Electronic Journal of Qualitative Theory of Differential Equations
2017, No. 41, 1–18; doi: 10.14232/ejqtde.2017.1.41 http://www.math.u-szeged.hu/ejqtde/

Existence, regularity and upper semicontinuity of pullback attractors for the evolution process associated to a neural field model

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Received 15 February 2017, appeared 21 May 2017
Communicated by Hans-Otto Walther

Abstract. In this work we study the pullback dynamics of a class of nonlocal non-autonomous evolution equations for neural fields in a bounded smooth domain Ω in $\mathbb{R}^N$

$$\begin{cases} \partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}^N} J(x,y) f(t,u(t,y))dy, \ t \geq \tau, \ x \in \Omega, \\ u(\tau,x) = u_\tau(x), \ x \in \Omega, \end{cases}$$

with $u(t,x) = 0$, $t \geq \tau$, $x \in \mathbb{R}^N \backslash \Omega$, where the integrable function $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable, $\int_{\mathbb{R}^N} J(x,y)dy = \int_{\mathbb{R}^N} J(x,y)dx = 1$ and symmetric i.e., $J(x,y) = J(y,x)$ for any $x, y \in \mathbb{R}^N$. Under suitable assumptions on the nonlinearity $f : \mathbb{R}^2 \to \mathbb{R}$, we prove existence, regularity and upper semicontinuity of pullback attractors for the evolution process associated to this problem.

Keywords: pullback attractors, neural fields, nonlocal evolution equation.

2010 Mathematics Subject Classification: 35B40, 35B41, 37B55.

1 Introduction

In this paper we study the pullback dynamics for a class of nonlocal non-autonomous evolution equations generated as continuum limits of computational models of neural fields theory. In short, neural field equations are tissue level models that describe the spatiotemporal evolution of coarse grained variables such as synaptic or firing rate activity in populations of neurons, see e.g. [1–3, 9, 20, 21, 24, 26, 28, 29].

1.1 Mathematical framework

To better present our results, we first introduce some terminology and notation. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain modelling the geometric configuration of the network, u :

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$\mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ a function modelling the mean membrane potential, $u(t,x)$ being the potential of a patch of tissue located at position $x \in \Omega$ at time $t \in \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a time dependent transfer function. Let also the integrable function $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the connection between locations, that is, $J(x,y)$ is the strength, or weight, of the connections of neuronal activity at location y on the activity of the neuron at location x. The strength of the connection is supposed to be symmetric, that is $J(x,y) = J(y,x)$, for any $x, y \in \mathbb{R}^N$. We also adopt a homogeneous and isotropic assumption for the layer so that, without loss of generality

$$\int_{\mathbb{R}^N} J(x,y)dy = \int_{\mathbb{R}^N} J(x,y)dx = 1.$$

We say that a neuron at a point x is active at time t if $f(t,u(t,x)) > 0$.

We thus analyze the following non-autonomous theoretical model for networks of nerve cells

$$\begin{cases} \partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}^N} Kf(t,u(t,y))dy, \ t \geqslant \tau, \ x \in \Omega, \\ u(\tau,x) = u_\tau(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

with the “boundary” condition

$$u(t,x) = 0, \quad t \geqslant \tau, \ x \in \mathbb{R}^N \backslash \Omega, \tag{1.2}$$

where the integral operator with symmetric kernel K is defined by

$$Kv(x) := \int_{\mathbb{R}^N} J(x,y)v(y)dy.$$

for all $v \in L^1(\mathbb{R}^N)$.

Also we will assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a sufficiently smooth function (some growth conditions about f are also assumed, as presented along the Section 3).

We are interested in showing existence of the pullback attractor for the evolution process associated to Cauchy problem (1.1)–(1.2) in an appropriated Banach space, as well as some of its properties such as regularity and upper semicontinuity with respect to the functional parameter f.

Our model is a generalization of the one analyzed by many authors, (e.g. [1,9,20,25,27,28]), which takes the form

$$\partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}^N} J(x,y)(f \circ u)(t,y)dy,$$

where the strength of the connection depends only on the distance between cells, that is, $J(x,y) = J(x-y)$ and the firing rate function is time-independent.

1.2 Outline of the paper

This paper is organized as follows. In Section 2 we recall some definitions from the theory of evolution process (or non-autonomous dynamical systems).

In Section 3, assuming the growth conditions (3.7), (3.8), (3.11) and (3.14), below for the nonlinearity f, we prove that (1.1)–(1.2) generates a $\mathcal{C}^1$ flow in the phase space

$$X_p = \left\{ u \in L^p(\mathbb{R}^N); \ u(x) = 0, \text{ if } x \in \mathbb{R}^N \backslash \Omega \right\} \tag{1.3}$$

with the induced norm, satisfying the "variation of constants formula"

$$u(t,x) = \begin{cases} e^{-(t-\tau)}u_\tau(x) + \displaystyle\int_\tau^t e^{-(t-s)}Kf(s,u(s,\cdot))(x)ds, & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

In Section 4, we prove existence of the pullback attractor in X_p and establish some regularity properties for it.

Finally, in section 5 we prove the upper semicontinuity of the pullback attractors with respect to the function f.

2 Functional setting and background results

In this section we recall some definitions from the theory of evolution processes (or infinite-dimensional non-autonomous dynamical systems); following [7], where full proofs and more details can be found, (see also [8, 15, 16, 22, 23], and references therein).

Definition 2.1. Let $\mathbb{X}$ be a complete metric space and $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be its metric. An evolution process in $\mathbb{X}$ is a family of maps $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ (or simply $S(\cdot,\cdot)$) from $\mathbb{X}$ into itself with the following properties:

- $S(t,t) = I$, for all $t \in \mathbb{R}$, where $I : \mathbb{X} \to \mathbb{X}$ is the identity map;
- $S(t,\tau) = S(t,s)S(s,\tau)$, for all $t \geq s \geq \tau$;
- the map $\{(t,\tau) \in \mathbb{R}^2;\ t \geq \tau\} \times \mathbb{X} \ni (t,\tau,x) \mapsto S(t,\tau)x \in \mathbb{X}$ is continuous.

Definition 2.2. A globally-defined solution (or simply a global solution) of the evolution process $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ is a function $\xi : \mathbb{R} \to \mathbb{X}$ such that for all $t \geq \tau$ we have $S(t,\tau)\xi(\tau) = \xi(t)$. A global solution $\xi : \mathbb{R} \to \mathbb{X}$ of the evolution process $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ is backward-bounded if there is a $\tau \in \mathbb{R}$ such that $\{\xi(t); t \leq \tau\}$ is a bounded subset of $\mathbb{X}$.

Definition 2.3. The subset B of $\mathbb{X}$ pullback absorbs bounded subsets of $\mathbb{X}$ at time $t \in \mathbb{R}$ under $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ if there exists $\tau_0 = \tau_0(t,D)$ with

$$S(t,\tau)D \subset B \quad \text{for any } \tau \leq \tau_0 \leq t.$$

The family $\{B(t); t \in \mathbb{R}\}$ of subsets of $\mathbb{X}$ pullback absorbs bounded sets if $B(t)$ pullback absorbs bounded sets in $\mathbb{X}$ at time t, for each $t \in \mathbb{R}$.

Definition 2.4. The subset K of $\mathbb{X}$ pullback attracts bounded subsets of $\mathbb{X}$ under $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ at time t if, for each bounded subset C of $\mathbb{X}$

$$\lim_{\tau \to -\infty} \text{dist}(S(t,\tau)C, K) = 0,$$

where $\text{dist}(\cdot,\cdot)$ denotes the Hausdorff semi-distance:

$$\text{dist}_H(A,B) = \sup_{a \in A} \inf_{b \in B} d(a,b).$$

The family $\{K(t);\ t \in \mathbb{R}\}$ of subsets of $\mathbb{X}$ pullback attracts bounded subsets of $\mathbb{X}$ under $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ if $K(t)$ pullback attracts bounded subsets of $\mathbb{X}$ at time t under the process $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$, for each $t \in \mathbb{R}$.

We observe that the Hausdorff semi-distance between A and B, $\mathrm{dist}_H(A,B)$, measures how far the set A is from being contained in the set B. For example, $\mathrm{dist}_H(A,B) = 0$ if and only if A is contained in the closure of the set B.

Now we remember the notion of an ω-limit for processes; we will build our pullback attractor as a union of ω-limit sets.

Definition 2.5. The pullback omega-limit set at time t of a subset B of $\mathbb{X}$ is defined by

$$\omega_{\wp}(B,t) := \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} S(t,\tau)B}.$$

or equivalently,

$$\omega_{\wp}(B,t) := \{y \in \mathbb{X};\ \text{there are sequences } \{\tau_k\},\ \tau \leq t,\ \tau_k \to -\infty\ k \to \infty, \text{ and } \{x_k\} \text{ in } B, \text{ such that } y = \lim_{k\to\infty} S(t,\tau_k)x_k\}.$$

Now, we introduce the central concept of pullback attractor.

Definition 2.6 (Pullback attractor)**.** A family $\{\mathcal{A}(t);\ t \in \mathbb{R}\}$ of compact subsets of $\mathbb{X}$ is said to be the pullback attractor for an evolution process $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ if it is invariant with respect to $S(\cdot,\cdot)$, i.e., $S(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geq \tau$, pullback attracts bounded subsets of $\mathbb{X}$, and is the minimal family of closed sets with property of pullback attraction, that is, if there is another family of closed sets $\{C(t);\ t \in \mathbb{R}\}$ which pullback attracts bounded subsets of $\mathbb{X}$, then $\mathcal{A}(t) \subset C(t)$, for all $t \in \mathbb{R}$.

Remark 2.7. The minimality requirement in the Definition 2.6 is an addition with respect to the theory of attractors for semigroups and is necessary to ensure uniqueness (see [7]). It can be dropped if we require that $\bigcup_{\tau\leq t} \mathcal{A}(\tau)$ is bounded for any $t \in \mathbb{R}$. In this case, we also have that each 'section' $\mathcal{A}(t)$ of the pullback attractor $\mathcal{A}(\cdot)$ of $S(\cdot,\cdot)$ satisfies

$$\mathcal{A}(t) = \{\xi(t);\ \xi : \mathbb{R} \to \mathbb{X} \text{ is a global backwards bounded solution of } S(t,\tau)\}.$$

Definition 2.8. An evolution process $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ in a Banach space $\mathbb{X}$ is pullback asymptotically compact if, for each $t \in \mathbb{R}$, each sequence $\{\tau_k\}_{k\in\mathbb{N}}$ in $(-\infty, t]$ such that $\tau_k \to -\infty$ as $k \to \infty$, and each bounded sequence $\{z_k\}_{k\in\mathbb{N}}$ in $\mathbb{X}$ with $\{S(t,\tau_k)z_k\}_{k\in\mathbb{N}}$ bounded, the sequence $\{S(t,\tau_k)z_k\}_{k\in\mathbb{N}}$ possesses a convergent subsequence.

Definition 2.9. A family of continuous operators $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ (which need not be a process) is called strongly compact if for each time t and each bounded $B \subset X$ there exists a $T_B \geq 0$ and a compact set $K \subset \mathbb{X}$ such that $S(s,\tau)B \subset K$ for all $\tau \leq s \leq t$ with $s - \tau \geq T_B$.

The following two results will be used to prove the existence of the pullback attractor for the evolution process generated by (1.1)–(1.2) in the Banach space X_p (defined in (1.3)).

Theorem 2.10. *Let $\mathbb{X}$ be a Banach space and $|\cdot|_{\mathbb{X}} : \mathbb{X} \to \mathbb{R}$ be its norm. If an evolution process $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ in $\mathbb{X}$ satisfies the properties*

$$S(t,\tau) = T(t,\tau) + U(t,\tau), \qquad t \geq \tau,$$

where $U(t,\tau)$ is a strongly compact operator and there exists a non-increasing function $k : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ with $k(\sigma,r) \to 0$ as $\sigma \to +\infty$, and for all $\tau \leq t$ and $z \in \mathbb{X}$ with $|z|_{\mathbb{X}} \leq r$, $|T(t,\tau)|_{\mathbb{X}} \leq k(t-\tau,r)$, then the process $\{S(t,\tau);\ t \geq \tau,\ \tau \in \mathbb{R}\}$ is pullback asymptotically compact.

Proof. See Theorem 2.37, Chapter 2 in [7]. □

Theorem 2.11. *If an evolution process $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ in a Banach space $\mathbb{X}$ is strongly pullback bounded dissipative and pullback asymptotically compact, then $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ possesses a compact pullback attractor $\{\mathcal{A}(t); t \in \mathbb{R}\}$. Moreover, the union $\bigcup_{\tau \leq t} \mathcal{A}(\tau)$ is bounded for each $t \in \mathbb{R}$, and each 'section' $\mathcal{A}(t)$ of the pullback attractor is given by*

$$\mathcal{A}(t) = \omega_{\wp}(\overline{\mathcal{B}(t)}, t),$$

where $\{B(t); t \in \mathbb{R}\}$ is a family of bounded subsets of $\mathbb{X}$ which for each $t \in \mathbb{R}$ pullback attracts bounded subsets of $\mathbb{X}$ at time τ, for any $\tau \leq t$.

Proof. See Theorem 2.23, Chapter 2 in [7]. □

The pullback attractor of strongly bounded dissipative process however, is always bounded in the past. To be more precise, for every $t \in \mathbb{R}$ the union $\bigcup_{\tau \leq t} \mathcal{A}(\tau)$ is bounded in $\mathbb{X}$.

3 Well-posedness of the problem

In this section we show the global well-posedness of the problem (1.1)–(1.2) in an appropriate Banach space, under suitable growth condition on the nonlinearity f.

Consider, for any $1 \leq p \leq \infty$, the subspace X_p of $L^p(\mathbb{R}^N)$ given by

$$X_p = \left\{u \in L^p(\mathbb{R}^N);\ u(x) = 0, \text{ if } x \in \mathbb{R}^N \backslash \Omega\right\}$$

with the induced norm. The Banach space X_p is canonically isometric to $L^p(\Omega)$ and we usually identify the two spaces, without further comment. We also use the same notation for a function in $\mathbb{R}^N$ and its restriction to Ω for simplicity, wherever we believe the intention is clear from the context.

In order to obtain well-posedness of (1.1)–(1.2) in X_p, we consider the Cauchy problem in the Banach space X_p

$$\begin{cases} \dfrac{du}{dt} = -u + F(t,u),\ t > \tau, \\ u(\tau) = u_\tau, \end{cases} \tag{3.1}$$

where the nonlinearity $F : \mathbb{R} \times X_p \to X_p$ is defined by

$$F(t,u)(x) = \begin{cases} Kf(t, u(t,\cdot))(x), & \text{if } t \in \mathbb{R},\ x \in \Omega, \\ 0, & \text{if } t \in \mathbb{R},\ x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.2}$$

where the map K given by

$$Kv(x) := \int_{\mathbb{R}^N} J(x,y) v(y) dy \tag{3.3}$$

is well defined as a bounded linear operator in various function spaces, depending on the properties assumed for J; for example, with J satisfying the hypotheses from introduction, K is well defined in X_p as shown below.

The following simple estimates will be useful in the sequel.

Lemma 3.1. *Let K be the map defined by (3.3) and $\|J\|_r := \sup_{x\in\Omega} \|J(x,\cdot)\|_{L^r(\Omega)}$, $1 \le r \le \infty$. If $u \in L^p(\Omega)$, $1 \le p \le \infty$, then $Ku \in L^\infty(\Omega)$, and*

$$|Ku(x)| \le \|J\|_q \|u\|_{L^p(\Omega)} \quad \text{for all } x \in \Omega, \tag{3.4}$$

where $1 \le q \le \infty$ is the conjugate exponent of p. Moreover,

$$\|Ku\|_{L^p(\Omega)} \le \|J\|_1 \|u\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)}. \tag{3.5}$$

If $u \in L^1(\Omega)$, then $Ku \in L^p(\Omega)$, $1 \le p \le \infty$, and

$$\|Ku\|_{L^p(\Omega)} \le \|J\|_p \|u\|_{L^1(\Omega)}. \tag{3.6}$$

Proof. Estimate (3.4) follows easily from Hölder's inequality. Estimate (3.5) follows from the generalized Young's inequality (see [12]). The proof of (3.6) is similar to (3.5), but we include it here for the sake of completeness. Suppose $1 < p < \infty$ and let q be its the conjugate exponent. Then, by Hölder's inequality

$$\begin{aligned}
|Ku(x)| &\le \left[\int_\Omega |J(x,y)u(y)^{\frac{1}{p}} u(y)^{\frac{1}{q}}|\, dy\right] \\
&\le \left[\int_\Omega |J(x,y)|^p |u(y)|\, dy\right]^{\frac{1}{p}} \left[\int_\Omega |u(y)|\, dy\right]^{\frac{1}{q}} \\
&\le \|u\|_{L^1(\Omega)}^{\frac{1}{q}} \left[\int_\Omega |J(x,y)|^p |u(y)|\, dy\right]^{\frac{1}{p}}.
\end{aligned}$$

Raising both sides to the p-th power and integrating, we obtain

$$\begin{aligned}
\int_\Omega |Ku(x)|^p\, dx &\le \|u\|_{L^1(\Omega)}^{\frac{p}{q}} \left[\int_\Omega \int_\Omega |J(x,y)|^p |u(y)|\, dx\, dy\right] \\
&\le \|u\|_{L^1(\Omega)}^{\frac{p}{q}} \left[\int_\Omega |u(y)| \int_\Omega |J(x,y)|^p\, dx\, dy\right] \\
&\le \|u\|_{L^1(\Omega)}^{\frac{p}{q}} \|u\|_{L^1(\Omega)} \|J\|_p^p \\
&\le \|u\|_{L^1(\Omega)}^{\frac{p+q}{q}} \|J\|_p^p.
\end{aligned}$$

The inequality (3.6) then follows by taking p-th roots.

The case $p = 1$ is similar but easier, and the case $p = \infty$ is trivial. □

Definition 3.2. If E is a normed space, and $I \subset \mathbb{R}$ is an interval, we say that a function $F : I \times E \to E$ is *locally Lipschitz continuous (or simply locally Lipschitz)* in the second variable if, for any $(t_0, x_0) \in I \times E$, there exists a constant C and a rectangle $R = \{(t,x) \in I \times E : |t - t_0| < b_1, \|x - x_0\| < b_2\}$ such that, if (t,x) and (t,y) belong to R, then

$$\|F(t,x) - F(t,y)\| \le C\|x - y\|.$$

Now we prove that the map F, given in (3.2), is well defined under appropriate growth conditions on f and is locally Lipschitz continuous (see Proposition 3.3).

Proposition 3.3. *Suppose, in addition to the hypotheses of Lemma 3.1, that the function f satisfies the growth condition*

$$|f(t,x)| \leq C_1(t)(1+|x|^p), \quad \text{for any } (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.7}$$

with $1 \leq p < \infty$, where $C_1 : \mathbb{R} \to \mathbb{R}$ is a locally bounded function. Then the function F given by (3.2) is well defined in $\mathbb{R} \times X_p$. If $f(t,\cdot)$ is locally bounded for any $t \in \mathbb{R}$, F is well defined in $\mathbb{R} \times L^\infty(\Omega)$.

Additionally, if f is continuous in the first variable,then F is also continuous in the first variable. If there exists a strictly positive function $C_2 : \mathbb{R} \to \mathbb{R}$ such that

$$|f(t,x) - f(t,y)| \leq C_2(t)(1+|x|^{p-1}+|y|^{p-1})|x-y|, \quad \text{for any } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,\ t \in \mathbb{R}, \tag{3.8}$$

then, for any $1 \leq p < \infty$ the function F is locally Lipschitz continuous in the second variable If $p = \infty$, this is true if f is locally Lipschitz in the second variable.

Proof. Initially, suppose $1 \leq p < \infty$. Let $u \in L^p(\Omega)$. We will use, henceforth, the notation $f(t,u)$ for the function $f(t,u)(x) = f(t,u(x))$. We have, for each $t \in \mathbb{R}$, from (3.7)

$$\begin{aligned} \|f(t,u)\|_{L^1(\Omega)} &\leq \int_\Omega C_1(t)(1+|u(x)|^p)dx \\ &\leq C_1(t)\left(|\Omega| + \|u\|^p_{L^p(\Omega)}\right). \end{aligned} \tag{3.9}$$

From estimates (3.6) and (3.9), it follows that

$$\begin{aligned} \|F(t,u)\|_{L^p(\Omega)} &\leq \|Kf(t,u)\|_{L^p(\Omega)} \\ &\leq C_1(t)\|J\|_p \|f(t,u)\|_{L^1(\Omega)} \\ &\leq C_1(t)\|J\|_p\left(|\Omega| + \|u\|^p_{L^p(\Omega)}\right), \end{aligned}$$

showing that F is well defined.

If $f(t,x)$ is also continuous in t, then for any $(t,u) \in \mathbb{R} \times X_p$ we have

$$\|f(t,u) - f(t+h,u)\|_{L^1(\Omega)} \leq \int_\Omega |f(t,u(x)) - f(t+h,u(x))|dx \tag{3.10}$$

for a small $h \in \mathbb{R}$. From (3.7), the integrand is bounded by $2C(1+|u(x)|^p)$, where C is a bound for $C(t)$ in a neighborhood of t and goes to 0 as $h \to 0$. Therefore, by Lebesgue's dominated convergence theorem, $\|f(t,u) - f(t+h,u)\|_{L^1(\Omega)} \to 0$ as $h \to 0$. Thus

$$\begin{aligned} \|F(t+h,u) - F(t,u)\|_{L^p(\Omega)} &\leq \|K(f(t+h,u) - f(t,u)\|_{L^p(\Omega)} \\ &\leq \|J\|_p \|f(t+h,u) - f(t,u)\|_{L^1(\Omega)} \end{aligned}$$

which goes to 0 as $h \to 0$, proving the continuity of F in t.

Suppose now that

$$|f(t,x) - f(t,y)| \leq C_2(t)(1+|x|^{p-1}+|y|^{p-1})|x-y|,$$

for some $1 < p < \infty$, where $C_2 : \mathbb{R} \to \mathbb{R}$ is a strictly positive function. Then, for u and v belonging to $L^p(\Omega)$ we get

$$\begin{aligned} \|f(t,u) - f(t,v)\|_{L^1(\Omega)} &\leq \int_\Omega C_2(t)(1+|u|^{p-1}+|v|^{p-1})|u-v|\,dx \\ &\leq C_2(t)\left[\int_\Omega (1+|u|^{p-1}+|v|^{p-1})^q dx\right]^{\frac{1}{q}} \left[\int_\Omega |u-v|^p dx\right]^{\frac{1}{p}} \\ &\leq C_2(t)\left[\|1\|_{L^q(\Omega)} + \|u^{p-1}\|_{L^q(\Omega)} + \|v^{p-1}\|_{L^q(\Omega)}\right]\|u-v\|_{L^p(\Omega)} \\ &\leq C_2(t)\left[|\Omega|^{\frac{1}{q}} + \|u\|^{\frac{p}{q}}_{L^p(\Omega)} + \|v\|^{\frac{p}{q}}_{L^p(\Omega)}\right]\|u-v\|_{L^p(\Omega)}, \end{aligned}$$

where q is the conjugate exponent of p.

Using (3.6) once again and the hypothesis on f, it follows that

$$\begin{aligned}\|F(t,u)-F(t,v)\|_{L^p(\Omega)} &\le \|K(f(t,u)-f(t,v))\|_{L^p(\Omega)}\\ &\le \|J\|_p\|f(t,u)-f(t,v)\|_{L^1(\Omega)}\\ &\le C_2(t)\|J\|_p\left[|\Omega|^{\frac{1}{q}}+\|u\|^{\frac{p}{q}}_{L^p(\Omega)}+\|v\|^{\frac{p}{q}}_{L^p(\Omega)}\right]\|u-v\|_{L^p(\Omega)},\end{aligned}$$

showing that F is Lipschitz in bounded sets of $L^p(\Omega)$ as claimed.

If $p=1$, the proof is similar, but simpler. Suppose finally that $\|u\|_{L^\infty(\Omega)}\le R$, $\|v\|_{L^\infty(\Omega)}\le R$ and let M be the Lipschitz constant of f in the interval $[-R,R]\subset\mathbb{R}$. Then

$$|f(t,u(x))-f(t,v(x))|\le M|u(x)-v(x)|,\quad \text{for any } x\in\Omega,$$

and this allows us to conclude that

$$\|f(t,u)-f(t,v)\|_{L^\infty(\Omega)}\le M\|u-v\|_{L^\infty(\Omega)}.$$

Thus, by (3.5) we have that

$$\begin{aligned}\|F(t,u)-F(t,v)\|_{L^\infty(\Omega)} &\le \|K(f(t,u)-f(t,v))\|_{L^\infty(\Omega)}\\ &\le M\|J\|_1\|u-v\|_{L^\infty(\Omega)},\end{aligned}$$

and this completes the proof. □

From Proposition 3.3, and well known results, it follows that the initial value problem (3.1) has a unique local solution for any initial condition in X_p. For the global existence, we need the following result (see [18, Theorem 5.6.1]).

Theorem 3.4. *Let X be a Banach space, and suppose that $\mathcal{G}:[t_0,+\infty)\times X\to X$ is continuous and*

$$\|\mathcal{G}(t,u)\|\le g(t,\|u\|),\quad \textit{for all } (t,u)\in[t_0,+\infty)\times X,$$

where $g:[t_0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous and $g(t,r)$ is non decreasing in $r\ge 0$, for each $t\in[t_0,+\infty)$. Then, if the maximal solution $r(t;t_0,r_0)$ of the scalar initial value problem

$$\begin{cases}\dfrac{dr}{dt}=g(t,r), & t>t_0,\\ r(t_0)=r_0,\end{cases}$$

exists throughout $[t_0,+\infty)$, the maximal interval of existence of any solution $u(t;t_0,y_0)$ of the initial value problem

$$\begin{cases}\dfrac{du}{dt}=\mathcal{G}(t,u), & t>t_0,\\ u(t_0)=u_0,\end{cases}$$

also contains $[t_0,+\infty)$.

Corollary 3.5. *Suppose, in addition to the hypotheses of Proposition 3.3, that f satisfies the dissipative condition*

$$\limsup_{|x|\to\infty}\frac{|f(t,x)|}{|x|}<k_1, \tag{3.11}$$

for some constant $k_1 \in \mathbb{R}$, independent of t. Then the problem (3.1) has a unique globally defined solution for any initial condition in X, which is given, for $t \geq \tau$, by the "variation of constants formula"

$$u(t,x) = e^{-(t-\tau)}u_\tau(x) + \int_\tau^t e^{-(t-s)}F(s,u(s,x))ds, \qquad t \geq \tau,\ x \in \mathbb{R}^N,$$

that is,

$$u(t,x) = \begin{cases} e^{-(t-\tau)}u_\tau(x) + \displaystyle\int_\tau^t e^{-(t-s)}Kf(s,u(s,\cdot))(x)\,ds, & t \geq \tau,\ x \in \Omega, \\ 0, & t \geq \tau,\ x \in \mathbb{R}^N \backslash \Omega. \end{cases} \tag{3.12}$$

Proof. From Proposition 3.3, it follows that the right-hand-side of (3.1) is Lipschitz continuous in bounded sets of X and, therefore, the Cauchy problem (3.1) is well posed in X_p, with a unique local solution $u(t,x)$, given by (3.12) (see [10]).

From condition (3.11) it follows that

$$|f(t,x)| \leq k_2(t) + k_1|x|, \quad \text{for any } (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.13}$$

where $k_2 : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly positive function.

If $1 \leq p < \infty$, we obtain from (3.5) and (3.13) the following estimate

$$\begin{aligned} \|Kf(t,u)\|_{L^p(\Omega)} &\leq \|f(t,u)\|_{L^p(\Omega)} \\ &\leq k_2(t)|\Omega|^{1/p} + k_1\|u\|_{L^p(\Omega)}. \end{aligned}$$

For $p = \infty$, we obtain by the same arguments (or by making $p \to \infty$), that

$$\|Kf(t,u)\|_{L^\infty(\Omega)} \leq k_2(t) + k_1\|u\|_{L^\infty(\Omega)}.$$

Now defining the function

$$g : [t_0, \infty) \times \mathbb{R}^+ \to \mathbb{R}^+, \qquad (t,r) \mapsto g(t,r) = |\Omega|^{1/p}k_2(t) + (k_1+1)r$$

it follows that problem (3.1) satisfies the hypothesis of Theorem 3.4 and the global existence follows immediately. The variation of constants formula can be verified by direct derivation. □

The result below can be found in [19].

Proposition 3.6. *Let Y and Z be normed linear spaces, $F : Y \to Z$ a map and suppose that the Gâteaux derivative of F, $DF : Y \to \mathcal{L}(Y,Z)$ exists and is continuous at $y \in Y$. Then the Fréchet derivative F' of F exists and is continuous at y.*

Proposition 3.7. *Suppose, in addition to the hypotheses of Corollary 3.5 that the function f is continuously differentiable in the second variable and $\partial_2 f$ satisfies the growth condition*

$$|\partial_2 f(t,x)| \leq C_1(t)(1+|x|^{p-1}), \quad \text{for any } (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.14}$$

if $1 \leq p < \infty$. Then $F(t,\cdot)$ is continuously Fréchet differentiable on X_p with derivative given by

$$DF(t,u)v(x) := \begin{cases} K(\partial_2 f(t,u)v)(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

Proof. From a simple computation, using the fact f is continuously differentiable in the second variable, it follows that the Gâteaux's derivative of $F(t,\cdot)$ is given by

$$DF(t,u)v(x) := \begin{cases} K(\partial_2 f(t,u)v)(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $(\partial_2 f(t,u)v)(x) := \partial_2 f(t,u(x)) \cdot v(x)$. The operator $D_2F(t,u)$ is clearly a linear operator in X_p.

Suppose $1 \leq p < \infty$ and q is the conjugate exponent of p. Then, for $u \in L^p(\Omega)$ we have that

$$\begin{aligned} \|\partial_2 f(t,u)\|_{L^q(\Omega)} &\leq \left\{ \int_\Omega [C_1(t)(1+|u|^{p-1})]^q \, dx \right\}^{\frac{1}{q}} \\ &\leq C_1(t)|\Omega|^{\frac{1}{q}} + C_1(t) \left\{ \int_\Omega |u|^p \, dx \right\}^{\frac{1}{q}} \\ &= C_1(t) \left(|\Omega|^{\frac{1}{q}} + \|u\|_{L^p}^{\frac{p}{q}} \right) \\ &= C_1(t) \left(|\Omega|^{\frac{1}{q}} + \|u\|_{L^p(\Omega)}^{p-1} \right). \end{aligned} \tag{3.15}$$

From Hölder's inequality and (3.15), it follows that

$$\|\partial_2 f(t,u) \cdot v\|_{L^1(\Omega)} \leq C_1(t)(|\Omega|^{\frac{1}{q}} + \|u\|_{L^p(\Omega)}^{p-1})\|v\|_{L^p(\Omega)}.$$

Now from estimate (3.6) we concluded that

$$\begin{aligned} \|DF(t,u) \cdot v\|_{L^p(\Omega)} &\leq \|K(\partial_2 f(t,u)v)\|_{L^p(\Omega)} \\ &\leq C_1(t)\|J\|_p \|\partial_2 f(t,u)v\|_{L^1(\Omega)} \\ &\leq C_1(t)\|J\|_p (|\Omega|^{\frac{1}{q}} + \|u\|_{L^p(\Omega)}^{p-1})\|v\|_{L^p(\Omega)}, \end{aligned}$$

showing that $DF(t,u)$ is a bounded operator. In the case $p = \infty$, we have that $|\partial_2 f(t,u)|$ is bounded by $C_2(t)$, for each $u \in L^\infty(\Omega)$. Therefore

$$\|\partial_2 f(t,u)v\|_{L^\infty(\Omega)} \leq C_2(t)\|v\|_{L^\infty(\Omega)}$$

and thus, from (3.5), we obtain

$$\begin{aligned} \|DF(t,u) \cdot v\|_{L^\infty(\Omega)} &\leq \|K(\partial_2 f(t,u)v)\|_{L^\infty(\Omega)} \\ &\leq \|J\|_1 \|\partial_2 f(t,u)v\|_{L^\infty(\Omega)} \\ &\leq C_2(t)\|J\|_1 \|v\|_{L^\infty(\Omega)} \end{aligned}$$

showing the boundedness of $DF(t,u)$ also in this case.

Suppose now that u_1 and u_2 and v belong to $L^p(\Omega)$, $1 \leq p < \infty$. From (3.6) and Hölder's inequality, it follows that

$$\begin{aligned} \|(DF(t,u_1) - DF(t,u_2))v\|_{L^p(\Omega)} &\leq \|K[(\partial_2 f(t,u_1) - \partial_2 f(t,u_2))v)]\|_{L^p(\Omega)} \\ &\leq \|J\|_p \|(\partial_2 f(t,u_1) - \partial_2 f(t,u_2))v\|_{L^1(\Omega)} \\ &\leq \|J\|_p \|\partial_2 f(t,u_1) - \partial_2 f(t,u_2)\|_{L^q(\Omega)} \|v\|_{L^p(\Omega)}. \end{aligned}$$

Thus to prove continuity of the derivative, we only have to show that

$$\|\partial_2 f(t,u_1) - \partial_2 f(t,u_2)\|_{L^q(\Omega)} \to 0$$

as $\|u_1 - u_2\|_{L^p(\Omega)} \to 0$. Now, from the growth condition we obtain

$$|\partial_2 f(t,u_1)(x) - \partial_2 f(t,u_2)(x)|^q \leq [C_1(t)(2 + |u_1(x)|^{p-1} + |u_2(x)|^{p-1})]^q$$

and a computation similar to (3.15) above shows that the right-hand side is integrable. The result then follows from Lebesgue's convergence theorem.

In the case $p = \infty$, we obtain from (3.5)

$$\begin{aligned} \|(DF(t,u_1) - DF(t,u_2))v\|_{L^\infty(\Omega)} &\leq \|K[(\partial_2 f(t,u_1) - \partial_2 f(t,u_2))v)]\|_{L^\infty(\Omega)} \\ &\leq \|J\|_1 \|\partial_2 f(t,u_1) - \partial_2 f(t,u_2)\|_{L^\infty(\Omega)} \|v\|_{L^\infty(\Omega)} \end{aligned}$$

and the continuity of DF follows from the continuity of $\partial_2 f(t,u)$.

Therefore, it follows from Proposition 3.6 above that $F(t,\cdot)$ is Fréchet differentiable with continuous derivative in X_p. □

Remark 3.8. Since, under the hypotheses of the Proposition 3.7 the right-hand side of (3.1) is continuous in t and $\mathcal{C}^1$ in the second variable, the process generated by (3.1) in X_p is $\mathcal{C}^1$ with respect to initial conditions, (see [10] and [13]).

From the results above, we have that, for each $t \in \mathbb{R}$ and $u_\tau \in X_p$, the unique solution of (3.1) with initial condition u_τ exists for all $t \geq \tau$ and this solution $(t,\tau,x) \mapsto u(t,x) = u(t;\tau,x,u_\tau)$ (defined by (3.12)) gives rise to a family of nonlinear $\mathcal{C}^1$ flow on X_p given by

$$S(t,\tau)u_\tau(x) := u(t,x), \qquad t \geq \tau \in \mathbb{R}.$$

4 Existence and regularity of the pullback attractor for $1 \leq p < \infty$

We prove the existence of a pullback attractor $\{\mathcal{A}(t); t \in \mathbb{R}\}$ in X_p for the evolution process $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ when $1 \leq p < \infty$.

Lemma 4.1. *Suppose that the hypotheses of Proposition 3.7 hold with the constant k_1 in (3.11) satisfying $k_1 < 1$. Then the ball of $L^p(\Omega)$, $1 \leq p < \infty$, centered at the origin with radius $R_\delta(t)$ defined by*

$$R_\delta(t) = \frac{1}{1-k_1}(1+\delta)k_2(t)|\Omega|^{\frac{1}{p}}, \tag{4.1}$$

which we denote by $\mathcal{B}(0, R_\delta(t))$, where k_1 and k_2 are derived from (3.13) and δ is any positive constant, pullback absorbs bounded subsets of X_p at time $t \in \mathbb{R}$ with respect to the process $S(\cdot,\cdot)$ generated by (3.1).

Proof. If $u(t,x)$ is the solution of (3.1) with initial condition u_τ then, for $1 \leq p < \infty$

$$\begin{aligned} \frac{d}{dt}\int_\Omega |u(t,x)|^p dx &= \int_\Omega p|u(t,x)|^{p-1}\mathrm{sgn}(u(t,x))u_t(t,x)dx \\ &= -p\int_\Omega |u|^p(t,x)dx + p\int_\Omega |u(t,x)|^{p-1}\mathrm{sgn}(u(t,x))Kf(t,u(t,x))dx. \end{aligned} \tag{4.2}$$

Using Hölder's inequality, estimate (3.5) and condition (3.11), we obtain

$$\begin{aligned}
&\int_\Omega |u(t,x)|^{p-1}\mathrm{sgn}(u(t,x))Kf(t,u(t,x))dx \\
&\leq \left(\int_\Omega |u(t,x)|^{q(p-1)}dx\right)^{\frac{1}{q}} \left(\int_\Omega |Kf(t,u(t,x))|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega |u(t,x)|^p dx\right)^{\frac{1}{q}} \|J\|_1 \|f(t,u(t,\cdot))\|_{L^p(\Omega)} \\
&\leq \|u(t,\cdot)\|^{p-1}_{L^p(\Omega)} \left(k_1\|u(t,\cdot)\|_{L^p(\Omega)} + k_2(t)|\Omega|^{\frac{1}{p}},\right)
\end{aligned} \tag{4.3}$$

where q is the conjugate exponent of p.

Hence, combining (4.2) with (4.3) we concluded that

$$\begin{aligned}
\frac{d}{dt}\|u(t,\cdot)\|^p_{L^p(\Omega)} &\leq -p\|u(t,\cdot)\|^p_{L^p(\Omega)} + pk_1\|u(t,\cdot)\|^p_{L^p(\Omega)} + pk_2(t)|\Omega|^{\frac{1}{p}}\|u(t,\cdot)\|^{p-1}_{L^p(\Omega)} \\
&= p\|u(t,\cdot)\|^p_{L^p(\Omega)}\left[-1 + k_1 + \frac{k_2(t)|\Omega|^{\frac{1}{p}}}{\|u(t,\cdot)\|_{L^p(\Omega)}}\right].
\end{aligned}$$

Let $\varepsilon = 1 - k_1 > 0$. Then, while

$$\|u(t,\cdot)\|_{L^p(\Omega)} \geq \frac{1}{\varepsilon}(1+\delta)(k_2(t)|\Omega|^{\frac{1}{p}}),$$

we have

$$\begin{aligned}
\frac{d}{dt}\|u(t,\cdot)\|^p_{L^p(\Omega)} &\leq p\|u(t,\cdot)\|^p_{L^p(\Omega)}\left(-\varepsilon + \frac{\varepsilon}{1+\delta}\right) \\
&= -\frac{\delta\varepsilon p}{1+\delta}\|u(t,\cdot)\|^p_{L^p(\Omega)}.
\end{aligned}$$

Therefore, while

$$\|u(t,\cdot)\|_{L^p(\Omega)} \geq \frac{1}{1-k_1}(1+\delta)(k_2(t)|\Omega|^{\frac{1}{p}}),$$

we have

$$\begin{aligned}
\|u(t,\cdot)\|^p_{L^p(\Omega)} &\leq e^{-\frac{\delta\varepsilon p}{(1+\delta)}(t-\tau)}\|u_\tau\|^p_{L^p(\Omega)} \\
&= e^{-\frac{\delta p}{(1+\delta)}(1-k_1)(t-\tau)}\|u_\tau\|^p_{L^p(\Omega)}.
\end{aligned} \tag{4.4}$$

From this, the result follows easily for $1 \leq p < \infty$, and this complete the proof of the lemma. □

Theorem 4.2. *In addition to the conditions of Lemma 4.1, suppose that*

$$\|J_x\|_{L^p(\Omega)} = \sup_{x\in\Omega}\|\partial_x J(x,\cdot)\|_{L^q(\Omega)} < \infty.$$

Then there exists a pullback attractor $\{\mathcal{A}(t); t \in \mathbb{R}\}$ for the process $\{S(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ generated by (3.1) in $X = L^p(\Omega)$ and the 'section' $\mathcal{A}(t)$ of the pullback attractor $\mathcal{A}(\cdot)$ of $S(\cdot,\cdot)$ is contained in the ball centered at the origin with radius $R_\delta(t)$ defined in (4.1), in $L^p(\Omega)$, for any $\delta > 0$, $t \in \mathbb{R}$ and $1 \leq p < \infty$.

Proof. We have proved that for each initial value $u(\tau, x) \in X$ and initial time $\tau \in \mathbb{R}$, (3.1) possesses a unique solution, which we now write as

$$S(t,\tau)u(\tau,x) = T(t,\tau)u(\tau,x) + U(t,\tau)u(\tau,x),$$

where from (3.12) we have that

$$T(t,\tau)u(\tau,x) := e^{-(t-\tau)}u(\tau,x),$$

and

$$U(t,\tau)u(\tau,x) := \int_\tau^t e^{-(t-s)}Kf(s,u(s,x))ds.$$

Now, using Theorem 2.10 (or Theorem 2.37, Chapter 2 in [7]), we prove that $S(\cdot,\cdot)$ is pullback asymptotically compact. For this, suppose $u \in B$, where B is a bounded subset of X_p. We may suppose that B is contained in the ball centered at the origin of radius $r > 0$. Then

$$\|T(t,\tau)u\|_{L^p(\Omega)} \le re^{-(t-\tau)}, \qquad t \ge \tau.$$

From (4.4), we have that $\|u(t,\cdot)\|_{L^p(\Omega)} \le M$, for $t \ge \tau$, where

$$M = \max\left\{ r, \frac{2k_2(t)|\Omega|^{\frac{1}{p}}}{1-k_1} \right\} > 0.$$

Hence, using (3.9), we obtain

$$\begin{aligned}\|f(t,u)\|_{L^1(\Omega)} &\le C_1(t)(|\Omega| + \|u\|^p_{L^p(\Omega)})\\ &\le C_1(t)(|\Omega| + M^p).\end{aligned}$$

From estimate (3.6) (applied to J_x in the place of J) it follows that

$$\begin{aligned}\|\partial_x Kf(t,u)\|_{L^p(\Omega)} &\le \|J_x\|_{L^p(\Omega)}\|f(t,u)\|_{L^1(\Omega)}\\ &\le C_1(t)\|J_x\|_{L^p(\Omega)}(|\Omega| + M^p).\end{aligned}$$

Thus, we get

$$\begin{aligned}\|\partial_x U(t,\tau)u\|_{L^p(\Omega)} &\le \int_\tau^t e^{-(t-s)}\|\partial_x Kf(s,u(s,\cdot))\|_{L^p(\Omega)}ds\\ &\le C_1(t)\|J_x\|_p(|\Omega| + M^p).\end{aligned} \tag{4.5}$$

Therefore, for $t > \tau$ and any $u \in B$, the value of $\|\partial_x U(t,\tau)u\|_{L^p(\Omega)}$ is bounded by a constant (independent of $u \in B$). It follows that $U(t,\tau)u$ belongs to a ball of $W^{1,p}(\Omega)$ for all $u \in B$. From Sobolev's Embedding Theorem, it follows that $U(t,\tau)$ is a compact operator, for any $t > \tau$.

Therefore it follows from Lemma 4.1 and Theorem 2.11 (or Theorem 2.23, Chapter 2 in [7]), that the pullback attractor $\{\mathcal{A}(t); t \in \mathbb{R}\}$ exists and each 'section' $\mathcal{A}(t)$ of the pullback attractor $\mathcal{A}(\cdot)$ is the pullback ω-limit set of any bounded subset of X_p containing the ball centered at the origin with radius R_δ, defined in (4.1), for any $\delta > 0$. From this, since the ball centered at the origin with radius R_δ pullback absorbs bounded subsets of X_p, it also follows that the set $\mathcal{A}(t)$ is contained in the ball centered at the origin of radius

$$\frac{k_2(t)|\Omega|^{\frac{1}{p}}}{1-k_1}$$

in $L^p(\Omega)$, for any $t \in \mathbb{R}$, $1 \le p < \infty$. $\square$

Theorem 4.3. *Assume the same conditions as in Theorem 4.2. Then there exists a bounded set of $W^{1,p}(\Omega)$, $1 \leq p < \infty$ containing the 'section' $\mathcal{A}(t)$ of the pullback attractor $\mathcal{A}(\cdot)$ of $S(\cdot,\cdot)$.*

Proof. From Theorem 4.2, we obtain that $\mathcal{A}(t)$ is contained in the ball centered at the origin and radius

$$\frac{k_2(t)|\Omega|^{\frac{1}{p}}}{1-k_1}$$

in $L^p(\Omega)$. Now, if $u(t,x)$ is a solution of (3.1) such that $u(\tau,x) \in \mathcal{A}(t)$ for all $t \in \mathbb{R}$, then

$$u(t,x) = \int_{-\infty}^{t} e^{-(t-s)} Kf(s,u(s,x))ds,$$

where the equality above is in the sense of $L^p(\mathbb{R}^N)$.

Proceeding as in the proof of the Theorem 4.2 (see estimate (4.5)), we obtain

$$\begin{aligned} \|\partial_x u(t,\cdot)\|_{L^p(\Omega)} &\leq \int_{-\infty}^{t} e^{-(t-s)} \|\partial_x Kf(s,u(s,\cdot))\|_{L^p(\Omega)} ds \\ &\leq \int_{-\infty}^{t} \|J_x\|_{L^p(\Omega)} \|f(s,u(s,\cdot))\|_{L^1(\Omega)} ds \\ &\leq C_1(t)\|J_x\|_{L^p(\Omega)}(|\Omega| + M^p), \end{aligned}$$

where now $M = \frac{2k_2(t)|\Omega|^{\frac{1}{p}}}{1-k_1}$.

It follows that $\mathcal{A}(t) = S(t,\tau)\mathcal{A}(\tau)$ is in a bounded set of $W^{1,p}(\Omega)$, as claimed. □

5 Upper semicontinuity of the pullback attractors for $1 \leq p < \infty$

In this section we will consider a sequence $\{f_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ of nonlinearities, $f_n : \mathbb{R}^2 \to \mathbb{R}$ satisfying the hypotheses of the Lemma 4.1 with f_n being locally Lipschitz continuous in the second variable with Lipschitz constant L_n such that

$$\ell := \limsup_{n\to\infty} L_n < \infty. \tag{5.1}$$

Let $\{S_n(t,\tau); t \geq \tau, \tau \in \mathbb{R}\}$ be the sequence of processes associated with the family of problems

$$\begin{cases} \partial_t u_n(t,x) = -u_n(t,x) + Kf_n(t,u(t,x)),\ t \geq \tau,\ x \in \Omega, \\ u_n(\tau,x) = u_\tau(x),\ x \in \Omega, \end{cases} \tag{5.2}$$

with

$$u_n(t,x) = 0, \qquad t \geq \tau,\ x \in \mathbb{R}^N \backslash \Omega. \tag{5.3}$$

In this section $\{\mathcal{A}_n(t); t \in \mathbb{R}\}$ denotes the pullback attractor for the process $S_n(\cdot,\cdot)$ for $n \in \mathbb{N} \cup \{\infty\}$.

Theorem 5.1. *Let $\{f_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence of nonlinearities $f_n : \mathbb{R}^2 \to \mathbb{R}$ satisfying the hypotheses of the Lemma 4.1. Moreover assume that*

$$f_n(t,\cdot) \text{ converges to } f_\infty(t,\cdot) \text{ in } X_p, \text{ as } n \to \infty.$$

If $S_n(\cdot,\cdot)$ denotes the process generates by the problem (5.2)–(5.3) for $n \in \mathbb{N} \cup \{\infty\}$. Then we have that

$$S_n(t,\tau)u_\tau \text{ converges to } S_\infty(t,\tau)u_\tau \text{ in } X_p, \text{ as } n \to \infty,$$

uniformly for $t \in [\tau, T]$, for any $T > \tau$.

Proof. Let $T > \tau$ and $u_n(t,x) = S_n(t,\tau)u_\tau(x)$ be the solution of the problem (5.2)–(5.3) for $t \in [\tau, T]$, given by (3.12). Then

$$u_n(t,x) - u_\infty(t,x) = \begin{cases} \displaystyle\int_\tau^t e^{-(t-s)} K(f_n(s,u_n(s,x)) - f_\infty(s,u_\infty(s,x)))ds, & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

It follows from Jensen's inequality and (3.5) that

$$\begin{aligned} \|u_n(t,\cdot) - u_\infty(t,\cdot)\|_{L^p(\Omega)} &\leq \int_\tau^t e^{-(t-s)} \|K(f_n(s,u_n(s,\cdot)) - f_\infty(s,u_\infty(s,\cdot)))\|_{L^p(\Omega)} ds \\ &\leq \int_\tau^t e^{-(t-s)} \|f_n(s,u_n(s,\cdot)) - f_\infty(s,u_\infty(s,\cdot))\|_{L^p(\Omega)} ds \\ &\leq \int_\tau^t e^{-(t-s)} \|f_n(s,u_n(s,\cdot)) - f_n(s,u_\infty(s,\cdot))\|_{L^p(\Omega)} ds \\ &\quad + \int_\tau^t e^{-(t-s)} \|f_n(s,u_\infty(s,\cdot)) - f_\infty(s,u_\infty(s,\cdot))\|_{L^p(\Omega)} ds. \end{aligned}$$

Let B a bounded subset of X_p such that $u_n(t,\cdot) \in B$ for all n and $t \in [\tau, T]$. Using (5.1), we have for n sufficiently large

$$\begin{aligned} &\int_\tau^t e^{-(t-s)} \|f_n(s,u_n(s,\cdot)) - f_n(s,u_\infty(s,\cdot))\|_{L^p(\Omega)} ds \\ &\qquad \leq \ell \int_\tau^t e^{-(t-s)} \|u_n(s,\cdot) - u_\infty(s,\cdot)\|_{L^p(\Omega)} ds, \end{aligned} \tag{5.4}$$

Now, for any $\varepsilon > 0$, we obtain

$$\int_\tau^t e^{-(t-s)} \|f_n(s,u_\infty(s,\cdot)) - f_\infty(s,u_\infty(s,\cdot))\|_{L^p(\Omega)} ds < \varepsilon, \tag{5.5}$$

if n is sufficiently large.

Combining (5.4) with (5.5) we conclude that

$$\|u_n(t,\cdot) - u_\infty(t,\cdot)\|_{L^p(\Omega)} \leq \varepsilon + \ell \int_\tau^t e^{-(t-s)} \|u_n(s,\cdot) - u_\infty(s,\cdot)\|_{L^p(\Omega)} ds,$$

for n sufficiently large and then, by Gronwall's inequality we get

$$\|u_n(t,\cdot) - u_\infty(t,\cdot)\|_{L^p(\Omega)} \leq \varepsilon e^{\ell t},$$

for $t \in [\tau, T]$ and n sufficiently large. □

For each value of the parameter $n \in \mathbb{N}$ we recall that $S_n(\cdot,\cdot)$ is the evolution process associated to problem (5.2)–(5.3). Now we prove the main result of this section.

Theorem 5.2. *Under same hypotheses of Theorem 5.1 the family of pullback attractors $\{\mathcal{A}_n(t); t \in \mathbb{R}\}_{n \in \mathbb{N} \cup \{\infty\}}$ is upper-semicontinuous in ∞.*

Proof. Note that, using the invariance of attractors, for each $t \geq \tau$, we have

$$\begin{aligned} &\mathrm{dist}_H(\mathcal{A}_n(t), \mathcal{A}_\infty(t)) \\ &\quad \leq \mathrm{dist}_H(S_n(t,\tau)\mathcal{A}_n(\tau), S_\infty(t,\tau)\mathcal{A}_n(\tau)) + \mathrm{dist}_H(S_\infty(t,\tau)\mathcal{A}_n(\tau), S_\infty(t,\tau)\mathcal{A}_\infty(\tau)) \\ &\quad = \sup_{a_n \in \mathcal{A}_n(\tau)} \mathrm{dist}(S_n(t,\tau)a_n, S_\infty(t,\tau)a_n) + \mathrm{dist}_H(S_\infty(t,\tau)\mathcal{A}_n(\tau), \mathcal{A}_\infty(t)). \end{aligned}$$

For each $\varepsilon > 0$, by the Theorem 5.1 there exists $n_{\varepsilon_1} \in \mathbb{N}$ such that

$$\sup_{a_n \in \mathcal{A}_n(\tau)} \operatorname{dist}(S_n(t,\tau)a_n, S_\infty(t,\tau)a_n) < \frac{\varepsilon}{2},$$

for all $n \geq n_{\varepsilon_1}$, by the definition of pullback attractor, and Theorem 4.2, there exists $n_{\varepsilon_2} \in \mathbb{N}$ such that

$$\operatorname{dist}_H(S_\infty(t,\tau)\mathcal{A}_n(\tau), \mathcal{A}_\infty(t)) \leq \operatorname{dist}_H\left(S_\infty(t,\tau)\bigcup_{\tau\in\mathbb{R}} \mathcal{A}_n(\tau), \mathcal{A}_\infty(t)\right) < \frac{\varepsilon}{2},$$

for all $n \geq n_{\varepsilon_2}$, and therefore, for $n \geq \max\{n_{\varepsilon_1}, n_{\varepsilon_2}\}$ we get

$$\operatorname{dist}_H(\mathcal{A}_n(t), \mathcal{A}_\infty(t)) < \varepsilon.$$

□

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