# On the solvability of a boundary value problem for $p$-Laplacian differential equations 

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#### Abstract

Using barrier strip conditions, we study the existence of $C^{2}[0,1]$-solutions of the boundary value problem $\left(\phi_{p}\left(x^{\prime}\right)\right)^{\prime}=f\left(t, x, x^{\prime}\right), x(0)=A, x^{\prime}(1)=B$, where $\phi_{p}(s)=s|s|^{p-2}, p>2$. The question of the existence of positive monotone solutions is also affected.


Keywords: boundary value problem, second order differential equation, $p$-Laplacian, existence, sign conditions.
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## 1 Introduction

This paper is devoted to the solvability of the boundary value problem (BVP)

$$
\begin{align*}
\left(\phi_{p}\left(x^{\prime}\right)\right)^{\prime} & =f\left(t, x, x^{\prime}\right), \quad t \in[0,1]  \tag{1.1}\\
x(0) & =A, \quad x^{\prime}(1)=B \tag{1.2}
\end{align*}
$$

Here $\phi_{p}(s)=s|s|^{p-2}, p>2$, the scalar function $f(t, x, y)$ is defined for $(t, x, y) \in[0,1] \times D_{x} \times$ $D_{y}$, where the sets $D_{x}, D_{y} \subseteq \mathbf{R}$ may be bounded, and $B \geq 1$. Besides, $f$ is continuous on a suitable subset of its domain.

The solvability of various singular and nonsingular BVPs with $p$-Laplacian has been studied, for example, in $[1-5,7-12,14]$. Conditions used in these works or do not allow the main nonlinearity to change sign, $[2,11]$, or impose a growth restriction on it, $[3,9,11]$, or require the existence of upper and lower solutions, $[1,3,5,8,9,12]$; other type conditions have been used in [7], where the main nonlinearity may changes its sign. As a rule, the obtained results guarantee the existence of positive solutions.

Another type of conditions have been used in [10] for studying the solvability of (1.1), (1.2) in the case $p \in(1,2)$. The existence of at least one positive and monotone $C^{2}[0,1]$-solution is established therein under the following barrier condition:

[^0]H. There are constants $L_{i}, F_{i}, i=1,2$, and a sufficiently small $\sigma>0$ such that
\[

$$
\begin{align*}
& F_{1} \geq F_{2}+\sigma, \quad F_{1}-\sigma>0, \quad L_{2}-\sigma \geq L_{1}, \\
& {[A-\sigma, L+\sigma] } \subseteq D_{x}, \quad\left[F_{2}, L_{2}\right] \subseteq D_{y}, \quad \text { where } L=L_{1}+|A|, \\
& f(t, x, y) \geq 0 \quad \text { for } \quad(t, x, y) \in[0,1] \times D_{x} \times\left[L_{1}, L_{2}\right],  \tag{1.3}\\
& f(t, x, y) \leq 0 \quad \text { for } \quad(t, x, y) \in[0,1] \times D_{A} \times\left[F_{2}, F_{1}\right], \tag{1.4}
\end{align*}
$$
\]

where the constants $m$ and $M$ are, respectively, the minimum and the maximum of $f(t, x, p)$ on $[0,1] \times[A-\sigma, L+\sigma] \times\left[F_{1}-\sigma, L_{1}+\sigma\right]$ and $D_{A}=(-\infty, L] \cap D_{x}$.

Let us recall, the strips $[0,1] \times\left[L_{1}, L_{2}\right]$ and $[0,1] \times\left[F_{2}, F_{1}\right]$ are called "barrier" because they limit the values of the first derivatives of all $C^{2}[0,1]$-solution of (1.1), (1.2) between themselves. Recently, it was shown in [13] that conditions of form (1.3) and (1.4) guarantee $C^{1}[0,1]$-solutions to the $\phi$-Laplacian equation

$$
\left(\phi\left(x^{\prime}\right)\right)^{\prime}=f\left(t, x, x^{\prime}\right), \quad t \in(0,1)
$$

with boundary conditions (1.2), where $\phi: \mathbf{R} \rightarrow \mathbf{R}$ is an increasing homeomorphism and $f:[0,1] \times \mathbf{R}^{2} \rightarrow \mathbf{R}$ is continuous.

It turned out that the cases $1<p<2$ and $p>2$ require different technical approaches for the use of $\mathbf{H}$ for studying the solvability of (1.1), (1.2). So, in the present paper we show that $\mathbf{H}$ with the additional requirement

$$
\begin{equation*}
B-M \geq F_{1} \tag{1.5}
\end{equation*}
$$

guarantees the existence of at least one monotone, and positive in the case $A \geq 0, C^{2}[0,1]$ solution to (1.1), (1.2) with $p>2$. In fact, our main result is the following.

Theorem 1.1. Let $\mathbf{H}$ and (1.5) hold, and $f(t, x, y)$ be continuous on the set $[0,1] \times[A-\sigma, L+\sigma]$ $\times\left[F_{1}-\sigma, L_{1}+\sigma\right]$. Then BVP (1.1), (1.2) has at least one strictly increasing solution in $C^{2}[0,1]$ for each $p \in(2, \infty)$.

The paper is organized as follows. In Section 2 we present preliminaries needed to formulate the Topological Transversality Theorem, which is our basic tool, and prove auxiliary results. In Section 3 we give the proof of Theorem 1.1, formulate a corollary and give an example.

## 2 Fixed point theorem, auxiliary results

Let $K$ be a convex subset of a Banach space $E$ and $U \subset K$ be open in $K$. Let $\mathbf{L}_{\partial u}(\bar{U}, K)$ be the set of compact maps from $\bar{U}$ to $K$ which are fixed point free on $\partial U$; here, as usual, $\bar{U}$ and $\partial U$ are the closure of $U$ and boundary of $U$ in $K$.

A map $F$ in $\mathbf{L}_{\partial U}(\bar{U}, K)$ is essential if every map $G$ in $\mathbf{L}_{\partial U}(\bar{U}, K)$ such that $G / \partial U=F / \partial U$ has a fixed point in $U$. It is clear, in particular, every essential map has a fixed point in $U$.

The following fixed point theorem due to A. Granas et al. [6].

Theorem 2.1 (Topological transversality theorem). Suppose:
(i) $F, G: \bar{U} \rightarrow K$ are compact maps;
(ii) $G \in \mathbf{L}_{\partial u}(\bar{U}, K)$ is essential;
(iii) $H(x, \lambda), \lambda \in[0,1]$, is a compact homotopy joining $G$ and $F$, i.e. $H(x, 0)=G(x)$ and $H(x, 1)=$ $F(x)$;
(iv) $H(x, \lambda), \lambda \in[0,1]$, is fixed point free on $\partial U$.

Then $H(x, \lambda), \lambda \in[0,1]$, has at least one fixed point in $U$ and in particular there is a $x_{0} \in U$ such that $x_{0}=F\left(x_{0}\right)$.

The following results is important for our consideration. It can be found also in [6].
Theorem 2.2. Let $l \in U$ be fixed and $F \in \mathbf{L}_{\partial u}(\bar{U}, K)$ be the constant map $F(x)=l$ for $x \in \bar{U}$. Then $F$ is essential.

Further, we need the following fact.
Proposition 2.3. Let the constants $B$ and $M$ be such that $B \geq 1$ and $B>M>0$. Then

$$
(B-M)^{r} \leq B^{r}-M \quad \text { for } r \in[1, \infty) .
$$

Proof. The inequality is evident for $r=1$. For $M \in(0, B)$ consider the function $g(r)=$ $(B-M)^{r}-B^{r}+M, r \in(1, \infty)$. First, let $B-M \in(0,1)$. Then $\ln (B-M)<0$ and so

$$
g^{\prime}(r)=(B-M)^{r} \ln (B-M)-B^{r} \ln B<0 \quad \text { for } r \in \mathbf{R} .
$$

Next, assume $B-M=1$. Now we get

$$
g^{\prime}(r)=-(1+M)^{r} \ln (1+M)<0 \quad \text { for } r \in \mathbf{R} .
$$

Finally, let $B-M \in(1, \infty)$. In this case from $B>B-M>0$ we have $B^{r} \geq(B-M)^{r}$ for $r \in[0, \infty)$ and so

$$
g^{\prime}(r) \leq B^{r} \ln (B-M)-B^{r} \ln B=B^{r} \ln \frac{B-M}{B}<0 \quad \text { for } r \in[0, \infty) .
$$

In summary, we have proved that $g^{\prime}(r)<0$ for each $r \in[0, \infty)$. Then, the result follows from the fact that $g(1)=0$.

Let us emphasize explicitly that we conduct the rest consideration of this section for an arbitrary fixed $p>2$.

For $\lambda \in[0,1]$ consider the family of BVPs

$$
\begin{cases}\left(\phi_{p}\left(x^{\prime}\right)\right)^{\prime}=\lambda f\left(t, x, x^{\prime}\right), & t \in[0,1],  \tag{2.1}\\ x(0)=A, x^{\prime}(1)=B, & B \geq 1,\end{cases}
$$

where $f:[0,1] \times D_{x} \times D_{y} \rightarrow \mathbf{R}, D_{x}, D_{y} \subseteq \mathbf{R}$. Since

$$
\phi_{p}(s)=s|s|^{p-2}= \begin{cases}s^{p-1}, & s \geq 0 \\ -(-s)^{p-1}, & s<0\end{cases}
$$

we have

$$
\phi_{p}^{\prime}(s)=\left\{\begin{array}{ll}
(p-1) s^{p-2}, & s \geq 0 \\
(p-1)(-s)^{p-2}, & s<0
\end{array}=(p-1)|s|^{p-2}\right.
$$

and $\left(\phi_{p}\left(x^{\prime}(t)\right)\right)^{\prime}=(p-1)\left|x^{\prime}(t)\right|^{p-2} x^{\prime \prime}(t)$, if $x^{\prime \prime}(t)$ exists. So, we can write (2.1) as

$$
\left\{\begin{array}{l}
(p-1)\left|x^{\prime}(t)\right|^{p-2} x^{\prime \prime}(t)=\lambda f\left(t, x, x^{\prime}\right), t \in[0,1],  \tag{2.1'}\\
x(0)=A, x^{\prime}(1)=B .
\end{array}\right.
$$

For convenience set

$$
m_{p}=\frac{m}{(p-1)\left(F_{1}-\sigma\right)^{p-2}} \quad \text { and } \quad M_{p}=\frac{M}{(p-1)\left(F_{1}-\sigma\right)^{p-2}},
$$

where $F_{1}, \sigma, m$ and $M$ are as in $\mathbf{H}$.
The next result gives a priori bounds for the $C^{2}[0,1]$-solutions of family (2.1') (as well as of (2.1)).

Lemma 2.4. Let $\mathbf{H}$ hold and $x \in C^{2}[0,1]$ be a solution to family (2.1'). Then

$$
A \leq x(t) \leq L, F_{1} \leq x^{\prime}(t) \leq L_{1} \text { and } m_{p} \leq x^{\prime \prime}(t) \leq M_{p} \text { for } t \in[0,1] .
$$

Proof. The proof of the bounds for $x$ and $x^{\prime}$ is the same as the corresponding part of the proof of [10, Lemma 3.1], but we will state it for completeness. So, assume on the contrary that

$$
\begin{equation*}
x^{\prime}(t) \leq L_{1} \quad \text { for } t \in[0,1] \tag{2.2}
\end{equation*}
$$

is not true. Then, $x^{\prime}(1)=B \leq L_{1}$ together with $x^{\prime} \in C[0,1]$ implies that

$$
S_{+}=\left\{t \in[0,1]: L_{1}<x^{\prime}(t) \leq L_{2}\right\}
$$

is not empty. Moreover, there exists an interval $[\alpha, \beta] \subset S_{+}$with the property

$$
\begin{equation*}
x^{\prime}(\alpha)>x^{\prime}(\beta) . \tag{2.3}
\end{equation*}
$$

Then, by the fundamental theorem of calculus applied to $x^{\prime}$, (2.3) implies that there is a $\gamma \in$ $(\alpha, \beta)$ such that

$$
x^{\prime \prime}(\gamma)<0 .
$$

We have $\left(\gamma, x(\gamma), x^{\prime}(\gamma)\right) \in S_{+} \times D_{x} \times\left(L_{1}, L_{2}\right]$, which yields

$$
f\left(\gamma, x(\gamma), x^{\prime}(\gamma)\right) \geq 0,
$$

by (1.3). Then,

$$
0>(p-1)\left|x^{\prime}(\gamma)\right|^{p-2} x^{\prime \prime}(\gamma)=\lambda f\left(\gamma, x(\gamma), x^{\prime}(\gamma)\right) \geq 0 \quad \text { for } \lambda \in[0,1],
$$

a contradiction. Thus, (2.2) is true.
By the mean value theorem, for each $t \in(0,1]$ there exists $\xi \in(0, t)$ such that $x(t)-x(0)=$ $x^{\prime}(\xi) t$, which yields

$$
x(t) \leq L \quad \text { for } t \in[0,1] .
$$

Arguing as above and using (1.4), we establish $x^{\prime}(t) \geq F_{1}$ for all $t \in[0,1]$ and, as a consequence, $x(t) \geq A$ on $[0,1]$.

To reach the bounds for $x^{\prime \prime}(t)$ from

$$
x^{\prime}(t)>F_{1}-\sigma>0, \quad t \in[0,1]
$$

we obtain firstly

$$
0<\frac{1}{(p-1)\left(x^{\prime}(t)\right)^{p-2}} \leq \frac{1}{(p-1)\left(F_{1}-\sigma\right)^{p-2}} .
$$

Next, multiplying both sides of this inequality by $\lambda M \geq 0$ and $\lambda m \leq 0$, for $t \in[0,1]$ obtain respectively

$$
\frac{\lambda M}{(p-1)\left(x^{\prime}(t)\right)^{p-2}} \leq \frac{\lambda M}{(p-1)\left(F_{1}-\sigma\right)^{p-2}} \leq \frac{M}{(p-1)\left(F_{1}-\sigma\right)^{p-2}}=M_{p}
$$

and

$$
\frac{\lambda m}{(p-1)\left(x^{\prime}(t)\right)^{p-2}} \geq \frac{\lambda m}{(p-1)\left(F_{1}-\sigma\right)^{p-2}} \geq \frac{m}{(p-1)\left(F_{1}-\sigma\right)^{p-2}}=m_{p}
$$

from $f\left(t, x, L_{1}\right) \geq 0$ for $(t, x) \in[0,1] \times[A-\sigma, L+\sigma]$ and $f\left(t, x, F_{1}\right) \leq 0$ for $(t, x) \in[0,1]$ $\times[A-\sigma, L+\sigma]$, it follows that $M \geq 0$ and $m \leq 0$.

On the other hand,

$$
m \leq f\left(t, x(t), x^{\prime}(t)\right) \leq M \quad \text { for } t \in[0,1]
$$

since $\left(x(t), x^{\prime}(t)\right) \in[A, L] \times\left[F_{1}, L_{1}\right]$ for each $t \in[0,1]$. Multiplying the last inequality by $\lambda(p-1)^{-1}\left(x^{\prime}(t)\right)^{2-p} \geq 0, \lambda, t \in[0,1]$, we arrive to

$$
m_{p} \leq \frac{\lambda m}{(p-1)\left(x^{\prime}(t)\right)^{p-2}} \leq \frac{\lambda f\left(t, x(t), x^{\prime}(t)\right)}{(p-1)\left(x^{\prime}(t)\right)^{p-2}} \leq \frac{\lambda M}{(p-1)\left|x^{\prime}(t)\right|^{p-2}} \leq M_{p}
$$

for all $\lambda, t \in[0,1]$, from where, keeping in mind that $x^{\prime}(t)>0$ on $[0,1]$, we get

$$
m_{p} \leq \frac{\lambda f\left(t, x(t), x^{\prime}(t)\right)}{(p-1)\left|x^{\prime}(t)\right|^{p-2}} \leq M_{p} \quad \text { for all } \lambda, t \in[0,1]
$$

which yields the required bounds for $x^{\prime \prime}(t)$.
Now, introduce sets

$$
C_{+}^{1}[0,1]=\left\{x \in C^{1}[0,1]: x(t)>0 \text { on }[0,1], x(1)=\phi_{p}(B)\right\}
$$

and, in case that $\mathbf{H}$ holds,

$$
V=\left\{x \in C^{1}[0,1]: A-\sigma \leq x \leq L+\sigma, F_{1}-\sigma \leq x^{\prime} \leq L_{1}+\sigma\right\} .
$$

Introduce also the map $\Lambda_{\lambda}: V \rightarrow C_{+}^{1}[0,1]$ defined by

$$
\Lambda_{\lambda} x=\lambda \int_{1}^{t} f\left(s, x(s), x^{\prime}(s)\right) d s+\phi_{p}(B) \quad \text { for } \lambda \in[0,1] .
$$

Lemma 2.5. Let $\mathbf{H}$ hold and

$$
\begin{equation*}
f(t, x, y) \in C\left([0,1] \times[A-\sigma, L+\sigma] \times\left[F_{1}-\sigma, L_{1}+\sigma\right]\right) \tag{2.4}
\end{equation*}
$$

Then $\Lambda_{\lambda}, \lambda \in[0,1]$, is well defined and continuous.

Proof. Clearly, because of (2.4), $\left(\Lambda_{\lambda} x\right)^{\prime}(t)=\lambda f\left(t, x(t), x^{\prime}(t)\right), x \in V$, is continuous on [0,1] for each $\lambda \in[0,1]$. Next, observe that for each $x \in V$ we have

$$
\lambda f\left(t, x(t), x^{\prime}(t)\right) \leq \lambda M \leq M \quad \text { for } \lambda, t \in[0,1]
$$

Integrating this inequality from 1 to $t, t \in[0,1)$, we get

$$
\lambda \int_{1}^{t} f\left(s, x(s), x^{\prime}(s)\right) d s \geq M(t-1), \quad t \in[0,1]
$$

from where it follows

$$
\lambda \int_{1}^{t} f\left(s, x(s), x^{\prime}(s)\right) d s \geq-M, \quad t \in[0,1]
$$

and

$$
-M+\phi_{p}(B) \leq\left(\Lambda_{\lambda} x\right)(t), \quad t \in[0,1]
$$

By (1.5) and Proposition 2.3, we have

$$
0<\left(F_{1}-\sigma\right)^{p-1}<(B-M)^{p-1} \leq-M+B^{p-1}=-M+\phi_{p}(B)
$$

and then,

$$
0<\left(F_{1}-\sigma\right)^{p-1}<\left(\Lambda_{\lambda} x\right)(t), \quad t \in[0,1]
$$

Obviously, $\left(\Lambda_{\lambda} x\right)(1)=\phi_{p}(B)$. Finally, (2.4) implies that the map $\Lambda_{\lambda}, \lambda \in[0,1]$, is continuous on $V$.

Further, introduce the sets

$$
\begin{aligned}
C_{B C}^{2}[0,1] & =\left\{x \in C^{2}[0,1]: x(0)=A, x^{\prime}(1)=B\right\} \\
K & =\left\{x \in C_{B C}^{2}[0,1]: x^{\prime}(t)>0 \text { on }[0,1]\right\}
\end{aligned}
$$

and the $\operatorname{map} \Phi_{p}: K \rightarrow C_{+}^{1}[0,1]$ defined by $\Phi_{p} x=\phi_{p}\left(x^{\prime}\right)$.
Lemma 2.6. The map $\Phi_{p}$ is well defined and continuous.
Proof. For each $x \in K$ we have $x^{\prime}(t)>0, t \in[0,1]$. Then,

$$
\begin{equation*}
\left(\Phi_{p} x\right)(t)=x^{\prime}(t)\left|x^{\prime}(t)\right|^{p-2}=x^{\prime}(t)^{p-1}>0 \quad \text { on }[0,1] \tag{2.5}
\end{equation*}
$$

and, obviously, $\left(\Phi_{p} x\right)^{\prime}(t)=(p-1)\left(x^{\prime}(t)\right)^{p-2} x^{\prime \prime}(t)$ is continuous on $[0,1]$. Also, $\left(\Phi_{p} x\right)(1)=$ $x^{\prime}(1)\left|x^{\prime}(1)\right|^{p-2}=\phi_{p}(B)$. So, $\Phi_{p} x \in C_{+}^{1}[0,1]$. The continuity of $\Phi_{p}$ follows from $x^{\prime} \in C[0,1]$ and (2.5).

It is well known that the inverse function of $\phi_{p}(s)$ is $\phi_{q}(s)=s|s|^{q-2}, q^{-1}+p^{-1}=1, p>1$. Using it, we introduce the map $\Phi_{q}: C_{+}^{1}[0,1] \rightarrow K$, defined by

$$
\left(\Phi_{q} y\right)(t)=\int_{0}^{t} \phi_{q}(y(s)) d s+A, \quad t \in[0,1]
$$

But, for $y \in C_{+}^{1}[0,1]$ we have $y(t)>0$ on $[0,1]$ and so

$$
\left(\Phi_{q} y\right)(t)=\int_{0}^{t}(y(s))^{\frac{1}{p-1}} d s+A, \quad t \in[0,1]
$$

Lemma 2.7. The map $\Phi_{q}: C_{+}^{1}[0,1] \rightarrow K$ is well defined, the inverse map of $\Phi_{p}$ and continuous.
Proof. For each fixed $y \in C_{+}^{1}[0,1]$ we get a unique $x(t)=\left(\Phi_{q} y\right)(t)=\int_{0}^{t}(y(s))^{\frac{1}{p-1}} d s+A$. In fact, to establish the veracity of the first two assertions, we have to show that $x \in K$ or, what is the same, to show that $x$ is a unique $C^{2}[0,1]$-solution to the BVP

$$
\begin{equation*}
x^{\prime}\left|x^{\prime}\right|^{p-2}=y, \quad t \in[0,1], \quad x(0)=A, \quad x^{\prime}(1)=B \tag{2.6}
\end{equation*}
$$

with $x^{\prime}(t)>0$ on $[0,1]$.
The last follows immediately from $x^{\prime}(t)=(y(t))^{\frac{1}{p-1}}$ on $[0,1]$. Then, $x^{\prime}\left|x^{\prime}\right|^{p-2}=\left(x^{\prime}(t)\right)^{p-1}=$ $y(t)$ for $t \in[0,1]$. Besides, $x^{\prime}(1)=(y(1))^{\frac{1}{p-1}}=\left(\phi_{p}(B)\right)^{\frac{1}{p-1}}=B$ and $x(0)=A$. Now, the continuity of $y^{\prime}(t)$ and $y(t)>0$ on $[0,1]$ imply that

$$
x^{\prime \prime}(t)=\frac{1}{p-1}(y(t))^{\frac{2-p}{p-1}} y^{\prime}(t)
$$

exists and is continuous on $[0,1]$. Thus, $x(t)$ is a solution to (2.6) and is in $C^{2}[0,1]$.
To complete the proof we just have to observe that the continuity of $\Phi_{q}$ follows from the continuity of $y^{1 /(p-1)}(t)$ on $[0,1]$.

## 3 Proof of main result

Proof of Theorem 1.1. We will prove the assertion for an arbitrary fixed $p>2$. Introduce the set

$$
U=\left\{x \in K: A-\sigma<x<L+\sigma, F_{1}-\sigma<x^{\prime}<L_{1}+\sigma, m_{p}-\sigma<x^{\prime \prime}(t)<M_{p}+\sigma\right\}
$$

and consider the homotopy

$$
H_{\lambda}: \bar{U} \times[0,1] \rightarrow K
$$

defined by $H_{\lambda}(x):=\Phi_{q} \Lambda_{\lambda} j$, where $j: \bar{U} \rightarrow C^{1}[0,1]$ is the embedding $j x=x$. To show that all assumptions of Theorem 2.1 are fulfilled observe firstly that $U$ is an open subset of $K$, and $K$ is a convex subset of the Banach space $C^{2}[0,1]$. For the fixed points of $H_{\lambda}, \lambda \in[0,1]$, we have

$$
\Phi_{q} \Lambda_{\lambda} j(x)=x
$$

and

$$
\Phi_{p} x=\Lambda_{\lambda} j(x)
$$

which is the operator form of the family

$$
\left\{\begin{array}{l}
\phi_{p}\left(x^{\prime}\right)=\lambda \int_{1}^{t} f\left(s, x(s), x^{\prime}(s)\right) d s+\phi_{p}(B), t \in(0,1)  \tag{3.1}\\
x(0)=A, x^{\prime}(1)=B
\end{array}\right.
$$

Thus, the fixed points of $H_{\lambda}$ coincide with the $C^{2}[0,1]$-solutions of (3.1). But, it is obvious that each $C^{2}[0,1]$-solution of (3.1) is a $C^{2}[0,1]$-solution of (2.1). So, all conclusions of Lemma 2.4 are valid in particular and for the $C^{2}[0,1]$-solutions of (3.1) which allow us to conclude that the $C^{2}[0,1]$-solutions of (3.1) do not belong to $\partial U$ and so the homotopy is fixed point free on $\partial U$. On the other hand, it is well known that $j$ is completely continuous, that is, it maps each bounded set to a compact one. Thus, $j(\bar{U})$ is a compact set. Besides, it is clear that $j(\bar{U}) \subset V$. Then, according to Lemma $2.5, \Lambda_{\lambda}(j(\bar{U})) \subseteq C_{+}^{1}[0,1]$ is compact. Finally, the set
$\Phi_{q}\left(\Lambda_{\lambda}(j(\bar{U})) \subset K\right.$ is compact, by Lemma 2.7. So, the homotopy is compact. Now, since for $x \in \bar{U}$ we have $\Lambda_{0} j(x)=\phi_{p}(B)=B^{p-1}$, the map $H_{0}$ maps each $x \in \bar{U}$ to the unique solution $l=B t+A \in K$ to the BVP

$$
\begin{aligned}
x^{\prime} & =B, \quad t \in(0,1) \\
x(0) & =A, \quad x^{\prime}(1)=B,
\end{aligned}
$$

i.e., it is a constant map and so is essential, by Theorem 2.2. So, we can apply Theorem 2.1. It infers that the map $H_{1}(x)$ has a fixed point in $U$. It is easy to see that it is a $C^{2}[0,1]$-solution of the BVPs of families (3.1) and (2.1) obtained for $\lambda=1$ and, what is the same, of (1.1), (1.2).

An elementary consequence of the just proved theorem is the following.
Corollary 3.1. Let $A \geq 0, \mathbf{H}$ and (1.5) hold, and $f(t, x, y)$ be continuous for $(t, x, y) \in[0,1] \times$ $[A-\sigma, L+\sigma] \times\left[F_{1}-\sigma, L_{1}+\sigma\right]$. Then for each $p>2$ BVP (1.1), (1.2) has at least one strictly increasing solution in $C^{2}[0,1]$ with positive values on $(0,1]$.

We illustrate this result by the following example.
Example 3.2. Consider the BVP

$$
\begin{gathered}
\left(\phi_{p}\left(x^{\prime}\right)\right)^{\prime}=\frac{\left(2 x^{\prime}-1\right)\left(x^{\prime}-10\right)}{\sqrt{x+1}+100}, \quad t \in(0,1), \\
x(0)=2, \quad x^{\prime}(1)=5
\end{gathered}
$$

where $p>2$ is fixed.
It is easy to check that $\mathbf{H}$ holds for $F_{2}=1, F_{1}=2.1, L_{1}=11.9, L_{2}=13$ and $\sigma=0.1$; moreover, we can take $L=14, m=-0.5$ and $M=0.5$. The function $f(t, x, y)=\frac{(2 y-1)(y-10)}{\sqrt{x+1}+100}$ is continuous for $(t, x, y) \in[0,1] \times[2,14] \times[2.1,11.9]$. Thus, we can apply Corollary 3.1 to conclude that this BVP has a positive strictly increasing solution in $C^{2}[0,1]$.

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