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Non-real eigenvalues of symmetric Sturm–Liouville problems with indefinite weight functions

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Abstract. The present paper deals with non-real eigenvalues of regular Sturm–Liouville problems with odd symmetry indefinite weight functions applying the two-parameter method. Sufficient conditions for the existence and non-existence of non-real eigenvalues are obtained. Furthermore, an explicit expression of the bound of non-real eigenvalues will be given in the paper.

Keywords: indefinite weight function, Sturm–Liouville problem, non-real eigenvalue, eigencurve.

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1 Introduction

Consider the Sturm–Liouville problem

$$-y''(x) - \mu y(x) = \lambda w(x) y(x), \qquad x \in [-1,1], \tag{1.1}$$

with the Dirichlet boundary condition

$$y(1) = y(-1) = 0, \tag{1.2}$$

where μ is real, λ is the spectral parameter and the weight function w is a real-valued integrable function satisfying the following conditions.

$$\begin{aligned} &\text{For a.e. } x \in [0,1],\ w(x) \text{ is a monotone nonincreasing function.} \\ &\text{For a.e. } x \in [0,1],\ w(x) = -w(-x) \text{ and } w(x) > 0. \end{aligned} \tag{1.3}$$

Set $T(y) := -y''$. Then we can rewrite problems (1.1), (1.2) in Hilbert space $L^2[-1,1]$, with the inner product $\langle f,g \rangle := \int_{-1}^{1} f\overline{g}$, as

$$Ty - \mu y = \lambda W y, \qquad y \in \mathcal{D}(T), \tag{1.4}$$

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where $\mathcal{D}(T)$ is the natural domain of T, i.e.,

$$\mathcal{D}(T) = \{y \in L^2[-1,1] : \; y, \; y' \in AC[-1,1]. \; Ty \in L^2[-1,1], \; y(\pm 1) = 0\}.$$

Here $AC[-1,1]$ is the set of absolutely continuous functions on $[-1,1]$ and W is the operator of multiplication by w. Then T is self-adjoint, bounded below with compact resolvents and W is self-adjoint in Hilbert space $L^2[-1,1]$.

Such problem is called to be *right-indefinite* if the weighted function $w(x)$ changes signs on $[-1,1]$ in the sense of

$$\text{mes}\{x : w(x) > 0, \; x \in (-1,1)\} > 0 \quad \text{and} \quad \text{mes}\{x : w(x) < 0, \; x \in (-1,1)\} > 0.$$

Hence, the problem (1.1), (1.2) or (1.4) is a right-indefinite problem. As a special case, the existence of non-real eigenvalues for the Richardson equation [19,20]

$$-y'' - \mu y = \lambda \, \text{sgn}(x) y, \qquad x \in [-1,1] \tag{1.5}$$

associated to the Dirichlet conditions $y(\pm 1) = 0$ was studied. Various authors have investigated such kind of equations, see, for example, Volkmer [22,23], Turyn [21], Fleckinger and Mingarelli [9].

We can regard μ as another spectral parameter. Meanwhile, we call (λ, μ) is an eigenpair of (1.1), (1.2) or (1.4). If $\lambda \in \mathbb{R}$ is fixed, then (1.1), (1.2) poses a regular Sturm–Liouville problem with the eigenvalue parameter μ. It is well known that it possesses exactly one real eigenvalue μ with an eigenfunction which has exactly $n-1$ zeros in $(-1,1)$ for $n = 1,2,\ldots$ We denote this eigenvalue by $\mu = \mu_n(\lambda)$, then $\mu_n(\lambda)$ is continuous on $\lambda \in \mathbb{R}$ (see Lemma 2.1). It also follows from the classical Sturm–Liouville theory (cf. [26]) that $\mu_1(\lambda) < \mu_2(\lambda) < \mu_3(\lambda) < \cdots$. Clearly, $\mu_{2m-1}(0) = (\frac{2m-1}{2})^2\pi^2$, $\mu_{2m}(0) = m^2\pi^2$, $m = 1,2,3,\ldots$ At this time, we call the graph of the continuous function $\mu = \mu_n(\lambda)$ is the *nth real eigencurve*. If λ is non-real but μ is real, we call such eigenpair (λ, μ) is a *non-real eigenpair*. If there exists an interval $J \subset \mathbb{R}$ for $\mu \in J \subset \mathbb{R}$ is fixed, there exists non-real eigenpair (λ, μ). Then we denote this non-real eigenvalue by $\lambda = \lambda(\mu)$ and if $\lambda(\mu)$ is continuous on $\mu \in J \subset \mathbb{R}$, we call the graph of the function $\lambda = \lambda(\mu)$ is the *non-real eigencurve*. For more details about eigencurve, we can see Binding and Volkmer [6,7], Binding and Browne [5].

Under the condition that the first two eigenvalues of

$$-y''(x) - \mu y(x) = \lambda y(x), \quad x \in [-1,1], \; y(\pm 1) = 0 \tag{1.6}$$

have contrary signs, papers [15,25] tell us that (1.1), (1.2) has exactly two non-real eigenvalues. In more general conditions, Volkmer [23, pp. 233–234] studies the existence of non-real eigenvalues for the Richardson equation (1.5) associated to the Dirichlet conditions $y(\pm 1) = 0$ (see Corollary 3.12). For the general Sturm–Liouville problems, Mingarelli [14] made a summary of regular indefinite Sturm–Liouville problems and posed many questions about the bounds and the existence of the non-real eigenvalues. Recently, Behrndt, Philipp and Trunk [3] and Behrndt, Schmitz and Trunk [4] studied the existence and obtained a bound on non-real eigenvalues in a special singular case. For the regular case, Behrndt, Chen, Philipp and Qi [1], Kikonko, Mingarelli [11] and other papers [10, 15, 24] got bounds on non-real eigenvalues. The existences of non-real eigenvalues were studied in [2, 18, 25]. Papers [16, 17] gave some applications about the non-real eigenvalues of indefinite Sturm–Liouville problems.

In this paper, we will prove the existence of non-real eigenvalues of problem (1.1), (1.2), for $\mu \in (\mu_{2m-1}(0), \mu_{2m}(0))$, $m = 1,2,\ldots$, (see Theorem 3.11). And a sufficient condition for the non-existence of non-real eigenvalues of (1.1), (1.2) is obtained in Theorem 4.3. The arrangement of the present paper is as follows. The next section gives some preliminary knowledge and some properties of real eigencurves. The main result of this paper, the existence of non-real eigenvalues, Theorems 3.11 and its proof are stated in Section 3. Furthermore, an explicit expression of non-real eigenvalues' bound will be given in Lemma 3.2 of Section 3. The last section, Section 4, gives the non-existence of non-real eigenvalues, Theorems 4.3 and its proof.

2 Properties of real eigencurves and preliminary knowledge

This section gives some preliminary knowledge and some properties of real eigencurves, $\mu = \mu_n(\lambda), n = 1, 2, 3, \dots$ In paper [6], Binding and Volkmer have made a comprehensive summary and further research on real eigencurves about two-parameter Sturm–Liouville problems. The following first five lemmas are from paper [6].

Lemma 2.1 (see [6, Theorem 2.1]). *For every positive integer n, the real eigencurve $\mu_n(\lambda)$ is (real-)analytic for $\lambda \in \mathbb{R}$.*

Lemma 2.2 (see [6, Theorem 2.2]). *For every positive integer n,*

$$\lim_{\lambda\to\infty} \frac{\mu_n(\lambda)}{\lambda} = -\operatorname{ess\,sup} w \quad \textit{and} \quad \lim_{\lambda\to-\infty} \frac{\mu_n(\lambda)}{\lambda} = -\operatorname{ess\,inf} w,$$

where the $\operatorname{ess\,sup} w$ *and* $\operatorname{ess\,inf} w$ *denote the essential supremum and essential infimum of w.*

From this lemma and $\operatorname{ess\,sup} w > 0$ and $\operatorname{ess\,inf} w < 0$, we can get

$$\mu_n(\pm\infty) := \lim_{\lambda\to\pm\infty} \mu_n(\lambda) = -\infty. \tag{2.1}$$

Lemma 2.3 (see [6, Theorem 2.5]). *Consider m distinct real numbers $\lambda_1, \dots, \lambda_m$ and m (not necessarily distinct) positive integers $n_1, \dots, n_m$ such that $\mu_{n_1}(\lambda_1) = \mu_{n_2}(\lambda_2) = \cdots = \mu_{n_m}(\lambda_m) = \mu^*$. If $\mu'_{n_j}(\lambda_j)(\lambda^* - \lambda_j) \le 0$ for some λ^* and for all $j = 1, \dots, m$, then $\mu_m(\lambda^*) \le \mu^*$.*

Lemma 2.4 (see [6, Corollary 2.6]). *The intersection of any straight line in the $(\operatorname{Re}\lambda, \mu)$-plane with the union of the first n eigencurves consists of at most $2n$ points for every positive integer n.*

Lemma 2.5 (see [6, Theorem 2.9]). *For $\lambda \in \mathbb{R}$, the order of $\mu_n(\lambda)$ is at most $2n$ for every positive integer n, i.e., $\mu_n^{(2n)}(\lambda) \neq 0$.*

We call the point λ_0 is a critical point of u, if $u'(\lambda_0) = 0$. If there are $2n$ critical points about $u_n(\lambda)$, then it can lead that there exists a point λ_0 such that $\mu_n^{(2n)}(\lambda_0) = 0$, by the mean value theorem. Applying (2.1) and Lemma 2.5 to real eigencurves, we can obtain the next result.

Lemma 2.6.

(i) For any $\lambda \in \mathbb{R}$, $\mu_1''(\lambda) < 0$.

(ii) For every positive integer n, there are at most $2n - 1$ critical points for $u_n(\lambda)$.

Next, we will prove 0 is an extreme point of every eigencurve.

Lemma 2.7. *0 is a minimum (resp. maximum) of the real eigencurve $\mu_{2m}(\lambda)$ (resp. $\mu_{2m-1}(\lambda)$) and $\mu''_{2m}(0) > 0$ (resp. $\mu''_{2m-1}(0) < 0$), $m = 1, 2, \dots$*

Proof. Since (λ,μ) is an eigenpair of the linear problem (1.1) and (1.2), we can suppose the corresponding eigenfunction $\phi(x;\lambda,\mu)$ satisfying $\phi(-1)=0$, $\phi'(-1)=1$. $\phi(x;\lambda,\mu)$ is continuously differentiable with respect to (λ,μ), and $\mu:=\mu_{2m}(\lambda)$ is an analytic function, hence we can denote $y_1:=\frac{\partial y}{\partial\lambda}$, where $y:=y(x,\lambda):=\phi(x;\lambda,\mu)$. From y is an eigenfunction, we can get $y,y_1\in\mathcal{D}(T)$.

Differentiating (1.1), (1.2) or (1.4) with respect to λ yields

$$(T-\lambda W-\mu(\lambda))y_1=(w+\mu'(\lambda))y.$$

Note T and W are self-adjoint, hence $\langle\cdot,y\rangle$, we obtain

$$\langle(w+\mu'(\lambda))y,y\rangle=\langle(T-\lambda W-\mu(\lambda))y_1,y\rangle=\langle y_1,(T-\lambda W-\mu(\lambda))y\rangle=0.$$

This gives

$$\mu'(\lambda)=-\frac{\langle wy,y\rangle}{\langle y,y\rangle}=-\frac{\int_{-1}^{1}wy^2}{\int_{-1}^{1}y^2}.$$

Then $\mu'(0)=0$ since at this time $y(x,0)=A\sin(m\pi x)$, where A is the constant satisfying $y'(x,0)|_{x=-1}=1$, and $y^2(x,0)=y^2(-x,0)$.

Repeating the differentiation, we have

$$(T-\lambda W-\mu(\lambda))y_2=2(w+\mu'(\lambda))y_1+\mu''(\lambda)y,$$

where $y_2:=\frac{\partial y_1}{\partial\lambda}$. With the same method above, using $\mu'(0)=0$ we obtain

$$\mu''(0)=-\frac{2\langle wy,y_1\rangle}{\langle y,y\rangle}=-\frac{2\int_{-1}^{1}wyy_1}{\int_{-1}^{1}y^2}.$$

To find the second derivative we calculate $y_1(x,0)$ by solving the linear inhomogeneous differential equation

$$-y_1''=wy+\mu(0)y_1,\qquad y_1(-1)=y_1'(-1)=0,$$

where $\mu(0)=\mu_{2m}(0)=m^2\pi^2$. Hence

$$y_1(x,0)=\frac{-1}{m\pi}\int_{-1}^{x}w(t)\sin m\pi t\sin m\pi(x-t)dt$$

and the sign of $\mu''(0)$ is the same as the sign of

$$\int_{-1}^{1}w(x)\sin m\pi x\int_{-1}^{x}w(t)\sin m\pi t\sin m\pi(x-t)dtdx. \tag{2.2}$$

Set $l=-t$, $s=x$, then we have

$$\begin{aligned}&\int_{-1}^{0}\int_{t}^{-t}w(x)\sin m\pi xw(t)\sin m\pi t\sin m\pi(x-t)dxdt\\&\quad=\int_{0}^{1}\int_{-l}^{l}w(s)\sin m\pi sw(l)\sin m\pi l\sin m\pi(s+l)dsdl\\&\quad=\int_{0}^{1}\int_{-l}^{l}w(s)\sin m\pi sw(l)\sin^2 m\pi l\cos m\pi sdsdl\\&\quad=\int_{0}^{1}\int_{-x}^{x}w(x)\sin m\pi xw(t)\sin m\pi t\sin m\pi(x-t)dtdx.\end{aligned}$$

Hence (2.2) can be written as

$$\begin{aligned}
&\int_{-1}^{1} w(x)\sin m\pi x \int_{-1}^{x} w(t)\sin m\pi t \sin m\pi(x-t)dtdx \\
&\quad = \int_{-1}^{0}\int_{t}^{-t} w(x)\sin m\pi x w(t)\sin m\pi t\sin m\pi(x-t)dxdt \\
&\qquad + \int_{0}^{1}\int_{-x}^{x} w(x)\sin m\pi x w(t)\sin m\pi t\sin m\pi(x-t)dtdx \\
&\quad = 2\int_{0}^{1}\int_{-x}^{x} w(x)\sin m\pi x w(t)\sin m\pi t\sin m\pi(x-t)dtdx \\
&\quad = 2\int_{0}^{1}\int_{-x}^{x} w(x)\sin m\pi x w(t)\sin m\pi t\sin m\pi x\cos m\pi t dtdx \\
&\quad = \int_{0}^{1} w(x)\sin^2 m\pi x\int_{-x}^{x} w(t)\sin 2m\pi t dtdx \\
&\quad = 2\int_{0}^{1} w(x)\sin^2 m\pi x\int_{0}^{x} w(t)\sin 2m\pi t dtdx.
\end{aligned}$$

From (1.3), we know that w is monotone non-increasing on $(0,1)$. Hence we can obtain $\int_0^x w(t)\sin 2m\pi t dt > 0$ for a.e. $x \in (0,1)$ and the formula (2.2) is greater than zero, i.e., $\mu''_{2m}(0) > 0$. This fact and $\mu'_{2m}(0) = 0$ can lead that 0 is the minimum of the real eigencurve $\mu_{2m}(\lambda)$.

With the same method, we can get the sign of $\mu''_{2m-1}(0)$ is as same as the sign of

$$-\int_0^1 w(x)\cos^2(m-\frac{1}{2})\pi x\int_0^x w(t)\sin(2m-1)\pi t dtdx.$$

Hence $\int_0^x w(t)\sin(2m-1)\pi t dt > 0$ for $x \in (0,1)$ and $\mu''_{2m-1}(0) < 0$, by this and $\mu'_{2m-1}(0) = 0$, we can get 0 is a maximum of real eigencurve $\mu_{2m-1}(\lambda)$. □

3 Existence of non-real eigenvalues

In this section, we will obtain sufficient conditions of the existence about non-real eigenvalues of problem (1.1), (1.2). In Lemma 3.2, we will give an a priori bound on the modulus of the largest non-real eigenvalue which might appear. For this purpose, the lower bound about μ for any non-real eigenpair (λ, μ) must be given first.

It is well known that if the indefinite problem (1.1), (1.2), is a left-definite problem, then the problem only has real eigenvalues (see [12,13,26]). Since $T \geq \frac{\pi^2}{4}$, hence as $\mu \leq \mu_1(0) = \frac{\pi^2}{4}$ the problem is left-definite and thus has real spectrum.

Lemma 3.1. $\lambda(\mu) \in \mathbb{R}$, *for any* $\mu \leq \mu_1(0) = \frac{\pi^2}{4}$.

This lemma means that if (λ, μ) is an eigenpair of (1.1), (1.2) and $\lambda \notin \mathbb{R}$, then $\mu > \mu_1(0) = \frac{\pi^2}{4}$. An explicit bound for the non-real eigenvalues will be obtained.

Lemma 3.2. *Suppose* $(\lambda,\ \mu)$ *is an eigenpair of* (1.1), (1.2) *and* $\lambda \notin \mathbb{R}$, *then for* $\mu > \mu_1(0) = \frac{\pi^2}{4}$,

$$|\lambda| \leq M(\mu) := \frac{16(2+\frac{1}{\sqrt{2}})\mu^2}{w(1-\frac{1}{4\mu})}. \tag{3.1}$$

Clearly, $M(\mu)$ *is a bounded function on any finite interval in* $(\frac{\pi^2}{4}, \infty)$.

Proof. Let $\varphi(x)$ be a normalized eigenfunction of (1.1), (1.2), i.e., $\int_{-1}^{1}|\varphi|^2 = 1$, corresponding to the eigenpair (λ, μ). Without loss of generality, we assume that $\int_0^1 |\varphi|^2 \geq \frac{1}{2}$. Multiplying both sides of the equations in (1.1) by $\overline{\varphi}$ and integrating by parts on the interval $[x, 1]$, we get

$$\varphi\overline{\varphi'}(x) + \int_x^1 |\varphi'|^2 = \lambda \int_x^1 w|\varphi|^2 + \mu \int_x^1 |\varphi|^2, \tag{3.2}$$

$$\operatorname{Im}(\varphi\overline{\varphi'})(x) = \operatorname{Im}\lambda \int_x^1 w|\varphi|^2. \tag{3.3}$$

Set $x = -1$ in (3.2), (3.3), we obtain

$$\int_{-1}^1 w|\varphi|^2 = 0, \qquad \int_{-1}^1 |\varphi'|^2 = \mu \int_{-1}^1 |\varphi|^2 = \mu, \tag{3.4}$$

by $\operatorname{Im}\lambda \neq 0$ and $\varphi(-1) = 0$. Clearly $\mu > 0$ by Lemma 3.1 and from the Cauchy inequality and (3.4), for $x \in [0, 1]$

$$|\varphi(x)| = \left|\int_x^1 \varphi'\right| \leq \sqrt{1-x}\left(\int_0^1 |\varphi'|^2\right)^{\frac{1}{2}} \leq \sqrt{1-x}\sqrt{\mu}. \tag{3.5}$$

This inequality together with (3.2) yields for $x \in [0, 1]$

$$|\lambda| \int_x^1 w|\varphi|^2 \leq \sqrt{\mu}\sqrt{1-x}|\varphi'(x)| + 2\mu.$$

Integrating this inequality on the interval $[0, 1]$ and using the Cauchy–Schwarz inequality again, it follows that

$$\begin{aligned} |\lambda| \int_0^1 xw(x)|\varphi(x)|^2 dx &= |\lambda| \int_0^1 \int_x^1 w|\varphi|^2 \leq \sqrt{\mu} \int_0^1 \sqrt{1-x}|\varphi'(x)|dx + 2\mu \\ &\leq \sqrt{\mu}\left(\int_0^1 (1-x)dx \int_0^1 |\varphi'|^2\right)^{\frac{1}{2}} + 2\mu \leq \left(2 + \frac{1}{\sqrt{2}}\right)\mu. \end{aligned} \tag{3.6}$$

Now, for every $a \in (0, \frac{1}{2})$,

$$\int_a^{1-a} |\phi(x)|^2 dx \geq \frac{1}{2} - \int_0^a (1-x)\mu dx - \int_{1-a}^1 (1-x)\mu dx = \frac{1}{2} - a\mu$$

by (3.5). Hence

$$\int_0^1 xw(x)|\varphi(x)|^2 dx \geq \left(\frac{1}{2} - a\mu\right)\int_a^{1-a} xw(x)dx \geq a\left(\frac{1}{2} - a\mu\right)w(1-a).$$

The function $a \mapsto a\left(\frac{1}{2} - a\mu\right)$ attains its maximum at $a = \frac{1}{4\mu}$. And so,

$$\int_0^1 xw(x)|\varphi(x)|^2 dx \geq \frac{1}{16\mu} w\left(1 - \frac{1}{4\mu}\right). \tag{3.7}$$

Note that for any non-real eigenpair (λ, μ), $\mu > \frac{\pi^2}{4}$ by Lemma 3.1, therefore $0 < \frac{1}{4\mu} < 1 - \frac{1}{4\mu} < 1$ and the last inequality is reasonable. (3.6) and (3.7) lead to

$$|\lambda| \leq \frac{16\left(2 + \frac{1}{\sqrt{2}}\right)\mu^2}{w\left(1 - \frac{1}{4\mu}\right)}.$$

The proof is finished. □

Let $\phi(x;\lambda,\mu)$ be the solution of (1.1) satisfying the initial conditions

$$\phi(-1) = 0, \qquad \phi'(-1) = 1.$$

Here λ and μ can be arbitrary complex numbers. By analytic parameter dependence, the function

$$D(\lambda,\mu) := \phi(1;\lambda,\mu) \tag{3.8}$$

is an entire function and the zeros (λ, μ) of D are the eigenpairs of (1.1), (1.2). Hence by the continuity of zeros of analytic functions (see [8, p. 248] or the next proposition), we can obtain the corresponding conclusion about the analytic function D, in Lemma 3.4.

Proposition 3.3 (The continuity of zeros of analytic functions)**.** *Let A be an open set in the complex plane $\mathbb{C}$, X a metric space, f a continuous complex valued function on $A \times X$ such that for each $\alpha \in X$, the map $z \to f(z,\alpha)$ is an analytic function on A. Let B be an open set of A whose closure $\overline{B}$ in $\mathbb{C}$ is compact and contained in A, and let $\alpha_0 \in X$ be such that no zero of $f(z,\alpha_0)$ is on the boundary of B. Then there exists a neighborhood W of α_0 in X such that*

(1) for any $\alpha \in W$, $f(z,\alpha)$ has no zero on the boundary of B;

(2) for any $\alpha \in W$, the sum of the order of the zeros of $f(z,\alpha)$ contained in B is independent of α.

Using Proposition 3.3, let the metric space X be $\mathbb{R}$, then

Lemma 3.4. *Let B be an open set of $\mathbb{C}$ whose closure $\overline{B}$ is compact, and let $\alpha_0 \in \mathbb{R}$ be such that no zero of $D(z,\alpha_0)$ is on the boundary of B. Then there exists a neighborhood W of α_0 in $\mathbb{R}$ such that*

(1) for any $\alpha \in W$, $D(z,\alpha)$ has no zero on the boundary of B;

(2) for any $\alpha \in W$, the sum of the order of the zeros of $D(z,\alpha)$ contained in B is independent of α.

In the sequel, we obtain the existence and multiplicity of non-real eigencurves nearby the extremum point of real eigencurves. First, we give the multiplicity of the function D about μ on the real eigencurve.

Lemma 3.5. *For the real eigenvalue (λ,μ), as a root of the μ-equation $D(\lambda,\mu) = 0$, the multiplicity of μ is exactly one, i.e., $\frac{\partial D}{\partial \mu}(\lambda,\mu) \neq 0$.*

Proof. See the proof of [6, Theorem 2.1, equation (2.3) p. 34]. □

Lemma 3.6. *Suppose λ_0 is a maximum (resp. minimum) of the nth real eigencurve $\mu_n(\lambda)$ on $\mathbb{R}$, satisfying $\mu_n''(\lambda_0) < 0$ (resp. $\mu_n''(\lambda_0) > 0$), $n = 1,2,3,\ldots$ Then for every $\varepsilon > 0$ sufficiently small, there exists $\delta > 0$ such that for each $\mu \in (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta)$ (resp. $\mu \in (\mu_n(\lambda_0) - \delta, \mu_n(\lambda_0))$), $\mathcal{O}(\lambda_0,\varepsilon)$ contains exactly two non-real eigenvalues(in the sense of multiplicity) of (1.1), (1.2). Here $\mathcal{O}(\lambda_0,\varepsilon) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon\}$.*

Proof. Suppose λ_0 is the maximum of real eigencurve $\mu_n(\lambda)$, satisfying $\mu_n''(\lambda_0) < 0$, $n = 1,2,3,\ldots$

First, we will prove, above nearby $(\lambda_0, \mu_n(\lambda_0))$, the existence of non-real eigenvalues. Since $D(\lambda_0, \mu_n(\lambda_0)) = 0$ and there is no intersection point between any two distinct real eigencurves, we have that for sufficient small $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ such that

$$\{\mathcal{O}(\lambda_0,\varepsilon) \times (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)\} \cap \{(\lambda, \mu_m(\lambda)) : \lambda \in \mathbb{R}\} = \emptyset, \qquad m \geq 1, \tag{3.9}$$

where $\mathcal{O}(\lambda_0, \varepsilon) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon\}$ and for each $\mu \in (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)$ the λ-equation $D(\lambda, \mu) = 0$ has roots in $\mathcal{O}(\lambda_0, \varepsilon)$ by Lemma 3.4. However, from (3.9) we know, for $\mu \in (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)$ the λ-equation $D(\lambda, \mu) = 0$ has no real roots in $\mathcal{O}(\lambda_0, \varepsilon)$. Therefore, there only exist non-real λ-roots and this proves the existence of non-real eigenvalues above nearby $(\lambda_0, \mu_n(\lambda_0))$.

The next, we will prove, nearby $(\lambda_0, \mu_n(\lambda_0))$, for any fixed μ there exactly exist two non-real eigenvalues above. We only need to prove that

$$\frac{\partial D}{\partial \lambda}(\lambda_0, \mu_n(\lambda_0)) = 0 \quad \text{and} \quad \frac{\partial^2 D}{\partial \lambda^2}(\lambda_0, \mu_n(\lambda_0)) \neq 0.$$

Differentiating $D(\lambda, \mu_n(\lambda)) = 0$ with respect to λ we have

$$\frac{\partial D}{\partial \lambda} + \mu_n'(\lambda)\frac{\partial D}{\partial \mu_n} = 0. \tag{3.10}$$

Set $\lambda = \lambda_0$ in (3.10) we get from $\mu_n'(\lambda_0) = 0$ that $\frac{\partial D}{\partial \lambda}(\lambda_0, \mu_n(\lambda_0)) = 0$. Differentiating twice $D(\lambda, \mu_n(\lambda)) = 0$ with respect to λ we have

$$\frac{\partial^2 D}{\partial \lambda^2} + 2\mu_n'(\lambda)\frac{\partial^2 D}{\partial \lambda \partial \mu_n} + \mu_n'(\lambda)\frac{\partial^2 D}{\partial \mu_n^2} + \mu_n''(\lambda)\frac{\partial D}{\partial \mu_n} = 0. \tag{3.11}$$

Set $\lambda = \lambda_0$ in (3.11) we get from $\mu_n'(\lambda_0) = 0$, $\mu_n''(\lambda_0) < 0$ and $\frac{\partial D}{\partial \mu_n} \neq 0$ by Lemma 3.5 that $\frac{\partial^2 D}{\partial \lambda^2}(\lambda_0, \mu_n(\lambda_0)) \neq 0$.

The proof of the other case is similar and the proof of this lemma is finished. □

Using Lemma 3.4 again, we will get that the point set of the non-real eigenvalues in Lemma 3.6 can compose two non-real eigencurves $\lambda(\mu)$, i.e., there exists an interval $J \subset \mathbb{R}$ such that $\lambda = \lambda(\mu)$ is continuous on $\mu \in J \subset \mathbb{R}$. We continue Lemma 3.6 the following way.

Lemma 3.7. *Suppose λ_0 is a maximum (resp. minimum) of the nth real eigencurve $\mu_n(\lambda)$ on $\mathbb{R}$, satisfying $\mu_n''(\lambda_0) < 0$ (resp. $\mu_n''(\lambda_0) > 0$), $n = 1, 2, 3, \ldots$ Then there exists ε, $\delta > 0$, such that there exactly exist two simple multiplicity non-real eigencurves $\lambda(\mu)$, $\overline{\lambda(\mu)} \in \mathcal{O}(\lambda_0, \varepsilon)$, for every $\mu \in (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta)$ (resp. $\mu \in (\mu_n(\lambda_0) - \delta, \mu_n(\lambda_0))$).*

Proof. Suppose (λ, μ) is an eigenpair and $\varphi(x)$ is a corresponding eigenfuntion(nontrivial complex-valued function) of problem (1.1) and (1.2), i.e.,

$$-\varphi(x)'' = (\lambda w(x) + \mu)\varphi(x), \qquad \varphi(\pm 1) = 0,$$

then

$$-\overline{\varphi(x)}'' = (\overline{\lambda} w(x) + \mu)\overline{\varphi(x)}, \qquad \overline{\varphi(\pm 1)} = 0.$$

Hence, if λ is non-real, then $(\overline{\lambda}, \mu)$ is another distinct non-real eigenpair.

Suppose λ_0 is a maximum of real eigencurve $\mu_n(\lambda)$, satisfying $\mu_n''(\lambda_0) < 0$, $n = 1, 2, 3, \ldots$ Following the proof of the last lemma, there exist $\varepsilon > 0$, $\delta_\varepsilon > 0$ such that for each $\mu \in (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)$, there are two distinct roots of the λ-equation $D(\lambda, \mu) = 0$ in $\mathcal{O}(\lambda_0, \varepsilon)$. These roots are $\lambda(\mu)$ and $\overline{\lambda(\mu)}$, by the discussion above. We may assume that $\operatorname{Im} \lambda(\mu) > 0$ for any $\mu \in (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)$ since either $\operatorname{Im} \lambda(\mu) > 0$ or $\operatorname{Im} \overline{\lambda(\mu)} > 0$. We will prove the nonreal value function $\lambda(\mu)$, $\mu \in (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)$ is a non-real eigencurve nearby above $(\lambda_0, \mu_n(\lambda_0))$. Clearly, we only need to prove that $\lambda = \lambda(\mu)$ is continuous on $(\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)$ by the definition of non-real eigencurves.

Suppose on the contrary, there exist $\nu_0 \in (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)$ such that $\lambda(\mu)$ is discontinuous at ν_0. We assume that $\lambda(\mu)$ is not right continuous at ν_0, without loss of generality. Then there exist $\{\nu_n, n = 1,2,3,\dots\} \subset (\nu_0, \mu_n(\lambda_0) + \delta_\varepsilon)$ and $\epsilon > 0$ such that $\nu_n \to \nu_0+$ as $n \to \infty$ and $|\lambda(\nu_n) - \lambda(\nu_0)| > \epsilon$, $n = 1,2,3,\dots$ However, there exist $\delta_\epsilon > 0$ such that for each $\mu \in (\nu_0, \nu_0 + \delta_\epsilon)$, the λ-equation $D(\lambda, \mu) = 0$ has exactly one root in $\mathcal{O}(\lambda(\nu_0), \epsilon)$. This is clearly a contradiction and leads to the non-real eigencurve $\lambda = \lambda(\mu)$ is continuous on $(\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)$.

The other case about the minimum of the real eigencurve is the same as the proof above. □

If $\lambda = 0$, we can calculate the eigenvalues of (1.1), (1.2),

$$\mu_{2m-1}(0) = \left(\frac{2m-1}{2}\right)^2 \pi^2, \qquad \mu_{2m}(0) = m^2\pi^2, \qquad m = 1,2,3,\dots$$

Lemma 3.8. *For enough small $\delta > 0$, there exactly exist two distinct simple multiplicity non-real (imaginary-valued) eigencurves $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ start at $(0, \mu_{2m-1}(0))$ (resp. $(0, \mu_{2m}(0))$) for $\mu \in (\mu_{2m-1}(0), \mu_{2m-1}(0) + \delta)$ (resp. $\mu \in (\mu_{2m}(0) - \delta, \mu_{2m}(0))$), $m = 1,2,3,\dots$*

Proof. We only need consider the $(2m-1)$th eigencurve. From Lemma 2.7, $\mu''_{2m-1}(0) < 0$, and hence there exists ε, $\delta > 0$, such that there exactly exist two distinct simple multiplicity non-real(imaginary-valued) eigencurves $\lambda(\mu)$, $\overline{\lambda(\mu)} \in \mathcal{O}(0, \varepsilon)$, for every $\mu \in (\mu_{2m-1}(0), \mu_{2m-1}(0) + \delta)$, by Lemma 3.7. Now, we will prove these two non-real eigencurves must be imaginary-valued.

Suppose (λ, μ) is an eigenpair and $\varphi(x)$ is a corresponding eigenfuntion of problem (1.1), (1.2), i.e.,

$$-\varphi(x)'' = (\lambda w(x) + \mu)\varphi(x), \qquad \varphi(\pm 1) = 0,$$

then by $w(x) = -w(-x)$,

$$\begin{aligned} -\varphi(-x)'' &= (-\lambda w(x) + \mu)\varphi(-x), &\quad \varphi(\mp 1) &= 0, \\ -\overline{\varphi(x)}'' &= (\overline{\lambda} w(x) + \mu)\overline{\varphi(x)}, &\quad \overline{\varphi(\pm 1)} &= 0, \\ -\overline{\varphi(-x)}'' &= (-\overline{\lambda} w(x) + \mu)\overline{\varphi(-x)}, &\quad \overline{\varphi(\mp 1)} &= 0. \end{aligned} \tag{3.12}$$

Hence, $(-\lambda, \mu)$, $(\overline{\lambda}, \mu)$, $(-\overline{\lambda}, \mu)$ are also eigenpairs. However, there exactly exist two distinct non-real eigencurves $\lambda(\mu)$, $\overline{\lambda(\mu)} \in \mathcal{O}(0, \varepsilon)$, for every $\mu \in (\mu_{2m-1}(0), \mu_{2m-1}(0) + \delta)$. This fact leads to $-\lambda(\mu) = \overline{\lambda(\mu)}$ and the proof is finished. □

Suppose λ_0 is a maximum of the real eigencurve $\mu_n(\lambda)$, satisfying $\mu''_n(\lambda_0) < 0$, $n = 1,2,3,\dots$ and $\lambda(\mu)$, $\mu \in (\mu_n(\lambda_0), \mu_n(\lambda_0) + \delta_\varepsilon)$ is a non-real eigencurve. Then we can get $\lim_{\mu \downarrow \mu_n(\lambda_0)} \lambda(\mu) = \lambda_0$ with the same method in the proof of Lemma 3.6. Moreover, for any non-real eigenpair (λ, μ), there exists at least one non-real eigencurve through it.

Lemma 3.9. *Suppose $\lambda = \lambda(\mu)$, $\mu \in J \subset \mathbb{R}$ is a non-real eigencurve of (1.1), (1.2), where J is a bounded interval, then for the right (resp. left) end-point of J, η, the limitation of $\lambda(\mu)$, $\lambda(\eta\pm)$, exists finitely as $\mu \to \eta\pm$. Clearly, $(\lambda(\eta\pm), \eta\pm)$ are also eigenpairs.*

Proof. Suppose η is the right end-point of J. Let Λ be the set of all limit points of $\lambda(\mu)$ as $\mu \to \eta-$,

$$\Lambda = \left\{\xi : \exists \mu^{(n)} \to \eta- \text{ such that } \lim_{n\to\infty} \lambda(\mu^{(n)}) = \xi\right\}.$$

From Lemma 3.2, Λ is bounded, by the boundedness of the interval J. Then it follows from $D(\lambda(\mu^{(n)}), \mu^{(n)}) = 0$ and the continuity of the function D that $D(\xi, \eta) = 0$ for any $\xi \in \Lambda$. We only need to prove that Λ has only one point.

Suppose on the contrary, if Λ has more than one point ξ_1, ξ_2. Then by the continuity of $\lambda(\mu)$, $\mu \in J$, we know that for any fixed r, $0 < r < |\xi_1 - \xi_2|$, and any $\delta > 0$ such that $(\eta - \delta, \eta] \subset J$, the set

$$\{(\lambda(\mu), \mu) : \mu \in (\eta - \delta, \eta]\} \cap S(\xi_1, r)$$

must contain infinite points, where $S(\xi_1, r)$ denotes the sphere in $\mathbb{C}$ with the center ξ_1 and the radius r, respectively. This means that the number of the λ-solutions about the η-equation $D(\lambda, \eta) = 0$ on the compact set $S(\xi_1, r)$ is infinite. Hence for any $0 < r < |\xi_1 - \xi_2|$ there exists at least one accumulation point λ_r for these λ-solutions and $D(\lambda_r, \eta) = 0$. That is to say that the zeros of $D(\lambda, \eta)$ are uncountable and hence $D(\lambda, \eta) = 0$ for any $\lambda \in \mathbb{R}$, since for the fixed η, $D(\lambda, \eta)$ is analytic about λ. Clearly, this is a contradiction since for the fixed η the eigenvalue problem has only countable eigenvalues. Therefore, Λ has only one point, say ξ_0, and ξ_0 is a finite point of $\mathbb{C}$. The proof about the left end-point of J is the same as the one above and Lemma 3.9 is proved. □

In the next lemma, we will give the existence of non-real eigencurves between the $2m$th and $(2m-1)$th real eigencurves $\mu_{2m-1}(\lambda)$ and $\mu_{2m}(\lambda)$, $m = 1, 2, 3, \ldots$

Lemma 3.10. *Consider the problem* (1.1) *and* (1.2). *There exist at least two non-real (imaginary-valued) eigencurves* $\lambda(\mu)$ *and* $\overline{\lambda(\mu)}$ *on* $\mu \in \left(\left(\frac{2m-1}{2}\right)^2 \pi^2, m^2\pi^2\right)$, $m = 1, 2, 3, \ldots$

Proof. For any fixed $m = 1, 2, 3, \ldots$, we will prove there exist two imaginary value eigencurves

$$\pm i\widetilde{\lambda}(\mu), \qquad \mu \in \left(\left(\frac{2m-1}{2}\right)^2 \pi^2, m^2\pi^2\right),$$

where $\widetilde{\lambda}$ is a real function. Let $\left(\left(\frac{2m-1}{2}\right)^2 \pi^2, \eta\right)$ be the maximal interval on which $i\widetilde{\lambda}(\mu) \in i\mathbb{R}$ is a non-real eigencurve, then $\eta > \left(\frac{2m-1}{2}\right)^2 \pi^2$ by Lemma 3.8. Without loss of generality, we assume that $\widetilde{\lambda}(\mu) > 0$ and $\eta < +\infty$. Note if $\eta = +\infty$ this theorem is true clearly. Then by Lemma 3.9, the limitation of $\widetilde{\lambda}(\mu)$ exists finitely as $\mu \to \eta-$, denoted as $\widetilde{\lambda}(\eta-)$. We only need to prove $\eta \geq m^2\pi^2$.

Suppose on the contrary, $\eta < m^2\pi^2$. Since $D(i\widetilde{\lambda}(\mu), \mu) \equiv 0$ on $\left(\left(\frac{2m-1}{2}\right)^2 \pi^2, \eta\right)$, we have $D(i\widetilde{\lambda}(\eta-), \eta) = 0$ and $\widetilde{\lambda}(\eta-) \geq 0$ by the continuity of D. If $\widetilde{\lambda}(\eta-) = 0$, it is a contrary for $\eta < \mu_{2m}(0) = m^2\pi^2$, hence $\widetilde{\lambda}(\eta-) > 0$. With the same method of Lemma 3.7, we can conclude that there exist $\delta > 0$ sufficiently small such that there exists an imaginary-valued eigencurve (for convenience, also writing $\widetilde{\lambda}$ here) $\widetilde{\lambda}(\mu)$, $\mu \in (\eta, \eta + \delta)$ and $\widetilde{\lambda}(\eta-) = \widetilde{\lambda}(\eta+)$. That is to say, the imaginary-valued function $\widetilde{\lambda}(\mu)$ can be defined continuously on $\left(\left(\frac{2m-1}{2}\right)^2 \pi^2, \eta + \delta\right)$. This clearly contradicts the choice of η and the proof is over. □

Lemma 3.10 can lead to the main result of this paper.

Theorem 3.11. *Suppose* $\mu \in \left(\left(\frac{2m-1}{2}\right)^2 \pi^2, m^2\pi^2\right)$, $m = 1, 2, 3, \ldots$ *Then* (1.1), (1.2) *has at least two non-real (imaginary value) eigenvalues.*

Applying Theorem 3.11 to the Richardson equation

$$-y'' - \mu y = \lambda \operatorname{sgn}(x) y, \qquad x \in [-1, 1] \tag{3.13}$$

associated to the Dirichlet conditions $y(\pm 1) = 0$, we immediately have

Corollary 3.12. *Suppose* $\mu \in \left(\left(\frac{2m-1}{2}\right)^2\pi^2, m^2\pi^2\right)$, $m = 1, 2, 3, \ldots$ *Then the Richardson problem* (3.13) *with Dirichlet boundary condition* (1.2) *has at least two non-real (imaginary value) eigenvalues.*

In fact, this conclusion has been contained in Volkmer [23, pp. 233–234].

Any non-real eigenpair must be contained in a non-real eigencurve. If a non-real eigencurve $\lambda(\mu)$ intersects a real eigencurve $\mu(\lambda)$, the intersection must be a critical point of $\mu(\lambda)$. Moreover, by Lemma 3.1, for any non-real eigenpair of (1.1), (1.2), (λ, μ), i.e., $\lambda \notin \mathbb{R}$, we have

$$\mu(\lambda) > \mu_1(0) = \frac{\pi^2}{4}, \quad \text{for any } \lambda \notin \mathbb{R}. \tag{3.14}$$

The following is a summary of the properties of non-real eigencurves.

Remark 3.13. Any non-real eigencurve $\lambda(\mu)$ must start from a maximum of a real eigencurve and go upwards, ending at a minimum of a real eigencurve or to $+\infty$, i.e.,

$$\sup\{\mu : \lambda(t) \text{ is non real, for any } t \in (\hat{\mu}, \mu)\} = \check{\mu} \text{ or } +\infty,$$

where $\hat{\mu}$ is a maximum of a real eigencurve and $\check{\mu}$ a minimum of a real eigencurve.

Another description can be given that for any non-real eigencurve, it must start from a minimum of a real eigencurve and downwards end at a maximum of a real eigencurve. In such case, any non-real eigencurve downwards at most arrives at $\mu_1(0) = \frac{\pi^2}{4}$.

4 Nonexistence of non-real eigenvalues

In this section, we will obtain sufficient conditions for the non-existence of non-real eigenvalues. The next two lemmas give some properties about the maximum and minimum of the real eigencurves.

Lemma 4.1. *For every positive integer n, the real eigencurve $\mu_n(\lambda)$, $\lambda \in \mathbb{R}$, is an even function in the* $(\operatorname{Re}\lambda, \mu)$*-plane, i.e., $\mu_n(-\lambda) = \mu_n(\lambda)$. Furthermore, for the 2nd real eigencurve $\mu_2(\lambda)$, there exactly exist two maxima(maximal points) and one minimum. And for the 3rd real eigencurve $\mu_3(\lambda)$, there exist either three maxima and two minima or one maximum (maximal point) and no minimum.*

Proof. Suppose (λ, μ) is a real eigenpair and $\varphi(x)$ is a corresponding eigenfuntion of problem (1.1), (1.2), i.e.,

$$-\varphi(x)'' = (\lambda w(x) + \mu)\varphi(x), \qquad \varphi(\pm 1) = 0,$$

then

$$-\varphi(-x)'' = (-\lambda w(-x) + \mu)\varphi(-x), \qquad \varphi(\mp 1) = 0.$$

Hence, $(-\lambda, \mu)$ is another real eigenpair. This fact leads to $\mu_n(-\lambda) = \mu_n(\lambda)$, for every positive integer n.

Furthermore, for the 2nd real eigencurve $\mu_2(\lambda)$, 0 is a minimum and $\mu_2''(0) > 0$ by Lemma 2.7. Therefore, there exist at least two maxima by $\mu_2(\pm\infty) = -\infty$, see Lemma 2.2. From Lemma 2.6 (ii), we know there are at most 3 critical points for $u_2(\lambda)$. Hence for $u_2(\lambda)$ there exactly exist two maxima, one minimum. By $\mu_2(\lambda)$ is an even function, these two maxima are equal and are the maximal points.

For the 3rd real eigencurve $\mu_3(\lambda)$, there are at most 5 critical points, by Lemma 2.6 (ii). This fact, $u_3''(0) < 0$, $\mu_3(\pm\infty) = -\infty$ and $\mu_3(\lambda)$ is an even function can lead that there exist either three maxima or one maximum for $\mu_3(\lambda)$. Moveover, from $\mu_3(\pm\infty) = -\infty$, we know there exist two minima when there are three maxima and there no minimum when there is one maximum. □

Lemma 4.2. *Suppose $\lambda_n \in \mathbb{R}$ is a minimum of the nth real eigencurve $\mu_n(\lambda), n > 2$ and $\pm\lambda_2 \in \mathbb{R}$ are the maxima (maximal points) of the 2nd real eigencurve $\mu_2(\lambda)$. Then $\mu_n(\lambda_n) \geq \mu_2(\lambda_2)(= \mu_2(-\lambda_2))$, $n > 2$.*

Proof. If $n = 2m-1$, $m \geq 2$, $\lambda_{2m-1} \neq 0$, since 0 is a maximum of μ_{2m-1}. From μ_{2m-1} is an even function, see Lemma 4.1, and $\mu_n(\pm\infty) = -\infty$, see (2.1), we know for every $\varepsilon > 0$ sufficiently small, the horizontal $\mu = \mu_{2m-1}(\lambda_{2m-1}) + \varepsilon$ intersect with the real eigencurve μ_{2m-1} on at least 6 points. This fact and Lemma 2.3 can lead to $\mu_2(\lambda_2) \leq \mu_{2m-1}(\lambda_{2m-1}) + \varepsilon$, hence $\mu_2(\lambda_2) \leq \mu_{2m-1}(\lambda_{2m-1})$.

Now we consider the case $n = 2m$, $m \geq 2$. In the case $\lambda_{2m} \neq 0$, the proof is the same as $n = 2m-1$. In the case $\lambda_{2m} = 0$, $\mu_{2m}(0) > \mu_3(0)$. By Lemma 4.1, there are also two cases for $\mu_3(\lambda)$. If for $\mu_3(\lambda)$ there exists a minimum λ_3 such that $\mu_3(\lambda_3) < \mu_3(0)$, then $\mu_{2m}(0) > \mu_3(0) > \mu_3(\lambda_3) \geq \mu_2(\lambda_2)(= \mu_2(-\lambda_2))$. If for $\mu_3(\lambda)$ there exists no minimum, then 0 is the only maximal point of $\mu_3(\lambda)$. Hence for any $\lambda \in \mathbb{R}$, $\mu_3(0) > \mu_2(\lambda)$ and $\mu_{2m}(0) > \mu_3(0) > \mu_2(\lambda_2)(= \mu_2(-\lambda_2))$. □

From Lemma 4.2, we know that below the maxima (maximal points) of the 2nd real eigencurve $\mu_2(\lambda)$, there is only one maximum of all real eigencurves, that is the first real eigencurve's maximum (maximal point), $(0, \mu_1(0)) = (0, \frac{\pi^2}{4})$.

Theorem 4.3. *Suppose $\pm\lambda_2 \in \mathbb{R}$ are the maxima (maximal points) of the 2nd real eigencurve $\mu_2(\lambda)$. Then the problem (1.1), (1.2) has no non-real eigenvalue λ for $\mu \in (\mu_2(0), \mu_2(\lambda_2))$.*

Proof. Suppose on the contrary, then there exists a non-real eigenpair $(\widetilde{\lambda}(\widetilde{\mu}), \widetilde{\mu})$ such that $\widetilde{\mu} \in (\mu_2(0), \mu_2(\lambda_2))$ and there exists a non-real eigencurve through $(\widetilde{\lambda}(\widetilde{\mu}), \widetilde{\mu})$. Since $\mu_1(0)$ is the only maximum of all real eigencurves below $\mu_2(\lambda_2)$, this non-real eigencurve must connect $(\lambda(\widetilde{\mu}), \widetilde{\mu})$ and $(0, \mu_1(0))$, by Remark 3.13. Hence we can set this non-real eigencurve as

$$\widetilde{\lambda}(\mu), \qquad \mu \in (\mu_1(0), \widetilde{\mu}).$$

Then

$$\overline{\widetilde{\lambda}(\mu)}, \qquad \mu \in (\mu_1(0), \widetilde{\mu})$$

is an other non-real eigencurve through $(0, \mu_1(0)) = (0, \frac{\pi^2}{4})$. By Lemma 3.10, we know

$$\lambda(\mu) \quad \text{and} \quad \overline{\lambda(\mu)}, \qquad \mu \in (\mu_1(0), \mu_2(0)) = \left(\frac{\pi^2}{4}, \pi^2\right)$$

are two other non-real eigencurves through $(0, \mu_1(0))$.

These non-real eigencurves may coincide, i.e., there may exists $\delta > 0$ such that

$$\widetilde{\lambda}(\mu) = \lambda(\mu), \qquad \mu \in (\mu_1(0), \mu_1(0) + \delta).$$

In this case, the multiplicity of this non-real eigencurve is two. Hence, in the sense of multiplicity, there exist at least four non-real eigencurves starting from $(0, \mu_1(0))$. However, Lemma 3.8 tells us there are only two distinct simple multiplicity non-real (imaginary-valued) eigencurves starting from $(0, \mu_1(0))$. This is a contradiction and the proof is finished. □

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