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On Morrey and BMO regularity for gradients of weak solutions to nonlinear elliptic systems with non-differentiable coefficients

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Abstract. We consider weak solutions to nonlinear elliptic systems with non-differentiable coefficients whose principal parts are split into linear and nonlinear ones. Assuming that the nonlinear part $g(x,u,z)$ is equipped by sub-linear growth in z only for big value of $|z|$ (but the growth is arbitrarily close to the linear one), we prove the Morrey and BMO regularity for gradient of weak solutions.

Keywords: nonlinear equations, regularity, Morrey–Campanato spaces.

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1 Introduction

In the paper, we consider the problem of interior everywhere regularity of gradients of weak solutions to the nonlinear elliptic system

$$-\operatorname{div} a(x,u,Du) = b(x,u,Du), \tag{1.1}$$

where $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$, $b : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ are Caratheodorian mappings, $\Omega \subset \mathbb{R}^n$ is a bounded open set, $N > 1$, $n \geq 3$. A function $u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is called a weak solution to (1.1) in Ω if

$$\int_\Omega \langle a(x,u,Du), D\varphi(x)\rangle \, dx = \int_\Omega \langle b(x,u,Du), \varphi(x)\rangle \, dx, \quad \forall \varphi \in C_0^\infty(\Omega,\mathbb{R}^N).$$

As it is shown by examples, in a case of general system (1.1), only partial regularity of weak solutions can be expected for $n \geq 3$ (see e.g. [2,7,12]). Under the assumptions specified below

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we prove, in Campanato spaces, $\mathcal{L}^{2,n}$–regularity (or, so called *BMO*-regularity) of gradient of weak solutions for the system (1.1) whose coefficients a can be written in the special form

$$a(x,u,Du) = A(x)Du + g(x,u,Du), \tag{1.2}$$

where $A = (A_{ij}^{\alpha\beta})$, i, $j = 1,\ldots,N$, α, $\beta = 1,\ldots,n$, is a matrix of functions, the following condition of strong ellipticity

$$\langle A(x)\xi,\xi\rangle \geq \nu|\xi|^2, \qquad \text{a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^{nN};\ \nu > 0 \tag{1.3}$$

holds, and $g = g(x,u,z)$ are functions with sub-linear growth in z. In what follows, we formulate the conditions on the smoothness and the growth of the functions A, g and b precisely.

It is well known that in the case of linear elliptic systems with continuous (see [2]) or with $VMO \cap L^\infty$ (see [8]) coefficients A, the gradient of weak solutions has the $L^{2,\lambda}$-regularity. Supposing that the coefficients A of the linear system belong to some Hölder class, the author of [2] proved that the gradient of weak solutions belongs to the *BMO*-class. The foregoing result has been refined in [1], where the coefficients A are supposed to belong to the class of so-called "small multipliers of *BMO*". The both mentioned results from [2] and [1] have been generalized in [8], where the coefficients A belong to some subclass of $VMO \cap L^\infty$ and in [13], where nonstandard growth conditions of $p(x)$-type are considered.

Similar regularity results ($L^{2,\lambda}$-regularity for continuous coefficients A and *BMO*-regularity for Hölder ones) were achieved in [2] for systems (1.1)–(1.2) in a case when $g = g(x,u)$ (but does not depend on Du). The last mentioned results are generalized in [4], where the first author has proved the $L^{2,\lambda}$-regularity of the gradient of weak solutions to (1.1)–(1.2) when the coefficients A are continuous and the *BMO*-regularity of gradient in the case of Hölder continuous coefficients A under an assumption that the function $g = g(x,u,z)$ grows sub-linearly in z and the growth is controlled by power function $|z|^\alpha$, $0 < \alpha < 1$. The $L^{2,\lambda}$-regularity result from [4] has been generalized to the $VMO \cap L^\infty$ coefficients A in [5].

The present paper extends the results from [4] and [5] in two directions. The first one consists in the fact that, while the sub-linear in z growth of the function $g(x,u,z)$ from (1.2) is controlled by the power function $|z|^\alpha$, $\alpha \in (0,1)$, the present paper offers the control by a function $|z|/\ln^{s/2}(\mathrm{e}+|z|^2)$, $s > 0$, which is closer to the linear function then the power one. The second extension is that in [4] and [5] the sub-linear growth is required for all $|z| > 0$ and, on the other hand, here we prescribe it only for big values of $|z|$ as it is visible in (3.2), (3.3), (3.5) below. The last mentioned assumption could be seen as a kind of asymptotic growth condition. Recently a few papers have appeared, which study regularity of weak solutions to nonlinear systems $\operatorname{div} a(Du) = 0$, where the coefficients $a = a(z)$ are so called asymptotically regular (for precise definitions and statements see [15] and references therein). Our growth condition is a bit different from the condition on asymptotic regularity of coefficients in [15] because of structure of the systems. Here it is useful to mention a paper [10], where the authors deal with (beside other problems) the partial $C^{1,\alpha}$-regularity of $W^{1,\infty}$-weak solutions to quasi-monotone systems $\operatorname{div} a(x,Du) = 0$, $a = a(x,z)$ is C^1 in variable z, where they provide upper bounds for the Hausdorff dimension of the singular set (see [10, Chapter 6]). If $a(x,z) = a(z)$ and the coefficients a satisfy an asymptotic condition, which requires the differentiability of a with respect to z, then weak solutions to the previous systems belong to $W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. A typical model example for reaching such a result is $a(z) = z + b(z)$, where the derivative $b_z(z) \to 0$ when $|z| \to \infty$ (see [10, Chapter 6] as well). In this paper we provide

$\mathcal{L}^{2,n}_{\rm loc}$-regularity of gradients of weak solutions because of special structure of the system (but here we have $a = a(x,u,Du)$) and $a = a(x,u,z)$ does not have to be differentiable in the variable z and so we can not suppose any condition of the type $g_z(x,u,z) \to 0$ for $|z| \to \infty$.

2 Notation and definitions

We consider the bounded open set $\Omega \subset \mathbb{R}^n$ with points $x = (x_1,\dots,x_n)$, $n \geq 3$, $u : \Omega \to \mathbb{R}^N$, $N > 1$, $u(x) = (u^1(x),\dots,u^N(x))$ is a vector-valued function, $Du = (D_1u,\dots,D_nu)$, $D_\alpha = \partial/\partial x_\alpha$. The meaning of $\Omega_0 \subset\subset \Omega$ is that the closure of Ω_0 is contained in Ω, i.e. $\overline{\Omega}_0 \subset \Omega$. For the sake of simplicity we denote by $|\cdot|$ the norm in $\mathbb{R}^n$ as well as in $\mathbb{R}^N$ and $\mathbb{R}^{nN}$. If $x \in \mathbb{R}^n$ and r is a positive real number, we write $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$, i.e., the open ball in $\mathbb{R}^n$ with radius $r > 0$, centered at x and $\Omega_r(x) = \Omega \cap B_r(x)$. Denote by $u_{x,r} = |\Omega_r(x)|_n^{-1} \int_{\Omega_r(x)} u(y)\,dy = \fint_{\Omega_r(x)} u(y)\,dy$ the mean value of the function $u \in L^1(\Omega,\mathbb{R}^N)$ over the set $\Omega_r(x)$, where $|\Omega_r(x)|_n$ is the n-dimensional Lebesgue measure of $\Omega_r(x)$. Beside the usually Sobolev spaces $W^{k,p}(\Omega,\mathbb{R}^N)$, $W^{k,p}_{\rm loc}(\Omega,\mathbb{R}^N)$, $W^{k,p}_0(\Omega,\mathbb{R}^N)$ (see, e.g. [11]), we use the following Morrey and Campanato spaces.

Definition 2.1. Let $\lambda \in (0,n)$, $q \in [1,\infty)$. A function $u \in L^q(\Omega,\mathbb{R}^N)$ is said to belong to the Morrey space $L^{q,\lambda}(\Omega,\mathbb{R}^N)$ if

$$\|u\|^q_{L^{q,\lambda}(\Omega,\mathbb{R}^N)} = \sup_{x\in\Omega, r>0} \frac{1}{r^\lambda} \int_{\Omega_r(x)} |u(y)|^q\,dy < \infty.$$

Let $\lambda \in [0,n+q]$, $q \in [1,\infty)$. The Campanato space $\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$ is the subspace of such functions $u \in L^q(\Omega,\mathbb{R}^N)$ for which

$$[u]^q_{\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)} = \sup_{r>0, x\in\Omega} \frac{1}{r^\lambda} \int_{\Omega_r(x)} |u(y) - u_{x,r}|^q\,dy < \infty.$$

Proposition 2.2. *For a domain $\Omega \subset \mathbb{R}^n$ of the class $C^{0,1}$ we have the following*

(a) With the norms $\|u\|_{L^{q,\lambda}}$ and $\|u\|_{\mathcal{L}^{q,\lambda}} = \|u\|_{L^q} + [u]_{L^{q,\lambda}}$, $L^{q,\lambda}(\Omega,R^N)$ and $\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$ are Banach spaces.

(b) $L^{q,\lambda}(\Omega,\mathbb{R}^N)$ is isomorphic to the $\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$, $1 \leq q < \infty$, $0 < \lambda < n$.

(c) $L^{q,n}(\Omega,\mathbb{R}^N)$ is isomorphic to the $L^\infty(\Omega,\mathbb{R}^N) \subsetneq \mathcal{L}^{q,n}(\Omega,\mathbb{R}^N)$, $1 \leq q < \infty$.

(d) $\mathcal{L}^{2,n}(\Omega,\mathbb{R}^N)$ is isomorphic to the $\mathcal{L}^{q,n}(\Omega,\mathbb{R}^N)$ and $\mathcal{L}^{q,n}(Q,\mathbb{R}^N) = BMO(Q,\mathbb{R}^N)$, Q being a cube, $1 \leq q < \infty$.

(e) If $u \in W^{1,2}_{\rm loc}(\Omega,\mathbb{R}^N)$ and $Du \in L^{2,\lambda}_{\rm loc}(\Omega,\mathbb{R}^{nN})$, $n-2 < \lambda < n$, then $u \in C^{0,(\lambda+2-n)/2}(\Omega,\mathbb{R}^N)$.

(f) $\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$ is isomorphic to the $C^{0,(\lambda-n)/q}(\overline{\Omega},\mathbb{R}^N)$ for $n < \lambda \leq n+q$.

(g) For $p \in [1,\infty)$, $\Omega' \subset\subset \Omega$, $0 < a \leq \operatorname{dist}(\Omega',\partial\Omega)$ and $u \in \mathcal{L}^{p,n}(\Omega,\mathbb{R}^N)$ set

$$\mathcal{N}_{p,a}(u;\Omega') = \sup_{x\in\Omega', r\leq a} \left(\fint_{B_r(x)} |u(y) - u_{x,r}|^p\,dy \right)^{1/p}.$$

Then we have for each $u \in \mathcal{L}^{p,n}(\Omega,\mathbb{R}^N)$

$$\mathcal{N}_{1,a}(u;\Omega') \leq \mathcal{N}_{p,a}(u;\Omega') \leq c(p,n) \sup_{x\in\Omega, r>0} \left(\fint_{\Omega_r(x)} |u(y) - u_{x,r}|^2\,dy \right)^{1/2}.$$

For more details see [2,7,11,16].

Definition 2.3 (see [14])**.** Let $f \in BMO(\mathbb{R}^n)$ and

$$\eta(f,R) = \sup_{\rho \le R} \fint_{B_\rho(x)} |f(y) - f_{x,r}|\, dy,$$

where $B_\rho(x)$ ranges over the class of the balls of $\mathbb{R}^n$ of radius ρ. We say that $f \in VMO(\mathbb{R}^n)$ if

$$\lim_{R \to 0} \eta(f,R) = 0.$$

We can observe that substituting $\mathbb{R}^n$ for Ω we obtain the definition of $VMO(\Omega)$. Some basic properties of the above-mentioned classes are formulated in [1,14,16].

3 Main results

Suppose that for almost all $x \in \Omega$ and all $u \in \mathbb{R}^N$, $z \in \mathbb{R}^{nN}$ the following conditions hold:

$$|b(x,u,z)| \le f(x) + M(|u|^{\delta_0} + |z|^{\gamma_0}), \tag{3.1}$$

$$|g(x,u,z)| \le F(x) + M\left(|u|^{\delta} + h(|z|)\right), \tag{3.2}$$

where

$$h(|z|) = \begin{cases} \frac{|z|}{\ln^{s/2}(\mathrm{e}+t_0^2)} & \text{if } |z| \le t_0, \\ \frac{|z|}{\ln^{s/2}(\mathrm{e}+|z|^2)} & \text{if } |z| > t_0. \end{cases} \tag{3.3}$$

Here $f \in L^{2q_0,\lambda q_0}(\Omega)$, $q_0 = n/(n+2)$, $0 < \lambda \le n$, M is a positive constant, $1 \le \delta_0 < (n+2)/(n-2)$, $1 \le \gamma_0 < 1/q_0$, $F \in L^{2,\lambda}(\Omega)$, $1 \le \delta < n/(n-2)$, $s > 0$, $t_0 > 0$. We remark that $t_0 = t_0(s)$ is chosen in such a way that, putting $h^2(|z|) = H(|z|^2)$, the function $H = H(t)$ is nondecreasing on $[0,\infty)$, absolutely continuous on every closed interval of finite length and $H(0) = 0$. The relationship between $t_0 > 0$ and s can be expressed through an inequality $s \le (\mathrm{e}+t_0)\ln(\mathrm{e}+t_0)/t_0$.

Now we can state a result for the continuous case.

Theorem 3.1. *Let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution to the system (1.1) with (1.2) and the conditions (1.3), (3.1), (3.2) be satisfied. Suppose further that $A \in C(\overline{\Omega},\mathbb{R}^{nN})$. Then*

$$Du \in \begin{cases} L^{2,\lambda}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN}), & \text{if } \lambda < n, \\ L^{2,\lambda'}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN}) \text{ with arbitrary } \lambda' < n, & \text{if } \lambda = n. \end{cases}$$

Therefore,

$$u \in \begin{cases} C^{0,(\lambda-n+2)/2}(\Omega,\mathbb{R}^N), & \text{if } n-2 < \lambda < n, \\ C^{0,\vartheta}(\Omega,\mathbb{R}^N) \text{ with arbitrary } \vartheta < 1, & \text{if } \lambda = n. \end{cases}$$

If the coefficients of the linear part of the system are supposed to be discontinuous, we have to modify the previous assumptions in the following way:

$$|b(x,u,z)| \le f(x) + M|z|^{\gamma_0}, \tag{3.4}$$

$$|g(x,u,z)| \le F(x) + M\,h(|z|), \tag{3.5}$$

$$\langle g(x,u,z), z\rangle \ge \nu_1 h^2(|z|) - l^2(x), \tag{3.6}$$

where $f \in L^{qq_0,\lambda q_0}(\Omega)$, $F \in L^{q,\lambda}(\Omega)$, $q > 2$, ν_1 is a positive constant, $l \in L^{q,\lambda}(\Omega)$ and the other constants and functions are supposed to be the same as in (3.1), (3.2).

The next theorem slightly extends the main result from [5].

Theorem 3.2. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1.1) with (1.2) and the conditions (1.3), (3.4), (3.5) and (3.6) be satisfied. Suppose further that $A \in L^\infty \cap VMO(\Omega, \mathbb{R}^{nN})$. Then*

$$Du \in \begin{cases} L^{2,\lambda}_{\text{loc}}(\Omega, \mathbb{R}^{nN}) & \text{if } \lambda < n, \\ L^{2,\lambda'}_{\text{loc}}(\Omega, \mathbb{R}^{nN}) \text{ with arbitrary } \lambda' < n & \text{if } \lambda = n. \end{cases}$$

Therefore,

$$u \in \begin{cases} C^{0,(\lambda-n+2)/2}(\Omega, \mathbb{R}^N) & \text{if } n-2 < \lambda < n, \\ C^{0,\vartheta}(\Omega, \mathbb{R}^N) \text{ with arbitrary } \vartheta < 1 & \text{if } \lambda = n. \end{cases}$$

To obtain $\mathcal{L}^{2,n}$-regularity for the first derivatives of the weak solution we strengthen the conditions on the coefficients g and b. Namely suppose that

$$|g(x,u,z_1) - g(y,v,z_2)| \le M\Big(|F(x) - F(y)| + (|u| + |v|)^\delta + h(|z_1 - z_2|)\Big) \tag{3.7}$$

for a.e. $x \in \Omega$ and all $u, v \in \mathbb{R}^N$, $z_1, z_2 \in \mathbb{R}^{nN}$. Here $F \in \mathcal{L}^{2,n}(\Omega)$, $g(\cdot,0,0) \in \mathcal{L}^{2,n}(\Omega, \mathbb{R}^{nN})$. It is not difficult to see that (3.7) implies (3.2) with $\lambda = n$.

Now we can formulate the main result of the paper.

Theorem 3.3. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1.1) with (1.2) and suppose that the conditions (1.3), (3.1) with $f \in L^{2q_0,nq_0}(\Omega)$ and (3.7) with $0 < s \le 1$ hold. Let further $A \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^{nN})$ for some $\alpha \in (0,1]$. Then $Du \in \mathcal{L}^{2,n}_{\text{loc}}(\Omega, \mathbb{R}^{nN})$.*

4 Some lemmas

In this section we present results needed for the proofs of the theorems. In $B_R(x) \subset \mathbb{R}^n$ we consider a linear elliptic system (here the summation convention over repeated indices is used)

$$-D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = 0, \qquad i = 1, \dots, N \tag{4.1}$$

with constant coefficients (according to the introduced denotation, the previous system can be written in the form $-\operatorname{div}(A \cdot Du) = 0$) for which (1.3) holds.

Lemma 4.1 ([2, pp. 54–55])**.** *Let $u \in W^{1,2}(B_R(x), \mathbb{R}^N)$ be a weak solution to the system (4.1). Then, for each $0 < \sigma \le R$,*

$$\int_{B_\sigma} |Du(y)|^2\, dy \le L_1 \left(\frac{\sigma}{R}\right)^n \int_{B_R} |Du(y)|^2\, dy,$$
$$\int_{B_\sigma} |Du(y) - (Du)_\sigma|^2\, dy \le L_2 \left(\frac{\sigma}{R}\right)^{n+2} \int_{B_R} |Du(y) - (Du)_R|^2\, dy$$

hold with constants L_1, L_2 independent of the homothety.

The following lemma is fundamental for proving the theorems.

Lemma 4.2 ([9, pp. 537–538])**.** *Let ϕ be a nonnegative function on $(0, d]$ and let E_1, E_2, D, α, β be nonnegative constants. Suppose that $\phi(d) < \infty$ and*

$$\phi(\sigma) \leq \left(E_1 \left(\frac{\sigma}{R}\right)^{\alpha} + E_2\right) \phi(R) + DR^{\beta}, \qquad \forall\, 0 < \sigma \leq R \leq d$$

hold. Further let the constant $k \in (0, 1)$ exist such that $\epsilon = E_1 k^{\alpha-\beta} + E_2 k^{-\beta} < 1$. Then

$$\phi(\sigma) \leq C\sigma^{\beta}, \qquad \forall\, \sigma \in (0, d],$$

where

$$C = \max\left\{\frac{D}{(1-\epsilon)\, k^{\beta}}, \sup_{\sigma\in[kd,d]} \frac{\phi(\sigma)}{\sigma^{\beta}}\right\}.$$

We set

$$v_0 = \min\left\{n\left(1 - \frac{n-2}{n+2}\delta_0\right), n\left(1 - q_0\gamma_0\right)\right\}. \tag{4.2}$$

Lemma 4.3 ([2, pp. 106–107])**.** *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, $Du \in L^{2,\eta}(\Omega, \mathbb{R}^{nN})$, $0 \leq \eta < n$ and (3.1) or (3.4) be satisfied. Then $b \in L^{2q_0,\lambda_0}(\Omega, \mathbb{R}^N)$ and for each ball $B_R(x) \subset \Omega$ we have*

$$\int_{B_R(x)} |b(y, u, Du)|^{2q_0}\, dy \leq c\, R^{\lambda_0}, \tag{4.3}$$

where $c = c(n, M, \delta_0, \gamma_0, q_0, \operatorname{diam}\Omega, \|f\|_{L^{2q_0,\lambda q_0}(\Omega)}, \|u\|_{L^1(\Omega,\mathbb{R}^N)}, \|Du\|_{L^{2,\eta}(\Omega,\mathbb{R}^{nN})})$, $\lambda_0 = \min\{\lambda q_0, v_0 + \eta q_0\}$ in the case (3.1) or $c = c(n, M, \gamma_0, q_0, \operatorname{diam}\Omega, \|f\|_{L^{qq_0,\lambda q_0}(\Omega)}, \|Du\|_{L^{2,\eta}(\Omega,\mathbb{R}^{nN})})$ and $\lambda_0 = \min\{n(1-2/q) + 2\lambda q_0/q, n - (n-\eta)q_0\gamma_0\}$ in the case (3.4).

In the case of discontinuous coefficients of the linear part of the system (1.1) with (1.2) we will use a result about higher integrability of the gradient of a weak solution to the system.

Proposition 4.4 ([7, p. 138])**.** *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1.1) with (1.2) and the conditions (1.3), (3.4)–(3.6) be satisfied. Then there exists an exponent $2 < r < q$ such that $u \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Moreover there exists a constant $c = c(\nu, \nu_1, L, \|A\|_{L^\infty})$ and $\widetilde{R} > 0$ such that, for all balls $B_R(x) \subset \Omega$, $R < \widetilde{R}$, the following inequality is satisfied*

$$\left(\fint_{B_{R/2}(x)} |Du|^r\, dy\right)^{1/r} \leq c\left\{\left(\fint_{B_R(x)} |Du|^2\, dy\right)^{1/2} + \left(\fint_{B_R(x)} (|l|^r + |F|^r)\, dy\right)^{1/r} + R\left(\fint_{B_R(x)} |f|^{rq_0} dy\right)^{1/rq_0}\right\}.$$

Lemma 4.5 ([17, p. 37])**.** *Let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function which is absolutely continuous on every closed interval of finite length, $\phi(0) = 0$. If $w \geq 0$ is measurable and $E(t) = \{y \in \mathbb{R}^n : w(y) > t\}$ then*

$$\int_{\mathbb{R}^n} \phi \circ w\, dy = \int_0^\infty m(E(t))\phi'(t)\, dt.$$

In the proof of the theorems we will use a modification of Natanson's lemma (for a proof see [6, pp. 8–9]). It can be read as follows.

Lemma 4.6. *Let $f : [a,\infty) \to \mathbb{R}$ be a nonnegative function which is integrable on $[a,b]$ for all $a < b < \infty$ and*

$$\mathcal{N} = \sup_{0<h<\infty} \frac{1}{h} \int_a^{a+h} f(t)\,dt < \infty$$

is satisfied. Let $g : [a,\infty) \to \mathbb{R}$ be an arbitrary nonnegative, non-increasing and integrable function. Then

$$\int_a^\infty f(t)g(t)\,dt$$

exists and

$$\int_a^\infty f(t)g(t)\,dt \le \mathcal{N} \int_a^\infty g(t)\,dt$$

holds.

Remark 4.7. The foregoing estimate is optimal because if we put $f(t) = 1$, $t \in [a,\infty)$ then an equality will be achieved.

5 Proofs of the theorems

Proof of Theorem 3.1. Let $\Omega_0 \subset\subset \Omega$, $d_0 = \operatorname{dist}(\Omega_0, \partial\Omega)$, $B_R = B_R(x_0) \subset \Omega$, $x_0 \in \Omega_0$ be an arbitrary ball and let $w \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$ be a solution to the system

$$\begin{aligned}\int_{B_{R/2}} \langle (A)_{R/2} Dw, D\varphi\rangle\, dx = &\int_{B_{R/2}} \langle ((A)_{R/2} - A(x))\, Du, D\varphi\rangle\, dx \\ &- \int_{B_{R/2}} \langle g(x,u,Du), D\varphi\rangle\, dx + \int_{B_{R/2}} \langle b(x,u,Du), \varphi\rangle\, dx\end{aligned}$$

for all $\varphi \in W_0^{1,2}(B_{R/2}, \mathbb{R}^N)$. It is known (according to the linear theory and the Lax–Milgram theorem) that, under the assumption of this theorem, such solution exists and it is unique for all $R < R'$ ($R' \le 1$ is sufficiently small). We can put $\varphi = w$ in the previous equation and, using ellipticity, Hölder and Sobolev inequalities, we get

$$\begin{aligned}\nu \int_{B_{R/2}} |Dw|^2\, dx \le c\Bigg(&\int_{B_{R/2}} |A_{R/2} - A(x)|^2\, |Du|^2\, dx + \int_{B_{R/2}} |g(x,u,Du)|^2\, dx \\ &+ \left(\int_{B_{R/2}} |b(x,u,Du)|^{2q_0}\, dx \right)^{1/q_0} \Bigg) =: c\,(I + II + III).\end{aligned} \tag{5.1}$$

Now we obtain

$$I \le \omega^2(R) \int_{B_{R/2}} |Du|^2\, dx, \tag{5.2}$$

where $\omega(R) = \sup_{x,y\in\Omega,\, |x-y|<R} |A(x) - A(y)|$.

From the assumption (3.2) (taking into account (3.3) and the comments below it), putting $m_R(t) = m\big(\{y \in B_R(x_0) : |Du|^2 > t\}\big)$, we can estimate II as follows.

$$\begin{aligned} II &\le 3\int_{B_{R/2}} |F|^2\, dx + 3M^2 \left(\int_{B_{R/2}} |u|^{2\delta}\, dx + \int_{B_{R/2}} h^2(|Du|)\, dx \right) \\ &\le c\left(R^\lambda + \left(\int_{B_{R/2}} |u|^{2n/(n-2)}\, dx \right)^{\delta(n-2)/n} R^{n(1-\delta(n-2)/n)} + \int_0^\infty \frac{d}{dt}\,(H(t))\, m_{R/2}(t)\, dt \right) \\ &\le c\left(R^\lambda + R^{n(1-\delta(n-2)/n)} + J \right), \end{aligned} \tag{5.3}$$

where $c = c(n, M, \delta, \operatorname{diam}\Omega, \|F\|_{L^{q,\lambda}}, \|u\|_{L^{2n/(n-2)}})$. By means of Lemma 4.5 and Lemma 4.6 we get

$$\begin{aligned}
J &= \int_0^{t_0} \frac{d}{dt}\left(\frac{t}{\ln^s(\mathrm{e}+t_0^2)}\right) m_{R/2}(t)\,dt + \int_{t_0}^{\infty} \frac{d}{dt}\left(\frac{t}{\ln^s(\mathrm{e}+t)}\right) m_{R/2}(t)\,dt \\
&\le \frac{\kappa_n t_0}{2^n \ln^s(\mathrm{e}+t_0^2)} R^n + \sup_{t_0<t<\infty}\left(\frac{1}{t-t_0}\int_{t_0}^{t} \frac{d}{dw}\left(\frac{w}{\ln^s(\mathrm{e}+w)}\right) dw\right) \int_{t_0}^{\infty} m_{R/2}(w)\,dw \\
&\le \frac{\kappa_n t_0}{2^n \ln^s(\mathrm{e}+t_0^2)} R^n + \sup_{t_0<\xi<\infty}\left[\frac{1}{\ln^s(\mathrm{e}+\xi)}\left(1-\frac{s\xi}{(\mathrm{e}+\xi)\ln(\mathrm{e}+\xi)}\right)\right] \int_{B_{R/2}} |Du|^2\,dy \\
&\le \frac{\kappa_n t_0}{2^n \ln^s(\mathrm{e}+t_0^2)} R^n + \frac{1}{\ln^s(\mathrm{e}+t_0)} \int_{B_{R/2}} |Du|^2\,dx \\
&\le \frac{1}{\ln^s(\mathrm{e}+t_0)} \int_{B_R} |Du|^2\,dx + \frac{t_0}{\ln^s(\mathrm{e}+t_0^2)} R^n.
\end{aligned} \tag{5.4}$$

From (5.3) and (5.4) we have

$$II \le c\left(\frac{1}{\ln^s(\mathrm{e}+t_0)} \int_{B_R} |Du|^2\,dx + \frac{t_0}{\ln^s(\mathrm{e}+t_0^2)} R^n + R^\lambda + R^{n(1-\delta(n-2)/n)}\right). \tag{5.5}$$

We can estimate III by means of Lemma 4.3 (with $\eta = 0$) and we have

$$III \le cR^{\lambda_0/q_0}. \tag{5.6}$$

Together we have

$$\begin{aligned}
\nu^2 \int_{B_{R/2}} |Dw|^2\,dx \le c\Bigg\{ &\left[\omega^2(R) + \frac{1}{\ln^s(\mathrm{e}+t_0)}\right] \int_{B_R} |Du|^2\,dx \\
&+ \frac{t_0}{\ln^s(\mathrm{e}+t_0^2)} R^n + R^\lambda + R^{n(1-\delta(n-2)/n)} + R^{\lambda_0/q_0} \Bigg\}.
\end{aligned} \tag{5.7}$$

The function $v = u - w \in W^{1,2}(B_{R/2}, \mathbb{R}^N)$ is the solution to the system

$$\int_{B_{R/2}} \langle (A)_{R/2} Dv, D\varphi\rangle\,dx = 0, \qquad \forall \varphi \in W_0^{1,2}(B_{R/2}, \mathbb{R}^N)$$

and from Lemma 4.1 we have, for $0 < \sigma \le R/2$,

$$\int_{B_\sigma} |Dv|^2\,dx \le c\left(\frac{\sigma}{R}\right)^n \int_{B_{R/2}} |Dv|^2\,dx.$$

By means of (5.7) and the last estimate we obtain, for all $0 < \sigma \le R$, the following estimate:

$$\begin{aligned}
\int_{B_\sigma} |Du|^2\,dx &\le c_1\left[\left(\frac{\sigma}{R}\right)^n + \omega^2(R) + \frac{1}{\ln^s(\mathrm{e}+t_0)}\right] \int_{B_R} |Du|^2\,dx \\
&\quad + c_2\left[\frac{t_0}{\ln^s(\mathrm{e}+t_0^2)} R^n + R^\lambda + R^{n(1-\delta(n-2)/n)} + R^{\lambda_0/q_0}\right] \\
&\le c_1\left[\left(\frac{\sigma}{R}\right)^n + \omega^2(R) + \frac{1}{\ln^s(\mathrm{e}+t_0)}\right] \int_{B_R} |Du|^2\,dx + c_2 R^{\lambda'},
\end{aligned}$$

where the constants c_1 and c_2 only depend on the above-mentioned parameters and $\lambda' = \min\{n, \lambda, n(1-\delta(n-2)/n), \lambda_0/q_0\}$ ($\lambda' < n$). For $\eta = 0$ (see Lemma 4.3) we have $\lambda' = \min\{\lambda, n-(n-2)\delta, (n+2)-(n-2)\delta_0, (n+2)-n\gamma_0\}$. Set

$$\phi(\sigma) = \int_{B_\sigma} |Du|^2\,dx, \qquad E_1 = c_1, \qquad E_2 = c_1\left(\omega^2(R) + \frac{1}{\ln^s(\mathrm{e}+t_0)}\right), \qquad D = c_2.$$

Further we can choose $k < 1$ such that $E_1 k^{n-\lambda'} < 1/2$. It is obvious (the coefficients A are continuous) that the constants $R_0 > 0$ and $t_0 > 0$ exist such that $E_2 k^{-\lambda'} < 1/2$, then $E_1 k^{n-\lambda'} + E_2 k^{-\lambda'} < 1$. For all $0 < \sigma \le R \le \min\{d_0, R_0\}$ the assumptions of Lemma 4.2 are satisfied and therefore

$$\int_{B_\sigma} |Du|^2\,dx \le c\sigma^{\lambda'}, \qquad \forall \sigma \le \min\{d_0, R_0\}.$$

If $\min\{d_0, R_0\} < \operatorname{diam}\Omega_0$, it is easy to check that for $\min\{d_0, R_0\} \le \sigma \le \operatorname{diam}\Omega_0$ we have

$$\int_{\Omega_\sigma(x_0)} |Du|^2\,dx \le c\left(\frac{\sigma}{\min\{d_0, R_0\}}\right)^{\lambda'} \int_\Omega |Du|^2\,dx,$$

and thus we get

$$\|Du\|_{L^{2,\lambda'}(\Omega_0,\mathbb{R}^{nN})} \le c\,\|Du\|_{L^2(\Omega,\mathbb{R}^{nN})}.$$

If $\lambda = \lambda'$ the Theorem is proved. If $\lambda' < \lambda$ the previous procedure can be repeated with $\eta = \lambda'$ in Lemma 4.3. It is clear that after a finite number of steps (since λ' increases in each step as it follows from Lemma 4.3) we obtain $\lambda' = \lambda$. □

Proof of Theorem 3.2. Using the same procedure as in the foregoing proof we get the inequality (5.1). The terms I, II and III we can estimate as follows.

From Proposition 4.4 with $2 < r < q$, Hölder inequality ($r' = r/(r-2)$) and from the fact that, for a BMO-function, all L^r norms, $1 \le r < \infty$ are equivalent (see Proposition 2.2 (g)) we obtain

$$\begin{aligned}
I &\le \left(\int_{B_{R/2}} |A(x) - A_{R/2}|^{2r'}\,dx\right)^{1/r'} \left(\int_{B_{R/2}} |Du|^r\,dx\right)^{2/r} \\
&\le c\left(\int_{B_{R/2}} |A(x) - A_{R/2}|^{2r'}\,dx\right)^{1/r'} \\
&\quad \times \left\{R^{-n/r'}\int_{B_R} |Du|^2\,dx + \left(\int_{B_R} (|l|^r + |F|^r)\,dx\right)^{2/r} + R^{2+2n(1-1/q_0)/r}\left(\int_{B_R} |f|^{rq_0}\,dx\right)^{2/rq_0}\right\} \\
&\le c\mathcal{N}^2_{2r',R}(A;\Omega_0)\left[\int_{B_R} |Du|^2\,dx + \left(R^{2n/r-2(n-\lambda)/q} + R^{2+2n(1/r-1/q)-4/q+2\lambda/q}\right)R^{n/r'}\right] \\
&\le c\mathcal{N}^2_{2r',R}(A;\Omega_0)\left[\int_{B_R} |Du|^2\,dx + R^{n-2(n-\lambda)/q}\right],
\end{aligned} \tag{5.8}$$

where $c = c(r, \|l\|_{L^{q,\lambda}}, \|F\|_{L^{q,\lambda}}, \|f\|_{L^{qq_0,\lambda q_0}})$.

From assumption (3.5) (taking into account (3.3) and the comments below it) we can estimate II as follows.

$$\begin{aligned}
II &\le 2\int_{B_{R/2}} |F|^2\,dx + 2M^2\int_{B_{R/2}} h^2(|Du|)\,dx \le c\left(R^{n-2(n-\lambda)/q} + \int_0^\infty \frac{d}{dt}(H(t))\,m_{R/2}(t)\,dt\right) \\
&=: c\left(R^{n-2(n-\lambda)/q} + J\right).
\end{aligned} \tag{5.9}$$

The term J in the previous inequality can be estimated in the same way as in (5.4) and so (5.9) and (5.4) give us

$$II \le c\left(\frac{1}{\ln^s(e+t_0)}\int_{B_R} |Du|^2\,dx + \frac{t_0}{\ln^s(e+t_0^2)}R^n + R^{n-2(n-\lambda)/q}\right). \tag{5.10}$$

We can estimate III by means of Lemma 4.3 (with $\eta = 0$) and we have

$$III \le cR^{\lambda_0/q_0}. \tag{5.11}$$

Now (5.1) implies

$$\nu^2 \int_{B_{R/2}} |Dw|^2\,dx \le c\left\{\left[\mathcal{N}^2_{2r',R}(A;\Omega_0) + \frac{1}{\ln^s(\mathrm{e}+t_0)}\right]\int_{B_R}|Du|^2\,dx \right. \\ \left. + \left(\mathcal{N}^2_{2r',R}(A;\Omega_0)+1\right)R^{n-2(n-\lambda)/q} + \frac{t_0}{\ln^s(\mathrm{e}+t_0^2)}R^n + R^{\lambda_0/q_0}\right\}. \tag{5.12}$$

The function $v = u - w \in W^{1,2}(B_{R/2},\mathbb{R}^N)$ is the solution to the system

$$\int_{B_{R/2}} \langle (A)_{R/2}Dv, D\varphi\rangle\,dx = 0, \qquad \forall \varphi \in W^{1,2}_0(B_{R/2},\mathbb{R}^N)$$

and Lemma 4.1 gives us, for $0 < \sigma \le R/2$,

$$\int_{B_\sigma} |Dv|^2\,dx \le c\left(\frac{\sigma}{R}\right)^n \int_{B_{R/2}} |Dv|^2\,dx.$$

Inequality (5.12) and the last estimate give us, for all $0 < \sigma \le R$, the following estimate:

$$\begin{aligned}\int_{B_\sigma} |Du|^2\,dx \le& c_1\left[\left(\frac{\sigma}{R}\right)^n + \mathcal{N}^2_{2r',R}(A;\Omega_0) + \frac{1}{\ln^s(\mathrm{e}+t_0)}\right]\int_{B_R}|Du|^2\,dx \\ &+ c_2\left[\left(\mathcal{N}^2_{2r',R}(A;\Omega_0)+1\right)R^{n-2(n-\lambda)/q} + \frac{t_0}{\ln^s(\mathrm{e}+t_0^2)}R^n + R^{\lambda_0/q_0}\right] \\ \le& c_1\left[\left(\frac{\sigma}{R}\right)^n + \mathcal{N}^2_{2r',R}(A;\Omega_0) + \frac{1}{\ln^s(\mathrm{e}+t_0)}\right]\int_{B_R}|Du|^2\,dx + c_2\,R^{\lambda'},\end{aligned}$$

where the constants c_1 and c_2 only depend on the above-mentioned parameters and $\lambda' = \min\{n - 2(n-\lambda)/q, \lambda_0/q_0\}$ ($\lambda' < n$). For $\eta = 0$ (see Lemma 4.3) we have $\lambda' = \min\{n - 2(n-\lambda)/q, 2 + n(1-\gamma_0)\}$. Set

$$\phi(\sigma) = \int_{B_\sigma} |Du|^2\,dx, \qquad E_1 = c_1, \qquad E_2 = c_1\left(\mathcal{N}^2_{2r',R}(A;\Omega_0) + \frac{1}{\ln^s(\mathrm{e}+t_0)}\right), \qquad D = c_2.$$

Further, we can choose $k < 1$ such that $E_1 k^{n-\lambda'} < 1/2$. It is obvious (the coefficients A are *VMO*) that the constants $R_0 > 0$ and $t_0 > 0$ exist such that $E_2 k^{-\lambda'} < 1/2$, then $E_1 k^{n-\lambda'} + E_2 k^{-\lambda'} < 1$. For all $0 < \sigma \le R \le \min\{d_0, R_0\}$ the assumptions of Lemma 4.2 are satisfied and therefore

$$\int_{B_\sigma} |Du|^2\,dx \le c\sigma^{\lambda'}, \qquad \forall\,\sigma \le \min\{d_0, R_0\}.$$

The remaining part of the proof is analogous to the corresponding part of the proof of Theorem 3.1. □

Proof of Theorem 3.3. Theorem 3.1 gives that $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN})$ for arbitrary $\lambda < n$ and, consequently, $u \in C^{0,\alpha}(\Omega,\mathbb{R}^N)$ for each $\alpha \in (0,1)$. Let $B_{R/2}(x_0) \subset B_R(x_0) \subset \Omega$ be an arbitrary ball and let $w \in W^{1,2}_0(B_{R/2}(x_0),\mathbb{R}^N)$ be a solution to the system (we denote $B_R = B_R(x_0)$ and $u_R = u_{x_0,R}$)

$$\int_{B_{R/2}} \langle (A)_{R/2} Dw, D\varphi \rangle \, dx = \int_{B_{R/2}} \langle ((A)_{R/2} - A(x))\, Du, D\varphi \rangle \, dx$$
$$- \int_{B_{R/2}} \langle g(x,u,Du) - (g(x,u,Du))_{R/2}, D\varphi \rangle \, dx$$
$$+ \int_{B_{R/2}} \langle b(x,u,Du), \varphi \rangle \, dx \tag{5.13}$$

for every $\varphi \in W_0^{1,2}(B_{R/2}, \mathbb{R}^N)$. It is known that, under the assumption of the theorem, such solution exists and, it is unique for all $R < R'$ (R' is sufficiently small, $R' \le 1$). We can put $\varphi = w$ in (5.13) and using the ellipticity, Hölder's and Sobolev's inequalities, we get

$$\nu^2 \int_{B_{R/2}} |Dw|^2 \, dx \le c \Bigg(\int_{B_{R/2}} |A_{R/2} - A(x)|^2 |Du|^2 \, dx + \int_{B_{R/2}} |g(x,u,Du) - (g(x,u,Du))_{R/2}|^2 \, dx$$
$$+ \left(\int_{B_{R/2}} |b(x,u,Du)|^{2q_0} \, dx \right)^{1/q_0} \Bigg) =: c(I + II + III). \tag{5.14}$$

The estimate of I is analogous to that in the proof of Theorem 3.1, but here we have to use the Hölder continuity of coefficients, which is the crucial assumption for obtaining some reasonable estimate (using the information at the beginning of the proof).

$$I \le cR^{2\alpha} \int_{B_{R/2}} |Du|^2 \, dx \le cR^n,$$

where $\alpha \in (0,1]$ is a given constant.

Further, we estimate the second integral on the right hand side of (5.14). From the assumption (3.7) and by means of the Young inequality, we obtain

$$II \le \fint_{B_{R/2}} \left(\int_{B_{R/2}} |g(x,u(x),Du(x)) - g(y,u(y),Du(y))|^2 \, dy \right) dx$$
$$\le 3M \fint_{B_{R/2}} \left(\int_{B_{R/2}} |F(x) - F(y)|^2 \, dy \right) dx + 3M \fint_{B_{R/2}} \left(\int_{B_{R/2}} (|u(x)| + |u(y)|)^{2\delta} \, dy \right) dx$$
$$+ 3M \fint_{B_{R/2}} \left(\int_{B_{R/2}} h^2(|Du(x) - Du(y)|) \, dy \right) dx$$
$$\le 12M \int_{B_{R/2}} |F(x) - F_{R/2}|^2 \, dx + 3 \cdot 2^{\delta - n} M \kappa_n \|u\|^{2\delta}_{C(B_{R/2}, \mathbb{R}^N)} R^n$$
$$+ 3M \fint_{B_{R/2}} \left(\int_{B_{R/2}} \frac{|Du(x) - Du(y)|^2}{\ln^s(\mathrm{e} + |Du(x) - Du(y)|^2)} \, dy \right) dx$$
$$= 12M \int_{B_{R/2}} |F(x) - F_{R/2}|^2 \, dx + 3 \cdot 2^{\delta - n} M \kappa_n \|u\|^{2\delta}_{C(B_{R/2}, \mathbb{R}^N)} R^n + 3M \fint_{B_{R/2}} \left(\int_{B_{R/2}} K(x,y) \, dy \right) dx.$$

Using the fact that the function $l(t) = t^2 / \ln^s(\mathrm{e} + t^2)$ is nondecreasing and convex on $[0, \infty)$ if $0 < s \le 1$ and so the function $k(z) = l(|z|)$ is convex on $\mathbb{R}^{nN}$, we can get the estimate

$$K(x,y) = \frac{|Du(x) - (Du)_{R/2} + (Du)_{R/2} - Du(y)|^2}{\ln^s(\mathrm{e} + |Du(x) - (Du)_{R/2} + (Du)_{R/2} - Du(y)|^2)}$$
$$\le \frac{1}{2} \frac{|2(Du(x) - (Du)_{R/2})|^2}{\ln^s(\mathrm{e} + |2(Du(x) - (Du)_{R/2})|^2)} + \frac{1}{2} \frac{|2(Du(y) - (Du)_{R/2})|^2}{\ln^s(\mathrm{e} + |2(Du(y) - (Du)_{R/2})|^2)}$$

which gives

$$\fint_{B_{R/2}} \left(\int_{B_{R/2}} K(x,y)\,dy \right) dx \le \int_{B_{R/2}} \frac{4|Du(x)-(Du)_{R/2}|^2}{\ln^s(\mathrm{e}+4|Du(x)-(Du)_{R/2}|^2)}\,dx =: J_A$$

and so

$$II \le 12M \int_{B_{R/2}} |F(x)-F_{R/2}|^2\,dx + 3.2^{\delta-n} M\kappa_n \|u\|^{2\delta}_{C(B_{R/2},\mathbb{R}^N)} R^n + 3M\,J_A. \tag{5.15}$$

The quantity J_A can be estimated in a way, analogous to that in (5.4). Putting $m_R(t) = m(\{y \in B_R(x_0) : 4\,|Du-(Du)_R|^2 > t\})$ and $H(t) = t/\ln^s(\mathrm{e}+t)$, $t \in [0,\infty)$, we get (through Lemmas 4.5 and 4.6)

$$\begin{aligned} J_A &= \int_0^\infty \frac{d}{dt}\,(H(t))\,m_{R/2}(t)\,dt \\ &= \int_0^{t_0} \frac{d}{dt}\left(\frac{t}{\ln^s(\mathrm{e}+t)}\right) m_{R/2}(t)\,dt + \int_{t_0}^\infty \frac{d}{dt}\left(\frac{t}{\ln^s(\mathrm{e}+t)}\right) m_{R/2}(t)\,dt \\ &\le \frac{4}{\ln^s(\mathrm{e}+t_0)} \int_{B_R} |Du-(Du)_R|^2\,dx + \frac{t_0}{\ln^s(\mathrm{e}+t_0)} R^n. \end{aligned} \tag{5.16}$$

From (5.15) and (5.16) we have

$$II \le c\left(\frac{4}{\ln^s(\mathrm{e}+t_0)} \int_{B_R} |Du-(Du)_R|^2\,dx + M\left(\frac{t_0}{M\ln^s(\mathrm{e}+t_0)} + \|F\|_{\mathcal{L}^{2,n}(\Omega)} + \|u\|^{2\delta}_{C(B_{R/2},\mathbb{R}^N)} \right) R^n \right).$$

The term III we can estimate in the following manner. In Lemma 4.3 (remember that $\lambda = n$), thanks to Theorem 3.1, the parameter $\eta < n$ can be chosen arbitrarily close to n. Consequently, λ_0 can be bigger than value nq_0 and so

$$III \le cR^n.$$

Now, using the estimates I, II, III, from (5.14) we have

$$\nu^2 \int_{B_{R/2}} |Dw|^2\,dx \le c\,\frac{1}{\ln^s(\mathrm{e}+t_0)} \int_{B_R} |Du(x)-(Du)_R|^2\,dx + c\,R^n. \tag{5.17}$$

The function $v = u - w \in W^{1,2}(B_{R/2},\mathbb{R}^N)$ is the solution to the system

$$\int_{B_{R/2}} \langle (A)_{R/2} Dv, D\varphi \rangle\ dx = 0, \qquad \forall \varphi \in W_0^{1,2}(B_{R/2},\mathbb{R}^N)$$

and Lemma 4.1 gives us, for $0 < \sigma \le R/2$,

$$\int_{B_\sigma} |Dv(x)-(Dv)_\sigma|^2\,dx \le c\left(\frac{\sigma}{R}\right)^{n+2} \int_{B_{R/2}} |Dv-(Dv)_{R/2}|^2\,dx.$$

Inequality (5.17) and the last estimate give us, for all $0 < \sigma \le R$, the following estimate:

$$\int_{B_\sigma} |Du(x)-(Du)_\sigma|^2\,dx \le c_1\left[\left(\frac{\sigma}{R}\right)^{n+2} + \frac{1}{\ln^s(\mathrm{e}+t_0)}\right] \int_{B_R} |Du(x)-(Du)_R|^2\,dx + c_2\,R^n,$$

where the constants c_1 and c_2 only depend on the above-mentioned parameters.

If we put $\phi(R) = \int_{B_R} |Du(x)-(Du)_R|^2\,dx$, $\alpha = n+2$, $\beta = n$, $E_1 = c_1$, $E_2 = c_1/\ln^s(\mathrm{e}+t_0)$, $D = c_2$ and use Lemma 4.2, the result follows in a standard way, analogous to those in the previous proofs. So we can conclude that $Du \in \mathcal{L}^{2,n}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN})$. □

Remark 5.1. It is known that for weak solutions $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ to the system (1.1) the Hölder continuity of their gradients is, broadly speaking, equivalent to the fact that the condition of Liouville type is satisfied (see [12, Chapter 6] for precise information). Later the first author of the paper proved in [3] that the same holds under the assumption that gradients of weak solutions belong to the class $\mathcal{L}^{2,n}(\Omega, \mathbb{R}^{nN})$. So the paper [4] and the statement of Theorem 3.3 could be seen as contributions to the above mentioned theory.

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