THETA OPERATORS ON v-ADIC MODULAR FORMS AND v-ADIC FAMILIES OF GOSS POLYNOMIALS AND EISENSTEIN SERIES

A Dissertation

by

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ABSTRACT

The first part of the dissertation is mainly from my first paper joint with my advisor Papanikolas and the second part will be our second paper.

In 1973, Serre introduced p-adic modular forms for a fixed prime p, which are defined to be p-adic limits of Fourier expansions of holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with rational coefficients. He also established fundamental results about families of p-adic modular forms by developing the theories of differential operators and Hecke operators acting on p-adic spaces of modular forms. In particular, he showed that the weight 2 Eisenstein series E_2 is also p-adic. If we let $\vartheta := \frac{1}{2\pi i}\frac{d}{dz}$ be Ramanujan's theta operator acting on holomorphic complex forms, then letting $\mathbf{q}(z) = e^{2\pi i z}$, we have $\vartheta = \mathbf{q}\frac{d}{d\mathbf{q}},\ \vartheta(\mathbf{q}^n) = n\mathbf{q}^n$. Although ϑ does not preserve spaces of complex modular forms, Serre proved the induced operation $\vartheta : \mathbb{Q}\otimes\mathbb{Z}_p[[\mathbf{q}]] \to \mathbb{Q}\otimes\mathbb{Z}_p[[\mathbf{q}]]$ does take p-adic modular forms to p-adic modular forms and preserves p-integrality. Moreover, the Bernoulli numbers B_m and the Eisenstein series E_m have p-adic limits as m goes to a p-adic limit.

To extend the theory to function fields, we investigate hyperderivatives of Drinfeld modular forms and determine formulas for these derivatives in terms of Goss polynomials for the kernel of the Carlitz exponential. As a consequence we prove that v-adic modular forms in the sense of Serre, as defined by Goss and Vincent, are preserved under hyperdifferentiation. Similar to the classical case, the false Eisenstein series E is a v-adic modular form, though it is not a Drinfeld modular form. Moreover, upon multiplication by a Carlitz factorial, hyperdifferentiation preserves v-integrality, which can be proved using Goss polynomials.

Furthermore, we can show that the Bernoulli-Carlitz numbers BC_{m_j} have a v-adic limit if m_j have the form $aq^{dj} + b$ with a, b non-negative. Using the same method, we can

also prove that the Goss polynomials have v-adic limits after multiplication by a Carlitz factorial. Because of this, we can also prove the limit of $\Pi_{m_j}\Theta^{m_j}$ exists. Therefore, since the Eisenstein series E_n can be expressed as the sum of Bernoulli-Carlitz numbers and Goss polynomials, we can derive that E_{m_j} also have a v-adic limit in $K \otimes_A A_v[[u]]$. Notice for the Eisenstein series in function fields, the result we get is different from the classical number fields. In the classical case, Serre proved that if m_j has a limit m in the p-adic topology and m_j goes to infinity in the Euclidean norm, then the classical Eisenstein series E_{m_j} has a p-adic limit only depending on m. However, for example in function fields, even if the two series $aq^{dj} + b$ and $(q-1)q^{2dj} + aq^{dj} + b$ satisfy the previous two condition and their corresponding Eisenstein series are non-zero, they do not have the same v-adic limit.

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The research of Chapter 2 was conducted in part by Matthew Papanikolas of the Department of Mathematics and will be published soon in an article listed in the Journal de Théorie des Nombres de Bordeaux.

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1. INTRODUCTION

1.1 Introduction of v-adic limits of modular forms

In [26], Serre defined p-adic modular forms for a fixed prime p, as p-adic limits of Fourier expansions of holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with rational coefficients. He established fundamental results about families of p-adic modular forms by developing the theories of differential operators and Hecke operators acting on p-adic spaces of modular forms, and in particular he showed that the weight 2 Eisenstein series E_2 is also p-adic. If we let $\vartheta := \frac{1}{2\pi i}\frac{d}{dz}$ be Ramanujan's theta operator acting on holomorphic complex forms, then letting $\mathbf{q}(z) = e^{2\pi iz}$, we have

$$\vartheta = \mathbf{q}\frac{d}{d\mathbf{q}}, \quad \vartheta(\mathbf{q}^n) = n\mathbf{q}^n. \tag{1.1.1}$$

Although ϑ does not preserve spaces of complex modular forms, Serre proved the induced operation $\vartheta : \mathbb{Q}\otimes\mathbb{Z}_p[[\mathbf{q}]] \to \mathbb{Q}\otimes\mathbb{Z}_p[[\mathbf{q}]]$ does take p-adic modular forms to p-adic modular forms and preserves p-integrality.

In this dissertation, we investigate differential operators on spaces of v-adic modular forms, where v is a finite place corresponding to a prime ideal of the polynomial ring $A = \mathbb{F}_q[\theta]$, for $\mathbb{F}_q$ a field with q elements. Drinfeld modular forms were first studied by Goss [12], [13], [14], [15] as rigid analytic functions,

$$f : \Omega \to C_\infty,$$

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on the Drinfeld upper half space Ω that transform with respect to the group $\Gamma = \mathrm{GL}_2(A)$ (see §2.3 for precise definitions). Here if we take $K = \mathbb{F}_q(\theta)$, then Ω is defined to be $C_\infty \setminus K_\infty$, where $K_\infty = \mathbb{F}_q((1/\theta))$ is the completion of K at its infinite place and C_∞ is the completion of an algebraic closure of K_∞. Goss showed that Drinfeld modular forms have expansions in terms of the uniformizing parameter $u(z) := 1/e_C(\widetilde{\pi} z)$ at the infinite cusp of Ω, where $e_C(z)$ is the exponential function of the Carlitz module and $\widetilde{\pi}$ is the Carlitz period. Each such form f is uniquely determined by its u-expansion,

$$f = \sum_{n=0}^{\infty} c_n u^n \in C_\infty[[u]].$$

If $k \equiv 0 \pmod{q-1}$, then the weight k Eisenstein series of Goss [13], has a u-expansion due to Gekeler [9, (6.3)] of the form

$$E_k = -\frac{\zeta_C(k)}{\widetilde{\pi}^k} - \sum_{a \in A,\, a \text{ monic}} G_k(u(az)), \quad \frac{\zeta_C(k)}{\widetilde{\pi}^k} \in K, \tag{1.1.2}$$

where $\zeta_C(k)$ is a Carlitz zeta value, $G_k(u)$ is a Goss polynomial of degree k for the lattice $\Lambda_C = A\widetilde{\pi}$ (see §2.2–2.3 and (2.3.2)), and $u(az)$ can be shown to be represented as a power series in u (see §2.3). Gekeler and Goss also show that spaces of forms for Γ are generated by forms with u-expansions with coefficients in A. Using this as a starting point, Goss [16] and Vincent [31] defined v-adic modular forms in the sense of Serre by taking v-adic limits of u-expansions and thus defining v-adic forms as power series in $K \otimes_A A_v[[u]]$ (see §2.4). Goss [16] constructed a family of v-adic forms based on forms with A-expansions due to Petrov [24] (see Theorem 2.5.3), and Vincent [31] showed that forms for the group $\Gamma_0(v) \subseteq \mathrm{GL}_2(A)$ with v-integral u-expansions are also v-adic modular forms.

It is natural to ask how Drinfeld modular forms and v-adic forms behave under differentiation, and since we are in positive characteristic it is favorable to use hyperdifferential operators ∂_z^r, rather than straight iteration $\frac{d^r}{dz^r} = \frac{d}{dz} \circ \cdots \circ \frac{d}{dz}$ (see §2.1 for definitions). Gekeler [9, §8] showed that if we define $\Theta := -\frac{1}{\tilde{\pi}} \frac{d}{dz} = -\frac{1}{\tilde{\pi}}\, \partial_z^1$, then we have the action on u-expansions determined by the equality

$$\Theta = u^2 \frac{d}{du} = u^2\, \partial_u^1. \tag{1.1.3}$$

Now as in the classical case, derivatives of Drinfeld modular forms are not necessarily modular, but Bosser and Pellarin [3], [4], showed that hyperdifferential operators ∂_z^r preserve spaces of quasi-modular forms, i.e., spaces generated by modular forms and the false Eisenstein series E of Gekeler (see Example 2.3.7), which itself plays the role of E_2.

For $r \geqslant 0$, following Bosser and Pellarin we define the operator Θ^r by

$$\Theta^r := \frac{1}{(-\tilde{\pi})^r}\, \partial_z^r.$$

Uchino and Satoh [29, Lem. 3.6] proved that Θ^r takes functions with u-expansions to functions with u-expansions, and Bosser and Pellarin [3, Lem. 3.5] determined formulas for the expansion of $\Theta^r(u^n)$. If we consider the r-th iterate of the classical ϑ-operator, $\vartheta^{\circ r} = \vartheta \circ \cdots \circ \vartheta$, then clearly by (1.1.1),

$$\vartheta^{\circ r}(q^n) = n^r q^n.$$

If we iterate Θ, taking $\Theta^{\circ r} = \Theta \circ \cdots \circ \Theta$, then by (1.1.3) we find

$$\Theta^{\circ r}(u^n) = r! \binom{n+r-1}{r} u^{n+r},$$

which vanishes identically when $r \geqslant p$. On the other hand, the factor of $r!$ is not the only discrepancy in comparing Θ^r and $\Theta^{\circ r}$, and in fact we prove two formulas in Corollary 2.3.10 revealing that Θ^r is intertwined with Goss polynomials for Λ_C:

$$\Theta^r(u^n) = u^n\, \partial_u^{n-1}\bigl(u^{n-2}G_{r+1}(u)\bigr), \quad \forall\, n \geqslant 1, \tag{1.1.4}$$

$$\Theta^r(u^n) = \sum_{j=0}^{r} \binom{n+j-1}{j} \beta_{r,j} u^{n+j}, \quad \forall\, n \geqslant 0, \tag{1.1.5}$$

where $\beta_{r,j}$ are the coefficients of $G_{r+1}(u)$. These formulas arise from general results (Theorem 2.2.4) on hyperderivatives of Goss polynomials for arbitrary $\mathbb{F}_q$-lattices in C_∞, which is the primary workhorse of this paper, and they induce formulas for hyperderivatives of u-expansions of Drinfeld modular forms (Corollary 2.3.12). It is important to note that (1.1.5) is close to a formula of Bosser and Pellarin [3, Eq. (28)], although the connections with coefficients of Goss polynomials appears to be new and the approaches are somewhat different.

Goss [16] defines the weight space of v-adic modular forms to be $\mathbb{S} = \mathbb{Z}/(q^d-1)\mathbb{Z} \times \mathbb{Z}_p$, where d is the degree of v, and if we take $\mathcal{M}_s^m \subseteq K \otimes_A A_v[[u]]$ to be the space of v-adic forms of weight $s \in \mathbb{S}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ (see §2.4), then we prove (Theorem 2.5.1) that Θ^r preserves spaces of v-adic modular forms,

$$\Theta^r : \mathcal{M}_s^m \to \mathcal{M}_{s+2r}^{m+r}, \quad r \geqslant 0.$$

Of particular importance here is proving that the false Eisenstein series E is a v-adic form (Theorem 2.5.5). Unlike in the classical case, Θ^r does not preserve v-integrality due to denominators coming from $G_{r+1}(u)$, but we show in §2.6 that this failure can be controlled, namely showing (Theorem 2.6.4) that

$$\Pi_r\Theta^r : \mathcal{M}_s^m(A_v) \to \mathcal{M}_{s+2r}^{m+r}(A_v), \quad r \geqslant 0,$$

where $\Pi_r \in A$ is the Carlitz factorial (see §2.1) and $\mathcal{M}_s^m(A_v) = \mathcal{M}_s^m \cap A_v[[u]]$.

1.2 Introduction to the v-adic limits of spectial values

The part of research will be the paper [22] joint with my advisor Papanikolas.

In the p-adic case, Serre showed that the classical Eisenstein series E_{m_j} have a p-adic limit only depending on the p-adic limit of m_j with $m_j \to \infty$ in the Euclidean topology. In function fields, we have a similar phenomenon, but it differs from in the p-adic case. Since we have proved that the Eisenstein series E_n can be expressed as the summation of the Goss polynomials and Bernoulli-Carlitz numbers BC_n, the question is mainly translated to "do Goss polynomials G_n and Bernoulli-Carlitz numbers BC_n have v-adic limits as n goes to some limit?". However, this statement turns out not to hold in complete generality. Nevertheless, if we make the statement a little bit weaker, we can find that the Goss polynomials $G_{aq^{dj}+b}$ do have a v-adic limit as j goes to infinity, though the limit depends on a and b in $aq^{dj}+b$, and not just on its p-adic limit. Here $d = \deg(v)$.

Moreover, Carlitz [6] and Goss [11] gave a formula for the Bernoulli-Carlitz numbers (3.1.2), which is closely related to the Goss polynomials. To be more precise, we can use their formulas to prove that

$$BC_m = \Pi_m \sum_{q^k \leqslant m+1} \frac{\beta_{m,q^k-1}}{L_k},$$

where β_{m,q^k-1} is the coefficients of the Goss polynomial $G_{m+1}(u) = \sum_{l=0}^{m} \beta_{m,l} u^{l+1}$. In the Theorem (3.1.15), for the sequence $BC_{m_j} = BC_{aq^{dj}+b}$, we mainly compute the difference of $\Pi_{m_j}\beta_{m_j,q^k-1}$ and $\Pi_{m_{j+1}}\beta_{m_{j+1},q^k-1}$ and prove the v-adic norm tends to 0 as j goes to infinity. This argument can be also applied to Goss polynomials $\Pi_{m_j} G_{m_j+1}$ (see Theorem 3.1.26). Moreover, recall the relation between Goss polynomials and the Θ operator

(2.3.12b)

$$\Theta^r(f) = \Theta^r(c_0) + \sum_{n=1}^{\infty} c_n u^n \, \partial_u^{n-1}\big(u^{n-2}G_{r+1}(u)\big).$$

Combining with Theorem (2.5.1), we show that if f is a v-adic modular form, then $\lim_{j\to\infty} \Pi_{m_j}\Theta^{m_j}(f)$ is still a v-adic modular form (Corollary (3.1.31)). Since the Eisenstein series E_n can be written as the sum of Bernoulli-Carlitz numbers and the Goss polynomials (1.1.2), it is natural to expect that Eisenstein series have the same property that E_{m_j} has a v-adic limit. After carefully dealing with the infinite summation, we find that this expectation is true.

In the last section of the dissertation, we give some examples of the limits of Bernoulli-Carlitz numbers. Moreover, we find the v-adic limit of the Bernoulli-Carlitz numbers BC_{m_j} can be computed explicitly, and the limit is in a constant field extension of K with degree d (Theorem 3.3.8).

Goss [15, §9.6] and Thakur [27, §4.2] talked about the interpolations at the finite places, which gives us the idea to compute the v-adic limits for each terms in the Bernoulli-Carlitz number. Various papers of Anglès, Ngo Dac, Tavares Ribeiro, Pellarin, Perkins and Thakur ([23], [1], [2] and [28]) discussed the properties of Bernoulli-Carlitz numbers, such as v-adic limits, high congruence. They mainly discussed the properties of BC_{q^j-s} with $s \equiv 1 \pmod{q-1}$ is fixed. First note $q^j - s$ tends to a negative number in the p-adic topology, however, our sequence $m_j = aq^{dj} + b$ tends to b, a positive number. If we assume that $q = p$ a prime, by [10, Thm. 6.12], we can know the difference between the two situations. For the first case, the lowest term in the Goss polynomial G_{q^j-s} will have higher degree as j goes larger. However, the sequence we used, G_{m_j}, the lowest term will have stable degree as j goes to infinity. Moreover, in the first case, although each of the coefficients have limits 0, we do not have any evidence to see whether they have v-adic limits.

2. THETA OPERATORS, GOSS POLYNOMIALS, AND v-ADIC MODULAR FORMS

2.1 Functions and hyperderivatives

Much of the exposition in this chapter is taken from the author's paper with Papanikolas [21].

Let $\mathbb{F}_q$ be the finite field with q elements, q a fixed power of a prime p. Let $A := \mathbb{F}_q[\theta]$ be a polynomial ring in one variable, and let $K := \mathbb{F}_q(\theta)$ be its fraction field. We let A_+ denote the monic elements of A, A_{d+} the monic elements of degree d, and $A_{(<d)}$ the elements of A of degree $< d$.

For each place v of K, we define an absolute value $|\cdot|_v$ and valuation ord_v, normalized in the following way. If v is a finite place, we fix $\wp \in A_+$ to be the monic generator of the prime ideal $\mathfrak{p}_v$ corresponding to v and we set $|\wp|_v = 1/q^{\deg \wp}$ and $\mathrm{ord}_v(\wp) = 1$. If $v = \infty$, then we set $|\theta|_\infty = q$ and $\mathrm{ord}_\infty(\theta) = -\deg(\theta) = -1$. For any place v we let A_v and K_v denote the v-adic completions of A and K. For the place ∞, we note that $K_\infty = \mathbb{F}_q((1/\theta))$, and we let C_∞ be a completion of an algebraic closure of K_∞. Finally, we let $\Omega := C_\infty \setminus K_\infty$ be the Drinfeld upper half-plane of C_∞.

For $i \geqslant 1$, we set

$$[i] = \theta^{q^i} - \theta, \quad D_i = [i][i-1]^q \cdots [1]^{q^{i-1}}, \quad L_i = (-1)^i [i][i-1]\cdots[1], \tag{2.1.1}$$

and we let $D_0 = L_0 = 1$. We have the recursions, $D_i = [i]D_{i-1}^q$ and $L_i = -[i]L_{i-1}$, and

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we recall [15, Prop. 3.1.6] that

$$[i] = \prod_{\substack{f \in A_+,\ \text{irred.} \\ \deg(f) \mid i}} f, \quad D_i = \prod_{a \in A_{i+}} a, \quad L_i = (-1)^i \cdot \operatorname{lcm}(f \in A_{i+}). \tag{2.1.2}$$

For $m \in \mathbb{Z}_+$, we define the Carlitz factorial Π_m as follows. If we write $m = \sum m_i q^i$ with $0 \leqslant m_i \leqslant q-1$, then

$$\Pi_m = \prod_i D_i^{m_i}. \tag{2.1.3}$$

For more information about Π_m the reader is directed to Goss [15, §9.1].

For an $\mathbb{F}_q$-algebra L, we let $\tau : L \to L$ denote the q-th power Frobenius map, and we let $L[\tau]$ denote the ring of twisted polynomials over L, subject to the condition that $\tau c = c^q \tau$ for $c \in L$. We then define as usual the Carlitz module to be the $\mathbb{F}_q$-algebra homomorphism $C : A \to A[\tau]$ determined by

$$C_\theta = \theta + \tau.$$

The Carlitz exponential is the $\mathbb{F}_q$-linear power series,

$$e_C(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i}. \tag{2.1.4}$$

The induced function $e_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is both entire and surjective, and for all $a \in A$,

$$e_C(az) = C_a(e_C(z)).$$

The kernel Λ_C of $e_C(z)$ is the A-lattice of rank 1 given by $\Lambda_C = A\widetilde{\pi}$, where for a fixed

$(q-1)$-st root of $-\theta$,

$$\widetilde{\pi} = \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} \Big(1 - \theta^{1-q^i}\Big)^{-1} \in K_\infty\big((-\theta)^{1/(q-1)}\big)$$

is called the Carlitz period (see [15, §3.2] or [20, §3.1]). Moreover, we have a product expansion

$$e_C(z) = z \prod_{\lambda \in \Lambda_C}' \left(1 - \frac{z}{\lambda}\right) = z \prod_{a \in A}' \left(1 - \frac{z}{a\widetilde{\pi}}\right), \tag{2.1.5}$$

where the prime indicates omitting the $a = 0$ term in the product. For more information about the Carlitz module, and Drinfeld modules in general, we refer the reader to [15, Chs. 3–4].

We will say that a function $f : \Omega \to C_\infty$ is holomorphic if it is rigid analytic in the sense of [8]. We set $\mathcal{H}(\Omega)$ to be the set of holomorphic functions on Ω. We define a holomorphic function $u : \Omega \to C_\infty$ by setting

$$u(z) := \frac{1}{e_C(\widetilde{\pi} z)}, \tag{2.1.6}$$

and we note that $u(z)$ is a uniformizing parameter at the infinite cusp of Ω (see [9, §5]), which plays the role of $\mathbf{q}(z) = e^{2\pi i z}$ in the classical case. The function $u(z)$ is A-periodic in the sense that $u(z+a) = u(z)$ for all $a \in A$. The imaginary part of an element $z \in C_\infty$ is set to be

$$|z|_i = \inf_{x \in K_\infty} |z - x|_\infty,$$

which measures the distance from z to the real axis $K_\infty \subseteq C_\infty$. We will say that an A-periodic holomorphic function $f : \Omega \to C_\infty$ is holomorphic at ∞ if we can write a convergent series,

$$f(z)=\sum_{n=0}^{\infty} c_n u(z)^n, \quad c_n \in C_\infty, \quad |z|_i \gg 0.$$

The function f is then determined by the power series $f=\sum c_n u^n \in C_\infty[[u]]$, and we call this power series the u-expansion of f and the coefficients c_n the u-expansion coefficients of f. We set $\mathcal{U}(\Omega)$ to be the subset of $\mathcal{H}(\Omega)$ comprising functions on Ω that are A-periodic and holomorphic at ∞. In other words, $\mathcal{U}(\Omega)$ consists of functions that have u-expansions.

We now define hyperdifferential operators and hyperderivatives (see [5], [7], [17], [29] for more details). For a field F and an independent variable z over F, for $j \geqslant 0$ we define the j-th hyperdifferential operator $\partial_z^j : F[z] \to F[z]$ by setting

$$\partial_z^j(z^n)=\binom{n}{j} z^{n-j}, \quad n \geqslant 0,$$

where $\binom{n}{j} \in \mathbb{Z}$ is the usual binomial coefficient, and extending F-linearly. (By usual convention $\binom{n}{j}=0$ if $0 \leqslant n < j$.) For $f \in F[z]$, we call $\partial_z^j(f) \in F[z]$ its j-th hyperderivative. Hyperderivatives satisfy the product rule,

$$\partial_z^j(fg)=\sum_{k=0}^{j} \partial_z^k(f)\partial_z^{j-k}(g), \quad f,g \in F[z], \tag{2.1.7}$$

and composition rule,

$$(\partial_z^j \circ \partial_z^k)(f)=(\partial_z^k \circ \partial_z^j)(f)=\binom{j+k}{j}\partial_z^{j+k}(f), \quad f \in F[z]. \tag{2.1.8}$$

Using the product rule one can extend to $\partial_z^j : F(z) \to F(z)$ in a unique way, and $F(z)$ together with the operators ∂_z^j form a hyperdifferential system. If F has characteristic 0, then $\partial_z^j = \frac{1}{j!}\frac{d^j}{dz^j}$, but in characteristic p this holds only for $j \leqslant p-1$. Furthermore, hyperderivatives satisfy a number of differentiation rules (e.g., product, quotient, power, chain rules), which aid in their description and calculation (see [17, §2.2], [20, §2.3], for a complete list of rules and historical accounts). Moreover, if $f \in F(z)$ is regular at $c \in F$, then so is $\partial_z^j(f)$ for each $j \geqslant 0$, and it follows that we have a Taylor expansion,

$$f(z) = \sum_{j=0}^{\infty} \partial_z^j(f)(c) \cdot (z-c)^j \in F[[z-c]]. \tag{2.1.9}$$

In this way we can also extend ∂_z^j uniquely to $\partial_z^j : F((z-c)) \to F((z-c))$.

For a holomorphic function $f : \Omega \to C_\infty$, it was proved by Uchino and Satoh [29, §2] that we can define a holomorphic hyperderivative $\partial_z^j(f) : \Omega \to C_\infty$ (taking $F = C_\infty$ in the preceding paragraph). That is,

$$\partial_z^j : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega).$$

Moreover they prove that the system of operators ∂_z^j on holomorphic functions inherits the same differentiation rules for hyperderivatives of polynomials and power series. Thus for $f \in \mathcal{H}(\Omega)$ and $c \in \Omega$, we have a Taylor expansion,

$$f(z) = \sum_{j=0}^{\infty} \partial_z^j(f)(c) \cdot (z-c)^j \in C_\infty[[z-c]].$$

We have the following crucial lemma for our later considerations in §2.3, where we find new identities for derivatives of functions in $\mathcal{U}(\Omega)$.

Lemma 2.1.10 (Uchino-Satoh [29, Lem. 3.6])**.** *If $f \in \mathcal{H}(\Omega)$ is A-periodic and holomorphic at ∞, then so is $\partial_z^j(f)$ for each $j \geqslant 0$. That is,*

$$\partial_z^j : \mathcal{U}(\Omega) \to \mathcal{U}(\Omega), \quad j \geqslant 0.$$

We recall computations involving $u(z)$ and $\partial_z^1(u(z))$ (see [9, §3]). First we see from (2.1.4) that $\partial_z^1(e_C(z)) = 1$, so using (2.1.5) and taking logarithmic derivatives,

$$u(z) = \frac{1}{e_C(\widetilde{\pi}z)} = \frac{1}{\widetilde{\pi}} \cdot \frac{\partial_z^1(e_C(\widetilde{\pi}z))}{e_C(\widetilde{\pi}z)} = \frac{1}{\widetilde{\pi}} \sum_{a \in A} \frac{1}{z+a}. \tag{2.1.11}$$

Furthermore,

$$\partial_z^1(u(z)) = \partial_z^1\left(\frac{1}{e_C(\widetilde{\pi}z)}\right) = \frac{-\partial_z^1(e_C(\widetilde{\pi}z))}{e_C(\widetilde{\pi}z)^2} = -\widetilde{\pi}u(z)^2. \tag{2.1.12}$$

Thus, $\partial_z^1(u) = -\widetilde{\pi}u^2 \in \mathcal{U}(\Omega)$. In §2.3 we generalize this formula and calculate $\partial_z^r(u^n)$ for $r, n \geqslant 0$.

We conclude this section by discussing some properties of hyperderivatives particular to positive characteristic. Suppose $\operatorname{char}(F) = p > 0$. If we write $j = \sum_{i=0}^{s} b_i p^i$, with $0 \leqslant b_i \leqslant p-1$ and $b_s \neq 0$, then (see [17, Thm. 3.1])

$$\partial_z^j = \partial_z^{b_0} \circ \partial_z^{b_1 p} \circ \cdots \circ \partial_z^{b_s p^s}, \tag{2.1.13}$$

which follows from the composition law and Lucas's theorem (e.g., see [3, Eq. (14)]). We note that for $0 \leqslant b \leqslant p-1$,

$$\partial_z^{bp^k} = \frac{1}{b!} \cdot \partial_z^{p^k} \circ \cdots \circ \partial_z^{p^k}, \quad (b \text{ times}).$$

Moreover the p-th power rule (see [5, §7], [17, §2.2]) says that for $f \in F((z-c))$,

$$\partial_z^j\big(f^{p^s}\big) = \begin{cases} \big(\partial_z^\ell(f)\big)^{p^s} & \text{if } j = \ell p^s, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1.14}$$

and so calculation using (2.1.13) and (2.1.14) can often be fairly efficient.

2.2 Goss polynomials and hyperderivatives

We review here results on Goss polynomials, which were introduced by Goss in [14, §6] and have been studied further by Gekeler [9, §3], [10]. We start first with an $\mathbb{F}_q$-vector space $\Lambda \subseteq C_\infty$ of dimension d. We define the exponential function of Λ,

$$e_\Lambda(z) = z \sideset{}{'}\prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right),$$

which is an $\mathbb{F}_q$-linear polynomial of degree q^d. If we take $t_\Lambda(z) = 1/e_\Lambda(z)$, then just as in (2.1.11) we have

$$t_\Lambda(z) = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}.$$

We can extend these definitions to any discrete lattice $\Lambda \subseteq C_\infty$, which is the union of nested finite dimensional $\mathbb{F}_q$-vector spaces $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$. We find that generally $e_\Lambda(z) = \lim_{i\to\infty} e_{\Lambda_i}(z)$ and $t_\Lambda(z) = \lim_{i\to\infty} t_{\Lambda_i}(z)$, where the convergence is coefficient-wise in $C_\infty((z))$.

Remark 2.2.1. If we take $\Lambda = \Lambda_C$, then $e_{\Lambda_C}(z) = e_C(z)$, whereas if we take $\Lambda = A$, then $e_A(z) = \frac{1}{\widetilde{\pi}} e_C(\widetilde{\pi} z)$. Thus

$$t_A(z) = \widetilde{\pi}\, t_{\Lambda_C}(\widetilde{\pi} z) = \frac{\widetilde{\pi}}{e_C(\widetilde{\pi} z)},$$

and $u(z)$, as defined in (2.1.6), is given by

$$u(z) = \frac{t_A(z)}{\widetilde{\pi}} = t_{\Lambda_C}(\widetilde{\pi} z).$$

This normalization of $u(z)$ is taken so that the u-expansions of some Drinfeld modular forms will have K-rational coefficients.

Theorem 2.2.2 (Goss [14, §6]; see also Gekeler [9, §3])**.** *Let* $\Lambda \subseteq C_\infty$ *be a discrete*

$\mathbb{F}_q$*-vector space. Let*

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda}{}' \left(1 - \frac{z}{\lambda}\right) = \sum_{j=0}^{\infty} \alpha_j z^{q^j},$$

and let $t_\Lambda(z) = 1/e_\Lambda(z)$*. For each* $k \geqslant 1$*, there is a monic polynomial* $G_{k,\Lambda}(t)$ *of degree* k *with coefficients in* $\mathbb{F}_q[\alpha_0, \alpha_1, \ldots]$ *so that*

$$S_{k,\Lambda}(z) := \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^k} = G_{k,\Lambda}\bigl(t_\Lambda(z)\bigr).$$

Furthermore the following properties hold.

(a) $G_{k,\Lambda}(t) = t(G_{k-1,\Lambda}(t) + \alpha_1 G_{k-q,\Lambda}(t) + \alpha_2 G_{k-q^2,\Lambda}(t) + \cdots)$.

(b) We have a generating series identity

$$\mathcal{G}_\Lambda(t,x) = \sum_{k=1}^{\infty} G_{k,\Lambda}(t) x^k = \frac{tx}{1 - t e_\Lambda(x)}.$$

(c) If $k \leqslant q$*, then* $G_{k,\Lambda}(t) = t^k$.

(d) $G_{pk,\Lambda}(t) = G_{k,\Lambda}(t)^p$.

(e) $t^2\, \partial_t^1\bigl(G_{k,\Lambda}(t)\bigr) = k G_{k+1,\Lambda}(t)$.

Gekeler [9, (3.8)] finds a formula for each $G_{k,\Lambda}(t)$,

$$G_{k+1,\Lambda}(t) = \sum_{j=0}^{k} \sum_{\underline{i}} \binom{j}{\underline{i}} \alpha^{\underline{i}} t^{j+1}, \tag{2.2.3}$$

where the sum is over all $(s+1)$-tuples $\underline{i} = (i_0, \ldots, i_s)$, with s arbitrary, satisfying $i_0 + \cdots + i_s = j$ and $i_0 + i_1 q + \cdots + i_s q^s = k$; $\binom{j}{\underline{i}} = j!/(i_0! \cdots i_s!)$ is a multinomial coefficient; and $\alpha^{\underline{i}} = \alpha_0^{i_0} \cdots \alpha_s^{i_s}$.

Part (e) of Theorem 2.2.2 indicates that there are interesting hyperderivative relations among Goss polynomials, with respect to t and to z, which we now investigate. All hyperderivatives we will take will be of polynomials and formal power series, but the considerations in §2.1 about holomorphic functions will play out later in the paper. The main result of this section is the following.

Theorem 2.2.4. *Let $\Lambda \subseteq C_\infty$ be a discrete $\mathbb{F}_q$-vector space, and let $t = t_\Lambda(z)$. For $r \geqslant 0$, we define $\beta_{r,j}$ so that*

$$G_{r+1,\Lambda}(t) = \sum_{j=0}^{r} \beta_{r,j} t^{j+1}.$$

Then

$$\partial_z^r(t^n) = (-1)^r \cdot t^n \, \partial_t^{n-1}\big(t^{n-2} G_{r+1,\Lambda}(t)\big), \quad \forall\, n \geqslant 1, \tag{2.2.4a}$$

$$\partial_z^r(t^n) = (-1)^r \sum_{j=0}^{r} \beta_{r,j} \, t^{j+1} \, \partial_t^j (t^{n+j-1}), \quad \forall\, n \geqslant 0, \tag{2.2.4b}$$

and

$$\binom{n+r-1}{r} G_{n+r,\Lambda}(t) = \sum_{j=0}^{r} \beta_{r,j} \, t^{j+1} \, \partial_t^j \big(t^{j-1} G_{n,\Lambda}(t)\big), \quad \forall\, n \geqslant 1. \tag{2.2.4c}$$

Remark 2.2.5. We see that (2.2.4a) and (2.2.4b) generalize (2.1.12) and that (2.2.4c) generalizes Theorem 2.2.2(e). In later sections (2.2.4a) and (2.2.4b) will be useful for taking derivatives of Drinfeld modular forms. The coefficients $\beta_{r,j}$ can be computed using the generating series $\mathcal{G}_\Lambda(t,x)$ or equivalently (2.2.3). The proof requires some preliminary lemmas.

Lemma 2.2.6 (cf. Petrov [25, §3])**.** *For $r \geqslant 0$ and $n \geqslant 1$,*

$$\partial_z^r\big(S_{n,\Lambda}(z)\big) = (-1)^r \binom{n+r-1}{r} G_{n+r,\Lambda}(t). \tag{2.2.6a}$$

Moreover, we have

$$\partial_z^r\big(S_{n,\Lambda}(z)\big) = (-1)^{n+r-1} \cdot \partial_z^{n-1}\big(S_{r+1,\Lambda}(z)\big) \tag{2.2.6b}$$

and

$$\partial_z^r(t) = (-1)^r G_{r+1,\Lambda}(t). \tag{2.2.6c}$$

Proof. Using the power and quotient rules [17, §2.2], we see that for $\lambda \in C_\infty$,

$$\partial_z^r\left(\frac{1}{(z-\lambda)^n}\right) = \binom{-n}{r}\frac{1}{(z-\lambda)^{n+r}} = (-1)^r\binom{n+r-1}{r}\frac{1}{(z-\lambda)^{n+r}}.$$

Therefore,

$$\partial_z^r\big(S_{n,\Lambda}(z)\big) = (-1)^r\binom{n+r-1}{r}S_{n+r,\Lambda}(z),$$

and combining with the defining property of $G_{n+r,\Lambda}(t)$ in Theorem 2.2.2, we see that (2.2.6a) follows. Now

$$\binom{n+r-1}{r} = \binom{(r+1)+(n-1)-1}{n-1},$$

and so (2.2.6b) follows from (2.2.6a). Finally, (2.2.6c) is a special case of (2.2.6a) with $n = 1$. □

Lemma 2.2.7. *For $n \geqslant 1$, we have an identity of rational functions in x,*

$$\frac{x}{(1-te_\Lambda(x))^n} = \partial_t^{n-1}\left(\frac{t^{n-1}x}{1-te_\Lambda(x)}\right) = \partial_t^{n-1}\big(t^{n-2}\mathcal{G}_\Lambda(t,x)\big).$$

Proof. Our derivatives with respect to t are taken while considering x to be a constant. We note that for $\ell \geqslant 0$,

$$\partial_t^\ell\left(\frac{1}{1-te_\Lambda(x)}\right) = \frac{e_\Lambda(x)^\ell}{(1-te_\Lambda(x))^{\ell+1}},$$

by the quotient and chain rules [17, §2.2]. Therefore, by the product rule,

$$\begin{aligned}\partial_t^{n-1}\left(\frac{t^{n-1}}{1-te_\Lambda(x)}\right) &= \sum_{k=0}^{n-1}\partial_t^k(t^{n-1})\partial_t^{n-1-k}\left(\frac{1}{1-te_\Lambda(x)}\right) \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{te_\Lambda(x)}{1-te_\Lambda(x)}\right)^{n-1-k}\cdot\frac{1}{1-te_\Lambda(x)} \\ &= \left(1+\frac{te_\Lambda(x)}{1-te_\Lambda(x)}\right)^{n-1}\cdot\frac{1}{1-te_\Lambda(x)}.\end{aligned}$$

A simple calculation yields that this is $1/(1-te_\Lambda(x))^n$, and the result follows. $\square$

Proof of Theorem 2.2.4. The chain rule [17, §2.2] and (2.2.6c) imply that

$$\begin{aligned}\partial_z^r(t^n) &= \sum_{k=1}^{r}\binom{n}{k}t^{n-k}\sum_{\substack{\ell_1,\dots,\ell_k\geqslant 1\\ \ell_1+\cdots+\ell_k=r}}\partial_z^{\ell_1}(t)\cdots\partial_z^{\ell_k}(t) \\ &= (-1)^r\sum_{k=1}^{r}\binom{n}{k}t^{n-k}\sum_{\substack{\ell_1,\dots,\ell_k\geqslant 1\\ \ell_1+\cdots+\ell_k=r}}G_{\ell_1+1,\Lambda}(t)\cdots G_{\ell_k+1,\Lambda}(t).\end{aligned}$$

By direct expansion (see [17, §2.2, Eq. (I)]), the final inner sum above is the coefficient of x^r in

$$\big(G_{2,\Lambda}(t)x+G_{3,\Lambda}(t)x^2+\cdots\big)^k,$$

and therefore by the binomial theorem,

$$\partial_z^r(t^n) = (-1)^r\cdot\Big(\text{coefficient of } x^r \text{ in } \big(t+G_{2,\Lambda}(t)x+G_{3,\Lambda}(t)x^2+\cdots\big)^n\Big).$$

Now $G_{1,\Lambda}(t)=t$, so

$$t+G_{2,\Lambda}(t)x+G_{3,\Lambda}(t)x^2+\cdots = \sum_{k=1}^{\infty}G_{k,\Lambda}(t)x^{k-1} = \frac{\mathcal{G}_\Lambda(t,x)}{x} = \frac{t}{1-te_\Lambda(x)}.$$

Therefore,

$$\partial_z^r(t^n) = (-1)^r \cdot \left(\text{coefficient of } x^{r+1} \text{ in } \frac{t^n x}{(1 - te_\Lambda(x))^n}\right).$$

From Lemma 2.2.7 we see that

$$\frac{t^n x}{(1 - te_\Lambda(x))^n} = t^n \sum_{k=1}^{\infty} \partial_t^{n-1}\big(t^{n-2} G_{k,\Lambda}(t)\big) x^k,$$

and so (2.2.4a) holds. To prove (2.2.4b), we first note that it holds when $n = 0$ by checking the various cases and using that $\beta_{r,0} = 0$ for $r \geqslant 1$, since $G_{r+1}(t)$ is divisible by t^2 for $r \geqslant 1$ by Theorem 2.2.2, and that $\beta_{0,0} = 1$. For $n \geqslant 1$, we use (2.2.4a) and write

$$\partial_z^r(t^n) = (-1)^r \cdot t^n \, \partial_t^{n-1}\big(t^{n-2} G_{r+1,\Lambda}(t)\big) = (-1)^r \cdot t^n \, \partial_t^{n-1}\left(\sum_{j=0}^{r} \beta_{r,j} \, t^{n+j-1}\right).$$

Noting that

$$\partial_t^{n-1}(t^{n+j-1}) = \binom{n+j-1}{n-1} t^j = t^{j-n+1} \, \partial_t^j(t^{n+j-1}),$$

we then have

$$\partial_z^r(t^n) = (-1)^r \sum_{j=0}^{r} \beta_{r,j} \, t^{j+1} \, \partial_t^j(t^{n+j-1}),$$

and so (2.2.4b) holds. Furthermore, by (2.2.6a) and (2.2.6b),

$$\binom{n+r-1}{r} G_{n+r,\Lambda}(t) = (-1)^{n-1} \cdot \partial_z^{n-1}\big(S_{r+1,\Lambda}(z)\big) = (-1)^{n-1} \cdot \partial_z^{n-1}\big(G_{r+1,\Lambda}(t)\big).$$

But then by (2.2.4a),

$$\partial_z^{n-1}\big(G_{r+1,\Lambda}(t)\big) = \sum_{j=0}^{r} \beta_{r,j} \partial_z^{n-1}(t^{j+1}) = (-1)^{n-1} \sum_{j=0}^{r} \beta_{r,j} t^{j+1} \, \partial_t^j\big(t^{j-1} G_{n,\Lambda}(t)\big),$$

which yields (2.2.4c). $\square$

2.3 Theta operators on Drinfeld modular forms

We recall the definition of Drinfeld modular forms for $\mathrm{GL}_2(A)$, which were initially studied by Goss [12], [13], [14]. We will also review results on u-expansions of modular forms due to Gekeler [9]. Throughout we let $\Gamma = \mathrm{GL}_2(A)$. A holomorphic function $f : \Omega \to C_\infty$ is a Drinfeld modular form of weight $k \geqslant 0$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ if

1. for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \Omega$,

$$f(\gamma z) = (\det \gamma)^{-m}(cz+d)^k f(z), \quad \gamma z = \frac{az+b}{cz+d};$$

2. and f is holomorphic at ∞, i.e., f has a u-expansion and so $f \in \mathcal{U}(\Omega)$.

We let M_k^m be the C_∞-vector space of modular forms of weight k and type m. We know that $M_k^m \cdot M_{k'}^{m'} \subseteq M_{k+k'}^{m+m'}$ and that $M = \bigoplus_{k,m} M_k^m$ and $M^0 = \bigoplus_k M_k^0$ are graded C_∞-algebras. Moreover, in order to have $M_k^m \neq 0$, we must have $k \equiv 2m \pmod{q-1}$. If L is a subring of C_∞, then we let $M_k^m(L)$ denote the space of forms with u-expansion coefficients in L, i.e., $M_k^m(L) = M_k^m \cap L[[u]]$. We note that if $f = \sum c_n u^n$ is the u-expansion of $f \in M_k^m$, then

$$c_n \neq 0 \quad \Rightarrow \quad n \equiv m \pmod{q-1}, \tag{2.3.1}$$

which can be seen by using $\gamma = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & 1 \end{smallmatrix}\right)$, for ζ a generator of $\mathbb{F}_q^\times$, in the definition above.

Certain Drinfeld modular forms can be expressed in terms of A-expansions, which we now recall. For $k \geqslant 1$, we set

$$G_k(t) = G_{k,\Lambda_C}(t) = \sum_{j=0}^{k-1} \beta_{k-1,j} t^{j+1}, \tag{2.3.2}$$

to be the Goss polynomials with respect to the lattice Λ_C. Since $e_C(z) \in K[[z]]$, it follows from Theorem 2.2.2 that the coefficients $\beta_{k-1,j} \in K$ for all k, j. As in (2.1.6) and Remark 2.2.1, we have $u(z) = 1/e_C(\widetilde{\pi}z)$, and for $a \in A$ we set

$$u_a(z) := u(az) = \frac{1}{e_C(\widetilde{\pi}az)}. \tag{2.3.3}$$

Since $e_C(\widetilde{\pi}az) = C_a(e_C(\widetilde{\pi}z))$, if we take the reciprocal polynomial for $C_a(z)$ to be $R_a(z) = z^{q^{\deg a}} C_a(1/z)$ then

$$u_a = \frac{u^{q^{\deg a}}}{R_a(u)} = u^{q^{\deg a}} + \cdots \in A[[u]]. \tag{2.3.4}$$

We say that a modular form f has an A-expansion if there exist $k \geqslant 1$ and c_0, $c_a \in C_\infty$ for $a \in A_+$, so that

$$f = c_0 + \sum_{a \in A_+} c_a G_k(u_a).$$

Example 2.3.5. For $k \equiv 0 \pmod{q-1}$, $k > 0$, the primary examples of Drinfeld modular forms with A-expansions come from Eisenstein series,

$$E_k(z) = \frac{1}{\widetilde{\pi}^k} \sum_{a,b \in A}{}' \frac{1}{(az+b)^k},$$

which is a modular form of weight k and type 0. Gekeler [9, (6.3)] showed that

$$E_k = \frac{1}{\widetilde{\pi}^k} \sum_{b \in A}{}' \frac{1}{b^k} - \sum_{a \in A_+} G_k(u_a) = -\frac{\zeta_C(k)}{\widetilde{\pi}^k} - \sum_{a \in A_+} G_k(u_a), \tag{2.3.6}$$

where $\zeta_C(k) = \sum_{a \in A_+} a^{-k}$ is a Carlitz zeta value. We know (see [15, §9.2]) that $\zeta_C(k)/\widetilde{\pi}^k \in K$. We also define Bernoulli-Carlitz numbers BC_k to be $\Pi_k \zeta_C(k)/\widetilde{\pi}^k$. I will talk about Bernoulli-Carlitz numbers in the next chapter.

For more information and examples on A-expansions the reader is directed to Gekeler [9], López [18], [19], and Petrov [24], [25].

Example 2.3.7. We can also define the false Eisenstein series $E(z)$ of Gekeler [9, §8] to be

$$E(z) := \frac{1}{\widetilde{\pi}} \sum_{a \in A_+} \sum_{b \in A} \frac{a}{az+b},$$

which is not quite a modular form but is a quasi-modular form similar to the classical weight 2 Eisenstein series [3], [9]. Gekeler showed that $E \in \mathcal{U}(\Omega)$ and that E has an A-expansion,

$$E = \sum_{a \in A_+} aG_1(u_a) = \sum_{a \in A_+} au_a. \tag{2.3.8}$$

We now define theta operators Θ^r on functions in $\mathcal{H}(\Omega)$ by setting for $r \geqslant 0$,

$$\Theta^r := \frac{1}{(-\widetilde{\pi})^r}\, \partial_z^r. \tag{2.3.9}$$

If we take $\Theta = \Theta^1$, then by (2.1.12), $\Theta u = u^2$, and Θ plays the role of the classical theta operator $\vartheta = \mathbf{q}\frac{d}{d\mathbf{q}}$. Just as in the classical case, Θ and more generally Θ^r do not take modular forms to modular forms. However, Bosser and Pellarin [3, Thm. 2] prove that Θ^r preserves quasi-modularity:

$$\Theta^r : C_\infty[E, g, h] \to C_\infty[E, g, h],$$

where E is the false Eisenstein series, $g = E_{q-1}$, and h is the cusp form of weight $q+1$ and type 1 defined by Gekeler [9, Thm. 5.13] as the $(q-1)$-st root of the discriminant function Δ. To prove their theorem, Bosser and Pellarin [3, Lem. 3.5] give formulas for $\Theta^r(u^n)$, which are ostensibly a bit complicated. From Theorem 2.2.4, we have the following corollary, which perhaps conceptually simplifies matters.

Corollary 2.3.10. *For $r \geqslant 0$,*

$$\Theta^r(u^n) = u^n\, \partial_u^{n-1}\big(u^{n-2}G_{r+1}(u)\big), \quad \forall\, n \geqslant 1, \tag{2.3.10a}$$

$$\Theta^r(u^n) = \sum_{j=0}^{r} \beta_{r,j} u^{j+1} \partial_u^j (u^{n+j-1}) = \sum_{j=0}^{r} \binom{n+j-1}{j} \beta_{r,j} u^{n+j}, \quad \forall\, n \geqslant 0, \tag{2.3.10b}$$

where $\beta_{r,j}$ are the coefficients of $G_{r+1}(t)$ in (2.3.2)*.*

Proof. The proof of (2.3.10a) is straightforward, but it is worth noting how the different normalizations of $u(z)$ and $t_{\Lambda_C}(z)$ work out. From Remark 2.2.1, we see that

$$\begin{aligned}\Theta^r(u^n) = \left(\frac{-1}{\widetilde{\pi}}\right)^r \partial_z^r\big(t_{\Lambda_C}(\widetilde{\pi}z)^n\big) &= \left(\frac{-1}{\widetilde{\pi}}\right)^r \cdot \widetilde{\pi}^r\, \partial_z^r\big(t_{\Lambda_C}\big)\big|_{z=\widetilde{\pi}z} \\ &= t^n\, \partial_t^{n-1}\big(t^{n-2}G_{r+1}(t)\big)\big|_{t=t_{\Lambda_C}(\widetilde{\pi}z)} = u^n\, \partial_u^{n-1}\big(u^{n-2}G_{r+1}(u)\big),\end{aligned}$$

where the third equality is (2.2.4a). The proof of (2.3.10b) is then the same as for (2.2.4b).

□

Remark 2.3.11. We see from (2.3.10a) that there is a duality of some fashion between the r-th derivative of u^n and the $(n-1)$-st derivative of $G_{r+1}(u)$, which dovetails with (2.2.6b).

We see from this corollary that Θ^r can be seen as the operator on power series in $C_\infty[[u]]$ given by the following result. Moreover, from (2.3.12b), we see that computation of $\Theta^r(f)$ is reasonably straightforward once the computation of the coefficients of $G_{r+1}(t)$ can be made.

Corollary 2.3.12. *Let $f = \sum c_n u^n \in \mathcal{U}(\Omega)$. For $r \geqslant 0$,*

$$\Theta^r(f) = \Theta^r(c_0) + \sum_{n=1}^{\infty} c_n u^n\, \partial_u^{n-1}\big(u^{n-2}G_{r+1}(u)\big), \tag{2.3.12a}$$

$$\Theta^r(f) = \sum_{j=0}^{r} \beta_{r,j}\, u^{j+1}\, \partial_u^j\big(u^{j-1}f\big), \tag{2.3.12b}$$

where $\beta_{r,j}$ are the coefficients of $G_{r+1}(t)$ in (2.3.2).

Finally we recall the definition of the r-th Serre operator $\mathcal{D}^r$ on modular forms in M_k^m for $r \geqslant 0$. We set

$$\mathcal{D}^r(f) := \Theta^r(f) + \sum_{i=1}^{r} (-1)^i \binom{k+r-1}{i} \Theta^{r-i}(f)\Theta^{i-1}(E). \tag{2.3.13}$$

The following result shows that $\mathcal{D}^r$ takes modular forms to modular forms.

Theorem 2.3.14 (Bosser-Pellarin [4, Thm. 4.1])**.** *For any weight k, type m, and $r \geqslant 0$,*

$$\mathcal{D}^r\left(M_k^m\right) \subseteq M_{k+2r}^{m+r}.$$

2.4 v-adic modular forms

In this section we review the theory of v-adic modular forms introduced by Goss [16] and Vincent [30], [31]. In [26], Serre defined p-adic modular forms as p-adic limits of Fourier series of classical modular forms and determined their properties, in particular their behavior under the ϑ-operator. For a fixed finite place v of K, Goss and Vincent recently transferred Serre's definition to the function field setting of v-adic modular forms, and Goss produced families of examples based on work of Petrov [24] (see Theorem 2.5.3). In §2.5, we show that v-adic modular forms are invariant under the operators Θ^r.

For our place v of K we fix $\wp \in A_+$, which is the monic irreducible generator of the ideal $\mathfrak{p}_v$ associated to v, and we let $d := \deg(\wp)$. As before we let A_v and K_v denote completions with respect to v.

We will write $K \otimes A_v[[u]]$ for $K \otimes_A A_v[[u]]$, and we recall that $K \otimes A_v[[u]]$ can be identified with elements of $K_v[[u]]$ that have bounded denominators. For $f = \sum_{n=0}^{\infty} c_n u^n \in$

$K \otimes A_v[[u]]$, we set

$$\operatorname{ord}_v(f) := \inf_n \{\operatorname{ord}_v(c_n)\} = \min_n \{\operatorname{ord}_v(c_n)\}. \tag{2.4.1}$$

If $\operatorname{ord}_v(f) \geqslant 0$, i.e., if $f \in A_v[[u]]$, then we say f is v-integral. For f, $g \in K \otimes A_v[[u]]$, we write that

$$f \equiv g \pmod{\wp^m},$$

if $\operatorname{ord}_v(f-g) \geqslant m$. We also define a topology on $K \otimes A_v[[u]]$ in terms of the v-adic norm,

$$\|f\|_v := q^{-\operatorname{ord}_v(f)}, \tag{2.4.2}$$

which is a multiplicative norm by Gauss' lemma.

Following Goss, we define the v-adic weight space $\mathbb{S} = \mathbb{S}_v$ by

$$\mathbb{S} := \varprojlim_{\ell} \mathbb{Z}/(q^d-1)p^\ell\mathbb{Z} = \mathbb{Z}/(q^d-1)\mathbb{Z} \times \mathbb{Z}_p. \tag{2.4.3}$$

We have a canonical embedding of $\mathbb{Z} \hookrightarrow \mathbb{S}$, by identifying $n \in \mathbb{Z}$ with $(\overline{n}, n)$, where $\overline{n}$ is the class of n modulo $q^d - 1$. For any $a \in A_+$ with $\wp \nmid a$, we can decompose a as $a = a_1 a_2$, where $a_1 \in A_v^\times$ is the (q^d-1)-st root of unity satisfying $a_1 \equiv a \pmod{v}$ and $a_2 \in A_v^\times$ satisfies $a_2 \equiv 1 \pmod{v}$. Then for any $s = (x, y) \in \mathbb{S}$, we define

$$a^s := a_1^x a_2^y. \tag{2.4.4}$$

This definition of a^s is compatible with the usual definition when s is an integer. Furthermore, it is easy to check that the function $s \mapsto a^s$ is continuous from $\mathbb{S}$ to $A_v^\times$.

Definition 2.4.5 (Goss [16, Def. 5])**.** We say a power series $f \in K \otimes A_v[[u]]$ is a v-

adic modular form of weight $s \in \mathbb{S}$, in the sense of Serre, if there exists a sequence of K-rational modular forms $f_i \in M_{k_i}^{m_i}(K)$ so that as $i \to \infty$,

(a) $\|f_i - f\|_v \to 0$,

(b) $k_i \to s$ in $\mathbb{S}$.

Moreover, if $f \neq 0$, then m_i is eventually a constant $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, and we say that m is the type of f. We say that f_i converges to f as v-adic modular forms.

It is easy to see that the sum and difference of two v-adic modular forms, both with weight s and type m, are also v-adic modular forms with the same weight and type. We set

$$\mathcal{M}_s^m = \bigl\{ f \in K \otimes A_v[[u]] \bigm| f \text{ a } v\text{-adic modular form of weight } s \text{ and type } m \bigr\}, \quad (2.4.6)$$

which is a K_v-vector space, and we note that

$$\mathcal{M}_{s_1}^{m_1} \cdot \mathcal{M}_{s_2}^{m_2} \subseteq \mathcal{M}_{s_1+s_2}^{m_1+m_2}.$$

We take $\mathcal{M}_s^m(A_v) := \mathcal{M}_s^m \cap A_v[[u]]$, which is an A_v-module. Moreover, any Drinfeld modular form in $M_k^m(K)$ is also a v-adic modular form as the limit of the constant sequence (u-expansion coefficients of forms in $M_k^m(K)$ have bounded denominators by [9, Thm. 5.13, §12], [13, Thm. 2.23]), and so for $k \in \mathbb{Z}$, $k \geqslant 0$,

$$M_k^m(K) \subseteq \mathcal{M}_k^m, \quad M_k^m(A) \subseteq \mathcal{M}_k^m(A_v).$$

The justification of the final part of Definition 2.4.5 is the following lemma.

Lemma 2.4.7. *Suppose that $f_i \in M_{k_i}^{m_i}(K)$ converge to a non-zero v-adic modular form f. Then there is some $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ so that except for finitely terms $m_i = m$.*

Proof. Since $\|f - f_i\|_v \to 0$, it follows that $\|f_i - f_j\|_v \to 0$ as $i, j \to \infty$. If $f = \sum c_n u^n$ and $c_n \neq 0$, then from (2.3.1) we see that for $i, j \gg 0$, $n \equiv m_i \equiv m_j \pmod{q-1}$. $\square$

Proposition 2.4.8. *Suppose $\{f_i\}$ is a sequence of v-adic modular forms with weights s_i. Suppose that we have $f_0 \in K \otimes A_v[[u]]$ and $s_0 \in \mathbb{S}$ satisfying,*

(a) $\|f_i - f_0\|_v \to 0$,

(b) $s_i \to s_0$ *in* $\mathbb{S}$.

Then f_0 is a v-adic modular form of weight s_0. The type of f_0 is the eventual constant type of the sequence $\{f_i\}$.

Proof. For each $i \geqslant 1$, we have a sequence of Drinfeld modular forms $g_{i,j} \to f_i$ as $j \to \infty$. Standard arguments show that the sequence of Drinfeld modular forms $\{g_{i,i}\}_{i=1}^{\infty}$ converges to f_0 with respect to the $\|\cdot\|_v$-norm and that the weights k_i of $g_{i,i}$ go to s_0 in $\mathbb{S}$. $\square$

We recall the definitions of Hecke operators on Drinfeld modular forms and their actions on u-expansions [9, §7], [14, §7]. For $\ell \in A_+$ irreducible of degree e, the Hecke operator $T_\ell : M_k^m \to M_k^m$ is defined by

$$(T_\ell f)(z) = \ell^k f(\ell z) + U_\ell f(z) = \ell^k f(\ell z) + \sum_{\beta \in A_{(<e)}} f\left(\frac{z+\beta}{\ell}\right).$$

Just as in the classical case the operators T_ℓ and U_ℓ are uniquely determined by their actions on u-expansions. We define $U_\ell, V_\ell : C_\infty[[u]] \to C_\infty[[u]]$ by

$$U_\ell\left(\sum_{n=0}^{\infty} c_n u^n\right) := \sum_{n=1}^{\infty} c_n G_{n,\Lambda_\ell}(\ell u), \tag{2.4.9}$$

where $\Lambda_\ell \subseteq C_\infty$ is the e-dimensional $\mathbb{F}_q$-vector space of ℓ-division points on the Carlitz module C, and

$$V_\ell\biggl(\sum_{n=0}^{\infty} c_n u^n\biggr) := \sum_{n=0}^{\infty} c_n u_\ell^n. \tag{2.4.10}$$

We find [9, Eq. (7.3)] that $T_\ell : C_\infty[[u]] \to C_\infty[[u]]$ of weight k is given by $T_\ell = \ell^k V_\ell + U_\ell$.

If $f \in \mathcal{M}_s^m$ for some weight $s \in \mathbb{S}$, then we define $U_\ell(f)$, $V_\ell(f) \in K \otimes A_v[[u]]$ as above, and if $\ell \neq \wp$, we set

$$T_\ell(f) = \ell^s V_\ell(f) + U_\ell(f), \tag{2.4.11}$$

where ℓ^s is defined as in (2.4.4) (note that if $\ell = \wp$, then (2.4.4) is not well-defined). Of importance to us is that Hecke operators preserve spaces of v-adic modular forms.

Proposition 2.4.12. *Let $\ell \in A_+$ be irreducible, $\ell \neq \wp$. For all v-adic weights s and types m, the operators T_ℓ, $U_\wp$, and $V_\wp$ preserve the spaces $\mathcal{M}_s^m$ and $\mathcal{M}_s^m(A_v)$.*

We first define a sequence of normalized Eisenstein series studied by Gekeler [9, §6]. For $d \geqslant 1$, we let

$$g_d(z) = -L_d \cdot E_{q^d-1}(z), \tag{2.4.13}$$

which is a Drinfeld modular form of weight $q^d - 1$ and type 0. By the following proposition we see that g_d plays the role of E_{p-1} for classical modular forms.

Proposition 2.4.14 (Gekeler [9, Prop. 6.9, Cor. 6.12])**.** *For $d \geqslant 1$, the following hold:*

(a) $g_d \in A[[u]]$;

(b) $g_d \equiv 1 \pmod{[d]}$.

Proof of Proposition 2.4.12. Let $f \in \mathcal{M}_s^m$. Once we establish that $T_\ell(f)$, $U_\wp(f)$, and $V_\wp(f)$ are elements of $\mathcal{M}_s^m$, we claim the statement about the operators preserving $\mathcal{M}_s^m(A_v)$ is a consequence of (2.4.9)–(2.4.11). Indeed, in either case $\ell \neq \wp$ or $\ell = \wp$ we have

$V_\ell(A_v[[u]]) \subseteq A_v[[u]]$, since in (2.4.10) the u_ℓ^n terms are in $A[[u]]$ by (2.3.4). Likewise for U_ℓ, the polynomials $G_{n,\Lambda_\ell}(\ell u)$ in (2.4.9) are in $A[u]$, as the $\mathbb{F}_q$-lattice Λ_ℓ has exponential function given by polynomials from the Carlitz action, namely $e_{\Lambda_\ell}(z) = C_\ell(z)/\ell$, and thus by Theorem 2.2.2(b),

$$\mathcal{G}_{\Lambda_\ell}(\ell u, x) = \sum_{n=1}^{\infty} G_{n,\Lambda_\ell}(\ell u)x^n = \frac{\ell u x}{1 - uC_\ell(x)} \in \ell \cdot A[u][[x]].$$

Additionally we recall that the cases of $U_\wp$ and $V_\wp$ preserving v-integrality were previously proved by Vincent [31, Cor. 3.2, Prop. 3.3].

Now by hypothesis we can choose a sequence $\{f_i\}$ of Drinfeld modular forms of weight k_i and type m so that $f_i \to f$ and $k_i \to s$. By Proposition 2.4.14(b), for any $i \geqslant 0$,

$$g_d^{q^i} \equiv 1 \pmod{\wp^{q^i}},$$

since $\operatorname{ord}_v([d]) = 1$. The form $g_d^{q^i}$ has weight $(q^d-1)q^i$ and type 0, and certainly $f_i g_d^{q^i} \to f$ with respect to the $\|\cdot\|_v$-norm. However, we also have that as real numbers,

$$\text{weight of } f_i g_d^{q^i} = k_i + (q^d-1)q^i \to \infty, \quad \text{as } i \to \infty.$$

Therefore, it suffices to assume that $k_i \to \infty$ as real numbers, as $i \to \infty$.

Suppose that $f = \sum c_n u^n$, $f_i = \sum c_{n,i} u^n \in K \otimes A_v[[u]]$. For $\ell \neq \wp$, since $\ell^{k_i} \to \ell^s$ and $c_{n,i} \to c_n$, we have

$$T_\ell(f_i) = \ell^{k_i} \sum_{n=0}^{\infty} a_{n,i} u_\ell^n + \sum_{n=0}^{\infty} c_{n,i} G_{n,\Lambda_\ell}(\ell u) \;\longrightarrow\; T_\ell(f).$$

Since $T_\ell(f_i) \in M_{k_i}^m(K)$, it follows that $T_\ell(f) \in \mathcal{M}_s^m$.

Now consider the case $\ell = \wp$. Since $k_i \to \infty$, we see that $|\wp^{k_i}|_v \to 0$. Therefore,

$$T_\wp(f_i) \to \sum_{n=0}^{\infty} c_n G_{n,\Lambda_\wp}(\wp u) = U_\wp(f),$$

and so $U_\wp(f) \in \mathcal{M}_s^m$. By the same argument each $U_\wp(f_i) \in \mathcal{M}_{k_i}^m$, starting with the constant sequence f_i in the first paragraph. By subtraction each

$$V_\wp(f_i) = \wp^{-k_i}\bigl(T_\wp(f_i) - U_\wp(f_i)\bigr) \tag{2.4.15}$$

is then an element of $\mathcal{M}_{k_i}^m$. Because $c_{n,i} \to c_n$, we see from (2.4.10) that $V_\wp(f_i) \to V_\wp(f)$ with respect to the $\|\cdot\|_v$-norm. Thus by Proposition 2.4.8, $V_\wp(f) \in \mathcal{M}_s^m$ as desired. $\square$

2.5 Theta operators on v-adic modular forms

As is well known the operators Θ^r do not generally take Drinfeld modular forms to Drinfeld modular forms [3], [29]. However, we will prove in this section that each Θ^r, $r \geqslant 0$, does preserve spaces of v-adic modular forms. Using the equivalent formulations in (2.3.12a) and (2.3.12b), we define K_v-linear operators

$$\Theta^r : K \otimes A_v[[u]] \to K \otimes A_v[[u]], \quad r \geqslant 0.$$

Theorem 2.5.1. *For any weight $s \in \mathbb{S}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, we have for $r \geqslant 0$,*

$$\Theta^r : \mathcal{M}_s^m \to \mathcal{M}_{s+2r}^{m+r}.$$

This can be seen as similar in spirit to the results of Bosser and Pellarin [3, Thm. 2], [4, Thm. 4.1] (see also Theorem 2.3.14), that Θ^r preserves spaces of Drinfeld quasi-modular forms, and our main arguments rely on essentially showing that quasi-modular forms with

K_v-coefficients are v-adic and applying Theorem 2.3.14. Consider first the operator $\Theta = \Theta^1$, which can be equated by (2.1.12) with the operation on u-expansions given by

$$\Theta = u^2 \, \partial_u^1.$$

We recall a formula of Gekeler [9, §8] (take $r = 1$ in (2.3.13)), which states that for $f \in M_k^m$,

$$\Theta(f) = \mathcal{D}^1(f) + kEf,$$

where E is the false Eisenstein series whose u-expansion is given in (2.3.8). Our first goal is to show that E is a v-adic modular form, for which we use results of Goss and Petrov. For k, $n \geqslant 1$ and $s \in \mathbb{S}$, we set

$$f_{k,n} := \sum_{a \in A_+} a^{k-n} G_n(u_a), \quad \hat{f}_{s,n} := \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a^s G_n(u_a). \tag{2.5.2}$$

The notation $f_{k,n}$ and $\hat{f}_{s,n}$ is not completely consistent, since $f_{k,n}$ is more closely related to $\hat{f}_{k-n,n}$ than $\hat{f}_{k,n}$, but this viewpoint is convenient in many contexts (see [16]).

Theorem 2.5.3 (Goss [16, Thm. 2], Petrov [24, Thm. 1.3])**.**

(a) (Petrov) *Let k, $n \geqslant 1$ be chosen so that $k - 2n > 0$, $k \equiv 2n \pmod{q-1}$, and $n \leqslant p^{\mathrm{ord}_p(k-n)}$. Then*

$$f_{k,n} \in M_k^m(K),$$

where $m \equiv n \pmod{q-1}$.

(b) (Goss) *Let $n \geqslant 1$. For $s = (x, y) \in \mathbb{S}$ with $x \equiv n \pmod{q-1}$ and $y \equiv 0 \pmod{q^{\lceil \log_q(n) \rceil}}$, we have*

$$\hat{f}_{s,n} \in \mathcal{M}_{s+n}^m,$$

where $m \equiv n \pmod{q-1}$.

We note that the statement of Theorem 2.5.3(b) is slightly stronger than what is stated in [16], but Goss' proof works here without changes. We then have the following corollary.

Corollary 2.5.4. *For any* $\ell \equiv 0 \pmod{q-1}$, *we have* $\hat{f}_{\ell+1,1} \in \mathcal{M}^1_{\ell+2}$.

If we take $\ell = 0$, we see that

$$\hat{f}_{1,1} = \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_a \in A_v[[u]]$$

is a v-adic modular form in $\mathcal{M}^1_2(A_v)$ and is a partial sum of E in (2.3.8). From this we can prove that E itself is a v-adic modular form.

Theorem 2.5.5. *The false Eisenstein series* E *is a* v*-adic modular form in* $\mathcal{M}^1_2(A_v)$.

Proof. Starting with the expansion in (2.3.8), we see that $E \in A_v[[u]]$. Also,

$$\begin{aligned} E = \sum_{a \in A_+} a u_a &= \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_a + \wp \sum_{a \in A_+} a u_{\wp a} \\ &= \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_a + \wp \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_{\wp a} + \wp^2 \sum_{a \in A_+} a u_{\wp^2 a}, \end{aligned}$$

and continuing in this way, we find

$$E = \sum_{j=0}^{\infty} \left(\wp^j \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_{\wp^j a} \right).$$

We note that

$$\sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_{\wp^j a} = V_\wp^{\circ j} \left(\sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_a \right) = V_\wp^{\circ j}(\hat{f}_{1,1}),$$

where $V_\wp^{\circ j}$ is the j-th iterate $V_\wp \circ \cdots \circ V_\wp$. By Proposition 2.4.12, we see that $V_\wp^{\circ j}(\hat{f}_{1,1}) \in \mathcal{M}_2^1(A_v)$ for all j. Moreover,

$$E = \sum_{j=0}^{\infty} \wp^j V_\wp^{\circ j}(\hat{f}_{1,1}),$$

the right-hand side of which converges with respect to the $\|\cdot\|_v$-norm, and so we are done by Proposition 2.4.8. □

Proof of Theorem 2.5.1. Let $f \in \mathcal{M}_s^m$ and pick $f_i \in M_{k_i}^m(K)$ with $f_i \to f$. It follows from the formulas in Corollary 2.3.12 that $\Theta^r(f_i) \to \Theta^r(f)$ with respect to the $\|\cdot\|_v$-norm for each $r \geqslant 0$, so by Proposition 2.4.8 it remains to show that each

$$\Theta^r(f_i) \in \mathcal{M}_{k_i+2r}^{m+r}.$$

We proceed by induction on r. If $r = 1$, then since $\mathcal{D}^1(f_i) \in M_{k_i+2}^{m+1}(K)$ for each i by Theorem 2.3.14, it follows from Theorem 2.5.5 that

$$\Theta(f_i) = \mathcal{D}^1(f_i) + k_i E f_i \in \mathcal{M}_{k_i+2}^{m+1},$$

for each i. Now by (2.3.13), for each i

$$\Theta^r(f_i) = \mathcal{D}^r(f_i) - \sum_{j=1}^{r} (-1)^j \binom{k_i + r - 1}{j} \Theta^{r-j}(f_i)\Theta^{j-1}(E).$$

By Theorem 2.3.14, $\mathcal{D}^r(f_i) \in M_{k_i+2r}^{m+r}(K)$, and by the induction hypothesis and Theorem 2.5.5 the terms in the sum are in $\mathcal{M}_{k_i+2r}^{m+r}$. □

2.6 Theta operators and v-adic integrality

We see from Theorem 2.5.1 that $\Theta^r : \mathcal{M}_s^m \to \mathcal{M}_{s+2r}^{m+r}$, and it is a natural question to ask whether Θ^r preserves v-integrality, i.e.,

$$\Theta^r : \mathcal{M}_s^m(A_v) \xrightarrow{?} \mathcal{M}_{s+2r}^{m+r}(A_v).$$

However, it is known that this can fail for r sufficiently large because of the denominators in $G_{r+1}(u)$ (e.g., see Vincent [32, Cor. 1]). Nevertheless, in this section we see that Θ^r is not far off from preserving v-integrality.

For an A-algebra R and a sequence $\{b_m\} \subseteq R$, we define an A-Hurwitz series over R (cf. [15, §9.1]) by

$$h(x) = \sum_{m=0}^{\infty} \frac{b_m}{\Pi_m} x^m \in (K \otimes_A R)[[x]], \tag{2.6.1}$$

where we recall the definition of the Carlitz factorial Π_m from (2.1.3). Series of this type were initially studied by Carlitz [6, §3] and further investigated by Goss [11, §3], [15, §9.1]. The particular cases we are interested in are when $R = A$ or $R = A[u]$, but we have the following general proposition whose proof can be easily adapted from [11, §3.2], [15, Prop. 9.1.5].

Proposition 2.6.2. *Let R be an A-algebra, and let $h(x)$ be an A-Hurwitz series over R.*

(a) If the constant term of $h(x)$ is 1, then $1/h(x)$ is also an A-Hurwitz series over R.

(b) If $g(x)$ is an A-Hurwitz series over R with constant term 0, then $h(g(x))$ is also an A-Hurwitz series over R.

We apply this proposition to the generating function of Goss polynomials.

Lemma 2.6.3. *For each $k \geqslant 1$, we have $\Pi_{k-1} G_k(u) \in A[u]$.*

Proof. Consider the generating series

$$\frac{\mathcal{G}(u,x)}{x} = \sum_{k=1}^{\infty} G_k(u) x^{k-1} = \frac{u}{1 - u e_C(x)}.$$

We claim that $\mathcal{G}(u,x)/x$ is an A-Hurwitz series over $A[u]$. Indeed certainly the constant series u itself is one, and

$$1 - u e_C(x) = 1 - \sum_{i=0}^{\infty} \frac{u x^{q^i}}{\Pi_{q^i}}$$

is an A-Hurwitz series over $A[u]$ with constant term 1, so the claim follows from Proposition 2.6.2(a). The result is then immediate. $\square$

Theorem 2.6.4. *For $r \geqslant 0$, if $f \in \mathcal{M}_s^m(A_v)$, then $\Pi_r \Theta^r(f) \in \mathcal{M}_{s+2r}^{m+r}(A_v)$. Thus we have a well-defined operator,*

$$\Pi_r \Theta^r : \mathcal{M}_s^m(A_v) \to \mathcal{M}_{s+2r}^{m+r}(A_v).$$

Proof. By (2.3.12a), we see that the possible denominators of $\Theta^r(f)$ come from the denominators of $G_{r+1}(u)$, which are cleared by Π_r using Lemma 2.6.3. $\square$

Remark 2.6.5. Once we see that $\Pi_r \Theta^r$ preserves v-integrality, the question of whether Π_r is the best possible denominator is important but subtle, and in general the answer is no. For example, taking $r = q^{d+1} - 1$, we see from Theorem 2.2.2(d), that

$$G_{q^{d+1}}(u) = u^{q^{d+1}},$$

and so $\Theta^{q^{d+1}-1} : A_v[[u]] \to A_v[[u]]$ already by (2.3.12a). However, $\Pi_{q^{d+1}-1}$ can be seen to be divisible by $\wp$.

Nevertheless, we do see that Π_r is the best possible denominator in many cases. For example, let $r = q^i$ for $i \geqslant 1$. Then from Theorem 2.2.2(a),

$$G_{q^i+1}(u) = u\left(G_{q^i}(u) + \frac{G_{q^i+1-q}(u)}{D_1} + \cdots + \frac{G_1(u)}{D_i}\right) = u\left(u^{q^i} + \frac{G_{q^i+1-q}(u)}{D_1} + \cdots + \frac{u}{D_i}\right).$$

From Theorem 2.2.2(b) we know that u^2 divides $G_k(u)$ for all $k \geqslant 2$, and so we find that the coefficient of u^2 in $G_{q^i+1}(u)$ is precisely $1/D_i$, which is the same as Π_{q^i}.

3. LIMITS OF BERNOULLI-CARLITZ NUMBERS AND v-ADIC FAMILIES OF EISENSTEIN SERIES

3.1 Bernoulli-Carlitz numbers

We define Bernoulli-Carlitz numbers BC_m using the generating series:

$$\frac{z}{e_C(z)} = \sum_{m=0}^{\infty} \frac{BC_m}{\Pi_m} z^m. \tag{3.1.1}$$

By Carlitz [6] and Goss [11], we have a formula for Bernoulli-Carlitz numbers,

$$BC_m = \Pi_m \sum_{k=1}^{\lfloor \log_q(m+1) \rfloor} \frac{A_m^{(k)}}{L_k}, \tag{3.1.2}$$

where $A_m^{(k)}$ is defined by the series

$$e_C(z)^{q^k-1} = \sum_{m=q^k-1}^{\infty} A_m^{(k)} z^m. \tag{3.1.3}$$

Since we have $e_C(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i}$, expanding (3.1.3), we have the explicit formula for $A_m^{(k)}$.

$$A_m^{(k)} = \sum_{\substack{d_0+d_1+\cdots+d_s=q^k-1 \\ d_0+d_1q+\cdots+d_sq^s=m}} \binom{q^k-1}{\underline{d}} \frac{1}{D_0^{d_0} D_1^{d_1} \cdots D_s^{d_s}}, \tag{3.1.4}$$

where $\underline{d} = (d_0, d_1, \ldots, d_s)$ is an $(s+1)$-tuples of non-negative integers (s arbitrary) and $\binom{q^k-1}{\underline{d}}$ is defined to be $(q^k-1)!/(d_0!d_1!\cdots d_s!)$.

We define

$$J_{m,k} := \Pi_m \frac{1}{L_k} \sum_{\substack{d_0+d_1+\cdots+d_s=q^k-1 \\ d_0+d_1q+\cdots+d_sq^s=m}} \binom{q^k-1}{\underline{d}} \frac{1}{D_0^{d_0} D_1^{d_1} \cdots D_s^{d_s}}.$$

Therefore, the Bernoulli-Carlitz numbers can be written as

$$BC_m = \Pi_m \sum_{k=1}^{\lfloor \log_q(m+1) \rfloor} \frac{1}{L_k} \sum_{\substack{d_0+d_1+\cdots+d_s=q^k-1 \\ d_0+d_1q+\cdots+d_sq^s=m}} \binom{q^k-1}{\underline{d}} \frac{1}{D_0^{d_0} D_1^{d_1} \cdots D_s^{d_s}} \tag{3.1.5}$$

$$= \sum_{k=1}^{\lfloor \log_q(m+1) \rfloor} J_{m,k}. \tag{3.1.6}$$

Lemma 3.1.7. *For any Bernoulli-Carlitz number* $BC_m = \sum_{k=1}^{\lfloor \log_q(m+1) \rfloor} J_{m,k}$*, there is a lower bound of* $\mathrm{ord}_v(J_{m,k})$ *which will increase as* k *increases. More specifically,*

$$\mathrm{ord}_v(J_{m,k}) \geqslant \mathrm{ord}_v(\Pi_m) - \frac{m}{q^d-1} + \frac{q^k-1}{q^d-1} - \left\lfloor \frac{k}{d} \right\rfloor.$$

Proof. By [20, Lemma 2.1.8], we have formulas for the valuations,

$$\mathrm{ord}_v(L_i) = \left\lfloor \frac{i}{d} \right\rfloor,$$
$$\mathrm{ord}_v(D_i) = \frac{q^i - q^{\bar{i}}}{q^d-1},$$

where $\bar{i}$ is a representative of i in $\{0, 1, \ldots, d-1\}$ modulo d. We only need to check the valuation of $\frac{1}{L_k D_0^{d_0} D_1^{d_1} \cdots D_s^{d_s}}$, and we find

$$\mathrm{ord}_v\left(\frac{1}{L_k D_0^{d_0} D_1^{d_1} \cdots D_s^{d_s}}\right) = -\left\lfloor \frac{k}{d} \right\rfloor - \frac{1}{q^d-1}(d_0(q^0 - q^{\bar{0}}) + d_1(q^1 - q^{\bar{1}}) + \cdots$$

$$
\begin{aligned}
+ d_s(q^s - q^{\bar{s}})) &= -\left\lfloor \frac{k}{d} \right\rfloor - \frac{1}{q^d-1}(m - (d_0 q^{\bar{0}} + d_1 q^{\bar{1}} + \cdots + d_s q^{\bar{s}})) \\
&= -\frac{m}{q^d-1} + \frac{1}{q^d-1}(d_0 q^{\bar{0}} + d_1 q^{\bar{1}} + \cdots + d_s q^{\bar{s}}) - \left\lfloor \frac{k}{d} \right\rfloor \\
&\geqslant -\frac{m}{q^d-1} + \frac{1}{q^d-1}(d_0 + d_1 + \cdots + d_s) - \left\lfloor \frac{k}{d} \right\rfloor \\
&= -\frac{m}{q^d-1} + \frac{q^k-1}{q^d-1} - \left\lfloor \frac{k}{d} \right\rfloor .
\end{aligned}
$$

Therefore, if k increases, the valuation will increase. In fact, we have the following inequality.

$$
\operatorname{ord}_v(J_{m,k}) \geqslant \operatorname{ord}_v(\Pi_m) - \frac{m}{q^d-1} + \frac{q^k-1}{q^d-1} - \left\lfloor \frac{k}{d} \right\rfloor .
$$

□

Remark 3.1.8. If we say $m = m_0 + m_1 q + \cdots + m_s q^s$, then the previous lemma yields

$$
\begin{aligned}
\operatorname{ord}_v(J_{m,k}) &\geqslant \operatorname{ord}_v(D_0^{m_0} \cdots D_s^{m_s}) - \frac{m}{q^d-1} + \frac{q^k-1}{q^d-1} - \left\lfloor \frac{k}{d} \right\rfloor \\
&= \frac{1}{q^d-1}(m_0(q^0 - q^{\bar{0}}) + \cdots + m_s(q^s - q^{\bar{s}})) - \frac{m}{q^d-1} + \frac{q^k-1}{q^d-1} - \left\lfloor \frac{k}{d} \right\rfloor \\
&= -\frac{1}{q^d-1}(m_0 q^{\bar{0}} + \cdots + m_s q^{\bar{s}}) + \frac{q^k-1}{q^d-1} - \left\lfloor \frac{k}{d} \right\rfloor \\
&\geqslant \frac{q^k-1}{q^d-1} - \left\lfloor \frac{k}{d} \right\rfloor - \frac{q^d}{q^{d-1}-1}(m_0 + \cdots + m_s).
\end{aligned}
$$

From now on, we let $m = m_j = aq^{dj} + b$, where $d = \deg v$, $a = a_0 + a_1 q + a_2 q^2 + \cdots + a_r q^r \in \mathbb{N}$ and $b = b_0 + b_1 q + b_2 q^2 + \cdots + b_t q^t \in \mathbb{N}$ ($0 \leqslant a_i, b_i < q$ and $a_r, b_t \neq 0$). Therefore, in calculating BC_m using the formulas (3.1.2) and (3.1.4), we can take $s = s_j = r + dj$.

Remark 3.1.9. By the formula (3.1.1), it is easy to see that if $(q-1) \not| m$, then $BC_m = 0$. Therefore if $(q-1) \not| (a+b)$, then $BC_{aq^{dj}+b} = 0$, which is trivial. In this chapter, we will assume that $(q-1) | (a+b)$.

Corollary 3.1.10. *For $m_j = aq^{dj} + b$ as defined above, we have*

$$\operatorname{ord}_v(J_{m_j,k}) \geqslant \frac{q^k - 1}{q^d - 1} - \left\lfloor \frac{k}{d} \right\rfloor - \frac{q^{d-1}(q-1)}{q^d - 1}(r + s + 2),$$

which is independent on j.

Furthermore, we set

$$\mathcal{I}_{j,l} = \left\{ (d_0, \ldots, d_{r+dj}) \in \mathbb{Z}_{\geq 0}^{r+dj+1} \;\middle|\; \begin{array}{l} d_0 + \cdots + d_{r+dj} = l \\ \\ d_0 + d_1 q + \cdots + d_{r+dj} q^{r+dj} = aq^{dj} + b \end{array} \right\}. \tag{3.1.11}$$

Lemma 3.1.12. *For $j > \frac{l}{q-1} + t + 1$, then for any element $(d_0, d_1, \ldots, d_{r+dj})$ in $\mathcal{I}_{j,l}$, we have $d_0 + d_1 q + \cdots + d_t q^t = b_0 + b_1 q + \cdots + b_t q^t$. Also there is a bijection ϕ between $\mathcal{I}_{j,l}$ and $\mathcal{I}_{j+1,l}$:*

$$\phi : \ (d_0, d_1, \ldots, d_{r+dj}) \mapsto (d_0, d_1, \ldots, d_t, \underbrace{0, 0, \ldots, 0}_{d \text{ terms}}, d_{t+1}, d_{t+2}, \ldots, d_{r+dj}). \tag{3.1.13}$$

Proof. First, for any element $(d_0, d_1, \ldots, d_{r+dj})$ in $\mathcal{I}_{j,l}$, we have $d_0 + d_1 q + \cdots + d_{r+dj} q^{r+dj} = aq^{dj} + b = b_0 + b_1 q + \cdots + b_t q^t + a_0 q^{dj} + a_1 q^{dj+1} + \cdots + a_r q^{dj+r}$. For $j > \frac{l}{q-1} + t$, if $d_0 + d_1 q + \cdots + d_t q^t \neq b_0 + b_1 q + \cdots + b_t q^t$, we know that $d_0 + d_1 q + \cdots + d_t q^t > b_0 + b_1 q + \cdots + b_t q^t$. If not, then $0 < b_0 + b_1 q + \cdots + b_t q^t - (d_0 + d_1 q + \cdots + d_t q^t) < q^{t+1}$. However this term can also be expressed as $d_{t+1} q^{t+1} + \cdots + d_{r+dj} q^{r+dj} - (a_0 q^{dj} + \cdots + a_r q^{r+dj})$, which is positive and divisible by q^{t+1}, so $b_0 + b_1 q + \cdots + b_t q^t - (d_0 + d_1 q + \cdots + d_t q^t) = d_{t+1} q^{t+1} + \cdots + d_{r+dj} q^{r+dj} - (a_0 q^{dj} + \cdots + a_r q^{r+dj}) \geqslant q^{t+1}$. Contradiction.

Therefore consider the gap between q^t and q^{dj}, it is not difficult to see that $d_0 + d_1 + \cdots + d_{jd-1} \geqslant b_0 + b_1 + \cdots + b_t + (q-1)(jd - 1 - t) > (q-1)(j - 1 - t) > (q-1)\frac{l}{q-1} = l$, which contradicts the first condition. Therefore, $d_0 + d_1 q + \cdots + d_t q^t = b_0 + b_1 q + \cdots + b_t q^t$.

Moreover, by the same idea, we can easily check that $d_{t+1} = d_{t+2} = \cdots = d_{jd-l/(q-1)-1} = 0$.

We set $\phi: \mathcal{I}_{j,l} \to \mathcal{I}_{j+1,l}$ by

$$\phi: (d_0, d_1, \ldots, d_{r+dj}) \mapsto (d_0, d_1, \ldots, d_t, \underbrace{0, 0, \ldots, 0}_{d \text{ terms}}, d_{t+1}, d_{t+2}, \ldots, d_{r+dj}), \tag{3.1.14}$$

which is clearly an injection. Also by the last sentence of the previous paragraph, we observe that ϕ is surjective. □

Theorem 3.1.15. *The Bernoulli-Carlitz numbers BC_{m_j} have a v-adic limit in K_v as j goes to infinity, where $m_j = a^{dj} + b$ with a and b chosen as above.*

Proof. We only need to prove that $\|BC_{m_j} - BC_{m_{j+1}}\|_v \to 0$. Following by the discussion above, we can compute the v-adic norm directly:

$$\begin{aligned} BC_{m_j} - BC_{m_{j+1}} = \Pi_{m_j} & \sum_{1<q^k \leqslant m_j+1} \frac{1}{L_k} \sum_{\underline{d} \in \mathcal{I}_{j,q^k-1}} \binom{q^k-1}{\underline{d}} \frac{1}{D_0^{d_0} D_1^{d_1} \cdots D_{jd+r}^{d_{jd+r}}} \\ & - \Pi_{m_{j+1}} \sum_{1<q^k \leqslant m_{j+1}+1} \frac{1}{L_k} \sum_{\underline{d'} \in \mathcal{I}_{j+1,q^k-1}} \binom{q^k-1}{\underline{d'}} \frac{1}{D_0^{d'_0} D_1^{d'_1} \cdots D_{(j+1)d+r}^{d'_{(j+1)d+r}}}. \end{aligned}$$

Moreover, for $j > t$, we know that $\Pi_{m_j} = D_0^{b_0} D_1^{b_1} \cdots D_t^{b_t} D_{dj}^{a_0} D_{dj+1}^{a_1} \cdots D_{dj+r}^{a_r}$ and $\Pi_{m_{j+1}} = D_0^{b_0} D_1^{b_1} \cdots D_t^{b_t} D_{d(j+1)}^{a_0} D_{d(j+1)+1}^{a_1} \cdots D_{d(j+1)+r}^{a_r}$ by the definition of Π_m (2.1.3). Notice that the first t terms are the same. Therefore, for any positive integer K (to be specified later), we have the following equation:

$$\begin{aligned} BC_{m_j} - BC_{m_{j+1}} = D_0^{b_0} D_1^{b_1} \cdots D_t^{b_t} \Bigg(& \sum_{q^k \leqslant q^K} \frac{1}{L_k} \sum_{\underline{d} \in \mathcal{I}_{j,q^k-1}} \binom{q^k-1}{\underline{d}} \frac{D_{dj}^{a_0} D_{dj+1}^{a_1} \cdots D_{dj+r}^{a_r}}{D_0^{d_0} D_1^{d_1} \cdots D_{jd+r}^{d_{jd+r}}} \\ & - \sum_{q^k \leqslant q^K} \frac{1}{L_k} \sum_{\underline{d'} \in \mathcal{I}_{j+1,q^k-1}} \binom{q^k-1}{\underline{d'}} \frac{D_{d(j+1)}^{a_0} D_{d(j+1)+1}^{a_1} \cdots D_{d(j+1)+r}^{a_r}}{D_0^{d'_0} D_1^{d'_1} \cdots D_{(j+1)d+r}^{d'_{(j+1)d+r}}} \Bigg) \end{aligned} \tag{3.1.16}$$

$$+\Pi_{m_j}\sum_{q^K<q^k\leqslant m_j+1}\frac{1}{L_k}\sum_{\underline{d}\in\mathcal{I}_{j,q^k-1}}\binom{q^k-1}{\underline{d}}\frac{1}{D_0^{d_0}D_1^{d_1}\cdots D_{jd+r}^{d_{jd+r}}}$$
$$-\Pi_{m_{j+1}}\sum_{q^K<q^k\leqslant m_{j+1}+1}\frac{1}{L_k}\sum_{\underline{d'}\in\mathcal{I}_{j+1,q^k-1}}\binom{q^k-1}{\underline{d'}}\frac{1}{D_0^{d'_0}D_1^{d'_1}\cdots D_{(j+1)d+r}^{d'_{(j+1)d+r}}}.$$

First, by Lemma (3.1.7), we have the following inequality:

$$\begin{aligned}\mathrm{ord}_v(J_{m_j,k}) &\geqslant \mathrm{ord}_v(\Pi_{m_j})-\frac{m_j}{q^d-1}+\frac{q^k-1}{q^d-1}-\left\lfloor\frac{k}{d}\right\rfloor\\
&=\frac{1}{q^d-1}(b_0(q^0-q^{\overline{0}})+b_1(q^1-q^{\overline{1}})+\cdots+b_t(q^t-q)+a_0(q^{dj}-q^{\overline{dj}})\\
&\quad+a_1(q^{dj+1}-q^{\overline{dj+1}})+\cdots+a_r(q^{dj+r}-q^{\overline{dj+r}}))\\
&\quad-\frac{aq^{dj}+b}{q^d-1}+\frac{q^k-1}{q^d-1}-\left\lfloor\frac{k}{d}\right\rfloor\\
&=\frac{1}{q^d-1}(aq^{dj}+b-(b_0q^{\overline{0}}+b_1q^{\overline{1}}+\cdots+b_tq^{\overline{t}}+a_0q^{\overline{dj}}\\
&\quad+a_1q^{\overline{dj+1}}+\cdots+a_sq^{\overline{dj+s}}))-\frac{aq^{dj}+b}{q^d-1}+\frac{q^k-1}{q^d-1}-\left\lfloor\frac{k}{d}\right\rfloor\\
&\geqslant\frac{q^k-1}{q^d-1}-\left\lfloor\frac{k}{d}\right\rfloor-\frac{q^{d-1}}{q^d-1}(b_0+b_1+\cdots+b_t+a_0+a_1+\cdots+a_s).\end{aligned}$$

It is clear that for any $N>0$, we can find a $K\in\mathbb{Z}$ such that for any $k>K$, $\frac{q^k-1}{q^d-1}-\lfloor\frac{k}{d}\rfloor-\frac{q^{d-1}}{q^d-1}(b_0+b_1+\cdots+b_t+a_0+a_1+\cdots+a_s)>N$. This means that for $k>K$,

$$\left\|-\Pi_{m_j}\sum_{q^K<q^k\leqslant m_j+1}\frac{1}{L_k}\sum_{\underline{d}\in\mathcal{I}_{j,q^k-1}}\binom{q^k-1}{\underline{d}}\frac{1}{D_0^{d_0}D_1^{d_1}\cdots D_{jd+r}^{d_{jd+r}}}\right\|_v<q^{-N}. \tag{3.1.17}$$

The same K also works for m_{j+1}, i.e.

$$\left\|-\Pi_{m_{j+1}}\sum_{q^K<q^k\leqslant m_{j+1}+1}\frac{1}{L_k}\sum_{\underline{d'}\in\mathcal{I}_{j+1,q^k-1}}\binom{q^k-1}{\underline{d'}}\frac{1}{D_0^{d_0}D_1^{d_1}\cdots D_{(j+1)d+r}^{d_{(j+1)d+r}}}\right\|_v<q^{-N}. \tag{3.1.18}$$

Now we consider the first term in the right hand side of (3.1.16), which we can compare term by term for a fixed k. For a given $k \leqslant m_j + 1$, compute

$$\sum_{\underline{d} \in \mathcal{I}_{j,q^k-1}} \binom{q^k-1}{\underline{d}} \frac{D_{dj}^{a_0} D_{dj+1}^{a_1} \cdots D_{dj+r}^{a_r}}{D_0^{d_0} D_1^{d_1} \cdots D_{jd+r}^{d_{jd+r}}} - \sum_{\underline{d'} \in \mathcal{I}_{j+1,q^k-1}} \binom{q^k-1}{\underline{d'}} \frac{D_{d(j+1)}^{a_0} D_{d(j+1)+1}^{a_1} \cdots D_{d(j+1)+r}^{a_r}}{D_0^{d'_0} D_1^{d'_1} \cdots D_{(j+1)d+r}^{d'_{(j+1)d+r}}}. \tag{3.1.19}$$

By Lemma (3.1.12), for $j > \frac{q^K-1}{q-1} + t + 1$, there is a bijection between $\mathcal{I}_{j,q^k-1}$ and $\mathcal{I}_{j+1,q^k-1}$ for all $k \leqslant K$. Furthermore, since the map ϕ is just a shifting, it keeps the multinomial $\binom{q^k-1}{\underline{d}}$ unchanged, i.e. $\binom{q^k-1}{\underline{d}} = \binom{q^k-1}{\phi(\underline{d})}$. Also by the proof in Lemma (3.1.12), we know that for $\underline{d} \in \mathcal{I}_{j,q^k-1}$, $d_{t+1} = d_{t+2} = \cdots = d_{jd-h-1} = 0$ and for $\underline{d'} \in \mathcal{I}_{j+1,q^k-1}$, $d'_{t+1} = d'_{t+2} = \cdots = d'_{(j+1)d-h-1} = 0$, where $h := h(k) := (q^k-1)/(q-1)$. That means the number of possible nonzero terms in $\underline{d}$ is same for all j, which allows us to consider only a fixed number of terms in (3.1.19). In fact, we can rewrite (3.1.19) to be

$$\sum_{\underline{d} \in \mathcal{I}_{j,q^k-1}} \binom{q^k-1}{\underline{d}} \frac{1}{D_0^{d_0} D_1^{d_1} \cdots D_t^{d_t}} \left(\frac{D_{dj}^{a_0} D_{dj+1}^{a_1} \cdots D_{dj+r}^{a_r}}{D_{jd-h}^{d_{jd-h}} D_{jd-h+1}^{d_{jd-h+1}} \cdots D_{jd+r}^{d_{jd+r}}} - \frac{D_{d(j+1)}^{a_0} D_{d(j+1)+1}^{a_1} \cdots D_{d(j+1)+r}^{a_r}}{D_{(j+1)d-h}^{d_{jd-h}} D_{(j+1)d-h+1}^{d_{jd-h+1}} \cdots D_{(j+1)d+r}^{d_{jd+r}}} \right). \tag{3.1.20}$$

Note that $d_{jd-h} q^{jd-h} + d_{jd-h+1} q^{jd-h+1} + \cdots + d_{jd+r} q^{jd+r} = a q^{dr}$. Therefore, the question is to prove that the difference inside the parentheses goes to 0 in the v-adic norm as j tends to infinity.

Replacing all D_i by $[i][i-1]^q \cdots [1]^{q^{i-1}}$, we can simplify the first term inside the parentheses as

$$\frac{D_{dj}^{a_0} D_{dj+1}^{a_1} \cdots D_{dj+r}^{a_r}}{D_{jd-h}^{d_{jd-h}} D_{jd-h+1}^{d_{jd-h+1}} \cdots D_{jd+r}^{d_{jd+r}}}$$
$$= \frac{([dj][dj-1]^q \cdots [1]^{q^{dj-1}})^{a_0} \cdots ([dj+r][dj+r-1]^q \cdots [1]^{q^{dj+r-1}})^{a_r}}{([jd-h][jd-h-1]^q \cdots [1]^{q^{jd-h-1}})^{d_{jd-h}} \cdots ([jd+r][jd+r-1]^q \cdots [1]^{q^{jd+r-1}})^{d_{jd+r}}}.$$

Remembering that $d_{dj-h}q^{dj-h} + d_{dj-h+1}q^{dj-h+1} + \cdots + d_{dj+r}q^{dj+r} = aq^{jd} = a_0q^{jd} + a_1q^{jd+1} + \cdots + a_rq^{jd+r}$, we can simplify this term to be the following:

$$\frac{[dj+r]^{a_r}[dj+r-1]^{a_{r-1}+qr} \cdots [dj-h+1]^{a_0q^{h-1}+a_1q^h+\cdots+a_rq^{h+r-1}}}{[dj+r]^{d_{dj+r}}[dj+r-1]^{d_{dj+r-1}+qd_{dj+r}} \cdots [dj-h+1]^{d_{dj-h+1}+d_{dj-h+2}q+\cdots+d_{dj+r}q^{r+h-1}}}.$$

We apply the same manipulations to the second term, and we find that the numerator of

$$\frac{D_{d(j+1)}^{a_0} D_{d(j+1)+1}^{a_1} \cdots D_{d(j+1)+r}^{a_r}}{D_{(j+1)d-h}^{d_{jd-h}} D_{(j+1)d-h+1}^{d_{jd-h+1}} \cdots D_{(j+1)d+r}^{d_{jd+r}}}$$

is

$$[dj+r+d]^{a_r}[dj+r-1+d]^{a_{r-1}+qr} \cdots [dj-h+1+d]^{a_0q^{h-1}+a_1q^h+\cdots+a_rq^{h+r-1}},$$

and the denominator is

$$[dj+r+d]^{d_{dj+r}}[dj+r-1+d]^{d_{dj+r-1}+qd_{dj+r}} \cdots [dj-h+1+d]^{d_{dj-h+1}+d_{dj-h+2}q+\cdots+d_{dj+r}q^{r+h-1}}.$$

Also, notice that $[m+d]-[m] = [d]^{q^m}$ and $\frac{1}{[m+d]} - \frac{1}{[m]} = -\frac{[d]^{q^m}}{[m+d][m]}$, which implies that $\mathrm{ord}_v([m+d]-[m]) = q^m$ and $\mathrm{ord}_v(\frac{1}{[m+d]} - \frac{1}{[m]}) \geqslant q^m - 2$. Thus in (3.1.19), the valuation of the difference inside the parentheses is at least $q^{dj-h+1} - 2$. Therefore, as j goes to infinity, (3.1.19) tends to 0 v-adically for any $k \leqslant K$. Together with (3.1.17) and (3.1.18), we have proved the theorem. $\square$

Remark 3.1.21. In Goss [11, Theorem 3.3.1], we have $\wp BC_{m_j} \in A_v$. Thus $\wp \lim_{j\to\infty} BC_{m_j} \in A_v$.

Recall the definition and properties of Goss polynomials G_{r+1} and coefficients $\beta_{r,j}$ (Theorem 2.2.4) in the last chapter. Now comparing $\beta_{r,j}$ and $A_m^{(k)}$, we notice that $A_m^{(k)} = \beta_{m,q^k-1}$.

Remark 3.1.22. Actually, in the proof of the Theorem 3.1.15, especially (3.1.20), we have proved that $\Pi_{m_j}\beta_{m_j,q^k-1}$ has a v-adic limit in K_v. Moreover, it is easy to prove that $\Pi_{m_j}\beta_{m_j,l}$ has a v-adic limit in K_v for any positive integer l.

Remark 3.1.23. Recall that the topology in $A_v[[u]]$ is defined in Section 2.4: $\|f\|_v := q^{-\operatorname{ord}_v(f)}$ and $\operatorname{ord}_v(f) := \inf_n\{\operatorname{ord}_v(c_n)\}$ for any $f = \sum_{n=0}^{\infty} c_n u^n \in A_v[[u]]$.

Proposition 3.1.24. *The space $A_v[[u]]$ with the v-adic topology is complete.*

Proof. Assume that the sequence $f_j(u) = \sum_{i\geqslant 0} a_i^{(j)} u^i \in A_v[[u]]$ is a Cauchy sequence with the v-adic topology. By the definition of the v-adic norm, we have for any $\epsilon > 0$, there exists $N > 0$, for any $j, j' \leqslant N$, s.t.

$$\sup_{i\geqslant 0}\|a_i^{(j)} - a_i^{(j')}\|_v < \epsilon. \tag{3.1.25}$$

Thus, for all i, we know the sequence $\{a_i^{(j)}\}_{j=1}^{\infty}$ is a Cauchy sequence in A_v. Since A_v is complete, there exists an $a_i \in A_v$ to be the limit of $a_i^{(j)}$. Define

$$f(u) := \sum_{i\geqslant 0} a_i u^i \in A_v[[u]].$$

We will show that f is the limit of f_j in the v-adic topology.

For any $\epsilon > 0$, we let $0 < \epsilon' < \epsilon$, then $\exists N > 0$ s.t. $\|a_i^{(j)} - a_i^{(j')}\|_v < \epsilon'$ for all i and $j, j' \leqslant N$. Thus, we have $\|a_i^{(j)} - a_i^{(N)}\|_v < \epsilon'$ for all i and $j > N$. Moreover, we have

$$\|a_i - a_i^{(N)}\|_v = \lim_{j\to\infty} \|a_i^{(j)} - a_i^{(N)}\|_v \leqslant \epsilon',$$

for all i.

Therefore, we have

$$\begin{aligned}\|a_i - a_i^{(j)}\|_v &= \|(a_i - a_i^{(N)}) + (a_i^{(N)} - a_i^{(j)})\|_v \\ &\leqslant \max\{\|a_i - a_i^{(N)}\|_v, \|a_i^{(N)} - a_i^{(j)}\|_v\} \leqslant \epsilon' < \epsilon,\end{aligned}$$

for any i and $j > N$. That is $\sup_{i\geqslant 0} \|a_i^{(j)} - a_i\|_v < \epsilon$, i.e. $\lim_{j\to\infty} \|f_j - f\|_v = 0$.

□

Noticing $\Pi_m G_{m+1}(u) \in A[u]$ by Lemma 2.6.3. Together with similar method above, we have the following theorem.

Theorem 3.1.26. *Goss polynomials $\Pi_{m_j} G_{m_j+1}(u)$ have a v-adic limit in $A_v[[u]]$ as j goes to infinity, for any m_j has the form $aq^{dj} + b$, with a and b chosen as above.*

Remark 3.1.27. The summation for $\beta_{m_j,l}$ (defined in Theorem 2.2.4) runs through $\mathcal{I}_{j,l}$.

Proof. We have the formula

$$\Pi_{m_j} G_{m_j+1}(u) = \sum_{l=0}^{m_j} \sum_{\substack{i_0+i_1+\cdots+i_{r+dj}=l \\ i_0+i_1q+\cdots+i_{r+dj}q^{r+dj}=m_j}} \binom{l}{\underline{i}} \frac{\Pi_{m_j}}{D_0^{i_0}\cdots D_{r+dj}^{i_{r+dj}}} u^{l+1}. \tag{3.1.28}$$

Using the same method in Theorem (3.1.15), for fixed l, we can show that $\mathrm{ord}_v(\Pi_{m_{j+1}} \beta_{m_{j+1},l} - \Pi_{m_j}\beta_{m_j,l}) \geqslant q^{dj-h+1} - 2 - t\frac{q^t-1}{q^d-1}$, where $h := h(l) := \frac{l}{q-1}$ which appears in (3.1.20). Therefore, each coefficient in $\Pi_{m_j} G_{m_j+1}$ has a limit c_l in A_v. Hence, we now only need to prove that $\Pi_{m_j} G_{m_j+1}$ is a Cauchy sequence in $A_v[[u]]$. We will use the same strategy as in Theorem (3.1.15):

$$\begin{aligned}
\operatorname{ord}_v(\Pi_{m_{j+1}} G_{m_{j+1}+1} - \Pi_{m_j} G_{m_j+1}) &= \operatorname{ord}_v\Big(\sum_{K<l\leqslant m_{j+1}} \Pi_{m_{j+1}}\beta_{m_{j+1},l}u^{l+1} \\
&\quad + \sum_{0\leqslant l\leqslant K}(\Pi_{m_{j+1}}\beta_{m_{j+1},l} - \Pi_{m_j}\beta_{m_j,l})u^{l+1} - \sum_{K<l\leqslant m_j}\Pi_{m_j}\beta_{m_j,l}u^{l+1}\Big) \\
&\geqslant \min\Big\{\min_{K<l\leqslant m_{j+1}}\{\operatorname{ord}_v(\Pi_{m_{j+1}}\beta_{m_{j+1},l}\}), \min_{0\leqslant l\leqslant K}\{\operatorname{ord}_v(\Pi_{m_{j+1}}\beta_{m_{j+1},l} - \Pi_{m_j}\beta_{m_j,l})\}, \\
&\quad \min_{K<l\leqslant m_j}\{\operatorname{ord}_v(\Pi_{m_j}\beta_{m_j,l}\})\Big\},
\end{aligned}$$

where $K < m_j$ is a positive integer to be specified later.

Using same argument in Lemma (3.1.7), we know that the lower bound of $\operatorname{ord}_v(\Pi_{m_j}\beta_{m_j,l})$ increases with l. To be precise, we can get

$$\operatorname{ord}_v(\Pi_m\beta_{m,l}) \geqslant \operatorname{ord}_v(\Pi_m) - \frac{m}{q^d-1} + \frac{l}{q^d-1}. \tag{3.1.29}$$

Thus, we have

$$\min_{K<l\leqslant m_{j+1}}\{\operatorname{ord}_v(\Pi_{m_{j+1}}\beta_{m_{j+1},l})\} \geqslant \operatorname{ord}_v(\Pi_{m_j}) - \frac{m_{j+1}}{q^d-1} + \frac{K+1}{q^d-1} > \frac{K-(a+b)q^{d-1}}{q^d-1},$$

where the last inequality comes from the same idea in Corollary (3.1.10). At the same time, we have

$$\min_{K<l\leqslant m_{j+1}}\{\operatorname{ord}_v(\Pi_{m_{j+1}}\beta_{m_{j+1},l})\} > \frac{K-(a+b)q^{d-1}}{q^d-1},$$

Therefore, we get

$$\mathrm{ord}_v(\Pi_{m_{j+1}} G_{m_{j+1}+1} - \Pi_{m_j} G_{m_j+1}) \geqslant \min\left\{\frac{K-(a+b)q^{d-1}}{q^d-1},\ q^{dj-h(K)+1}-2-t\frac{q^t-1}{q^d-1}\right\}.$$

For any $M > 0$, we can find K_0 such that $\frac{K_0-(a+b)q^{d-1}}{q^d-1} > M$ and find $J > 0$ such that $q^{dJ-h(K_0)+1} - 2 - t\frac{q^t-1}{q^d-1} > M$. Therefore, for all $j, j' > J$, we have $\mathrm{ord}_v(\Pi_{m_{j'}} G_{m_{j'}+1} - \Pi_{m_j} G_{m_j+1}) > M$, which proves the sequence $\Pi_{m_j} G_{m_j+1}$ is Cauchy. By Proposition (3.1.24), we know $\lim\limits_{j\to\infty} \Pi_{m_j} G_{m_j+1} = \sum\limits_{l\geqslant 0} c_l u^{l+1} \in A_v[[u]]$.

$\square$

Remark 3.1.30. By Gekeler [10, Thm. 6.12], for $q = p$ a prime, we know that the multiplicity of 0 as a zero of $G_{m_j}(u)$, which now is considered as a polynomial in u, will stay stable if j is larger enough. Also by computing of the valuation of the lowest degree term, the valuation will stay stable too. Therefore, the limit of Goss polynomials $\Pi_{m_j} G_{m_j}$ is not trivial.

Corollary 3.1.31. *For $m_j = aq^{dj} + b$ defined as above and $f \in \mathcal{M}_s^m(A_v)$, we have*

$$\lim_{j\to\infty} \Pi_{m_j} \Theta^{m_j}(f) \in \mathcal{M}_{s+2b}^{m+b}(A_v).$$

Proof. Combine the Theorem (2.6.4), Corollary (2.3.12b) and the Theorem (3.1.26). $\square$

Remark 3.1.32. Although $m_j \to b$ as j goes to infinity in the p-adic topology and the Bernoulli-Carlitz numbers BC_{m_j} have a v-adic limit in K_v, the situation is different from the case of Kummer's congruences in the classic p-adic case. However, $\lim\limits_{j\to\infty} BC_{m_j} \neq BC_b$ even if $m_j \to b$ q-adically.

By Gekeler [9, (6.3)], we have the following formula for Eisenstein series:

$$E_{m+1} = -\frac{\zeta_C(m+1)}{\widetilde{\pi}^{m+1}} - \sum_{\mathfrak{a}\in A_+} G_{m+1}(u_{\mathfrak{a}}) = -\frac{BC_{m+1}}{\Pi_{m+1}} - \sum_{\mathfrak{a}\in A_+} G_{m+1}(u_{\mathfrak{a}}). \tag{3.1.33}$$

To avoid any confussion, we change the notation a little bit. In this chapter, we use $\mathfrak{a}$ to represent a polynomial in A. Therefore, multiplying by Π_m, we obtain

$$\Pi_m E_{m+1} = -\frac{\Pi_m}{\Pi_{m+1}} BC_{m+1} - \Pi_m \sum_{\mathfrak{a}\in A_+} G_{m+1}(u_{\mathfrak{a}}). \tag{3.1.34}$$

Notice that $\mathrm{ord}_v(\frac{\Pi_{m_j}}{\Pi_{m_j+1}}) > -\mathrm{ord}_v(D_{t+1}) \geqslant -\frac{q^{t+1}-1}{q-1}$, which is a fixed number. Together with Theorem (3.1.15) and (3.1.26), we can then prove the following theorem.

Theorem 3.1.35. *The Eisenstein series $\Pi_{m_j} E_{m_j+1}(u)$ have a v-adic limit in $K \otimes_A A_v[[u]]$ as j goes to infinity, for any m_j has the form $aq^{dj} + b$, with a and b chosen as above.*

Proof. At first, we need to prove that $\Pi_{m_j} G_{m_j+1}(u_{\mathfrak{a}})$ has a limit in $A_v[[u]]$. Recall that $u^{q^{\deg \mathfrak{a}}} + \cdots \in A[[u]]$ as defined in the Equation (2.3.4). We may assume that $u_{\mathfrak{a}} = \sum_{i \geqslant q^{\deg \mathfrak{a}}} \gamma_i u^i$ with $\gamma_i \in A$. Therefore, we have

$$\mathrm{ord}_v(\gamma_i) \geqslant 0,$$

for all i, i.e.

$$\|\gamma_i\|_v \leqslant 1, \tag{3.1.36}$$

for all i. Note that $\gamma_{q^{\deg \mathfrak{a}}} = 1$.

By Theorem (3.1.26), we have $\Pi_{m_j} G_{m_j+1}(u) = \sum_{l\geqslant 0} \Pi_{m_j} \beta_{m_j,l} u^{l+1}$ is convergent to $f := \sum_{l \geqslant 0} c_l u^{l+1}$ in $A_v[[u]]$, which tells us that for any $\epsilon > 0$, there exists $J > 0$, such that

$$\|\Pi_{m_j} \beta_{m_j,l} - c_l\|_v < \epsilon, \tag{3.1.37}$$

for all $l \geqslant 0$, $j \geqslant J$.

Moreover, by the proof of Lemma (3.1.7), there exists $L > 0$, such that

$$\|c_l\|_v < \epsilon, \tag{3.1.38}$$

for all $l \geqslant L$.

Consider

$$\begin{aligned}
\Pi_{m_j} G_{m_j+1}(u_{\mathfrak{a}}) &= \sum_{l=0}^{m_j} \Pi_{m_j} \beta_{m_j,l} u_{\mathfrak{a}}^{l+1} \\
&= \sum_{l=0}^{m_j} \Pi_{m_j} \beta_{m_j,l} \Bigg(\sum_{i \geqslant q^{\deg \mathfrak{a}}} \gamma_i u^i \Bigg)^{l+1} \\
&= \sum_{l=0}^{m_j} \Pi_{m_j} \beta_{m_j,l} \sum_{n \geqslant (l+1) q^{\deg \mathfrak{a}}} \Bigg(\sum_{\substack{i_1, i_2, \cdots, i_{l+1} \geqslant q^{\deg \mathfrak{a}} \\ i_1 + i_2 + \cdots + i_{l+1} = n}} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{l+1}} \Bigg) u^n \\
&= \sum_{n \geqslant q^{\deg \mathfrak{a}}} \Bigg(\sum_{l=0}^{l_{j,n}} \Pi_{m_j} \beta_{m_j,l} \sum_{\substack{i_1, i_2, \cdots, i_{l+1} \geqslant q^{\deg \mathfrak{a}} \\ i_1 + i_2 + \cdots + i_{l+1} = n}} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{l+1}} \Bigg) u^n,
\end{aligned}$$

where $l_{j,n} := \min\{m_j, \lfloor n/q^{\deg \mathfrak{a}} \rfloor - 1\}$. If $l_{j,n} < 0$, then the coefficient of u^n will be 0. For convenience, we set

$$\gamma_{\mathfrak{a},l,n} := \sum_{\substack{i_1, i_2, \cdots, i_{l+1} \geqslant q^{\deg \mathfrak{a}} \\ i_1 + i_2 + \cdots + i_{l+1} = n}} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{l+1}}.$$

Since the v-adic norm is non-Archimedean, then $\|\gamma_{\mathfrak{a},l,n}\|_v \leqslant 1$. Notice that the coefficients of u^n is a finite summation, since l has at most $\lfloor n/q^{\deg \mathfrak{a}} \rfloor < n$ choices. Thus, for each n, we have

$$\lim_{j \to \infty} \sum_{l=0}^{l_{j,n}} \Pi_{m_j} \beta_{m_j,l} \gamma_{\mathfrak{a},l,n} = \sum_{l=0}^{\lfloor n/q^{\deg a} \rfloor - 1} c_l \gamma_{\mathfrak{a},l,n}.$$

For any n, let us compute the difference:

$$\begin{aligned}&\sum_{l=0}^{l_{j,n}} \Pi_{m_j}\beta_{m_j,l}\gamma_{\mathfrak{a},l,n} - \sum_{l=0}^{\lfloor n/q^{\deg \mathfrak{a}}\rfloor - 1} c_l\gamma_{\mathfrak{a},l,n} \\ &= \sum_{l=0}^{l_{j,n}} \left(\Pi_{m_j}\beta_{m_j,l} - c_l\right)\gamma_{\mathfrak{a},l,n} - \sum_{l=l_{j,n}+1}^{\lfloor n/q^{\deg \mathfrak{a}}\rfloor - 1} c_l\gamma_{\mathfrak{a},l,n}. \end{aligned} \tag{3.1.39}$$

Notice that the second summation will be 0 if $m_j \geqslant \lfloor n/q^{\deg \mathfrak{a}}\rfloor - 1$. Therefore, we get

$$\begin{aligned}&\left\|\sum_{l=0}^{l_{j,n}} \Pi_{m_j}\beta_{m_j,l}\gamma_{\mathfrak{a},l,n} - \sum_{l=0}^{\lfloor n/q^{\deg \mathfrak{a}}\rfloor - 1} c_l\gamma_{\mathfrak{a},l,n}\right\|_v \\ &\leqslant \max\left\{\max_{0\leqslant l\leqslant l_{j,n}}\{\|(\Pi_{m_j}\beta_{m_j,l} - c_l)\gamma_{\mathfrak{a},l,n}\|_v\},\ \max_{l_{j,n}+1\leqslant l\leqslant \lfloor n/q^{\deg \mathfrak{a}}\rfloor - 1}\{\|c_l\gamma_{\mathfrak{a},l,n}\|_v\}\right\} \\ &\leqslant \max\left\{\max_{0\leqslant l\leqslant l_{j,n}}\{\|\Pi_{m_j}\beta_{m_j,l} - c_l\|_v\},\ \max_{l\geqslant l_{j,n}+1}\{\|c_l\|_v\}\right\}.\end{aligned}$$

For any $\epsilon > 0$, there exist $J' > 0$ satisfying Equations (3.1.37) for all $l \geqslant 0$, $j \geqslant J'$. Now find $J'' > 0$ and $N > 0$, such that $l_{j,n} \geqslant l_{J'',N} > L$, which satisfies Equation (3.1.38). Set $J = \max\{J',\ J''\}$, then it is easy to see that $\max\limits_{0\leqslant l\leqslant l_{j,n}}\{\|\Pi_{m_j}\beta_{m_j,l} - c_l\|_v\} < \epsilon$ and $\max\limits_{l\geqslant l_{j,n}+1}\{\|c_l\|_v\} < \epsilon$ for all $j \geqslant J$ and $n \geqslant N$.

Now for $n < N$, we can set $\hat{J} = \max\{J, \log_q(N)\}$, then

$$\begin{aligned}\left\|\sum_{l=0}^{l_{j,n}} \Pi_{m_j}\beta_{m_j,l}\gamma_{\mathfrak{a},l,n} - \sum_{l=0}^{\lfloor n/q^{\deg \mathfrak{a}}\rfloor - 1} c_l\gamma_{\mathfrak{a},l,n}\right\|_v &= \left\|\sum_{l=0}^{l_{j,n}} \left(\Pi_{m_j}\beta_{m_j,l}\gamma_{\mathfrak{a},l,n} - c_l\gamma_{\mathfrak{a},l,n}\right)\right\|_v \\ &= \left\|\sum_{l=0}^{l_{j,n}} \left(\Pi_{m_j}\beta_{m_j,l} - c_l\right)\gamma_{\mathfrak{a},l,n}\right\|_v \\ &\leqslant \left\|\sum_{l=0}^{l_{j,n}} \left(\Pi_{m_j}\beta_{m_j,l} - c_l\right)\right\|_v < \epsilon,\end{aligned}$$

for any $j \geqslant \hat{J}$. Note that since $j > \log_q(N)$, the second summation in Equation (3.1.39) is just 0 for all $n < N$.

Therefore, we have

$$\left\| \sum_{l=0}^{l_{j,n}} \Pi_{m_j} \beta_{m_j,l} \gamma_{\mathfrak{a},l,n} - \sum_{l=0}^{\lfloor n/q^{\deg \mathfrak{a}} \rfloor - 1} c_l \gamma_{\mathfrak{a},l,n} \right\|_v < \epsilon,$$

for any $j \geqslant \hat{J}$, which is independent on n and $\mathfrak{a}$. Therefore, $\Pi_{m_j} G_{m_j+1}(u_{\mathfrak{a}})$ is a Cauchy sequence in $A_v[[u]]$. Then by Proposition (3.1.24), it has a limit in $A_v[[u]]$, say the limit is $\sum_{l \geqslant q^{\deg \mathfrak{a}} - 1} c_{\mathfrak{a},l} u^{l+1} \in A_v[[u]]$. That is, for any $\mathfrak{a} \in A_+$, $\sum_{l=0}^{l_{j,n}} \Pi_{m_j} \beta_{m_j,l} \gamma_{\mathfrak{a},l,n}$ tends to $c_{\mathfrak{a},n-1}$ uniformly on n as well as $\mathfrak{a}$.

As for $\sum_{\mathfrak{a} \in A_+} \Pi_{m_j} G_{m_j+1}(u_{\mathfrak{a}})$, we get

$$\begin{aligned} \sum_{\mathfrak{a} \in A_+} \Pi_{m_j} G_{m_j+1}(u_{\mathfrak{a}}) &= \sum_{\mathfrak{a} \in A_+} \sum_{n \geqslant q^{\deg \mathfrak{a}}} \left(\sum_{l=0}^{l_{j,n}} \Pi_{m_j} \beta_{m_j,l} \gamma_{\mathfrak{a},l,n} \right) u^n \\ &= \sum_{n \geqslant 1} \left(\sum_{\substack{\mathfrak{a} \in A_+ \\ \deg \mathfrak{a} \leqslant \log_q(n)}} \sum_{l=0}^{l_{j,n}} \Pi_{m_j} \beta_{m_j,l} \gamma_{\mathfrak{a},l,n} \right) u^n. \end{aligned}$$

Comparing the coefficients and using the same method as for $\Pi_{m_j} G_{m_j+1}(u_{\mathfrak{a}})$, we can easily show that $\sum_{\mathfrak{a} \in A_+} \Pi_{m_j} G_{m_j+1}(u_{\mathfrak{a}}) \to \sum_{n \geqslant 1} \sum_{\substack{\mathfrak{a} \in A_+ \\ \deg \mathfrak{a} \leqslant \log_q(n)}} c_{\mathfrak{a},n-1} u^n$ with respect to the v-adic norm.

Now consider the constant term $\frac{\Pi_{m_j}}{\Pi_{m_j+1}} BC_{m_j+1}$. We have

$$\frac{\Pi_{m_j}}{\Pi_{m_j+1}} = \frac{\Pi_b}{\Pi_{b+1}},$$

for $dj > \log_q(b)$. Together with Theorem (3.1.15), we know $\frac{\Pi_{m_j}}{\Pi_{m_j+1}} BC_{m_j+1}$ has a limit in K_v.

Now we can compute the v-adic limit for the Eisenstein series:

$$\begin{aligned}
\Pi_{m_j} E_{m_j+1} &= -\frac{\Pi_{m_j}}{\Pi_{m_j+1}} BC_{m_j+1} - \sum_{\mathfrak{a} \in A_+} G_{m_j+1}(u_{\mathfrak{a}}) \\
&\to -\frac{\Pi_b}{\Pi_{b+1}} BC_{m_j+1} - \sum_{\mathfrak{a} \in A_+} \sum_{l \geqslant q^{\deg \mathfrak{a}} - 1} c_{\mathfrak{a},l} u^{l+1} \\
&= -\frac{\Pi_b}{\Pi_{b+1}} BC_{m_j+1} - \sum_{l \geqslant 0} \Bigg(\sum_{\substack{\mathfrak{a} \in A_+ \\ \deg \mathfrak{a} \leqslant \log_q(l)}} c_{\mathfrak{a},l} \Bigg) u^{l+1}. \qquad (3.1.40)
\end{aligned}$$

Remark 3.1.41. By Remark (3.1.30) and the computation above (3.1.40), we know that the v-adic limit will not be zero for $\Pi_{m_j} G_{m_j+1}(u_{\mathfrak{a}})$ as well as $\Pi_{m_j} E_{m_j+1}$. Moreover, next section will tell us the constant term $-\frac{\Pi_b}{\Pi_{b+1}} BC_{m_j+1}$ is not trivial.

□

3.2 Limits of Bernoulli-Carlitz numbers

Proposition 3.2.1. *For any prime polynomial $\wp \in A_+$ of degree d, $\lim_{j \to \infty} \theta^{q^{dj}}$ exists in A_v. Moreover, the limit is in an algebraic extension of $\mathbb{F}_q$ of degree d.*

Proof. We set Teichmüller character $w : A_v \to A_v$ by $w(\alpha) := \lim_{j \to \infty} \alpha^{q^{dj}}$. Therefore, when α has v-adic expansion $\alpha = a_0 + a_1 \wp + \cdots$, then $w(\alpha) = \lim_{j \to \infty} a_0^{q^{dj}}$. Moreover, $\mathbb{F}_v := Im(w)$ is isomorphic to $A/\wp$ as a field with order q^d by $\psi(\bar{a}) = \lim_{j \to \infty} a^{q^{dj}}$. Now we have the following commutative diagram,

$$\begin{tikzcd}
A_v \arrow[r, "\omega"] \arrow[dr] & \mathbb{F}_v \\
 & A/\wp \arrow[u, "\psi"]
\end{tikzcd}$$

where the map from A_v to $A/\wp$ is the canonical projective which maps θ to $\overline{\theta}$. Therefore, notice that $\mathfrak{p}$ is a polynomial with coefficients in $\mathbb{F}_q$, we have

$$\wp(w(\theta)) = \wp(\psi(\overline{\theta})) = \psi(\wp(\overline{\theta})) = \psi(\overline{\wp(\theta)}) = \psi(\overline{0}) = 0, \tag{3.2.2}$$

i.e. $w(\theta) = \lim_{j\to\infty} \theta^{q^{dj}}$ is a root of a prime polynomial $\wp$ of degree d. $\square$

From now on, we set $\alpha := \lim_{j\to\infty} \theta^{q^{dj}} \in A_v$.

Remark 3.2.3. $Gal(K(\alpha)/K)$ is a cyclic group with the Frobenius map $\sigma(\alpha) = \alpha^q$.

Corollary 3.2.4. $\lim_{j\to\infty}[dj+n] = -\theta + \alpha^{q^n}$ *for all* $n \in \mathbb{Z}$.

Proof. For j large enough, we have $dj + n > 0$, then $\lim_{j\to\infty}[dj+n] = \lim_{j\to\infty} \theta^{q^{dj+n}} - \theta = (\lim_{j\to\infty} \theta^{q^{dj}})^{q^n} - \theta = -\theta + \alpha^{q^n}$. $\square$

Remark 3.2.5. We can assume that n is a non-negative integer less than d in the corollary above since that $\lim_{j\to\infty}[dj+n] = \lim_{j\to\infty}[d(j+1)+n] = \lim_{j\to\infty}[dj+(n+d)]$.

We set $\alpha_n := \lim_{j\to\infty}[dj+n]$. Actually, $\alpha^{q^d} = \alpha$, $\alpha_n = -\theta + \alpha^{q^n}$, $\sigma(\alpha_n) = \alpha_{n+1}$ and $\alpha_m = \alpha_n$ for $m \equiv n \pmod d$. Also, we fix the choice of (q^d-1)-th root of $\alpha_1, \alpha_2, \ldots, \alpha_d$, and for $i \in \mathbb{Z}$, we set

$$\mathcal{A}_i := (\alpha_{d-i}^{q^{d-1}} \alpha_{d-i+1}^{q^{d-2}} \cdots \alpha_{2d-i-1})^{\frac{1}{q^d-1}} \tag{3.2.6}$$

and

$$F_i(z) := \sum_{j\geqslant 0} \frac{z^{q^{dj+i}}}{L_{dj+i}}. \tag{3.2.7}$$

Usually we may assume that $0 \leqslant i \leqslant d-1$, since $\alpha_m = \alpha_n$ for $m \equiv n \pmod d$. In other words, $\mathcal{A}_m = \mathcal{A}_n$ and $F_m(z) = F_n(z)$ for $m \equiv n \pmod d$. By these notations, we have $\log_C(z) = \sum_{0\leqslant i\leqslant d-1} F_i(z)$ and $\mathcal{A}_i^q = \alpha_{d-i}\mathcal{A}_{i-1}$.

Proposition 3.2.8. *In the v-adic topology,* $\log_C(\sum_{0\leqslant n\leqslant d-1} \mathcal{A}_n) = 0$.

Proof. At first, $\mathrm{ord}_v(\mathcal{A}_n) \geqslant \frac{1}{q^d-1} > 0$ for all n, so the series converges as well as $F_i(\mathcal{A}_n)$ for all i and n. It is a standard fact that $\theta \log_C(z) - \log_C(\theta z) = \log_C(z^q)$ (see [15, §3.4]). Therefore, we have the following equality,

$$\theta F_i(z) - F_i(\theta z) = F_{i-1}(z^q).$$

Thus we can compute

$$\theta(\sum_{i=0}^{d-1} F_i(\mathcal{A}_i)) = \sum_{i=0}^{d-1} F_i(\theta \mathcal{A}_i) + F_{i-1}(\mathcal{A}_i^q).$$

Noting that $\mathcal{A}_i^q = \alpha_{d-i}\mathcal{A}_{i-1} = -\theta\mathcal{A}_{i-1} + \alpha^{q^{d-i}}\mathcal{A}_{i-1}$ and $F_i(z)$ is linear over $\mathbb{F}_q$, we can simplify the equation as the following,

$$\theta(\sum_{i=0}^{d-1} F_i(\mathcal{A}_i)) = \sum_{i=0}^{d-1} F_i(\alpha^{q^{d-1-i}}\mathcal{A}_i) = \alpha^{q^{d-1}}(\sum_{i=0}^{d-1} F_i(\mathcal{A}_i)).$$

Then $\sum_{i=0}^{d-1} F_i(\mathcal{A}_i) = 0$, since $\theta \neq \alpha^{q^{d-1}}$. Similarly, one can show that $\sum_{i=0}^{d-1} F_i(\mathcal{A}_{i+n}) = 0$ for any integer n. Note that $\mathcal{A}_m = \mathcal{A}_n$ for $m \equiv n \pmod{d}$. We can sum these equation up for n from 0 to $d-1$, which is

$$0 = \sum_{n=0}^{d-1}\sum_{i=0}^{d-1} F_i(\mathcal{A}_{i+n}) = \sum_{n=0}^{d-1}\sum_{i=0}^{d-1} F_i(\mathcal{A}_n) = \log_C(\sum_{n=0}^{d-1} \mathcal{A}_n). \tag{3.2.9}$$

□

Remark 3.2.10. Since we have $\mathcal{A}_i^q = \alpha_{d-i}\mathcal{A}_{i-1}$, if we fix a root of $\mathcal{A}_i^{\frac{1}{q^d-1}}$, then all root of $\mathcal{A}_n^{\frac{1}{q^d-1}}$ will be fixed. In other words, if $\zeta \in \mathbb{F}_{q^d}^\times$ is a (q^d-1)-th root of unity, then $\log_C(\sum_{n=0}^{d-1} \zeta^{lq^{nd}}\mathcal{A}_{d-n-1}) = 0$ for any integer l.

Remark 3.2.11. By [33, Section. 3], $\sum_{n=0}^{d-1} \zeta^{lq^{nd}} \mathcal{A}_{d-n-1}$ are Carlitz $\wp$-torsion points, i.e. $C_\wp(\sum_{n=0}^{d-1} \zeta^{lq^{nd}} \mathcal{A}_{d-n-1}) = 0$, this fact can also be derived directly. Together with the trivial zero 0, we have q^d different zeros for $C_\wp$. Note that the degree of $C_\wp$ is q^d, which means that we have found all the zeros.

3.3 Some examples for the limits of Bernoulli-Carlitz numbers

Unless otherwise stated, all limits in this section are taken with respect to the v-adic topology. Also all hyperdifferential operators ∂^m are with respect to x.

First, let us consider a simplest case: the prime polynomial $\wp$ is linear, say $\wp = \theta + a$ with $a \in \mathbb{F}_q$. In this case, we have $\alpha = \lim_{j\to\infty} \theta^{q^{dj}} = \lim_{j\to\infty} (\theta + a)^{q^{dj}} - a^{q^{dj}} = -a$. We can also rewrite $\lim_{j\to\infty} [j]$ to be $\mathcal{A}_0 = -\theta - a$, which means that the limits of all bracket polynomials are the same. Before we start to compute the limits, there is a useful lemma.

Lemma 3.3.1. *Let $g(x) = x + x^{q^d} + x^{q^{2d}} + \cdots \in \mathbb{F}_q[[x]]$ and $j > n$. For $0 \leqslant m < q^d - 1$ we have*

$$\partial^{q^{dj}}(g^{m+n(q^d-1)})|_{x=0} = \begin{cases} 1 & m = 1, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. If $m = 1$, the statement is clearly true for $n = 0, 1$. Now we consider $F(x) := g^{1+n(q^d-1)} - g^{q^d+n(q^d-1)} = xg^{n(q^d-1)}$. Using induction, we have the following calculation,

$$\partial^{q^{dj}}(g^{1+n(q^d-1)})|_{x=0} - \partial^{q^{dj}}(g^{q^d+n(q^d-1)})|_{x=0} = \partial^{q^{dj}}(F)|_{x=0} = \partial^{q^{dj}-1}(g^{n(q^d-1)})|_{x=0}.$$

Therefore, we have

$$\partial^{q^{dj}}(F)|_{x=0} = \partial^{q^{dj}-1}(g^{n(q^d-1)})|_{x=0},$$

which is the coefficient of $x^{q^{dj}-1}$ in $g^{n(q^d-1)}$. That is, the summation $\sum_{\underline{i}} \binom{n(q^d-1)}{\underline{i}}$, where

$\underline{i} = (i_0, i_1, \ldots, i_s)$ satisfies the following two equations,

$$i_0 + i_1 + \cdots + i_s = n(q^d - 1)$$

and

$$i_0 + q^d i_1 + \cdots + q^{ds} i_s = q^{dj} - 1.$$

By our assumption, $j > n$, which means that $i_0 + i_1 + \cdots + i_s > n(q^d - 1)$ by taking the q^d-expension for $q^{dj} - 1$. That is, $\partial^{q^{dj}}(F)|_{x=0} = 0$, which tells us that

$$\partial^{q^{dj}}(g^{1+n(q^d-1)})|_{x=0} = \partial^{q^{dj}}(g^{q^d+n(q^d-1)})|_{x=0}.$$

Thus we can use induction to prove the first statement.

If $m \neq 1$, then $\partial^{q^{dj}}(g^{m+n(q^d-1)})|_{x=0}$ is the coefficient of $x^{q^{dj}}$ in $g^{m+n(q^d-1)}$, which is $\sum\limits_{\underline{i}} \binom{m+n(q^d-1)}{\underline{i}}$, where $\underline{i} = (i_0, i_1, \ldots, i_s)$ satisfies the following two equations,

$$i_0 + i_1 + \cdots + i_s = m + n(q^d - 1)$$

and

$$i_0 + q^d i_1 + \cdots + q^{ds} i_s = q^{dj}.$$

Taking both equations modulo $q^d - 1$, we obtain

$$\begin{aligned} i_0 + i_1 + \cdots + i_s &\equiv m \qquad \pmod{q^d - 1}, \\ i_0 + i_1 + \cdots + i_s &\equiv 1 \qquad \pmod{q^d - 1}, \end{aligned}$$

which is contradiction since $m \neq 1$. Thus $\partial^{q^{dj}}(g^{m+n(q^d-1)})|_{x=0} = 0$ with $m \neq 1$. $\square$

Proposition 3.3.2. *If we assume that $m_j = q^j + q - 2$ and $\wp = \theta + a \in A$ with $a \in \mathbb{F}_q$,* *then* $\lim\limits_{j\to\infty} BC_{m_j} = -(\theta + a)^{-1}$.

Proof. From the computation in the Theorem (3.1.15), we can compute the limit for BC_{m_j}.

$$\lim_{j\to\infty} BC_{m_j} = \sum_{k=1}^{\infty} \frac{1}{L_k} \sum_{\underline{i}} \binom{q^k - 1}{\underline{i}} \mathcal{A}_0^{\frac{q^k-q}{q-1}}, \tag{3.3.3}$$

where $\underline{i} = (i_0, i_1, \dots, i_j)$ satisfies $i_0 + i_1 + \cdots + i_j = q^k - 1$ and $i_0 + i_1 q + \cdots + i_j q^j = m_j = q^j + q - 2$. Notice that the term $(-\theta - a)^{\frac{q^k-q}{q-1}}$ is independent on the choice of $\underline{i}$, so that we only need to consider the summation $\sum\limits_{\underline{i}} \binom{q^k-1}{\underline{i}}$. Defining power series g as in lemma (3.3.1), we can see that the summation is equal to the coefficient of x^{q^j+q-2} in $g^{q^k} - 1$, which is $\partial^{q^j+q-2}(g^{q^k-1})|_{x=0}$. Use formula $\partial^{q^j} \circ \partial^{q-2} = \binom{q^j+q-2}{q^j} \partial^{q^j+q-2}$ ([20, Prop. 2.3.7]) and Lucas's theorem, we have

$$\begin{aligned}
\partial^{q^j+q-2}(g^{q^k-1})|_{x=0} &= \frac{1}{\binom{q^j+q-2}{q^j}} \partial^{q^j} \circ \partial^{q-2}(g^{q^k-1})|_{x=0} \\
&\equiv \frac{1}{1} \partial^{q^j} \circ \partial^{q-2}(g^{q^k-1})|_{x=0} \\
&= (q^k - 1)\partial^{q^j}(g^{q^k-1-(q-2)})|_{x=0} \\
&= -\partial^{q^j}(g^{q^k-q+1})|_{x=0}.
\end{aligned}$$

Since $q^k - q + 1 \equiv 1 \pmod{q-1}$ and $d = 1$ in lemma (3.3.1), we have

$$\partial^{q^j}(g^{q^k-q+1})|_{x=0} = 1.$$

Therefore we can simplify (3.3.3) according to Proposition (3.2.8),

$$\begin{aligned}
\lim_{j\to\infty} BC_{m_j} &= -\sum_{k=1}^{\infty} \frac{1}{L_k} \mathcal{A}_0^{\frac{q^k-q}{q-1}} \\
&= -\mathcal{A}_0^{-\frac{q}{q-1}} \sum_{k=1}^{\infty} \frac{1}{L_k} (\mathcal{A}_0^{\frac{1}{q-1}})^{q^k}
\end{aligned}$$

$$
\begin{aligned}
&= -\mathcal{A}_0^{-\frac{q}{q-1}} \big(\sum_{k=0}^{\infty} \frac{1}{L_k} (\mathcal{A}_0^{\frac{1}{q-1}})^{q^k} - \frac{1}{L_0} A_0^{\frac{1}{q-1}} \big) \\
&= \mathcal{A}_0^{-\frac{q}{q-1}} \mathcal{A}_0^{\frac{1}{q-1}} \\
&= \mathcal{A}_0^{-1} \\
&= (-\theta - a)^{-1}.
\end{aligned}
$$

□

Notice that in the case $d = 1$, we can pull the term

$$
\frac{\Pi_{m_j}}{D_0^{i_0} \cdots D_{r+dj}^{i_{r+dj}}}
$$

outside of the summation, which makes the computation much easier. However, this does not work any more if $d > 1$. To prove the general result, a stronger version of Lemma (3.3.1) is required.

Proposition 3.3.4. *Let g be a power series as in Lemma (3.3.1) and $m \geqslant 0$ and $n \geqslant 0$ be integers. For $j > n + \frac{a}{q^d - 1}$ and $a = a_0 + a_1 q^d + \cdots + a_s q^{ds} > 0$, the base q^d expansion of a, we have*

$$
\partial^{aq^{dj}} (g^{m+n(q^d-1)})|_{x=0} = \begin{cases} \displaystyle\sum_{\substack{1 \leqslant k_i \leqslant a_i q^{dj+di} \\ k_i \equiv a_i \pmod{q^d - 1}}} \binom{a}{k_1, k_2, \ldots, k_s, \widetilde{a}} & m \equiv a \pmod{q^d - 1}, \\ 0 & otherwise, \end{cases}
$$

where $\widetilde{a} = a - (k_1 + k_2 + \cdots + k_s)$.

Remark 3.3.5. Therefore, Lemma (3.3.1) is a special case of the proposition by taking $a = 1$.

Remark 3.3.6. If $j > s$, the sum will be stable. To be precise, we have

$$\sum_{\substack{1\leqslant k_i\leqslant a_i q^{dj+di} \\ k_i\equiv a_i \pmod{q^d-1}}} \binom{a}{k_1, k_2, \ldots, k_s, \widetilde{a}} = \sum_{\substack{1\leqslant k_i\leqslant a_i q^{ds+1} \\ k_i\equiv a_i \pmod{q^d-1}}} \binom{a}{k_1, k_2, \ldots, k_s, \widetilde{a}},$$

since $k_i \leqslant a$ for all i.

Proof. At first, we can use the similar method to prove part of the proposition. If $m \not\equiv a \pmod{q^d-1}$, we know that $\partial^{aq^{dj}}(g^{m+n(q^d-1)})|_{x=0} = \sum_{\underline{i}} \binom{m+n(q^d-1)}{\underline{i}}$, where $\underline{i} = (i_0, i_1, \ldots, i_s)$ satisfying that

$$i_0 + i_1 + \cdots + i_s = m + n(q^d - 1)$$

and

$$i_0 + q^d i_1 + \cdots + q^{ds} i_s = aq^{dj}.$$

Reducing the two equations modulo $q^d - 1$, we get

$$\begin{aligned} i_0 + i_1 + \cdots + i_s &\equiv m \qquad (mod\ q^d - 1) \\ i_0 + i_1 + \cdots + i_s &\equiv a \qquad (mod\ q^d - 1). \end{aligned}$$

This is contradiction since $m \not\equiv a \pmod{q^d-1}$, which means that $\partial^{q^{dj}}(g^{m+n(q^d-1)})|_{x=0} = 0$ with $m \not\equiv a \pmod{q^d - 1}$.

Assuming that we know the value of $\partial^{aq^{dj}}(g^a)|_{x=0}$, we will show that $\partial^{aq^{dj}}(g^a)|_{x=0} = \partial^{aq^{dj}}(g^{a+n(q^d-1)})|_{x=0}$ for all integers n satisfying $a + n(q^d - 1) > 0$ and $n < j - \frac{a}{q^d-1}$. Actually, the idea is almost the same as in Lemma (3.3.1). Setting $f := g^{a+n(q^d-1)} - g^{a+(n+1)(q^d-1)} = xg^{a-1+n(q^d-1)}$, we can show that $\partial^{aq^{dj}}(F)|_{x=0} = 0$, which proves that $\partial^{aq^{dj}}(g^{a+n(q^d-1)})|_{x=0} = \partial^{aq^{dj}}(g^{a+(n+1)(q^d-1)})|_{x=0}$.

Now we claim that that $\partial^{aq^{dj}}(g^a)|_{x=0} = 1$ if $a < q^d$. Since we have $g^a = (x + x^{q^d} + \cdots + x^{q^{dj}} + \cdots)^a$, the only term with degree aq^{dj} comes from $x^{q^{dj}}$, if we multiply out the exponent of the power series, and add up the terms with degree aq^{dj}. If there is a term X that contains a monomial with degree higher than aq^{dj}, then $\deg(X) > \deg(x^{q^{d(j+1)}}) = q^{d(j+1)} > aq^{dj}$, which is impossible. If there is a term X that contains monomials with degree no greater than q^{dj} and at least one monomial is less than q^{dj}, then $\deg(X) < aq^{dj}$. Therefore, the only term with degree q^{dj} comes from $x^{q^{dj}}$ in each parenthesis. This shows our claim.

For $a \geqslant q^d$, we prove the result by induction. If $a = a_0 + a_1^{q^d}$, taking advantage of the properties of hyperderivative ([20, Prop. 2.3.12], [17, §2]), we get

$$
\begin{aligned}
\partial^{aq^{dj}}(g^a) &= \partial^{a_0 q^{dj}} \circ \partial^{a_1 q^{dj+d}}(g^a) \\
&= \partial^{a_0 q^{dj}} \Bigg(\sum_{k=1}^{a_1 q^{dj+d}} \binom{a}{k} g^{a-k} \sum_{\substack{l_1, l_2, \dots l_{a_1 q^{dj+d}} \geqslant 0 \\ l_1 + l_2 + \cdots + l_{a_1 q^{dj+d}} = k \\ l_1 + 2l_2 + \cdots + a_1 q^{dj+d} l_{a_1 q^{dj+d}} = a_1 q^{dj+d}}} \binom{k}{l_1, l_2, \dots, l_{a_1 q^{dj+d}}} \\
&\qquad \cdot (\partial^1(g))^{l_1} (\partial^2(g))^{l_2} \cdots (\partial^{a_1 q^{dj+d}}(g))^{l_{a_1 q^{dj+d}}} \Bigg) \\
&= \sum_{k=1}^{a_1 q^{dj+d}} \binom{a}{k} \partial^{a_0 q^{dj}}(g^{a-k}) \sum_{\underline{l}} \binom{k}{\underline{l}} (\partial^1(g))^{l_1} (\partial^2(g))^{l_2} \cdots (\partial^{a_1 q^{dj+d}}(g))^{l_{a_1 q^{dj+d}}}.
\end{aligned}
$$

By the discussion above, we know that $\partial^{a_0 q^{dj}}(g^{a-k})|_{x=0} = 1$ if and only if $a - k \equiv a_0 \pmod{q^d - 1}$, and otherwise it is 0. This condition can be written as

$$
k \equiv a - a_0 = a_0 + a_1 q^d - a_0 = a_1 q^d \equiv a_1 \pmod{q^d - 1}.
$$

Also considering that $\partial^h(g) = 1$ if and only if h is a power of q^d, we can simplify the equation to be

$$\partial^{aq^{dj}}(g^a)|_{x=0} = \sum_{\substack{1\leqslant k\leqslant a_1q^{dj+d}\\ k\equiv a_1 \pmod{q^d-1}}} \binom{a}{k} \sum_{\underline{l}} \binom{k}{\underline{l}} (\partial^1(g))^{l_1}(\partial^2(g))^{l_2}\cdots(\partial^{a_1q^{dj+d}}(g))^{l_{a_1q^{dj+d}}}\Big|_{x=0}$$

$$= \sum_{\substack{1\leqslant k\leqslant a_1q^{dj+d}\\ k\equiv a_1 \pmod{q^d-1}}} \binom{a}{k} \sum_{\substack{l_1,l_{q^d},\dots l_{q^{dj+d}}\geqslant 0\\ l_1+l_{q^d}+\cdots+l_{q^{dj+d}}=k\\ l_1+q^dl_{q^d}+\cdots+q^{dj+d}l_{q^{dj+d}}=a_1q^{dj+d}}} \binom{k}{l_1,l_{q^d},\dots,l_{q^{dj+d}}}.$$

Notice in the sum, k has the form $a_1+n(q^d-1)$ for some nonnegative n. Therefore, from the above result, we have

$$\sum_{\substack{l_1,l_{q^d},\dots l_{q^{dj+d}}\geqslant 0\\ l_1+l_{q^d}+\cdots+l_{q^{dj+d}}=k\\ l_1+q^dl_{q^d}+\cdots+q^{dj+d}l_{q^{dj+d}}=a_1q^{dj+d}}} \binom{k}{l_1,l_{q^d},\dots,l_{q^{dj+d}}} = \text{ coefficient of } x^{a_1q^{dj+d}} \text{ in } g^k$$

$$= \partial^{a_1q^{dj+d}}(g^k)|_{x=0} = 1.$$

Hence we have

$$\partial^{aq^{dj}}(g^a)|_{x=0} = \sum_{\substack{1\leqslant k\leqslant a_1q^{dj+d}\\ k\equiv a_1\ (mod\ (q^d-1))}} \binom{a}{k}.$$

Now we consider when $a = a_0 + a_1q^d + \cdots + a_sq^{ds}$ and assume the result

$$\partial^{a'q^{dj}}(g^{a'})|_{x=0} = \sum_{\substack{1\leqslant k_i\leqslant a_iq^{dj+di}\\ k_i\equiv a_i \pmod{q^d-1}}} \binom{a'}{k_1,k_2,\dots,k_{s-1},\widetilde{a'}}$$

is true for $a' = a_0 + a_1 q^d + \cdots + a_{s-1} q^{d(s-1)}$. Then by the previous argument, we have

$$\partial^{a' q^{dj}}(g^{\hat{a}})|_{x=0} = \sum_{\substack{1 \leqslant k_i \leqslant a_i q^{dj+di} \\ k_i \equiv a_i \pmod{q^d - 1}}} \binom{\hat{a}}{k_1, k_2, \ldots, k_{s-1}, \widetilde{\hat{a}}}, \tag{3.3.7}$$

where $\hat{a} \equiv a' \pmod{q^d - 1}$. We then have

$$\begin{aligned}
\partial^{aq^{dj}}(g^a) &= \partial^{a'q^{dj}}\left(\partial^{a_s q^{dj+ds}}(g^a)\right) \\
&= \partial^{a'q^{dj}}\Bigg(\sum_{k_s=1}^{a_s q^{dj+ds}} \binom{a}{k_s} g^{a-k_s} \sum_{\substack{l_1, l_2, \ldots l_{a_s q^{dj+ds}} \geqslant 0 \\ l_1 + l_2 + \cdots + l_{a_s q^{dj+ds}} = k_s \\ l_1 + 2l_2 + \cdots + a_s q^{dj+ds} l_{a_s q^{dj+ds}} = a_s q^{dj+ds}}} \binom{k_s}{l_1, l_2, \ldots, l_{a_s q^{dj+ds}}} \\
&\qquad \cdot (\partial^1(g))^{l_1} (\partial^2(g))^{l_2} \cdots (\partial^{a_s q^{dj+ds}}(g))^{l_{a_s q^{dj+ds}}}\Bigg) \\
&= \sum_{k_s=1}^{a_s q^{dj+ds}} \binom{a}{k_s} \partial^{a'q^{dj}}(g^{a-k_s}) \sum_{\underline{l}} \binom{k_s}{\underline{l}} (\partial^1(g))^{l_1} (\partial^2(g))^{l_2} \cdots (\partial^{a_s q^{dj+ds}}(g))^{l_{a_s q^{dj+ds}}}.
\end{aligned}$$

By (3.3.7) and same argument above for the second summation, we get

$$\begin{aligned}
\partial^{aq^{dj}}(g^a)|_{x=0} &= \sum_{\substack{1 \leqslant k_s \leqslant a_s q^{dj+ds} \\ k_s \equiv a_s \pmod{q^d - 1}}} \binom{a}{k_s} \sum_{\substack{1 \leqslant k_i \leqslant a_i q^{dj+di} \\ k_i \equiv a_i \pmod{q^d - 1}}} \binom{a'}{k_1, k_2, \ldots, k_{s-1}, \widetilde{a'}} \\
&= \sum_{\substack{1 \leqslant k_i \leqslant a_i q^{dj+di} \\ k_i \equiv a_i \pmod{q^d - 1}}} \binom{a}{k_s} \binom{a - k_s}{k_1, k_2, \ldots, k_{s-1}, \widetilde{a - k_s}} \\
&= \sum_{\substack{1 \leqslant k_i \leqslant a_i q^{dj+di} \\ k_i \equiv a_i \pmod{q^d - 1}}} \binom{a}{k_1, k_2, \ldots, k_{s-1}, \widetilde{a}}.
\end{aligned}$$

Then by the result we obtained previously, $\partial^{aq^{dj}}(g^{a+n(q^d-1)})|_{x=0} = \partial^{aq^{dj}}(g^{a+(n+1)(q^d-1)})|_{x=0}$, and we have proved the proposition. $\square$

Now consider for general case $m_j = aq^{dj} + b$, where $a, b \geqslant 0$ and $a + b \equiv 0 \pmod{q -$

1). For convenience, we may assume that $a = a_0 + a_1 q + \cdots + a_{dr} q^{dr}$ and $b = b_0 + b_1 q + \cdots + b_{dt} q^{dt}$ for all a_i and b_i are nonnegative integers less than q. Note we allow a_{dr} or b_{dt} be 0.

Theorem 3.3.8. *If we assume that $m_j = aq^{dj} + b$, where $a = a_0 + a_1 q + \cdots + a_{dr} q^{dr}$ and $b = b_0 + b_1 q + \cdots + b_{dt} q^{dt}$ are the base q^d expansion of a and b respectively. Then $\lim_{j \to \infty} BC_{m_j} \in K(\alpha)$.*

Proof. Recall that the formula for Bernoulli-Carlitz numbers (3.1.5),

$$BC_{m_j} = \sum_{k=1}^{dj+dr} \frac{1}{L_k} \sum_{\underline{i}} \binom{q^k - 1}{\underline{i}} \frac{\Pi_{m_j}}{D_0^{i_0} D_1^{i_1} \cdots D_{dj+dr}^{i_{dj+dr}}},$$

where $\underline{i} = (i_0, i_1, \ldots, i_{dj+dr})$ satisfies

$$i_0 + i_1 + \cdots + i_{dj+dr} = q^k - 1,$$
$$i_0 + q i_1 + \cdots + q^{dj+dr} i_{dj+dr} = m_j.$$

We set

$$\mathcal{I}'_{j,k} = \left\{ (i_0, \ldots, i_{dj+dr}) \in \mathbb{Z}_{\geq 0}^{dj+dr+1} \;\middle|\; \begin{array}{l} i_0 + \cdots + i_{dj+dr} = q^k - 1 \\ i_0 + i_1 q + \cdots + i_{dj+dr} q^{dj+dr} = aq^{dj} + b \end{array} \right\},$$

$$S_b = \{(i_0, i_1, \cdots, i_{dt}) | i_0 + i_1 q + \cdots + i_{dt} q^{dt} = b\}$$

and for any $\underline{\sigma} \in S_b$, set

$$I_\sigma = \{\underline{i} = (i_0, i_1, \ldots, i_{dj+dr}) \in \mathcal{I}'_{j,k} | (i_0, i_1, \cdots, i_{dt}) = \underline{\sigma}\}.$$

Note that the definition of $\mathcal{I}'_{j,k}$ is similar as $\mathcal{I}_{j,l}$ defined in (3.1.11). Also note that $|S_b| <$

∞. We may assume that $j > dt$. Using Lemma (3.1.12), we know that for all integers $k < K_j' := \log_q((j-dt-1)(q-1)+1)$, $\bigcup_{\sigma \in S_b} I_\sigma = \mathcal{I}'_{j,k}$.

Therefore we can rewrite the Bernoulli-Carlitz number BC_{m_j} as

$$
\begin{aligned}
BC_{m_j} &= \sum_{\sigma \in S_b} \sum_{k=1}^{K_j'-1} \frac{1}{L_k} \sum_{\underline{i} \in I_\sigma} \binom{q^k-1}{\underline{i}} \frac{\Pi_{m_j}}{D_0^{i_0} D_1^{i_1} \cdots D_{dj+dr}^{i_{dj+dr}}} \\
&\quad + \sum_{k=K_j'}^{dj+dr} \frac{1}{L_k} \sum_{\underline{i} \in \mathcal{I}'_{j,k}} \binom{q^k-1}{\underline{i}} \frac{\Pi_{m_j}}{D_0^{i_0} D_1^{i_1} \cdots D_{dj+dr}^{i_{dj+dr}}} \\
&= \sum_{\sigma \in S_b} \frac{\Pi_b}{D_0^{i_0} \cdots D_{dt}^{i_{dt}}} \sum_{k=1}^{K_j'-1} \frac{1}{L_k} \sum_{\underline{i} \in I_\sigma} \binom{q^k-1}{\underline{i}} \frac{\Pi_{aq^{dj}}}{D_{dt+1}^{i_{dt+1}} D_{dt+2}^{i_{dt+2}} \cdots D_{dj+dr}^{i_{dj+dr}}} + O(j),
\end{aligned}
$$

where

$$
O(j) := \sum_{k=K_j'}^{dj+dr} \frac{1}{L_k} \sum_{\underline{i} \in \mathcal{I}'_{j,k}} \binom{q^k-1}{\underline{i}} \frac{\Pi_{m_j}}{D_0^{i_0} D_1^{i_1} \cdots D_{dj+dr}^{i_{dj+dr}}}.
$$

Note that $O(j)$ tends to 0 as j goes to infinity by (3.1.8). Since the order of the set S_b is finite, we can reach the maximum of $i_0 + i_1 + \cdots + i_{dt}$, denoted as $\widetilde{b}$. Define K to be a positive integer such that $q^{K-1} > 1 + \widetilde{b}$, which makes the binomial $\binom{q^k-1-(i_0+\cdots+i_{dt})}{\underline{i}}$ well-defined for all $\underline{i} \in I_\sigma$. Hence we can express the multinomial $\binom{q^k-1}{\underline{i}}$ as $\binom{q^k-1}{i_0}\binom{q^k-1-i_0}{i_1} \cdots \cdot \binom{q^k-1-i_0-i_1-\cdots-i_{dt-1}}{i_{dt}} \binom{q^k-1-(i_0+\cdots+i_{dt})}{\underline{i'}}$, where $\underline{i'} = (i_{dt+1}, i_{dt+2}, \ldots, i_{dj+dr})$. Moreover, by Lucas's theorem, we have that each term in the product $\binom{q^k-1}{i_0}\binom{q^k-1-i_0}{i_1} \cdots \cdot \binom{q^k-1-i_0-i_1-\cdots-i_{dt-1}}{i_{dt}}$ stays stable when $k \geqslant K$. Therefore we let $j > \frac{q^K-1}{q-1} + dj + 1$, which makes K_j' larger than K. Now we split the summation into three parts,

$$
\begin{aligned}
BC_{m_j} = \sum_{\sigma \in S_b} \frac{\Pi_b}{D_0^{i_0} \cdots D_{dt}^{i_{dt}}} \binom{q^K-1}{i_0} \cdots \binom{q^K-1-i_0-i_1-\cdots-i_{dt-1}}{i_{dt}} \sum_{k=K}^{K_j'} \frac{1}{L_k} \\
\cdot \sum_{\underline{i} \in I_\sigma} \binom{q^k-1-(i_0+\cdots+i_{dt})}{\underline{i'}} \frac{\Pi_{aq^{dj}}}{D_{dt+1}^{i_{dt+1}} \cdots D_{dj+dr}^{i_{dj+dr}}}
\end{aligned}
$$

$$+\sum_{\sigma\in S_b}\frac{\Pi_b}{D_0^{i_0}\cdots D_{dt}^{i_{dt}}}\sum_{1\leqslant k<K}\frac{1}{L_k}\sum_{\underline{i}\in I_\sigma}\binom{q^k-1}{\underline{i'}}\frac{\Pi_{aq^{dj}}}{D_{dt+1}^{i_{dt+1}}\cdots D_{dj+dr}^{i_{dj+dr}}}+O(j).$$

We first compute the limit for the term

$$\frac{\Pi_{m_j}}{D_0^{i_0}D_1^{i_1}\cdots D_{dj+dr}^{i_{dj+dr}}}$$

as we did in Lemma (3.3.2),

$$\begin{aligned}\lim_{j\to\infty}\frac{\Pi_{aq^{dj}}}{D_{dt+1}^{i_{dt+1}}\cdots D_{dj+dr}^{i_{dj+dr}}}&=\lim_{j\to\infty}\frac{D_{dj}^{a_0}\cdots D_{dj+dr}^{a_dr}}{D_{dt+1}^{i_{dt+1}}\cdots D_{dj+dr}^{i_{dj+dr}}}\\&=\lim_{j\to\infty}\frac{([dj]\cdots[1]^{q^{dj-1}})^{a_0}\cdots([dj+dr]\cdots[1]^{q^{dj+dr-1}})^{a_{dr}}}{([dt+1]\cdots[1]^{q^{dt}})^{i_{dt+1}}\cdots([dj+dr]\cdots[1]^{q^{dj+dr-1}})^{i_{dj+dr}}}.\end{aligned}\tag{3.3.9}$$

Note we have $\lim\limits_{j\to\infty}[dj+n]=\alpha_m$, where $0\leqslant m\leqslant d-1$ and $m\equiv n\ (mod\ d)$. The most important observation is that we can consider all $[n]$ to have limit α_m if $m\equiv n\ (mod\, d)$. Since we know from the computation in the proof of Lemma (3.1.12) and Theorem (3.1.15), for $n<dj-h$ (where $h=\frac{q^k-1}{q-1}+1$) the exponents are actually 0. However, for the convenience of computation, we still add them up. Therefore we can simplify the limit in (3.3.9) as

$$\begin{aligned}\mathrm{Lim}(I,P)=&\alpha_0^{\frac{1}{q^d-1}(I_0+qI_1+\cdots+q^{d-1}I_{d-1}-(P_0+qP_1+\cdots+q^{d-1}P_{d-1}))}\\&\cdot\alpha_1^{\frac{1}{q^d-1}(q^{d-1}I_0+I_1+\cdots+q^{d-2}I_{d-1}-(q^{d-1}P_0+P_1+\cdots+q^{d-2}P_{d-1}))}\cdots\\&\cdot\alpha_{d-1}^{\frac{1}{q^d-1}(qI_0+q^2I_1+\cdots+I_{d-1}-(qP_0+q^2P_1+\cdots+P_{d-1}))},\end{aligned}$$

where

$$I=(I_0,I_1,\ldots,I_{d-1}),\qquad P=(P_0,P_1,\ldots,P_{d-1}),$$

and

$$I_n := \sum_{\substack{td+1 \leqslant m \leqslant dj+dr \\ m \equiv n \ (mod \ d)}} i_m,$$

$$P_n := \sum_{\substack{0 \leqslant m \leqslant dr \\ m \equiv n \ (mod \ d)}} a_m.$$

Let us focus on

$$\sum_{\underline{i} \in I_\sigma} \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{\underline{i}'} \frac{\Pi_{aq^{dj}}}{D_{dt+1}^{i_{dt+1}} \cdots D_{dj+dr}^{i_{dj+dr}}}.$$

Right now, we can regroup the summation to be

$$\begin{aligned} &\sum_{\underline{i} \in I_\sigma} \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{\underline{i}'} \frac{\Pi_{aq^{dj}}}{D_{dt+1}^{i_{dt+1}} \cdots D_{dj+dr}^{i_{dj+dr}}} \\ &\to \sum_{\underline{i} \in I_\sigma} \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{\underline{i}'} \operatorname{Lim}(I, P) \\ &= \sum_{I_0 + \cdots + I_{d-1} = q^k - 1 - (i_0 + \cdots + i_{dt})} \sum_{\underline{i} \in I_B} \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{\underline{i}'} \operatorname{Lim}(I, P), \end{aligned}$$

where $I_B = \{\underline{i} \in I_\sigma | \sum_{\substack{td+1 \leqslant m \leqslant dj+dr \\ m \equiv n \ (mod \ d)}} i_m = I_n \text{ for all } I_n\}$. It is easy to see that

$$\sum_{\underline{i} \in I_B} \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{\underline{i}'} = \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{I_0, I_1, \ldots, I_{d-1}} \cdot \text{ coefficient of}$$
$$x^{aq^{dj}} \text{ in} \left(\sum_{h \geqslant 0} x^{q^{dt+hd+d}} \right)^{I_0} \left(\sum_{h \geqslant 0} x^{q^{dt+hd+1}} \right)^{I_1} \cdots \left(\sum_{h \geqslant 0} x^{q^{dt+hd+d-1}} \right)^{I_{d-1}}.$$

If we set g to be the same as in Lemma (3.3.1), the power series can be expressed as $g^{I_0 q^{dt+d} + I_1 q^{dt+1} + \cdots + I_{d-1} q^{dt+d-1}}$. Thus the coefficient of $x^{aq^{dj}}$ in the power series is

$\partial^{aq^{dj}}(g^{I_0q^{dt+d}+I_1q^{dt+1}+\cdots+I_{d-1}q^{dt+d-1}})|_{x=0}$. By Proposition (3.3.4), we know that the value is

$$\sum_{\substack{1\leqslant k_i\leqslant a_iq^{dj+di}\\ k_i\equiv a_i \pmod{q^d-1}}} \binom{a}{k_1,k_2,\ldots,k_s,\widetilde{a}}$$

if $I_0q^{dt+d}+I_1q^{dt+1}+\cdots+I_{d-1}q^{dt+d-1}\equiv a \pmod{q^d-1}$, which is only dependant on a. Notice that the condition is equivalent as $I_0+qI_1+\cdots+q^{d-1}I_{d-1}\equiv a \pmod{q^d-1}$. Therefore we can simplify our expression further, i.e.

$$\begin{aligned}
&\sum_{\underline{i}\in I_\sigma}\binom{q^k-1-(i_0+\cdots+i_{dt})}{\underline{i}'}\frac{\Pi_{aq^{dj}}}{D_{dt+1}^{i_{dt+1}}\cdots D_{dj+dr}^{i_{dj+dr}}}\\
&\to \sum_{\substack{1\leqslant k_i\leqslant a_iq^{ds+1}\\ k_i\equiv a_i \pmod{q^d-1}}} \binom{a}{k_1,k_2,\ldots,k_s,\widetilde{a}}\\
&\qquad\cdot \sum_{\substack{I_0+\cdots+I_{d-1}=q^k-1-(i_0+\cdots+i_{dt})\\ I_0+qI_1+\cdots+q^{d-1}I_{d-1}\equiv a \pmod{q^d-1}}}\binom{q^k-1-(i_0+\cdots+i_{dt})}{I_0,I_1,\ldots,I_{d-1}}\operatorname{Lim}(I,P).
\end{aligned}$$

Notice that the condition $I_0+qI_1+\cdots+q^{d-1}I_{d-1}\equiv a \pmod{q^d-1}$ means that the power of each α_n in $\operatorname{Lim}(I,P)$ is an integer, and vice versa. Say

$$f(\underline{\alpha}):=\sum_{\substack{I_0+\cdots+I_{d-1}=q^k-1-(i_0+\cdots+i_{dt})\\ I_0+qI_1+\cdots+q^{d-1}I_{d-1}\equiv a \pmod{q^d-1}}}\binom{q^k-1-(i_0+\cdots+i_{dt})}{I_0,I_1,\ldots,I_{d-1}}\operatorname{Lim}(I,P),$$

where $\underline{\alpha}=(\alpha_0,\ldots,\alpha_{d-1})$. Also define

$$F(\underline{x}):=\sum\binom{q^k-1-(i_0+\cdots+i_{dt})}{I_0,I_1,\ldots,I_{d-1}}x_0^{I_0-P_0}\cdots x_{d-1}^{I_{d-1}-P_{d-1}},$$

where $\underline{x}=(x_0,\ldots,x_{d-1})$.

Let $\zeta \in \mathbb{F}_{q^d}^{\times}$ be a primitive root, i.e. ζ has exact order $q^d - 1$. Let us compute

$$\sum_{l=0}^{q^d-2} F(\zeta^l x_0, \zeta^{lq} x_1, \zeta^{lq^2} x_2, \ldots, \zeta^{lq^{d-1}} x_{d-1}) = \sum_{l=0}^{q^d-2} \sum \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{I_0, \ldots, I_{d-1}}$$
$$\cdot\, x_0^{I_0 - P_0} x_1^{I_1 - P_1} \cdots x_{d-1}^{I_{d-1} - P_{d-1}} \zeta^{l(I_0 - P_0 + q(I_1 - P_1) + \cdots + q^{d-1}(I_{d-1} - P_{d-1}))}$$
$$= \sum \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{I_0, \ldots, I_{d-1}} x_0^{I_0 - P_0} x_1^{I_1 - P_1} \cdots x_{d-1}^{I_{d-1} - P_{d-1}}$$
$$\cdot \sum_{l=0}^{q^d-2} (\zeta^{I_0 - P_0 + q(I_1 - P_1) + \cdots + q^{d-1}(I_{d-1} - P_{d-1})})^l$$
$$= - \sum_{I_0 - P_0 + q(I_1 - P_1) + \cdots + q^{d-1}(I_{d-1} - P_{d-1}) \equiv 0 \pmod{q^d - 1}} \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{I_0, \ldots, I_{d-1}}$$
$$\cdot\, x_0^{I_0 - P_0} x_1^{I_1 - P_1} \cdots x_{d-1}^{I_{d-1} - P_{d-1}} x_0^{I_0 - P_0} x_1^{I_1 - P_1} \cdots x_{d-1}^{I_{d-1} - P_{d-1}}$$
$$= - \sum_{I_0 + qI_1 + \cdots + q^{d-1} I_{d-1} \equiv a \pmod{q^d - 1}} \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{I_0, \ldots, I_{d-1}}$$
$$\cdot\, x_0^{I_0 - P_0} x_1^{I_1 - P_1} \cdots x_{d-1}^{I_{d-1} - P_{d-1}} x_0^{I_0 - P_0} x_1^{I_1 - P_1} \cdots x_{d-1}^{I_{d-1} - P_{d-1}}$$

If we input $x_i = \zeta^{lq^i} \mathcal{A}_{d-i-1}$ for $i = 0, 1, \ldots, d-1$ as $\mathcal{A}_i$ defined in (3.2.6), it is easy to check that $f(\underline{\alpha}) = -\sum_{l=0}^{q^d-2} F(\zeta^l \mathcal{A}_{d-1}, \zeta^{lq} \mathcal{A}_{d-2}, \ldots, \zeta^{lq^{d-1}} \mathcal{A}_0)$. On the other hand, we also have $F(\underline{x}) = x_0^{-P_0} \cdots x_{d-1}^{-P_{d-1}} (x_0 + \cdots + x_{d-1})^{q^k - 1 - (i_0 + \cdots + i_{d-1})}$. Therefore we have

$$\sum_{\underline{i} \in I_\sigma} \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{\underline{i'}} \operatorname{Lim}(I, P) = - \sum_{l=0}^{q^d-2} x_0^{-P_0} \cdots x_{d-1}^{-P_{d-1}}$$
$$\cdot\, (x_0 + \cdots + x_{d-1})^{q^k - 1 - (i_0 + \cdots + i_{d-1})}.$$

Now as j goes to ∞, we can start to compute

$$\sum_{k=K}^{K'_j - 1} \frac{1}{L_k} \sum_{\underline{i} \in I_\sigma} \binom{q^k - 1 - (i_0 + \cdots + i_{dt})}{\underline{i'}} \frac{\Pi_{aq^{dj}}}{D_{dt+1}^{i_{dt+1}} \cdots D_{dj+dr}^{i_{dj+dr}}}$$

$$
\begin{aligned}
\to\ & -\sum_{k\geqslant K}\frac{1}{L_k}\sum_{l=0}^{q^d-2} x_0^{-P_0}\cdots x_{d-1}^{-P_{d-1}}(x_0+\cdots+x_{d-1})^{q^k-1-(i_0+\cdots+i_{d-1})}\\
&= -\sum_{l=0}^{q^d-2} x_0^{-P_0}\cdots x_{d-1}^{-P_{d-1}}(x_0+\cdots+x_{d-1})^{-1-(i_0+\cdots+i_{d-1})}\sum_{k\geqslant K}\frac{1}{L_k}(x_0+\cdots+x_{d-1})^{q^k}\\
&= -\sum_{l=0}^{q^d-2} x_0^{-P_0}\cdots x_{d-1}^{-P_{d-1}}(x_0+\cdots+x_{d-1})^{-1-(i_0+\cdots+i_{d-1})}\\
&\quad\cdot\left(\log_C(x_0+\cdots+x_{d-1})-\sum_{k<K}\frac{1}{L_k}(x_0+\cdots+x_{d-1})^{q^k}\right).
\end{aligned}
$$

By the remark (3.2.10), we know that $\log_C(x_0+\cdots+x_{d-1})=0$. Also, we have $x_i=\frac{x_0^{q^i}}{\alpha_1^{q^{i-1}}\alpha_2^{q^{i-2}}\cdots\alpha_i}$, and considering the trace for the field extension $K(\alpha)(x_0)/K(\alpha)$, we have

$$
\begin{aligned}
&\sum_{k=K}^{K'_j-1}\frac{1}{L_k}\sum_{\underline{i}\in I_\sigma}\binom{q^k-1-(i_0+\cdots+i_{dt})}{\underline{i'}}\frac{\Pi_{aq^{dj}}}{D_{dt+1}^{i_{dt+1}}\cdots D_{dj+dr}^{i_{dj+dr}}}\\
&\to \sum_{l=0}^{q^d-2} x_0^{-P_0}\cdots x_{d-1}^{-P_{d-1}}(x_0+\cdots+x_{d-1})^{-1-(i_0+\cdots+i_{d-1})}\\
&\qquad\cdot\sum_{k<K}\frac{1}{L_k}(x_0+\cdots+x_{d-1})^{q^k}\\
&= Tr\Bigg(x_0^{-P_0}\cdots x_{d-1}^{-P_{d-1}}(x_0+\cdots+x_{d-1})^{-1-(i_0+\cdots+i_{d-1})}\\
&\qquad\cdot\sum_{k<K}\frac{1}{L_k}(x_0+\cdots+x_{d-1})^{q^k}\Bigg)\in K(\alpha).
\end{aligned}
$$

For the other part of Bernoulli-Carlitz numbers, $\sum\limits_{1\leqslant k<K}\frac{1}{L_k}\sum\limits_{\underline{i}}\binom{q^k-1}{\underline{i}}\frac{\Pi_{m_j}}{D_0^{i_0}D_1^{i_1}\cdots D_{dj+dr}^{i_{dj+dr}}}$ is a finite sum and each of the limits for the bracket polynomials are in $K(\alpha)$, which means the term is in $K(\alpha)$.

Therefore, both parts of Bernoulli-Carlitz numbers are elements in $K(\alpha)$, so we have proved the theorem. $\square$

Proposition 3.3.10. *If $q = 3$ and $\wp = \theta^2 + 1$, then we have*

$$\lim_{j\to\infty} BC_{3^{2j}+1} = \alpha, \qquad \textit{and} \qquad \lim_{j\to\infty} BC_{3^{2j+1}+1} = -\alpha.$$

We can also compute the following two limits:

$$\lim_{j\to\infty} BC_{3^{2j}+5} = -\frac{\theta^3-\theta}{\theta^2+1}(1+\alpha\theta)(-\alpha-\theta^3-\theta) - \alpha\theta + 1,$$

and

$$\lim_{j\to\infty} BC_{3^{2j+1}+5} = -\frac{\theta^3-\theta}{\theta^2+1}(1-\alpha\theta)(\alpha-\theta^3-\theta) + \alpha\theta + 1.$$

Proof. Use the theorem above, and set $K = 1$. As j goes to infinity, using the same notation as above, we have

$$\begin{aligned}
BC_{3^{2j}+1} &\to \sum_{k=1}^{\infty} \frac{1}{L_k} \sum_{l=0}^{7} F(\zeta^l \mathcal{A}_1, \zeta^{3l}\mathcal{A}_0) \\
&= \sum_{s=0}^{7} \frac{1}{L_k} \sum_{l=0}^{7} \zeta^{-l}\mathcal{A}_1^{-1}(\zeta^l\mathcal{A}_1 + \zeta^{3l}\mathcal{A}_0)^{3^k-2} \\
&= \sum_{s=0}^{7} \frac{1}{\zeta^l\mathcal{A}_1(\zeta^l\mathcal{A}_1+\zeta^{3l}\mathcal{A}_0)^2} \cdot (\log(\zeta^l\mathcal{A}_1+\zeta^{3l}\mathcal{A}_0) - \frac{1}{L_0}(\zeta^l\mathcal{A}_1+\zeta^{3l}\mathcal{A}_0)^{3^0}) \\
&= -\sum_{s=0}^{7} \frac{1}{\zeta^l\mathcal{A}_1(\zeta^l\mathcal{A}_1+\zeta^{3l}\mathcal{A}_0)^2} \cdot (\zeta^l\mathcal{A}_1+\zeta^{3l}\mathcal{A}_0) \\
&= -\sum_{s=0}^{7} \frac{1}{\zeta^l\mathcal{A}_1(\zeta^l\mathcal{A}_1+\zeta^{3l}\mathcal{A}_0)} \\
&= -\sum_{s=0}^{7} \frac{1}{\zeta^l\mathcal{A}_1(\zeta^l\mathcal{A}_1+\zeta^{3l}\mathcal{A}_0)} \cdot \frac{\zeta^{-2l}(\mathcal{A}_1-\zeta^{2l}\mathcal{A}_0)(\mathcal{A}_1^2+\zeta^{4l}\mathcal{A}_0^2)}{\zeta^{-2l}(\mathcal{A}_1-\zeta^{2l}\mathcal{A}_0)(\mathcal{A}_1^2+\zeta^{4l}\mathcal{A}_0^2)} \\
&= -\sum_{s=0}^{7} \frac{\zeta^{-2l}\mathcal{A}_1^3 + \zeta^{2l}\mathcal{A}_1\mathcal{A}_0^2 - \mathcal{A}_1^2\mathcal{A}_0 - \zeta^{4l}\mathcal{A}_0^3}{\mathcal{A}_1(\mathcal{A}_1^4-\mathcal{A}_0^4)} \\
&= \sum_{s=0}^{7} \frac{\mathcal{A}_0\mathcal{A}_1}{\mathcal{A}_1^4-\mathcal{A}_0^4}
\end{aligned}$$

$$= \sum_{s=0}^{7} \frac{\alpha_0^{\frac{1}{2}}\alpha_1^{\frac{1}{2}}}{\alpha_0^{\frac{1}{2}}\alpha_1^{\frac{3}{2}} - \alpha_0^{\frac{3}{2}}\alpha_1^{\frac{1}{2}}}$$
$$= \sum_{s=0}^{7} \frac{1}{\alpha_1 - \alpha_0} = \sum_{s=0}^{7}(-\alpha) = 8(-\alpha) = \alpha \in K(\alpha),$$

where ζ is a primative root in $\mathbb{F}_9^{\times}$, $\mathcal{A}_0 = (\alpha_0^3\alpha_1)^{\frac{1}{8}}$, $\mathcal{A}_1 = (\alpha_1^3\alpha_0)^{\frac{1}{8}}$, $\alpha_0 = -\theta + \alpha$ and $\alpha_1 = -\theta + \alpha^3$. Using the similar computation, we can also get that $\lim_{j\to\infty} BC_{3^{2j+1}+1} = -\alpha$ and the other two limits.

□

Remark 3.3.11. In the previous proposition, notice that we have $\alpha^3 = -\alpha$. By Remark (3.2.3), Theorem (3.3.8) and Propsition (3.3.10), we find that

$$\lim_{j\to\infty} BC_{3^{2j}+1} = \sigma\left(\lim_{j\to\infty} BC_{3^{2j+1}+1}\right),$$

where $\sigma \in Gal(K(\alpha)/K) \cong \mathbb{Z}/2\mathbb{Z}$, the Frobenius map. More generally, observing the computation in Theorem (3.3.8), we have

$$\lim_{j\to\infty} BC_{aq^{dj}+b} = \sigma\left(\lim_{j\to\infty} BC_{aq^{dj+1}+b}\right),$$

where $\sigma \in Gal(K(\alpha)/K) \cong \mathbb{Z}/d\mathbb{Z}$, the Frobenius map.

4. SUMMARY AND FUTURE WORK

In this dissertation, we have shown that Θ operators map a v-adic modular form to another v-adic modular form (see Theorem (2.5.1)). Moreover, by multiplying a Carlitz factorial, we can restrict the coefficients in A_v (see Theorem (2.6.4)). Then we let $m_j = aq^{dj} + b = (a_0 + a_1 q + \cdots + a_r q^r)q^{dj} + (b_0 + b_1 + \cdots + b_t q^t)$, we show the v-adic limits exists for Bernoulli-Carlitz numbers BC_{m_j}, Goss polynomials $\Pi_{m_j} G_{m_j+1}$ and Eisenstein series $\Pi_{m_j} E_{m_j+1}$ (see Theorem (3.1.15), (3.1.26) and (3.1.35)). At last, I show that the limit of the Bernoulli-Carlitz numbers BC_{m_j} is actually in an algebraic extension of K by explicitly computing the limit (see Theorem (3.3.8)). Also, I give two examples of the limits of some sequence of Bernoulli-Carlitz numbers (see Proposition (3.3.2) and (3.3.10)).

One natural direction is to compute the v-adic limit for Bernoulli-Carlitz numbers for more general sequences m_j. Moreover, an important plan for future work is to try to understand why v-adic limits exist for Bernoulli-Carlitz numbers, and more important, why the limits will lie in $K(\alpha)$. Papanikolas and I proved the theorem by calculation, but we believe there is something intrinsic that we have not discovered. Since the Bernoulli-Carlitz numbers are so close to the Carlitz zeta function, any property of Bernoulli-Carlitz numbers may help us understand the Carlitz zeta function better. The ideal situation will be whether we can find clear analogies between the Bernoulli numbers and Bernoulli-Carlitz numbers, as well as the classical zeta function and the Carlitz zeta function. In my research, it is very possible that we can find the corresponding properties in function fields.

In the near future, I also plan to extend these arguments to Drinfeld modules. In the Drinfeld module case, the exponential function is different from that in the Carlitz module, which is much easier to compute. Moreover, the corresponding lattice can have dimension more than 1, which makes the calculations much more difficult and many elementary results are no longer correct. However, I believe that similar ideas can still be applied to Drinfeld modules and there may be similar Bernoulli-Carlitz numbers for Drinfeld modules.

REFERENCES

[1] B. Anglès and F. Pellarin, *Functional identities for L-series values in positive characteristic*, J. Number Theory **142** (2014), 223-251.

[2] B. Anglès, T. Ngo Dac and F. Tavares Riveiro, *Exceptional Zeros of L-series and Bernoulli-Carlitz Numbers*, arXiv:1511.06209 [math.NT], (2015).

[3] V. Bosser and F. Pellarin, *Hyperdifferential properties of Drinfeld quasi-modular forms*, Int. Math. Res. Not. IMRN **2008** (2008), Art. ID rnn032, 56 pp.

[4] V. Bosser and F. Pellarin, *On certain families of Drinfeld quasi-modular forms*, J. Number Theory **129** (2009), no. 12, 2952–2990.

[5] W. D. Brownawell, *Linear independence and divided derivatives of a Drinfeld module. I.* in: Number Theory in Progress, Vol. 1 (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, 1999, pp. 47–61.

[6] L. Carlitz, *An analogue of the von Staudt-Clausen theorem*, Duke Math. J. **3** (1937), no. 3, 503–517.

[7] K. Conrad, *The digit principle*, J. Number Theory **84** (2000), no. 2, 230–257.

[8] J. Fresnel and M. van der Put, *Rigid Analytic Geometry and its Applications*, Birkhäuser, Boston, 2004.

[9] E.-U. Gekeler, *On the coefficients of Drinfeld modular forms*, Invent. Math. **93** (1988), no. 3, 667–700.

[10] E.-U. Gekeler, *On the zeroes of Goss polynomials*, Trans. Amer. Math. Soc. **365** (2013), no. 3, 1669–1685.

[11] D. Goss, *von Staudt for* $\mathbf{F}_q[T]$, Duke Math. J. **45** (1978), no. 4, 885–910.

[12] D. Goss, *Modular forms for* $\mathbf{F}_r[T]$, J. reine Angew. Math. **317** (1980), 16–39.

[13] D. Goss, π*-adic Eisenstein series for function fields*, Compos. Math. **41** (1980), no. 1, 3–38.

[14] D. Goss, *The algebraist's upper half-plane*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), no. 3, 391–415.

[15] D. Goss, *Basic Structures of Function Field Arithmetic*, Springer-Verlag, Berlin, 1996.

[16] D. Goss, *A construction of* $\mathfrak{v}$*-adic modular forms*, J. Number Theory **136** (2014), 330–338.

[17] S. Jeong, *Calculus in positive characteristic* p, J. Number Theory **131** (2011), no. 6, 1089–1104.

[18] B. López, *A non-standard Fourier expansion for the Drinfeld discriminant function*, Arch. Math. (Basel) **95** (2010), no. 2, 143–150.

[19] B. López, *Action of Hecke operators on two distinguished Drinfeld modular forms*, Arch. Math. (Basel) **97** (2011), no. 5, 423–429.

[20] M. A. Papanikolas, *Log-algebraicity on tensor powers of the Carlitz module and special values of Goss* L*-functions*, in preparation.

[21] M. A. Papanikolas and G. Zeng, *Theta operators, Goss polynomials, and* v*-adic modular forms*, arXiv:1605.04235, J. Théor. Nombres Bordeaux **29**, (2017), no. 3.

[22] M. A. Papanikolas and G. Zeng, *Limits of Bernoulli-Carlitz numbers and* v*-adic families of Eisenstein series*, in preparation.

[23] F.Pellarin and R. Perkins, *On twisted* A*-harmonic sums and Carlitz finite zeta values*, arXiv:1512.05953v2, (2015).

[24] A. Petrov, *A-expansions of Drinfeld modular forms*, J. Number Theory **133** (2013), no. 7, 2247–2266.

[25] A. Petrov, *On hyperderivatives of single-cuspidal Drinfeld modular forms with $\mathcal{A}$-expansions*, J. Number Theory **149** (2015), 153–165.

[26] J.-P. Serre, *Formes modulaires et fonctions zêta p-adiques*, in: Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), Springer, Berlin, 1973, pp. 191–268.

[27] D. S. Thakur, *Gamma functions for function fields and Drinfeld modules*, Ann. of Math. (2) **134** (1991), no. 1, 25-64.

[28] D. S. Thakur, *A note on numerators of Bernoulli numbers*, Proc. Amer. Math. Soc. **140** (2012), no. 11, 3673-3676.

[29] Y. Uchino and T. Satoh, *Function field modular forms and higher-derivations*, Math. Ann. **311** (1998), no. 3, 439–466.

[30] C. Vincent, *Drinfeld modular forms modulo $\mathfrak{p}$*, Proc. Amer. Math. Soc. **138** (2010), no. 12, 4217–4229.

[31] C. Vincent, *On the trace and norm maps from $\Gamma_0(\mathfrak{p})$ to $\mathrm{GL}_2(A)$*, J. Number Theory **142** (2014), 18–43.

[32] C. Vincent, *Weierstrass points on the Drinfeld modular curve $X_0(\mathfrak{p})$*, Res. Math. Sci. **2** (2015), 2:10, 40 pp.

[33] J. Yu, *Transcendence and special zeta values in characteristic p*, Ann. of Math. (2) **134** (1991), no. 1, 1-23.