




## Distributional, differential and integral problems: equivalence and existence results

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**Abstract.** We are interested in studying the matter of equivalence of the following problems:

$$\begin{aligned} Dx &= f(t, x)Dg \\ x(0) &= x_0 \end{aligned} \quad (1)$$

where  $Dx$  and  $Dg$  stand for the distributional derivatives of  $x$  and  $g$ , respectively;

$$\begin{aligned} x'_g(t) &= f(t, x(t)), \quad m_g\text{-a.e.} \\ x(0) &= x_0 \end{aligned} \quad (2)$$

where  $x'_g$  denotes the  $g$ -derivative of  $x$  (in a sense to be specified in Section 2) and  $m_g$  is the variational measure induced by  $g$ ; and

$$x(t) = x_0 + \int_0^t f(s, x(s))dg(s), \quad (3)$$

where the integral is understood in the Kurzweil–Stieltjes sense.

We prove that, for regulated functions  $g$ , (1) and (3) are equivalent if  $f$  satisfies a bounded variation assumption. The relation between problems (2) and (3) is described for very general  $f$ , though, more restrictive assumptions over the function  $g$  are required. We provide then two existence results for the integral problem (3) and, using the correspondences established with the other problems, we deduce the existence of solutions for (1) and (2).

**Keywords:** distribution, derivative with respect to functions, regulated primitive integral, Kurzweil–Stieltjes integral, regulated function, variational measure, solution.

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## 1 Introduction

This paper deals with three types of equations aiming to investigate the equivalence of their solvability, that is, whether the existence of solutions to one of the equations leads to the existence of solutions to the other two. Among the problems to be studied here, the distributional differential equations of the form

$$\begin{aligned} Dx &= f(t, x)Dg \\ x(0) &= x_0, \end{aligned} \tag{1}$$

certainly represent a very general formulation of differential problems. Evidently, when  $g$  is absolutely continuous, then its distributional derivative coincides with the usual derivative and we retrieve the classical differential equation. Besides, recalling that the distributional derivative of a function of bounded variation originates a Borel measure, it is clear that measure-driven equations can be regarded as a particular case of (1); see [3], [31] and the references therein. Accordingly, equation (1) covers a broad range of problems for the theory of measure differential equations has been an effective tool in the study of impulsive systems, retarded equations and equations on time scales (e.g. [13], [14] and [25]).

A novel feature in the present study is that the function  $g$  in the distributional problem (1) is not assumed to be of bounded variation, but only regulated. To treat such a problem we will make use of the regulated primitive integral introduced in [38]. This integral somehow inverts the distributional derivatives of regulated functions allowing us to convert a distributional equation to an integral equation. This method has been used in many papers recently; see, for instance, [20], [21] and [22]. In our approach, though, we take advantage of the connection between the regulated primitive integral and the Kurzweil–Stieltjes integral (cf. [38, Definition 12] or Theorem 2.15). Therefore, we investigate problem (1) by reducing it to an integral equation (3). It is important to remark that, to avoid paradoxes, extra attention is required when defining solutions for (1) as functions satisfying (3); see [19] for more details.

The study of derivatives with respect to functions and its connection with integrals is not exactly new in analysis (cf. [41] and [4]). A rather recent idea, though, is presented in [27] together with an interesting applicability of such a differentiation process. In [27], the authors consider derivatives with respect to non-decreasing left-continuous functions; however, nothing really prevents the study of such a notion in a more general setting. Besides, for monotone  $g$ , in most cases we can reduce the differentiation with respect to  $g$  to ordinary differentiation. This motivated us to define  $g$ -derivative for left-continuous regulated functions  $g$ . The generality of such a derivative asks for a notion of measure which can be meaningfully applied to more general functions, thus the use of a variational measure in the present paper (see Definition 2.4). In the case when  $g$  is the identity, it is known even in the abstract setting that the equivalence between (2) and (3) is always possible by appropriately choosing the integration process and respectively the type of derivative (see [2]). In our case, the investigation of  $g$ -differentiation problems of the type (2) via integral equations (3) is due to new versions of the Fundamental Theorems of Calculus we proved under quite weak assumptions.

At last, we provide two existence results for the integral problem (3) which, unlike other results available in the literature (cf. [13, Theorem 5.3]), do not rely on the assumption of  $g$  being monotone. We conclude the paper by using the correspondences established with the other problems to deduce the existence of solutions for (1) and (2).

## 2 Preliminary results

Recall that a function  $g : [a, b] \rightarrow \mathbb{R}$  is regulated if the one sided-limits exist, more precisely:

$$g(t^+) = \lim_{r \rightarrow t^+} g(r), \quad t \in [a, b), \quad g(s^-) = \lim_{r \rightarrow s^-} g(r), \quad s \in (a, b].$$

It is well-known that regulated functions are bounded and they can have at most a countable number of points of discontinuity (see [23, Corollary I.3.2]). The space  $G([a, b])$  of real-valued regulated functions on  $[a, b]$  is a Banach space when endowed with the norm

$$\|g\|_\infty = \sup_{t \in [a, b]} |g(t)|, \quad g \in G([a, b]).$$

Moreover, the set of all left-continuous regulated functions on  $[a, b]$  and right-continuous at  $a$  is a closed subspace of  $G([a, b])$  and it will be denoted by  $G_-([a, b])$ .

The following notion is important when investigating compactness in the space of regulated functions.

**Definition 2.1** ([15]). A set  $\mathcal{F} \subset G([a, b])$  is said to be equiregulated if for every  $\varepsilon > 0$  and every  $t_0 \in [a, b]$  there exists  $\delta > 0$  such that, for all  $x \in \mathcal{F}$  we have:

- i)  $|x(t) - x(t_0^-)| < \varepsilon$  for every  $t_0 - \delta < t < t_0$ ;
- ii)  $|x(s) - x(t_0^+)| < \varepsilon$  for every  $t_0 < s < t_0 + \delta$ .

**Lemma 2.2** ([15]). Let  $f : [a, b] \rightarrow \mathbb{R}$  and  $f_n \in G([a, b]), n \in \mathbb{N}$ , be such that

$$\lim_{n \rightarrow \infty} f_n(t) = f(t) \quad \text{for every } t \in [a, b].$$

If the set  $\{f_n : n \in \mathbb{N}\}$  is equiregulated, then  $f_n$  converges uniformly to  $f$ .

For regulated functions, the analogous to Arzelà–Ascoli theorem reads as follows.

**Lemma 2.3** ([15, Corollary 2.4]). Let  $\mathcal{F} \subset G([a, b])$  be equiregulated. If for each  $t \in [a, b]$ , the set  $\{x(t) : x \in \mathcal{F}\}$  is bounded, then  $\mathcal{F}$  is relatively compact in  $G([a, b])$ .

Given  $A \subseteq [a, b]$ , a system  $S$  on  $A$  is a finite collection of tagged intervals

$$a \leq a_1 < b_1 \leq \dots \leq a_m < b_m \leq b$$

with  $c_j \in [a_j, b_j] \cap A, j = 1, \dots, m$ ; we write  $S = \{(c_j, [a_j, b_j]) : j = 1, \dots, m\}$ .

Given a gauge  $\delta$  on  $A$ , i.e.  $\delta : A \rightarrow \mathbb{R}_+$ , a system  $S = \{(c_j, [a_j, b_j]) : j = 1, \dots, m\}$  is said to be  $\delta$ -fine if

$$[a_j, b_j] \subset (c_j - \delta(c_j), c_j + \delta(c_j)) \quad \text{for every } j = 1, \dots, m.$$

The set of all of  $\delta$ -fine systems on  $A$  will be denoted by  $\mathcal{S}(A, \delta)$ .

A partition of the interval  $[a, b]$  is a system  $S = \{(c_j, [a_j, b_j]) : j = 1, \dots, m\}$  satisfying  $b_j = a_{j+1}, j = 1, \dots, m$ , where  $a_1 = a$  and  $a_{m+1} = b$ . We remark that for an arbitrary gauge  $\delta$  on  $[a, b]$  there always exists a  $\delta$ -fine partition of  $[a, b]$ . This is stated by the Cousin lemma (see [32, Lemma 1.4]).

Throughout this paper,  $\lambda(E)$  denotes the Lebesgue measure of  $E$ , for Lebesgue measurable sets  $E \subset \mathbb{R}$ . The following definition corresponds to the notion of variational measure which figures in the problem (2).

**Definition 2.4.** Let  $g : [a, b] \rightarrow \mathbb{R}$ . For each  $A \subseteq [a, b]$ , we define the  $g$ -outer measure of  $A$  by

$$m_g(A) = \inf_{\delta} \sup \{W_g(S) : S \in \mathcal{S}(A, \delta)\},$$

where  $W_g(S) = \sum_{j=1}^m |g(b_j) - g(a_j)|$  for  $S = \{(c_j, [a_j, b_j]) : j = 1, \dots, m\}$ .

Note that  $m_g$  is actually the Thomson's variational measure  $\mathcal{S}_0\text{-}\mu_g$  defined in [39] (see [7, Proposition 4.2(xiv)]). In the case when  $g$  is the identity function, the definition above leads to the Lebesgue outer measure (see [10, Proposition 3.4] for details). The next proposition summarizes some of the properties of  $m_g$  and ensures that it defines a metric outer measure (see [39, p. 87] and [10, Proposition 3.3] for the proofs).

**Proposition 2.5.** Let  $g : [a, b] \rightarrow \mathbb{R}$ . The functional  $m_g$  satisfies:

- i)  $m_g(A) \geq 0$  for every  $A \subseteq [a, b]$  and  $m_g(\emptyset) = 0$ ;
- ii) if  $A \subseteq B$ , then  $m_g(A) \leq m_g(B)$ ;
- iii)  $m_g(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m_g(A_n)$  for any sequence of sets  $A_n \subseteq [a, b]$ ;
- iv)  $m_g(A \cup B) = m_g(A) + m_g(B)$  whenever  $A$  and  $B$  are contained in two disjoint open subsets of  $[a, b]$ ;
- v)  $m_g(\{c\}) = \limsup_{h \rightarrow 0^+} |g(c+h) - g(c)| + \limsup_{h \rightarrow 0^-} |g(c+h) - g(c)|$  for every  $c \in [a, b]$ .

Proposition 2.5(v) shows that the variational measure over singletons provides information on the 'size' of the discontinuity of the function at a point. More important, a function  $g$  is continuous at  $c$  if and only if  $m_g(\{c\}) = 0$ .

**Remark 2.6.** Regarding the outer measure  $m_g$ , we will say that a property holds  $m_g$ -almost everywhere (shortly,  $m_g$ -a.e.) if it is valid except for a set  $N \subset [a, b]$  with  $m_g(N) = 0$ .

Note that, given  $A \subset [a, b]$ , for a fixed gauge  $\gamma : A \rightarrow \mathbb{R}_+$  we have

$$m_g(A) \leq \inf_{\delta \leq \gamma} \sup \{W_g(S) : S \in \mathcal{S}(A, \delta)\}.$$

Thus, in order to prove that a set  $A$  has  $m_g$ -measure zero, it is enough to show that given  $\varepsilon > 0$ , there exists  $\gamma_\varepsilon : A \rightarrow \mathbb{R}_+$  such that

$$W_g(S) < \varepsilon \quad \text{for every } S \in \mathcal{S}(A, \gamma_\varepsilon).$$

**Definition 2.7.** Let  $g : [a, b] \rightarrow \mathbb{R}$ . Given a function  $F : [a, b] \rightarrow \mathbb{R}$ , we say that  $F$  is  $g$ -normal, if  $m_F(A) = 0$  whenever  $m_g(A) = 0$ ,  $A \subset [a, b]$ .

The definition above was presented in [11] for functions  $g$  which are continuous and BVG. In the particular case when  $g$  is the identity function (and consequently  $m_g$  is the Lebesgue outer measure) the notion of  $g$ -normal coincides with the so-called (strong) Lusin condition (see [29] or [33]). The interested reader can find more details on the relation between these two notions in [12, Section 5] and [9, Section 4].

The following result is a particular case of [11, Lemma 3].

**Lemma 2.8.** Let  $H : [a, b] \rightarrow \mathbb{R}$  be an increasing function. If  $H$  is continuous on a set  $A \subset [a, b]$ , then  $m_H(A) \leq \lambda(H(A))$ .

Recall that a function  $g : [a, b] \rightarrow \mathbb{R}$  is said to be of bounded variation (or a *BV*-function) if its total variation

$$\text{var}_a^b(g) = \sup \sum_{i=1}^m |g(t_i) - g(t_{i-1})|$$

is finite, where the supremum is taken over all finite divisions

$$D : a = t_0 < t_1 < \cdots < t_m = b$$

of the interval  $[a, b]$ . Enclosing this subsection we will discuss two other general notions of variation and some of their properties.

**Definition 2.9** ([12]). Let  $g : [a, b] \rightarrow \mathbb{R}$ . We say that:

- i)  $g$  is  $BV^\circ$  on a set  $A \subseteq [a, b]$  if  $m_g(A) < \infty$ ;
- ii)  $g$  is *generalized  $BV^\circ$*  (shortly,  $BVG^\circ$ ) if there exists a decomposition  $[a, b] = \bigcup_{n=1}^{\infty} E_n$  such that  $g$  is  $BV^\circ$  on  $E_n$  for every  $n \in \mathbb{N}$ .

It is easy to see that any function  $g$  of bounded variation on  $[a, b]$  is  $BV^\circ$  on any  $A \subset [a, b]$  and  $m_g(A) \leq \text{var}_a^b(g)$ . In particular, if  $g$  is a continuous *BV*-function,  $\text{var}_J(g) = m_g(J)$  for any subinterval  $J \subseteq [a, b]$  (cf. [33, Lemma 3.2]).

We can draw an analogy connecting the concept of  $BVG^\circ$  functions and  $\sigma$ -finite measure. Indeed, if  $g$  is  $BVG^\circ$ , this means that the outer measure  $m_g$  is  $\sigma$ -finite on  $[a, b]$ . Thus, in view of [39, Theorem 40.1], the relation between  $BVG^\circ$  and the notion of generalized bounded variation,  $VBG_*$  in the sense of Saks [29], reads as follows: *a function is  $BVG^\circ$  if and only if it is bounded and  $VBG_*$* .

From the remarks above, we can see that a  $BVG^\circ$  function is bounded; moreover, it is not hard to show that such a function has at most countably many points of discontinuity (see [39, p. 93]). Although the class  $BVG^\circ$  encompasses the functions of bounded variation, a  $BVG^\circ$  function need not even be regulated. A simple example of this fact is the function  $g : [0, 1] \rightarrow \mathbb{R}$  given by  $g(1/n) = 1$  for  $n \in \mathbb{N}$ , and  $g(t) = 0$  otherwise.

The following proposition provides a useful estimate for  $BV^\circ$  functions.

**Proposition 2.10** ([12, Lemma 3.5]). Let  $g : [a, b] \rightarrow \mathbb{R}$  be  $BV^\circ$  on a set  $E \subseteq [a, b]$ . Then, there exist a strictly increasing function  $H : [a, b] \rightarrow \mathbb{R}$  and a gauge  $\delta$  on  $E$  such that for every  $t \in E$  we have

$$|g(s) - g(t)| \leq |H(s) - H(t)| \quad \text{whenever } |s - t| < \delta(t).$$

**Remark 2.11.** It is worth emphasizing that in Proposition 2.10 the increasing function  $H$  can be chosen left-continuous when  $g$  is supposed to be left-continuous. Indeed, let us recall from the proof of Lemma 3.5 in [12] that

$$H(t) = t + \sup\{W_g(S) : S \in \mathcal{S}(E, \delta), S \subset [a, t]\},$$

where the gauge  $\delta$  on  $E$  is chosen so that  $W_g(S) < m_g(E) + 1$  for every  $\delta$ -fine system  $S$  on  $E$ . It is not hard to see that for any  $t \in [a, b]$  and  $\varepsilon > 0$

$$\begin{aligned} H(t) - H(t - \varepsilon) &\leq \varepsilon + \sup\{W_g(S) : S \in \mathcal{S}(E, \delta), S \subset (t - \varepsilon, t]\} \\ &\leq \varepsilon + \sup\{W_g(S) : S \in \mathcal{S}(E, \delta), S \subset (t - \varepsilon, t)\} + |g(t) - g(t^-)|. \end{aligned}$$

Thus, assuming that  $g$  is left-continuous, the left-continuity of  $H$  at an arbitrary point  $t \in (a, b]$  holds once we show that

$$\lim_{\varepsilon \rightarrow 0^+} \sup \{W_g(S) : S \in \mathcal{S}(E, \delta), S \subset (t - \varepsilon, t)\} = 0.$$

To prove this fact we follow the method of the proof of Lemma 16 at page 140 in [5]. More precisely, reasoning by contradiction, suppose that there exists  $\eta > 0$  such that for every  $\varepsilon > 0$ , there exists  $S_\varepsilon \in \mathcal{S}(E, \delta)$ ,  $S_\varepsilon \subset (t - \varepsilon, t)$  with  $W_g(S_\varepsilon) \geq \eta$ .

Considering  $0 < \varepsilon_1 < t - a$ , let  $S_1 \in \mathcal{S}(E, \delta)$  be such that  $S_1 \subset (t - \varepsilon_1, t)$  and  $W_g(S_1) \geq \eta$ . Writing  $S_1 = \{(c_j^{(1)}, [a_j^{(1)}, b_j^{(1)}]) : j = 1, \dots, n_1\}$ , choose  $0 < \varepsilon_2 < t - b_{n_1}^{(1)}$  and let  $S_2 \in \mathcal{S}(E, \delta)$  be such that  $S_2 \subset (t - \varepsilon_2, t)$  and  $W_g(S_2) \geq \eta$ . If we proceed in this way, we obtain a decreasing sequence of positive numbers  $\varepsilon_k$ ,  $k \in \mathbb{N}$ , and systems  $S_k \in \mathcal{S}(E, \delta)$ ,  $k \in \mathbb{N}$ , such that

$$S_{k+1} \subset (t - \varepsilon_k, t - \varepsilon_{k+1}) \quad \text{and} \quad W_g(S_k) \geq \eta.$$

Therefore, for every  $k \in \mathbb{N}$ ,  $T_k = \bigcup_{\ell=1}^k S_\ell$  is a  $\delta$ -fine system on  $E$  which satisfies

$$k\eta \leq \sum_{j=1}^k W_g(S_j) = W_g(T_k) < m_g(E) + 1;$$

a direct contradiction to  $g$  being  $BV^\circ$  on  $E$ .

In view of the above remark, next assertion provides a characterization of  $BVG^\circ$  functions borrowed from [12, Lemma 3.6].

**Proposition 2.12.** *Let  $g : [a, b] \rightarrow \mathbb{R}$  be given. Then,  $g$  is  $BVG^\circ$  if and only if there exists a strictly increasing function  $H : [a, b] \rightarrow \mathbb{R}$  such that*

$$\limsup_{s \rightarrow t} \left| \frac{g(s) - g(t)}{H(s) - H(t)} \right| < \infty \quad \text{for every } t \in [a, b].$$

*If, in addition,  $g$  is left-continuous, then  $H$  can be chosen left-continuous.*

The following result will be useful later.

**Lemma 2.13** ([12, Lemma 3.8]). *Let  $H : [a, b] \rightarrow \mathbb{R}$  be strictly increasing and let  $E \subset [a, b]$  be such that  $m_H(E) = 0$ . If  $g : [a, b] \rightarrow \mathbb{R}$  satisfies*

$$\limsup_{s \rightarrow t} \left| \frac{g(s) - g(t)}{H(s) - H(t)} \right| < \infty \quad \text{for every } t \in E,$$

*then  $m_g(E) = 0$ .*

## 2.1 Integrals and derivatives

This subsection is devoted to the notions of integrals and their related derivatives which will be used in our work. We recall some of their basic properties and prove a few new ones which, to our knowledge, are not available in the literature. As problem (1) is related to the theory of distributions, we will begin with a short introduction into this setting (see [35, 36] for more details).

A distribution on  $[a, b]$  is a linear continuous functional on the topological vector space  $\mathcal{D}$  of test functions, namely, functions  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  which have continuous derivative  $\phi^{(j)}$  of any order  $j \in \mathbb{N}$  vanishing on  $\mathbb{R} \setminus (a, b)$ . The space  $\mathcal{D}$  is endowed with the topology induced by the following convergence of sequences:

$$\phi_n \rightarrow \phi \iff \phi_n^{(j)} \rightarrow \phi^{(j)} \quad \text{uniformly on } (a, b), \text{ for every } j \in \mathbb{N}.$$

The distributional derivative of a distribution  $G$ , denoted by  $DG$ , is itself a distribution defined by

$$\langle DG, \phi \rangle = -\langle G, \phi' \rangle \quad \text{for every } \phi \in \mathcal{D}.$$

In particular, if  $f : [a, b] \rightarrow \mathbb{R}$  is a left-continuous BV-function, then its distributional derivative corresponds to the Stieltjes measure associated to  $f$ , defined by

$$Df([c, d]) = f(d) - f(c) \quad \text{for } [c, d] \subset [a, b]$$

and then extended to all Borel subsets of  $[a, b]$  in the standard way (for details, see [28, Example 6.14]).

To deal with the problem (1) we will make use of the notion of regulated primitive integral introduced in [38]. Hence, we will restrict ourselves to distributions which correspond to the distributional derivative of a regulated function, i.e., distributions  $g$  on  $[a, b]$  such that  $g = DG$  for some left-continuous regulated function  $G : [a, b] \rightarrow \mathbb{R}$ . Note that, in this case, for any test function  $\phi \in \mathcal{D}$

$$\langle g, \phi \rangle = \langle DG, \phi \rangle = -\langle G, \phi' \rangle = -\int_a^b G(t)\phi'(t)dt.$$

These distributions are called RP-integrable in the sense to be specified in the following definition.

**Definition 2.14.** Let  $g$  be a distribution on  $[a, b]$  and  $G \in G_-([a, b])$  be such that  $g = DG$ . The regulated primitive integral of  $g$  is defined by

$$r \int_s^t g = G(t) - G(s), \quad a \leq s \leq t \leq b.$$

and we say that  $g$  is RP-integrable with primitive  $G$ . The space of RP-integrable distributions on  $[a, b]$  is denoted by  $\mathcal{A}_{\mathcal{R}}([a, b])$ .

We remark that the definition above can be regarded as a particular case of the notion introduced in [38] – which is concerned with distributions on the extended real line. It is shown in [38] that the RP-integral is more general than Riemann, Lebesgue and Henstock–Kurzweil integrals. Moreover,  $\mathcal{A}_{\mathcal{R}} := \mathcal{A}_{\mathcal{R}}(\overline{\mathbb{R}})$  is a Banach space when endowed with the Alexiewicz norm and, consequently, the completion of the space of signed Radon measures (see [38, Theorem 4]).

In the sequel we borrow some of the results presented in [38] with an obvious adaptation to compact intervals.

**Proposition 2.15.** *The multipliers of the space  $\mathcal{A}_{\mathcal{R}}([a, b])$  are the functions of bounded variation. Moreover, if  $f : [a, b] \rightarrow \mathbb{R}$  is a BV-function and  $G \in G_-([a, b])$ , the RP-integral of the product  $f DG$  is defined by*

$$r \int_a^b f DG = \int_a^b f(t) dG(t),$$

where the integral on the right-hand side is the Kurzweil–Stieltjes integral (see Definition 2.18).

**Remark 2.16.** The expression of the product presented in Proposition 2.15, defined via the integration by parts formula [38, Definition 12], agrees with Definition 11 in the same paper. In this regard, it is also worth mentioning [26] where, along with a discussion on the product of distributions, we find the following identity

$$\langle f DG, \phi \rangle = \int_a^b f(t) \phi(t) dG(t), \quad \phi \in \mathcal{D},$$

for the case when  $f$  is a BV-function,  $G$  is regulated and both functions are assumed to be right-continuous.

The connection between the RP-integral and the distributional derivative is described in the following Fundamental Theorem of Calculus.

**Theorem 2.17** ([38, Theorem 6]). *If  $g \in \mathcal{A}_{\mathcal{R}}([a, b])$ , then the function*

$$F(t) = \int_a^t g, \quad t \in [a, b]$$

*satisfies  $DF = g$ .*

Now we present a short overview on Kurzweil–Stieltjes integral, which is the integral found in problem (3). For a more comprehensive study of this topic, see [32] or [34] for instance.

**Definition 2.18.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be Kurzweil–Stieltjes integrable with respect to (shortly, KS-integrable w.r.t.)  $g : [a, b] \rightarrow \mathbb{R}$  if there exists  $\int_a^b f(s)dg(s) \in \mathbb{R}$  such that, for every  $\varepsilon > 0$ , there is a gauge  $\delta_\varepsilon$  on  $[a, b]$  satisfying

$$\left| \sum_{j=1}^m f(\tau_j)(g(t_j) - g(t_{j-1})) - \int_a^b f(s)dg(s) \right| < \varepsilon$$

for every  $\delta_\varepsilon$ -fine partition  $\{(\tau_j, [t_{j-1}, t_j]), j = 1, \dots, m\}$  of  $[a, b]$ .

Notice that when  $g(t) = t$ ,  $t \in [a, b]$ , the definition above reduces to the notion of Henstock–Kurzweil integral (for which the reader is referred to [16], see also [24]). Recall that such an integral generalizes the Lebesgue integral and integrates all derivatives. When it comes to Stieltjes-type integrals, it is known that, for integrators of bounded variation, Lebesgue–Stieltjes integrability implies Kurzweil–Stieltjes integrability (cf. [29, Theorem VI.8.1]), while the equivalence relies on stronger assumptions (see [6, Theorem 2.71]).

The following result is a special case of [32, Theorem 1.16] (see also [40, Proposition 2.3.16]).

**Proposition 2.19.** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be such that  $f$  is KS-integrable w.r.t.  $g$ . If  $g \in G([a, b])$ , then the function  $F : [a, b] \rightarrow \mathbb{R}$  given by*

$$F(t) = \int_a^t f(s)dg(s), \quad t \in [a, b],$$

*is regulated and satisfies*

$$F(t^+) - F(t) = f(t) [g(t^+) - g(t)] \quad \text{and} \quad F(t) - F(t^-) = f(t) [g(t) - g(t^-)].$$

*If, in addition,  $g$  is a BV-function and  $f$  is bounded, then  $F$  is a BV-function.*



In order to investigate some properties of the Kurzweil–Stieltjes integral regarding the variational measure defined previously, we recall the following lemma.

**Lemma 2.20** (Saks–Henstock lemma). *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be such that  $f$  is KS-integrable w.r.t.  $g$ . Let  $\varepsilon > 0$  be given and assume that  $\delta$  is a gauge on  $[a, b]$  such that*

$$\left| \sum_{j=1}^m f(\tau_j)(g(t_j) - g(t_{j-1})) - \int_a^b f(s)dg(s) \right| < \varepsilon,$$

for every  $\delta$ -fine partition  $\{(\tau_j, [t_{j-1}, t_j]), j = 1, \dots, m\}$  of  $[a, b]$ . Then,

$$\sum_{j=1}^{\ell} \left| f(c_j)(g(b_j) - g(a_j)) - \int_{a_j}^{b_j} f(s)dg(s) \right| \leq \varepsilon,$$

for any system  $S \in \mathcal{S}([a, b], \delta)$ , with  $S = \{(c_j, [a_j, b_j]) : j = 1, \dots, \ell\}$ .

Next proposition presents two additional properties of the indefinite Kurzweil–Stieltjes integral (for a similar result in the framework of functions  $VBG_*$  see [37, Lemma 3.12]).

**Proposition 2.21.** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be such that  $f$  is KS-integrable w.r.t.  $g$ . Consider the function  $F : [a, b] \rightarrow \mathbb{R}$  given by*

$$F(t) = \int_a^t f(s)dg(s), \quad t \in [a, b],$$

Then,  $F$  is  $g$ -normal.

If, in addition,  $g$  is a  $BVG^\circ$  function, then  $F$  is  $BVG^\circ$ .

*Proof.* To prove that  $F$  is  $g$ -normal, let  $A \subset [a, b]$  be such that  $m_g(A) = 0$ . For each  $n \in \mathbb{N}$ , consider the set  $A_n := \{t \in A : |f(t)| \leq n\}$ . Since  $A = \bigcup_{n \in \mathbb{N}} A_n$ , in view of Proposition 2.5 (iii), it is enough to show that  $m_F(A_n) = 0$ , for  $n \in \mathbb{N}$ .

Given  $\varepsilon > 0$  and fixed  $n \in \mathbb{N}$ , there is a gauge  $\gamma_1 : A_n \rightarrow \mathbb{R}_+$  such that

$$W_g(S) < \frac{\varepsilon}{n} \quad \text{for every } S \in \mathcal{S}(A_n, \gamma_1).$$

Let  $\gamma_2 : [a, b] \rightarrow \mathbb{R}_+$  be a gauge as in the Saks–Henstock lemma (Lemma 2.20) and consider the gauge  $\gamma(t) = \min\{\gamma_1(t), \gamma_2(t)\}$ ,  $t \in A_n$ . Bearing all these in mind, for any system  $S \in \mathcal{S}(A_n, \gamma)$ , with  $S = \{(c_j, [a_j, b_j]) : j = 1, \dots, m\}$ , we have

$$\begin{aligned} \sum_{j=1}^m |F(b_j) - F(a_j)| &\leq \sum_{j=1}^m |F(b_j) - F(a_j) - f(c_j)[g(b_j) - g(a_j)]| \\ &\quad + \sum_{j=1}^m |f(c_j)| |g(b_j) - g(a_j)| \\ &\leq \varepsilon + n W_g(S) < 2\varepsilon. \end{aligned}$$

Therefore,  $W_F(S) < 2\varepsilon$  for every  $S \in \mathcal{S}(A_n, \gamma)$ , which implies that

$$m_F(A_n) \leq \inf_{\delta \leq \gamma} \sup\{W_F(S) : S \in \mathcal{S}(A_n, \delta)\} \leq 2\varepsilon$$

(see Remark 2.6). Since  $\varepsilon > 0$  is arbitrary, it follows that  $m_F(A_n) = 0$ .

The second statement can be proved in a similar way observing that: if  $g$  is  $BV^\circ$  on a set  $E \subseteq [a, b]$ , then  $F$  is  $BV^\circ$  on  $E_n := \{x \in E : |f(x)| \leq n\}$  for each  $n \in \mathbb{N}$ .  $\square$

The following result is contained in [12, Proposition 2.9].

**Lemma 2.22.** *Let  $g : [a, b] \rightarrow \mathbb{R}$ , and assume that  $f : [a, b] \rightarrow \mathbb{R}$  is null, except on a set  $N \subset [a, b]$  with  $m_g(N) = 0$ . Then  $f$  is KS-integrable w.r.t.  $g$  and  $\int_a^t f(s) dg(s) = 0$  for every  $t \in [a, b]$ .*

Convergence theorems are essential when working with integral equations. In our study we will need a result based on the following notion.

**Definition 2.23.** Let  $g : [a, b] \rightarrow \mathbb{R}$  and let  $\mathcal{F}$  be a family of real functions defined in  $[a, b]$ . We say that  $\mathcal{F}$  is equiintegrable with respect to  $g$  if for every  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that

$$\left| \sum_{j=1}^{\ell} f(\tau_j)[g(t_j) - g(t_{j-1})] - \int_a^b f(s) dg(s) \right| < \varepsilon,$$

for every  $f \in \mathcal{F}$  and every  $\delta$ -fine partition  $\{(\tau_j, [t_{j-1}, t_j]), j = 1, \dots, \ell\}$  of  $[a, b]$ .

**Proposition 2.24** ([30, Proposition 3.4]). *Let  $g \in G([a, b])$  and assume that  $\mathcal{F}$  is a family of real functions defined in  $[a, b]$  equiintegrable w.r.t.  $g$ . If for each  $t \in [a, b]$ , the set  $\{f(t), f \in \mathcal{F}\}$  is bounded, then*

$$\left\{ \int_a^{\cdot} f(s) dg(s) : f \in \mathcal{A} \right\}$$

is equiregulated.

The proof of the following theorem follows the same approach used in [32, Theorem 1.28] (see also [18, Theorem 3.28]).

**Theorem 2.25.** *Let  $g, f, f_n : [a, b] \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ , be such that*

$$\lim_{n \rightarrow \infty} f_n(t) = f(t) \quad \text{for } t \in [a, b].$$

If  $\{f_n : n \in \mathbb{N}\}$  is equiintegrable w.r.t.  $g$ , then  $f$  is KS-integrable w.r.t.  $g$  and

$$\int_a^t f(s) dg(s) = \lim_{n \rightarrow \infty} \int_a^t f_n(s) dg(s) \quad \text{for every } t \in [a, b].$$

In [27] a notion of differentiability connected to Stieltjes-type integral was introduced for non-decreasing left-continuous functions  $g$ . In this work, we will consider the  $g$ -derivative as defined in [27], but assuming simply that  $g$  is regulated and left-continuous.

**Definition 2.26.** Let  $g \in G_-([a, b])$ . The derivative with respect to  $g$  (or the  $g$ -derivative) of a function  $f : [a, b] \rightarrow \mathbb{R}$  at a point  $t \in [a, b]$  is given by

$$f'_g(t) = \lim_{s \rightarrow t} \frac{f(s) - f(t)}{g(s) - g(t)} \quad \text{if } g \text{ is continuous at } t,$$

$$f'_g(t) = \lim_{s \rightarrow t^+} \frac{f(s) - f(t)}{g(s) - g(t)} \quad \text{if } g \text{ is discontinuous at } t,$$

provided the limit exists. In this case, we say that  $f$  is  $g$ -differentiable at  $t$ . If  $f$  is  $g$ -differentiable at  $t$ , for every  $t \in [a, b]$ , we say that  $f$  is  $g$ -differentiable on  $[a, b]$ .

Given a function  $g \in G_-([a, b])$ , we consider the following sets:

$$C_g = \{t \in [a, b] : g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\} \quad (2.1)$$

$$J_g^+ = \{t \in [a, b] : g(t^+) - g(t) > 0\}. \quad (2.2)$$

It is not hard to see that for  $t \in J_g^+$ , the  $g$ -derivative  $f'_g(t)$  exists if and only if  $f(t^+)$  exists. In particular, we have the following proposition.

**Proposition 2.27.** *If  $f, g \in G([a, b])$  and  $g$  is left-continuous, then  $f$  is  $g$ -differentiable at the points of  $J_g^+$ .*

In [12], a notion of differentiation with respect to another function is defined in terms of limit superior and limit inferior. It is worth highlighting that, at the points of continuity of  $g$ , our Definition 2.26 coincides with [12, Definition 3.1].

As it was observed in [27], for the points  $t \in [a, b]$  in which  $g$  is continuous, the definition above has sense only if  $t \notin C_g$ . However, the next theorem shows that this set is rather ‘small’, not representing a real drawback to our purposes in this work.

**Theorem 2.28.** *If  $g \in G_-([a, b])$ , then  $m_g(C_g) = 0$ .*

*Proof.* Since  $C_g$  is open, it can be written as a countable union of disjoint open intervals. Hence, due to Proposition 2.5 (iii), it is enough to prove that  $m_g((u, v)) = 0$ , where  $(u, v)$  is assumed to be one of those open intervals.

For  $n \in \mathbb{N}$ , consider the interval  $J_n := [u + \frac{1}{n}, v - \frac{1}{n}]$  and let  $\gamma(t) = \frac{1}{2n}$ ,  $t \in J_n$ . Note that, for  $S \in \mathcal{S}(J_n, \gamma)$ ,  $S = \{(c_j, [a_j, b_j]) : j = 1, \dots, \ell\}$ , we have  $[a_j, b_j] \subset (u, v)$ . By the fact that  $g$  is constant on  $(u, v)$ , it follows that  $W_g(S) = 0$  for any system  $S \in \mathcal{S}(J_n, \gamma)$ , and consequently

$$m_g(J_n) \leq \inf_{\delta \leq \gamma} \sup \{W_g(S) : S \in \mathcal{S}(J_n, \delta)\} = 0,$$

Since  $(u, v) = \bigcup_{n \in \mathbb{N}} J_n$ , it follows from Proposition 2.5 (iii) that  $m_g((u, v)) = 0$ .  $\square$

The following is a direct consequence of Proposition 2.5 (v).

**Proposition 2.29.** *Let  $g \in G_-([a, b])$ . If  $C_g = \bigcup_{n \in \mathbb{N}} (u_n, v_n)$  is a disjoint decomposition of  $C_g$  and*

$$N_g = \{u_n, v_n : n \in \mathbb{N}\} \setminus J_g^+,$$

*then  $m_g(N_g) = 0$ .*

**Remark 2.30.** In view of Propositions 2.28 and 2.29, whenever a property holds  $m_g$ -almost everywhere in some set  $E \subseteq [a, b]$ , without loss of generality, we can assume that it holds excluding also the sets  $C_g$  and  $N_g$ , that is,  $m_g$ -almost everywhere in  $E \setminus (C_g \cup N_g)$ .

The following proposition is the corresponding to [27, Lemma 6.1].

**Proposition 2.31.** *Let  $g \in G_-([a, b])$  and assume that  $F : [a, b] \rightarrow \mathbb{R}$  is  $g$ -differentiable at  $t_0 \in [a, b]$ .*

*i) If  $t_0 \in J_g^+$ , then for every  $\varepsilon > 0$  there exists  $\rho(t_0) > 0$  such that*

$$|F(t) - F(t_0) - F'_g(t_0)(g(t) - g(t_0))| \leq \varepsilon |g(t) - g(t_0)|$$

*for  $t_0 < t < t_0 + \rho(t_0)$ .*

ii) If  $t_0 \notin J_g^+ \cup N_g$ , then for every  $\varepsilon > 0$  there exists  $\rho(t_0) > 0$  such that

$$|F(s) - F(t) - F'_g(t_0)(g(s) - g(t))| \leq \varepsilon |g(s) - g(t)|$$

for  $t_0 - \rho(t_0) < t \leq t_0 \leq s < t_0 + \rho(t_0)$ .

In order to give conditions ensuring the differentiability with respect to increasing functions, we will need the following classical result from real analysis.

**Proposition 2.32** ([11, Proposition 2]). *Let  $F : [a, b] \rightarrow \mathbb{R}$  be a given function and consider the set*

$$R(F) = \{t \in [a, b] : F(s) \leq F(t), \text{ for } s \leq t, \text{ and } F(s) \geq F(t), \text{ for } s \geq t\}. \quad (2.3)$$

*Then,  $F$  is differentiable on  $R(F) \setminus A$ , where  $A \subset R(F)$  with  $\lambda(A) = 0$ .*

The following result contains a variant of [11, Proposition 4] as well as an analogous to [8, Lemma 5.2] in the case of functions  $BV^\circ$ .

**Proposition 2.33.** *Let  $H : [a, b] \rightarrow \mathbb{R}$  be a strictly increasing and left-continuous function.*

i) *If  $F \in G_-([a, b])$ , then  $F$  is  $H$ -differentiable  $m_H$ -a.e. on the set  $R(F)$  defined in (2.3).*

ii) *If  $F \in G_-([a, b])$  is  $BVG^\circ$ , then  $F$  is  $H$ -differentiable  $m_H$ -a.e.*

*Proof.* i) Let  $G : [H(a), H(b)] \rightarrow \mathbb{R}$  be given by

$$G(t) = \inf\{s \in [a, b] : H(s) \geq t\}.$$

It is not hard to see that  $G$  is increasing, continuous and  $G(H(t)) = t$  for  $t \in [a, b]$ . By Proposition 2.32,  $F \circ G$  is differentiable on  $R(F \circ G) \setminus A$ , where  $A \subset R(F \circ G)$  with  $\lambda(A) = 0$ . Considering the set

$$N = \{t \in R(F) \setminus J_H^+ : F \circ G \text{ is not differentiable at } H(t)\}$$

and observing that  $H(R(F)) \subset R(F \circ G)$ , it is clear that for all  $t \in N$  we must have  $H(t) \in A$ . Thus  $\lambda(H(N)) = 0$ , and applying Lemma 2.8 we obtain that  $m_H(N) = 0$ .

If  $t \in R(F) \setminus N$  and  $t \in J_H^+$ , Proposition 2.27 implies that  $F$  is  $H$ -differentiable at  $t$ . On the other hand, for  $t \in R(F) \setminus (N \cup J_H^+)$ , making use of the chain rule found in [27, Theorem 2.3(1)], for  $h = F \circ G$  and  $f = g = H$ , we have

$$(F \circ G \circ H)'_H(t) = (F \circ G)'(H(t)) H'_H(t)$$

that is,  $F'_H(t) = (F \circ G \circ H)'_H(t) = (F \circ G)'(H(t))$ . In summary,  $F$  is  $H$ -differentiable on  $R(F) \setminus N$ , which proves (i).

ii) Since, by Proposition 2.27,  $F$  is  $H$ -differentiable on  $J_H^+$ , it suffices to prove that  $F$  is  $H$ -differentiable  $m_H$ -a.e. on  $[a, b] \setminus J_H^+$ . The key point in the proof of this assertion is the fact that  $H$  is continuous in  $[a, b] \setminus J_H^+$  and  $BVG^\circ$  (due to its monotonicity); therefore, the  $H$ -differentiability of  $F$  can be understood as the differentiability in the sense of [12, Definition 3.1]. In view of this and recalling that  $F$  is  $BVG^\circ$ , we can apply [12, Proposition 3.10] deducing that  $F$  is differentiable relatively to  $H$  in  $([a, b] \setminus J_H^+) \setminus U$  in the sense of [12, Definition 3.1], where  $U \subset [a, b] \setminus J_H^+$  can be written as a union  $U = U_1 \cup U_2$ , with  $m_H(U_1) = 0$  and  $U_2$  at most countable.

Applying Lemma 2.8 we obtain  $m_H(U_2) \leq \lambda(H(U_2))$ . As  $U_2$  is at most countable, so is  $H(U_2)$ ; therefore  $\lambda(H(U_2)) = 0$  from whence  $m_H(U_2) = 0$ . Consequently, by Proposition 2.5 (iii) we get  $m_H(U) = 0$  and the assertion is proved.  $\square$

**Proposition 2.34.** *Let  $H : [a, b] \rightarrow \mathbb{R}$  be strictly increasing and left-continuous,  $g \in G_-([a, b])$  and  $F : [a, b] \rightarrow \mathbb{R}$ . If  $F'_H$  and  $g'_H$  exist on  $A \subseteq [a, b]$ , then  $F$  is  $g$ -differentiable on  $A \setminus (C_g \cup Z)$  and*

$$F'_g(t) = \frac{F'_H(t)}{g'_H(t)}, \quad t \in A \setminus (C_g \cup Z),$$

where  $Z = \{t \in [a, b] \setminus C_g : g'_H(t) = 0\}$ . Moreover,  $m_g(Z) = 0$ .

*Proof.* Given  $t \in A \setminus (C_g \cup Z)$ , note that

$$\lim_{s \rightarrow t} H(s) - H(t) = \frac{1}{g'_H(t)} \lim_{s \rightarrow t} g(s) - g(t).$$

This shows that:  $g$  is continuous at  $t$  if and only if  $H$  is continuous at  $t$ .

Therefore, if  $t$  is a point of continuity of  $g$ ,

$$\lim_{s \rightarrow t} \frac{F(s) - F(t)}{g(s) - g(t)} = \lim_{s \rightarrow t} \frac{F(s) - F(t)}{H(s) - H(t)} \frac{H(s) - H(t)}{g(s) - g(t)} = \frac{F'_H(t)}{g'_H(t)},$$

which shows that  $F$  is  $g$ -differentiable at  $t$  and  $F'_g(t) = \frac{F'_H(t)}{g'_H(t)}$ . On the other hand, for  $t \in A \setminus (C_g \cup Z)$  such that  $t \in J_g^+$ , we have that  $t$  is a point of discontinuity of  $H$  and

$$\lim_{s \rightarrow t+} \frac{F(s) - F(t)}{g(s) - g(t)} = \lim_{s \rightarrow t+} \frac{F(s) - F(t)}{H(s) - H(t)} \frac{H(s) - H(t)}{g(s) - g(t)} = \frac{F'_H(t)}{g'_H(t)}.$$

Hence  $F'_g(t) = \frac{F'_H(t)}{g'_H(t)}$ .

Let us prove that  $m_g(Z) = 0$ . Fixed an arbitrary  $\varepsilon > 0$ , by Proposition 2.31, for  $t \in Z \setminus J_H^+$ , there exists  $\rho(t) > 0$  such that

$$|g(s) - g(r)| < \varepsilon |H(s) - H(r)| \quad \text{for } t - \rho(t) < r \leq t \leq s < t + \rho(t);$$

while for  $t \in Z \cap J_H^+$ , there is  $\rho(t) > 0$  such that

$$|g(s) - g(t)| < \varepsilon |H(s) - H(t)| \quad \text{for } t < s < t + \rho(t).$$

Put  $Z \cap J_H^+ = \{\tau_i : i \in \Gamma\}$ , where  $\Gamma \subseteq \mathbb{N}$ , with  $\tau_i \neq \tau_j$  for  $i \neq j$ . The left-continuity of  $g$  implies that, for each  $\tau_i \in Z \cap J_H^+$ , we can find  $\eta_i > 0$  such that

$$|g(s) - g(\tau_i)| < \frac{\varepsilon}{2^i} \quad \text{for } \tau_i - \eta_i < t \leq \tau_i.$$

Define the gauge  $\gamma_\varepsilon : Z \rightarrow \mathbb{R}_+$

$$\gamma_\varepsilon(t) = \begin{cases} \rho(t), & \text{if } t \in Z \setminus J_H^+, \\ \min\{\eta_i, \rho(t)\}, & \text{if } t = \tau_i \in Z \text{ for some } i \in \Gamma. \end{cases}$$

Given  $S \in \mathcal{S}(A_n, \gamma_\varepsilon)$ , with  $S = \{(c_j, [a_j, b_j]) : j = 1, \dots, k\}$ , using the inequalities above we obtain

$$\begin{aligned} W_g(S) &= \sum_{c_j \in Z \setminus J_H^+} |g(b_j) - g(a_j)| + \sum_{c_j \in Z \cap J_H^+} |g(b_j) - g(a_j)| \\ &\leq \sum_{c_j \in Z \setminus J_H^+} \varepsilon (H(b_j) - H(a_j)) + \sum_{c_j \in Z \cap J_H^+} (|g(b_j) - g(c_j)| + |g(c_j) - g(a_j)|) \\ &\leq 2\varepsilon (H(b) - H(a)) + \sum_{i \in \Gamma} \frac{\varepsilon}{2^i} < \varepsilon (2(H(b) - H(a)) + 1). \end{aligned}$$

This, together with Remark 2.6, proves that  $m_g(Z) = 0$ . □

In [27], we find two Fundamental Theorems of Calculus (Theorems 6.2 and 6.5) connecting the KS-integral w.r.t.  $g$  with the  $g$ -derivative in the case when  $g$  is non-decreasing left-continuous. In Subsection 2.2 we will provide similar results for the case when  $g$  is a function in  $G_-([a, b])$  which is  $BVG^\circ$ .

## 2.2 Fundamental Theorem of Calculus

A descriptive characterization of the Kurzweil–Henstock integral in terms of variational measures is given in [33]. Concerning Stieltjes-type integral, we can mention the results in [12]; though, some continuity assumption is required. The content of this subsection, devoted to Fundamental Theorem of Calculus, somehow provides a descriptive characterization of the Kurzweil–Stieltjes integral.

The first Fundamental Theorem of Calculus to be presented extends the result from [27, Theorem 6.5] to a more general class of functions  $g$ , namely, regulated functions which are  $BVG^\circ$ . The passage to a  $BVG^\circ$  integrator is based on the notion of  $g$ -normal function, in connection with some elements from [12] and [11]. We mention that this result also generalizes [12, Corollary 4.8] proved for continuous  $BVG^\circ$  functions.

**Theorem 2.35.** *Let  $g \in G_-([a, b])$  be a  $BVG^\circ$  function. If  $f : [a, b] \rightarrow \mathbb{R}$  is KS-integrable w.r.t.  $g$  and  $F : [a, b] \rightarrow \mathbb{R}$  is given by*

$$F(t) = \int_a^t f(s) d g(s), \quad t \in [a, b],$$

*then,  $F'_g = f$  on  $[a, b] \setminus N$ , where  $N \subset [a, b]$  and  $m_g(N) = 0$ .*

*Proof.* Note that  $F \in G_-([a, b])$  is a  $BVG^\circ$  function (see Propositions 2.19 and 2.21). Let  $H_1, H_2 : [a, b] \rightarrow \mathbb{R}$  be strictly increasing left-continuous functions which exist by Proposition 2.12 for  $F$  and  $g$ , respectively. Defining  $H = H_1 + H_2$ , from Proposition 2.33 we know that the derivatives  $F'_H$  and  $g'_H$  exist on  $[a, b] \setminus U$ , where  $U \subset [a, b]$  and  $m_H(U) = 0$ . Applying Proposition 2.34 for  $A = [a, b] \setminus U$ , we conclude that  $F$  is  $g$ -differentiable on  $[a, b] \setminus (U \cup C_g \cup Z)$ , where  $Z = \{t \in [a, b] \setminus C_g : g'_H(t) = 0\}$ . Taking  $N = U \cup C_g \cup Z \cup N_g$ , since  $m_g(Z) = m_g(C_g) = m_g(N_g) = 0$  (see Theorem 2.28 and Proposition 2.34), it remains to show that  $m_g(U) = 0$ .

Recalling that  $H, H_1$  and  $H_2$  are increasing, it is not hard to see that for any system  $S$  on  $N$  we have  $W_H(S) = W_{H_1}(S) + W_{H_2}(S)$ . Thus,  $m_H(U) = 0$  implies  $m_{H_2}(U) = 0$  and the result is then a consequence of Lemma 2.13.

Clearly,  $F'_g(t) = f(t)$  for  $t \in [a, b] \setminus N$  with  $t \in J_g^+$  (see Proposition 2.19). Let us prove the equality for points  $t_0 \in [a, b] \setminus N$  in which  $g$  is continuous. Given  $\varepsilon > 0$ , let  $\delta : [a, b] \rightarrow \mathbb{R}$  be a gauge as in Saks–Henstock lemma (Lemma 2.20). Using Proposition 2.31, we can choose  $0 < \rho(t_0) < \delta(t_0)$  so that

$$|F(t) - F(s) - F'_g(t_0)(g(t) - g(s))| \leq \varepsilon |g(t) - g(s)|$$

for  $[s, t] \subset (t_0 - \rho(t_0), t_0 + \rho(t_0))$ . Since  $t_0 \notin C_g$ , we can find  $t \in [a, b]$  so that  $|t - t_0| < \rho(t_0)$  and  $|g(t) - g(t_0)| = M > 0$ . Without loss of generality, assume  $t_0 < t$ . Thus, applying

Saks–Henstock lemma together with the inequality above we obtain

$$\begin{aligned}
 |f(t_0) - F'_g(t_0)| &= \frac{1}{M} |f(t_0) - F'_g(t_0)| |g(t) - g(t_0)| \\
 &\leq \frac{1}{M} \left| f(t_0)(g(t) - g(t_0)) - \int_{t_0}^t f(\sigma) d g(\sigma) \right| \\
 &\quad + \frac{1}{M} |F(t) - F(t_0) - F'_g(t_0)(g(t) - g(t_0))| \\
 &\leq \varepsilon \left( \frac{1}{M} + 1 \right).
 \end{aligned}$$

Since  $\varepsilon$  is arbitrary, we conclude that  $F'_g(t_0) = f(t_0)$  and the result follows.  $\square$

Also connecting  $g$ -derivatives and the KS-integral, next Fundamental Theorem of Calculus somehow generalizes a similar result given for non-decreasing left-continuous functions in [27, Theorem 6.2]. The method of proof combines ideas from [27] and [12].

**Theorem 2.36.** *Let  $g \in G_-([a, b])$  be a  $BVG^\circ$  function. Assume that  $F : [a, b] \rightarrow \mathbb{R}$  satisfies the following conditions:*

- i)  $F$  is left-continuous at the points of  $J_g^+$ ;
- ii)  $F$  is  $g$ -differentiable on  $[a, b] \setminus N$ , where  $N \subset [a, b]$  and  $m_g(N) = 0$ ;
- iii)  $F$  is  $g$ -normal.

Then,

$$F(t) - F(a) = \int_a^t h(s) d g(s), \quad \text{for every } t \in [a, b], \quad (2.4)$$

where  $h(s) = F'_g(s)$  for  $s \in [a, b] \setminus N$  and  $h(s) = 0$  otherwise.

*Proof.* Consider a disjoint decomposition  $[a, b] = \bigcup_{n=1}^{\infty} E_n$  such that  $g$  is  $BV^\circ$  on  $E_n$ ,  $n \in \mathbb{N}$ . By Proposition 2.10, for each  $n \in \mathbb{N}$ , there exists a strictly increasing function  $H_n : [a, b] \rightarrow \mathbb{R}$  and a gauge  $\psi_n : E_n \rightarrow \mathbb{R}_+$  such that for  $t \in E_n$  we have

$$|g(s) - g(t)| \leq |H_n(s) - H_n(t)| \quad \text{whenever } |s - t| < \psi_n(t). \quad (2.5)$$

Let  $\varepsilon > 0$  be given. Without loss of generality, by Remark 2.30, we can assume that  $N_g \subset N$ . Since  $F$  is  $g$ -normal, we have  $m_F(N) = 0$  and we can choose a gauge  $\gamma : N \rightarrow \mathbb{R}_+$  such that

$$W_F(S) < \varepsilon \quad \text{for every } S \in \mathcal{S}(N, \gamma). \quad (2.6)$$

Recalling that  $g$  has at most a countable number of points of discontinuity we can write  $J_g^+ = \{\tau_i : i \in \Gamma\}$ ,  $\Gamma \subseteq \mathbb{N}$ , with  $\tau_i \neq \tau_j$  for  $i \neq j$ . Due to the left continuity of the functions  $F$  and  $g$ , for each  $i \in \mathbb{N}$  there exists  $\eta_i > 0$  such that

$$|F(s) - F(\tau_i)| \leq \frac{\varepsilon}{2^{i+2}} \quad \text{and} \quad |g(s) - g(\tau_i)| \leq \frac{\varepsilon}{2^{i+2}(|F'_g(\tau_i)| + 1)} \quad (2.7)$$

for  $\tau_i - \eta_i < s \leq \tau_i$ .

For each  $n \in \mathbb{N}$ , let  $Z_n := E_n \setminus N$  and  $\varepsilon_n = \frac{\varepsilon}{2^{n+1}(H_n(b) - H_n(a))}$ .

Given  $t \in [a, b] \setminus N$ , we know that  $t \in Z_n$  for some  $n \in \mathbb{N}$  and we have then two cases to consider: if  $t \in J_g^+$  or not. If  $t \in Z_n \cap J_g^+$ , by Proposition 2.31, there exists  $\rho_n(t) > 0$  such that

$$|F(s) - F(t) - F'_g(t)(g(s) - g(t))| \leq \varepsilon_n |g(s) - g(t)| \quad \text{for } t < s < t + \rho_n(t).$$

Assuming that  $\rho_n(t) < \psi_n(t)$ , it follows from (2.5) that

$$|F(s) - F(t) - F'_g(t)(g(s) - g(t))| \leq \varepsilon_n (H_n(s) - H_n(t)) \quad \text{for } t < s < t + \rho_n(t). \quad (2.8)$$

Analogously, for  $t \in Z_n \setminus J_g^+$ , we can choose  $0 < \rho_n(t) < \psi_n(t)$  so that

$$|F(s) - F(r) - F'_g(t)(g(s) - g(s'))| \leq \varepsilon_n (H_n(s) - H_n(r)), \quad (2.9)$$

whenever  $t - \rho_n(t) < r \leq t \leq s < t + \rho_n(t)$ .

Consider the gauge  $\delta : [a, b] \rightarrow \mathbb{R}$  defined by

$$\delta(t) = \begin{cases} \gamma(t), & \text{if } t \in N, \\ \rho_n(t), & \text{if } t \in Z_n \setminus J_g^+ \text{ for some } n \in \mathbb{N}, \\ \min\{\eta_i, \rho_n(t)\}, & \text{if } t = \tau_i \in Z_n \text{ for some } i \in \Gamma \text{ and } n \in \mathbb{N}. \end{cases}$$

and let  $\{(c_j, [t_{j-1}, t_j]) : j = 1, \dots, \ell\}$  be a  $\delta$ -fine partition of  $[a, b]$ . Thus,

$$\begin{aligned} & \sum_{j=1}^{\ell} |F(t_j) - F(t_{j-1}) - h(c_j)(g(t_j) - g(t_{j-1}))| \\ &= \sum_{c_j \in N} |F(t_j) - F(t_{j-1})| + \sum_{n=1}^{\infty} \sum_{c_j \in Z_n} |F(t_j) - F(t_{j-1}) - F'_g(c_j)(g(t_j) - g(t_{j-1}))|, \end{aligned}$$

(where the series is actually a sum with finitely many terms). In view of (2.6), it follows that  $\sum_{c_j \in N} |F(t_j) - F(t_{j-1})| < \varepsilon$ . In order to analyse the remaining sum, let us fix an arbitrary  $n \in \mathbb{N}$ . If  $Z_n \cap \{c_j : j = 1, \dots, \ell\} = \emptyset$  there is nothing to be proved, otherwise, at least one of the sets

$$\begin{aligned} \Lambda_n &= \{j \in \{1, \dots, \ell\} : c_j \in Z_n \setminus J_g^+\} \\ \Gamma_n &= \{j \in \{1, \dots, \ell\} : c_j = \tau_{i_j} \in Z_n \text{ for some } i_j \in \Gamma\} \end{aligned}$$

is non-empty. It is not hard to see that the sum over  $c_j \in Z_n$  is obtained by combining the sums over  $\Lambda_n$  and  $\Gamma_n$ . Clearly, by (2.9) we obtain

$$\sum_{j \in \Lambda_n} |F(t_j) - F(t_{j-1}) - F'_g(c_j)(g(t_j) - g(t_{j-1}))| \leq \varepsilon_n \sum_{j \in \Lambda_n} (H_n(t_j) - H_n(t_{j-1})).$$

On the other hand, (2.8) together with (2.7) imply

$$\begin{aligned} & \sum_{j \in \Gamma_n} |F(t_j) - F(t_{j-1}) - F'_g(\tau_{i_j})(g(t_j) - g(t_{j-1}))| \\ & \leq \sum_{j \in \Gamma_n} |F(t_j) - F(\tau_{i_j}) - F'_g(\tau_{i_j})(g(t_j) - g(\tau_{i_j}))| \\ & \quad + \sum_{j \in \Gamma_n} \left( |F(\tau_{i_j}) - F(t_{j-1})| + |F'_g(\tau_{i_j})| |g(\tau_{i_j}) - g(t_{j-1})| \right) \\ & \leq \varepsilon_n \sum_{j \in \Gamma_n} (H_n(t_j) - H_n(\tau_{i_j})) + \sum_{j \in \Gamma_n} \left( \frac{\varepsilon}{2^{i_j+2}} + |F'_g(\tau_{i_j})| \frac{\varepsilon}{2^{i_j+2} (|F'_g(\tau_{i_j})| + 1)} \right) \\ & \leq \varepsilon_n \sum_{j \in \Gamma_n} (H_n(t_j) - H_n(t_{j-1})) + \sum_{j \in \Gamma_n} \frac{\varepsilon}{2^{i_j+1}}. \end{aligned}$$



Combining the inequalities above we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{c_j \in Z_n} |F(t_j) - F(t_{j-1}) - F'_g(c_j)(g(t_j) - g(t_{j-1}))| \\ & \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} \sum_{c_j \in Z_n} \frac{H_n(t_j) - H_n(t_{j-1})}{H_n(b) - H_n(a)} + \sum_{n=1}^{\infty} \sum_{j \in \Gamma_n} \frac{\varepsilon}{2^{j+1}} \leq 2 \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} < \varepsilon \end{aligned}$$

wherefrom we conclude that (2.4) holds.  $\square$

**Remark 2.37.** At first glance, assumption (iii) on Theorem 2.36 might seem too restrictive. However, in view of Proposition 2.21, we observe that assumption (iii) simply pinpoints properties that one might expect from a function satisfying the equality in (2.4). Moreover, in the case  $N = C_g$ , condition (iii) is clearly satisfied if, as in [27, Theorem 6.2], we assume that  $F$  is constant on every subinterval where  $g$  is.

### 3 Main results

#### 3.1 Equivalence of distributional, differential and integral problems

In this section, we investigate the relation between the following three problems:

the distributional equation

$$\begin{aligned} Dx &= f(t, x)Dg, \\ x(0) &= x_0; \end{aligned} \tag{1}$$

the  $g$ -differential equation

$$\begin{aligned} x'_g(t) &= f(t, x(t)), \text{ } m_g\text{-a.e.}, \\ x(0) &= x_0; \end{aligned} \tag{2}$$

and the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))d_g(s). \tag{3}$$

We start by presenting the definition of solution for each of these problems.

**Definition 3.1.** Let  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g \in G_-([0, 1])$  and  $x_0 \in \mathbb{R}$  be given.

1. A function  $x \in G_-([0, 1])$  is a solution of the problem (1) if the distributional derivative of  $x$  satisfies

$$Dx = f(t, x)Dg \quad \text{for every } t \in [0, 1],$$

and  $x(0) = x_0$ .

2. A function  $x \in G_-([0, 1])$  is a solution of the problem (2) if  $x$  is  $g$ -normal and there exists  $N \subset [0, 1]$ , with  $m_g(N) = 0$ , such that the  $g$ -derivative of  $x$  satisfies

$$x'_g(t) = f(t, x(t)) \quad \text{for every } t \in [0, 1] \setminus N$$

and  $x(0) = x_0$ .

3. A function  $x \in G_-([0, 1])$  is a solution of the problem (3) if

$$x(t) = x_0 + \int_0^t f(s, x(s)) dg(s) \quad \text{for every } t \in [0, 1].$$

**Remark 3.2.** In the definition of a solution of the problem (2), we can always assume that  $C_g \subset N$  (see Remark 2.30).

To convert a distributional differential equation to an integral equation in the space of primitive functions, the Fundamental Theorem of Calculus relatively to the regulated primitive integral found in [38, Theorem 6] is a very useful tool. This approach appears, for instance, in [20] and [21]. In order to obtain an equivalence between problems (1) and (3), besides the aforementioned result we take into account the relation described in Remark 2.16.

**Theorem 3.3.** Let  $g \in G_-([0, 1])$  and  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following condition:

$$t \mapsto f(t, x(t)) \quad \text{is a BV-function for every } x \in G_-([0, 1]).$$

Then  $x : [0, 1] \rightarrow \mathbb{R}$  is a solution of problem (1) if and only if it is a solution of problem (3).

*Proof.* Suppose that  $x$  is a solution of (1). Since  $h_x(t) = f(t, x(t))$ ,  $t \in [0, 1]$ , is a function of bounded variation and  $Dg \in \mathcal{A}_{\mathcal{R}}([0, 1])$ , by Proposition 2.15 we have  $h_x Dg \in \mathcal{A}_{\mathcal{R}}([0, 1])$  and

$${}^r \int_0^t h_x Dg = \int_0^t h_x(s) dg(s) \quad \text{for every } t \in [0, 1].$$

(where the integral on the right hand side is the Kurzweil–Stieltjes integral). Moreover, Theorem 2.17 implies that for all  $t \in [0, 1]$ :

$${}^r \int_0^t h_x Dg = {}^r \int_0^t Dx = x(t) - x(0).$$

Combining these two facts we conclude that  $x$  is a solution of (3).

Let now  $x$  be a solution of (3). Using the equality found in Proposition 2.15 we get

$$x(t) - x_0 = \int_0^t f(s, x(s)) dg(s) = {}^r \int_0^t h_x Dg, \quad t \in [0, 1],$$

where  $h_x(s) = f(s, x(s))$ ,  $s \in [0, 1]$ . Thus, by Theorem 2.17, the distributional derivative of  $x$  is  $Dx = f(t, x)Dg$  and the result follows.  $\square$

**Remark 3.4.** The superposition assumption on  $f$  in Theorem 3.3 ensures the equivalence of the mentioned problems for a very large class of functions  $g$ , namely for every left-continuous regulated function. We remark that this assumption on  $f$  could be weakened if we require stronger assumptions on  $g$ ; that is, if  $g$  is a left-continuous BV-function. Indeed, let us recall that [38, Definition 11] introduces the product  $h_x Dg$  as the distributional derivative of the function defined in [38, Proposition 10] as follows:

$$\Xi(t) = h_x(t)g(t) - \int_0^t g(s)dh_x(s) - \sum_{c_n < t} (h_x(c_n) - h_x(c_n^+))(g(c_n) - g(c_n^+)),$$

where  $\{c_n, n \in \mathbb{N}\}$  denotes the set of common discontinuity points of  $g$  and  $h_x$ . It can be seen that  $\Xi$  is also BV when  $g$  and  $h_x$  are BV, thus  $h_x Dg$  is in this case the distributional derivative of a BV-function. As by Definition 3.1.1. a solution  $x$  of problem (1) satisfies the equality

$$Dx = h_x Dg,$$

our solutions  $x$  will be in the space of BV-functions. It turns out that in the case when  $g$  is BV, the main assumption in the previous equivalence result can be replaced with the following (weaker) assumption:

$$t \mapsto f(t, x(t)) \text{ is a BV-function for every left-continuous BV-function } x.$$

Conditions ensuring this property of the superposition operator can be found, for instance, in [1].

On the other hand, recalling that for left-continuous functions of bounded variation the distributional derivative is a Borel measure, this result ensures the equivalence between measure differential equations and  $g$ -differential equations.

In [27], the authors briefly illustrate the applicability of the  $g$ -derivative showing that ordinary differential equations, dynamic equations on a time scale and impulsive equations can be regarded as a  $g$ -differential problem (2). Next theorem is concerned with the relation between problem (2) and an integral equation, allowing us to explore other aspects of the  $g$ -differential equation.

**Theorem 3.5.** *Let  $g \in G_-([0, 1])$  be a BV $G^\circ$  function,  $x_0 \in \mathbb{R}$  and  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ . Then  $x : [0, 1] \rightarrow \mathbb{R}$  is a solution of problem (2) if and only if it is a solution of problem (3).*

*Proof.* Let  $x$  be a solution of (2) and, without loss of generality, assume that  $C_g \subset N$  (see Remark 2.30). This means that  $x$  is  $g$ -normal and  $x'_g(t) = f(t, x(t))$  for  $t \in [0, 1] \setminus N$ . Therefore, by the second Fundamental Theorem of Calculus, Theorem 2.36,

$$x(t) = x_0 + \int_0^t \tilde{f}(s, x(s)) dg(s), \quad t \in [0, 1].$$

where  $\tilde{f}$  is the function given by

$$\tilde{f}(t, y) = \begin{cases} f(t, y), & \text{if } t \in [0, 1] \setminus N, y \in \mathbb{R} \\ 0, & \text{otherwise} \end{cases}$$

By Lemma 2.22, one can see that  $\int_0^t \tilde{f}(s, x(s)) dg(s) = \int_0^t f(s, x(s)) dg(s)$  for every  $t \in [a, b]$ , proving that  $x$  is a solution of (3).

Conversely, assume that  $x$  is a solution of problem (3), that is

$$x(t) = x_0 + \int_0^t f(s, x(s)) dg(s) \quad \text{for every } t \in [0, 1].$$

Using the Fundamental theorem of calculus, Theorem 2.35, we obtain that  $x$  is  $g$ -differentiable  $m_g$ -a.e. and that

$$x'_g(t) = f(t, x(t)), \quad t \in [0, 1] \setminus N$$

where  $N \subset [0, 1]$  is such that  $m_g(N) = 0$ . In summary,  $x$  is a solution of problem (2), which concludes the proof.  $\square$

Let us note that if we restrict ourselves to the case when  $g$  is non-decreasing and left-continuous, the equivalence in Theorem 3.5 can be obtained by applying the Fundamental Theorems of Calculus found in [27]. In this particular case, the role of the outer measure is played by the Stieljes measure associated to  $g$ . Moreover, the relation provided by Theorem 3.5 suggests that problem (2) can be regarded as the differential counterpart of the notion of measure differential equation in the sense introduced in [13].

From Remark 3.4 and Theorem 3.5 we can deduce the following equivalence result.

**Corollary 3.6.** *Let  $g : [0, 1] \rightarrow \mathbb{R}$  be a left-continuous BV-function,  $x_0 \in \mathbb{R}$  and  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following condition:*

$$t \mapsto f(t, x(t)) \text{ is a BV-function for every left-continuous BV-function } x.$$

*Then the problems (1), (2) and (3) are equivalent.*

### 3.2 Existence results

We present in the sequel an existence result for the integral problem (3) based on a Leray–Schauder alternative which reads as follows.

**Theorem 3.7** ([17]). *Let  $E$  be a normed linear space and  $B_\rho$  the closed ball in  $E$  centered at 0 with radius  $\rho$ . Then, each compact map  $F : B_\rho \rightarrow E$  has at least one of the following two properties:*

- a)  *$F$  has a fixed point;*
- b) *there exists  $x \in \partial B_\rho$  and  $\lambda \in (0, 1)$  such that  $x = \lambda F(x)$ .*

The first existence result is proved in the very general setting of a left-continuous regulated function  $g$ . In what follows,  $B_R(y)$  stands for the closed ball in  $G_-([0, 1])$  with radius  $R$  and centered at the constant  $y$  function. If  $y = 0$ , the null function, we write simply  $B_R$ .

**Theorem 3.8.** *Let  $g \in G_-([0, 1])$ ,  $x_0 \in \mathbb{R}$  and  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following conditions:*

- i) *for each  $t \in [0, 1]$ ,  $f(t, \cdot)$  is continuous on  $\mathbb{R}$ ;*
- ii) *for every  $R > 0$ , the family  $\{f(\cdot, x(\cdot)) : x \in B_R(x_0)\}$  is equiintegrable w.r.t.  $g$ ;*
- iii) *for each  $R > 0$ , the set  $\{\int_0^t f(s, x(s))dg(s) : x \in B_R(x_0)\}$  is bounded for each  $t \in [0, 1]$ ;*
- iv) *there exists  $R_0 > 0$  such that for every  $x \in G_-([0, 1])$  with  $\|x - x_0\|_\infty = R_0$ , we have*

$$x(t) - x_0 - \lambda \int_0^t f(s, x(s))dg(s) \neq 0$$

*for every  $\lambda \in (0, 1)$  and some  $t \in [0, 1]$ .*

*Then the integral problem (3) has at least one solution.*

*Proof.* Let  $B := B_{R_0}(x_0)$ , where  $R_0 > 0$  is the number from assumption (iv). For  $x \in B$  and  $t \in [0, 1]$ , let

$$Tx(t) = x_0 + \int_0^t f(s, x(s))dg(s).$$

By Proposition 2.19,  $Tx$  defines a left-continuous, regulated function, that is  $T : B \rightarrow G_-([0, 1])$ . Note that a solution of (3) is necessarily a fixed point of the operator  $T$ . We will prove that the assumptions of the nonlinear alternative, Theorem 3.7, are satisfied for  $F : B_{R_0} \rightarrow G_-([0, 1])$ ,  $F = T - x_0$ .

*Step 1.* Let us check that the operator  $T$  is continuous. To this end, consider a sequence  $(x_n)_n$  converging uniformly to  $x$  in  $B$ . Hypothesis (i) implies that  $(f(\cdot, x_n(\cdot)))_n$  converges pointwisely to  $f(\cdot, x(\cdot))$  while assumption (ii) ensures the equiintegrability of the sequence. Using Theorem 2.25 we get

$$\lim_{n \rightarrow \infty} \int_0^t f(s, x_n(s)) dg(s) = \int_0^t f(s, x(s)) dg(s) \quad \text{for each } t \in [0, 1],$$

and consequently,  $\lim_{n \rightarrow \infty} Tx_n(t) = Tx(t)$ . On the other hand, it yields from the convergence of  $(f(\cdot, x_n(\cdot)))_n$  that the sequence is pointwisely bounded. Since, it is equiintegrable by assumption (ii), an application of Proposition 2.24 brings us to the equiregularity of the family of their primitives, that is  $\{Tx_n : n \in \mathbb{N}\}$  is equiregulated. Then, by Lemma 2.2  $Tx_n$  converges to  $Tx$  uniformly on  $[0, 1]$ , which implies the continuity of  $T$  at  $x \in B$ .

*Step 2.* We claim that  $T(B)$  satisfies that assumptions of Lemma 2.3. Indeed, assumption (iii) states that  $\{Tx(t) : x \in B_{R_0}\}$  is bounded for each  $t \in [0, 1]$ , so only the equiregularity of the family  $\{Tx : x \in B\}$  remains to be proved. As before, this is a consequence of Proposition 2.24, using (ii) and the fact that, due to the continuity in the second argument, for a fixed  $t \in [0, 1]$ , there is  $M_t > 0$  such that  $|f(t, x(t))| \leq M_t$  for all  $x \in B$ . Thus, applying Lemma 2.3, we conclude that  $T(B)$  is relatively compact in  $G_-([0, 1])$ .

Combining steps 1 and 2, we can see that  $T : B \rightarrow G_-([0, 1])$  is a compact mapping. Since assumption (iv) asserts that the alternative in Theorem 3.7 is not possible, we conclude that the operator  $F = T - x_0$  has a fixed point.  $\square$

In the case when  $g$  is a  $BVG^\circ$  function, we can deduce another existence result under weaker assumptions on  $f$ .

**Theorem 3.9.** *Let  $g \in G_-([0, 1])$  be a  $BVG^\circ$  function,  $x_0 \in \mathbb{R}$  and  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the assumptions (i), (ii) and (iv) in Theorem 3.8. Then, the integral problem (3) has at least one solution.*

**Proof.** We will prove that if  $g \in G_-([0, 1])$  is a  $BVG^\circ$  function, then the assumption (iii) in Theorem 3.8 is a consequence of assumptions (i) and (ii). Let  $R > 0$  be given and consider  $B := B_R(x_0) \subset G_-([0, 1])$ . Note that assumption (ii) together with Saks-Henstock lemma imply that there exists a gauge  $\gamma : [0, 1] \rightarrow \mathbb{R}_+$  such that

$$\sum_j \left| f(\xi_j, x(\xi_j))(g(\alpha_j) - g(\alpha_{j-1})) - \int_{\alpha_{j-1}}^{\alpha_j} f(s, x(s)) dg(s) \right| \leq 1, \quad (3.1)$$

for every  $\gamma$ -fine system  $(\xi_j, [\alpha_{j-1}, \alpha_j])$  and for all  $x \in B$ .

Since  $g$  is a  $BVG^\circ$  function, the interval  $[0, 1]$  can be written as a disjoint countable union of sets  $E_n$  such that  $m_g(E_n) < \infty$ ,  $n \in \mathbb{N}$ . For each  $n$ , by the definition of  $m_g$ , there exists a gauge  $\delta_n : E_n \rightarrow \mathbb{R}_+$  (which we can assume to be bounded from above by  $\gamma$ ) such that

$$W_g(S) < m_g(E_n) + 1 \quad \text{for every } S \in \mathcal{S}(E_n, \delta_n). \quad (3.2)$$

By the continuity of  $f$  in the second argument, assumption (i), for each  $t \in [0, 1]$ , we can find  $M_t > 0$  such that  $|f(t, x(t))| \leq M_t$  for all  $x \in B$ . Define now for each  $n, k \in \mathbb{N}$ :

$$\tilde{E}_{n,k} = \{t \in E_n : M_t \leq k\}.$$

Given  $t \in [0, 1]$ , there exists  $n(t) \in \mathbb{N}$  such that  $t \in E_{n(t)}$  and there exists also  $k(t) \in \mathbb{N}$  satisfying

$$k(t) - 1 < M_t \leq k(t),$$

wherefrom we get  $t \in \tilde{E}_{n(t),k(t)}$ . Let

$$J_t = \left( t - \delta_{n(t)}(t), t + \delta_{n(t)}(t) \right).$$

Note that, for  $s \in J_t$ , the system  $S = (t, [s, t])$ , if  $s < t$ , (or  $S = (t, [t, s])$ , if  $t < s$ ) is  $\delta_{n(t)}$ -fine. Hence, by (3.1) and (3.2), for  $x \in B$  we have

$$\begin{aligned} \left| \int_s^t f(\tau, x(\tau)) dg(\tau) \right| &\leq \left| \int_s^t f(\tau, x(\tau)) dg(\tau) - f(t, x(t))(g(t) - g(s)) \right| \\ &\quad + |f(t, x(t))(g(t) - g(s))| \\ &\leq 1 + M_t |g(t) - g(s)| \\ &\leq 1 + k(t) W_g(S) \\ &< 1 + k(t) m_g(E_{n(t)}) + k(t), \end{aligned}$$

that is,

$$\left| \int_s^t f(\tau, x(\tau)) dg(\tau) \right| < k(t) m_g(E_{n(t)}) + k(t) + 1, \quad (3.3)$$

for every  $x \in B$  and  $|s - t| < \delta_n(t)$ .

We use now (3.3) and the compactness of  $[0, 1]$  in order to get the boundedness property (iii) of Theorem 3.8.

Obviously,  $\{J_t : t \in [0, 1]\}$  is a cover of  $[0, 1]$ , thus there exists a finite set  $\{t_1, \dots, t_N\} \subset [0, 1]$  such that  $[0, 1] \subset \bigcup_{i=1, \dots, N} J_i$ , where  $J_i = J_{t_i}$ . Without loss of generality we may assume that  $t_i \notin J_j$  if  $i \neq j$ .

Let us denote  $K_i = k(t_i) m_g(E_{n(t_i)}) + k(t_i) + 1$  for each  $i = 1, \dots, N$ . Choosing,  $s_i \in J_{i-1} \cap J_i$  for  $i = 2, \dots, N$ , we have

$$|s_i - t_{i-1}| < \delta_{n(t_{i-1})}(t_{i-1}) \quad \text{and} \quad |s_i - t_i| < \delta_{n(t_i)}(t_i),$$

which together with (3.3) imply

$$\begin{aligned} \left| \int_{t_{i-1}}^{t_i} f(\tau, x(\tau)) dg(\tau) \right| &\leq \left| \int_{t_{i-1}}^{s_i} f(\tau, x(\tau)) dg(\tau) \right| + \left| \int_{s_i}^{t_i} f(\tau, x(\tau)) dg(\tau) \right| \\ &\leq K_{i-1} + K_i, \end{aligned} \quad (3.4)$$

for every  $x \in B$ .

Let  $M = \sum_{i=1}^N K_i$ . Without loss of generality, we can assume that  $0 < t_1$  and  $t_N < 1$ ; consequently  $[0, t_1] \subset J_1$  and  $[t_N, 1] \subset J_N$ . We will show that

$$\left| \int_0^t f(\tau, x(\tau)) dg(\tau) \right| \leq 2M, \quad \text{for every } t \in [0, 1] \text{ and } x \in B. \quad (3.5)$$

Given an arbitrary  $x \in B$ , if  $t \in (0, t_1]$ , by (3.3) we have

$$\left| \int_0^t f(\tau, x(\tau)) dg(\tau) \right| \leq \left| \int_0^{t_1} f(\tau, x(\tau)) dg(\tau) \right| + \left| \int_{t_1}^t f(\tau, x(\tau)) dg(\tau) \right| \leq 2K_1,$$

and (3.5) holds. Now, consider  $t \in (t_1, 1]$  and let  $p := \max \{i \in \{1, \dots, N\} : t_i < t\}$ . Using (3.3) and (3.4) we obtain

$$\begin{aligned} & \left| \int_0^t f(\tau, x(\tau)) dg(\tau) \right| \\ & \leq \left| \int_0^{t_1} f(\tau, x(\tau)) dg(\tau) \right| + \sum_{j=2}^p \left| \int_{t_{j-1}}^{t_j} f(\tau, x(\tau)) dg(\tau) \right| + \left| \int_{t_p}^t f(\tau, x(\tau)) dg(\tau) \right| \\ & \leq K_1 + \sum_{j=2}^p (K_{j-1} + K_j) + \left| \int_{t_p}^t f(\tau, x(\tau)) dg(\tau) \right|, \end{aligned}$$

that is,

$$\left| \int_0^t f(\tau, x(\tau)) dg(\tau) \right| \leq 2 \sum_{j=1}^{p-1} K_j + K_p + \left| \int_{t_p}^t f(\tau, x(\tau)) dg(\tau) \right|.$$

In order to estimate the last term we have to distinguish two cases: if  $t$  belongs or not to  $J_p$ . In case  $t \in J_p$ , we can apply (3.3) again concluding that

$$\left| \int_0^t f(\tau, x(\tau)) dg(\tau) \right| \leq 2 \sum_{j=1}^p K_j \leq 2M.$$

On the other hand, if  $t \notin J_p$ , since  $t \leq t_{p+1}$  we must have  $t \in J_{p+1}$ . Therefore,

$$\begin{aligned} \left| \int_{t_p}^t f(\tau, x(\tau)) dg(\tau) \right| & \leq \left| \int_{t_p}^{t_{p+1}} f(\tau, x(\tau)) dg(\tau) \right| + \left| \int_{t_{p+1}}^t f(\tau, x(\tau)) dg(\tau) \right| \\ & \leq K_p + 2K_{p+1}, \end{aligned}$$

and consequently

$$\left| \int_0^t f(\tau, x(\tau)) dg(\tau) \right| \leq 2 \sum_{j=1}^{p+1} K_j \leq 2M.$$

In summary, for each  $t \in [0, 1]$  and  $x \in B$ , (3.5) holds, which shows that condition (iii) from Theorem 3.8 is satisfied.  $\square$

**Remark 3.10.** Both existence results, Theorem 3.8 and 3.9, might throw a new light in the study of measure functional differential equations in the sense of [13]. Actually, problem (3) is an example of the so-called measure differential equations; however, unlike the theory which has been developed up to now, here we deal with a more general class of integrators.

In view of the equivalence stated in Theorem 3.3, from our first existence result we derive the following.

**Theorem 3.11.** Let  $g \in G_-([0, 1])$ ,  $x_0 \in \mathbb{R}$  and  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following conditions:

i) for each  $t \in [0, 1]$ ,  $f(t, \cdot)$  is continuous on  $\mathbb{R}$  and

$$t \mapsto f(t, x(t)) \quad \text{is a BV-function for every } x \in G_-([0, 1]).$$

- ii) for every  $R > 0$ , the family  $\{f(\cdot, x(\cdot)) : x \in B_R(x_0) \subset G_-[0, 1]\}$  is equiintegrable w.r.t.  $g$ ;
- iii) for each  $R > 0$ , the family  $\{\int_0^t f(s, x(s))dg(s) : x \in B_R(x_0)\}$  is bounded for each  $t \in [0, 1]$ ;
- iv) there exists  $R_0 > 0$  such that for every  $x \in G_-[0, 1]$  with  $\|x - x_0\|_\infty = R_0$ ,

$$x(t) - x_0 - \lambda \int_0^t f(s, x(s))dg(s) \neq 0$$

for every  $\lambda \in (0, 1)$  and some  $t \in [0, 1]$ .

Then the distributional problem (1) has at least one solution.

By combining Theorems 3.5 and 3.9 we obtain the following existence result for problem (2).

**Theorem 3.12.** *Let  $g \in G_-([0, 1])$  be a BVG $^\circ$  function,  $x_0 \in \mathbb{R}$  and  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the assumptions (i), (ii) and (iv) in Theorem 3.8. Then the problem (2) has at least one solution.*

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