields in the adjoint of $U(N)$. The matter kinetic terms read

$$\mathcal{L}_{kin} = \frac{1}{2\pi}\mathrm{tr}\left\{-D^\mu X^A D_\mu X_A + i\tilde{\Psi}_A\gamma^\mu D_\mu \Psi^A\right\} \ , \tag{2.5}$$

where $A = 1, \ldots, 4$ is an $Sp(2)$ index; X_A is in the bifundamental $(\bar{N}, N)$ while X^A is in the $(N, \bar{N})$, and similarly for Ψ_A, Ψ^A. Also we adopt the following convention for the spinors:

$$\widetilde{\psi} \equiv \psi^T C^{-1} \ . \tag{2.6}$$

†We follow closely the notation of [28], to which the reader is referred for more details.

The most general quartic interaction terms can be written in the form $\mathcal{L}_4 = \mathcal{L}_{4a} + \mathcal{L}_{4b} + \mathcal{L}_{4c} + \mathcal{L}'$, where

$$\begin{aligned}
\mathcal{L}_{4a} &= \frac{1}{2\pi} i\mathrm{tr}\{\bar{\alpha}_1 \varepsilon^{ABCD} \tilde{\Psi}_A X_B \Psi_C X_D - \alpha_1 \varepsilon_{ABCD} \tilde{\Psi}^A X^B \Psi^C X^D\} \\
\mathcal{L}_{4b} &= \frac{1}{2\pi} i\mathrm{tr}\{\alpha_{2,1} \tilde{\Psi}^A \Psi_A X_B X^B - \alpha_{2,2} \tilde{\Psi}_A \Psi^A X^B X_B\} \\
\mathcal{L}_{4c} &= \frac{1}{2\pi} 2i\mathrm{tr}\{\alpha_{3,1} \tilde{\Psi}_A \Psi^B X^A X_B - \alpha_{3,2} \tilde{\Psi}^B \Psi_A X_B X^A\} \\
\mathcal{L}' &= \frac{1}{2\pi} \mathrm{tr}\{a_1 \Omega^{AD} \Omega_{BC} \tilde{\Psi}_A \Psi^B X^C X_D + a_2 \Omega_{AD} \Omega^{BC} \tilde{\Psi}^A \Psi_B X_C X^D \\
&\quad + a_3 \Omega^{AC} \Omega^{BD} \tilde{\Psi}_A X_B \Psi_C X_D + \bar{a}_3 \Omega_{AC} \Omega_{BD} \tilde{\Psi}^A X^B \Psi^C X^D \\
&\quad + a_4 \Omega^{AB} \Omega^{CD} \tilde{\Psi}_A X_B \Psi_C X_D + \bar{a}_4 \Omega_{AB} \Omega_{CD} \tilde{\Psi}^A X^B \Psi^C X^D\} \, .
\end{aligned} \tag{2.7}$$

The sextic potential consists of two terms $\mathcal{L}_6 = \mathcal{L}_{pot} + \mathcal{L}''$, where

$$\begin{aligned}
\mathcal{L}_{pot} =& \frac{1}{2\pi}\frac{1}{3} \mathrm{tr}\{\alpha_{4,1} X^A X_A X^B X_B X^C X_C + \alpha_{4,2} X_A X^A X_B X^B X_C X^C \\
&\quad + 4\alpha_{4,3} X_A X^B X_C X^A X_B X^C - 6\alpha_{4,4} X^A X_B X^B X_A X^C X_C\} \\
\mathcal{L}'' =& \frac{1}{2\pi} \Omega^{BC} \Omega_{DE} \mathrm{tr}\{n X_B X^A X_C X^D X_A X^E\} \\
&+ \frac{1}{2\pi} \Omega^{BC} \Omega_{DE} \mathrm{tr}\{m X_B X^A X_A X^D X_C X^E\} \\
&+ \frac{1}{2\pi} \Omega_{BC} \Omega^{DE} \mathrm{tr}\{\bar{m} X^B X_A X^A X_D X^C X_E\} \, .
\end{aligned} \tag{2.8}$$

Here Ω_{AB} is the $Sp(2)$-invariant antisymmetric tensor, which satisfies $\Omega^{AB}\Omega_{AC} = \delta^B_C$. One can show that the theory is invariant under the following $\mathcal{N} = 1$ Poincaré supersymmetry

$$
\begin{aligned}
\delta X_A =& i\Omega_{AB}\tilde{\epsilon}\Psi^B \\
\delta X^A =& i\Omega^{AB}\tilde{\epsilon}\Psi_B \\
\delta \Psi_A =& \Omega_{AB}\gamma^\mu\epsilon D_\mu X^B + \{\Omega_{AB}(\alpha_{2,2}X^C X_C X^B \\
& - \alpha_{2,1}X^B X_C X^C) - 2\alpha_3\Omega_{BC}X^B X_A X^C\}\epsilon \\
\delta \Psi^A =& \Omega^{AB}\gamma^\mu\epsilon D_\mu X_B + \{\Omega^{AB}(-\alpha_{2,1}X_C X^C X_B \\
& + \alpha_{2,2}X_B X^C X_C) + 2\alpha_3\Omega^{BC}X_B X^A X_C\}\epsilon \\
\delta A_\mu =& \frac{1}{k_1}[\Omega_{AB}\tilde{\epsilon}\gamma_\mu\Psi^A X^B + \Omega^{AB}X_B\tilde{\Psi}_A\gamma_\mu\epsilon] \\
\delta \hat{A}_\mu =& \frac{1}{k_2}[\Omega_{AB}X^B\tilde{\epsilon}\gamma_\mu\Psi^A + \Omega^{AB}\tilde{\Psi}_A\gamma_\mu\epsilon X_B] \ ,
\end{aligned}
\tag{2.9}
$$

provided that the coefficients satisfy the relations

$$
\begin{aligned}
& a_1 = -2i(\frac{1}{k_1} + \bar{\alpha}_1) \ , \ \ a_2 = 2i(\frac{1}{k_2} + \alpha_1) \ , \\
& a_3 = -\bar{a}_3 - i(\alpha_1 - \bar{\alpha}_1) \ , \ \ a_4 = i(\alpha_1 - \bar{\alpha}_1) \ , \\
& \alpha_{2,1} = -\frac{1}{k_1} - 2\bar{\alpha}_1 \ , \ \ \alpha_{2,2} = -\frac{1}{k_2} - 2\alpha_1 \ , \ \ \alpha_3 = i\bar{a}_3 - \alpha_1 \ , \\
& \alpha_{4,1} = -3\alpha_{2,2}^2 + 4\alpha_{2,2}\alpha_3 + m \ , \ \ \alpha_{4,2} = -3\alpha_{2,1}^2 + 4\alpha_{2,2}\alpha_3 + m \ , \\
& \alpha_{4,3} = \alpha_{2,2}\alpha_3 + \frac{m}{4} \ , \ \ \alpha_{4,4} = -\alpha_{2,1}\alpha_{2,2} + 2\alpha_{2,2}\alpha_3 + \frac{m}{2} \ , \\
& \bar{m} = 4(\alpha_{2,2} - \alpha_{2,1})\alpha_3 + m \ , \ \ n = 4(\alpha_3 - \alpha_{2,2})\alpha_3 - m \ .
\end{aligned}
\tag{2.10}
$$

In addition to the CS levels k_1, k_2, the theory has four independent parameters. One can choose them to be α_1, $\bar{\alpha}_1$, $\bar{a}_3$ and m.

2.2.2 Off-Shell

In the previous section we studied the on-shell formulation of the theory. However to carry out the localization procedure one needs off-shell supersymmetry. For that purpose we introduce the auxiliary scalar fields F and the gaugini λ, $\hat{\lambda}$ in the scalar and gauge multiplets, respectively. The off-shell action reads

$$\mathcal{L} = \mathcal{L}_{CS} + \mathcal{L}_{kin} + \mathcal{L}_{\text{potential}} \,, \tag{2.11}$$

where

$$\begin{aligned}\mathcal{L}_{CS} =& \frac{k_1}{2\pi}\mathrm{tr}\left\{\varepsilon^{\mu\nu\rho}(\frac{1}{2}A_\mu\partial_\nu A_\rho + \frac{i}{3}A_\mu A_\nu A_\rho) + \frac{i}{2}\tilde{\lambda}\lambda\right\} \\ & - \frac{k_2}{2\pi}\mathrm{tr}\left\{\varepsilon^{\mu\nu\rho}(\frac{1}{2}\hat{A}_\mu\partial_\nu \hat{A}_\rho + \frac{i}{3}\hat{A}_\mu \hat{A}_\nu \hat{A}_\rho) + \frac{i}{2}\tilde{\hat{\lambda}}\hat{\lambda}\right\} \,,\end{aligned} \tag{2.12}$$

$$\mathcal{L}_{kin} = \frac{1}{2\pi}\mathrm{tr}\left\{-D^\mu X^A D_\mu X_A + i\tilde{\Psi}_A\gamma^\mu D_\mu \Psi^A - F^A F_A\right\} \,, \tag{2.13}$$

$$\begin{aligned}\mathcal{L}_{\text{potential}} =& \frac{1}{2\pi}\mathrm{tr}\{i[(-\alpha_{2,1}X_B X^B X_A + \alpha_{2,2}X_A X^B X_B) - 2\alpha_3\Omega_{AB}\Omega^{CD}X_C X^B X_D]F^A \\ & + iF_A[(-\alpha_{2,1}X^A X_B X^B + \alpha_{2,2}X^B X_B X^A) + 2\alpha_3\Omega^{AB}\Omega_{CD}X^C X_B X^D]\} \\ & + \frac{1}{2\pi}\mathrm{tr}\{\Omega_{AB}\tilde{\lambda}\Psi^A X^B - \Omega^{AB}X_B\tilde{\Psi}_A\lambda - \Omega_{AB}X^B\tilde{\Psi}^A\hat{\lambda} + \Omega^{AB}\tilde{\hat{\lambda}}\Psi_A X_B\} \\ & + \frac{1}{2\pi}\mathrm{tr}\{i\alpha_{2,1}\Omega^{AD}\Omega_{BC}\tilde{\Psi}_A\Psi^B X^C X_D - i\alpha_{2,2}\Omega_{AD}\Omega^{BC}\tilde{\Psi}^A\Psi_B X_C X^D \\ & - \frac{i}{2}\alpha_{2,2}\Omega^{AB}\Omega^{CD}\tilde{\Psi}_A X_B\Psi_C X_D + \frac{i}{2}\alpha_{2,1}\Omega_{AB}\Omega_{CD}\tilde{\Psi}^A X^B\Psi^C X^D \\ & + i\alpha_3\Omega^{AC}\Omega^{BD}\tilde{\Psi}_A X_B\Psi_C X_D - i\alpha_3\Omega_{AC}\Omega_{BD}\tilde{\Psi}^A X^B\Psi^C X^D \\ & - \frac{i}{2}\alpha_{2,1}\Omega^{AD}\Omega^{BC}\tilde{\Psi}_A X_B\Psi_C X_D + \frac{i}{2}\alpha_{2,2}\Omega_{AD}\Omega_{BC}\tilde{\Psi}^A X^B\Psi^C X^D\} \\ & + \frac{1}{2\pi}i\mathrm{tr}\{\alpha_{2,1}\tilde{\Psi}^A\Psi_A X_B X^B - \alpha_{2,2}\tilde{\Psi}_A\Psi^A X^B X_B\} \\ & + \frac{1}{2\pi}2i\mathrm{tr}\{\alpha_3\tilde{\Psi}_A\Psi^B X^A X_B - \alpha_3\tilde{\Psi}^B\Psi_A X_B X^A\} \,.\end{aligned} \tag{2.14}$$

This can be rewritten compactly in superspace formalism, see e.g. (3.8) of [29] which we reproduce here

$$
\begin{aligned}
S = & \frac{k_1}{2\pi} S_{CS}(A) - \frac{k_2}{2\pi} S_{CS}(\hat{A}) + \frac{1}{2\pi} \int d^2\theta \mathrm{tr}\{D_a \Phi_A^\dagger D^a \Phi^A \\
& + (c_1 \Phi_A^\dagger \Phi^A \Phi_B^\dagger \Phi^B + c_2 \Phi_A^\dagger \Phi^B \Phi_B^\dagger \Phi^A + c_3 \Omega^{AB} \Omega_{CD} \Phi_A^\dagger \Phi^C \Phi_B^\dagger \Phi^D)\} ,
\end{aligned} \tag{2.15}
$$

where Φ_A is a superfield, and the connection with the component formulation discussed previously is provided by the relations

$$
c_1 = -i\bar{\alpha}_1 - \frac{i}{2k_1} ; \quad c_2 = i\alpha_1 + \frac{i}{2k_2} ; \quad c_3 = i\alpha_1 + \bar{a}_3 . \tag{2.16}
$$

The action is invariant under the off-shell supersymmetry transformations

$$
\begin{aligned}
\delta X_A &= i\Omega_{AB}\tilde{\epsilon}\Psi^B \\
\delta X^A &= i\Omega^{AB}\tilde{\epsilon}\Psi_B \\
\delta \Psi_A &= \Omega_{AB}\gamma^\mu \epsilon D_\mu X^B - i\Omega_{AB} F^B \epsilon \\
\delta \Psi^A &= \Omega^{AB}\gamma^\mu \epsilon D_\mu X_B - i\Omega^{AB} F_B \epsilon \\
\delta F_A &= -\Omega_{AB}\tilde{\epsilon}\gamma^\mu D_\mu \Psi^B - iX_A(\tilde{\epsilon}\hat{\lambda}) + i(\tilde{\epsilon}\lambda)X_A \\
\delta F^A &= -\Omega^{AB}\tilde{\epsilon}\gamma^\mu D_\mu \Psi_B - iX^A(\tilde{\epsilon}\lambda) + i(\tilde{\epsilon}\hat{\lambda})X^A \\
\delta A_\mu &= -i\tilde{\epsilon}\gamma_\mu \lambda \\
\delta \hat{A}_\mu &= -i\tilde{\epsilon}\gamma_\mu \hat{\lambda} \\
\delta \lambda &= -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} \\
\delta \hat{\lambda} &= -\frac{1}{2}\gamma^{\mu\nu}\epsilon \hat{F}_{\mu\nu} .
\end{aligned} \tag{2.17}
$$

We note that besides k_1, k_2 the off-shell theory has only three free parameters, as can be seen from (2.16). This is one fewer parameter than in the on-shell formulation. Specifically, after replacing the auxiliary field F and gaugini λ, $\hat{\lambda}$ by the solutions of their respective equations of motion, the Lagrangian (2.11) goes back to (2.3), but with $\alpha_{4,3} = 0$ in $\mathcal{L}_{pot}$. In other words, for the on-shell theory obtained by starting from (2.11) and then eliminating the auxiliary fields, m is not an independent parameter but is equal to $-4\alpha_{2,2}\alpha_3$, which in its turn can be expressed in terms of α_1, $\bar{\alpha}_1$ and $\bar{a}_i$. This can be understood from the fact that the sextic potential $X_A X^B X_C X^A X_B X^C$ in $\mathcal{L}_{pot}$ cannot be obtained from the off-shell Lagrangian by replacing F by its solution.

In the following we will put the theory on a curved manifold. More specifically, to go from flat to curved spacetime one needs to:

- covariantize all derivatives,
- introduce additional terms $\frac{1}{3}\Omega_{AB}X^B\gamma^\mu\nabla_\mu\epsilon$ and $\frac{1}{3}\Omega^{AB}X_B\gamma^\mu\nabla_\mu\epsilon$ in the transformations of Ψ_A and Ψ^A, respectively,
- have ϵ satisfy the conformal Killing spinor equation

$$\nabla_\mu\epsilon = \gamma_\mu\eta \ , \tag{2.18}$$

  where η is some arbitrary spinor,
- add a scalar-curvature coupling term, $-\frac{1}{8}RX^AX_A$, to the Lagrangian.

Explicitly:

$$\begin{aligned}
\delta\Psi_A \to \delta\Psi_A &= \Omega_{AB}\gamma^\mu\epsilon D_\mu X^B + \frac{1}{3}\Omega_{AB}X^B\gamma^\mu\nabla_\mu\epsilon - i\Omega_{AB}F^B\epsilon \ , \\
\delta\Psi^A \to \delta\Psi^A &= \Omega^{AB}\gamma^\mu\epsilon D_\mu X_B + \frac{1}{3}\Omega^{AB}X_B\gamma^\mu\nabla_\mu\epsilon - i\Omega^{AB}F_B\epsilon \ ,
\end{aligned} \tag{2.19}$$

$$\mathcal{L}_{kin} \to \mathcal{L}_{kin} = \frac{1}{2\pi}\mathrm{tr}\left\{-D^\mu X^A D_\mu X_A - \frac{1}{8}RX^AX_A + i\tilde{\Psi}_A\gamma^\mu D_\mu\Psi^A - F^AF_A\right\} \ . \tag{2.20}$$

The resulting curved-space Lagrangian will be used in the next section.

2.3 Localization

In order to apply the localization procedure, the theory must be invariant under the action of a fermionic symmetry δ which is nilpotent, $\delta^2 = 0$, or more generally squares to a symmetry of the theory. Deforming the action by a δ-exact term,

$$S \longrightarrow S + t\delta V \ , \tag{2.21}$$

leaves invariant the expectation values of δ-closed operators. Hence we may take the limit $t \to \infty$, upon which the theory localizes to the set Σ of critical points of δV [27]. In this limit the path integral can be performed by restricting S to Σ and computing a one-loop determinant describing the fluctuations normal to Σ. This procedure was first carried out in detail in [5] for the case of SYM on the round S^4.

In order for the path integral to be well-defined, we will consider the theory in Euclidean signature. All fields are then complexified, while the action becomes a holomorphic functional in the space of complexified fields. This procedure is known under the name of "holomorphic complexification" and ensures that supersymmetry is preserved, see e.g. [30]. Following [5] our strategy will be to choose a path-integration contour in the space of fields, such that when restricted to that contour the deformation δV becomes a sum of positive semi-definite terms. The locus Σ will then be determined by the condition that each term in the sum vanishes.

2.3.1 Setup

As explained above, in order to apply the localization procedure we need to pass from Lorentzian to Euclidean signature, where all fields become complex. Moreover $\epsilon^{\mu\nu\rho}$ in the CS piece of the Lagrangian becomes $i\epsilon^{\mu\nu\rho}$.

We then deform the action by adding a term $t\delta V$ such that $\delta^2 V = 0$. For theories with

$\mathcal{N} \geqslant 2$ supersymmetry, one can have $\delta^2 = 0$ on all fields of the theory. However, this is not possible for the $\mathcal{N} = 1$ superalgebra. Instead, as we will show later, for $\mathcal{N} = 1$ we can require that δ squares to a transformation in the isometry group of the manifold, which in turn leads to $\delta^2 V = 0$ upon volume integration.

Furthermore we must restrict the supersymmetry parameter ϵ to satisfy the Killing spinor equation‡

$$\nabla_\mu \epsilon = S \gamma_\mu \epsilon \ , \tag{2.22}$$

where S is in general a complex function. The reason for restricting to this Killing spinor equation instead of the more general one (2.18) is the following. Equation (2.18) would in general imply that δ^2 induces not only a translation, a rotation and a gauge transformation but also a dilatation, which would break the invariance of the deformation δV.

Under the assumption of smoothness, any solution to the Killing spinor equation which is not identically zero is nowhere-vanishing on the manifold. This follows from the fact that (2.22) is a first-order differential equation, hence if the Killing spinor vanishes at any one point it must vanish everywhere.

Given a nowhere-vanishing Killing spinor ϵ, any spinor Ψ can be decomposed as follows

$$\Psi = \Psi_+ \epsilon + \Psi_- \epsilon^c \ , \tag{2.23}$$

where $\Psi_\pm$ are anticommuting scalars, and in Euclidean signature we have defined $\epsilon^c \equiv C\epsilon^*$. From now on we require the supersymmetry parameters to be commuting. The off-shell Lagrangian given in section 2 remains invariant under supersymmetry with these commuting parameters. With the above definitions the supersymmetric transformations

‡A detailed analysis of this Killing spinor equation in Lorentzian signature is given in section 3 of [31].

can be rewritten as

$$
\begin{aligned}
\delta X_A =& ia\Omega_{AB}\Psi^B_-\\
\delta X^A =& ia\Omega^{AB}\Psi_{B-}\\
\delta\Psi_{A-} =& \frac{1}{a}\Omega_{AB}V^\mu D_\mu X^B\\
\delta\Psi_{A+} =& \frac{1}{a}\Omega_{AB}U^\mu D_\mu X^B + S\Omega_{AB}X^B - i\Omega_{AB}F^B\\
\delta\Psi^A_- =& \frac{1}{a}\Omega^{AB}V^\mu D_\mu X_B\\
\delta\Psi^A_+ =& \frac{1}{a}\Omega^{AB}U^\mu D_\mu X_B + S\Omega^{AB}X_B - i\Omega^{AB}F_B\\
\delta F_A =& -\Omega_{AB}V^\mu D_\mu\Psi^B_+ + \Omega_{AB}U^\mu D_\mu\Psi^B_-\\
&+ 3S^* a\Omega_{AB}\Psi^B_- - iaX_A\hat{\lambda}_- + ia\lambda_- X_A\\
\delta F^A =& -\Omega^{AB}V^\mu D_\mu\Psi_{B+} + \Omega^{AB}U^\mu D_\mu\Psi_{B-}\\
&+ 3S^* a\Omega^{AB}\Psi_{B-} - iaX^A\lambda_- + ia\hat{\lambda}_- X^A \ ,
\end{aligned}
\tag{2.24}
$$

and for the gauge multiplets

$$
\begin{aligned}
\delta A_\mu =& -iV_\mu\lambda_+ + iU_\mu\lambda_-\\
\delta\hat{A}_\mu =& -iV_\mu\hat{\lambda}_+ + iU_\mu\hat{\lambda}_-\\
\delta\lambda_+ =& -\frac{1}{2a}i\epsilon^{\mu\nu\rho}U_\rho F_{\mu\nu}\\
\delta\lambda_- =& -\frac{1}{2a}i\epsilon^{\mu\nu\rho}V_\rho F_{\mu\nu}\\
\delta\hat{\lambda}_+ =& -\frac{1}{2a}i\epsilon^{\mu\nu\rho}U_\rho\hat{F}_{\mu\nu}\\
\delta\hat{\lambda}_- =& -\frac{1}{2a}i\epsilon^{\mu\nu\rho}V_\rho\hat{F}_{\mu\nu} \ ,
\end{aligned}
\tag{2.25}
$$

where

$$
\begin{aligned}
&a \equiv \epsilon^\dagger \epsilon = \tilde{\epsilon}\epsilon^c = -\tilde{\epsilon^c}\epsilon\ , \ \ V^\mu \equiv \tilde{\epsilon}\gamma^\mu \epsilon\ , \\
&U^\mu \equiv \epsilon^\dagger \gamma^\mu \epsilon = -\tilde{\epsilon}\gamma^\mu \epsilon^c\ , \ \ \nabla_\mu \epsilon^c = -S^* \gamma_\mu \epsilon^c\ .
\end{aligned}
\tag{2.26}
$$

Note that Ψ_-^A, Ψ_+^A, Ψ_{A-}, Ψ_{A+}, λ_- and λ_+ are anticommuting; so is the supersymmetry transformation δ. With the above setup, we find

$$
\begin{aligned}
&\delta^2 X_A = -iV^\mu D_\mu X_A \\
&\delta^2 \Psi_-^A = -iV^\mu D_\mu \Psi_-^A \\
&\delta^2 \Psi_+^A = -iV^\mu D_\mu \Psi_+^A - 2ia(S-S^*)\Psi_-^A \\
&\delta^2 F_A = -V^\mu D_\mu F_A + V^\mu \partial_\mu S X_A\ ,
\end{aligned}
\tag{2.27}
$$

and

$$
\begin{aligned}
&\delta^2 A_\mu = -iV^\nu F_{\nu\mu} \\
&\delta^2 \hat{A}_\mu = -iV^\nu \hat{F}_{\nu\mu} \\
&\delta^2 \lambda_- = -iV^\mu D_\mu \lambda_- \\
&\delta^2 \lambda + = -iV^\mu D_\mu \lambda_+ - 2ia(S-S^*)\lambda_- \\
&\delta^2 \hat{\lambda}_- = -iV^\mu D_\mu \hat{\lambda}_- \\
&\delta^2 \hat{\lambda} + = -iV^\mu D_\mu \hat{\lambda}_+ - 2ia(S-S^*)\hat{\lambda}_-\ .
\end{aligned}
\tag{2.28}
$$

Equivalently, written in terms of the original fields, two supersymmetry transformations give

$$\begin{aligned}\delta^2 X_A &= -iV^\mu D_\mu X_A \\ \delta^2 \Psi^A &= -iV^\mu D_\mu \Psi^A - iSV^\mu \gamma_\mu \Psi^A \\ \delta^2 F_A &= -iV^\mu D_\mu F_A + V^\mu \partial_\mu S X_A \ ,\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\delta^2 A_\mu &= -iV^\nu F_{\nu\mu} \\ \delta^2 \hat{A}_\mu &= -iV^\nu \hat{F}_{\nu\mu} \\ \delta^2 \lambda &= -iV^\mu D_\mu \lambda - iSV^\mu \gamma_\mu \lambda \\ \delta^2 \hat{\lambda} &= -iV^\mu D_\mu \hat{\lambda} - iSV^\mu \gamma_\mu \hat{\lambda} \ .\end{aligned} \tag{2.30}$$

V^μ can be identified as part of the orthonormal frame that trivializes the tangent bundle of the manifold. Therefore, apart from additional terms which can be interpreted as gauge transformations or rotations, δ^2 acting on each field gives a translation along V^μ.

In the next subsection we will ultimately set $a = 1$ and $S = 0$, upon which the above equations simplify further.

2.3.2 Deformations

Matter Sector:

To localize the matter sector, we first consider the deformation,

$$\delta V = \int \sqrt{g} d^3x \delta[(\delta \Psi_A)^\dagger \Psi_A] \ , \tag{2.31}$$

where we have *defined*

$$(\delta \Psi_A)^\dagger \equiv \Omega^{AB} \epsilon^\dagger \gamma^\mu D_\mu X_B + S^* \Omega^{AB} X_B \epsilon^\dagger + i\Omega^{AB} F_B \epsilon^\dagger \ . \tag{2.32}$$

Note that at generic points in field space $(\delta\Psi_A)^\dagger$ is *not* the adjoint of $\delta\Psi_A$, and δV as defined in (2.31) is a holomorphic functional in the space of complexified fields.

As explained in the beginning of section 2.3, we will choose a path-integration contour $\mathcal{C}$ in the space of fields such that when restricted to $\mathcal{C}$ the deformation δV becomes a sum of positive semi-definite terms. This requirement selects $\mathcal{C}$ as the subspace where the fields satisfy the reality condition

$$\text{Contour } \mathcal{C} : \quad \begin{gathered} X^{A\dagger} = X_A \;, \quad F^{A\dagger} = F_A \;, \\ A_\mu^\dagger = A_\mu \;, \quad \hat{A}_\mu^\dagger = \hat{A}_\mu \;. \end{gathered} \tag{2.33}$$

Moreover the integrand in (2.31) is given by

$$\delta[(\delta\Psi_A)^\dagger \Psi_A] = \delta(\delta\Psi_A)^\dagger \Psi_A + (\delta\Psi_A)^\dagger \delta\Psi_A \;. \tag{2.34}$$

Recall that the supersymmetry transformation δ is anticommuting; the relative sign on the right-hand side is positive since $(\delta\Psi_A)^\dagger$ is bosonic.

Let us now verify that the deformation is δ-closed. From (2.34) we obtain

$$\begin{aligned} \delta^2[(\delta\Psi_A)^\dagger \Psi_A] =& \delta^2(\delta\Psi_A)^\dagger \Psi_A - \delta(\delta\Psi_A)^\dagger \delta\Psi_A + \delta(\delta\Psi_A)^\dagger \delta\Psi_A + (\delta\Psi_A)^\dagger \delta^2\Psi_A \\ =& \delta^2(\delta\Psi_A)^\dagger \Psi_A + (\delta\Psi_A)^\dagger \delta^2\Psi_A \;. \end{aligned} \tag{2.35}$$

The second term in the second line can be read off from (2.29). One can obtain the first term in the second line from (2.29) and (2.32)

$$\begin{aligned} \delta^2(\delta\Psi_A)^\dagger \Psi_A =& - iV^\mu D_\mu[(\delta\Psi_A)^\dagger]\Psi_A + iS^* V^\mu (\delta\Psi_A)^\dagger \gamma_\mu \Psi_A \\ & + 2iS^* \Omega^{AB} V_\mu D_\nu X_B \epsilon^\dagger \gamma^{\mu\nu} \Psi_A - 2iS\Omega^{AB} V_\mu D_\nu X_B \epsilon^\dagger \gamma^{\mu\nu} \Psi_A \;, \end{aligned} \tag{2.36}$$

where we used $\nabla_\mu \epsilon^c = -S^* \gamma_\mu \epsilon^c$ and chose S to be a constant. Finally,

$$\begin{aligned}\delta^2[(\delta\Psi_A)^\dagger \Psi_A] = & - iV^\mu \partial_\mu [(\delta\Psi_A)^\dagger \Psi_A] \\ & + iS^* V^\mu (\delta\Psi_A)^\dagger \gamma_\mu \Psi_A - iSV^\mu (\delta\Psi_A)^\dagger \gamma_\mu \Psi_A \\ & + 2iS^* \Omega^{AB} V_\mu D_\nu X_B \epsilon^\dagger \gamma^{\mu\nu} \Psi_A - 2iS\Omega^{AB} V_\mu D_\nu X_B \epsilon^\dagger \gamma^{\mu\nu} \Psi_A \, .\end{aligned} \tag{2.37}$$

This vanishes under the volume integration if and only if S is real constant. On the other hand the integrability condition of the Killing spinor (2.22) relates the constant S to the curvature scalar of the manifold

$$R = -24 S^2 \, . \tag{2.38}$$

If S is nonvanishing, this would allow hyperbolic space as a solution. In the following we will discard this possibility and instead demand that the manifold should be compact, in order to ensure that the partition function is well-defined.

On T^3, the curvature scalar vanishes and so does S. This implies that the Killing spinor is constant and nowhere-vanishing. Moreover, in (2.29) and (2.30), with vanishing S terms, δ^2 gives a translation and a gauge transformation on all fields. δ-exactness and δ-closedness of the deformation are thus guaranteed.

We will henceforth restrict the manifold to be T^3. We normalize the constant Killing spinor such that $\tilde{\epsilon}\epsilon^c = 1$. The bosonic part of the deformation (2.34) is

$$\begin{aligned}(\delta\Psi_A)^\dagger \delta\Psi_A = & D_\mu X_A D^\mu X^A + i\epsilon^{\mu\nu\rho} U_\rho D_\mu X_A D_\nu X^A + F_A F^A \\ & + iU^\mu D_\mu X^A F_A - iU^\mu D_\mu X_A F^A \, ,\end{aligned} \tag{2.39}$$

where U_μ is a real unit vector, which we may choose to be along the third direction of T^3 without loss of generality. When restricted to the contour $\mathcal{C}$, cf. (2.33), the bosonic part of the deformation is positive semi-definite, and the saddle points where it vanishes are given by

$$D_1 X_A + iD_2 X_A = 0 \ , \quad D_3 X - iF = 0 \ . \tag{2.40}$$

Hence with this deformation alone the theory does not reduce to a matrix integral with discrete saddle points: one can always choose some nontrivial functions for X_A and F so that (2.40) is satisfied. We therefore add another term $\delta[(\delta\Psi^A)^\dagger \Psi^A]$ to the original deformation

$$\begin{aligned}(\delta\Psi_A)^\dagger \delta\Psi_A + (\delta\Psi^A)^\dagger \delta\Psi^A =& D_\mu X_A D^\mu X^A + i\epsilon^{\mu\nu\rho} U_\rho D_\mu X_A D_\nu X^A + F_A F^A \\ &+ iU^\mu D_\mu X^A F_A - iU^\mu D_\mu X_A F^A \\ &+ D_\mu X_A D^\mu X^A - i\epsilon^{\mu\nu\rho} U_\rho D_\mu X_A D_\nu X^A + F_A F^A \\ &- iU^\mu D_\mu X^A F_A + iU^\mu D_\mu X_A F^A \\ =& 2\{D_\mu X_A D^\mu X^A + F_A F^A\} \ . \end{aligned} \tag{2.41}$$

When restricted to the contour $\mathcal{C}$, the two terms in the last line are both positive semi-definite, and the critical points are given by

$$D_\mu X_A = F_A = 0 \ . \tag{2.42}$$

Gauge Sector:

A δ-closed deformation for the gauge sector is

$$\int d^3x \left\{ \delta[(\delta\lambda)^\dagger \lambda] + \delta[(\delta\hat{\lambda})^\dagger \hat{\lambda}] \right\} \ , \tag{2.43}$$

where we have *defined*

$$(\delta\lambda)^\dagger \equiv -\epsilon^\dagger\gamma_{\mu\nu}F^{\mu\nu}\ ; \quad (\delta\hat{\lambda})^\dagger \equiv -\epsilon^\dagger\gamma_{\mu\nu}\hat{F}^{\mu\nu}\ , \tag{2.44}$$

so that the deformation (2.43) is a holomorphic functional of the complexified fields. Note in particular that $(\delta\lambda)^\dagger$ is *not* the adjoint of $\delta\lambda$ at generic points in field space, but only when restricted to the contour $\mathcal{C}$, cf. (2.33).

The bosonic part of the deformation (2.43) is given by

$$(\delta\lambda)^\dagger\delta\lambda + (\delta\hat{\lambda})^\dagger\delta\hat{\lambda} = \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}\ . \tag{2.45}$$

When restricted to the contour $\mathcal{C}$ this becomes a sum of positive semi-definite terms, with critical points given by

$$F_{\mu\nu} = \hat{F}_{\mu\nu} = 0\ . \tag{2.46}$$

2.3.3 Gauge Fixing

We now introduce the usual ghost and anti-ghost action to fix the infinite degrees of freedom of the gauge fields. The ghost term is not invariant under supersymmetry, so one cannot immediately proceed to do localization. To deal with this, we follow [5, 6], and introduce a new fermionic symmetry Δ

$$\Delta \equiv \delta_Q + \delta_B\ , \tag{2.47}$$

where δ_Q stands for supersymmetry and δ_B for BRST transformation.

Under a BRST transformation, we have

$$\delta_B A_\mu = \partial_\mu C + i[A_\mu, C]\ , \quad \delta_B\lambda = -i\{\lambda, C\}\ . \tag{2.48}$$

and similarly for $\hat{A}$, $\hat{\lambda}$. Here C is the usual anti-commuting ghost field. It transforms under supersymmetry and BRST as

$$\delta_Q C = 0\ ,\ \ \Delta C = \delta_B C = a_0 - \frac{i}{2}\{C, C\}\ ,\ \ \Delta a_0 = 0\ , \tag{2.49}$$

where a_0 is a constant ghost-for-ghost field that takes care of the zero mode of C. With this combined transformation, one can verify that

$$\begin{aligned} \Delta^2 A_\mu =& - iV^\nu F_{\nu\mu} + i[A_\mu, a_0]\ , \\ \Delta^2 \lambda =& - iV^\mu D_\mu \lambda + i[\lambda, a_0]\ , \\ \Delta^2 C =& i[C, a_0]\ . \end{aligned} \tag{2.50}$$

The rest of the ghost complex transforms under Δ as

$$\begin{aligned} &\Delta\bar{C} = b\ ,\ \ \Delta b = -iV\cdot D\bar{C} + i[\bar{C}, a_0]\ , \\ &\Delta\bar{a}_0 = \bar{C}_0\ ,\ \ \Delta\bar{C}_0 = i[\bar{a}_0, a_0]\ , \\ &\Delta b_0 = C_0\ ,\ \ \Delta C_0 = [V\cdot A, b_0] + [A_\mu, \partial^\mu(V\cdot A)] - i\,\Box(V\cdot A) + i[b_0, a_0]\ , \end{aligned} \tag{2.51}$$

where $\bar{C}$ is the anti-ghost, and b is the Lagrangian multiplier; $\bar{a}_0$,b_0,C_0 and $\bar{C}_0$ are constant fields needed to fix the zero modes of the ghosts and b.

The gauge-fixing action is

$$\begin{aligned} & i\int d^3x \mathrm{tr}\{\Delta[\bar{C}(\partial^\mu A_\mu + b_0) - C\bar{a}_0]\} \\ =& i\int d^3x \mathrm{tr}\{b(\partial^\mu A_\mu + b_0) - \bar{C}(\partial^\mu D_\mu C + \partial^\mu \delta_Q A_\mu + C_0) \\ & - (a_0 - \frac{i}{2}\{C, C\})\bar{a}_0 + C\bar{C}_0\}\ . \end{aligned} \tag{2.52}$$

Note that the ghost, the anti-ghost and the transformation Δ are all anti-commuting.

It can be shown that the integration over all fields in the ghost complex gives the Lorentz gauge. Now this action is invariant under Δ transformation

$$\begin{aligned}
&\Delta^2[\bar{C}(\partial^\mu A_\mu + b_0) - C\bar{a}_0] \\
=&\Delta^2(\bar{C})(\partial^\mu A_\mu + b_0) + \bar{C}(\partial^\mu \Delta^2(A_\mu) + \Delta^2(b_0)) \\
&- \Delta^2(C)\bar{a}_0 - C\Delta^2(\bar{a}_0) \\
=&\Delta^2(\bar{C})(\partial^\mu A_\mu + b_0) + \bar{C}(\partial^\mu \Delta^2(A_\mu) + \Delta^2(b_0)) \\
&- i[C, a_0]\bar{a}_0 - iC[\bar{a}_0, a_0]\ .
\end{aligned} \tag{2.53}$$

The last two terms cancel under the trace. The first two can also be shown to cancel

$$\begin{aligned}
&\int d^3x \mathrm{tr}\{\Delta^2(\bar{C})(\partial^\mu A_\mu + b_0) + \bar{C}(\partial^\mu \Delta^2(A_\mu) + \Delta^2(b_0))\} \\
=&\int d^3x \mathrm{tr}\{(-iV\cdot D\bar{C} + i[\bar{C}, a_0])(\partial^\mu A_\mu + b_0) \\
&+ \bar{C}\partial^\mu(-iV^\nu F_{\nu\mu} + i[A_\mu, a_0]) \\
&+ \bar{C}([V\cdot A, b_0] + [A_\mu, \partial^\mu(V\cdot A)] - i\,\Box(V\cdot A) + i[b_0, a_0])\} \\
=&\int d^3x \mathrm{tr}\{i[\bar{C}, a_0](\partial\cdot A + b_0) + i\bar{C}([\partial\cdot A + b_0, a_0]) \\
&- iV\cdot\partial[\bar{C}(\partial\cdot A + b_0)] + [V\cdot A, \bar{C}](\partial\cdot A + b_0) \\
&+ \bar{C}[V\cdot A, \partial\cdot A + b_0] + i\bar{C}\,\Box(V\cdot A) - i\bar{C}\,\Box(V\cdot A) \\
&\bar{C}[\partial^\mu(V\cdot A), A_\mu] + \bar{C}[A_\mu, \partial^\mu(V\cdot A)]\} \\
=&0\ .
\end{aligned} \tag{2.54}$$

2.3.4 Saddle Points

For the gauge sector we replace δ by Δ in (2.43) and modify the deformation as follows

$$\begin{aligned}\Delta V_{\text{gauge}} &= \int dx^3 \Delta \mathrm{tr}\{\frac{1}{2}\epsilon^\dagger \gamma_{\mu\nu} F_{\mu\nu} \lambda\} \\ &= \int dx^3 \mathrm{tr}\{\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - i\tilde{\lambda} \not{D} \lambda\}\ ,\end{aligned} \tag{2.55}$$

and similarly for the hatted fields. This deformation is Δ-exact and Δ-closed. For the matter sector, Δ is defined to be the same as δ, and the deformation is

$$\begin{aligned}\Delta V_{\text{matter}} =& \int dx^3 \mathrm{tr}\{\Delta[(\Delta\Psi^A)^\dagger \Psi^A + (\Delta\Psi_A)^\dagger \Psi_A]\} \\ =& 2\int dx^3 \mathrm{tr}\{D_\mu X_A D^\mu X^A + F_A F^A - i\tilde{\Psi}^A \not{D} \Psi_A \\ &+ \Omega^{AB} \tilde{\lambda} X_B \Psi_A + \Omega_{AB} \tilde{\hat{\lambda}} X^B \Psi^A - \Omega^{AB} X_B \tilde{\hat{\lambda}} \Psi_A - \Omega_{AB} X^B \tilde{\lambda} \Psi^A\}\ .\end{aligned} \tag{2.56}$$

The gauge sector localizes to

$$F_{\mu\nu} = 0\ ; \quad \lambda = 0\ , \tag{2.57}$$

where we have restricted to the contour $\mathcal{C}$, cf. (2.33). In particular the saddle points of the gauge field correspond to flat gauge connections over the Euclidean three-torus. For a simply-connected gauge group $\pi_1(G) = 0$, such as $G = SU(N) \times SU(N)$, this implies that

$$A_\mu = c^i_\mu H_i\ , \tag{2.58}$$

where c^i's are constants and $\{H_i\}$, $i = 1, \cdots, \mathrm{rank}(G)$, is the Cartan subalgebra of G. This can be seen as follows (see e.g. [32, 33]): Since A_μ is a flat connection there exists a group element $U \in G$ such that $A_\mu = -i\partial_\mu U U^{-1}$, at least locally. In other words U needs not to be globally defined but is allowed to undergo G-valued jumps as we wind around each of the three circles of the torus. More explicitly, suppose we have a square torus of radius L parameterized by $\{x^\mu \in [0, L]\}$. The group element $U(x^1, x^2, x^3)$ obeys nontrivial, in general, boundary conditions which may be parameterized as follows,

$$\begin{aligned} U(x^1 + L, x^2, x^3) &= U(x^1, x^2, x^3)\Omega_1 \; ; \\ U(x^1, x^2 + L, x^3) &= U(x^1, x^2, x^3)\Omega_2 \; ; \\ U(x^1, x^2, x^3 + L) &= U(x^1, x^2, x^3)\Omega_3 \; , \end{aligned} \tag{2.59}$$

for some constant $\Omega_\mu \in G$. In addition, for consistency, Ω_μ must mutally commute. Indeed going once around the circle parameterized by x^μ and then once around the circle parameterized by x^ν must produce the same jump in U as when going first around the x^ν direction and then along x^μ. This implies, taking (2.59) into account,

$$[\Omega_\mu, \Omega_\nu] = 0 \; . \tag{2.60}$$

For a unitary group G, as is the case in the present work, this implies that Ω_μ can be put in the form

$$\Omega_\mu = \exp(iLc^j_\mu H_j) \; , \tag{2.61}$$

up to similarity transformation. Recalling the relation between A_μ and U we are thus led to the result cited in (2.58), provided we can show that for any set of mutally commuting Ω_μ's we can always construct a group element $U \sim \exp(ix^\mu c^j_\mu H_j)$ obeying (2.59).

The proof of the last step proceeds by showing that there is no obstruction in constructing an element $U(x^1 x^2, x^3)$ on the edges of a cube of side L such that (2.59) is satisfied. Then U can be continued on the faces of the cube provided $\pi_1(G) = 0$, and finally in the interior provided $\pi_2(G) = 0$, which holds true for $G = SU(N) \times SU(N)$.

An important observation is that the constants c^i_μ should be understood as periodic variables with periodic identification,

$$c^i_\mu \sim c^i_\mu + \frac{2\pi}{L} \ . \tag{2.62}$$

This can be seen by performing a gauge transformation generated by $U = \exp(\frac{2\pi i}{L} x^\mu H_i)$, which shifts A_μ in accordance with (2.62). On the other hand the element U thus defined is periodic[§], i.e. as we wind around the x^μ direction of the torus it forms a closed loop in group space. But since the group is simply connected U may be continuously deformed to the identity, and the gauge transformation generated by U should act trivially on all fields of the theory. We thus arrive at the identification (2.62).

It follows from the above that the c^i_μ's can be constrained to take values in $[0, \frac{2\pi}{L}]$. In particular taking the infinite-volume limit of the torus, $L \to \infty$, we conclude that the only solution to (2.57) is the trivial flat connection $A_\mu = 0$. Of course on $\mathbb{R}^3$ there is no obstruction to gauging away any flat connection of the form (2.58). The point is that we can formally reproduce this result by considering $\mathbb{R}^3$ as the infinite-volume limit of T^3.

The case of $G = U(N) \times U(N)$ presents one crucial difference: $\pi_1(U(N)) \cong \mathbb{Z}$ and thus G is not simply connected. By considering the decomposition of the algebra-valued connection along the G-generators it is not very difficult to see that we may still put the most general flat connection in the form (2.58),

$$A_\mu = c^i_\mu H_i + d_\mu J + e_\mu K \ , \tag{2.63}$$

where the first term on the right-hand side is as in the case of $SU(N) \times SU(N)$; d_μ, e_μ are constants; J, K are the two additional $u(1)$ Cartan generators coming from the

[§]We are adopting the normalization $\exp(2\pi i H_i) = 1$.

decomposition

$$u(N) \oplus u(N) \cong su(N) \oplus su(N) \oplus u(1) \oplus u(1) \ . \tag{2.64}$$

Now the previous argument which allowed us to conclude that c^i_μ are periodic does not go through for the variables d_μ, e_μ. The reason is that the gauge transformations generated by $U = \exp(\frac{2\pi i}{L} x^\mu J)$ and $U = \exp(\frac{2\pi i}{L} x^\mu K)$ form closed loops in the group space which are not contractible to the identity. Hence the gauge transformations generated by U need not act trivially on all fields of the theory.

In particular our argument that in the infinite-volume limit the only flat connection is the trivial one, does not go through in this case without additional assumptions. If we wish to recover $A = 0$ as the unique (up to gauge transformations) solution to (2.63) in the infinite-volume limit, we must impose by hand that $U = \exp(\frac{2\pi i}{L} x^\mu J)$ and $U = \exp(\frac{2\pi i}{L} x^\mu K)$ act trivially on all fields of the theory.

Finally, the matter sector localizes to the following field configurations

$$F_A = 0 \ ; \quad \Psi_A = \Psi^A = 0 \ ; \quad X_A = \text{const} \ , \tag{2.65}$$

where we have restricted to the contour $\mathcal{C}$, cf. (2.33).

2.3.5 One-loop Determinant

We will now compute the one-loop determinant from the quadratic fluctuations around the following saddle points,

$$\begin{aligned} &A_\mu = 0 \ ; \quad \lambda = 0 \ ; \\ &F_A = 0 \ ; \quad \Psi_A = \Psi^A = 0 \ ; \quad X_A = \text{const} \ , \end{aligned} \tag{2.66}$$

and similarly for $\hat{A}$, $\hat{\lambda}$, i.e. we will ignore the contributions from non-vanishing flat gauge connections, as discussed in the previous section.

The full path integral is of the form

$$\int d\varphi \exp\{iS + iS_{\text{g.f.}} - t(\Delta V_{\text{gauge}} + \frac{1}{2}\Delta V_{\text{matter}})\} \ , \tag{2.67}$$

where $iS_{\text{g.f.}}$ is the gauge-fixing action (2.52), and $\int d\varphi$ stands for integrations over all fields and ghosts; ΔV_{gauge} contains deformations for both hatted and unhatted gauge multiplets.

Next we expand the fields around the saddle points

$$X_A \to X_A^0 + \frac{1}{\sqrt{t}} X'_A \ , \quad \phi \to 0 + \frac{1}{\sqrt{t}}\phi \ . \tag{2.68}$$

Here X_A^0 is a constant field and X'_A represents the nonzero mode of X_A; ϕ stands for all fields other than X_A. The path integral (2.67) is t-independent thanks to localization. On the other hand, taking $t \to \infty$ allows us to keep only the quadratic terms in the deformation

$$\begin{aligned}
&t(\Delta V_{\text{gauge}} + \frac{1}{2}\Delta V_{\text{matter}}) \\
=&\int dx^3 \text{tr}\{\frac{1}{2}F^A_{\mu\nu}F^{A\mu\nu} - i\tilde{\lambda}\partial\!\!\!/\lambda\} + \int dx^3 \text{tr}\{\frac{1}{2}\hat{F}^A_{\mu\nu}\hat{F}^{A\mu\nu} - i\tilde{\hat{\lambda}}\partial\!\!\!/\hat{\lambda}\} \\
&+ \int dx^3 \text{tr}\{\partial_\mu X'_A \partial^\mu X'^A + X^{0A}A_\mu A^\mu X^0_A + X^0_A \hat{A}_\mu \hat{A}^\mu X^{0A} - 2X^0_A \hat{A}_\mu X^{0A} A^\mu \\
&+ F_A F^A - i\tilde{\Psi}^A \partial\!\!\!/\Psi_A + \Omega^{AB}\tilde{\lambda}X^0_B\Psi_A + \Omega_{AB}\tilde{\hat{\lambda}}X^{0B}\Psi^A - \Omega^{AB}X^0_B\tilde{\hat{\lambda}}\Psi_A - \Omega_{AB}X^{0B}\tilde{\lambda}\Psi^A\} \ ,
\end{aligned} \tag{2.69}$$

where $F^A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the linearized field strength; some terms have been eliminated using Lorentz gauge.

Determinant from Bosons:

We start with the calculation of the one-loop determinant of the bosonic part. Under

Lorentz gauge, we have

$$
\begin{aligned}
&\int d^3x \mathrm{tr}\{\frac{1}{2}F^A_{\mu\nu}F^{A\mu\nu}\} + \int dx^3 \mathrm{tr}\{\frac{1}{2}\hat{F}_{A\mu\nu}\hat{F}^{A\mu\nu}\} + \int d^3x \mathrm{tr}\{\partial_\mu X'_A \partial^\mu X'^A \\
&+ X^{A0}A_\mu A^\mu X^0_A + X^0_A \hat{A}_\mu \hat{A}^\mu X^{A0} - 2X^0_A \hat{A}_\mu X^{A0} A^\mu + F_A F^A\} \\
= &\int d^3x \mathrm{tr}\{-A_\mu \,\square A^\mu\} + \int dx^3 \mathrm{tr}\{-\hat{A}_\mu \,\square \hat{A}^\mu\} + \int d^3x \mathrm{tr}\{-X'_A \,\square X'^A \\
&+ X^{A0}A_\mu A^\mu X^0_A + X^0_A \hat{A}_\mu \hat{A}^\mu X^{A0} - 2X^0_A \hat{A}_\mu X^{A0} A^\mu + F_A F^A\} \ .
\end{aligned}
\tag{2.70}
$$

On T^3 with periodic conditions, any field φ can be expanded in terms of Fourier modes

$$
\varphi = \sum_{\vec{n}} \varphi_{\vec{n}} \exp\{i2\pi\vec{n}\cdot\vec{x}\} \ , \tag{2.71}
$$

where $\vec{n} = (n_x, n_y, n_z)$ and each n_μ runs over all integers. In addition, for the gauge field the Lorentz gauge implies that for each $\vec{n}$,

$$
n_x A_{x,\vec{n}} + n_y A_{y,\vec{n}} + n_z A_{z,\vec{n}} = 0 \ . \tag{2.72}
$$

Let us first assume $n_z \neq 0$. (We will come back to the case $n_z = 0$ in the following). Then the previous equation can be used to eliminate $A_{z,\vec{n}}$ via

$$
A_{z,\vec{n}} = -\frac{n_x}{n_z}A_{x,\vec{n}} - \frac{n_y}{n_z}A_{y,\vec{n}} \ . \tag{2.73}
$$

The gauge fields are in the adjoint representation, $A_\mu = A^a_\mu t_a$, where the generators t_a

are normalized so that $\mathrm{tr}\{t_a t_b\} = \delta_{ab}$. The gauge kinetic action becomes

$$
\begin{aligned}
&\int d^3x \mathrm{tr}\{-A_\mu \Box A^\mu\} \\
&= \int d^3x \sum_a \sum_{\vec{n}, n_z \neq 0} 4\pi^2 \vec{n}^2 \{ (\frac{n_x^2 + n_z^2}{n_z^2}) A^a_{x,-\vec{n}} A^a_{x,\vec{n}} + (\frac{n_y^2 + n_z^2}{n_z^2}) A^a_{y,-\vec{n}} A^a_{y,\vec{n}} \\
&\quad + \frac{n_x n_y}{n_z^2} A^a_{x,-\vec{n}} A^a_{y,\vec{n}} + \frac{n_x n_y}{n_z^2} A^a_{y,-\vec{n}} A^a_{x,\vec{n}} \} \ .
\end{aligned}
\tag{2.74}
$$

By symmetrizing $\vec{n}$ and $-\vec{n}$, for each pair of $(\vec{n}, -\vec{n})$ and each a, this can be written in matrix notation as follows

$$
\begin{array}{cc}
 & \begin{array}{cccc} A^a_{x,\vec{n}} & A^a_{x,-\vec{n}} & A^a_{y,\vec{n}} & A^a_{y,-\vec{n}} \end{array} \\
\begin{array}{c} A^a_{x,\vec{n}} \\ A^a_{x,-\vec{n}} \\ A^a_{y,\vec{n}} \\ A^a_{y,-\vec{n}} \end{array} &
\begin{pmatrix}
0 & \frac{n_x^2+n_z^2}{n_z^2} & 0 & \frac{n_x n_y}{n_z^2} \\
\frac{n_x^2+n_z^2}{n_z^2} & 0 & \frac{n_x n_y}{n_z^2} & 0 \\
0 & \frac{n_x n_y}{n_z^2} & 0 & \frac{n_y^2+n_z^2}{n_z^2} \\
\frac{n_x n_y}{n_z^2} & 0 & \frac{n_y^2 n_z^2}{n_z^2} & 0
\end{pmatrix}
\end{array}
\times 4\pi^2 (\vec{n} \cdot \vec{n}) \ .
\tag{2.75}
$$

Similarly, for each $(\vec{n}, -\vec{n})$ and a, b, the potentials involving the gauge fields are

$$
\begin{aligned}
X^{A0} A^a \cdot A^b t_a t_b X^0_A : &\ \ \Gamma \times X^{A0} t_a t_b X^0_A \ , \\
X^0_A \hat{A}^a \cdot \hat{A}^b \hat{t}_a \hat{t}_b X^{A0} : &\ \ \Gamma \times X^0_A \hat{t}_a \hat{t}_b X^{A0} \ , \\
-2 X^0_A \hat{A}^a_\mu \hat{t}_a X^{A0} A^{b\mu} t_b : &\ \ \Gamma \times -2 X^0_A \hat{t}_a X^{A0} t_b \ ,
\end{aligned}
\tag{2.76}
$$

where

$$
\Gamma \equiv
\begin{pmatrix}
0 & \frac{n_x^2+n_z^2}{n_z^2} & 0 & \frac{n_x n_y}{n_z^2} \\
\frac{n_x^2+n_z^2}{n_z^2} & 0 & \frac{n_x n_y}{n_z^2} & 0 \\
0 & \frac{n_x n_y}{n_z^2} & 0 & \frac{n_y^2+n_z^2}{n_z^2} \\
\frac{n_x n_y}{n_z^2} & 0 & \frac{n_y^2 n_z^2}{n_z^2} & 0
\end{pmatrix} \ .
\tag{2.77}
$$

The matter fields are in the bifundamental representation of the gauge group $U(N) \times$

$U(N)$. Moreover $X = \sum_{(\rho,\hat{\rho})} X^{(\rho,\hat{\rho})} |\rho\rangle \otimes |\hat{\rho}\rangle$, where $|\rho\rangle$, $|\hat{\rho}\rangle$ are representatives of the weights in each weight space; we choose the normalization so that $\langle\rho|\rho'\rangle = \delta_{\rho,\rho'}$ and $\langle\hat{\rho}|\hat{\rho}'\rangle = \delta_{\hat{\rho},\hat{\rho}'}$, in some gauge-invariant contraction of the relevant color indices. We then have

$$\begin{aligned}
X^{A0} t_a t_b X^0_A &= \sum_{(\rho,\hat{\rho})} \sum_{(\rho',\hat{\rho}')} X^{A0(\rho,\hat{\rho})} \langle\hat{\rho}| \otimes \langle\rho| \, t_a t_b \, |\rho'\rangle \, |\hat{\rho}'\rangle \, X_A^{0(\rho',\hat{\rho}')} \\
&= \sum_{\rho,\rho',\hat{\rho},\rho''} X^{A0(\rho,\hat{\rho})} \langle\rho| \, t_a \, |\rho''\rangle \langle\rho''| \, t_b \, |\rho'\rangle \, X_A^{0(\rho',\hat{\rho})} \\
&= \sum_{\rho,\rho',\hat{\rho},\rho''} X^{A0(\rho,\hat{\rho})} \sigma_a^{(\rho,\rho'')} \sigma_b^{(\rho'',\rho')} X_A^{0(\rho',\hat{\rho})} \; , \\
X^0_A \hat{t}_a \hat{t}_b X^{A0} &= \sum_{\hat{\rho},\hat{\rho}',\rho,\hat{\rho}''} X_A^{0(\rho,\hat{\rho})} \hat{\sigma}_a^{(\hat{\rho},\hat{\rho}'')} \hat{\sigma}_b^{(\hat{\rho}'',\hat{\rho}')} X^{A0(\rho,\hat{\rho}')} \; , \\
X^0_A \hat{t}_a X^{A0} t_b &= \sum_{\rho,\rho',\hat{\rho},\hat{\rho}'} X_A^{0(\rho,\hat{\rho})} \hat{\sigma}_a^{(\hat{\rho},\hat{\rho}')} X^{A0(\rho',\hat{\rho}')} \sigma_b^{(\rho',\rho)} \; ,
\end{aligned} \tag{2.78}$$

where $\sigma_a^{(\rho,\rho')} \equiv \langle\rho| \, t_a \, |\rho'\rangle$ and we used the fact that $\sum_\rho |\rho\rangle \langle\rho| = \mathbb{1}$. We then define the following matrices

$$\begin{aligned}
\mathbb{B}_{ab} &= X^{A0} t_{(a} t_{b)} X^0_A \; , \\
\mathbb{C}_{ab} &= X^0_A \hat{t}_{(a} \hat{t}_{b)} X^{A0} \; , \\
\mathbb{D}_{ab} &= -X^0_A \hat{t}_a X^{A0} t_b \; ,
\end{aligned} \tag{2.79}$$

and the deformations that are quadratic in gauge fields can be represented as

$$\begin{array}{cc} & \begin{array}{cc} A & \hat{A} \end{array} \\ \begin{array}{c} A \\ \hat{A} \end{array} & \begin{pmatrix} \mathbb{B} + 4\pi^2(\vec{n}\cdot\vec{n}) \times \mathbb{1} & \mathbb{D}^{Tr} \\ \mathbb{D} & \mathbb{C} + 4\pi^2(\vec{n}\cdot\vec{n}) \times \mathbb{1} \end{pmatrix} \end{array} \otimes \Gamma \; . \tag{2.80}$$

The determinant of the tensor product of two matrices A and B is given by

$$\det(A \otimes B) = (\det A)^{\dim B} (\det B)^{\dim A} \ . \tag{2.81}$$

Therefore, when $n_z \neq 0$, we have

$$\begin{aligned}
\det(A, \hat{A})|_{n_z \neq 0} &= \prod_{(\vec{n}, -\vec{n}), n_z \neq 0} \{(\det \mathbb{A})^4 \times (\prod_a \det \Gamma)^2\} \\
&= \prod_{(\vec{n}, -\vec{n}), n_z \neq 0} \{(\det \mathbb{A})^4 \times (\prod_a \frac{(\vec{n} \cdot \vec{n})^2}{n_z^4})^2\} \\
&= \prod_{(\vec{n}, -\vec{n}), n_z \neq 0} \{(\det[\frac{\mathbb{A}}{4\pi^2 (\vec{n} \cdot \vec{n})}])^4 \times (\prod_a \frac{(4\pi^2)^4 (\vec{n} \cdot \vec{n})^6}{n_z^4})^2\} \\
&= \prod_{\vec{n}, n_z \neq 0} \{(\det[\frac{\mathbb{A}}{4\pi^2 (\vec{n} \cdot \vec{n})}])^2 \times (\prod_a \frac{16\pi^4 (\vec{n} \cdot \vec{n})^3}{n_z^2})^2\} \ ,
\end{aligned} \tag{2.82}$$

where

$$\mathbb{A} \equiv \begin{pmatrix} \mathbb{B} + 4\pi^2 (\vec{n} \cdot \vec{n}) \times \mathbb{1} & \mathbb{D}^{Tr} \\ \mathbb{D} & \mathbb{C} + 4\pi^2 (\vec{n} \cdot \vec{n}) \times \mathbb{1} \end{pmatrix} \ . \tag{2.83}$$

For the case where $n_z = 0$, but n_x or n_y are not equal to zero, the procedure is similar. The determinant coming from integrating over A_μ reads

$$\begin{aligned}
\det(A, \hat{A}) = &\prod_{\vec{n}} (\det(\frac{\mathbb{A}}{4\pi^2 (\vec{n} \cdot \vec{n})}))^2 \prod_a \{ \prod_{\vec{n}, n_z \neq 0} [16\pi^4 \frac{(\vec{n} \cdot \vec{n})^3}{n_z^2}]^2 \\
&\times \prod_{\vec{n}, n_z = 0, n_x \neq 0} [16\pi^4 \frac{(\vec{n} \cdot \vec{n})^3}{n_x^2}]^2 \prod_{\vec{n}, n_z = n_x = 0, n_y \neq 0} [16\pi^4 \frac{(\vec{n} \cdot \vec{n})^3}{n_y^2}]^2 \} \ .
\end{aligned} \tag{2.84}$$

The contribution to the one-loop determinant coming from the terms involving gauge fields is thus

$$
\begin{aligned}
Z_{1-loop}(A,\hat{A}) =& \frac{\prod_a\{\prod_{\vec{n},n_z\neq 0} n_z^2 \prod_{\vec{n},n_z=0,n_x\neq 0} n_x^2 \prod_{\vec{n},n_z=n_x=0,n_y\neq 0} n_y^2\}}{\prod_a\prod_{\vec{n}} 16\pi^4(\vec{n}\cdot\vec{n})^3} \\
&\times \prod_{\vec{n}} (\det[\frac{\mathbb{A}}{4\pi^2(\vec{n}\cdot\vec{n})}])^{-1} \ .
\end{aligned} \tag{2.85}
$$

One may worry about regularizing the numerator. However, we note that the gauge-fixing delta function also gives a Jacobian factor to the one-loop determinant. Indeed in the ghost action we have

$$
\begin{aligned}
&\exp\{i\int d^3x \mathrm{tr}(b\partial^\mu A_\mu)\} \\
=&\exp\{i2\pi\sum_{\vec{n}}\sum_a b^a_{-\vec{n}}(\vec{n}\cdot\vec{A}^a_{\vec{n}})\} \ .
\end{aligned} \tag{2.86}
$$

After integrating out $b^a_{\vec{n}}$ we obtain

$$
\prod_{\vec{n}}\prod_a \delta(\vec{n}\cdot\vec{A}^a_{\vec{n}}) \ . \tag{2.87}
$$

This product of delta functions imposes the gauge-fixing Lorentz condition and, upon integrating out A_μ, $\hat{A}_\mu$, gives a Jacobian factor which cancels the numerator of (2.85).

The integral over F_A simply contributes an overall constant factor. Finally we are left with the integration over X'_A

$$
\begin{aligned}
&\int d^3x \mathrm{tr}\{-X'_A \Box X'^A\} \\
=&\sum_{\vec{n}} 4\pi^2\vec{n}^2 \mathrm{tr}\{X'_{A,-\vec{n}}X'^A_{\vec{n}}\} \\
=&\sum_{\vec{n}} 2\pi^2\vec{n}^2 \mathrm{tr}\{X'_{A,-\vec{n}}X'^A_{\vec{n}} + X'^A_{\vec{n}}X'_{A,-\vec{n}}\} \\
=&\sum_{(\vec{n},-\vec{n})} 2\pi^2\vec{n}^2 \mathrm{tr}\{X'_{A,-\vec{n}}X'^A_{\vec{n}} + X'_{A,\vec{n}}X'^A_{-\vec{n}} \\
&+ X'^A_{\vec{n}}X'_{A,-\vec{n}} + X'^A_{-\vec{n}}X'_{A,\vec{n}}\}\ .
\end{aligned}
\tag{2.88}
$$

This integration is Gaussian, and the corresponding determinant is

$$
\det X'_A = \prod_A \prod_{(\rho,\hat{\rho})} \prod_{\vec{n}} (2\pi^2\vec{n}^2)^2\ ,
\tag{2.89}
$$

where $(\rho,\hat{\rho})$ runs over the weights of the bifundamental representation. Therefore, the total contribution of the bosonic part to the one-loop determinant reads

$$
\begin{aligned}
Z_{1-loop}(Boson) =& \frac{1}{\{\prod_a \prod_{\vec{n}} 16\pi^4(\vec{n}^2)^3\}|_{A,\hat{A}}\{\prod_A \prod_{(\rho,\hat{\rho})} \prod_{\vec{n}} 2\pi^2\vec{n}^2\}|_{X'}} \\
&\times \prod_{\vec{n}} (\det[\frac{\mathbb{A}}{4\pi^2(\vec{n}\cdot\vec{n})}])^{-1} \\
=& \frac{1}{\{\prod_{\vec{n}}[16\pi^4(\vec{n}^2)^3]^d\}\{\prod_A \prod_{\vec{n}}(2\pi^2\vec{n}^2)^{w^2}\}} \\
&\times \prod_{\vec{n}} (\det[\frac{\mathbb{A}}{4\pi^2(\vec{n}\cdot\vec{n})}])^{-1}\ .
\end{aligned}
\tag{2.90}
$$

Here d is the dimension of the gauge group and w is the dimension of its fundamental representation. For $U(N)$ in particular we have $d = N^2$, $w = N$.

Determinant from Fermions:

The fermionic part of the deformation is

$$\begin{aligned}&\int dx^3 \mathrm{tr}\{-i\tilde{\lambda}\partial\!\!\!/\lambda\} + \int dx^3 \mathrm{tr}\{-i\tilde{\hat{\lambda}}\partial\!\!\!/\hat{\lambda}\} + \int dx^3 \mathrm{tr}\{-i\tilde{\Psi}^A\partial\!\!\!/\Psi_A \\ &+ \Omega^{AB}\tilde{\lambda}X^0_B\Psi_A + \Omega_{AB}\tilde{\hat{\lambda}}X^{0B}\Psi^A - \Omega^{AB}X^0_B\tilde{\hat{\lambda}}\Psi_A - \Omega_{AB}X^{0B}\tilde{\lambda}\Psi^A\}\ .\end{aligned} \tag{2.91}$$

Using the expansion $\lambda = \lambda_+\epsilon + \lambda_-\epsilon^c$ for the gaugino kinetic term, we have

$$\begin{aligned}&\int dx^3 \mathrm{tr}\{-i\tilde{\lambda}\partial\!\!\!/\lambda\} \\ =&\int dx^3 \mathrm{tr}\{-i(\lambda_+ V\cdot\partial\lambda_+ - \lambda_-\bar{V}\cdot\partial\lambda_- - \lambda_- U\cdot\partial\lambda_+ - \lambda_+ U\cdot\partial\lambda_-)\} \\ =&2\pi\sum_a\sum_{\vec{n}}\{V\cdot\vec{n}\lambda^a_{+,-\vec{n}}\lambda^a_{+,\vec{n}} - \bar{V}\cdot\vec{n}\lambda^a_{-,-\vec{n}}\lambda^a_{-,\vec{n}} - U\cdot\vec{n}\lambda^a_{-,-\vec{n}}\lambda^a_{+,\vec{n}} \\ &- U\cdot\vec{n}\lambda^a_{+,-\vec{n}}\lambda^a_{-,\vec{n}}\} \\ =&2\pi\sum_a\sum_{(\vec{n},-\vec{n})}\{(V\cdot\vec{n}\lambda^a_{+,-\vec{n}}\lambda^a_{+,\vec{n}} - \bar{V}\cdot\vec{n}\lambda^a_{-,-\vec{n}}\lambda^a_{-,\vec{n}} - U\cdot\vec{n}\lambda^a_{-,-\vec{n}}\lambda^a_{+,\vec{n}} \\ &- U\cdot\vec{n}\lambda^a_{+,-\vec{n}}\lambda^a_{-,\vec{n}}) + (-V\cdot\vec{n}\lambda^a_{+,\vec{n}}\lambda^a_{+,-\vec{n}} + \bar{V}\cdot\vec{n}\lambda^a_{-,\vec{n}}\lambda^a_{-,-\vec{n}} \\ &+ U\cdot\vec{n}\lambda^a_{-,\vec{n}}\lambda^a_{+,-\vec{n}} + U\cdot\vec{n}\lambda^a_{+,\vec{n}}\lambda^a_{-,-\vec{n}})\}\ ,\end{aligned} \tag{2.92}$$

where we symmetrized the indices $+, -$ and $\vec{n}, -\vec{n}$ of the gaugini in the last equation. For each pair of $(\vec{n}, -\vec{n})$ and each a, this can be written in matrix notation as

$$\begin{array}{cc} & \begin{array}{cccc} \lambda^a_{+,\vec{n}} & \lambda^a_{-,\vec{n}} & \lambda^a_{+,-\vec{n}} & \lambda^a_{-,-\vec{n}} \end{array} \\ \begin{array}{c} \lambda^a_{+,\vec{n}} \\ \lambda^a_{-,\vec{n}} \\ \lambda^a_{+,-\vec{n}} \\ \lambda^a_{-,-\vec{n}} \end{array} & \begin{pmatrix} 0 & 0 & -2\pi V\cdot\vec{n} & 2\pi U\cdot\vec{n} \\ 0 & 0 & 2\pi U\cdot\vec{n} & 2\pi\bar{V}\cdot\vec{n} \\ 2\pi V\cdot\vec{n} & -2\pi U\cdot\vec{n} & 0 & 0 \\ -2\pi U\cdot\vec{n} & -2\pi\bar{V}\cdot\vec{n} & 0 & 0 \end{pmatrix} \end{array}\ . \tag{2.93}$$

Similarly for the matter fermion kinetic term

$$
\begin{aligned}
&2\pi \sum_{(\vec{n},-\vec{n})} \operatorname{tr}\{(V\cdot\vec{n}\Psi^A_{+,-\vec{n}}\Psi_{A+,\vec{n}} - \bar{V}\cdot\vec{n}\Psi^A_{-,-\vec{n}}\Psi_{A-,\vec{n}} - U\cdot\vec{n}\Psi^A_{-,-\vec{n}}\Psi_{A+,\vec{n}} \\
&- U\cdot\vec{n}\Psi^A_{+,-\vec{n}}\Psi_{A-,\vec{n}}) + (-V\cdot\vec{n}\Psi^A_{+,\vec{n}}\Psi_{A+,-\vec{n}} + \bar{V}\cdot\vec{n}\Psi^A_{-,\vec{n}}\Psi_{A-,-\vec{n}} \\
&+ U\cdot\vec{n}\Psi^A_{-,\vec{n}}\Psi_{A+,-\vec{n}} + U\cdot\vec{n}\Psi^A_{+,\vec{n}}\Psi_{A-,-\vec{n}})\} \\
=&\pi \sum_{(\vec{n},-\vec{n})} \operatorname{tr}\{(V\cdot\vec{n}\Psi^A_{+,-\vec{n}}\Psi_{A+,\vec{n}} - \bar{V}\cdot\vec{n}\Psi^A_{-,-\vec{n}}\Psi_{A-,\vec{n}} - U\cdot\vec{n}\Psi^A_{-,-\vec{n}}\Psi_{A+,\vec{n}} \\
&- U\cdot\vec{n}\Psi^A_{+,-\vec{n}}\Psi_{A-,\vec{n}}) + (-V\cdot\vec{n}\Psi^A_{+,\vec{n}}\Psi_{A+,-\vec{n}} + \bar{V}\cdot\vec{n}\Psi^A_{-,\vec{n}}\Psi_{A-,-\vec{n}} \\
&+ U\cdot\vec{n}\Psi^A_{-,\vec{n}}\Psi_{A+,-\vec{n}} + U\cdot\vec{n}\Psi^A_{+,\vec{n}}\Psi_{A-,-\vec{n}})\} + (-1)\Psi^A \leftrightarrow \Psi_A\,.
\end{aligned}
\tag{2.94}
$$

The last term arises due to the symmetrization of Ψ^A and Ψ_A. When decomposed into the weight spaces, this becomes

$$
\begin{aligned}
&\pi \sum_{(\rho,\hat{\rho})}\sum_{(\vec{n},-\vec{n})} \{(V\cdot\vec{n}\Psi^{A(\rho,\hat{\rho})}_{+,-\vec{n}}\Psi^{(\rho,\hat{\rho})}_{A+,\vec{n}} - \bar{V}\cdot\vec{n}\Psi^{A(\rho,\hat{\rho})}_{-,-\vec{n}}\Psi^{(\rho,\hat{\rho})}_{A-,\vec{n}} - U\cdot\vec{n}\Psi^{A(\rho,\hat{\rho})}_{-,-\vec{n}}\Psi^{(\rho,\hat{\rho})}_{A+,\vec{n}} \\
&- U\cdot\vec{n}\Psi^{A(\rho,\hat{\rho})}_{+,-\vec{n}}\Psi^{(\rho,\hat{\rho})}_{A-,\vec{n}}) + (-V\cdot\vec{n}\Psi^{A(\rho,\hat{\rho})}_{+,\vec{n}}\Psi^{(\rho,\hat{\rho})}_{A+,-\vec{n}} + \bar{V}\cdot\vec{n}\Psi^{A(\rho,\hat{\rho})}_{-,\vec{n}}\Psi^{(\rho,\hat{\rho})}_{A-,-\vec{n}} \\
&+ U\cdot\vec{n}\Psi^{A(\rho,\hat{\rho})}_{-,\vec{n}}\Psi^{(\rho,\hat{\rho})}_{A+,-\vec{n}} + U\cdot\vec{n}\Psi^{A(\rho,\hat{\rho})}_{+,\vec{n}}\Psi^{(\rho,\hat{\rho})}_{A-,-\vec{n}})\} + (-1)\Psi^A \leftrightarrow \Psi_A\,.
\end{aligned}
\tag{2.95}
$$

For each pair of weights $(\rho,\hat{\rho})$ and each pair of $(\vec{n},-\vec{n})$, these terms can be written with the help of two matrices

$$
\begin{array}{c c}
 & \begin{array}{cccc} \Psi^{(\rho,\hat{\rho})}_{A+,\vec{n}} & \Psi^{(\rho,\hat{\rho})}_{A-,\vec{n}} & \Psi^{(\rho,\hat{\rho})}_{A+,-\vec{n}} & \Psi^{(\rho,\hat{\rho})}_{A-,-\vec{n}} \end{array} \\
\begin{array}{c} \Psi^{A(\rho,\hat{\rho})}_{+,\vec{n}} \\ \Psi^{A(\rho,\hat{\rho})}_{-,\vec{n}} \\ \Psi^{A(\rho,\hat{\rho})}_{+,-\vec{n}} \\ \Psi^{A(\rho,\hat{\rho})}_{-,-\vec{n}} \end{array} &
\begin{pmatrix}
0 & 0 & -\pi V\cdot\vec{n} & \pi U\cdot\vec{n} \\
0 & 0 & \pi U\cdot\vec{n} & \pi\bar{V}\cdot\vec{n} \\
\pi V\cdot\vec{n} & -\pi U\cdot\vec{n} & 0 & 0 \\
-\pi U\cdot\vec{n} & -\pi\bar{V}\cdot\vec{n} & 0 & 0
\end{pmatrix}
\end{array},
\tag{2.96}
$$

and

$$
\begin{array}{c} \\ \Psi^{(\rho,\hat\rho)}_{A+,\vec n} \\ \Psi^{(\rho,\hat\rho)}_{A-,\vec n} \\ \Psi^{(\rho,\hat\rho)}_{A+,-\vec n} \\ \Psi^{(\rho,\hat\rho)}_{A-,-\vec n} \end{array}
\begin{array}{c}
\begin{array}{cccc} \Psi^{A(\rho,\hat\rho)}_{+,\vec n} & \Psi^{A(\rho,\hat\rho)}_{-,\vec n} & \Psi^{A(\rho,\hat\rho)}_{+,-\vec n} & \Psi^{A(\rho,\hat\rho)}_{-,-\vec n} \end{array} \\
\begin{pmatrix} 0 & 0 & -\pi V\cdot\vec n & \pi U\cdot\vec n \\ 0 & 0 & -\pi V\cdot\vec n & \pi U\cdot\vec n \\ \pi V\cdot\vec n & -\pi U\cdot\vec n & 0 & 0 \\ -\pi U\cdot\vec n & -\pi\bar V\cdot\vec n & 0 & 0 \end{pmatrix}
\end{array} . \tag{2.97}
$$

Similarly, the Yukawa interactions can be written as

$$
\begin{aligned}
&\int dx^3 \mathrm{tr}\{\Omega^{AB}\tilde\lambda X^0_B\Psi_A + \Omega_{AB}\tilde{\hat\lambda}X^{0B}\Psi^A - \Omega^{AB}X^0_B\tilde{\hat\lambda}\Psi_A - \Omega_{AB}X^{0B}\tilde\lambda\Psi^A\} \\
&= a\sum_{\vec n}\mathrm{tr}\{(\Psi_{A+,-\vec n}\lambda_{-,\vec n} - \Psi_{A-,-\vec n}\lambda_{+,\vec n})\Omega^{AB}X^0_B + (\Psi^A_{+,-\vec n}\hat\lambda_{-,\vec n} - \Psi^A_{-,-\vec n}\hat\lambda_{+,\vec n})\Omega_{AB}X^{0B} \\
&- \Omega^{AB}X^0_B(\hat\lambda_{+,-\vec n}\Psi_{A-,\vec n} - \hat\lambda_{-,-\vec n}\Psi_{A+,\vec n}) - \Omega_{AB}X^{0B}(\lambda_{+,-\vec n}\Psi^A_{-,\vec n} - \lambda_{-,-\vec n}\Psi^A_{+,\vec n})\} \\
&= a\sum_{(\vec n,-\vec n)}\mathrm{tr}\{(\Psi_{A+,-\vec n}\lambda_{-,\vec n} - \Psi_{A-,-\vec n}\lambda_{+,\vec n})\Omega^{AB}X^0_B + (\Psi^A_{+,-\vec n}\hat\lambda_{-,\vec n} - \Psi^A_{-,-\vec n}\hat\lambda_{+,\vec n})\Omega_{AB}X^{0B} \\
&- \Omega^{AB}X^0_B(\hat\lambda_{+,-\vec n}\Psi_{A-,\vec n} - \hat\lambda_{-,-\vec n}\Psi_{A+,\vec n}) - \Omega_{AB}X^{0B}(\lambda_{+,-\vec n}\Psi^A_{-,\vec n} - \lambda_{-,-\vec n}\Psi^A_{+,\vec n}) \\
&+ (\Psi_{A+,\vec n}\lambda_{-,-\vec n} - \Psi_{A-,\vec n}\lambda_{+,-\vec n})\Omega^{AB}X^0_B + (\Psi^A_{+,\vec n}\hat\lambda_{-,-\vec n} - \Psi^A_{-,\vec n}\hat\lambda_{+,-\vec n})\Omega_{AB}X^{0B} \\
&- \Omega^{AB}X^0_B(\hat\lambda_{+,\vec n}\Psi_{A-,-\vec n} - \hat\lambda_{-,\vec n}\Psi_{A+,-\vec n}) - \Omega_{AB}X^{0B}(\lambda_{+,\vec n}\Psi^A_{-,-\vec n} - \lambda_{-,\vec n}\Psi^A_{+,-\vec n})\} .
\end{aligned} \tag{2.98}
$$

Each term, such as $\mathrm{tr}\{\Psi_{A+,-\vec n}\lambda_{-,\vec n}\Omega^{AB}X^0_B\}$ for example, can be written in terms of the algebra representations as follows

$$
\begin{aligned}
&\sum_{(\rho,\hat\rho)}\sum_{(\rho',\hat\rho')}\sum_a \Psi^{(\rho,\hat\rho)}_{A+,-\vec n}\langle\hat\rho|\langle\rho|\lambda^a_{-,\vec n}t_a|\rho'\rangle|\hat\rho'\rangle\,\Omega^{AB}X^{0(\rho',\hat\rho')}_B \\
&= \sum_{\rho,\rho',\hat\rho}\sum_a \Psi^{(\rho,\hat\rho)}_{A+,-\vec n}\lambda^a_{-,\vec n}\sigma^{(\rho,\rho')}_a\Omega^{AB}X^{0(\rho',\hat\rho)}_B ,
\end{aligned} \tag{2.99}
$$

where $\sigma^{(\rho,\rho')}_a \equiv \langle\rho|\,t_a\,|\rho'\rangle$ ($\hat\sigma^{(\hat\rho,\hat\rho')}_a \equiv \langle\hat\rho|\,\hat t_a\,|\hat\rho'\rangle$). Therefore the matrix elements for each

$\Psi_A^{(\rho,\hat\rho)}$ and each λ^a are

$$
\begin{array}{c c}
 & \begin{array}{cccc} \lambda^a_{+,\vec n} & \lambda^a_{-,\vec n} & \lambda^a_{+,-\vec n} & \lambda^a_{-,-\vec n} \end{array} \\
\begin{array}{c} \Psi^{(\rho,\hat\rho)}_{A+,\vec n} \\ \Psi^{(\rho,\hat\rho)}_{A-,\vec n} \\ \Psi^{(\rho,\hat\rho)}_{A+,-\vec n} \\ \Psi^{(\rho,\hat\rho)}_{A-,-\vec n} \end{array} &
\begin{pmatrix} 0 & 0 & 0 & [\sigma X] \\ 0 & 0 & -[\sigma X] & 0 \\ 0 & [\sigma X] & 0 & 0 \\ -[\sigma X] & 0 & 0 & 0 \end{pmatrix}
\end{array}, \tag{2.100}
$$

$$
\begin{array}{c c}
 & \begin{array}{cccc} \Psi^{(\rho,\hat\rho)}_{A+,\vec n} & \Psi^{(\rho,\hat\rho)}_{A-,\vec n} & \Psi^{(\rho,\hat\rho)}_{A-,\vec n} & \Psi^{(\rho,\hat\rho)}_{A-,-\vec n} \end{array} \\
\begin{array}{c} \lambda^a_{+,\vec n} \\ \lambda^a_{-,\vec n} \\ \lambda^a_{+,-\vec n} \\ \lambda^a_{-,-\vec n} \end{array} &
\begin{pmatrix} 0 & 0 & 0 & [\sigma X] \\ 0 & 0 & -[\sigma X] & 0 \\ 0 & [\sigma X] & 0 & 0 \\ -[\sigma X] & 0 & 0 & 0 \end{pmatrix}
\end{array}, \tag{2.101}
$$

where $[\sigma X] \equiv \frac{1}{2}\sigma_a^{(\rho,\rho')}\Omega^{AB}X_B^{0(\rho',\hat\rho)}$ and λ, Ψ are symmetrized. This explains the factor $\frac{1}{2}$ in each entry. A summation over ρ' is understood in $\sigma_a^{(\rho,\rho')}\Omega^{AB}X_B^{0(\rho',\hat\rho)}$.

The fermionic part of the deformation for each pair of $(\vec n, -\vec n)$ can be written in matrix notation as

$$
\begin{array}{c c}
 & \begin{array}{cccc} \lambda^a & \hat\lambda^{a'} & \Psi^{A(\rho,\hat\rho)} & \Psi_A^{(\rho,\hat\rho)} \end{array} \\
\begin{array}{c} \lambda^a \\ \hat\lambda^{a'} \\ \Psi^{A(\rho,\hat\rho)} \\ \Psi_A^{(\rho,\hat\rho)} \end{array} &
\begin{pmatrix} M & 0 & (X\sigma)_A & -(\sigma X)^A \\ 0 & M & -\widehat{(\sigma X)}_A & \widehat{(X\sigma)}^A \\ (X\sigma)_A & -\widehat{(\sigma X)}_A & 0 & N \\ -(\sigma X)^A & \widehat{(X\sigma)}^A & N & 0 \end{pmatrix}
\end{array}, \tag{2.102}
$$

where

$$
\begin{aligned}
M = 2N &\equiv \begin{pmatrix} 0 & 0 & -2\pi V\cdot\vec{n} & 2\pi U\cdot\vec{n} \\ 0 & 0 & 2\pi U\cdot\vec{n} & 2\pi\bar{V}\cdot\vec{n} \\ 2\pi V\cdot\vec{n} & -2\pi U\cdot\vec{n} & 0 & 0 \\ -2\pi U\cdot\vec{n} & -2\pi\bar{V}\cdot\vec{n} & 0 & 0 \end{pmatrix} , \\
(\sigma X)^A &\equiv \frac{1}{2}\sigma_a^{(\rho,\rho')}\Omega^{AB}X_B^{0(\rho',\hat{\rho})} \times \$, \\
(X\sigma)_A &\equiv \frac{1}{2}\Omega_{AB}X^{B0(\rho',\hat{\rho})}\sigma_a^{(\rho',\rho)} \times \$, \\
\widehat{(\sigma X)}_A &\equiv \frac{1}{2}\hat{\sigma}_a^{(\hat{\rho},\hat{\rho}')}\Omega_{AB}X^{B0(\rho,\hat{\rho}')} \times \$, \\
\widehat{(X\sigma)}^A &\equiv \frac{1}{2}\Omega^{AB}X_B^{0(\rho,\hat{\rho}')}\hat{\sigma}_a^{(\hat{\rho}',\hat{\rho})} \times \$,
\end{aligned}
\tag{2.103}
$$

and

$$
\$ \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{2.104}
$$

As before we have $a, a' = 1, \ldots, d$; $\rho, \hat{\rho} = 1, \ldots, w$; $A = 1, \ldots, 4$, where d is the dimension of the gauge group and w is the dimension of its fundamental representation. Therefore (2.102) is a $2d + 8w^2$ by $2d + 8w^2$ block matrix: each entry is given by one of the above four by four matrices.

The matrix (2.102) can be partitioned into four blocks

$$\begin{pmatrix} A_{8d\times 8d} & B_{8d\times 32w^2} \\ C_{32w^2\times 8d} & D_{32w^2\times 32w^2} \end{pmatrix} := \left(\begin{array}{cc:cc} M & 0 & (X\sigma)_A & -(\sigma X)^A \\ 0 & M & -\widehat{(\sigma X)}_A & \widehat{(X\sigma)}^A \\ \hdashline (X\sigma)_A & -\widehat{(\sigma X)}_A & 0 & N \\ -(\sigma X)^A & \widehat{(X\sigma)}^A & N & 0 \end{array}\right) , \tag{2.105}$$

so that the determinant reads¶

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det D \det[\mathbb{1} - D^{-1}CA^{-1}B] \,. \tag{2.106}$$

The determinants $\det A$ and $\det D$ are straightforward to compute

$$\det A = (\det M)^{2d} = [16\pi^4(\vec{n}^2)^2]^{2d} \,, \tag{2.107}$$

$$\det D = \prod_A (\det N)^{2w^2} = \prod_A [\pi^4(\vec{n}^2)^2]^{2w^2} \,. \tag{2.108}$$

Their combined contribution to the one-loop determinant is

$$\prod_{\vec{n}} \{(4\pi^2\vec{n}^2)^d \prod_A (\pi^2\vec{n}^2)^{w^2}\} \,. \tag{2.109}$$

Furthermore the integrations over the ghosts and anti-ghosts for the two gauge groups contribute $(\det \Box)^2 = \{\prod_{\vec{n}}(4\pi^2\vec{n}^2)^d\}^2$. When combined with (2.109) this gives

$$\prod_{\vec{n}} \{(4\pi^2\vec{n}^2)^{3d} \prod_A (\pi^2\vec{n}^2)^{w^2}\} \,. \tag{2.110}$$

Up to a constant factor, this partially cancels the one-loop determinant from the boson

¶We use the notation $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ for the matrices in the bosonic sector, while the matrices A, B, C, D are used for the fermion fields. We hope this doesn't cause any confusion with the $Sp(2)$ indices.

sector, (2.90). We are thus left with only X^0-dependent contributions from both boson and fermion sectors.

Inserting the localization conditions (2.66) into the off-shell Lagrangian (2.11) gives a vanishing classical contribution. Therefore the partition function is given purely by the one-loop determinant

$$Z = \int \prod_A \prod_{(\rho,\hat{\rho})} dX_A^{0(\rho,\hat{\rho})} \prod_B \prod_{(\rho',\hat{\rho}')} dX^{B0(\rho',\hat{\rho}')} \frac{\prod_{(\vec{n},-\vec{n})} \{\det[\mathbb{1} - D^{-1}CA^{-1}B]\}^{\frac{1}{2}}}{\prod_{\vec{n}} \det[\frac{\mathbb{A}}{4\pi^2(\vec{n}\cdot\vec{n})}]} \ . \quad (2.111)$$

We now make use of the Sylvester identity

$$\det[\mathbb{1} - D^{-1}CA^{-1}B] = \det[\mathbb{1} - BD^{-1}CA^{-1}] \ , \quad (2.112)$$

where the matrix on the left-hand side above is $32w^2 \times 32w^2$, while the matrix on the right-hand side is $8d \times 8d$. Using the definitions in (2.103) and (2.104), one can show that

$$\begin{aligned}
&\det[\mathbb{1} - BD^{-1}CA^{-1}] \\
=&\det[\mathbb{1} + C^{Tr}D^{-1}CA^{-1}] \\
=&\det[\mathbb{1} + \begin{pmatrix} \mathbb{B} & \mathbb{D}^{Tr} \\ \mathbb{D} & \mathbb{C} \end{pmatrix} \otimes \frac{\mathbb{S}N^{-1}\mathbb{S}M^{-1}}{2}] \\
=&\det[\mathbb{1} + \begin{pmatrix} \mathbb{B} & \mathbb{D}^{Tr} \\ \mathbb{D} & \mathbb{C} \end{pmatrix} \otimes \frac{\mathbb{1}_{4\times 4}}{4\pi^2(\vec{n}\cdot\vec{n})}] \\
=&\{\det[\frac{\mathbb{A}}{4\pi^2(\vec{n}\cdot\vec{n})}]\}^4 \ .
\end{aligned} \quad (2.113)$$

Putting this back into the one-loop determinant, we see that the fermion and boson determinants cancel exactly against each other.

2.4 Discussion

We have partially carried out the localization procedure for the $\mathcal{N} = 1$ Chern-Simons matter theory on T^3 with periodic boundary conditions. In particular we computed the contributions to the partition function from the locus of saddle points with vanishing gauge connection. As expected, restricting to this locus gives a trivial contribution to the partition function, i.e. the bosonic and fermionic contributions exactly cancel each other. Indeed evaluating the partition function on the flat torus at the trivial vacuum (vanishing gauge connection) simply counts the degrees of freedom of the theory, and for a supersymmetric theory one expects a complete cancellation. Of course the full partition function should receive contributions also from saddle points with nonvanishing flat gauge connections, which we have not computed here. We hope to return to this in the future.

Another potentially interesting direction in which this work may be generalized is by allowing for a more general Killing spinor equation than the eq. (2.18) which was used for the present analysis. This may be achieved by coupling to a supergravity background and could provide additional possibilities for spaces on which the theory localizes.

We then conclude this chapter. We will move on to a new topic, the study of HS/CFT dualities, in the next chapter.

3. ONE LOOP TESTS OF SUPERSYMMETRIC HIGHER SPIN AdS_4/CFT_3*

The problem of constructing field theories describing the consistent propagation and interaction of HS fields has a long history and is a highly non-trivial one. Especially, in the tensionless limit of string theory on flat background all the massive tower of states gets squeezed to a common zero mass level and the free theory is described by an infinite amount of massless free fields with arbitrary integer high spin [34]. Several years ago the consistent cubic vertices of massless HS fields in (A)dS were explicitly constructed by Fradkin and Vasiliev [15] and, remarkably, a fully non-linear theory of interacting higher spins in (A)dS was found by Vasiliev [35–38]. In the context of AdS/CFT, the HS theories have precisely the right structure to be dual to simple vector models at the boundary [16, 18].

In the following we briefly introduce the general Konstein-Vasiliev theories, and then proceed to the study of HS/CFT duality by carrying out the one loop tests of the free energy.

This chapter is based on the work [25] in collaboration with Dr. Yi Pang and Dr. Ergin Sezgin.

3.1 Main idea and General Konstein-Vasiliev HS theories

An important test of the HS/CFT holography is to match the free energy of the bulk theory with that of the CFT on the boundary. Assuming the bulk HS theory possesses an action formulation, the partition function evaluated on Euclidean AdS_4 can be expanded

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in terms of the Newton's constant G_N as

$$F_{\text{bulk}} = \frac{1}{G_N} F^{(0)}_{\text{bulk}} + F^{(1)}_{\text{bulk}} + G_N F^{(2)}_{\text{bulk}} + \cdots \ . \tag{3.1}$$

When the bulk Euclidean AdS_4 is the hyperbolic space H_4 whose conformal boundary is a round S^3, the free energy of the bulk HS theory should match with that of a free CFT on a round S^3. The free energy of a free CFT on S^3 takes the simple form [39]

$$F_{\text{CFT}} = N F^{(0)}_{\text{CFT}} \ , \tag{3.2}$$

where $F^{(0)}_{\text{CFT}}$ is the free energy of a single component in $U(N)$ vector model. The zeroth-order contribution $F^{(0)}_{\text{bulk}}$ has not been computed so far due to the lack of an action for Vasiliev theory with all the required properties. Matching F_{bulk} with F_{CFT} necessarily requires that F_{bulk} is proportional to $F^{(0)}_{\text{CFT}}$ at each order in the small G_N expansion and that G_N is identified in terms of N as

$$G_N^{-1} \to \gamma(N + \Delta N) \ , \tag{3.3}$$

with γ and ΔN being constants, and ΔN should be a fixed integer for a given bulk/boundary dual pair. Assuming Fronsdal type quadratic action for the massless HS fields, one loop computations have shown that these requirements are fulfilled in the conjectured duality between non-minimal and minimal Type-A theory and the bosonic $U(N)$ and $O(N)$ vector model [20]. In particular, for minimal Type-A model it is shown that $F^{(1)}_{\text{bulk}} = F^{(0)}_{\text{CFT}}$, and thus indicating $\Delta N = -1$ [20]. However, for the conjectured duality between Type-B theories in Euclidean AdS_4 and the fermionic $U(N)$ and $O(N)$ vector models on the boundary S^3 [18], these requirements are not satisfied since $F^{(1)}_{\text{bulk}}$ and $F^{(0)}_{\text{CFT}}$ are not proportional to each other. Driven by this mismatch, we extend the previous one loop tests to a wider class of Konstein-Vasiliev HS theories [24] and the corresponding free CFT duals. In particular we study the consequences of supersymmetry which combine Type-A and Type-B spectra with an infinite tower of massless HS fermions.

Spectra of Konstein-Vasiliev HS theories:

The building block for the construction of the physical spectra are the singleton representations of $SO(3,2)$

$$(\mathrm{Rac}, m)\ ,\quad (\mathrm{Di}, n)\ , \tag{3.4}$$

where m labels the fundamental representations of internal symmetry group $u(m)$, $usp(m)$ or $o(m)$ and same for n. It has been shown that the physical spectra of three types of HS theories, based on HS algebras denoted by $hu(m;n|4), ho(m;n|4), husp(m;n|4)$, are obtained from the following tensor products of the singletons

$$hu(m;n|4):\quad S\otimes\bar{S} \tag{3.5}$$

$$ho(m;n|4):(S\otimes S)_S \tag{3.6}$$

$$husp(m;n|4):(S\otimes S)_A \tag{3.7}$$

where $(\cdot)_S$ and $(\cdot)_A$ stand for symmetric and antisymmetric tensor products, respectively, and we defined

$$S:=(\mathrm{Rac}, m)\oplus(\mathrm{Di}, n)\ . \tag{3.8}$$

The resulting spectra are as follows [24]

$$
\begin{array}{lll}
hu(m;n|4) & : (m^2-1,1)\oplus(1,n^2-1)\oplus(1,1)\oplus(1,1) & s=0,1,2,3,\dots \\
& (m,\bar{n})\oplus(\bar{m},n) & s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\dots \\
ho(m;n|4) & : (\frac{1}{2}m(m-1),1)\oplus(1,\frac{1}{2}n(n-1)) & s=1,3,\dots \\
& (\frac{1}{2}m(m+1)-1,1)\oplus(1,\frac{1}{2}n(n+1)-1)\oplus(1,1)\oplus(1,1) & s=0,2,4,\dots \\
& (m,n) & s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\dots \\
husp(m;n|4) & : (\frac{1}{2}m(m+1),1)\oplus(1,\frac{1}{2}n(n+1)) & s=1,3,\dots \\
& (\frac{1}{2}m(m-1)-1,1)\oplus(1,\frac{1}{2}n(n-1)-1)\oplus(1,1)\oplus(1,1) & s=0,2,4,\dots \\
& (m,n) & s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\dots\ ,
\end{array}
\tag{3.9}
$$

These algebras contain finite dimensional superalgebras only when $m=n=2^{N/2-1}$ or $m=n=2^{(N-1)/2}$ for N being even or odd. In these cases we have the isomorphisms

$$
shs^E(\mathcal{N}|4)\cong\begin{cases} hu\left(2^{\frac{\mathcal{N}}{2}-1};2^{\frac{\mathcal{N}}{2}-1}\middle|4\right) & \mathcal{N}=2\ \mathrm{mod}\ 4\ , \\ husp\left(2^{\frac{\mathcal{N}}{2}-1};2^{\frac{\mathcal{N}}{2}-1}\middle|4\right) & \mathcal{N}=4\ \mathrm{mod}\ 8\ , \\ ho\left(2^{\frac{\mathcal{N}}{2}-1};2^{\frac{\mathcal{N}}{2}-1}\middle|4\right) & \mathcal{N}=8\ \mathrm{mod}\ 8\ . \end{cases} \tag{3.10}
$$

and

$$
shs^E(\mathcal{N}|4)\cong\begin{cases} ho\left(2^{(\mathcal{N}-1)/2};2^{(\mathcal{N}-1)/2}\middle|4\right) & \mathcal{N}=1\ \mathrm{mod}\ 8\ , \\ husp\left(2^{(\mathcal{N}-1)/2};2^{(\mathcal{N}-1)/2}\middle|4\right) & \mathcal{N}=5\ \mathrm{mod}\ 8\ . \end{cases} \tag{3.11}
$$

As for the case of $\mathcal{N}$=3 mod 4, it has been shown in [40] that it is equivalent to the case of $\mathcal{N}$=4 mod 4.

3.2 Free energies of Konstein-Vasiliev higher spin theories in AdS_4 with S^3 boundary

In this section we shall compute the free energy of Konstein-Vasiliev HS theories in AdS_4 with S^3 boundary, imposing the HS symmetry preserving boundary conditions. Free energy of bosonic HS fields in AdS_4 has been studied in [20,21,41,42]. The regularization scheme that has been used in summing over infinite tower of HS fields, however, is very complicated. Here, we employ a simpler alternate method which utilizes the character of irreducible representation of $SO(2,3)$. As an important consequence, the regularized individual spin contributions are such that the subsequent sum over infinite tower of higher spins is finite, thereby avoiding the need for additional regularization of this sum. This method was introduced in [43] to compute the one loop free energy of massive HS fields, but was not applied to the computation of the above free energies to exhibit the contributions of the infinite tower of odd and even spins separately. In what follows we shall use the alternate method to compute these contributions separately. We then generalize the method and apply it to the computation in bulk fermion sector in the subsequent subsection.

The one loop correction to the free energy is defined as $F^{(1)} = -\log Z^{(1)}$ where $Z^{(1)}$ is the one loop partition function. For HS theory with n_S real scalars, n_P pseudoscalars, n_1 copies of fields with $s = 1, 3, ..., \infty$, n_2 copies of fields with $s = 2, 4, ..., \infty$ fields and

n_F copies of spin $1/2, 3/2, ..., \infty$ fields, we have

$$
\begin{aligned}
&F^{(1)}(n_S, n_P, n_1, n_2, n_F) = \tfrac{1}{2} n_S \log\det{}_1 \mathcal{D}_B(1,0) + \tfrac{1}{2} n_P \log\det{}_2 \mathcal{D}_B(2,0) \\
&+\tfrac{1}{2} n_1 \sum_{k=0}^{\infty} \Big[\log\det\, \mathcal{D}_B(2k+2, 2k+1) - \log\det\, \mathcal{D}_B(2k+3, 2k)\Big] \qquad (3.12)\\
&+\tfrac{1}{2} n_2 \sum_{k=1}^{\infty} \Big[\log\det\, \mathcal{D}_B(2k+1, 2k) - \log\det\, \mathcal{D}_B(2k+2, 2k-1)\Big] \\
&-\tfrac{1}{2} n_F \log\det\, \mathcal{D}_F(\tfrac{3}{2}, \tfrac{1}{2}) - \tfrac{1}{2} n_F \sum_{k=1}^{\infty} \Big[\log\det\, \mathcal{D}_F(k+\tfrac{3}{2}, k+\tfrac{1}{2}) - \log\det\, \mathcal{D}_F(k+\tfrac{5}{2}, k-\tfrac{1}{2})\Big] ,
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
\mathcal{D}_B(\Delta, s) &= \left[-\nabla^2 + \Delta(\Delta-3) - s\right] , \\
\mathcal{D}_F(\Delta, s) &= \left[-\nabla\!\!\!/^2 + \Delta(\Delta-3) + \tfrac{9}{4}\right] . \qquad (3.13)
\end{aligned}
$$

The negative contributions in the bosonic sector and the positive contributions in the fermionic sector are due to ghosts. In computing $\det_1$ and $\det_2$, the irregular ($\Delta_- = 1$) and regular ($\Delta_+ = 2$) boundary conditions are to be used.

For a differential operator of the form $\mathcal{D} = -\nabla^2 + X$, or $\mathcal{D} = -\nabla\!\!\!/^2 + Y$, writing

$$
-\log\det\mathcal{D} = \int_0^\infty \frac{dt}{t} K_{\mathcal{D}}(t) , \qquad K_{\mathcal{D}}(t) := \mathrm{Tr}\left[e^{-t\mathcal{D}}\right] , \qquad (3.14)
$$

and defining the spectral zeta function

$$
\zeta_{\mathcal{D}}(z) := \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1} K_{\mathcal{D}}(t) , \qquad (3.15)
$$

one finds the standard result [44]

$$-\log\det\mathcal{D} = \zeta_{\mathcal{D}}(0)\log(\ell^2\Lambda^2) + \zeta'_{\mathcal{D}}(0)\ , \tag{3.16}$$

where ℓ is the AdS radius and Λ is the renormalization scale. For fields of aribrary spins in hyperbolic space H_4, the spectral zeta function technique has been developed in [45,46] to compute their one loop effective potentials.

3.2.1 Bosons

Upon Euclideanization of AdS_4 to H_4, the boundary is S^3 and in this setting various free energies of the bosonic HS theory are given by

$$\begin{aligned}
F^{(1)}_{\text{even}\,1} &= -\frac{1}{2}\Big[\zeta^{B}_{(1,0)}(0) + \sum_{s=2,4,\cdots}^{\infty}\Big(\zeta^{B}_{(s+1,s)}(0) - \zeta^{B}_{(s+2,s-1)}(0)\Big)\Big]\log(\ell^2\Lambda^2)\\
&\quad -\frac{1}{2}\Big[\zeta^{B\prime}_{(1,0)}(0) + \sum_{s=2,4,\cdots}^{\infty}\Big(\zeta^{B\prime}_{(s+1,s)}(0) - \zeta^{B\prime}_{(s+2,s-1)}(0)\Big)\Big]\ ,\\
F^{(1)}_{\text{even}\,2} &= -\frac{1}{2}\Big[\zeta^{B}_{(2,0)}(0) + \sum_{s=2,4,\cdots}^{\infty}\Big(\zeta^{B}_{(s+1,s)}(0) - \zeta^{B}_{(s+2,s-1)}(0)\Big)\Big]\log(\ell^2\Lambda^2)\\
&\quad -\frac{1}{2}\Big[\zeta^{B\prime}_{(2,0)}(0) + \sum_{s=2,4,\cdots}^{\infty}\Big(\zeta^{B\prime}_{(s+1,s)}(0) - \zeta^{B\prime}_{(s+2,s-1)}(0)\Big)\Big]\ ,\\
F^{(1)}_{\text{odd}} &= -\frac{1}{2}\sum_{s=1,3,\cdots}^{\infty}\Big(\zeta^{B}_{(s+1,s)}(0) - \zeta^{B}_{(s+2,s-1)}(0)\Big)\log(\ell^2\Lambda^2)\\
&\quad -\frac{1}{2}\sum_{s=1,3,\cdots}^{\infty}\Big(\zeta^{B\prime}_{(s+1,s)}(0) - \zeta^{B\prime}_{(s+2,s-1)}(0)\Big)\ ,
\end{aligned} \tag{3.17}$$

where $F^{(1)}_{\text{even}\,1}$ and $F^{(1)}_{\text{even}\,2}$ denote the total free energy of all even spin fields $s = 0, 2, 4\cdots$, in which the scalar satisfies $\Delta = 1$ and $\Delta = 2$ boundary conditions, respectively, and $F^{(1)}_{\text{odd}}$ denotes the total free energy of all odd spin fields $s = 1, 3, 5\cdots$.

As stated earlier, we now employ a simpler method than those used previously, utilizing the character of irreducible representation of $SO(2,3)$. The method is based on the observation that the spectral zeta function of a bosonic spin-s field can be recast in the form [43]

$$\zeta^B_{(\Delta,s)}(z) = \frac{1}{\Gamma(z)} \int_0^\infty d\beta \Big[\mu(z,\beta) + \nu(z,\beta)\frac{\partial^2}{\partial\alpha^2}\Big] \chi_{\Delta,s}(\beta,\alpha) \Big|_{\alpha=0} , \tag{3.18}$$

in which

$$\begin{aligned}
&\chi_{\Delta,s}(\beta,\alpha) = \frac{e^{-\beta(\Delta-\frac{3}{2})}\sin[(s+\frac{1}{2})\alpha]}{4\sinh\frac{\beta}{2}\sin\frac{\alpha}{2}(\cosh\beta-\cos\alpha)} , \\
&\mu(z,\beta) = \tfrac{1}{3}\sinh\tfrac{\beta}{2}\Big[f_1(z,\beta)\Big(-6+\sinh^2\tfrac{\beta}{2}\Big) + 4f_3(z,\beta)\sinh^2\tfrac{\beta}{2}\Big] , \\
&\nu(z,\beta) = -4f_1(z,\beta)\sinh^3\tfrac{\beta}{2} , \\
&f_n(z,\beta) = \sqrt{\pi}\int_0^\infty du u^n \tanh(\pi u)(\tfrac{\beta}{2u})^{z-\frac{1}{2}} J_{z-1/2}(u\beta) ,
\end{aligned} \tag{3.19}$$

where $\chi_{\Delta,s}(\beta,\alpha)$ is the character of a representation of $SO(3,2)$ labeled by $D(\Delta,s)$. Owing to the $e^{-\beta(\Delta-\frac{3}{2})}$ factor in the character, $\sum_s \zeta_{(\Delta,s)}(z)$ is convergent. Therefore, no regularization is needed in performing the sum over infinitely many spins. This is the desired feature for computing the one loop free energy of HS theory where the summation over infinitely many spins is encountered. It was also noticed by [43] that since the one loop free energy depends only on $\zeta(0)$ and $\zeta'(0)$, an alternate zeta function $\widetilde{\zeta}(z)$ is physically equivalent to the original $\zeta(z)$, provided that $\widetilde{\zeta}(0) = \zeta(0)$, and $\widetilde{\zeta}'(0) = \zeta'(0)$. Thus, for the convenience of calculation, one can in fact utilize an alternate zeta function which is physically equivalent to the original zeta function. For bosonic HS fields, one choice of the alternate zeta function takes the form [43]

$$\widetilde{\zeta}^B_{(\Delta,s)}(z) = \frac{1}{\Gamma(2z)} \int_0^\infty d\beta\, \beta^{2z-1} \coth\tfrac{\beta}{2} \Big[1 + \big(\sinh^2\tfrac{\beta}{2}\big)\ \partial^2_\alpha\Big] \chi_{\Delta,s}(\beta,\alpha) \Big|_{\alpha=0} . \tag{3.20}$$

The physical equivalence between the alternate spectral zeta function and the original one (3.18) is shown in the appendix. The total character of all even spin fields and that of all odd spin fields are computed as

$$\begin{aligned}
\chi_{\text{even }1}(\beta,\alpha) &= \chi_{1,0}(\beta,\alpha) + \sum_{s=2,4,\cdots}\left(\chi_{s+1,s}(\beta,\alpha) - \chi_{s+2,s-1}(\beta,\alpha)\right) \\
&= \frac{1+\cos\alpha+\cosh\beta+\cosh 2\beta}{4(\cos\alpha-\cosh\beta)^2(\cos\alpha+\cosh\beta)} , \qquad (3.21) \\
\chi_{\text{even }2}(\beta,\alpha) &= \chi_{2,0}(\beta,\alpha) + \sum_{s=2,4,\cdots}\left(\chi_{s+1,s}(\beta,\alpha) - \chi_{s+2,s-1}(\beta,\alpha)\right) \\
&= \frac{1+\cos\alpha+\cos 2\alpha+\cosh\beta}{4(\cos\alpha-\cosh\beta)^2(\cos\alpha+\cosh\beta)} , \qquad (3.22) \\
\chi_{\text{odd}}(\beta,\alpha) &= \sum_{s=1,3,\cdots}\left(\chi_{s+1,s}(\beta,\alpha) - \chi_{s+2,s-1}(\beta,\alpha)\right) \\
&= \frac{\cos\alpha+\cosh\beta+2\cos\alpha\cosh\beta}{4(\cos\alpha-\cosh\beta)^2(\cos\alpha+\cosh\beta)} . \qquad (3.23)
\end{aligned}$$

Substituting the results above into (3.20), we find

$$\begin{aligned}
\widetilde{\zeta}^{B}_{\text{even},1}(z) &= \frac{1}{\Gamma(2z)}\int_0^\infty d\beta\beta^{2z-1}\frac{\cosh^2\beta}{4\sinh^3\beta} , \\
\widetilde{\zeta}^{B}_{\text{even},2}(z) &= -\frac{1}{\Gamma(2z)}\int_0^\infty d\beta\beta^{2z-1}\frac{1+2\cosh\beta}{4\sinh^3\beta} , \\
\widetilde{\zeta}^{B}_{\text{odd}}(z) &= -\widetilde{\zeta}^{B}_{\text{even }1}(z) . \qquad (3.24)
\end{aligned}$$

With the help of the following identities

$$\begin{aligned}
\frac{1}{\sinh^3\frac{\beta}{2}} &= \frac{2}{\beta^2}\frac{\partial^2}{\partial x^2}\frac{1}{\sinh\frac{\beta x}{2}}\ |_{x=1} - \frac{1}{2\sinh\frac{\beta}{2}} , \\
4^{-z}\zeta(2z,\frac{a}{2}) &= \frac{1}{\Gamma(2z)}\int_0^\infty d\beta\beta^{2z-1}\frac{e^{-a\beta}}{1-e^{-2\beta}} , \qquad (3.25)
\end{aligned}$$

where $\zeta(a,b)$ is the Hurwitz zeta function, we finally obtain

$$\begin{aligned}
\widetilde{\zeta}^{B}_{\text{even }1}(z) &= 4^{-(2+z)}\Big[3\zeta(2z,-\tfrac{1}{2}) + 4\zeta(2z-2,-\tfrac{1}{2}) + 8\zeta(2z-1,-\tfrac{1}{2}) \\
&\qquad + (4^z-1)\zeta(2z) + 3(4^z-4)\zeta(2z-2) - 4(4^z-2)\zeta(2z-1)\Big] \ , \\
\widetilde{\zeta}^{B}_{\text{even }2}(z) &= 4^{-(1+z)}\Big[-4\zeta(2z-2,0) - 4\zeta(2z-1,0) + (4^z-1)\zeta(2z) \\
&\qquad - 4^z\zeta(2z-2) + 4\zeta(2z-1)\Big] \ .
\end{aligned} \tag{3.26}$$

By using the relation between $F^{(1)}$ and spectral zeta function, one arrives at the results

$$\begin{aligned}
F^{(1)}_{\text{even }1} &= \frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \ , \qquad F^{(1)}_{\text{even }2} = \frac{1}{16}\left(2\log 2 - \frac{5\zeta(3)}{\pi^2}\right) \ , \\
F^{(1)}_{\text{odd}} &= -F^{(1)}_{\text{even }1} \ .
\end{aligned} \tag{3.27}$$

Note that the potential logarithmic divergences in $F^{(1)}_{\text{even }1}$ and $F^{(1)}_{\text{even }2}$ have canceled out, and the above finite results are from $\widetilde{\zeta}^{B\prime}(0)$ terms, in agreement with [20]. Furthermore, these results can be used as building blocks for the computation of the free energies of the Konstein-Vasiliev models we are interested in, thanks to the observation that for all those models discussed in Section 3.1, it is always the case that

$$n_2 = n_S + n_P \ , \tag{3.28}$$

where we recall that n_2 is number of copies of even fields with $s = 2, 4, \ldots \infty$, n_S is the number of scalars and n_P is the number of pseudoscalars.

3.2.2 Fermions

We now compute the one loop free energy of all fermionic HS fields. The spectral zeta function of a spin-s fermionic fields is given by

$$\zeta^F_{(\Delta,s)}(z) = \frac{1}{\Gamma(z)} \int_0^\infty d\beta \Big[\mu(z,\beta) + \nu(z,\beta)\frac{\partial^2}{\partial\alpha^2}\Big]\chi_{\Delta,s}(\beta,\alpha)\,\Big|_{\alpha=0}\,, \tag{3.29}$$

where

$$\begin{aligned}
\chi_{\Delta,s}(\beta,\alpha) &= \frac{e^{-\beta(\Delta-\frac{3}{2})}\sin[(s+\frac{1}{2})\alpha]}{4\sinh\frac{\beta}{2}\sin\frac{\alpha}{2}(\cosh\beta-\cos\alpha)}\,,\\
\mu(z,\beta) &= \tfrac{1}{3}\sinh\tfrac{\beta}{2}\Big[f_1(z,\beta)\Big(-6+\sinh^2\tfrac{\beta}{2}\Big)+4f_3(z,\beta)\sinh^2\tfrac{\beta}{2}\Big]\,,\\
\nu(z,\beta) &= -4f_1(z,\beta)\sinh^3\tfrac{\beta}{2}\,,\\
f_n(z,\beta) &= \sqrt{\pi}\int_0^\infty du u^n\coth(\pi u)(\tfrac{\beta}{2u})^{z-\frac{1}{2}}J_{z-1/2}(u\beta)\,.
\end{aligned} \tag{3.30}$$

To compute the one loop free energy of all fermionic HS fields, we propose the following alternate spectral zeta function, which is much easier to use. The physical equivalence between the alternate spectral zeta function (3.31) and the original one (3.29) is shown in the appendix.

$$\widetilde{\zeta}^F_{(\Delta,s)}(z) = \frac{1}{\Gamma(2z)}\int_0^\infty d\beta\beta^{2z-1}\Big[\tfrac{1}{4}\sinh\tfrac{\beta}{2}+\frac{1}{\sinh\frac{\beta}{2}}+\sinh\tfrac{\beta}{2}\partial_\alpha^2\Big]\chi_{\Delta,s}(\beta,\alpha)\,\Big|_{\alpha=0} \tag{3.31}$$

The sum of characters of all fermionic HS fields is computed as

$$\chi_{\frac{3}{2},\frac{1}{2}}(\beta,\alpha)+\sum_{s=3/2}^{\infty}\Big[\chi_{s+1,s}(\beta,\alpha)-\chi_{s+2,s-1}(\beta,\alpha)\Big] = \frac{\cos\frac{\alpha}{2}\cosh\frac{\beta}{2}}{(\cos\alpha-\cosh\beta)^2}\,. \tag{3.32}$$

It is straightforward to check that

$$\left[\tfrac{1}{4}\sinh\tfrac{\beta}{2}+\frac{1}{\sinh\frac{\beta}{2}}+\left(\sinh\tfrac{\beta}{2}\right)\partial_\alpha^2\right]\times$$
$$\left(\chi_{\frac{3}{2},\frac{1}{2}}(\beta,\alpha)+\sum_{s=3/2}^{\infty}\Big[\chi_{s+1,s}(\beta,\alpha)-\chi_{s+2,s-1}(\beta,\alpha)\Big]\right)\Big|_{\alpha=0}=0\;, \tag{3.33}$$

which indicates that the total one loop free energy of fermionic HS fields in fact vanishes.

3.2.3 Summary

For a Konstein-Vasiliev higher theory consisting of n_S real scalars, n_P pseudoscalars, n_1 copies of fields with $s=1,3,...,\infty$, $n_2=n_S+n_P$ copies of fields with $s=2,4,...,\infty$ fields and n_F copies of spin $1/2,3/2,...,\infty$ fields, we have

$$F^{(1)}(n_S,n_P,n_1,n_2,n_F)=\frac{\log 2}{8}(n_S+n_P-n_1)-\frac{\zeta(3)}{16\pi^2}(3n_S+5n_P-3n_1)\;, \tag{3.34}$$

where we have used the relation $n_2=n_S+n_P$. The values of n_S, n_P and n_1 can be read off from (3.9) for various Konstein-Vasiliev models. Substituting them into the equation above, we obtain

$$hu(m;n|4):\;F^{(1)}_{hu}=-\frac{\zeta(3)}{8\pi^2}n^2\;, \tag{3.35}$$

$$ho(m;n|4):\;F^{(1)}_{ho}=\frac{\log 2}{8}(m+n)-\frac{\zeta(3)}{16\pi^2}(3m+4n+n^2)\;, \tag{3.36}$$

$$husp(m;n|4):\;F^{(1)}_{husp}=-\frac{\log 2}{8}(m+n)+\frac{\zeta(3)}{16\pi^2}(3m+4n-n^2)\;. \tag{3.37}$$

The one loop free energy of $husp(m;n|4)$ model is related to the one of $ho(m;n|4)$ model via $m\to -m$, $n\to -n$. The ordinary supersymmetric HS models correspond to the cases $m=n=2^{\frac{\mathcal{N}}{2}-1}$ for even $\mathcal{N}$ and $m=n=2^{(\mathcal{N}-1)/2}$ for odd $\mathcal{N}$.

As for the ordinary supersymmetric HS models with internal symmetries, we recall that their spectra can be obtained by assigning fundamental representations of the internal symmetry group to the $OSp(\mathcal{N}|4)$ singletons, and working out the their two-fold tensor products. The resulting spectra are provided in Table 5 of [40]. In particular, the number of fermions with $s = \frac{1}{2} \bmod 2$ and $s = \frac{3}{2} \bmod 2$ are the same. As a consequence, the contributions of the fermions to the one loop free energy will continue to vanish since in (3.31) we found that fermions with each half integer spin occurring once give vanishing contribution. Consequently, the bulk free energy becomes the sum of free energies of Type-A and Type-B models with the desired internal symmetries, and both $\log 2$ and $\zeta(3)$ terms will show up in the one loop free energy. This information is sufficient to perform the one loop test by means of comparing the bulk and boundary free energies, as we shall see at the end of next section.

3.3 Free energies of free CFT's on S^3 and comparison

The free energies of free scalars and free fermions which are conformally coupled to S^3 have been studied in [39]. A conformally coupled free scalar and a free fermion on S^3 are described by the following two actions respectively

$$S_S = \frac{1}{2}\int d^3x\sqrt{g}\Big[(\nabla\phi)^2 + \frac{3}{4L^2}\phi^2\Big] \ , \quad S_D = \frac{1}{2}\int d^3x\sqrt{g}\psi^\dagger(\mathrm{i}\not{D}\psi) \ , \tag{3.38}$$

where L is the radius of the round S^3. Free energies of the above two theories are defined as usual

$$\begin{aligned} F_S &= -\log Z_S = \frac{1}{2}\log\det[\Lambda^{-2}\mathcal{O}_S] \ , \quad \mathcal{O} = -\nabla^2 + \frac{3}{4L^2} \ , \\ F_D &= -\log Z_D = -\log\det[\Lambda^{-1}\mathcal{O}_D] \ , \quad \mathcal{O} = \mathrm{i}\not{D} \ . \end{aligned} \tag{3.39}$$

Using zeta function, F_S and F_D can be computed straightforwardly and the results are [39]

$$F_S = \frac{1}{16}\Big(2\log 2 - \frac{3\zeta(3)}{\pi^2}\Big)\,, \quad F_D = \frac{1}{8}\Big(2\log 2 + \frac{3\zeta(3)}{\pi^2}\Big)\,. \tag{3.40}$$

Notice that the free energy of a Majorana fermion on S^3 is $\frac{1}{2}F_D$.

A bulk HS theory is conjectured to be dual to a free vector model when the boundary conditions of the bulk fields preserve the HS symmetry [16, 17], which is the case here. Assuming the bulk HS theory possesses an action, its free energy associated with AdS_4 should have the form displayed in (3.1) where G_N is the Newton's constant. In cases where the boundary of AdS_4 is S^3, the bulk free energy should be compared with that of a free vector model on S^3 order by order in $1/N$ expansion. Hence the comparison requires an identification between G_N and N. It was suggested by [20] that in general the relation between G_N and N is of the form given in (3.3) where γ and ΔN are constants and especially ΔN should be an integer. The basic fields in the vector model constitute a vector in the fundamental representation of a classical Lie group, which can be $U(N)$, $O(N)$ or $USp(N)$ in our cases. The free energy of a free vector model can be computed exactly and be put in the form†

$$F_{\rm CFT} = N F^{(0)}_{\rm CFT}\,, \tag{3.41}$$

where we use $F^{(0)}_{\rm CFT}$ to denote the contribution of a single component in the vector. For $F_{\rm bulk}$ to match with $F_{\rm CFT}$, it is clear that the bulk free energy at each order in G_N expansion should all be proportional to $F^{(0)}_{\rm CFT}$.

Various one loop tests of HS holography have been carried out in the literature [20,41].

†Strictly speaking, the bulk HS theory is dual to the $U(N)$, $O(N)$ or $USp(N)$ singlet sector of a free CFT. The partition function of a free CFT on S^3 is evaluated in the vacuum which is already a singlet state under the corresponding symmetry group in each case. Thus, imposing the singlet constraint should not affect the free energy.

For instance, the non-minimal Type-A model is conjectured to be dual to the $U(N)$ singlet sector of N complex scalars. When HS symmetry is preserved by the boundary condition, $F^{(1)}_{\text{bulk}}$ was found to be 0, indicating that G_N^{-1} is identified with N at one loop order. For minimal A model, the conjectured dual CFT is the $O(N)$ singlet sector of N real scalars. In this case, $F^{(1)}_{\text{bulk}}$ is equal to F_S, the free energy of a real free scalar (3.40). Thus, matching the bulk and boundary free energies at one loop order requires G_N^{-1} being identified with $N-1$. The $husp(2;0|4)$ Vasiliev theory is conjectured to be dual to the $USp(N)$ singlet sector of N complex scalars and $F^{(1)}_{\text{bulk}}$ is equal to $-F_S$. Therefore, for $husp(2;0|4)$ higher spin theory, G_N^{-1} is identified with $N+1$ at one loop order.

In this section, we consider the cases in which the bulk HS symmetry is preserved by the boundary condition, thus the CFT duals are certain singlet sectors of free CFTs composed by free scalars and free fermions. For the $hu(m;n|4)$ theory, the dual CFT consists of Nm complex free scalars ϕ^{ia}, $i=1,2,...N$, $a=1,2,...m$ and Nn Dirac fermions ψ^{ir}, $r=1,2,...n$. The m^2 $\Delta=1$ scalars and n^2 $\Delta=2$ pseudoscalars correspond to the operators

$$\bar{\phi}_{ia}\phi^{ib}\,, \qquad \tilde{\psi^c}_{ia}\psi^{ib}\,. \tag{3.42}$$

Free energy of this theory is given by

$$F_{\text{CFT}}=NF^{(0)}_{\text{CFT}}\,,\quad F^{(0)}_{\text{CFT}}=2mF_S+nF_D\,, \tag{3.43}$$

where F_S and F_D are given in (3.40).

For the $ho(m;n|4)$ theory, the dual CFT consists of Nm real free scalars ϕ^{ia}, $i=1,2,...N$, $a=1,2,...m$ and Nn majorana fermions ψ^{ir}, $r=1,2,...n$. The m^2 $\Delta=1$

scalar fields and n^2 $\Delta = 2$ pseudoscalars correspond to the operators

$$\phi^{ia}\phi^{jb}\delta_{ij}\ , \qquad \tilde{\psi}^{ia}\psi^{jb}\delta_{ij}\ . \tag{3.44}$$

The free energy is given by

$$F_{\rm CFT} = NF^{(0)}_{\rm CFT}\ , \quad F^{(0)}_{\rm CFT} = mF_S + \tfrac{1}{2}nF_D\ . \tag{3.45}$$

For the $husp(m;n|4)$ theory, the dual CFT consists of Nm complex free scalars ϕ^{ia}, $i = 1,2,...N$, $a = 1,2,...m$ and Nn Dirac fermions ψ^{ir}, $r = 1,2,...n$, subject to the symplectic reality condition. The m^2 $\Delta = 1$ scalar fields and n^2 $\Delta = 2$ pseudoscalars correspond to the operators

$$\phi^{ia}\phi^{jb}\Omega_{ij}\ , \qquad \tilde{\psi}^{ia}\psi^{jb}\Omega_{ij}\ , \tag{3.46}$$

where Ω_{ij} is the $USp(N)$ invariant tensor. Free energy of this theory is given by

$$F_{\rm CFT} = NF^{(0)}_{\rm CFT}\ , \quad F^{(0)}_{\rm CFT} = mF_S + \tfrac{1}{2}nF_D\ . \tag{3.47}$$

Since supersymmetric HS theories can be mapped to special cases of Konstein-Vasiliev models, we will not give separate discussions on them.

As discussed before, duality between the bulk HS theory and boundary free CFT may be achieved only if $F^{(1)}_{\rm bulk}$ is proportional to $F^{(0)}_{\rm CFT}$. Using (3.34), (3.40), (3.43), (3.45) and (3.47), we find that this requirement amounts to

$$(m+n)(3n_S + 5n_P - 3n_1) = 3(m-n)(n_S + n_P - n_1)\ , \tag{3.48}$$

obtained by setting the ratios of $\log 2$ and $\xi(3)$ dependent terms equal to each other. Taking the values of n_S, n_P and n_1 from (3.9), these ratios for the bulk sides can be read off from (3.35), (3.36) and (3.37) in terms of m and n. One can show that for all three Konstein-Vasiliev models, the only solution to the equation above is given by $n = 0$, which implies bosonic Type-A models. In this case the $\log 2$ and $\zeta(3)$ dependent terms arise in the same ratio as of a single real scalar field, and we have the result

$$F^{(1)}_{hu(m;0|4)} = 0 \, , \qquad F^{(1)}_{ho(m;0|4)} = mF_S \, , \qquad F^{(1)}_{ho(m;0|4)} = -mF_S \, . \tag{3.49}$$

Therefore, assuming that $F^{(0)}_{\text{bulk}} = F^{(0)}_{\text{CFT}}$, the bulk and boundary free energies match with each other provided that

$$\begin{aligned} &hu(m;0|4) : \; G_N^{-1} \to N \, , \\ &ho(m;0|4) : \; G_N^{-1} \to N-1 \, , \\ &husp(m;0|4) : \; G_N^{-1} \to N+1 \, . \end{aligned} \tag{3.50}$$

The holographic dictionaries relating G_N to N in various HS models have been put forward in [20] via testing the holography of $hu(1;0|4)$, $ho(1;0|4)$ and $husp(2;0|4)$ models at one loop level. Here, we have extended the validity of these holographic mappings to $hu(m;0|4)$, $ho(m;0|4)$ and $husp(m;0|4)$ Konstein-Vasiliev models. We see that the inclusion of infinite tower of bulk fermions does not cure the problem with the mismatch of the free energies in the Type-B model, which corresponds to the case in which $m = 0$ and $n \neq 0$, and its conjectured dual.

Finally, we consider the ordinary supersymmetric models with internal symmetry discussed earlier, whose spectra are given in Table 5 of [40]. In Section 3 we found that the contributions of the bulk fermions give vanishing contributions to one loop free energy and consequently the bulk one loop free energy becomes the sum of the ones of Type-A and Type-B models with the desired internal symmetries. In particular, there is still a non-vanishing $\zeta(3)$ term. On the other hand it is easy to show that the $\zeta(3)$ dependent terms on the CFT side vanish. Therefore, we conclude the problem of free energy mismatch will persist in ordinary supersymmetric HS theories with internal symmetry.

3.4 One loop free energies of supersymmetric higher spin theories in AdS_4 with $S^1_\beta \times S^2$ boundary

In thermal AdS_4, the one loop free energy of the bulk theory takes the form [21]

$$F^{(1)}_{\rm bulk} = F(\beta)_{\rm bulk} + \beta E_{c\,\rm bulk} + a_{\rm bulk} \log \Lambda \ , \tag{3.51}$$

where β is the period of the imaginary time, $F(\beta)_{\rm bulk}$ is the thermal free energy which can be computed by taking the log of the thermal partition function as $F(\beta)_{\rm bulk} \equiv \beta^{-1} \log Z_{\rm bulk}$ with $Z_{\rm bulk} \equiv {\rm tr}\, e^{-\beta H_{\rm bulk}}$, and $a_{\rm bulk}$ is the anomaly coefficient related to the Seeley coefficient. The trace denotes the sum over all HS particle states. $a_{\rm bulk}$ is proportional to the integral of local curvature invariants, and should be the same for AdS_4 with S^3 boundary and for the thermal AdS_4. Thus, after summing over spins the total $a_{\rm bulk}$ should vanish as shown in previous sections. $E_{c\,\rm bulk}$ is the one loop contribution to the Casimir energy which can be extracted from the thermal free energy in a standard way (cf. (3.55), (3.56)).

The free energy of the $U(N)$, $O(N)$ or $USp(N)$ singlet sector of a free vectorial CFT on $S^1_\beta \times S^2$ takes similar form

$$F_{\rm CFT} = F^{\rm singlet}(\beta)_{\rm CFT} + \beta E_{c\,\rm CFT} + a_{\rm CFT} \log \Lambda \, , \tag{3.52}$$

in which $F(\beta)_{\rm CFT}$ is the free energy of the subsector in Hilbert space consisting of only the states that are invariant under the required symmetry group. The Casimir energy $E_{c\,\rm CFT}$ is given by NE_0, where E_0 is the Casimir energy of a single conformally invariant free field on $S^1_\beta \times S^2$. The anomaly coefficient $a_{\rm CFT}$ vanishes on $S^1_\beta \times S^2$, which is conformally flat and has vanishing Euler number. Therefore, there are no logarithmic divergent terms on both the bulk and the boundary sides. There remains comparison of the thermal part of the free energies and the Casimir energies on both sides. The thermal part of the free energies are expected to match since, by definition, the bulk and boundary thermal partition functions which give rise to the corresponding thermal free energies are both equal to the character of the HS algebra associated with the spectrum of the HS theory. The comparison between the bulk and boundary Casimir energies, however, is not straightforward, since different from $E_{c\,\rm bulk}$, the Casimir energy on the CFT side is not directly related to the thermal free energy of the singlet sector through (3.55). Holographic matching of the free energies at $\mathcal{O}(N^0)$ demands that $E_{c\,\rm bulk}$ is an integer times the Casimir energy of a single conformally invariant free field on $S^1_\beta \times S^2$.

In this section, we first study the one loop free energy of Konstein-Vasiliev theory in thermal AdS_4 with $S^1_\beta \times S^2$ boundary. We then compare the bulk result with the free energy of the corresponding dual CFT at $\mathcal{O}(N^0)$. Recall that there exist generalizations of $d > 4$ Vasiliev theory which are dual to the $U(N)$ or $O(N)$ singlet sector of free scalars or fermions [47]. Free energy of this type of HS theory in thermal AdS_d has been calculated in [21] and compared with $O(N^0)$ term in the free energy of the large N $U(N)$ or $O(N)$ vectorial free CFT. It was found that the matching of free energy implies shifts in the relation between G_N^{-1} and N at leading order by an integer.

Different from [21] where the bulk theories are purely bosonic, in our case the bulk theory includes also fermionic HS fields. Accordingly, the dual CFT consists of both scalars and fermions. In particular, the fermionic HS fields are dual to the bilinear conserved currents built out of both scalars and fermions. State operator correspondence then implies the existence of scalar-fermion mixed states in the Hilbert space that are singlet under the required symmetry group. These scalar-fermion mixed states contribute to the thermal free energy of the singlet sector nontrivially, which means that the $F^{\rm singlet}(\beta)$ for a CFT involving both scalars and fermions cannot be obtained by a simple sum of the $F^{\rm singlet}(\beta)$'s of a pure-scalar CFT and of a pure-fermion CFT.

Below we start with the computation of the free energies in Konstein-Vasiliev models, which include supersymmetric HS theories as special cases. The story is far more elaborate in higher dimensions. In particular, we refer the readers to [48, 49] and [50] for the case of 5D, and [51] for the case of 7D.

3.4.1 The bulk side

As stated earlier, the one loop free energy of a massless field in thermal AdS_4 has the structure displayed in (3.51) with the vanishing log divergence. $F(\beta)$ can be obtained from the grand canonical partition function as

$$\text{For bosons:}\ \ F(\beta)_{\rm bulk} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}(m\beta)\ , \tag{3.53}$$

$$\text{For fermions:}\ \ F(\beta)_{\rm bulk} = \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\mathcal{Z}(m\beta)\ . \tag{3.54}$$

Here $\mathcal{Z}(\beta)$ is the one-particle canonical partition function. The Casimir energy $E_{c\,\rm bulk}$ can be obtained from the energy ζ-function as

$$E_{c\,\rm bulk} = \pm\frac{1}{2}\zeta_E(-1)\ , \tag{3.55}$$

where $\pm$ correspond to bosonic and fermionic cases respectively. The energy ζ-function is related to the one-particle partition function by a Mellin transform

$$\zeta_E(z) = \frac{1}{\Gamma(z)}\int_0^{\infty} d\beta\beta^{z-1}\mathcal{Z}(\beta)\ . \tag{3.56}$$

In $D=4$, the thermal one-particle partition function for a scalar field is given by

$$\mathcal{Z}_0^{(\Delta)} = \frac{q^\Delta}{(1-q)^3} \quad \Delta > \frac{1}{2}\,, \tag{3.57}$$

where Δ is the AdS energy and $q = e^{-\beta}$ [52]. Thermal one-particle partition function for $s \geq \frac{1}{2}$ massless field takes the form

$$\mathcal{Z}_s(\beta) = \frac{q^{s+1}}{(1-q)^3}\big[2s+1-(2s-1)q\big]\,. \tag{3.58}$$

From the results derived in [21], we deduce the useful formulae[‡]

$$\begin{aligned}
F^{(1)}_{\text{even}\,1} &= F(\beta)_{\text{even}\,1} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}_{\text{even}\,1}(m\beta)\,,\\
\mathcal{Z}_{\text{even}\,1}(\beta) &= \tfrac{1}{2}\frac{q(1+q)^2}{(1-q)^4} + \tfrac{1}{2}\frac{q(1+q^2)}{(1-q^2)^2} = \tfrac{1}{2}[\widetilde{\mathcal{Z}}_0(\beta)]^2 + \tfrac{1}{2}\widetilde{\mathcal{Z}}_0(2\beta)\,,\\
F^{(1)}_{\text{even}\,2} &= F(\beta)_{\text{even}\,2} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}_{\text{even}\,2}(m\beta)\,,\\
\mathcal{Z}_{\text{even}\,2}(\beta) &= \frac{2q^2}{(1-q)^4} - \frac{q^2}{(1-q^2)^2} = \tfrac{1}{2}[\widetilde{\mathcal{Z}}_{\frac{1}{2}}(\beta)]^2 - \tfrac{1}{2}\widetilde{\mathcal{Z}}_{\frac{1}{2}}(2\beta)\,,\\
F^{(1)}_{\text{odd}\,1} &= F(\beta)_{\text{odd}} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}_{\text{odd}}(m\beta)\,,\\
\mathcal{Z}_{\text{odd}}(\beta) &= \tfrac{1}{2}\frac{q(1+q)^2}{(1-q)^4} - \tfrac{1}{2}\frac{q(1+q^2)}{(1-q^2)^2} = \tfrac{1}{2}[\widetilde{\mathcal{Z}}_0(\beta)]^2 - \tfrac{1}{2}\widetilde{\mathcal{Z}}_0(2\beta)\,,
\end{aligned} \tag{3.59}$$

where for later convenience we express the results in terms of the characters $\widetilde{\mathcal{Z}}_0(\beta)$ and $\widetilde{\mathcal{Z}}_{\frac{1}{2}}(\beta)$ of the conformally coupled free scalar and the free real fermion which realize the spin-0 and spin-$\frac{1}{2}$ singleton representations of the $SO(3,2)$, respectively

$$\widetilde{\mathcal{Z}}_0(\beta) = \frac{q^{\frac{1}{2}}(1+q)}{(1-q)^2}\,, \qquad \widetilde{\mathcal{Z}}_{\frac{1}{2}}(\beta) = \frac{2q}{(1-q)^2}\,. \tag{3.60}$$

[‡]In the rest of this subsection the thermal free energies and partition functions refer to those of the bulk theory.

By using (3.55) and (3.56), one can show that $\mathcal{Z}_{\text{even}\,1}(\beta)$, $\mathcal{Z}_{\text{even}\,2}(\beta)$ and $\mathcal{Z}_{\text{odd}}(\beta)$ all lead to vanishing Casimir energy [21][§]. Therefore we simply dropped E_c term in (3.59). Also one should note that

$$\tfrac{1}{2}[\widetilde{\mathcal{Z}}_{\frac{1}{2}}(\beta)]^2 + \tfrac{1}{2}\widetilde{\mathcal{Z}}_{\frac{1}{2}}(2\beta) = \tfrac{1}{2}[\widetilde{\mathcal{Z}}_0(\beta)]^2 - \tfrac{1}{2}\widetilde{\mathcal{Z}}_0(2\beta)\ . \tag{3.61}$$

For all the fermionic fields, we find that the total one-particle canonical partition function is given by

$$\mathcal{Z}^F(\beta) = \sum_{s=\frac{1}{2}}^{\infty} \frac{q^{s+1}}{(1-q)^3}\Big[2s+1-(2s-1)q\Big] = \frac{2q^{\frac{3}{2}}(1+q)}{(1-q)^4} = \widetilde{\mathcal{Z}}_0(\beta)\widetilde{\mathcal{Z}}_{\frac{1}{2}}(\beta)\ . \tag{3.62}$$

Using the total one-particle canonical partition function, we can construct the energy ζ-function for fermions

$$\begin{aligned}
\zeta_E^F(z) &= \frac{1}{\Gamma(z)}\int_0^\infty d\beta\beta^{z-1}\frac{2e^{-\frac{3}{2}\beta}(1+e^{-\beta})}{(1-e^{-\beta})^4} \\
&= 2\sum_{n=1}^{\infty}\binom{n+2}{3}[(n+\tfrac{1}{2})^{-z} + (n+\tfrac{3}{2})^{-z}] \\
&= \tfrac{1}{8}\zeta(z,\tfrac{5}{2}) - \tfrac{1}{12}\zeta(z-1,\tfrac{5}{2}) - \tfrac{1}{2}\zeta(z-2,\tfrac{5}{2}) + \tfrac{1}{3}\zeta(z-3,\tfrac{5}{2}) \\
&\quad -\tfrac{1}{8}\zeta(z,\tfrac{3}{2}) - \tfrac{1}{12}\zeta(z-1,\tfrac{3}{2}) + \tfrac{1}{2}\zeta(z-2,\tfrac{3}{2}) + \tfrac{1}{3}\zeta(z-3,\tfrac{3}{2})\ .
\end{aligned} \tag{3.63}$$

This vanishes at $z=-1$. Therefore, the total Casimir energy for fermionic HS fields vanishes in thermal AdS_4 as well, and the correspoding one loop free energy is simply

$$F^{(1)\,F} = F(\beta)^F_{\text{bulk}} = \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\mathcal{Z}^F(m\beta)\ . \tag{3.64}$$

[§]Similar technique using $SO(3,2)$ character has been applied to compute one-loop free energy of HS theories constructed using higher-order singleton [53], where vanishing of Casimir energy was also observed.

Summarizing the results above and using the spectra given in (3.9), we find that the one loop free energies for generic Konstein-Vasiliev HS theories are given by

$$hu(m;n|4)\,:\,F^{(1)}_{hu}=-\sum_{k=1}^{\infty}\frac{1}{k}\Big[m\,\widetilde{\mathcal{Z}}_0(k\beta)+n\,(-)^{k+1}\,\widetilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)\Big]^2\,, \tag{3.65}$$

$$\begin{aligned} ho(m;n|4)\,:\,F^{(1)}_{ho}=-\sum_{k=1}^{\infty}\frac{1}{2k}\Big(&\Big[m\,\widetilde{\mathcal{Z}}_0(k\beta)+n\,(-)^{k+1}\,\widetilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)\Big]^2\\ &+m\,\widetilde{\mathcal{Z}}_0(2k\beta)-n\,\widetilde{\mathcal{Z}}_{\frac{1}{2}}(2k\beta)\Big)\,, \end{aligned} \tag{3.66}$$

$$\begin{aligned} husp(m;n|4)\,:\,F^{(1)}_{husp}=-\sum_{k=1}^{\infty}\frac{1}{2k}\Big(&\Big[m\,\widetilde{\mathcal{Z}}_0(k\beta)+n\,(-)^{k+1}\,\widetilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)\Big]^2\\ &-m\,\widetilde{\mathcal{Z}}_0(2k\beta)+n\,\widetilde{\mathcal{Z}}_{\frac{1}{2}}(2k\beta)\Big)\,. \end{aligned} \tag{3.67}$$

The free energy of $husp(m;n|4)$ theory can be obtained from that of the $ho(m;n|4)$ theory by $m\to -m\,, n\to -n$.

3.4.2 The CFT side and comparison

In this section, we calculate the partition function of the singlet sector of free CFTs on $S^1_\beta\times S^2$. We closely follow the technique developed in [54,55]. The partition function of a CFT on $S^1_\beta\times S^2$ is equal to the thermal partition function due to the vanishing of Casimir energy and logarithmic divergence. Therefore, we have

$$Z(\beta)=\sum_{i\in\mathrm{physical\,states}} q^{E_i}\,,\quad q=e^{-\beta}\,, \tag{3.68}$$

where the $\mathrm{physical\,states}$ are restricted to be the singlet states of $U(N)$, $O(N)$ or $USp(N)$ for our purpose. We have also used the fact that there is no non-trivial chemical potential in the system. The thermal partition functions of the $U(N)$ and $O(N)$ singlet sectors of free scalar and free fermion theories have been studied in [21,56]. We generalize their results to the cases with both scalars and fermions. We first consider the $U(N)$ singlet sector of a free CFT with Nm complex free scalars and Nn Dirac fermions. As shown in [21, 56], the thermal partition function can be expressed as a path integral localized on the eigenvalues of $U(N)$ matrix

$$
\begin{aligned}
&Z_{U(N)}(\beta) = e^{-F(\beta)_{U(N)}} = \int \prod_{i=1}^{N} d\alpha_i e^{-S(\alpha_1,...\alpha_N)} \, , \\
&S(\alpha_1, ...\alpha_N) = -\tfrac{1}{2} \sum_{i\neq j=1}^{N} \log\sin^2 \frac{\alpha_i - \alpha_j}{2} + 2\sum_{i=1}^{N} f_\beta(\alpha_i) \, , \\
&f_\beta(\alpha) = \sum_{k=1}^{N} c_k(\beta)\cos(k\alpha) \, , \quad c_k(\beta) = -\frac{1}{k}\Big[m\, \widetilde{\mathcal{Z}}_0(k\beta) + n\,(-)^{k+1}\widetilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)\Big] \, ,
\end{aligned} \tag{3.69}
$$

where the matter contents affect the effective action through $c_k(\beta)$. In the large N limit, the integral over α_i can be replaced by the path integral over the eigenvalue density $\rho(\alpha)$, $\alpha \in (-\pi, \pi)$. $\rho(\alpha)$ satisfies the standard normalization

$$
\int_{-\pi}^{\pi} d\alpha\rho(\alpha) = 1 \, . \tag{3.70}
$$

The effective action in terms of $\rho(\alpha)$ takes the form

$$
\begin{aligned}
S(\rho) &= N^2 \int d\alpha d\alpha' K(\alpha-\alpha')\rho(\alpha)\rho(\alpha') + 2N \int d\alpha\rho(\alpha) f_\beta(\alpha) \, , \\
K(\alpha-\alpha') &= -\tfrac{1}{2}\log(2 - 2\cos\alpha) \, , \qquad f_\beta(\alpha) = \sum_{k=1}^{N} c_k(\beta)\cos(k\alpha) \, .
\end{aligned} \tag{3.71}
$$

Integrating out ρ, one obtains

$$
F(\beta)_{U(N)} = -\sum_{k=1}^{\infty} k[c_k(\beta)]^2 = -\sum_{k=1}^{\infty} \frac{1}{k}\Big[m\, \widetilde{\mathcal{Z}}_0(k\beta) + n\,(-)^{k+1}\widetilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)\Big]^2 \, , \tag{3.72}
$$

which coincides with one loop free energy for $hu(m;n|4)$ higher spin theory (3.65). Next, we study the $O(N)$ singlet sector of a free CFT with Nm real free scalars and Nn Majorana fermions. This is a generalization of the results in [21], where the free CFT consists of only scalars or fermions. It is suggested in [21] that, one can choose N to be even, namely N=2N for simplicity in the large N. The difference between even N and odd N cases is at the next order in $1/N$ expansion. Free energy of the $O(2\mathrm{N})$ singlet sector of a free CFT with Nm real free scalars and Nn Majorana fermions can again be written as a path integral over the eigenvalues of $O(N)$ matrix. The effective potential of the $O(N)$ singlet sector is given by [21]

$$S(\alpha_1, ...\alpha_{\mathrm{N}}) = -\tfrac{1}{2}\sum_{i\neq j=1}^{\mathrm{N}} \log\sin^2\frac{\alpha_i-\alpha_j}{2} - \tfrac{1}{2}\sum_{i\neq j=1}^{\mathrm{N}} \log\sin^2\frac{\alpha_i+\alpha_j}{2} + 2\sum_{i=1}^{\mathrm{N}} f_\beta(\alpha_i), \tag{3.73}$$

where f_β is the same as the one in (3.69). The effective potential for the $O(N)$ singlet sector differs from that of the $U(N)$ by the $\log\sin^2\alpha$ terms which come from the Van der Monde determinant or the Haar measure. In the large N limit, the path integral over α_i can again be recast into an integral over the eigenvalue density $\rho(\alpha)$. After integrating out ρ, one obtains

$$\begin{aligned} F(\beta)_{O(N)} &= -\sum_{k=1}^{\infty}\frac{k}{2}\Big([c_k(\beta)]^2 - \tfrac{2}{k}\, c_{2k}(\beta)\Big) \qquad (3.74)\\ &= -\sum_{k=1}^{\infty}\frac{1}{2k}\Big(\Big[m\,\widetilde{\mathcal{Z}}_0(k\beta) + n\,(-)^{k+1}\,\widetilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)\Big]^2 + m\,\widetilde{\mathcal{Z}}_0(2k\beta) - n\,\widetilde{\mathcal{Z}}_{\frac{1}{2}}(2k\beta)\Big)\,, \end{aligned}$$

which matches the one loop free energy of $ho(m;n|4)$ HS theory in (3.66). In the last case, we consider the $USp(N)$ singlet sector of a free CFT with Nm complex free scalars ϕ^{ia}, $i = 1, 2, ...N$, $a = 1, 2, ...m$ and Nn Dirac fermions subject to the symplectic real condition. Since N is even in this case, we denote N by 2N. The effective potential of the

$USp(N)$ singlet sector takes the form

$$S(\alpha_1, ...\alpha_{\rm N}) = -\tfrac{1}{2}\sum_{i\neq j=1}^{\rm N} \log\sin^2\frac{\alpha_i-\alpha_j}{2} - \tfrac{1}{2}\sum_{i,j=1}^{\rm N} \log\sin^2\frac{\alpha_i+\alpha_j}{2}$$
$$-\tfrac{1}{2}\sum_{i=1}^{\rm N} \log\sin^2\alpha_i + 2\sum_{i=1}^{\rm N} f_\beta(\alpha_i)\ . \qquad (3.75)$$

In the large N limit, the path integral over α_i can be evaluated by using the same technique as before. The free energy of the $USp(N)$ singlet sector of a free CFT is obtained as

$$F(\beta)_{USp(N)} = -\sum_{k=1}^{\infty}\frac{k}{2}\Big([c_k(\beta)]^2 + \tfrac{2}{k}\,c_{2k}(\beta)\Big) \qquad (3.76)$$
$$= -\sum_{k=1}^{\infty}\frac{1}{2k}\Big(\Big[m\,\widetilde{\mathcal{Z}}_0(k\beta) + n\,(-)^{k+1}\,\widetilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)\Big]^2 - m\,\widetilde{\mathcal{Z}}_0(2k\beta) + n\,\widetilde{\mathcal{Z}}_{\frac{1}{2}}(2k\beta)\Big)\ ,$$

which matches one loop free energy of $husp(m;n|4)$ HS theory in (3.67).

3.5 Mixed boundary conditions in bulk and interacting $\mathcal{N}=1$ SCFT

In $\mathcal{N}=1$ HS theory, the $OSp(1|4)$ invariant boundary conditions are given in [18]. To describe this, we write the boundary behavior $(\rho\to 0)$ of the complex scalar $\phi = A + {\rm i}B$ as

$$A = \rho\alpha_+ + \rho^2\beta_+\ ,\quad B = \rho\alpha_- + \rho^2\beta_-\ , \qquad (3.77)$$

and define the 3D, $\mathcal{N}=1$ superfields

$$\Phi_- = \alpha_- + {\rm i}\tilde\theta\eta_- - \frac{\tilde\theta\theta}{2{\rm i}}\beta_+\ ,\quad \Phi_+ = \alpha_+ + {\rm i}\tilde\theta\eta_+ + \frac{\tilde\theta\theta}{2{\rm i}}\beta_-\ . \qquad (3.78)$$

The boundary conditions preserving $OSp(1|4)$ take the form

$$\Phi_- = \lambda\Phi_+\ , \qquad (3.79)$$

where λ is an arbitrary real number. In terms of the new scalar fields we have

$$A' = \sin\vartheta A - \cos\vartheta B \ , \quad B' = \cos\vartheta A + \sin\vartheta B \ , \tag{3.80}$$

where $\tan\vartheta = \lambda$, and the boundary condition (3.79) is equivalent to

$$\alpha'_+ = 0 \ , \quad \beta'_- = 0 \ . \tag{3.81}$$

The linearized bulk scalar field equations would remain the same form under the $SO(2)$ rotation, thus the newly defined scalar fields A' and B' possess the same Feffer-Graham expansion as the original scalar fields A and B. The boundary condition (3.81) implies that near the boundary

$$A' = \rho^2\beta'_+ \ , \quad B' = \rho\alpha'_- \ . \tag{3.82}$$

Therefore, in computing the one loop free energy, A' should have $\Delta = 2$, while B' should have $\Delta = 1$, which does not affect the $\mathcal{N} = 1$ HS spectrum and the corresponding one loop calculation. On the CFT side, the boundary condition (3.79) implies the $\mathcal{N} = 1$ free CFT being deformed by a supersymmetric double-trace term

$$\Delta S = \frac{\lambda}{2}\int d^3x d^2\theta \mathcal{O}^2 \ , \tag{3.83}$$

where $\mathcal{O}$ is given by

$$\mathcal{O} = \frac{1}{\sqrt{N}} W^2 \ , \quad W = \varphi + \mathrm{i}\tilde{\theta}\psi + \frac{\tilde{\theta}\theta}{2\mathrm{i}} f \ . \tag{3.84}$$

We compute the difference between the free energy of the deformed CFT and that of the free CFT, following the procedure adopted in [39, 57]. Denoting the partition function of the free CFT by Z_0, we calculate

$$\Delta F = -\log \frac{Z}{Z_0} \ . \tag{3.85}$$

Using the Hubbard-Stratonovich transformation, we have

$$\frac{Z}{Z_0} = \frac{1}{\int D\Sigma \exp(\frac{1}{2\lambda}\int dz'\Sigma^2)} \int D\Sigma \Big\langle \exp\Big[\int dz \Big(\frac{1}{2\lambda}\Sigma^2 + \Sigma\mathcal{O}\Big)\Big]\Big\rangle_0 \ , \tag{3.86}$$

where Σ is an auxiliary superfield and z denotes the supercoordinate. In the large N limit, the higher point functions of $\mathcal{O}$ are suppressed. This allows us to write

$$\Big\langle \exp\Big[\int dz \Sigma\mathcal{O}\Big]\Big\rangle_0 = \exp\Big[\frac{1}{2}\Big\langle\Big(\int dz\Sigma\mathcal{O}\Big)^2\Big\rangle_0 + o(1/N)\Big] \ . \tag{3.87}$$

Note that Σ and $\mathcal{O}$ are single-trace operators of $\mathcal{N}=1$ superfields, say M and W respectively, each with component fields A^i, λ^i, B^i and ϕ^i, ψ^i, f^i, where B and f are auxiliary fields, and the index i stands for the representation of $O(N)$. The component fields obey the following superconformal transformations

$$\delta A = \frac{1}{4}\tilde{\xi}\lambda \qquad\qquad \delta\phi = \frac{1}{4}\tilde{\xi}\psi \tag{3.88}$$

$$\delta\lambda = \not{\partial} A\xi - \frac{1}{4}B\xi + A\eta \qquad \delta\psi = \not{\partial}\phi\xi - \frac{1}{4}f\xi + \phi\eta \tag{3.89}$$

$$\delta B = -\tilde{\xi}\not{\nabla}\lambda \qquad\qquad \delta f = -\tilde{\xi}\not{\nabla}\psi \tag{3.90}$$

where ξ and η are spinors satisfying the conformal Killing spinor equation $\nabla_\mu\xi = \gamma_\mu\eta$.

Integrating out the spinor coordinates θ and $\tilde{\theta}$, we obtain

$$\begin{aligned}\int dz \frac{1}{2\lambda}\Sigma^2 =& \frac{1}{\lambda}\int dx^3\sqrt{g}(B^i A^i A^j A^j + \frac{1}{2}\tilde{\lambda}^i\lambda^i A^j A^j + \tilde{\lambda}^i\lambda^j A^i A^j) \\ =& \frac{1}{\lambda}\int dx^3\sqrt{g}(\Sigma_2\Sigma_1 + \Sigma_{3/2}\Sigma_{3/2}) \ ,\end{aligned} \tag{3.91}$$

$$\begin{aligned}\int dz\Sigma\mathcal{O} =& \int dx^3\sqrt{g}(f^i\phi^i A^j A^j + \frac{1}{2}\tilde{\psi}^i\psi^i A^j A^j + B^i A^i\phi^j\phi^j + \frac{1}{2}\tilde{\lambda}^i\lambda^i\phi^j\phi^j + 2\tilde{\psi}^i\lambda^j\phi^i A^j) \\ =& \int dx^3\sqrt{g}(\mathcal{O}_2\Sigma_1 + \Sigma_2\mathcal{O}_1 + 2\mathcal{O}_{3/2}\Sigma_{3/2}) \ ,\end{aligned} \tag{3.92}$$

where we defined

$$\begin{aligned}&\Sigma_1 = A^i A^i \ , \ \ \mathcal{O}_1 = \phi^i\phi^i, \ \ \Sigma_{3/2} = A^i\lambda^i \ , \ \ \mathcal{O}_{3/2} = \phi^i\psi^i \ , \\ &\Sigma_2 = B^i A^i + \frac{1}{2}\tilde{\lambda}^i\lambda^i \ , \ \ \mathcal{O}_2 = f^i\phi^i + \frac{1}{2}\tilde{\psi}^i\psi^i \ ,\end{aligned} \tag{3.93}$$

with the lower indices labeling the dimension of the single-trace operators.

With the above preparation the second factor of (3.86) at large N is

$$\begin{aligned}&\int D\Sigma \exp\Big[\frac{1}{2\lambda}\int dz\Sigma^2 + \frac{1}{2}\Big\langle\Big(\int dz\Sigma\mathcal{O}\Big)^2\Big\rangle_0\Big] \\ =& \int D\Sigma \exp\Big[\frac{1}{\lambda}\int dx^3\sqrt{g}(\Sigma_2\Sigma_1 + \Sigma_{3/2}\Sigma_{3/2}) \\ &+ \frac{1}{2}\Big\langle\Big(\int dx^3\sqrt{g}(\mathcal{O}_2\Sigma_1 + \Sigma_2\mathcal{O}_1 + 2\mathcal{O}_{3/2}\Sigma_{3/2})\Big)^2\Big\rangle_0\Big] \\ =& \int D\Sigma \exp\Big[\frac{1}{\lambda}\int dV(\Sigma_2\Sigma_1 + \Sigma_{3/2}\Sigma_{3/2}) \\ &+ \frac{1}{2}\int\int dV dV'\Big(\Sigma_1(x)\Sigma_1(x')\Big\langle\mathcal{O}_2(x)\mathcal{O}_2(x')\Big\rangle_0 + \Sigma_2(x)\Sigma_2(x')\Big\langle\mathcal{O}_1(x)\mathcal{O}_1(x')\Big\rangle_0 \\ &+ 4\Sigma_{3/2}(x)\Sigma_{3/2}(x')\Big\langle\mathcal{O}_{3/2}(x)\mathcal{O}_{3/2}(x')\Big\rangle_0\Big)\Big] \ ,\end{aligned} \tag{3.94}$$

where $dV \equiv dx^3\sqrt{g}$, and we dropped vanishing terms in the two-point function to reach the last line.

The integral in (3.86) then becomes gaussian, which integrates to give

$$\frac{Z}{Z_0} = \frac{\det\left(\mathbb{1} + 2\lambda\langle\mathcal{O}_{3/2}\mathcal{O}_{3/2}\rangle_0\right)}{\left\{\det\left(\frac{\lambda}{2}\langle\mathcal{O}_2\mathcal{O}_2\rangle_0\right)\det\left(\frac{\lambda}{2}\langle\mathcal{O}_1\mathcal{O}_1\rangle_0\right)\det\left(\mathbb{1} - (\frac{\lambda}{4}\langle\mathcal{O}_2\mathcal{O}_2\rangle_0)^{-1}(\frac{\lambda}{4}\langle\mathcal{O}_1\mathcal{O}_1\rangle_0)^{-1}\right)\right\}^{\frac{1}{2}}} . \tag{3.95}$$

At $\lambda \to \infty$, the change of the free energy compared to the free theory is

$$\begin{aligned}\Delta F = -\log\frac{Z}{Z_0} = &-\operatorname{tr}\log\left(2\langle\mathcal{O}_{3/2}\mathcal{O}_{3/2}\rangle_0\right) + \frac{1}{2}\operatorname{tr}\log\left(\frac{1}{2}\langle\mathcal{O}_2\mathcal{O}_2\rangle_0\right)\\ &+ \frac{1}{2}\operatorname{tr}\log\left(\frac{1}{2}\langle\mathcal{O}_1\mathcal{O}_1\rangle_0\right) .\end{aligned} \tag{3.96}$$

The two-point functions $\langle\mathcal{O}_1\mathcal{O}_1\rangle_0$ and $\langle\mathcal{O}_2\mathcal{O}_2\rangle_0$ can be expanded in terms of scalar harmonics on S^3 [57]

$$\langle\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(x')\rangle_0 = \sum_{\ell m} g_\ell^\Delta Y^*_{\ell m}(x)Y_{\ell m}(x') , \tag{3.97}$$

where g_ℓ^Δ is given by

$$g_\ell^\Delta = R^{3-2\Delta}\pi^{\frac{3}{2}}2^{3-\Delta}\frac{\Gamma(\frac{3}{2}-\Delta)}{\Gamma(\Delta)}\frac{\Gamma(\ell+\Delta)}{(3+\ell-\Delta)} . \tag{3.98}$$

Since the harmonics satisfy orthonormal relations, we have

$$\int\sqrt{g}d^3y\langle\mathcal{O}_2(x)\mathcal{O}_2(y)\rangle_0\langle\mathcal{O}_1(y)\mathcal{O}_1(x')\rangle_0 = \sum_{\ell m} g_\ell^{\Delta=2}g_\ell^{\Delta=1}Y^*_{\ell m}(x)Y_{\ell m}(x') . \tag{3.99}$$

It is straightforward to see that $g_\ell^{\Delta=2}g^{\Delta=1}$ is independent of ℓ, and therefore according to [57], $\operatorname{tr}\log\langle\mathcal{O}_2\mathcal{O}_2\rangle_0 + \operatorname{tr}\log\langle\mathcal{O}_1\mathcal{O}_1\rangle_0$ does not contribute to ΔF.

Similarly, for fermionic two-point function, it is shown in [39] that $\operatorname{tr}\log\langle\mathcal{O}_{3/2}\mathcal{O}_{3/2}\rangle_0$

is also zero. Therefore, in the IR there is no modification to the free energy given by the double-trace deformation.

When λ is small, one can apply perturbation theory to compute ΔF induced by the deformation. As shown in [39] the change of free energy caused by the deformation is proportional to the beta function of the deformation coupling. The deformation appearing here is exactly marginal in the $N \to \infty$ limit, which implies that the beta function of the coupling constant is suppressed by $1/N$. Thus, at small coupling it can also be seen that the deformation does not affect the $\mathcal{O}(N^0)$ free energy. In summary, although we have not computed the free energy of the deformed theory for arbitrary λ, the vanishing of ΔF at $\mathcal{O}(N^0)$ in both the strong and weak coupling limits provides strong evidence that ΔF does not receive $\mathcal{O}(N^0)$ contribution from the supersymmetric double-trace deformation, which is exactly marginal in the $N \to \infty$ limit.

3.6 Discussion

We have carried out one loop tests of the conjectured dualities between Konstein-Vasiliev HS theories in AdS_4 with S^3 and $S^1_\beta \times S^2$ boundaries. Our results for the free energies extend previous ones [20, 21, 41] by inclusion of fermionic bulk degrees of freedom. In computing the one loop free energies of bosonic and fermionic HS fields in AdS_4 with S^3 boundary, we have adopted the modified spectral zeta function method suggested by [43], thereby reproducing the one loop free energy for bosonic HS fields in a much simpler way without the ambiguities encountered in [20, 41]. We also find that the total one loop free energy of an infinite tower of bulk fermionic fields vanishes.

Matching the bulk fields with boundary operators suggests that the possible CFT duals of Konstein-Vasiliev theories based on $hu(m;n|4)$, $ho(m;n|4)$ and $husp(m;n|4)$, and subject to HS symmetry preserving boundary conditions, are respectively the $U(N)$, $O(N)$ and $USp(N)$ singlet sectors of free scalars and free fermions vector representations of the bosonic subalgebras conformally coupled to S^3. We find that the free energy of the HS theory may match with that of the free CFT only when the bulk theories are $hu(m;0|4)$, $ho(m;0|4)$, $husp(m;0|4)$ Konstein-Vasiliev theories, and with identifications $G_N^{-1} = \gamma(N + \Delta N)$ with suitable integers ΔN. These are generalized Type-A theories with bosonic scalars on the boundary and bosonic bulk HS fields containing even parity scalars. Thus, in particular, the free energies for generalized Type-B models with fermions on the S^3 boundary and bosonic HS fields including odd parity scalar fields do not match. The mismatch in the case of $m = 0, n = 1$ corresponding to the simplest Type-B model has already been noted in [20] where the one loop free energy $F^{(1)} = -\zeta(3)/(8\pi^2)$ obtained in the bulk does not agree with the free energy of Dirac fermions on S^3 boundary. We have also calculated the free energies of Konstein-Vasiliev theories in AdS_4 with $S^1_\beta \times S^2$ boundary. In this case, we find that the free energies of all three families of Konstein-Vasiliev theories match those of the conjectured dual free CFTs.

Turning to the problem of mismatch in free energies of Type-B model and its conjectured dual, one may have to take into account the issue of how to impose the $O(N)$ invariance condition on the CFT side. A natural way of implementing it is to gauge the $O(N)$ symmetry by means of vector gauge field with level k Chern-Simons kinetic term. This term breaks parity but the result for the free energy of the parity invariant model can be obtained in a limit in which the CS gauge field decouples. It has been suggested in [20] that as the fermions coupled to CS on the boundary give rise to a shift in the level k, it may not be justified to obtain the result for parity-preserving case by a naive subtraction of CS contribution from the free energy on the CFT side. However, one expects that this effect becomes irrelevant in the decoupling limit in which $k \to \infty$. In fact, we have examined the procedure of decoupling CS in the large k limit by evaluating the S^3 free energies for ABJ model based on $U(N)_k \times U(1)_{-k}$ [6, 58] and a few $\mathcal{N} = 3$ CS matter theories in which the matter sector consists of fundamental hypermultiplets [59–61]. After subtracting the contribution from pure CS term, we indeed obtain the free energies of free vector models. Therefore, the puzzle of free energy mismatch in Type-B remains unresolved and its solution requires deeper understanding of HS/vector model holography.

We conclude the chapter at this point. In next chapter we will focus on a parallel test of the HS/CFT duality, by comparing the three-point functions on both sides.

4. THREE-POINT FUNCTION TESTS OF HS/CFT DUALITIES

In this chapter we compute the tree-level three-point Witten diagrams of HS theories in AdS_4 without supersymmetry, and compare them with the known results on the CFT side [62–64]. More specifically, due to the non-localities present in the HS theories, we are only able to calculate the three-point functions with one scalar leg. In the pioneered work [22] the authors have adopted certain special kinematics in the computation of the three-point Witten diagrams with spins $s_1 - s_2 - 0$ for parity-preserving HS theories, and have shown that the coefficients of the leading terms (which are functions of s_1 and s_2 and thus nontrivial) match with the ones on CFT side. We will sharpen and expand considerably this test. In particular, we provide simplifications of the computations, and produce the full structures of the HS correlation functions. We also establish nontrivial new tests of the holography for the parity-violating HS theories.

In the following we first give out our convention for AdS_4 and review the structure of the CFT correlators in Chern-Simons matter theories, and discuss the general structure of HS interactions and relation to the Vasiliev equations. We then carry out the computation of the cubic Witten diagrams and compare the results with the ones in CFTs.

This chapter is based on the work [26] in collaboration with Dr. Ergin Sezgin and Dr. Evgeny Skvortsov.

4.1 Notation and Conventions in AdS_4

We adopt the mostly plus convention for the metric $\eta_{\underline{m}} = (-+++)$, which makes the Euclidean rotation easier to implement. Choosing $x^{\underline{m}} = (\vec{\mathrm{x}}, z)$ to be Poincare coordinates with z being the radial coordinate and $\vec{\mathrm{x}}$ the three coordinates along the boundary, the AdS_4-background can be described by vierbein $h^{\alpha\dot{\alpha}}_{\underline{m}}$ and spin-connection that splits into (anti) self-dual parts ${\omega_{(0)\underline{m}}}^{\alpha\beta}$ and ${\bar{\omega}_{(0)\underline{m}}}^{\dot{\alpha}\dot{\beta}}$:

$$h^{\alpha\dot{\alpha}} = \frac{1}{2z}\sigma^{\alpha\dot{\alpha}}_{\underline{m}} dx^{\underline{m}}\,, \qquad \omega^{\alpha\beta}_{(0)} = \frac{i}{2z}\vec{\sigma}^{\alpha\beta}\cdot d\vec{x}\,, \qquad \bar{\omega}^{\dot{\alpha}\dot{\beta}}_{(0)} = -\frac{i}{2z}\vec{\sigma}^{\alpha\beta}\cdot d\vec{x}\,. \tag{4.1}$$

The matrices $\sigma^{\alpha\dot{\alpha}}_{\underline{m}}$ are constant and in our convention they are given by $\sigma^{\alpha\dot{\beta}}_{\underline{m}} = (\vec{\sigma}^{\alpha\beta}, i\epsilon^{\alpha\beta})$. We have the relations

$$\mathrm{x}^{\alpha\beta} = \vec{\sigma}^{\alpha\beta}\cdot\vec{\mathrm{x}}\,, \qquad \mathrm{x}^2 = -\tfrac{1}{2}\mathrm{x}^{\alpha\beta}\mathrm{x}_{\alpha\beta}\,, \qquad \sigma^{\alpha\dot{\alpha}}_{\underline{m}}\sigma_{\underline{n}\,\alpha\dot{\alpha}} = -2\eta_{\underline{mn}}\,, \tag{4.2}$$

$$x^{\alpha\dot{\alpha}} = \mathrm{x}^{\alpha\dot{\alpha}} + iz\epsilon^{\alpha\dot{\alpha}}\,, \qquad x^2 = \mathrm{x}^2 + z^2\,, \qquad \mathrm{x}_{ij} = |\mathrm{x}_i - \mathrm{x}_j|\,. \tag{4.3}$$

The inverse vierbein $h^{\underline{m}}_{\alpha\dot{\alpha}} = -z\sigma^{\underline{m}}_{\alpha\dot{\alpha}}$ obeys the relations

$$h^{\alpha\dot{\alpha}}_{\underline{m}} h^{\underline{n}}_{\alpha\dot{\alpha}} = \delta^{\underline{n}}_{\underline{m}}\,, \qquad h^{\alpha\dot{\alpha}}_{\underline{m}} h^{\underline{m}}_{\beta\dot{\beta}} = \delta^{\alpha}_{\beta}\delta^{\dot{\alpha}}_{\dot{\beta}}\,, \tag{4.4}$$

and the AdS_4 metric tensor and spin connection are given by

$$g_{\underline{mn}} = -h^{\alpha\dot{\alpha}}_{\underline{m}} h_{\underline{n}\,\alpha\dot{\alpha}} dx^{\underline{m}} dx^{\underline{n}} = \frac{1}{2z^2}\eta_{\underline{mn}} dx^{\underline{m}} dx^{\underline{n}}\,, \tag{4.5}$$

$$\Omega = \frac{1}{4i}\left[\,\omega_{(0)\,\alpha\beta} y^{\alpha} y^{\beta} + \bar{\omega}_{(0)\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} y^{\dot{\beta}} + 2h_{\alpha\dot{\beta}} y^{\alpha}\bar{y}^{\dot{\beta}}\right]\,. \tag{4.6}$$

We use the raising and lowering conventions: $X^{\alpha} = \epsilon^{\alpha\beta}X_{\beta}$ and $X_{\alpha} = X^{\beta}\epsilon_{\beta\alpha}$ with $\epsilon_{\alpha\gamma}\epsilon^{\beta\gamma} = \delta^{\beta}_{\alpha}$, and similar conventions for the dotted indices. We use the convention $\epsilon_{12} = -\epsilon_{21} = 1$.

4.2 CFT

In this section we review the general structure of correlation function in three dimensions and list the results available in the literature for free theories and, more generally, for Chern-Simons matter theories. The advantage of three dimensions is that one can benefit from the fact that $so(2,1) \sim sp(2)$.

4.2.1 General Structure of the Correlators

In three dimensions a traceless rank-s $so(2,1)$-tensor is equivalent to a symmetric rank-$2s$ spin-tensor. It is convenient to contract the Lorentz indices of tensor operators with auxiliary polarization vectors that are now replaced by polarization spinors, which we denote by $\eta \equiv \eta^{\alpha}$. Therefore, a weight-$\Delta$, rank-$s$ tensor operator $O_{\Delta}^{a_1...a_s}$ is replaced by a generating function $O_{\Delta}(\mathrm{x}, \eta)$:

$$O_{\Delta}^{a_1...a_s}(\mathrm{x}) \qquad \longrightarrow \qquad O_{\Delta}^{\alpha_1...\alpha_{2s}}(\mathrm{x}^{\beta\gamma}) \qquad \longrightarrow \qquad O_{\Delta}(\mathrm{x}, \eta) = O_{\Delta}^{\alpha_1...\alpha_{2s}}(\mathrm{x})\eta_{\alpha_1}...\eta_{\alpha_{2s}} \,. \tag{4.7}$$

Suppose we are given a number of operators $O(\mathrm{x}_i, \eta_i)$ that are inserted at points x_i and whose tensor indices are contracted with polarization spinors η^i_{α}. The conformal group acts in the usual way. In particular, Lorentz transformations correspond to an $Sp(2)$ matrix $A_{\alpha}{}^{\beta}$ that acts both on coordinates x_i and polarization spinors η^i:

$$\mathrm{x}^{\beta\delta} \to A_{\alpha}{}^{\beta}\, A_{\gamma}{}^{\delta}\, \mathrm{x}^{\alpha\gamma} \,, \qquad\qquad \eta^i_{\alpha} \to A_{\alpha}{}^{\beta}\, \eta^i_{\beta} \,. \tag{4.8}$$

It is useful to define the inversion map as

$$R\vec{\mathrm{x}} = \frac{\vec{\mathrm{x}}}{\mathrm{x}^2} \,, \qquad R\eta^a_{\alpha} = \frac{\mathrm{x}_{\alpha}{}^{\beta}\eta^a_{\beta}}{\mathrm{x}^2} \,, \qquad Rx^{\alpha\dot{\alpha}} = \frac{x^{\alpha\dot{\alpha}}}{x^2} = \frac{\mathrm{x}^{\alpha\dot{\alpha}} + iz\epsilon^{\alpha\dot{\alpha}}}{\mathrm{x}^2 + z^2} \,, \tag{4.9}$$

Then, it is not hard to see that the following structures are conformally invariant [63]:

$$P_{ij} = \eta^i_{\alpha} R[\mathrm{x}_i - \mathrm{x}_j]^{\alpha\beta} \eta^j_{\beta} \,, \qquad\qquad RP_{ij} = P_{ij} \,, \tag{4.10}$$

$$Q^i_{jk} = \eta^i_{\alpha} \left(R[\mathrm{x}_j - \mathrm{x}_i] - R[\mathrm{x}_k - \mathrm{x}_i]\right)^{\alpha\beta} \eta^i_{\beta} \,, \qquad\qquad RQ^i_{jk} = Q^i_{jk} \,. \tag{4.11}$$

There is also one more structure that is parity-odd:

$$S^i_{jk} = \frac{\eta^k_\alpha (\mathrm{x}_{ki})^\alpha{}_\beta\, (\mathrm{x}_{ij})^{\beta\gamma} \eta^j_\gamma}{\mathrm{x}_{ij}\mathrm{x}_{ik}\mathrm{x}_{jk}}\,, \qquad\qquad RS^i_{jk} = -S^i_{jk}\,. \tag{4.12}$$

Any three-point correlation function $\langle O_1(\mathrm{x}_1, \eta^1) O_2(\mathrm{x}_2, \eta^2) O_3(\mathrm{x}_3, \eta^3)\rangle$ can be decomposed into an obvious prefactor times a polynomial in Q, P, S structures:

$$\langle O_1(\mathrm{x}_1, \eta^1) O_2(\mathrm{x}_2, \eta^2) O_3(\mathrm{x}_3, \eta^3)\rangle = \frac{1}{\mathrm{x}_{12}^{\Delta_1+\Delta_2-\Delta_3}\mathrm{x}_{13}^{\Delta_1+\Delta_3-\Delta_2}\mathrm{x}_{23}^{\Delta_2+\Delta_3-\Delta_1}} f(P,Q,S)\,. \tag{4.13}$$

The function f must comply with the spin of the operators and also should not contain any redundant structures that are possible due to not all of Q, P, S being independent. As we will need only two- and three-point correlation functions, it is convenient to introduce the following notation

$$Q_1 \equiv Q^1_{32}\,, \qquad Q_2 \equiv Q^2_{13}\,, \qquad Q_3 \equiv Q^3_{21}\,, \qquad S_3 \equiv S^3_{21}\,. \tag{4.14}$$

The even power of any odd structure is even, which is manifested by [63]

$$S_3^2 + Q_1 Q_2 - P_{12}^2 \equiv 0\,. \tag{4.15}$$

This is the only relation we need for the $s_1 - s_2 - 0$ correlators. The even structures P, Q can be identified as the building blocks of the simplest correlators that are completely fixed by conformal symmetry

$$\langle j_{s_1}(\mathrm{x}_1,\eta_1)j_{s_2}(\mathrm{x}_2,\eta_2)\rangle \sim \frac{1}{\mathrm{x}_{12}^2}\delta_{s_1,s_2}(P_{12})^{s_1+s_2}\,, \tag{4.16}$$

$$\langle j_{s_1}(\mathrm{x}_1,\eta_1)j_0(\mathrm{x}_2)j_0(\mathrm{x}_3)\rangle \sim \frac{1}{\mathrm{x}_{12}\mathrm{x}_{23}\mathrm{x}_{31}}(Q_1)^{s_1}\,, \tag{4.17}$$

where we assumed that j_s is the spin-s conserved tensor and the weight of scalar operator j_0 is $\Delta = 1$. The conservation of currents can be imposed with the help of a simple third order operator:

$$\mathrm{div} = \frac{\partial^2}{\partial\eta_\alpha\partial\eta_\beta}\frac{\partial}{\partial\mathrm{x}^{\alpha\beta}}\,. \tag{4.18}$$

4.2.2 Free Boson

The simplest example of duality is between the (non)-minimal the Type-A HS gravity and ($U(N)$) $O(N)$ free scalars. Dropping the canonical normalization factors, the two-point functions for the $U(N)$ case are

$$U(N): \qquad\qquad \langle\bar{\phi}_a(\mathrm{x})\phi^b(0)\rangle = \delta^b_a\frac{1}{|\mathrm{x}|}\,, \tag{4.19}$$

where $a, b = 1, ..., N$. As is well known [22, 65, 66], it is convenient to pack the HS currents into generating functions

$$j(\mathrm{x},\eta) = f(u,v)\bar{\phi}_a(\mathrm{x}_1)\phi^a(\mathrm{x}_2)\Big|_{\mathrm{x}_i=\mathrm{x}}\,, \qquad u = \tfrac{1}{2}\eta^\alpha\eta^\beta\partial^1_{\alpha\beta}\,, \quad v = \tfrac{1}{2}\eta^\alpha\eta^\beta\partial^2_{\alpha\beta}\,. \tag{4.20}$$

The conservation of the current implies a simple differential equation for f. The most convenient formulae are obtained [66,67] with the help of an auxiliary generating function $C^a(\eta|x)$ for the derivatives of the scalar field. It obeys $\partial_{\alpha\beta}C^a = \frac{i}{2}\partial_\alpha\partial_\beta C^a$, which is solved by $C^a = \cos\left[2e^{i\pi/4}\sqrt{u}\right]\phi^a(\mathrm{x})$. The generating function of the HS currents then has a simple factorized form:

$$j(\mathrm{x},\eta) = C_a(u)C^a(-v) = \cos[2e^{i\pi/4}\sqrt{u}]\cos[2e^{i\pi/4}\sqrt{-v}]\bar{\phi}_a(\mathrm{x}_1)\phi^a(\mathrm{x}_2)\Big|_{\mathrm{x}_i=\mathrm{x}} . \quad (4.21)$$

The two-point function of the currents is then

$$\langle j_s j_s\rangle_{f.b.} \equiv \langle j(\mathrm{x}_1,\eta_1)j(\mathrm{x}_2,\eta_2)\rangle = \frac{1}{2}\frac{1}{\mathrm{x}_{12}^2}\cosh\left(2P_{12}\right) , \quad (4.22)$$

where we introduced a shorthand notation j_s for $j(\mathrm{x}_i,\eta_i)$ and the arguments are in accordance with its position inside the brackets; $f.b.$ refers to "free boson". Sometimes we have to distinguish the first member of the family $j_0 \equiv \bar{\phi}_a(\mathrm{x})\phi^a(\mathrm{x})$ from the others. The simplest three-point function, which is fixed by the symmetry, is

$$\langle j_s j_0 j_0\rangle_{f.b.} = \frac{2}{\mathrm{x}_{12}\mathrm{x}_{13}\mathrm{x}_{23}}\cos\left(\tfrac{1}{2}Q_1\right) . \quad (4.23)$$

The three-point functions with two HS currents are assembled into

$$\langle j_s j_s j_0\rangle_{f.b.} = \frac{2}{\mathrm{x}_{12}\mathrm{x}_{13}\mathrm{x}_{23}}\cos\left(\tfrac{1}{2}Q_1 + \tfrac{1}{2}Q_2\right)\cos P_{12} . \quad (4.24)$$

The three-point functions of the three HS currents are [68] (see also [63, 67, 69–71]):

$$\langle j_s j_s j_s\rangle_{f.b.} = \frac{2}{\mathrm{x}_{12}\mathrm{x}_{23}\mathrm{x}_{31}}\cos\left(\tfrac{1}{2}Q_1 + \tfrac{1}{2}Q_2 + \tfrac{1}{2}Q_3\right)\cos(P_{12})\cos(P_{23})\cos(P_{31}) . \quad (4.25)$$

It is also useful to consider the case of $U(N)$-singlet constraint with leftover $U(M)$ global symmetry. The generating function is

$$\langle j_s j_s j_0\rangle_{f.b.} = \frac{2}{\mathrm{x}_{12}\mathrm{x}_{13}\mathrm{x}_{23}} \exp\left[-\tfrac{i}{2}(Q_1+Q_2)\right]\, \cos P_{12}\ . \tag{4.26}$$

This is the most general case, all others being simple truncations. Imposing the bose symmetry we get (4.24). Truncation to even spins only gives the $O(N)$ case.

4.2.3 Free Fermion

The second example is the duality of the Type-B theory and a theory of free $U(N)$ or $USp(N)$ fermions, of which we consider the former. The two-point function is

$$U(N):\qquad\qquad \langle\tilde{\psi}^c{}_{a\alpha}(\mathrm{x})\psi^b_\beta(0)\rangle = \delta^b_a \frac{1}{2}\frac{\vec{\mathrm{x}}\cdot\vec{\sigma}_{\alpha\beta}}{|\mathrm{x}|^3} = \delta^b_a\partial_{\alpha\beta}|\mathrm{x}^2|^{-\frac{1}{2}}\ . \tag{4.27}$$

HS currents are constructed analogously

$$j(\eta,\mathrm{x}) = f(u,v)\eta^\alpha\eta^\beta\psi^a_\alpha(\mathrm{x}_1)\psi_{a\beta}(\mathrm{x}_2)\Big|_{\mathrm{x}_i=\mathrm{x}}\ , \tag{4.28}$$

and again the useful trick is to pack the derivatives into C^a as follows

$$\partial_{\alpha\beta}C^a = \tfrac{i}{2}\partial_\alpha\partial_\beta C^a\ ,\qquad\qquad C^a = \tfrac{1}{\sqrt{u}}\sin\left[2e^{i\pi/4}\sqrt{u}\right]\,\psi^a_\alpha(\mathrm{x})\eta^\alpha\ . \tag{4.29}$$

The generating function of the HS currents has the factorized form:

$$j(\eta,\mathrm{x}) = C^a(u)C_a(-v) = \frac{1}{\sqrt{-uv}}\sin\left[2e^{i\pi/4}\sqrt{u}\right]\sin\left[2e^{i\pi/4}\sqrt{-v}\right]\,\psi^a_\alpha(\mathrm{x}_1)\psi_{a\beta}(\mathrm{x}_2)\Big|_{\mathrm{x}_i=\mathrm{x}}\ . \tag{4.30}$$

The two-point function is normalized in the same way as that of the free boson:

$$\langle j_s j_s\rangle_{f.f.} = \frac{1}{2}\frac{1}{\mathrm{x}_{12}^2}\cosh\left(2P_{12}\right)\ , \qquad\qquad \langle\tilde{j}_0\tilde{j}_0\rangle_{f.f.} = \frac{1}{4\mathrm{x}^4}\ , \tag{4.31}$$

where $f.f.$ refers to "free fermion". Here the scalar singlet operator

$$\tilde{j}_0 = \tilde{\psi^c}_a\psi^a \tag{4.32}$$

has dimension 2 and is not captured by the generating function above. The three-point function of $\tilde{j}_0$ vanishes due to parity

$$\langle\tilde{j}_0\tilde{j}_0\tilde{j}_0\rangle_{f.f.} = 0\,. \tag{4.33}$$

For the other cases we find (see also [63, 67, 69–71]):

$$\langle j_s j_s\tilde{j}_0\rangle_{f.f.} = \frac{2\cos(\frac{1}{2}Q_1+\frac{1}{2}Q_2)}{\mathrm{x}_{23}^2\mathrm{x}_{13}^2}S_3\sin P_{12}\ , \tag{4.34}$$

$$\langle j_s\tilde{j}_0\tilde{j}_0\rangle_{f.f.} = \frac{2\sin\frac{1}{2}Q_1}{\mathrm{x}_{12}\mathrm{x}_{23}^3\mathrm{x}_{31}}Q_1\ , \tag{4.35}$$

and

$$\langle j_s j_s j_s\rangle_{f.f.} = \frac{2}{\mathrm{x}_{12}\mathrm{x}_{23}\mathrm{x}_{31}}\sin(\tfrac{1}{2}Q_1+\tfrac{1}{2}Q_2+\tfrac{1}{2}Q_3)\sin(P_{12})\sin(P_{23})\sin(P_{31})\ . \tag{4.36}$$

Note that the expression for the correlators of j_s is valid for $s\geq 1$ only, while any insertion of $\tilde{j}_0$ should be treated separately. Again, it is useful to work with the free fermion with leftover $U(M)$ global symmetry and the generating function is

$$\langle j_s j_s\tilde{j}_0\rangle_{f.f.} = \frac{2}{\mathrm{x}_{23}^2\mathrm{x}_{13}^2}\exp(\tfrac{i}{2}Q_1+\tfrac{i}{2}Q_2)S_3\sin P_{12}\ . \tag{4.37}$$

4.2.4 Critical Boson

Critical Boson theory is the IR fixed point under a double-trace deformation $(\phi^2)^2$. The regime that is relevant for AdS/CFT is the large-N, see e.g. [72,73] for the systematic $1/N$ expansion. To the leading order in $1/N$ the correlation function of HS currents j_s, $s > 0$ stays the same as in the free boson theory, i.e. (4.25). The dimension of the lowest singlet operator ϕ^2, which is usually dubbed σ, jumps from 1 for free boson to $2 + \mathcal{O}(1/N)$ for critical one. Therefore, the spectrum of the singlet operators in the critical boson theory looks like that of the free fermion in $N = \infty$ limit. For this reason the scalar singlet is denoted as $\tilde{j}_0$. Correlation functions with a number of $\tilde{j}_0$ insertions are related to those in the free boson theory by attaching propagators of the σ-field. In [64] the three-point functions $\langle j_{s_1} j_{s_2} \tilde{j}_0 \rangle$ were fixed by employing the non-conservation equation in the Chern-Simons matter theories. The result is

$$\langle j_{s_1} j_{s_2} \tilde{j}_0 \rangle_{c.b.} = f_{s_1,s_2} \frac{1}{\mathrm{x}_{13}^2 \mathrm{x}_{23}^2} (Q_1)^{s_1} (Q_2)^{s_2} \sum_k A_k \left(\frac{P_{12}^2}{Q_1 Q_2} \right)^k \,, \tag{4.38}$$

where $c.b.$ refers to “critical boson", f_{s_1,s_2} is a overall spin-dependent factor and the coefficients obey $A_{-1} = 0$, $A_0 = 1$ and the rest are generated via

$$A_n = \frac{A_{n-1} \left(s_1 s_2 + (2n-1-s_1)(2n-1-s_2) - 5n + 4 \right) - 2A_{n-2}(n - s_1 - 2)(n - s_2 - 2)}{n(2n-1)} \,.$$

The coefficients f_{s_1,s_2} depend on normalization of $\langle j_s j_s \rangle_{c.b.}$ two-point functions. We choose the same normalization as in the free boson theory (4.22), which gives:

$$f_{s_1,s_2} = -\frac{i^{s_1+s_2} 2^{-s_1-s_2+2}}{\Gamma\left(s_1 + \frac{1}{2}\right) \Gamma\left(s_2 + \frac{1}{2}\right)} \,. \tag{4.39}$$

4.2.5 Critical Fermion

Critical fermion is the Gross-Neveu model, i.e. free fermion with $(\tilde{\psi}^c\psi)^2$-deformation added. The large-N expansion works fine despite the apparent non-renormalizability of the interactions, see e.g. [74,75]. Again, the correlation functions of HS currents j_s, $s>0$ are the same as in the free fermion theory (4.36) to the leading order in $1/N$, while the dimension of $\sigma=\bar{\psi}\psi$ jumps from 2 in the free theory to $1+\mathcal{O}(1/N)$ in the Gross-Neveu one. Therefore, the spectrum of singlet operators looks like that of the free boson theory in the $N=\infty$ limit and we use the same notation j_0 for $\bar{\psi}\psi$. The three-point functions $\langle j_{s_1}j_{s_2}j_0\rangle$ were found in [64] to have the form:

$$\langle j_{s_1}j_{s_2}j_0\rangle_{c.f.}=g_{s_1,s_2}\frac{1}{\mathrm{x}_{13}\mathrm{x}_{23}\mathrm{x}_{12}}S_3P_{12}(Q_1)^{s_1-1}(Q_2)^{s_2-1}\sum_k A_k\left(\frac{P_{12}^2}{Q_1Q_2}\right)^k\,, \tag{4.40}$$

where $c.f.$ refers to “critical fermion", $A_{-1}=0$, $A_0=1$ and the rest of A_k is generated via

$$A_n=-\frac{8A_{n-2}(-n+s_1+1)(-n+s_2+1)-4A_{n-1}((2n-s_1-1)(2n-s_2-1)+n+s_1s_2)}{4n(2n+1)}\,.$$

The overall factor in our normalization is

$$g_{s_1,s_2}=-\frac{i^{s_1+s_2}2^{-s_1-s_2+2}}{\Gamma\left(s_1+\frac{1}{2}\right)\Gamma\left(s_2+\frac{1}{2}\right)}\,. \tag{4.41}$$

4.2.6 Chern-Simons Matter Theories

There are four theories that are obtained by coupling the free/critical boson/fermion theories discussed above to Chern-Simons gauge field. The conjecture of three-dimensional bosonization is that they are equivalent in pairs under an appropriate identification of parameters N and $\lambda=N/k$. We are interested in $\langle j_{s_1}j_{s_2}j_0\rangle$ and $\langle j_{s_1}j_{s_2}\tilde{j}_0\rangle$ correlators, which are constrained to be [62]:

$$\langle j_{s_1} j_{s_2} j_0 \rangle_{B.} = \tilde{N} \left[\cos\theta \langle j_{s_1} j_{s_2} j_0 \rangle_{f.b.} + \sin\theta \langle j_{s_1} j_{s_2} j_0 \rangle_{odd} \right] \,, \tag{4.42a}$$

$$\langle j_{s_1} j_{s_2} \tilde{j}_0 \rangle_{F.} = \tilde{N} \left[\cos\theta \langle j_{s_1} j_{s_2} \tilde{j}_0 \rangle_{f.f.} + \sin\theta \langle j_{s_1} j_{s_2} \tilde{j}_0 \rangle_{odd} \right] \,, \tag{4.42b}$$

where $B.$ and $F.$ refer to CS-boson and CS-fermion, respectively. Here $\tilde{N}$ and θ are two macroscopical parameters and we recall that $\cos^2\theta = 1/(1+\tilde{\lambda}^2)$. This result holds true to the leading order in $\tilde{N}$ but to all orders in $\tilde{\lambda}$. The expressions in terms of the microscopical parameters depend on type of theory and on N, $\lambda = N/k$. We can work with $\tilde{N}$ and θ since their microscopical origin is invisible from the bulk. Note that the odd structures are different in the two cases (they even have different conformal dimensions due to j_0 and $\tilde{j}_0$). In fact the two odd structures can be found as the regular correlators on the opposite side of the bosonization duality:

$$\langle j_{s_1} j_{s_2} j_0 \rangle_{odd} = \langle j_{s_1} j_{s_2} j_0 \rangle_{c.f.} \,, \qquad \langle j_{s_1} j_{s_2} \tilde{j}_0 \rangle_{odd} = \langle j_{s_1} j_{s_2} \tilde{j}_0 \rangle_{c.b.} \,. \tag{4.43}$$

As far as $\langle j_{s_1} j_{s_2} \tilde{j}_0 \rangle$ correlators are concerned, all the results reviewed in the previous sections follow from (4.42) and this is the structure we would like to reproduce from the bulk side, including the details of the correlators and normalization factors. Note that the term 'parity odd correlator' corresponding to the second part of (4.42) may not have anything to do with parity, in view of (4.43). For example, the usual parity even correlator $\langle j_s j_s \tilde{j}_0 \rangle_{c.b.}$ in the critical boson theory appears to be odd from the point of view of the dual fermionic theory.

4.3 Higher Spin Interactions

We briefly review the structure of the equations that results from the Vasiliev equations [35]. For more detailed reviews, we refer to [38, 42, 76]. The corrections to the free equations that are bilinear in the fields were worked out in [22, 76–78]. The main conclusion that is based on [22, 78] is that, up to quadratic order, the Klein-Gordon equation is sourced by two type of terms. One part is fixed by the HS algebra and is local enough for the computation of the correlation functions using field theory tools. A second part gives rise to infinities, though a proposal has been made [79] on how to obtain finite results by a set of field redefinitions. While our computations will mostly be based on the use of the first part, we shall nonetheless test this proposal as well in Section 4.5, and comment further about it in Section 4.7.

The convenient field variables that a HS theory can be built with are the Fronsdal fields. The Vasiliev equations yield first order differential equations, which upon solving for the auxiliary fields give equations in term of the Fronsdal fields and an infinitely many of differential consequences of these equations, which we drop.

For the purpose of computing tree-level Witten diagrams it is sufficient to impose the transverse and traceless gauge. Then the $4d$ free Fronsdal equations for spins $s \geq 1$ read:

$$(\Box + 2(s^2 - 2s - 2))\Phi_{\underline{m}_1 \dots \underline{m}_s} = 0 \,, \quad \Phi^{\underline{n}}{}_{\underline{n}\underline{m}_3 \dots \underline{m}_s} = 0 \,, \quad \nabla^{\underline{n}}\Phi_{\underline{n}\underline{m}_2 \dots \underline{m}_s} = 0 \,. \tag{4.44}$$

To make a link to the frame-like and then to the unfolded formulation of the $4d$ HS theories we replace a set of the traceless world tensors $\Phi_{\underline{m}_1 \dots \underline{m}_s}$ with the generating function in the spinorial language as

$$\Phi(y, \bar{y}|x) = \sum_s \frac{1}{s!s!} \Phi_{\alpha_1 \dots \alpha_s, \dot{\alpha}_1 \dots \dot{\alpha}_s}(x)\, y^{\alpha_1} \dots y^{\alpha_s}\, \bar{y}^{\dot{\alpha}_1} \dots \bar{y}^{\dot{\alpha}_s} \,. \tag{4.45}$$

Then the gauge-fixed Fronsdal equations, as recovered from the $4d$ HS theory, read:

$$\big(\Box + 2(N^2 - 2N - 2)\big)\, \Phi = 0 \,, \qquad (\partial \nabla \bar{\partial})\Phi = 0 \,, \tag{4.46}$$

where N is the number operator $N = y^\alpha \partial_\alpha$ that counts spin (equivalently we can use $\bar{N} = \bar{y}^{\dot{\alpha}} \partial_{\dot{\alpha}}$). The last equation manifests the transverse gauge.

In the unfolded approach [80], a specific multiplet of HS fields is packed into generating functions ω and C that take values in a HS algebra. In our case the relevant HS algebra [81] is the even part of the Weyl algebra A_2. It is convenient to split the four generators of A_2 into (anti)-fundamentals of $sl(2,\mathbb{C})$

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}\,, \qquad\qquad [\hat{\bar{y}}_{\dot{\alpha}}, \hat{\bar{y}}_{\dot{\beta}}] = 2i\epsilon_{\dot{\alpha}\dot{\beta}}\,. \tag{4.47}$$

In practice, non-commuting operators $\hat{y}_\alpha$ and $\hat{\bar{y}}_{\dot{\alpha}}$ are replaced by commuting variables y_α, $\bar{y}_{\dot{\alpha}}$ while the product is replaced with the star product

$$(f \star g)(y,\bar{y}) = f(y,\bar{y}) \exp i \left[\frac{\overleftarrow{\partial}}{\partial y^\alpha} \epsilon^{\alpha\beta} \frac{\overrightarrow{\partial}}{\partial y^\beta} + \frac{\overleftarrow{\partial}}{\partial \bar{y}^{\dot{\alpha}}} \epsilon^{\dot{\alpha}\dot{\beta}} \frac{\overrightarrow{\partial}}{\partial \bar{y}^{\dot{\beta}}} \right] g(y,\bar{y})\,. \tag{4.48}$$

The bosonic higher spin algebra is defined as the even subalgebra, i.e. $f(y,\bar{y}) \in hs$ implies $f(y,\bar{y}) = f(-y,-\bar{y})$. It is straightforward to tensor any higher spin algebra with matrix algebra as to get $U(N)$ extension, for example. The SUSY case are studied in [24].

The field content consists of one-form $\omega = \omega_{\underline{m}}(y,\bar{y}|x)\, dx^{\underline{m}}$ and zero-form $C = C(y,\bar{y}|x)$. In free theory the Fronsdal fields can be identified with certain components of ω, which in the traceless gauge is

$$h^{\alpha\dot{\alpha}} \partial_\alpha \partial_{\dot{\alpha}} \Phi(y,\bar{y}|x) \in \omega(y,\bar{y}|x) \tag{4.49}$$

ω and C obey the following linearized equations, known as the on mass-shell theorem

(OMST) [80],

$$\mathsf{D}\omega = \mathcal{V}(\Omega,\Omega,C)\ , \qquad \widetilde{\mathsf{D}}C = 0\ , \tag{4.50}$$

where $\mathcal{V}(\Omega,\Omega,C)$ is a star function quadratic in Ω and linear in C, and

$$\mathsf{D}\omega \equiv d\omega - \Omega\star\omega - \omega\star\Omega = \nabla\omega - h^{\alpha\dot\alpha}(y_\alpha\bar\partial_{\dot\alpha} + \bar y_{\dot\alpha}\partial_\alpha)\,\omega\ , \tag{4.51}$$

$$\widetilde{\mathsf{D}}C \equiv dC - \Omega\star C + C\star\pi(\Omega) = \nabla C + ih^{\alpha\dot\alpha}(y_\alpha\bar y_{\dot\alpha} - \partial_\alpha\bar\partial_{\dot\alpha})\,C\ . \tag{4.52}$$

The AdS_4 connection Ω is defined in section 4.1, and the Lorentz covariant derivative acts in the same way on $\omega(y,\bar y|x)$ and $C(y,\bar y|x)$ as

$$\nabla \equiv d - \omega^{\alpha\beta}_{(0)}y_\alpha\partial_\beta - \bar\omega^{\dot\alpha\dot\beta}_{(0)}\bar y_{\dot\alpha}\bar\partial_{\dot\beta}\ . \tag{4.53}$$

The difference between D and $\widetilde{\mathsf{D}}$ is due to automorphism π: $\pi(f)(y,\bar y) = f(y,-\bar y) = f(-y,\bar y)$. The vertex that relates the order-s curl of the Fronsdal field to the HS Weyl tensors reads

$$\mathcal{V}(\Omega,\Omega,C) = A\big[H^{\alpha\beta}\partial_\alpha\partial_\beta C(y,0)e^{-i\theta} + \bar H^{\dot\alpha\dot\beta}\bar\partial_{\dot\alpha}\bar\partial_{\dot\beta}C(0,\bar y)e^{i\theta}\big]\ , \tag{4.54}$$

where $H^{\alpha\beta} = h^\alpha{}_\gamma \wedge h^{\beta\gamma}$, and analogously for $\bar H^{\dot\alpha\dot\beta}$. The constant A is an arbitrary normalization factor that we choose to be $A = i/4$. The linearized equations (4.50) are invariant under the linearized HS gauge transformations

$$\delta\omega = \mathsf{D}\xi\ , \qquad \delta C = 0\ . \tag{4.55}$$

Equations (4.50) are equivalent to Fronsdal equations supplemented with differential consequences thereof. Up to the second order the unfolded equations should have the following schematic form

$$d\omega = \omega \star \omega + \mathcal{V}(\omega, \omega, C) + \mathcal{V}^2(\omega, \omega, C, C) + \mathcal{O}(C^3)\,, \tag{4.56a}$$

$$dC = \omega \star C - C \star \pi(\omega) + \mathcal{U}(\omega, C, C) + \mathcal{O}(C^3)\,, \tag{4.56b}$$

where the vertices $\mathcal{V}^2$ and $\mathcal{U}$ need to be specified. The free equations result upon substituting $\omega \rightarrow \Omega + \omega$ and picking the terms that are linear in ω and C:

$$\mathsf{D}\omega^{(1)} = \mathcal{V}(\Omega, \Omega, C^{(1)})\,, \qquad\qquad \widetilde{\mathsf{D}}C^{(1)} = 0\,. \tag{4.57}$$

At the second order the weak-field expansion over the AdS background leads to

$$\mathsf{D}\omega^{(2)} - \mathcal{V}(\Omega, \Omega, C^{(2)}) = \omega^{(1)} \star \omega^{(1)} + \mathcal{V}(\Omega, \omega^{(1)}, C^{(1)}) + \mathcal{V}(\Omega, \Omega, C^{(1)}, C^{(1)})\,, \tag{4.58}$$

$$\widetilde{\mathsf{D}}C^{(2)} = \omega^{(1)} \star C^{(1)} - C^{(1)} \star \pi(\omega^{(1)}) + \mathcal{V}(\Omega, C^{(1)}, C^{(1)})\,. \tag{4.59}$$

We see that the second order fluctuations are sourced by the terms that are bilinear in the first order fluctuations. The terms that are bilinear in the zero-forms, the CC-terms for short, can be non-local. The part of the equations that does not have any problems with locality comes from the commutator in the HS algebra

$$\widetilde{\mathsf{D}}C^{(2)} = \omega^{(1)} \star C^{(1)} - C^{(1)} \star \pi(\omega^{(1)}) + \mathcal{O}(C^2)\,. \tag{4.60}$$

For some values of spins we do not expect the CC-terms to contribute. Indeed, when one of the legs is scalar, there is a unique (parity-preserving) coupling $s_1 - s_2 - 0$, [82]. It is non-abelian whenever $s_1 \neq s_2$ and is abelian for $s_1 = s_2$.

Let us note that the computation of correlators from the equations of motion that are not derived from an action principle does not guarantee the bose symmetry of the correlators under the permutation of the points. The same correlator $\langle j_{s_1} j_{s_2} j_0 \rangle$ can be obtained in two different ways: either by treating j_{s_1}, j_{s_2} as sources and then solving for the scalar field or by treating j_{s_1}, j_0 as sources and solving for the spin-s_2 Weyl tensor. Both computations are possible with (4.60). It was checked in [22] for the leading coefficients that the two ways of getting the same correlator give a bose symmetric correlator.

Lastly, Eq.(4.60) remains consistent if the fields ω, C are extended to matrix-valued fields. This way one can introduce Yang-Mills groups on top of the HS algebra. On the CFT side this should correspond to leftover global symmetries, i.e. those symmetries that remain after the singlet constraint is imposed.

4.4 Kinematics of the Boundary-to-Bulk Propagators

In this section we discuss the boundary-to-bulk propagators for HS fields. The problem has been extensively studied starting from the lower spin fields, see e.g. [83]. Specifically, we need the unfolded propagators, i.e. the propagators for the Fronsdal fields supplemented with derivatives thereof as to obey the free equations (4.50). In some form the unfolded propagators were found in [22]. In [70] it was observed that the propagators are simple functions that depend on a few universal geometrical data, which is the spinorial analog of [84].

4.4.1 Definitions

One of the basic objects is the Witten bulk-to-boundary propagator for the scalar field from the bulk point $x^{\underline{m}} = (\mathrm{x}^i, z)$ to the boundary point x_a

$$K_a = \frac{z}{(\mathrm{x} - \mathrm{x}_a)^2 + z^2}\,. \tag{4.61}$$

We often set $\mathrm{x}_a = 0$ in this section as the generic point can be recovered thanks to the three-dimensional Poincaré invariance. The propagator is the boundary limit of the geodesic distance, which the bulk-to-bulk propagator should depend on. It obeys the regular ($\Delta = 1$) boundary condition. The propagator K can be used to define a wave-vector, which turns into bi-spinor $F^{\alpha\dot{\alpha}}$ in the $4d$ spinorial language:

$$d \ln K = F_{\alpha\dot{\alpha}} h^{\alpha\dot{\alpha}} \,, \tag{4.62}$$

where

$$F^{\alpha\dot{\alpha}} = \left(\frac{2z}{\mathrm{x}^2 + z^2} \mathrm{x}^{\alpha\dot{\alpha}} - \frac{\mathrm{x}^2 - z^2}{\mathrm{x}^2 + z^2}\, i\epsilon^{\alpha\dot{\alpha}} \right) \,. \tag{4.63}$$

It will play the same role as the on-shell momentum p plays for e^{ipx} and is the boundary limit of the vector that is tangent to the geodesic connecting two bulk points. There also exist the parallel-transport bi-spinors $\Pi^{\alpha\beta}$ and $\bar{\Pi}^{\dot{\alpha}\beta}$ given by

$$\Pi^{\alpha\beta} = K \left(\frac{1}{\sqrt{z}}\, \mathrm{x}^{\alpha\beta} + \sqrt{z}\, i\epsilon^{\alpha\beta} \right) \,, \quad \bar{\Pi}^{\dot{\alpha}\beta} = K \left(\frac{1}{\sqrt{z}}\, \mathrm{x}^{\dot{\alpha}\beta} - \sqrt{z}\, i\epsilon^{\dot{\alpha}\beta} \right) = (\Pi^{\alpha\beta})^\dagger \,, \tag{4.64}$$

that allow one to propagate the boundary polarization spinors into the bulk:

$$\xi^{\alpha} = \Pi^{\alpha\beta} \eta_\beta \, e^{+i\frac{\pi}{4}} \,, \qquad \bar{\xi}^{\dot{\alpha}} = (\xi^\alpha)^\dagger = \bar{\Pi}^{\dot{\alpha}\beta} \eta_\beta \, e^{-i\frac{\pi}{4}} \,. \tag{4.65}$$

This is the full set of the data that any propagator can depend on. The set is closed under covariant derivatives:

$$\nabla F^{\alpha\dot{\alpha}} = h^{\alpha\dot{\alpha}} + F^{\alpha}{}_{\dot{\gamma}}\, h^{\delta\dot{\gamma}} F_{\delta}{}^{\dot{\alpha}} = 0 \,, \tag{4.66}$$

and the parallel transported spinors obey

$$\nabla\xi^{\alpha} - F^{\alpha}{}_{\dot{\gamma}}\,\xi_{\delta} h^{\delta\dot{\gamma}} = 0\,, \qquad \nabla\xi^{\dot{\alpha}} - F_{\delta}{}^{\dot{\alpha}}\,\xi_{\dot{\gamma}} h^{\delta\dot{\gamma}} = 0\,. \tag{4.67}$$

In practice it is useful to rewrite the Lorentz-covariant derivatives with all indices being explicit:

$$\nabla_{\alpha\dot{\alpha}} K = K F_{\alpha\dot{\alpha}}\,, \quad \nabla_{\alpha\dot{\alpha}}\xi_{\beta} = F_{\beta\dot{\alpha}}\xi_{\alpha}\,, \quad \nabla_{\alpha\dot{\alpha}}\bar{\xi}_{\dot{\beta}} = F_{\alpha\dot{\beta}}\xi_{\dot{\alpha}}\,, \quad \nabla_{\alpha\dot{\alpha}} F_{\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + F_{\alpha\dot{\alpha}} F_{\beta\dot{\beta}}\,. \tag{4.68}$$

As a consequence of the differential constraints above one also finds

$$(\Box - 4)K = 0\,, \quad (\Box - 6)F^{\alpha\dot{\alpha}} = 0\,, \quad (\Box - 4)\xi^{\alpha} = 0\,, \quad (\Box - 4)\bar{\xi}^{\dot{\alpha}} = 0\,. \tag{4.69}$$

4.4.2 Algebraic Identities

The quantities defined above obey several algebraic identities. The wave-vector satisfies

$$F^{\alpha\dot{\alpha}} F^{\beta}{}_{\dot{\alpha}} = \epsilon^{\alpha\beta}\,, \qquad F^{\alpha\dot{\alpha}} F_{\alpha}{}^{\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}\,. \tag{4.70}$$

It behaves in many respects as a 'symplectic' structure that converts the dotted and undotted indices to each other as follows

$$F^{\alpha}{}_{\dot{\gamma}}\,\bar{\Pi}^{\dot{\gamma}\beta} = i\Pi^{\alpha\beta}\,, \qquad F_{\gamma}{}^{\dot{\alpha}}\,\Pi^{\gamma\beta} = -i\bar{\Pi}^{\dot{\alpha}\beta}\,, \qquad \Pi^{\alpha}{}_{\gamma}\,\bar{\Pi}^{\dot{\alpha}\gamma} = -iKF^{\alpha\dot{\alpha}}\,, \tag{4.71}$$

$$\xi^{\alpha} = F^{\alpha}{}_{\dot{\alpha}}\,\bar{\xi}^{\dot{\alpha}}\,, \qquad \bar{\xi}^{\dot{\alpha}} = \xi^{\alpha} F_{\alpha}{}^{\dot{\alpha}}\,. \tag{4.72}$$

The parallel-transport bi-spinors $\Pi^{\alpha\beta}$ and $\bar{\Pi}^{\dot{\alpha}\beta}$ obey identities similar to (4.70):

$$\Pi^{\alpha\beta}\Pi^{\gamma}{}_{\beta} = K\epsilon^{\alpha\gamma}\,, \qquad \bar{\Pi}^{\dot{\alpha}\beta}\bar{\Pi}^{\dot{\gamma}}{}_{\beta} = K\epsilon^{\dot{\alpha}\dot{\gamma}}\,, \tag{4.73}$$

$$\Pi^{\beta\alpha}\Pi_{\beta}{}^{\gamma} = K\epsilon^{\alpha\gamma}\,, \qquad \bar{\Pi}^{\dot{\beta}\alpha}\bar{\Pi}_{\dot{\beta}}{}^{\gamma} = K\epsilon^{\alpha\gamma}\,. \tag{4.74}$$

There are also useful identities involving the one-form $h^{\alpha\dot{\alpha}} = dx^{\underline{m}}h^{\alpha\dot{\alpha}}_{\underline{m}}$:

$$(F\cdot h)F^{\alpha\dot{\alpha}} + h^{\alpha\dot{\alpha}} = F^{\alpha}{}_{\dot{\beta}}\,h^{\beta\dot{\beta}}F_{\beta}{}^{\dot{\alpha}}\,, \tag{4.75}$$

$$(F\cdot h)\xi^{\alpha} + (F^{\alpha}{}_{\dot{\gamma}}\,h_{\beta}{}^{\dot{\gamma}} - F_{\beta\dot{\gamma}}h^{\alpha\dot{\gamma}})\xi^{\beta} = 0\,, \tag{4.76}$$

which result from the fact that anti-symmetrization over any three spinorial indices vanishes identically.

4.4.3 Inversion Map

The basic computational tool we will employ is based on the inversion trick [85]. For that reason it is important to know the transformation properties of the variables defined above under the inversion isometry of AdS_4. Since the boundary-to-bulk objects K, F, ξ and $\bar{\xi}$ depend both on the bulk point, on the boundary point and polarization spinor η, the inversion map on the AdS-side should be accompanied by the inversion map on the boundary. Using the inversion map rules (4.9), we derive the following transformation properties:

$$K(R(\mathrm{x},z);R\mathrm{x}_i) = \mathrm{x}_i^2 K(\mathrm{x},z;\mathrm{x}_i)\ , \tag{4.77a}$$

$$\Pi^{\alpha\beta}(R(\mathrm{x},z);R(\mathrm{x}_i)) = -J^{\alpha}{}_{\dot\gamma}\,\bar\Pi^{\dot\gamma}{}_{\delta}\,(\mathrm{x},z;\mathrm{x}_i)\mathrm{x}_i^{\delta\beta}\ , \tag{4.77b}$$

$$\xi^{\alpha}(R(\mathrm{x},z);R(\mathrm{x}_i,\eta_i)) = +iJ^{\alpha}{}_{\dot\gamma}\,\bar\xi^{\dot\gamma}(\mathrm{x},z;\mathrm{x}_i,\eta_i)\ , \tag{4.77c}$$

$$\bar\xi^{\dot\alpha}(R(\mathrm{x},z);R(\mathrm{x}_i,\eta_i)) = -iJ_{\gamma}{}^{\dot\alpha}\,\xi^{\gamma}(\mathrm{x},z;\mathrm{x}_i,\eta_i)\ , \tag{4.77d}$$

$$F^{\alpha\dot\alpha}(R(\mathrm{x},z);R(\mathrm{x}_i)) = J^{\alpha}{}_{\dot\beta}\,J_{\beta}{}^{\dot\alpha}\,F^{\beta\dot\beta}(\mathrm{x},z;\mathrm{x}_i)\ , \tag{4.77e}$$

where we defined $J^{\alpha\dot\alpha}$ as

$$J^{\alpha\dot\alpha} = \frac{x^{\alpha\dot\alpha}}{\sqrt{x^2}} = \frac{\mathrm{x}^{\alpha\dot\alpha}+iz\epsilon^{\alpha\dot\alpha}}{\sqrt{\mathrm{x}^2+z^2}}\ , \qquad\qquad J^{\alpha}{}_{\dot\gamma}\,J^{\beta\dot\gamma} = -\epsilon^{\alpha\beta}\ . \tag{4.78}$$

4.5 Vertices and Propagators

In this section we discuss the boundary-to-bulk propagators for HS fields and evaluate the vertex (4.60) on the propagators.

4.5.1 Propagators

With the help of the geometric objects introduced in Section 4.4 it is very easy to construct propagators. First of all, there is a unique expression for the Fronsdal field propagator:

$$\Phi_{\alpha(s),\dot\alpha(s)} = -2\sigma^{2s}A\frac{\Gamma[s]\Gamma[s]}{\Gamma[2s]}\left[K\xi_{\alpha(s)}\bar\xi_{\dot\alpha(s)}\right]\ , \tag{4.79}$$

where A is the factor from the free unfolded equations (4.54), σ is a parameter that counts spin, the Γ-functions will be explained later as the most convenient normalization.

In practice we need the propagators for ω and C fields that enter the unfolded equations. These fields encode derivatives of the Fronsdal field. The propagators can be written in a very compact form as†

$$\begin{aligned}
\omega &= -2\sigma^2 AKh_{\alpha\dot{\alpha}}\xi^{\alpha}\bar{\xi}^{\dot{\alpha}} \int_0^1 dt\, \exp i[\sigma t y^{\alpha}\xi_{\alpha} - (1-t)\sigma\bar{y}^{\dot{\alpha}}\bar{\xi}_{\dot{\alpha}}]\,, \\
C &= K\exp i[-y_{\alpha}F^{\alpha\dot{\alpha}}\bar{y}_{\dot{\alpha}} + \sigma y^{\alpha}\xi_{\alpha} + \theta] + K\exp i[-y_{\alpha}F^{\alpha\dot{\alpha}}\bar{y}_{\dot{\alpha}} - \sigma\bar{y}^{\dot{\alpha}}\bar{\xi}_{\alpha} - \theta]\,, \\
&= K\exp i[-y_{\alpha}F^{\alpha\dot{\alpha}}\bar{y}_{\dot{\alpha}} + \sigma y^{\alpha}\xi_{\alpha} + \theta] + h.c.\,,
\end{aligned} \tag{4.80}$$

where A is a constant that is related to the normalization of the free unfolded equations (4.54). The $h.c.$ operation is defined as

$$h.c.(\xi) = -\bar{\xi}\,, \qquad\qquad h.c.(\theta) = -\theta \tag{4.81}$$

and σ is just a factor that counts spins, so can be put to one or any $\sigma(s)$. We will fix this normalization later.

4.5.2 Vertices

Let us remind that the equation we will extract the correlation function from is

$$\widetilde{\mathsf{D}}C^{(2)} = \omega \star C - C \star \pi(\omega) + \mathcal{O}(C^2)\,, \tag{4.82}$$

where we evaluate the r.h.s. on the propagators and then solve for the Klein-Gordon equation. Expanding the master zero-form C as

$$C(y,\bar{y}|x) = \phi(x) + \phi_{\alpha\dot{\alpha}}(x)y^{\alpha}\bar{y}^{\dot{\alpha}} + \cdots\,, \tag{4.83}$$

†The unfolded propagators were first found in [22] in a different form, especially the ω propagator where the gauge ambiguity is essential. The C propagator was cast into the form below in [70], see also [68,69,71].

the Klein-Gordon equation with a source is hidden in the first order form:

$$d\phi - ih_{\alpha\dot{\alpha}}\phi^{\alpha\dot{\alpha}} = h^{\alpha\dot{\alpha}}P_{\alpha\dot{\alpha}}\ , \tag{4.84}$$

$$\nabla\phi^{\alpha\dot{\alpha}} + h^{\alpha\dot{\alpha}}\phi + ... = h^{\beta\dot{\beta}}P^{\alpha\dot{\alpha}}_{\beta\dot{\beta}}\ , \tag{4.85}$$

where the one-form $\mathcal{P} = h^{\alpha\dot{\alpha}}P_{\alpha\dot{\alpha}}$ denotes the source built out of the free fields:

$$\mathcal{P} = \omega \star C - C \star \pi(\omega) \tag{4.86}$$

and we need the first two Taylor coefficients only:

$$\mathcal{P} = h^{\alpha\dot{\alpha}}\left[P_{\alpha\dot{\alpha}} + P^{\beta\dot{\beta}}_{\alpha\dot{\alpha}}\, y_\beta \bar{y}_{\dot{\beta}} + \cdots\right]\ . \tag{4.87}$$

The first constraint can be solved for the auxiliary field $\phi^{\alpha\dot{\alpha}}$ and the result then plugged into the second one to get the Klein-Gordon equation with a source [18]:

$$(\Box - 4)\phi = \left[\nabla^{\alpha\dot{\alpha}}P_{\alpha\dot{\alpha}} + i\partial^\alpha\partial^{\dot{\alpha}}P_{\alpha\dot{\alpha}}\right]\Big|_{y,\bar{y}=0}\ . \tag{4.88a}$$

Since the propagators are known, we can get the source explicitly as

$$(\Box - 4)\phi = -4AK_1K_2\int_0^1 dt\,\left[(1+h.c.)(1+\pi_{\xi_1})\left(w + it(1-t)w^2 + itw\xi_1\xi_2\right)B\right]\ , \tag{4.88b}$$

$$B = \exp i[t(1-t)w + t\xi_1\xi_2 + \theta]\ , \tag{4.88c}$$

$$w = (\xi_1 F_2 \bar{\xi}_1)\ , \tag{4.88d}$$

where K_1, ξ_1 refer to ω and K_2, ξ_2, F_2 to C, and

$$(\xi_1\xi_2) \equiv \xi_1^\alpha \xi_{2\alpha} \,, \qquad (\xi_1 F_2 \bar{\xi}_1) \equiv \xi_{1\alpha} F^{\alpha\dot{\alpha}} \xi_{2\dot{\alpha}} \,. \tag{4.89}$$

Here π_ξ is the twist map that is now realized on ξ (or $\bar{\xi}$ due to the bosonic projection), $\pi(\omega) = \omega(-\xi, \bar{\xi}) = \omega(\xi, -\bar{\xi})$. The appearance of π_{ξ_1} is responsible for the difference between HS fields with and without additional Yang-Mills groups. If we keep π_{ξ_1} then in the dual CFT the correlators $\langle j_{s_1} j_{s_2} j_0 \rangle$ will vanish for $s_1 + s_2$ odd. If we drop π_{ξ_1} then on the CFT side $\langle j_{s_1} j_{s_2} j_0 \rangle$ does not vanish for $s_1 + s_2$ odd and therefore we have a leftover global symmetry group. The latter case is more general and is easier to deal with.

On expanding the generating function (4.88b) and picking the terms of spins s_1 from ω and s_2 from C we find the vertex evaluated on the propagators in a very simple form[‡]

$$(\Box - 4)\phi = -4AK_1K_2 \sum_{s_1,s_2} V_{s_1,s_2,0} \,, \tag{4.90a}$$

$$V_{s_1,s_2,0} = v_{s_1,s_2,0} \left[(\xi_1 F_2 \bar{\xi}_1)^{s_1-s_2} (\xi_1\xi_2)^{2s_2} e^{i\theta} + h.c. \right] \,, \tag{4.90b}$$

$$v_{s_1,s_2,0} = \frac{i^{s_1+s_2-1}\Gamma(s_1+s_2+1)}{\Gamma(2s_1)\Gamma(2s_2+1)} \,, \tag{4.90c}$$

for $s_1 > s_2$. As we already commented in the Introduction, the vertex above can be used to obtain $0 - s_1 - s_2$ correlators for $s_1 \neq s_2$, but extrapolation to $s_1 = s_2$ will give the correct answer too since the correlation function depends smoothly on spins. Nevertheless, the $0 - s - s$ correlators should originate from the CC-terms in (4.56b), which previously gave infinite result [22, 86]. Therefore, we would like to use the proposal of [79], where

[‡]This vertex is also present in [22], but it does not seem to have such a simple form as below, which should be related to the ω propagator being in a different gauge.

the new CC-terms are

$$\mathcal{U}(h,C,C) = \frac{1}{4}\int_0^1 d\tau \int \frac{d\bar{u}d\bar{v}}{(2\pi)^2}\, e^{i\bar{u}_{\dot\beta}\bar{v}^{\dot\beta}}\, h_{\alpha\dot\alpha}y^\alpha\,[\tau\bar{u}+(1-\tau)\bar{v}]^{\dot\alpha}\; C(\tau y,\bar{y}+\bar{u})C((1-\tau)y,\bar{y}+\bar{v}) \\ +\,\mathrm{h.c}\,. \tag{4.91}$$

A simple computation along the lines above gives[§]

$$(\Box - 4)\phi = -4AK_1K_2\sum_s V_{s,s,0}\,. \tag{4.92}$$

Therefore, the $0-s-s$ vertex turns out to have the same form as the naive extrapolation of (4.90) and will give the correct answer without any additional computation needed once (4.90) is shown to be correct.

4.6 Computation of the Cubic Amplitude

From the bulk vertex (4.90b), the Witten diagram amplitude for $\langle J_{s_1}J_{s_2}J_0\rangle$ for $s_1 > s_2$ is obtained as

$$\begin{aligned}\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.} &= (-4A)c_{s_1}c_{s_2}c_0\, v_{s_1,s_2,0}\times \\ &\times\int \frac{d^3\mathrm{x}\,dz}{z^4}K_1K_2(K_3)^\Delta(\xi_1F_2\bar\xi_1)^{s_1-s_2}[(\xi_1\xi_2)^{2s_2}e^{i\theta}+(\bar\xi_1\bar\xi_2)^{2s_2}e^{-i\theta}]\,,\end{aligned} \tag{4.93}$$

where

$$K_i = K(\mathrm{x}-\mathrm{x}_i,z)\,,\quad F_i^{\alpha\dot\alpha} = F^{\alpha\dot\alpha}(\mathrm{x}-\mathrm{x}_i,z)\,,\quad \xi_i^\alpha = \xi^\alpha(\mathrm{x}-\mathrm{x}_i,z;\eta_i)\,,\quad \bar\xi_i^{\dot\alpha} = \bar\xi^{\dot\alpha}(\mathrm{x}-\mathrm{x}_i,z;\eta_i)\,. \tag{4.94}$$

[§]The issue of θ-dependence is unclear to us since in [79] it was proposed to take the $\mathcal{N}=2$ supersymmetric HS model and truncate it to the bosonic one using the boundary conditions that contain θ-dependence. Nevertheless, our bosonic truncation which does not rely on imposition of θ-dependent boundary conditions gives the right answer.

The factor $(-4A)$ in (4.93) does not have any physical meaning and is an arbitrary normalization factor between ω and C in (4.54). For convenience, we reproduce the factor

$$v_{s_1,s_2,0} = \frac{i^{s_1+s_2-1}\Gamma(s_1+s_2+1)}{\Gamma(2s_1)\Gamma(2s_2+1)} \,. \tag{4.95}$$

Also, we introduced normalization factors c_s for each of the three fields. These cannot be fixed from the equations of motion and correspond to a freedom on the CFT side to normalize at will the two-point functions $\langle j_s j_s\rangle$. Lastly, there are two options for boundary conditions on the scalar fields: $\Delta = 1$ and $\Delta = 2$.

The three-point integrals are doable in principle due to the fact that one can always ‘scalarize’ the integrand by representing all $x^{\alpha\dot{\alpha}}$-factors as derivatives with respect to the boundary points x_i. The scalar three-point integral was done long ago in [85]. The problem is to scalarize in the most efficient way as to break as less symmetries as possible. We extend the inversion method of [85] to our case. Firstly, using the translation invariance we can set $\mathrm{x}_1 = 0$. Then, we apply the inversion map both to the boundary and bulk data. As a result the basic structures that enter the integrand drastically simplify:

$$\frac{d^3\mathrm{x}dz}{z^4} \to \frac{d^3\mathrm{x}dz}{z^4} \,, \tag{4.96a}$$

$$K_1 \to z \,, \tag{4.96b}$$

$$K_{2,3} \to \mathrm{x}_{2,3}^2 K_{2,3} \,, \tag{4.96c}$$

$$(\xi_1 F_2 \bar{\xi}_1) \to 2zK_2[\eta_1(\mathrm{x}-\mathrm{x}_2)\eta_1] = -2zK_2T_{11} \,, \tag{4.96d}$$

$$(\xi_2 F_1 \bar{\xi}_2) \to 2K_2^2[\eta_2(\mathrm{x}-\mathrm{x}_2)\eta_2] = -2K_2^2T_{22} \,, \tag{4.96e}$$

$$(\xi_1\xi_2) + (\bar{\xi}_1\bar{\xi}_2) \to -2zK_2(\eta_1\eta_2) \,, \tag{4.96f}$$

$$(\xi_1\xi_2) - (\bar{\xi}_1\bar{\xi}_2) \to 2iK_2[\eta_1(\mathrm{x}-\mathrm{x}_2)\eta_2] = -2iK_2T_{12} \,, \tag{4.96g}$$

where we defined

$$T_{ij} = -[\eta^i_\alpha (\mathrm{x} - \mathrm{x}_2)^{\alpha\beta} \eta^j_\beta] \,, \tag{4.97}$$

and we will use the same notation for the variables after the inversion is applied. Our strategy is to rewrite the integrand in terms of simple differential operators acting on a scalar integrand

$$\int \frac{d^3\mathrm{x} dz}{z^4} z^a (K_2)^b (K_3)^\Delta = (\mathrm{x}_{23})^{a-b-\Delta} I_{a,b,\Delta} \,, \tag{4.98}$$

where

$$I_{a,b,\Delta} = \frac{\pi^{3/2}\Gamma\left(\frac{1}{2}(a+b-\Delta)\right)\Gamma\left(\frac{1}{2}(a-b+\Delta)\right)\Gamma\left(\frac{1}{2}(-a+b+\Delta)\right)\Gamma\left(\frac{1}{2}(a+b+\Delta-3)\right)}{2\Gamma(a)\Gamma(b)\Gamma(\Delta)} \tag{4.99}$$

There are three operators that can be immediately observed to generate the same structures that occur under the integral sign:

$$O_{11} = (\eta_1 \partial_2 \eta_1) \equiv \eta_1^\alpha \frac{\partial}{\partial \mathrm{x}_2^{\alpha\beta}} \eta_1^\beta \,, \qquad O_{12} = (\eta_1 \partial_2 \eta_2) \,, \qquad O_{22} = (\eta_2 \partial_2 \eta_2) \,. \tag{4.100}$$

The operators act on K_2 factors only and yield:

$$O_{ij} f(K_2) = \frac{(K_2)^2}{z} \frac{\partial}{\partial K_2} f(K_2)\, T_{ij} \,. \tag{4.101}$$

There is one relation between T_{ij} that is the bulk analog of the $S_3^2 + Q_1 Q_2 - P_{12}^2 \equiv 0$ relation:

$$(T_{12})^2 = T_{11} T_{22} + \mathrm{x}_{23}^2 (\eta_1 \eta_2)^2 \,. \tag{4.102}$$

The integrand can be represented as a function of O_{ij} and $(\eta_1\eta_2)$ acting on the scalar integrand, and there is more than one way to do so due to the identity above. Then, the integral can be done and one is left with the same functional acting on some powers of $|\mathrm{x}_{23}|$, which clearly generates some function of the conformally invariant structures resulting from setting $\mathrm{x}_1 = 0$ followed by the inversion map:

$$\mathrm{x}_{2,3} \to \frac{1}{\mathrm{x}_{2,3}} \,, \tag{4.103a}$$

$$\mathrm{x}_{23} \to \frac{\mathrm{x}_{23}}{\mathrm{x}_2\mathrm{x}_3} \,, \tag{4.103b}$$

$$P_{12} \to (\eta_1\eta_2) \,, \tag{4.103c}$$

$$Q_1 \to -[\eta_1\mathrm{x}_{23}\eta_1] \,, \tag{4.103d}$$

$$Q_2 \to +[\eta_2\mathrm{x}_{23}\eta_2]\,\frac{1}{\mathrm{x}_{23}^2} \,, \tag{4.103e}$$

$$S_3 \to +[\eta_1\mathrm{x}_{23}\eta_2]\,\frac{1}{\mathrm{x}_{23}} \,. \tag{4.103f}$$

It is convenient to use the same notation T_{ij} for the corresponding structures on the boundary $T_{ij} = [\eta_i\mathrm{x}_{23}\eta_j]$ since they arise under the action of O_{ij} on x_{23} resulting from the integral. In other words, if O_{ij} applied to the l.h.s. (4.98), it generates bulk T_{ij} defined in (4.97), and if O_{ij} acts on the r.h.s. of (4.98), it generates the boundary T_{ij}.

4.6.1 Leading Coefficients

Our goal is to reproduce the full structure of the three-point functions. However, it is useful to perform a few simple checks of the duality that do not require establishing a full dictionary between bulk and boundary. It is clear that O_{ij} operators when applied one after another produce

$$O_{11}^{n_1}O_{12}^{n_2}O_{22}^{n_3}(K_2)^a = \frac{\Gamma[a+n]}{z^n\Gamma[a]}(K_2)^{a+n}T_{11}^{n_1}T_{12}^{n_2}T_{22}^{n_3} + \mathcal{O}(\eta_1\eta_2) \,, \tag{4.104}$$

where $n = n_1 + n_2 + n_3$. Therefore, up to P_{12}-terms, which are represented by $\mathcal{O}(\eta_1\eta_2)$ after the inversion, the bulk computation amounts to pulling out the T_{ij} structures as powers of O_{ij}, computing the scalar integral and then pushing the O_{ij} factors in. In the last step operators O_{ij} act on x_{23} resulting from the bulk integral and generate $Q_{1,2}$ modulo $Q_1Q_2 \sim -S_3^2$.

Type-A, $\Delta = 1$*:*

In the case of $\Delta = 1$ and $\theta = 0$ a simple computation gives:

$$\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.} = (-4A)c_{s_1}c_{s_2}c_0\, v_{s_1,s_2,0}I^{\Delta=1}_{s_1,s_2,0} \times \left[(Q_1)^{s_1}(Q_2)^{s_2} + \mathcal{O}(P_{12})\right] \,, \tag{4.105}$$

where $I_{s_1,s_2,0}$ is the factor that comes from the bulk

$$I^{\Delta=1}_{s_1,s_2,0} = \frac{\pi^3\left(-\frac{1}{2}\right)^{s_1+s_2}(-)^{s_1}\Gamma\left(s_1+\frac{1}{2}\right)}{s_1\Gamma\left(\frac{1}{2}-s_2\right)\Gamma(s_1+s_2+1)}\,. \tag{4.106}$$

This should be compared with the generating function (4.24) in free boson theory:¶

$$\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{f.b.} = \frac{2}{\mathrm{x}_{12}\mathrm{x}_{23}\mathrm{x}_{13}}\left(\frac{i}{2}\right)^{s_1+s_2}\frac{1}{s_1!s_2!}\left[(Q_1)^{s_1}(Q_2)^{s_2} + \mathcal{O}(P_{12})\right] \,. \tag{4.107}$$

Since the normalization of the boundary-to-bulk propagators is not yet fixed, we should compare two function of s_1, s_2, the bulk result and the CFT result, up to a product of functions of s_1 and s_2 separately. In doing so we can find that the normalization factor is

$$c_s = -\frac{4^s}{\pi}\,. \tag{4.108}$$

¶What we compute would correspond to the dual theory with $U(M)$ global symmetry (4.26). The same comment applies to all correlators below.

With this normalization we have a perfect match up to the terms of order $(\eta_1\eta_2) = P_{12}$, namely

$$\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.} = \langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{f.b.} + \mathcal{O}(P_{12})\ . \tag{4.109}$$

Type-B, $\Delta = 2$*:*

The same computation but for $\Delta = 2$ and $\theta = \pi/2$ leads to

$$\begin{aligned}&\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.}\\ =&(-4A)c_{s_1}c_{s_2}\tilde{c}_0\, v_{s_1,s_2,0}I^{\Delta=2}_{s_1,s_2,0}\left[(Q_1)^{s_1-1}(Q_2)^{s_2-1}P_{12}S_3 + \mathcal{O}(P_{12}^2)\right]\ .\end{aligned} \tag{4.110}$$

Here we assumed that the normalization of the scalar field propagator $\tilde{c}_0$ can be different from c_0 due to the change in boundary conditions. The bulk integral and other factors combine into $I^{\Delta=2}_{s_1,s_2,0}$ as follows

$$I^{\Delta=2}_{s_1,s_2,0} = -\frac{\pi^3 s_2\left(-\frac{1}{2}\right)^{s_1+s_2-1}(-)^{s_1}\Gamma\left(s_1+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-s_2\right)\Gamma(s_1+s_2+1)}\ . \tag{4.111}$$

We note that one factor of P_{12} jumps out of the integral and the subleading terms are of order P_{12}^2. The result should be compared with the generating function (4.34) in free fermion theory:

$$\begin{aligned}&\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{f.f.}\\ =&\frac{i^{s_1+s_2-2}}{\mathrm{x}_{23}^2\mathrm{x}_{13}^2 2^{s_1+s_2-2}}\frac{2}{(s_1-1)!(s_2-1)!}\left[(Q_1)^{s_1-1}(Q_2)^{s_2-1}S_3P_{12} + \mathcal{O}(P_{12}^2)\right]\ .\end{aligned} \tag{4.112}$$

We find the two results to agree, namely

$$\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.}=-\frac{\tilde{c}_0}{c_0}\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{f.f.}+\mathcal{O}(P_{12})\ . \tag{4.113}$$

Once the normalization factors c_s are fixed by the Type-A duality the match just observed is even more nontrivial because the only freedom that we have is an overall spin-independent factor.[‖] The canonical normalization of the scalar field propagators [85] is such that $\tilde{c}_0/c_0=-2$.

4.6.2 Complete Dictionary

We would like to reproduce the full structure of CFT correlators. The idea is to take the subleading terms into account and express the bulk integrand as the action of a differential operator in O_{ij} on the scalar integrand. The operators O_{ij} produce T_{ij} and the subleading terms are obtained due to

$$O_{11}T_{22}=(\eta_1\eta_2)^2\ ,\qquad O_{22}T_{11}=(\eta_1\eta_2)^2\ ,\qquad O_{12}T_{12}=-\frac{1}{2}(\eta_1\eta_2)^2\ . \tag{4.114}$$

Operators O_{ij} commute with each other. Moreover, O_{11} does not produce any subleading terms at all. Then the action of any power of, say O_{12}, can be evaluated starting from ($\mathrm{x}\equiv\mathrm{x}_{23}$)

$$O_{12}f(\mathrm{x}^2,T_{12})=\left(-T_{12}\frac{\partial}{\partial\mathrm{x}^2}-\frac{1}{2}(\eta_1\eta_2)^2\frac{\partial}{\partial T_{12}}\right)f(\mathrm{x}^2,T_{12})\ , \tag{4.115}$$

[‖]The generating functions of the HS currents built out of free bosons and fermions, which were introduced in Section 4.2, are components of the supermultiplet of HS currents. Therefore, the normalization of the HS currents in free boson and free fermion are naturally related to each other.

and exponentiating it as

$$\exp[tO_{12}]\,(\mathrm{x}^2)^{-a} = \left(\mathrm{x}^2 - tT_{12} + \frac{t^2}{4}(\eta_1\eta_2)^2\right)^{-a} , \tag{4.116}$$

which leads to Gegenbauer polynomials:

$$(O_{12})^n(\mathrm{x}^2)^{-a} = \sum_k A_k^{a,n}(\eta_1\eta_2)^{2k}(T_{12})^{n-2k}(\mathrm{x}^2)^{-(a+n-k)} , \quad A_k^{a,n} = \frac{(-)^k\, n!\Gamma(a-k+n)}{4^k k!\Gamma(a)(n-2k)!} . \tag{4.117}$$

In particular, we find (4.104)

$$(O_{12})^n(\mathrm{x}^2)^{-a} = \frac{\Gamma[a+n]}{\Gamma[a]}(\mathrm{x}^2)^{-(a+n)}(T_{12})^n + \mathcal{O}(\eta_1\eta_2) . \tag{4.118}$$

In fact, we will use the inversion formula:

$$\sum_{k=0} B_k^{a,n}(\eta_1\eta_2)^{2k}(O_{12})^{n-2k}(\mathrm{x}^2)^{-(a+k-n)} = (\mathrm{x}^2)^{-a}(T_{12})^n , \quad B_k^{a,n} = \frac{\Gamma[a+k-n]n!}{\Gamma[a]4^k k!(n-2k)!}(\eta_1\eta_2)^{2k} . \tag{4.119}$$

The same formula works in the bulk if we replace $|\mathrm{x}^2|^{-1}$ by K_2/z and T_{ij} by the bulk T_{ij} (4.97) which we happen to denote by the same symbol, as explained below (4.103). Therefore, any T_{12} structure can also be factored out.

4.6.3 Complete Three-Point Functions

We can now compute (4.93) for general θ. The choice $\Delta = 1$ should be compared with CS-boson and $\Delta = 2$ with CS-fermion.

The computation is now reduced to the following simple steps: (i) express the integrand in terms of $(\xi_1\xi_2) \pm (\bar{\xi}_1\bar{\xi}_2)$; (ii) apply the inversion map and express the integrand in terms of T_{ij}; (iii) T_{11} is factored out immediately, while T_{12} is pull out with the help of (4.119) as certain polynomial in O_{12}; (iv) the integral can be done; (v) the operators O_{ij} should be evaluated in terms of T_{ij}, which is easy for O_{11} and we use (4.117) for O_{12}; (vi) powers of T_{ij} should be replaced by Q_1, Q_2, P_{12}, S_3.

As the first step we need the following simple identity, which reveals the dependence on θ of the final result as well as how it splits into parity-even and parity-odd structures:

$$\begin{aligned}
&[(\xi_1\xi_2)^{2s_2}e^{i\theta}+(\bar{\xi}_1\bar{\xi}_2)^{2s_2}e^{-i\theta}]\\
=&\sum_k\frac{2\cos\theta}{4^{s_2}}C^{2s_2}_{2k}[(\xi_1\xi_2)+(\bar{\xi}_1\bar{\xi}_2)]^{2s_2-2k}[(\xi_1\xi_2)-(\bar{\xi}_1\bar{\xi}_2)]^{2k}\\
+&\sum_k\frac{2i\sin\theta}{4^{s_2}}C^{2s_2}_{2k+1}[(\xi_1\xi_2)+(\bar{\xi}_1\bar{\xi}_2)]^{2s_2-2k-1}[(\xi_1\xi_2)-(\bar{\xi}_1\bar{\xi}_2)]^{2k+1}\,.
\end{aligned}\tag{4.120}$$

Since the θ-dependence is fixed, in computing the correlation functions it is sufficient to choose either $\theta=0$ or $\theta=\pi/2$ and for each of these cases to consider the $\Delta=1$ and $\Delta=2$ boundary conditions for the scalar field. Therefore, we need to compute four bulk integrals. In fact, these terms can be computed in the parity even Type-A,B theories but with different choice of boundary conditions for the scalar field, see also the end of Section 4.2.6.

The comment that applies to all the cases considered below is that the generating functions of correlators depend smoothly on spins and therefore it should be possible to extrapolate the result in the case of $s_1>s_2$ to the case of $s_1=s_2$, which is an argument used in [22]. The assumption $s_1>s_2$ was used to pick only $\omega_{s_1}C_{s_2}$ terms and for the opposite situation we find contribution from $\omega_{s_2}C_{s_1}$, which is the same. Therefore, the computation below covers all possible s_1 and s_2.

Another comment is that the results we obtain in the bulk are valid for the case of $U(N)$ CFT's that have HS currents with all integer spins (there are only HS currents with even spin in the $O(N)$ case). Also, the bulk results are valid for the case of the leftover global symmetry on the CFT side. The projection onto the singlet sector is trivial and can be obtained by taking the bose symmetric part of the correlator, as discussed in Section 4.5.2. It should be noted that the result of [64], which we will compare the AdS/CFT correlators with, were obtained assuming that the prediction of the slightly broken HS symmetry [62] extends to all integers spins and possibly to the case of leftover global symmetries.

Type-A, Free Boson:

The integral corresponding to $\Delta = 1$ and $\theta = 0$ is given by

$$\begin{aligned}&\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.}^{\Delta=1,\theta=0} = (-4A)c_{s_1}c_{s_2}c_0\, v_{s_1,s_2,0}\\ &\times\int \frac{d^3\mathrm{x}dz}{z^4}K_1K_2K_3(\xi_1F_2\bar{\xi}_1)^{s_1-s_2}\sum_k \frac{2}{4^{s_2}}C_{2k}^{2s_2}[(\xi_1\xi_2)+(\bar{\xi}_1\bar{\xi}_2)]^{2s_2-2k}[(\xi_1\xi_2)-(\bar{\xi}_1\bar{\xi}_2)]^{2k}\,.\end{aligned} \tag{4.121}$$

The final result for the right hand side of this equation is the following triple sum

$$\begin{aligned}\sum_{k=0}^{s_2}\sum_{i=0}^{k}\sum_{j=0}^{k-i}&\frac{(-1)^{k-j}i^{s_1+s_2}2^{s_1-s_2+1}\Gamma(2(-k+s_1+s_2))\Gamma\left(-i+k+s_1-s_2+\frac{1}{2}\right)}{\sqrt{\pi}i!j!(2s_1-1)!(2s_2-2k)!(-i+2s_1)!(-i-j+k)!}\\ &\times\frac{Q_1^{s_1-s_2}P_{12}^{2(i+j-k+s_2)}\left(P_{12}^2-Q_1Q_2\right)^{-i-j+k}}{\mathrm{x}_{12}\mathrm{x}_{13}\mathrm{x}_{23}}\,,\end{aligned} \tag{4.122}$$

where we undid the inversion map. This expression can be supplemented with the $s_2 > s_1$ contribution, extended to diagonal $s_1 = s_2$, as we explained above, then summed over spins as to build a generating function. This gives the result:

$$\frac{2}{\mathrm{x}_{12}\mathrm{x}_{13}\mathrm{x}_{23}}\exp\left(\tfrac{i}{2}Q_1+\tfrac{i}{2}Q_2\right)\cos P_{12}\,. \tag{4.123}$$

This matches exactly the CFT three-point function:

$$\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.}^{\Delta=1,\theta=0} = \langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{f.b.}\ . \tag{4.124}$$

This result is slightly new as we managed to reproduce the full structure of the CFT correlators from the local HS equations where the field theory methods can be applied, and we note that no regularization of infinities is required. The leading coefficients were found in [22] and the same generating function resulted from the non-local CC-terms after some regularization in [68].

Type-B, Free Fermion:

The integral corresponding to $\Delta = 2$ and $\theta = \pi/2$ reads

$$\begin{aligned}&\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.}^{\Delta=2,\theta=\frac{\pi}{2}} = (-4A)c_{s_1}c_{s_2}\tilde{c}_0\, v_{s_1,s_2,0}\\ &\times\int\frac{d^3\mathrm{x}dz}{z^4}K_1K_2K_3^2(\xi_1F_2\bar{\xi}_1)^{s_1-s_2}\sum_k\frac{2i}{4^{s_2}}C^{2s_2}_{2k+1}[(\xi_1\xi_2)+(\bar{\xi}_1\bar{\xi}_2)]^{2s_2-2k-1}[(\xi_1\xi_2)-(\bar{\xi}_1\bar{\xi}_2)]^{2k+1}\ .\end{aligned} \tag{4.125}$$

The right hand side of this equation can be brought to the form

$$\begin{aligned}&\frac{\tilde{c}_0}{c_0}\sum_{k=0}^{s_2-1}\sum_{i=0}^{k}\sum_{j=0}^{k-i}\frac{(-1)^{j+k}2^{s_1-s_2+1}i^{s_1+s_2}\Gamma(-2k+2s_1+2s_2-1)\Gamma\left(-i+k+s_1-s_2+\frac{3}{2}\right)}{\sqrt{\pi}i!j!\Gamma(2s_1)\Gamma(-i+2s_1+1)\Gamma(2s_2-2k)\Gamma(-i-j+k+1)}\\ &\qquad\times\frac{S_3Q_1^{s_1-s_2}P_{12}^{2i+2j-2k+2s_2-1}\left(P_{12}^2-Q_1Q_2\right)^{-i-j+k}}{\mathrm{x}_{23}^2\mathrm{x}_{13}^2}\ .\end{aligned} \tag{4.126}$$

As before, this expression can be supplemented with the $s_2 > s_1$ contribution, extended to diagonal $s_1 = s_2$, then summed over spins as to build a generating function. The result is:

$$-\frac{\tilde{c}_0}{c_0}\frac{\exp(\frac{i}{2}Q_1+\frac{i}{2}Q_2)}{\mathrm{x}_{23}^2\mathrm{x}_{13}^2}S_3\sin P_{12}\ , \tag{4.127}$$

and matches exactly the CFT three-point function (4.37):[**]

$$\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.}^{\Delta=2,\theta=\frac{\pi}{2}} = \langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{\text{f.f.}} \,. \tag{4.128}$$

where we recall that the prefactor accounts for the difference in the boundary conditions for the scalar field and is equal to -2.

Type-A, Critical Boson:

We take the $\Delta = 1$ Type-A expression (4.121) and simply set $\Delta = 2$. This gives

$$\begin{aligned}&\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.}^{\Delta=2,\theta=0} = (-4A)c_{s_1}c_{s_2}\tilde{c}_0\, v_{s_1,s_2,0}\\ &\times\int\frac{d^3\mathrm{x}dz}{z^4}K_1K_2K_3^2(\xi_1F_2\bar{\xi}_1)^{s_1-s_2}\sum_k\frac{2}{4^{s_2}}C_{2k}^{2s_2}[(\xi_1\xi_2)+(\bar{\xi}_1\bar{\xi}_2)]^{2s_2-2k}[(\xi_1\xi_2)-(\bar{\xi}_1\bar{\xi}_2)]^{2k}\,.\end{aligned} \tag{4.129}$$

The right hand side of this equation is evaluated to yield

$$\begin{aligned}\frac{\tilde{c}_0}{c_0}\sum_{k=0}^{s_2}\sum_{i=0}^{k}\sum_{j=0}^{k-i}&\frac{(-1)^{k-j}2^{s_1-s_2+1}i^{s_1+s_2}\Gamma(-2k+2s_1+2s_2)\Gamma(-i+k+s_1-s_2+1)}{\sqrt{\pi}i!j!\Gamma(2s_1)\Gamma(-i+2s_1+1)\Gamma(-2k+2s_2+1)\Gamma\left(-i-j+k+\frac{1}{2}\right)}\\ &\times\frac{Q_1^{s_1-s_2}P_{12}^{2(i+j-k+s_2)}\left(P_{12}^2-Q_1Q_2\right)^{-i-j+k}}{\mathrm{x}_{23}^2\mathrm{x}_{13}^2}\,.\end{aligned} \tag{4.130}$$

The three-point function in the CS-fermion theory was found in a form of recurrence relations in [64] (see also Section 4.2.4). We find that the triple sum above is an explicit solution to those recursion relations. The leading coefficient is

$$\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.}^{\Delta=2,\theta=0} = -\frac{1}{2}\frac{\tilde{c}_0}{c_0}\frac{i^{s_1+s_2}2^{-s_1-s_2+2}}{\Gamma\left(s_1+\frac{1}{2}\right)\Gamma\left(s_2+\frac{1}{2}\right)}\frac{Q_1^{s_1}Q_2^{s_2}}{\mathrm{x}_{23}^2\mathrm{x}_{13}^2}+\mathcal{O}(P_{12})\,. \tag{4.131}$$

[**]We again note that the answer is formally for the free fermion with leftover $U(M)$ global symmetry.

The sum (4.130) can be evaluated explicitly:

$$
\begin{aligned}
&\frac{\tilde{c}_0}{c_0}\sum_{l=0}^{s_2}\frac{(-1)^l i^{s_1+s_2}2^{1+s_1-s_2}\Gamma(1+l+s_1-s_2)\Gamma\left(2(-l+s_1+s_2)\right)}{\sqrt{\pi}\Gamma(\frac{1}{2}+l)\Gamma(2s_1)\Gamma(1+2s_1)\Gamma(1-2l+2s_2)}\\
&\qquad\times {}_2F_1(l-s_2,\frac{1}{2}+l-s_2,\frac{1}{2}+l-s_1-s_2,1)\\
&\qquad\times\frac{(Q_1)^{s_1-s_2}(P_{12})^{2(s_2-l)}(S_3)^{2l}}{\mathrm{x}_{23}^2\mathrm{x}_{13}^2}\quad (\text{Type-A, }\Delta=2)\,. \qquad (4.132)
\end{aligned}
$$

The computation shows that it is more convenient to use the S_3 variable. Indeed, the bulk integral gives S_3 after O_{12} is applied. We expect this to be the case for the most general case of three non-zero spins too: it may be more convenient to express even structures in terms of even functions of the odd S structures.

Type-B, Critical Fermion:

Using the same technique as before we can compute the parity odd integral for $\Delta=1$, which for $\Delta=2$ reproduced the free fermion result, the only modification required being $K_3^2\to K_3$:

$$
\begin{aligned}
&\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle_{h.s.}^{\Delta=1,\theta=\frac{\pi}{2}}=(-4A)c_{s_1}c_{s_2}c_0\,v_{s_1,s_2,0}\\
&\times\int\frac{d^3\mathrm{x}dz}{z^4}K_1K_2K_3(\xi_1F_2\bar{\xi}_1)^{s_1-s_2}\sum_k\frac{2i}{4^{s_2}}C^{2s_2}_{2k+1}[(\xi_1\xi_2)+(\bar{\xi}_1\bar{\xi}_2)]^{2s_2-2k-1}[(\xi_1\xi_2)-(\bar{\xi}_1\bar{\xi}_2)]^{2k+1}\,.
\end{aligned}
$$

(4.133)

As a result, the right hand side of this equation yields the following triple sum representation for the critical fermion:

$$
\begin{aligned}
&\sum_{k=0}^{s_2-1}\sum_{i=0}^{k}\sum_{j=0}^{k-i}\frac{(-1)^{j+k}i^{s_1+s_2}2^{s_1-s_2+1}\Gamma(-2k+2s_1+2s_2-1)\Gamma(-i+k+s_1-s_2+1)}{\sqrt{\pi}i!j!\Gamma(2s_1)(-2k+2s_2-1)!\Gamma(-i+2s_1+1)\Gamma\left(-i-j+k+\frac{3}{2}\right)}\\
&\qquad\times\frac{S_3Q_1^{s_1-s_2}P_{12}^{2i+2j-2k+2s_2-1}\left(P_{12}^2-Q_1Q_2\right)^{-i-j+k}}{\mathrm{x}_{12}\mathrm{x}_{13}\mathrm{x}_{23}}\,. \qquad (4.134)
\end{aligned}
$$

In [64], see also Section 4.2.5, the generating function of the correlators was found in CS-boson theory in an implicit form of a recurrence relation. The triple sum above provides an explicit solution to this system. The leading coefficient is easy to find, and gives

$$\langle J_{s_1}(\mathrm{x}_1,\eta_1)J_{s_2}(\mathrm{x}_2,\eta_2)J_0(\mathrm{x}_3)\rangle^{\Delta=1,\theta=\frac{\pi}{2}}_{h.s.} = -\frac{i^{s_1+s_2}2^{-s_1-s_2+2}}{\Gamma\left(s_1+\frac{1}{2}\right)\Gamma\left(s_2+\frac{1}{2}\right)}\frac{Q_1^{s_1-1}Q_2^{s_2-1}S_3P_{12}}{\mathrm{x}_{12}\mathrm{x}_{23}\mathrm{x}_{13}}+\mathcal{O}(P_{12}^2)\,. \tag{4.135}$$

The sum (4.134) can be evaluated explicitly, giving:

$$\begin{aligned}\sum_{l=0}^{s_2-1}&\frac{(-1)^l i^{s_1+s_2}2^{s_1-s_2+1}\Gamma(1+l+s_1-s_2)\Gamma(2s_1+2s_2-2l-1)}{\sqrt{\pi}\Gamma(\frac{3}{2}+l)\Gamma(2s_1)\Gamma(2s_1+1)\Gamma(2s_2-2l)}\\&\times{}_2F_1(\frac{1}{2}+l-s_2,1+l-s_2,\frac{3}{2}+l-s_1-s_2,1)\Big)\\&\times\frac{(Q_1)^{s_1-s_2}(P_{12})^{2(s_2-l)-1}(S_3)^{2l+1}}{\mathrm{x}_{12}\mathrm{x}_{13}\mathrm{x}_{23}}\quad\text{(Type-B, }\Delta=1)\,.\end{aligned}\tag{4.136}$$

Summary:

Combining all the four cases together with the θ dependence (4.120) we have confirmed that the structure of $\langle j_{s_1}j_{s_2}j_0\rangle$ is in accordance with the CFT result (4.42):

$$\langle j_{s_1}j_{s_2}j_0\rangle_{B.} = \tilde{N}\left[\cos\theta\langle j_{s_1}j_{s_2}j_0\rangle_{f.b.}+\sin\theta\langle j_{s_1}j_{s_2}j_0\rangle_{odd}\right]\,, \tag{4.137a}$$

$$\langle j_{s_1}j_{s_2}\tilde{j}_0\rangle_{F.} = \tilde{N}\left[\cos\theta\langle j_{s_1}j_{s_2}\tilde{j}_0\rangle_{f.f.}+\sin\theta\langle j_{s_1}j_{s_2}\tilde{j}_0\rangle_{odd}\right]\,. \tag{4.137b}$$

Here we restored the bulk coupling constant $G = 1/\tilde{N}$. The θ-dependence results from the fact that the (anti)-selfdual HS Weyl tensors are identified with the order s-curls of the

Fronsdal field with a phase shift (4.54)

$$C_{\alpha_1...\alpha_{2s}} = e^{+i\theta}\, \nabla_{\alpha_1}{}^{\dot{\alpha}_1} \ldots \nabla_{\alpha_s}{}^{\dot{\alpha}_s}\, \phi_{\alpha_{s+1}...\alpha_{2s},\dot{\alpha}_1...\dot{\alpha}_s}\ , \tag{4.138a}$$

$$C_{\dot{\alpha}_1...\dot{\alpha}_{2s}} = e^{-i\theta}\, \nabla^{\alpha_1}{}_{\dot{\alpha}_1} \ldots \nabla^{\alpha_s}{}_{\dot{\alpha}_s}\, \phi_{\alpha_1...\alpha_s,\dot{\alpha}_{s+1}...\dot{\alpha}_{2s}}\ . \tag{4.138b}$$

Given the θ-dependence, the four structures in (4.137) can be found by evaluating the bulk integral for $\Delta = 1, 2$ and $\theta = 0, \pi/2$, as we did.

4.7 Discussion

By isolating the scalar field equation in the Vasiliev theory up to quadratic terms in fields, we managed to extract the ωC-correction to the scalar field equation and produce the full structure of the correlators. The perfect agreement with the CFT results is found, including the θ-dependence and the parity-violating structures, which were recently derived in [64] on the CFT side.

Notice that, different from the previous one loop test of the free energies, here we found the perfect match between the tree-level three-point functions of Type-B model and its corresponding dual. The latter result concretely supports the duality between Type-B theory/fermionic vector model. We believe that the mismatch in the one loop test requires deeper understanding of the structure of the free energy, and will propose a few possible solutions to this mismatch in the conclusion. We also address that in this work we only obtained the cubic Witten diagrams involving one scalar field. The calculation of the correlators with three general spins is obstructed by the non-localities present is HS theories, and remains to be an open problem. We shall comment further on the issue of non-localities in the conclusions below.

5. CONCLUSIONS

In this dissertation we have mainly discussed two topics: application of localization to $\mathcal{N}=1$ theory in 3D and the tests of HS/CFT dualities.

In the work on the first topic, we considered the most general $\mathcal{N}=1$ superconformal Chern-Simons matter theory with global symmetry $Sp(2)$ and gauge group $U(N)\times U(N)$. We have shown that the Lagrangian in the on-shell formulation of the theory admits one more free parameter as compared to the theory formulated in off-shell $\mathcal{N}=1$ superspace. We found that the vanishing of the deformation under supersymmetry transformation requires the curvature of the 3D space being zero or negative, therefore the theory on T^3 can be formally localized. Localization procedure has then been partially carried out for the theory on T^3 with periodic boundary conditions. In particular we have shown that restricting to the saddle points with vanishing gauge connection gives a trivial contribution to the partition function, i.e. the bosonic and fermionic contributions exactly cancel each other.

Our results have the following implication for the partition function of the ABJM model on T^3. Our analysis of the saddle points shows that the classical CS action vanishes on the locus of flat gauge connections on T^3. Since the one-loop determinant around the saddle points does not introduce any dependence on the two CS levels, it follows by the localization argument that the partition function is independent of the level $k\equiv k_1=-k_2$. Hence we may compute the partition function in the limit $k\to\infty$ with N fixed, which corresponds to vanishing 't Hooft coupling. In this limit the matter sector becomes free and decouples from the CS action. Therefore the resulting partition function factorizes into a pure supersymmetric CS partition function and a free matter piece. The latter is trivial, i.e. the bosonic and fermionic contributions exactly cancel each other. Moreover our localization results can be applied to the pure CS partition function to show that the contribution from the saddle points with vanishing gauge connection is also trivial. As mentioned above, this is consistent with what one expects for a supersymmetric theory.

Moving on to the second topic, we first computed the one loop free energy for $4D$ Vasiliev higher spin gravities based on Konstein-Vasiliev algebras $hu(m;n|4)$, $ho(m;n|4)$ or $husp(m;n|4)$ and subject to higher spin preserving boundary conditions, which are conjectured to be dual to the $U(N)$, $O(N)$ or $USp(N)$ singlet sectors, respectively, of free CFTs on the boundary of AdS_4. Ordinary supersymmetric higher spin theories appear as special cases of Konstein-Vasiliev theories, when the corresponding higher spin algebra contains $OSp(\mathcal{N}|4)$ as subalgebra. In AdS_4 with S^3 boundary, we utilized a regularization scheme for individual spins that employs their character such that the subsequent sum over all spins is finite, thereby avoiding the need for additional regularization. Interestingly the contribution of the infinite tower of bulk fermions vanishes, and as a result, the free energy is the sum of those which arise in Type-A and Type-B models with internal symmetries. Thus the known mismatch between the bulk and boundary free energies for Type-B model persists, and ordinary supersymmetric higher spin theories exhibit the mismatch as well. The only models that have a match are Type-A models with internal symmetries, corresponding to $n = 0$. The matching requires identification of the inverse Newton's constant G_N^{-1} with N plus a proper integer as was found previously for special cases. In AdS_4 with $S^1 \times S^2$ boundary, the bulk one loop free energies match those of the dual free CFTs for arbitrary m and n. We have also shown that a supersymmetric double-trace deformation of free CFT based on $OSp(1|4)$ does not contribute to the $\mathcal{O}(N^0)$ free energy, as expected from the bulk.

Turning to the problem of mismatch in free energies of Type-B model and its conjectured dual, one may have to take into account the issue of how to impose the $O(N)$ invariance condition on the CFT side. A natural way of implementing it is to gauge the $O(N)$ symmetry by means of vector gauge field with level k Chern-Simons kinetic term.

This term breaks parity but the result for the free energy of the parity invariant model can be obtained in a limit in which the CS gauge field decouples. It has been suggested in [20] that as the fermions coupled to CS on the boundary give rise to a shift in the level k, it may not be justified to obtain the result for parity-preserving case by a naive subtraction of CS contribution from the free energy on the CFT side. However, one expects that this effect becomes irrelevant in the decoupling limit in which $k \to \infty$. In fact, we have examined the procedure of decoupling CS in the large k limit by evaluating the S^3 free energies for ABJ model based on $U(N)_k \times U(1)_{-k}$ [6, 58] and a few $\mathcal{N} = 3$ CS matter theories in which the matter sector consists of fundamental hypermultiplets [59–61]. After subtracting the contribution from pure CS term, we indeed obtain the free energies of free vector models. Therefore, the puzzle of free energy mismatch in Type-B remains unresolved and its solution requires deeper understanding of HS/vector model holography. In this context, it has been suggested by [87] and explored further in [88] that the vector-like limit of ABJ model based on $U(N)_k \times U(M)_{-k}$ is given by

$$N,\, k \to \infty \quad \text{with} \quad \lambda \equiv \frac{N}{k} \quad \text{and} \quad M \quad \text{finite}\,. \tag{5.1}$$

In this limit, the ABJ theory effectively behaves like a $\mathcal{N} = 6$ CS gauged vector model with $U(M)$ flavor symmetry [87]. Its bulk dual is conjectured to be the parity violating $\mathcal{N} = 6$ $U(M)$ gauged Vasiliev theory [87]. The parity violating angle θ_0 is conjectured to be related to the CFT 't Hooft coupling by $\theta_0 = \pi\lambda/2$ [87][†].

Complementary to the one loop test of the free energy, we conducted another test of

[†]Besides the Newton constant which is small in the limit described above, there is also a bulk 't Hooft coupling $g^2_{\text{bulk}} M \sim M/N \ll 1$. String theory emerges when $M/N \sim 1$. Due to strong interactions, the HS particles form $U(M)$ singlet states which are described by the color neutral string states. Since the M theory circle $R_{11} \sim (M/k^5)^{1/6}$ shrinks and $\sqrt{\alpha'}/R_{\text{AdS}} \sim (k/M)^{1/4} \to \infty$, this is type IIA string in the high energy limit. The $\mathcal{N} = 6$ parity violating $U(M)$ gauged Vasiliev theory can be perceived as a deconfinement phase of type IIA string when $M/N \ll 1$, in which the string states fragment into HS particles colored under $U(M)$ [87].

the HS/CFT duality by computing the parity-preserving and parity-violating three-point amplitudes with one scalar leg in higher spin gravity and compare results with those of Chern-Simons matter theories. The three-point correlators of the free boson, free fermion, critical vector model and Gross-Neveu model are reproduced including the dependence on the Chern-Simons coupling. We have also performed a simple test of the modified higher spin equations proposed in [79] and found that the results are consistent with the AdS/CFT correspondence.

In this work we could not extract the correlators for general three spins due to non-localities present in the additional CC-terms [22, 68, 78] that need to be taken into account. These non-localities make the coefficient of the bulk vertex infinite, resulting in infinite correlators. The infinity is due to the presence of infinitely many of higher derivative copies of the same vertex that are stacked in the CC-terms. The divergences at tree level must not arise in any field theory, including HS theories.[‡] Indeed, the correlation functions on the CFT side are finite. Moreover, there is a one-to-one correspondence between all possible three-point correlators and cubic vertices in AdS, which allows one to manufacture an action up to the cubic terms that will yield any given three-point correlation functions and no infinities can arise. This was explicitly done for some of the cubic vertices in [89–91] (full cubic action of Type-A in any dimension was obtained in [91]), see also [92] for the quartic results. In the recent paper [79] it was conjectured how to modify the CC-terms as to make them local. The field redefinition that does the job is non-local and changes the coefficient of the correlator from an infinite number to a finite one. Moreover, the CC-terms correspond to the abelian vertices at this order (in the sense that are not fixed by the HS symmetry) and can in principle be arbitrary. Also, non-local redefinitions can result in any given coefficient [78, 93–96]. Nonetheless, the proposal

[‡]There exist extremal correlators and the divergences observed in [22, 68, 78] have nothing to do with those.

of [79] involves a specific type of field redefinition, and as such it can be treated as a conjecture. Taking this point of view, we have tested this redefinition, which involves the CC terms, and we have found that it does produce an answer for the spin $0 - s - s$ amplitude that agrees with the CFT result. Further tests of this conjecture will require the study of more three-point functions, namely those involving three arbitrary spins, and then of higher order amplitudes, which may possibly require higher order extension of the proposed field redefinition as well.

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