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Nonlocal elliptic hemivariational inequalities

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Abstract. This paper is devoted to the existence of solutions for hemivariational inequalities involving fractional Laplace operator by means of the well-known surjectivity result for pseudomonotone mappings.

Keywords: nonlocal elliptic hemivariational inequalities, pseudomonotone mappings, existence results.

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1 Introduction

The aim of this paper is to prove the existence of at least one solution for the nonlocal elliptic hemivariational inequalities as follows:

$$\begin{cases} (-\Delta)^s u + \partial J(u) \ni f \ \text{ in } \Omega, \\ u = 0 \ \text{ in } \Omega^c := \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n > 2s$, is an open bounded set with Lipschitz boundary, $s \in (0,1)$ is fixed and $(-\Delta)^s$ stands for the fractional Laplace operator, which (up to normalization factor) is given by

$$-(-\Delta)^s u(x) := \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy, \qquad \forall x \in \mathbb{R}^n.$$

Moreover, $f : \Omega \to \mathbb{R}$, the integral functional is given by

$$J(v) := \int_\Omega j(x, v(x))\,dx \quad \text{for all } v \in L^p(\Omega)$$

and $\partial J(\cdot)$ denotes the generalized subdifferential in the sense of Clarke (cf. [4,9]).

We remark that the Dirichlet datum is given in $\Omega^c = \mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, consistently with the non-local character of the operator $(-\Delta)^s$.

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More precisely, we seek the weak solution u and $\eta \in \partial J(u)$ as follows

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[(u(x)-u(y))(v(x)-v(y))}{|x-y|^{(n+2s)}}\,dydx+\int_{\mathbb{R}^n}\eta(x)v(x)\,dx\\ =\int_{\mathbb{R}^n}f(x)v(x)\,dx,\qquad \forall v\in H^s(\mathbb{R}^n)\text{ with } v=0 \text{ in } \Omega^c,\end{aligned}$$

here $H^s(\mathbb{R}^n)$ the usual fractional Sobolev space and notice that

$$\begin{aligned}\int_{\mathbb{R}^n} v(x)(-\Delta)^s u(x)dx &= -\frac{1}{2}\int_{\mathbb{R}^n} v(x)\,dx\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy\\
&= -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(x+y)-u(x)]v(x)}{|y|^{n+2s}}\,dydx\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(x-y)-u(x)]v(x)}{|y|^{n+2s}}\,dydx\\
&= -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(y)-u(x)]v(x)}{|y-x|^{n+2s}}\,dydx\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(y)-u(x)]v(x)}{|y-x|^{n+2s}}\,dydx\\
&= -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(y)-u(x)]v(x)}{|y-x|^{n+2s}}\,dydx\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(x)-u(y)]v(y)}{|y-x|^{n+2s}}\,dydx\\
&= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[(u(x)-u(y))(v(x)-v(y))}{|x-y|^{-(n+2s)}}\,dydx\\
&\quad \forall v\in H^s(\mathbb{R}^n)\text{ with } v=0\text{ in }\Omega^c.\end{aligned}$$

In recent years, nonlocal variational problems attract a lot of interest since the fractional powers of the Laplacian play an important role in physics, mathematical finance, which also describe anomalous diffusion in collective dynamics, extended heterogeneities, and other sources of long-range correlations.

Hemivariational inequalities arise in variational expressions for some mechanical problems with nonsmooth and nonconvex energy superpotentials. The derivative of hemivariational inequality is based on the mathematical notion of the generalized gradient of Clarke [2,5,9].

Recently, Teng [12] and Xi et al. [15] established multiplicity of weak solution to nonlocal elliptic hemivariational inequalities with Dirichlet boundary condition by using the nonsmooth critical point theory [7] and nonsmooth version of the three-critical-points theorem under the framework of the nonsmooth functional.

However, in some cases, the nonsmooth critical point theory and the nonsmooth variational methods cannot be applied, because the formulated problems have not in general a variational structure. Then we have to look for other methods, for example, topological degree, theory for pseudomonotone operators, method of sub-supersolutions and fixed point theory and so on.

In this paper, we show the existence of at least one solution for the nonlocal hemivariational inequalities. The basic tools used in our paper are the surjectivity result for pseudomonotone and coercive operators (cf. [10]), properties of the generalized subdifferential in the sense of Clarke. In our hypotheses we only require a general growth condition with respect to the solution. We believe that our result gives a natural approach to the theory of the nonlocal nonlinear hemivariational inequalities. Furthermore, the hypotheses we assume on the nonlinear term are general and verifiable.

2 Mathematical framework

In this section, collect some important notations and useful results on nonlocal operators, nonsmooth analysis and operators of monotone type.

We recall the function spaces related to the fractional Laplacian (see, e.g., [1, 6, 11, 13]). Given $s \in (0,1)$. Let $\Omega \subset \mathbb{R}^n, n > 2s$ be an open bounded set with Lipschitz boundary. Define the fractional Sobolev space

$$E := \left\{ u \in L^2(\mathbb{R}^n) \mid u \equiv 0 \text{ in } \Omega^c, \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy < \infty \right\} \tag{2.1}$$

endowed with the Gagliardo norm

$$\|u\|_E = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy \right)^{1/2}$$

with the inner product for $u, v \in E$

$$\langle u, v\rangle_E := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy.$$

We stress that $C_0^2(\Omega) \subseteq E$ (see, e.g., [14]). So E is non-empty and dense in $L^2(\Omega)$. We may collect the useful facts on the space E (for more details, see [13]) as follows.

Lemma 2.1. *E is a Hilbert space and for $p \in [1, 2^*]$,there exists a positive constant $c(p)$ such that*

$$\|u\|_{L^p(\mathbb{R}^n)} \le c(p)\|u\|_E, \qquad \forall u \in E, \tag{2.2}$$

where $2^* = \frac{2n}{n-2s}$. Furthermore, the embedding is compact if $p \in [1, 2^*)$.

We recall some preliminary material of the pseudomonotone operator. Let X be a reflexive Banach space and X^* be its dual space with the dual paring $\langle \cdot, \cdot \rangle_X$.

Definition 2.2. We say that the multivalued operator $A : X \to 2^{X^*}$ is pseudomonotone if:

(i) for each $u \in X$, the set Au is nonempty, bounded, closed and convex in X^*;

(ii) A is upper semicontinuous from each finite-dimensional subspace of X to X^* endowed with the weak topology;

(iii) if $\{u_k\} \subset X$ with $u_k \to u$ weakly in X, and $u_k^* \in Au_k$ is such that

$$\limsup_{n\to\infty} \langle u_k^*, u_k - u\rangle_X \le 0,$$

then for every $y \in X$, there exists $u^*(y) \in Au$ such that

$$\liminf_{k\to\infty} \langle u_k^*, u_k - y\rangle_X \ge \langle u^*(y), u - y\rangle_X.$$

In what follows we introduce the notion of coercivity.

Definition 2.3. Let X be a Banach space and $A : X \to 2^{X^*}$ be an operator. We say that A is coercive if either $D(A)$ is bounded or $D(A)$ is unbounded and

$$\lim_{\|u\|_X \to \infty,\, u \in D(A)} \frac{\inf\{\langle u^*, u\rangle_X | u^* \in Au\}}{\|u\|_X} = +\infty.$$

The following is the main surjectivity result for pseudomonotone and coercive operators.

Theorem 2.4 ([9]). *Let X be a reflexive Banach space and $A : X \to 2^{X^*}$ be pseudomonotone and coercive. Then A is surjective, i.e., for every $f^* \in X^*$, there exists $u \in X$ such that $f^* \in A(u)$.*

Let us recall $h^0(u, v)$ the Clarke generalized directional derivative of a locally Lipschitz functional $h : V \to \mathbb{R}$ at $u \in V$ in the direction $v \in V$

$$h^0(u, v) = \limsup_{\lambda \to 0^+,\, w \to u} \frac{h(w + \lambda v) - h(w)}{\lambda}$$

and the generalized Clarke subdifferential of h at $u \in V$

$$\partial h(u) := \{u^* \in V^* \mid h^0(u, v) \geq \langle u^*, v\rangle \text{ for all } v \in V\}.$$

The next proposition provides basic properties of the generalized directional derivative and the generalized gradient.

Proposition 2.5 ([4,9]). *If $h : U \to \mathbb{R}$ is a locally Lipschitz function on a subset U of X, then*

(i) for every $u \in U$ the gradient $\partial h(x)$ is a nonempty, convex,and weakly* compact subset of X^* which is bounded by the Lipschitz constant $K_x > 0$ of h near u;

(ii) for each $y \in X$, there exists $z_x \in \partial h(x)$ such that

$$h^0(x; y) = \max\{\langle z, y\rangle_X, \mid z \in \partial h(x)\} = \langle z_x, y\rangle_X;$$

(iii) the graph of the generalized gradient ∂h is closed in $X \times (w^* - X^*)$ topology, i.e., if $\{x_k\} \subset U$ and $\{\zeta_k\} \subset X^*$ are sequences such that $\zeta_k \in \partial h(x_k)$ and $x_k \to x$ in X, $\zeta_k \to \zeta$ weakly* in X^*, then $\zeta \in \partial h(x)$ where, recall, $w^* - X^*$ denotes the space X^* equipped with weak* topology;

(iv) the multifunction $U \ni x \to \partial h(x) \subseteq X^*$ is upper semicontinuous from U into $w^* - X^*$.

We assume that $j : \Omega \times \mathbb{R} \to \mathbb{R}$, $j(\cdot, 0) \in L^1(\Omega)$ satisfying the assumption (H):

(i) $j(\cdot, s) : \Omega \to \mathbb{R}$ is measurable for all $s \in \mathbb{R}$;

(ii) $j(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz for a.e. $x \in \Omega$;

(iii) there exist $p \geq 1$, $c > 0$ and $b \in L^{\frac{p}{p-1}}(\Omega)$ such that

$$|z| \leq b(x) + c|s|^{p-1}, \qquad \forall x \in \Omega,\ \forall z \in \partial_s j(x, s).$$

Define the integral functional

$$J(v) := \int_\Omega j(x, v(x))\, dx \quad \text{for all } v \in L^p(\Omega). \tag{2.3}$$

Lemma 2.6 ([8]). *Under the assumption (H), the functional J in (2.3) is locally Lipschitz and the following inequalities hold:*

$$J^0(u, v) \leq c_1(1 + \|u\|_{L^p}^{p-1})\|v\|_{L^p} \qquad \forall u, v \in L^p(\Omega) \tag{2.4}$$

and

$$\|w\|_{L^{p'}} \leq c_1(1 + \|u\|_{L^p}^{p-1}), \qquad \forall w \in \partial(J|_{L^p})(u),\ u \in L^p(\Omega), \tag{2.5}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and c_1 is a positive constant.

3 Main results

Lemma 3.1. *$(-\Delta)^s : E \to E^*$ is a linear bounded strongly monotone operator.*

Proof. We observe that for all $u, v \in E$

$$\langle (-\Delta)^s u, u\rangle_E = \frac{1}{2}\|u\|_E^2$$

and

$$\begin{aligned}\langle (-\Delta)^s u, v\rangle &= \frac{1}{2}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dydx \\ &\leq \frac{1}{2}\left\{\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dydx\right\}^{1/2}\left\{\int_{\mathbb{R}^n} \frac{(v(x)-v(y))^2}{|x-y|^{n+2s}}\,dydx\right\}^{1/2} \\ &= \frac{1}{2}\|u\|_E\|v\|_E,\end{aligned}$$

which implies that

$$\|(-\Delta)^s u\|_{E^*} \leq \frac{1}{2}\|u\|_E.$$

The proof is complete. □

Proposition 3.2. *Under the assumption (H), $(-\Delta)^s + \partial J : E \to E^*$ is pseudomonotone. Furthermore, if $1 \leq p < 2$ or $p = 2$ such that $2c_1c(p) < 1$ in the assumption (H), where $c(p)$ and c_1 are the constants in (2.2) and (2.5), respectively, then $(-\Delta)^s + \partial J : E \to E^*$ is coercive.*

Proof. By Proposition 2.5, we observe that ∂J is nonempty, convex, weak-compact subset of E^*. Then for each $u \in E$, $(-\Delta)^s u + \partial J(u)$ is nonempty, bounded, closed and convex subset of E^*. Moreover, $(-\Delta)^s u + \partial J(u)$ is upper semicontinuous from E to $w - E^*$.

Let u_k be a sequence in E converging weakly to u, and $w_k \in \partial J(u_k)$ such that

$$\limsup_{k\to\infty}\langle (-\Delta)^s u_k + w_k, u_k - u\rangle_E \leq 0 \tag{3.1}$$

which implies

$$\limsup_{k\to\infty}\langle (-\Delta)^s u_k, u_k - u\rangle_E + \liminf_{k\to\infty}\langle w_k, u_k - u\rangle_E \leq 0. \tag{3.2}$$

By Lemma 2.1., we have $E \subseteq L^p(\Omega) \subseteq E^*, p \in [1, 2^*]$ and the embedding $E \hookrightarrow L^p(\Omega), p \in [1, 2^*)$ is compact. Therefore,

$$u_k \to u \quad \text{strongly in } L^p(\Omega).$$

Applying Theorem 2.2 in [3], we have

$$\partial(J|_E)(u) \subset \partial(J|_{L^p(\Omega)})(u), \qquad \forall u \in E.$$

Therefore,

$$|\langle w_k, u_k - u\rangle_E| \leq \text{const}\|w_k\|_{L^{p'}(\Omega)}\|u_k - u\|_{L^p(\Omega)}.$$

Thus

$$|\langle w_k, u_k - u\rangle_E| \to 0 \quad \text{as } k \to \infty.$$

Then, from (3.2) we have

$$\limsup_{k\to\infty}\langle(-\Delta)^s u_k, u_k - u\rangle_E \le 0.$$

We have from $u_k \to u$ weakly in E

$$\limsup_{k\to\infty}\langle(-\Delta)^s u_k - (-\Delta)^s u, u_k - u\rangle_E \le 0.$$

By Lemma 3.1, we get

$$\limsup_{k\to\infty}\frac{1}{2}\|u_k - u\|_E = \limsup_{k\to\infty}\langle(-\Delta)^s u_k - (-\Delta)^s u, u_k - u\rangle_E \le 0.$$

Therefore we obtain

$$u_k \to u \quad \text{strongly in } E, \tag{3.3}$$

$$(-\Delta)^s u_k \to (-\Delta)^s u \quad \text{strongly in } E^*. \tag{3.4}$$

Then, by Proposition 2.5,

$$w \in \partial J(u).$$

So, we have

$$\lim_{k\to\infty}\langle(-\Delta)^s u_k + w_k, u_k - v\rangle_E = \langle(-\Delta)^s u - w, u - v\rangle_E, \tag{3.5}$$

which implies $(-\Delta)^s + \partial J : E \to E^*$ is pseudomonotone.

In the following, we show that $(-\Delta)^s + \partial J : E \to E^*$ is coercive, i.e.,

$$\lim_{\|u\|_E\to\infty}\frac{\inf\{\langle(-\Delta)^s u + w, u\rangle_E | w \in \partial J(u)\}}{\|u\|_E} = +\infty.$$

Since

$$\begin{aligned}\inf\{\langle(-\Delta)^s u + w, u\rangle_E | w \in \partial J(u)\} &= \langle(-\Delta)^s u, u\rangle_E + \inf\{\langle w, u\rangle_{L^p(\Omega)} | w \in \partial J(u)\}\\ &\ge \frac{1}{2}\|u\|_E^2 - \sup\{\|w\|_{L^{p'}(\Omega)} | w \in \partial J(u)\}\|u\|_{L^p(\Omega)}\\ &\ge \frac{1}{2}\|u\|_E^2 - c_1\|u\|_{L^p(\Omega)} - c_1\|u\|_{L^p}^p \quad \text{(by Lemma 2.6)}\\ &\ge \frac{1}{2}\|u\|_E^2 - c_1 c(p)\|u\|_E - c_1 c(p)\|u\|_E^p,\end{aligned}$$

if $1 \le p < 2$, or $p = 2$ with $2c_1c(p) < 1$, the above inequality implies that $(-\Delta)^s + \partial J : E \to E^*$ is coercive. The proof is complete. □

Therefore, from Theorem 2.4, we get the following theorem.

Theorem 3.3. *Under the assumption (H) with $1 \le p < 2$ or $p = 2$ such that $2c_1c(p) < 1$, the nonlocal hemivariational inequality (1.1) has at least a weak solution.*

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