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On nonexistence of solutions to some nonlinear inequalities with transformed argument

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Abstract. We obtain results on nonexistence of nontrivial solutions for several classes of nonlinear partial differential inequalities and systems of such inequalities with transformed argument.

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1 Introduction

In recent years conditions for nonexistence of solutions to nonlinear partial differential equations and inequalities attract the attention of many mathematicians. This problem is not only of interest of its own, but also has important mathematical and physical applications. In particular, Liouville type theorems of nonexistence of nontrivial positive solutions to nonlinear equations in the whole space or half-space can be used for obtaining a priori estimates of solutions to respective problems in bounded domains [1, 4].

In [5–7] (see also references therein) sufficient conditions for nonexistence of solutions were obtained for different classes of nonlinear elliptic and parabolic inequalities using the nonlinear capacity method developed by S. Pohozaev [8]. On the other hand, there exists an elaborated theory of partial differential equations with transformed argument due to A. Skubachevskii [9]. But the problem of sufficient conditions for nonexistence of solutions to respective inequalities with transformed argument remained open. Some special cases of such problems were treated in [2, 3].

In this paper we obtain sufficient conditions for nonexistence of solutions to several classes of elliptic and parabolic inequalities with transformed argument and for systems of elliptic inequalities of this type.

The structure of the paper is as follows. In §2, we prove nonexistence theorems for semilinear elliptic inequalities of higher order; in §3, for quasilinear elliptic inequalities; in §4, for

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systems of quasilinear elliptic inequalities; and in §5, for nonlinear parabolic inequalities with a shifted time argument.

The letter c with different subscripts or without them denotes positive constants that may depend on the parameters of the inequalities and systems under consideration.

2 Semilinear elliptic inequalities

Let $k \in \mathbb{N}$. Consider a semilinear elliptic inequality

$$(-\Delta)^k u(x) \geq |u(g(x))|^q \qquad (x \in \mathbb{R}^n), \tag{2.1}$$

where $g \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is a mapping such that

(g1) there exists a constant $c > 0$ such that $|J_g^{-1}(x)| \geq c > 0$ for all $x \in \mathbb{R}^n$;

(g2) $|g(x)| \geq |x|$ for all $x \in \mathbb{R}^n$.

Example 2.1. The dilatation transform $g(x) = \gamma x$ with any $\gamma \in \mathbb{R}$ such that $|\gamma| > 1$ satisfies assumptions (g1) with $c = |\gamma|^{-n}$ and (g2).

Example 2.2. The rotation transform $g(x) = Ax$, where A is a $n \times n$ unitary matrix (and therefore $|g(x)| = |x|$ for all $x \in \mathbb{R}^n$), satisfies assumptions (g1) with $c = 1$ and (g2).

In some situations assumption (g2) can be replaced by a weaker one:

(g′2) there exist constants $c_0 > 0$ and $\rho > 0$ such that $|g(x)| \geq c_0|x|$ for all $x \in \mathbb{R}^n \setminus B_\rho(0)$.

Remark 2.3. We assume without loss of generality that $c_0 \leq 1$.

Example 2.4. The contraction transform $g(x) = \gamma x$ with $0 < |\gamma| \leq 1$ satisfies assumptions (g1) with $c = |\gamma|^{-n}$ and (g′2) with $c_0 = |\gamma|$ and any $\rho > 0$.

Example 2.5. So does the shift transform $g(x) = x - x_0$ for a fixed $x_0 \in \mathbb{R}^n$ with $c = 1$, $c_0 = 1/2$ and $\rho = 2|x_0|$.

Lemma 2.6. *There exists a nonincreasing function $\varphi(s) \geq 0$ in $C^{2k}[0, \infty)$, satisfying conditions*

$$\varphi(s) = \begin{cases} 1 & (0 \leq s \leq 1), \\ 0 & (s \geq 2), \end{cases} \tag{2.2}$$

and

$$\int_1^2 \frac{|\varphi'(s)|^{q'}}{\varphi^{q'-1}(s)}\, ds < \infty, \tag{2.3}$$

$$\int_1^2 \frac{|\Delta^k \varphi(s)|^{q'}}{\varphi^{q'-1}(s)}\, ds < \infty. \tag{2.4}$$

Here and below $q' = \frac{q}{q-1}$.

Proof. Take $\varphi(s)$ equal to $(2-s)^\lambda$ with a sufficiently large $\lambda > 0$ in a left neighborhood of 2 (see [7]). □

Theorem 2.7. *Let either $n \le 2k$ and $q > 1$, or $n > 2k$ and $1 < q \le \frac{n}{n-2k}$. Suppose that g satisfies assumptions (g1) and (g2). Then inequality (2.1) has no nontrivial solutions $u \in L_{q,\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. Assume for contradiction that a nontrivial solutions of (2.1) does exist. Let $0 < R < \infty$ (in particular, the case $R = 1$ is possible). The function

$$\varphi_R(x) = \varphi\left(\frac{|x|}{R}\right),$$

where $\varphi(s)$ is from Lemma 2.6, will be used as a *test function* for inequality (2.1). Multiplying both sides of (2.1) by the test function φ_R and integrating by parts $2k$ times, we get

$$\int_{\mathbb{R}^n} |u(x)| \cdot |\Delta^k \varphi_R(x)|\, dx \ge \int_{\mathbb{R}^n} |u(g(x))|^q \varphi_R(x)\, dx. \tag{2.5}$$

Using (g1), (g2), and the monotonicity of φ_R, one can estimate the right-hand side of (2.5) from below as

$$\int_{\mathbb{R}^n} |u(g(x))|^q \varphi_R(x)\, dx = \int_{\mathbb{R}^n} |u(x)|^q \varphi_R(g^{-1}(x))|J_g^{-1}(x)|\, dx \ge c \int_{\mathbb{R}^n} |u(x)|^q \varphi_R(x)\, dx. \tag{2.6}$$

On the other hand, applying the parametric Young inequality to the left-hand side of (2.5), we get

$$\begin{aligned} &\int_{\mathbb{R}^n} |u(x)| \cdot \left|\Delta^k \varphi_R(x)\right|\, dx \\ &\quad \le \frac{c}{2} \int_{\mathbb{R}^n} |u(x)|^q \varphi_R(x)\, dx + c_1 \int_{\mathbb{R}^n} \left|\Delta^k \varphi_R(x)\right|^{q'} \varphi_R^{1-q'}(x)\, dx \\ &\quad = \frac{c}{2} \int_{\mathbb{R}^n} |u(x)|^q \varphi_R(x)\, dx + c_1 R^{n-2kq'} \int_{\mathbb{R}^n} \left|\Delta^k \varphi_1(x)\right|^{q'} \varphi_1^{1-q'}(x)\, dx \\ &\quad = \frac{c}{2} \int_{\mathbb{R}^n} |u(x)|^q \varphi_R(x)\, dx + c_2 R^{n-2kq'} \end{aligned} \tag{2.7}$$

with some constants $c_1, c_2 > 0$. Combining (2.5)–(2.7), we have

$$\frac{c}{2} \int_{\mathbb{R}^n} |u(x)|^q \varphi_R(x)\, dx \le c_2 R^{n-2kq'}.$$

Restricting the integration domain in the left-hand side of the inequality, we obtain

$$\frac{c}{2} \int_{B_R(0)} |u(x)|^q\, dx \le c_2 R^{n-2kq'}.$$

Taking $R \to \infty$, we get a contradiction for $n - 2kq' < 0$, which proves the theorem in all cases except the critical one (where $n - 2kq' = 0$).

In the critical case we get

$$\int_{\mathbb{R}^n} |u(x)|^q\, dx < \infty$$

and hence

$$\int_{\operatorname{supp} \Delta^k \varphi_R} |u(x)|^q\, dx \le \int_{B_{2R}(0) \setminus B_R(0)} |u(x)|^q\, dx \to 0 \quad \text{as } R \to \infty.$$

But (2.5), (2.6) and the Hölder inequality imply

$$c \int_{B_R(0)} |u(x)|^q\, dx \le \left(\int_{\operatorname{supp} \Delta^k \varphi_R} |u(x)|^q\, dx\right)^{\frac{1}{q}} \cdot \left(\int_{\operatorname{supp} \Delta^k \varphi_R} \left|\Delta^k \varphi_R(x)\right|^{q'} \varphi_R^{1-q'}(x)\, dx\right)^{\frac{1}{q'}} \tag{2.8}$$

and therefore

$$\int_{B_R(0)} |u(x)|^q\, dx \le c \left(\int_{\operatorname{supp} \Delta^k \varphi_R} |u(x)|^q\, dx \right)^{\frac{1}{q}} \to 0 \quad \text{as } R \to \infty$$

since the second factor on the right-hand side of (2.8) can be estimated from above by $c_2 R^{n-2kq'}$ as before, where $n - 2kq' = 0$. Thus for a nontrivial u we obtain a contradiction in this case as well. This completes the proof. □

Theorem 2.8. *Let either $n \le 2k$ and $q > 1$, or $n > 2k$ and $1 < q \le \frac{n}{n-2k}$. Suppose that g satisfies assumptions (g1) and (g′2). Then inequality (2.1) has no nontrivial solutions $u \in L_{q,\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$l_\rho := \lim_{R\to\infty} \frac{\int_{B_{2R}(0)} |u(x)|^q\, dx}{\int_{B_{c_0 R}(0)\setminus B_\rho(0)} |u(x)|^q\, dx} < \infty \tag{2.9}$$

(in particular, $u \in L_q(\mathbb{R}^n)$).

Proof. Similarly to estimate (2.6), for $R > \rho$ we get

$$\begin{aligned} \int_{\mathbb{R}^n} |u(g(x))|^q \varphi_R(x)\, dx &= \int_{\mathbb{R}^n} |u(x)|^q \varphi_R(g^{-1}(x)) |J_g^{-1}(x)|\, dx \\ &\ge c \int_{\mathbb{R}^n \setminus B_\rho(0)} |u(x)|^q \varphi_R\left(\frac{x}{c_0}\right) dx \ge c \int_{B_{c_0 R}(0)\setminus B_\rho(0)} |u(x)|^q\, dx. \end{aligned} \tag{2.10}$$

Then (2.5) and (2.7)–(2.10) imply

$$\int_{B_{c_0 R}(0)\setminus B_\rho(0)} |u(x)|^q\, dx \le c_1 \int_{B_{2R}(0)} |u(x)|^q\, dx + c_2 R^{n-2kq'},$$

where $c_1, c_2 > 0$, and the constant c_1 can be chosen arbitrarily small. Hence by assumption (2.9) for $c_1 < \frac{1}{2l_\rho + 1}$ and sufficiently large R we have

$$\int_{B_{c_0 R}(0)\setminus B_\rho(0)} |u(x)|^q\, dx \le 2c_2 R^{n-2kq'},$$

i.e., the conclusion of Theorem 2.7 for any subcritical q remains valid in this case as well. The critical case can be treated similarly to the previous theorem. □

Further we consider the inequality

$$(-\Delta)^k u(x) \ge |Du(g(x))|^q \qquad (x \in \mathbb{R}^n). \tag{2.11}$$

Theorem 2.9. *Let either $n \le 2k - 1$ and $q > 1$, or $n > 2k - 1$ and $1 < q \le \frac{n}{n-2k+1}$. Suppose that g satisfies assumptions (g1) and (g2). Then inequality (2.11) has no nontrivial solutions $u \in W^1_{q,\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. Multiplying both sides of (2.11) by the test function φ_R and integrating by parts $2k - 1$ times, we get

$$\int_{\mathbb{R}^n} (Du(x), D(\Delta^{k-1}\varphi_R(x)))\, dx \ge \int_{\mathbb{R}^n} |Du(g(x))|^q \varphi_R(x)\, dx,$$

which implies

$$\int_{\mathbb{R}^n} |Du(x)| \cdot |D(\Delta^{k-1}\varphi_R(x))|\, dx \ge \int_{\mathbb{R}^n} |Du(g(x))|^q \varphi_R(x)\, dx. \tag{2.12}$$

Using (g1) and (g2), we can estimate the right-hand side of (2.12) from below as

$$\begin{aligned}\int_{\mathbb{R}^n} |Du(g(x))|^q \varphi_R(x)\,dx &= \int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(g^{-1}(x))|J_g^{-1}(x)|\,dx \\ &\geq c\int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(x)\,dx.\end{aligned} \tag{2.13}$$

On the other hand, applying the parametric Young inequality to the left-hand side of (2.12), we get

$$\begin{aligned}&\int_{\mathbb{R}^n} |Du(x)| \cdot \left|D(\Delta^{k-1}\varphi_R(x))\right|\,dx \\ &\quad\leq \frac{c}{2}\int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(x)\,dx + c_1\int_{\mathbb{R}^n} \left|D(\Delta^{k-1}\varphi_R(x))\right|^{q'} \varphi_R^{1-q'}(x)\,dx \\ &\quad\leq \frac{c}{2}\int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(x)\,dx + c_2 R^{n-(2k-1)q'}\end{aligned} \tag{2.14}$$

with some constants $c_1, c_2 > 0$. Combining (2.12)–(2.14), we have

$$\frac{c}{2}\int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(x)\,dx \leq c_2 R^{n-(2k-1)q'}.$$

Restricting the integration domain in the left-hand side of the inequality, we obtain

$$\frac{c}{2}\int_{B_R(0)} |Du(x)|^q\,dx \leq c_2 R^{n-(2k-1)q'}.$$

Taking $R \to \infty$, we get a contradiction for $n - (2k-1)q' < 0$. The critical case can be treated similarly to the previous theorems. □

Theorem 2.10. *Let either $n \leq 2k-1$ and $q > 1$, or $n > 2k-1$ and $1 < q \leq \frac{n}{n-2k+1}$. Suppose that g satisfies assumptions (g1) and (g′2). Then inequality (2.11) has no nontrivial solutions $u \in W^1_{q,\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$m_\rho := \lim_{R\to\infty} \frac{\int_{B_{2R}(0)} |Du(x)|^q\,dx}{\int_{B_{c_0R}(0)\setminus B_\rho(0)} |Du(x)|^q\,dx} < \infty \tag{2.15}$$

(in particular, $u \in W^1_q(\mathbb{R}^n)$).

Proof. It is similar to that of Theorem 2.8. □

3 Quasilinear elliptic inequalities

Further consider the inequality

$$-\Delta_p u(x) \geq u^q(g(x)) \qquad (x \in \mathbb{R}^n), \tag{3.1}$$

where $g \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ satisfies assumptions (g1) and (g′2).

Theorem 3.1. *Let $p > 1$ and $p - 1 < q \leq \frac{n(p-1)}{n-p}$. Suppose that g satisfies assumptions (g1) and (g′2). Then inequality (3.1) has no nontrivial nonnegative solutions $u \in W^1_{p,\mathrm{loc}}(\mathbb{R}^n) \cap L_{q,\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. We use the same test functions φ_R as in the previous section. Choose λ so that $1-p<\lambda<0$. Multiplying both sides of (3.1) by $u^\lambda(x)\varphi_R(x)$, integrating by parts, and applying the parametric Young inequality with $\eta>0$, we get

$$\begin{aligned}\lambda \int_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p\varphi_R(x)\,dx &+ \int_{\mathbb{R}^n} u^{\lambda}(x)|Du(x)|^{p-1}|D\varphi_R(x)|\,dx\\ &\geq \int_{\mathbb{R}^n} u^q(g(x))u^\lambda(x)\varphi_R(x)\,dx\\ &\geq c_\eta \int_{\mathbb{R}^n} u^{q+\lambda}(g(x))\varphi_R(x)\,dx-\eta\int_{\mathbb{R}^n} u^{q+\lambda}(x)\varphi_R(x)\,dx.\end{aligned}\tag{3.2}$$

Since $-\Delta_p u\geq 0$, u satisfies the weak Harnack inequality

$$\int_{B_{2R}(0)} u^{q+\lambda}(x)\,dx \leq cR^n \inf_{x\in B_R(0)} u^{q+\lambda}(x)$$

and by iteration

$$\int_{B_{2R}(0)} u^{q+\lambda}(x)\,dx \leq cR^n \inf_{x\in B_{c_0R}(0)} u^{q+\lambda}(x)\leq c\int_{B_{c_0R}(0)} u^{q+\lambda}(x)\,dx,$$

possibly with a different $c>0$ (see [10]). Therefore

$$\begin{aligned}\int_{\mathbb{R}^n} u^{q+\lambda}(x)\varphi_R(g^{-1}(x))|J_g^{-1}(x)|\,dx &\geq c\int_{B_{c_0R}(0)} u^{q+\lambda}(x)\,dx\\ &\geq c\int_{B_{2R}(0)} u^{q+\lambda}(x)\,dx \geq c\int_{\mathbb{R}^n} u^{q+\lambda}(x)\varphi_R(x)\,dx.\end{aligned}\tag{3.3}$$

For a sufficiently small $\eta>0$ (note that $c_\eta\to+\infty$ as $\eta\to+0$), we can estimate the right-hand side of (3.2) from below, since due to (g1), (g′2), and (3.3)

$$\begin{aligned}c_\eta \int_{\mathbb{R}^n} & u^{q+\lambda}(g(x))\varphi_R(x)\,dx-\eta\int_{\mathbb{R}^n} u^{q+\lambda}(x)\varphi_R(x)\,dx\\ &= c_\eta \int_{\mathbb{R}^n} u^{q+\lambda}(x)\varphi_R(g^{-1}(x))|J_g^{-1}(x)|\,dx-\eta\int_{\mathbb{R}^n} u^{q+\lambda}(x)\varphi_R(x)\,dx\\ &\geq (c_\eta c-\eta)\int_{\mathbb{R}^n} u^{q+\lambda}(x)\varphi_R(x)\,dx \geq c_1\int_{B_{c_0R}(0)\setminus B_\rho(0)} u^{q+\lambda}(x)\,dx\\ &\geq c_2\int_{B_{2R}(0)\setminus B_\rho(0)} u^{q+\lambda}(x)\,dx\end{aligned}\tag{3.4}$$

with some constants $c_1, c_2>0$.

On the other hand, applying the parametric Young inequality with exponents $\frac{p}{p-1}$ and p to the integrand at the left-hand side of (3.2) represented as

$$(p\varepsilon\varphi_R)^{\frac{p-1}{p}}u^{\frac{(\lambda-1)(p-1)}{p}}\cdot(p\varepsilon\varphi_R)^{\frac{1-p}{p}}|D\varphi_R|$$

with $0<\varepsilon<|\lambda|$, and then applying it again with exponents $\frac{q+\lambda}{\lambda+p-1}$ and $\frac{q+\lambda}{q-p+1}$ (note that these exponents are greater than 1 for a sufficiently small $|\lambda|$ due to the assumption $q>p-1$) to $u^{\lambda+p-1}(x)|D\varphi_R(x)|^p\varphi_R^{1-p}(x)$ represented as

$$\left(\frac{c_2(q+\lambda)}{2(\lambda+p-1)}\varphi_R\right)^{\frac{\lambda+p-1}{q+\lambda}}u^{\lambda+p-1}\cdot\left(\frac{c_2(q+\lambda)}{2(\lambda+p-1)}\varphi_R\right)^{-\frac{\lambda+p-1}{q+\lambda}}|D\varphi_R|^p\varphi_R^{1-p},$$

we get

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p \varphi_R(x)\,dx + \int_{\mathbb{R}^n} u^{\lambda}(x)|Du(x)|^{p-1}|D\varphi_R(x)|\,dx \\
&\quad\le (\lambda+\varepsilon) \int_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p \varphi_R(x)\,dx + c_3(\varepsilon) \int_{\mathbb{R}^n} u^{\lambda+p-1}(x)|D\varphi_R(x)|^p \varphi_R^{1-p}(x)\,dx \\
&\quad\le (\lambda+\varepsilon) \int_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p \varphi_R(x)\,dx + \frac{c_2}{2} \int_{\mathbb{R}^n} u^{q+\lambda}(x)\varphi_R(x)\,dx \\
&\qquad + c_4(\varepsilon) \int_{\mathbb{R}^n} |D\varphi_R(x)|^{\frac{p(q+\lambda)}{q-p+1}} \varphi_R^{1-\frac{p(q+\lambda)}{q-p+1}}(x)\,dx \le (\lambda+\varepsilon) \int_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p \varphi_R(x)\,dx \\
&\qquad + \frac{c_2}{2} \int_{\mathbb{R}^n} u^{q+\lambda}(x)\varphi_R(x)\,dx + c_5(\varepsilon) R^{n-\frac{p(q+\lambda)}{q-p+1}}
\end{aligned} \tag{3.5}$$

with some constants $\varepsilon, c_3(\varepsilon), c_4(\varepsilon), c_5(\varepsilon) > 0$. Combining (3.2)–(3.5), we have

$$\frac{c_2}{2} \int_{B_{2R}(0)\setminus B_\rho(0)} u^{q+\lambda}(x)\varphi_R(x)\,dx \le c_5(\varepsilon) R^{n-\frac{p(q+\lambda)}{q-p+1}}.$$

Choosing λ sufficiently close to 0 and taking $R \to \infty$, we obtain a contradiction for $n - \frac{pq}{q-p+1} < 0$, i.e., $p-1 < q < \frac{n(p-1)}{n-p}$. The critical case can be treated similarly to the previous theorems. □

Further we consider the inequality

$$-\Delta_p u(x) \ge |Du(g(x))|^q \qquad (x \in \mathbb{R}^n). \tag{3.6}$$

Theorem 3.2. *Let $p-1 < q \le \frac{n(p-1)}{n-1}$. Suppose that g satisfies assumptions (g1) and (g2). Then inequality (3.6) has no nontrivial nonnegative solutions $u \in W^1_{p,\mathrm{loc}}(\mathbb{R}^n) \cap W^1_{q,\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. Multiplying both parts of (3.6) by the test function φ_R and integrating by parts, we get

$$\int_{\mathbb{R}^n} |Du(x)|^{p-2}(Du(x), D\varphi_R(x))\,dx \ge \int_{\mathbb{R}^n} |Du(g(x))|^q \varphi_R(x)\,dx,$$

which implies

$$\int_{\mathbb{R}^n} |Du(x)|^{p-1} \cdot |D\varphi_R(x)|\,dx \ge \int_{\mathbb{R}^n} |Du(g(x))|^q \varphi_R(x)\,dx. \tag{3.7}$$

Using (g1) and (g2), one can estimate the right-hand side of (3.7) from below as

$$\begin{aligned}
\int_{\mathbb{R}^n} |Du(g(x))|^q \varphi_R(x)\,dx &= \int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(g^{-1}(x))|J_g^{-1}(x)|\,dx \\
&\ge c_0 \int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(x)\,dx.
\end{aligned} \tag{3.8}$$

On the other hand, applying the Hölder inequality to the left-hand side of (3.7), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |Du(x)| \cdot |D\varphi_R(x)|\,dx \\
&\quad\le \left(\int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(x)\,dx \right)^{\frac{p-1}{q}} \left(\int_{\mathbb{R}^n} |D\varphi_R(x)|^{\frac{q}{q-p+1}} \varphi_R^{1-\frac{q}{q-p+1}}(x)\,dx \right)^{\frac{q-p+1}{q}}.
\end{aligned} \tag{3.9}$$

Combining (3.7)–(3.9), we have

$$\int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(x)\,dx \le c_1 \int_{B_{2R}(0)} |D\varphi_R(x)|^{\frac{q}{q-p+1}} \varphi_R^{1-\frac{q}{q-p+1}}(x)\,dx$$

and hence

$$\int_{B_R(0)} |Du(x)|^q \, dx \leq c_2 R^{n-\frac{q}{q-p+1}}$$

with some constants $c_1, c_2 > 0$. Taking $R \to \infty$, we obtain a contradiction for $n - \frac{q}{q-p+1} < 0$. The critical case can be treated similarly to the previous theorems. □

Remark 3.3. If g satisfies (g′2) instead of (g2), a version of Theorem 3.2 can be proven for a class of solutions that satisfy (2.15) (in particular, $u \in W^1_{p,\mathrm{loc}}(\mathbb{R}^n) \cap W^1_q(\mathbb{R}^n)$) similarly to Theorems 2.8 and 2.10.

4 Systems of quasilinear elliptic inequalities

Now consider a system of quasilinear elliptic inequalities

$$\begin{cases} -\Delta_p u(x) \geq v^{q_1}(g_1(x)) & (x \in \mathbb{R}^n), \\ -\Delta_q v(x) \geq u^{p_1}(g_2(x)) & (x \in \mathbb{R}^n), \end{cases} \tag{4.1}$$

where $g_1, g_2 \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ are mappings that satisfy (g1) and (g′2).

Introduce the quantities

$$\begin{aligned} \sigma_1 &= n - \frac{qq_1}{q_1 - q + 1}, \\ \sigma_2 &= n - \frac{pp_1}{p_1 - p + 1}, \\ \sigma &= \sigma_1(p-1)(q_1 - q + 1) + \sigma_2 q_1 (p_1 - p + 1), \\ \tau &= \sigma_1 p_1 (q_1 - q + 1) + \sigma_2 (q-1)(p_1 - p + 1). \end{aligned} \tag{4.2}$$

Then there holds the following.

Theorem 4.1. *Let $p, q, p_1, q_1 > 1$, $p - 1 < p_1$, $q - 1 < q_1$, and $\min(\sigma, \tau) \leq 0$. Suppose that g_1 and g_2 satisfy assumptions (g1) and (g′2). Then system (4.1) has no nontrivial nonnegative solutions*

$$(u, v) \in (W^1_{p,\mathrm{loc}}(\mathbb{R}^n) \cap L_{p_1,\mathrm{loc}}(\mathbb{R}^n)) \times (W^1_{q,\mathrm{loc}}(\mathbb{R}^n) \cap L_{q_1,\mathrm{loc}}(\mathbb{R}^n)).$$

Proof. Assume that there exists (u, v) – a nontrivial nonnegative solution of system (4.1). Let $\{\varphi_R\}$ be the same family of test functions as in Sections 2 and 3.

Multiplying the first inequality (4.1) by $u_\varepsilon^\lambda \varphi_R$ and the second one by $v_\varepsilon^\lambda \varphi_R$, where $u_\varepsilon = u + \varepsilon$, $v_\varepsilon = v + \varepsilon$, $\varepsilon > 0$ and $\max\{1-p, 1-q\} < \lambda < 0$, we get

$$\int v^{q_1}(g_1(x)) u_\varepsilon^\lambda(x) \varphi_R(x) \, dx \leq \lambda \int |Du|^p u_\varepsilon^{\lambda-1} \varphi_R \, dx + \int |Du|^{p-1} |D\varphi_R| u_\varepsilon^\lambda \, dx,$$

$$\int u^{p_1}(g_2(x)) v_\varepsilon^\lambda(x) \varphi_R(x) \, dx \leq \lambda \int |Dv|^q v_\varepsilon^{\lambda-1} \varphi_R \, dx + \int |Dv|^{q-1} |D\varphi_R| v_\varepsilon^\lambda \, dx,$$

which can be rewritten as (note that $|\lambda| = -\lambda$ since $\lambda < 0$)

$$\int v^{q_1}(g_1(x)) u_\varepsilon^\lambda(x) \varphi_R(x) \, dx + |\lambda| \int |Du|^p u_\varepsilon^{\lambda-1} \varphi_R \, dx \leq \int |Du|^{p-1} |D\varphi_R| u_\varepsilon^\lambda \, dx, \tag{4.3}$$

$$\int u^{p_1}(g_2(x)) v_\varepsilon^\lambda(x) \varphi_R(x) \, dx + |\lambda| \int |Dv|^q v_\varepsilon^{\lambda-1} \varphi_R \, dx \leq \int |Dv|^{q-1} |D\varphi_R| v_\varepsilon^\lambda \, dx. \tag{4.4}$$

Here and below we omit the argument x if it is the only one in a certain integral, and $\mathbb{R}^n$ if it is the integration domain. Application of the Young inequality to the right-hand sides of the obtained relations results in

$$\int v^{q_1}(g_1(x))u_\varepsilon^\lambda(x)\varphi_R(x)\,dx + \frac{|\lambda|}{2}\int |Du|^p u_\varepsilon^{\lambda-1}\varphi_R\,dx \le c_\lambda \int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_\varepsilon^{\lambda+p-1}\,dx, \tag{4.5}$$

$$\int u^{p_1}(g_2(x))v_\varepsilon^\lambda(x)\varphi_R(x)\,dx + \frac{|\lambda|}{2}\int |Dv|^q v_\varepsilon^{\lambda-1}\varphi_R\,dx \le d_\lambda \int \frac{|D\varphi_R|^q}{\varphi_R^{q-1}} v_\varepsilon^{\lambda+q-1}\,dx, \tag{4.6}$$

where constants c_λ and d_λ depend only on p, q, and λ. Further, multiplying each inequality (4.1) by φ_R and integrating by parts, we obtain

$$\int v^{q_1}(g_1(x))\varphi_R(x)\,dx \le \left(\int |Du|^p u_\varepsilon^{\lambda-1}\varphi_R\,dx\right)^{\frac{p-1}{p}} \left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_\varepsilon^{(1-\lambda)(p-1)}\,dx\right)^{\frac{1}{p}}, \tag{4.7}$$

$$\int u^{p_1}(g_2(x))\varphi_R(x)\,dx \le \left(\int |Dv|^q v_\varepsilon^{\lambda-1}\varphi_R\,dx\right)^{\frac{q-1}{q}} \left(\int \frac{|D\varphi_R|^q}{\varphi_R^{q-1}} v_\varepsilon^{(1-\lambda)(q-1)}\,dx\right)^{\frac{1}{q}}. \tag{4.8}$$

Note that the integrals on the left-hand sides of these inequalities can be estimated from below by $\int_{B_{2R}(0)} v^{q_1}(x)\,dx$ and $\int_{B_{2R}(0)} u^{p_1}(x)\,dx$ (with some positive multiplicative constants) respectively, similarly to the proofs of Theorems 2.7 and 3.1. Thus, combining (4.5)–(4.8) and taking $\varepsilon \to 0$, we arrive to a priori estimates

$$\int_{B_{2R}(0)} v^{q_1}\,dx \le D_\lambda \left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u^{\lambda+p-1}\,dx\right)^{\frac{p-1}{p}} \left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u^{(1-\lambda)(p-1)}\,dx\right)^{\frac{1}{p}}, \tag{4.9}$$

$$\int_{B_{2R}(0)} u^{p_1}\,dx \le E_\lambda \left(\int \frac{|D\varphi_R|^q}{\varphi_R^{q-1}} v^{\lambda+q-1}\,dx\right)^{\frac{q-1}{q}} \left(\int \frac{|D\varphi_R|^q}{\varphi_R^{q-1}} v^{(1-\lambda)(q-1)}\,dx\right)^{\frac{1}{q}}, \tag{4.10}$$

where D_λ and $E_\lambda > 0$ depend only on p, q, and λ.

Apply the Hölder inequality with exponent $r > 1$ to the first integral on the right-hand side of (4.9):

$$\left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u^{\lambda+p-1}\,dx\right)^{\frac{p-1}{p}} \le \left(\int u^{(\lambda+p-1)r}\varphi_R\,dx\right)^{\frac{p-1}{pr}} \left(\frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\right)^{\frac{p-1}{pr'}}, \tag{4.11}$$

where $\dfrac{1}{r} + \dfrac{1}{r'} = 1$.

Choosing the exponent r so that $(\lambda + p - 1)r = p_1$, from (4.9) and (4.11) we get

$$\begin{aligned} &\int_{B_{2R}(0)} v^{q_1}\,dx \\ &\le D_\lambda \left(\int u^{p_1}\varphi_R\,dx\right)^{\frac{p-1}{pr}} \left(\int \frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\right)^{\frac{p-1}{pr'}} \left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u^{(1-\lambda)(p-1)}\,dx\right)^{\frac{1}{p}}. \end{aligned} \tag{4.12}$$

Applying the Hölder inequality with exponent $y > 1$ to the last integral on the right-hand side of (4.12), we obtain

$$\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u^{(1-\lambda)(p-1)}\,dx \le \left(\int u^{(1-\lambda)(p-1)y}\varphi_R\,dx\right)^{\frac{1}{y}} \left(\int \frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}}\,dx\right)^{\frac{1}{y'}}, \tag{4.13}$$

where $\frac{1}{y}+\frac{1}{y'}=1$.

Choosing y in (4.13) so that $(1-\lambda)(p-1)y=p_1$ and taking into account (4.12), we get the estimate

$$\int_{B_{2R}(0)} v^{q_1}\,dx \le D_\lambda \left(\int_{B_{2R}(0)} u^{p_1}\,dx\right)^{\frac{p-1}{pr}+\frac{1}{py}} \left(\int \frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\right)^{\frac{p-1}{pr'}} \left(\int \frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}}\,dx\right)^{\frac{1}{py'}}, \tag{4.14}$$

where the exponents r and y are chosen so that

$$\begin{cases} \dfrac{1}{y}+\dfrac{1}{y'}=1, & (1-\lambda)(p-1)y=p_1, \\ \dfrac{1}{r}+\dfrac{1}{r'}=1, & (\lambda+p-1)r=p_1. \end{cases} \tag{4.15}$$

Note that this choice of $r>1$ and $y>1$ is possible due to our assumptions on p and p_1 provided that $\lambda<0$ is sufficiently small in absolute value. Similarly, choosing s and z such that

$$\begin{cases} \dfrac{1}{z}+\dfrac{1}{z'}=1, & (1-\lambda)(q-1)z=q_1, \\ \dfrac{1}{s}+\dfrac{1}{s'}=1, & (\lambda+q-1)s=q_1, \end{cases} \tag{4.16}$$

and estimating the right-hand side of (4.10) by the Hölder inequality, we get

$$\int_{B_{2R}(0)} u^{p_1}\,dx \le E_\lambda \left(\int_{B_{2R}(0)} v^{q_1}\,dx\right)^{\frac{q-1}{qs}+\frac{1}{qz}} \left(\int \frac{|D\varphi_R|^{qs'}}{\varphi_R^{qs'-1}}\,dx\right)^{\frac{q-1}{qs'}} \left(\int \frac{|D\varphi_R|^{qz'}}{\varphi_R^{qz'-1}}\,dx\right)^{\frac{1}{qz'}}. \tag{4.17}$$

Combining (4.14) and (4.17), we finally arrive at

$$\begin{aligned} \left(\int_{B_{2R}(0)} v^{q_1}\,dx\right)^{1-\mu\nu} \le{}& D_\lambda E_\lambda^{\nu} \left(\int \frac{|D\varphi_R|^{qs'}}{\varphi_R^{qs'-1}}\,dx\right)^{\frac{\nu(q-1)}{qs'}} \left(\int \frac{|D\varphi_R|^{qz'}}{\varphi_R^{qz'-1}}\,dx\right)^{\frac{\nu}{qz'}} \\ &\times \left(\int \frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\right)^{\frac{p-1}{pr'}} \left(\int \frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}}\,dx\right)^{\frac{1}{py'}} \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} \left(\int_{B_{2R}(0)} u^{p_1}\,dx\right)^{1-\mu\nu} \le{}& E_\lambda D_\lambda^{\mu} \left(\int \frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\right)^{\frac{\mu(p-1)}{pr'}} \left(\int \frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}}\,dx\right)^{\frac{\mu}{py'}} \\ &\times \left(\int \frac{|D\varphi_R|^{qs'}}{\varphi_R^{qs'-1}}\,dx\right)^{\frac{q-1}{qs'}} \left(\int \frac{|D\varphi_R|^{qz'}}{\varphi_R^{qz'-1}}\,dx\right)^{\frac{1}{qz'}}, \end{aligned} \tag{4.19}$$

where

$$\mu := \frac{q-1}{qs}+\frac{1}{qz}, \qquad \nu := \frac{p-1}{pr}+\frac{1}{py}. \tag{4.20}$$

Simple calculations using (4.15) and (4.16) yield explicit values of μ and ν, namely,

$$\mu = \frac{q-1}{q_1}, \qquad \nu = \frac{p-1}{p_1}. \tag{4.21}$$

Our assumptions imply that the exponents on the left-hand side of (4.18) and (4.19) are such that

$$1 - \mu\nu = \frac{p_1 q_1 - (p-1)(q-1)}{p_1 q_1} > 0.$$

Thus from (4.19) we have

$$\int_{B_{2R}(0)} v^{q_1}\,dx \le CR^{\frac{\sigma}{p_1 q_1 - (p-1)(q-1)}}, \qquad \int_{B_{2R}(0)} u^{p_1}\,dx \le CR^{\frac{\tau}{p_1 q_1 - (p-1)(q-1)}}. \tag{4.22}$$

Taking $R \to \infty$ in (4.22), under the hypotheses of the theorem we arrive at a contradiction, which completes the proof. □

Similarly one can consider a system

$$\begin{cases} -\Delta_p u(x) \ge |Dv(g(x))|^{q_1} & (x \in \mathbb{R}^n), \\ -\Delta_q v(x) \ge |Du(g(x))|^{p_1} & (x \in \mathbb{R}^n), \end{cases} \tag{4.23}$$

where $p, q, p_1, q_1 > 1$ and $p - 1 < p_1$, $q - 1 < q_1$.

Introduce the quantities

$$\begin{aligned} \sigma &= n - \frac{(p_1 + p - 1)q_1}{p_1 q_1 - (p-1)(q-1)}, \\ \tau &= n - \frac{(q_1 + q - 1)p_1}{p_1 q_1 - (p-1)(q-1)}. \end{aligned} \tag{4.24}$$

Then one has

Theorem 4.2. *Let* $\min(\sigma, \tau) \le 0$. *Then system* (4.23) *has no nontrivial solutions.*

We leave the proof to the interested reader.

5 Nonlinear parabolic inequalities

Now let $\tau > 0$. Consider the semilinear parabolic inequality

$$\frac{\partial u(x,t)}{\partial t} + (-\Delta)^k u(x,t) \ge |u(x, g(t))|^q \qquad (x \in \mathbb{R}^n; t \in \mathbb{R}_+) \tag{5.1}$$

with initial condition

$$u(x, 0) = u_0(x) \qquad (x \in \mathbb{R}^n), \tag{5.2}$$

where $u_0 \in C(\mathbb{R}^n)$ is a function that satisfies the condition

$$\int_{\mathbb{R}^n} u_0(x)\,dx \ge 0, \tag{5.3}$$

and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that

(g3) $t \le g(t)$ and $g'(t) \ge 1$ for any $t \ge 0$.

Let $0 < R, T < \infty$. We will use as a test function the product of two functions

$$\Phi(x,t) = \varphi\left(\frac{|x|}{R}\right) \cdot \varphi\left(\frac{t}{T}\right),$$

where the function $\varphi(s)$ is the one from Lemma 2.6.

Theorem 5.1. *Problem* (5.1)–(5.2) *with* u_0 *that satisfies* (5.3) *and* g *that satisfies (g3) has no nontrivial solutions if* $1 < q \le 1 + \frac{2k}{n}$.

Proof. Multiplying both sides of (5.1) by the test function Φ and integrating by parts, we get

$$\begin{aligned} - \int_{\mathbb{R}^n} u_0(x)\Phi(x,0)\,dx + \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)| \cdot \left| \frac{\partial \Phi(x,t)}{\partial t} \right| \,dx\,dt \\ + \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)| \cdot \left| \Delta^k \Phi(x,t) \right| \,dx\,dt \\ \ge \int_0^\infty \int_{\mathbb{R}^n} |u(x,g(t))|^q \Phi(x,t)\,dx. \end{aligned} \tag{5.4}$$

Since the function $\varphi(t/T)$ monotonically decreases, using (g3) and the monotonic decay of $\Phi(x,t)$ in t for each $x \in \mathbb{R}^n$, one can estimate the right-hand side of (5.4) from below as

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} |u(x,g(t))|^q \Phi(x,t)\,dx\,dt &= \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^q \Phi(x,g^{-1}(t))(g^{-1})'(t)\,dx\,dt \\ &\ge \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^q \Phi(x,t)\,dx\,dt. \end{aligned} \tag{5.5}$$

On the other hand, applying the parametric Young inequality and Lemma 2.6 to the second and third terms of the left-hand side of (5.4), we obtain

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^n} |u(x,t)| \cdot \left| \frac{\partial \Phi(x,t)}{\partial t} \right| \,dx\,dt \\ &\quad \le \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^q \Phi(x,t)\,dx\,dt + c_1 \int_0^\infty \int_{\mathbb{R}^n} \left| \frac{\partial \Phi(x,t)}{\partial t} \right|^{q'} \Phi^{1-q'}(x,t)\,dx\,dt \\ &\quad \le \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^q \Phi(x,t)\,dx\,dt + c_2 R^n T^{1-q'} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^n} |u(x,t)| \cdot \left| \Delta^k \Phi(x,t) \right| \,dx\,dt \\ &\quad \le \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^q \Phi(x,t)\,dx\,dt + c_3 \int_0^\infty \int_{\mathbb{R}^n} \left| \Delta^k \Phi(x,t) \right|^{q'} \Phi^{1-q'}(x,t)\,dx\,dt \\ &\quad \le \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^q \Phi(x,t)\,dx\,dt + c_4 R^{n-2kq'} T \end{aligned} \tag{5.7}$$

with some constants $c_1, \dots, c_4 > 0$. Combining (5.4)–(5.7) and taking into account (5.3), we have

$$\frac{1}{2} \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^q \Phi(x,t)\,dx\,dt \le c_2 R^n T^{1-q'} + c_4 R^{n-2kq'} T.$$

Taking $T = R^{2k}$ and $R \to \infty$, we obtain a contradiction for $n - 2k(q'-1) < 0$, i.e., for $1 < q < 1 + \frac{2k}{n}$. The critical case can be considered similarly to the previous theorems. □

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