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On the Fučík type problem with integral nonlocal boundary conditions

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Abstract. The Fučík equation $x'' = -\mu x^+ + \lambda x^-$ with integral nonlocal boundary value conditions $x(0) = x(1) = \gamma \int_0^1 x(s)ds$ is considered where $\mu, \lambda, \gamma \in \mathbb{R}$. The Fučík spectrum for this problem is constructed. The visualization of the spectrum for some values of γ are provided.

Keywords: Fučík type problem, spectrum, integral nonlocal condition.

2010 Mathematics Subject Classification: 34B15.

1 Introduction

There is intensive literature on boundary value problems for the second order ordinary differential equations which depend on two parameters, see for example [1,4,6,7,11]. One of the pioneering works in this field is [3]. In this work the classical Fučík problem

$$x'' = -\mu x^+ + \lambda x^-, \qquad x(0) = 0, \ \ x(1) = 0, \tag{1.1}$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$ and $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}$, is considered and the spectrum of this problem is described.

The Fučík spectrum is a set of (μ, λ) such that the problem (1.1) has nontrivial solutions, it consists of infinite set of curves F_i^+ and F_i^- $(i = 0, 1, 2, \ldots)$ that look like hyperbolas. Several branches of the spectrum are shown in the Figure 1.1. The classical Fučík spectrum (for Dirichlet boundary conditions) is obtained by solving two linear equations $x'' + \mu x = 0$ and $x'' + \lambda x = 0$ with positive μ and λ and combining a solution $x(t)$ of the problem of (multiple) fragments $\pm \sin \sqrt{\mu} t$ and $\pm \sin \sqrt{\lambda} t$ looking for smooth gluing at zero points of $x(t)$.

The Fučík spectrum first appeared in the works by S. Fučík (and independently E. Dancer). It is a relatively simple semilinear equation and the spectrum of the problem can be obtained analytically. Since this spectrum contains all eigenvalue of the related linear problem $x'' + \lambda x = 0$, $x(0) = 0$, $x(1) = 0$ it can be considered as a generalization of the classical discreet spectrum. The knowledge of the Fučík spectrum is important for nonlinear problems of

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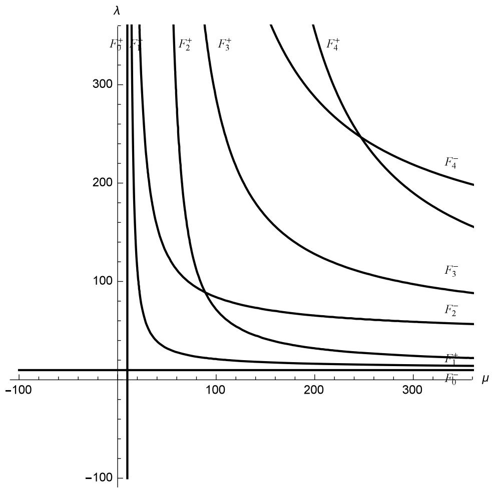


Figure 1.1: Some branches of spectrum of the classical Fučík problem (1.1).

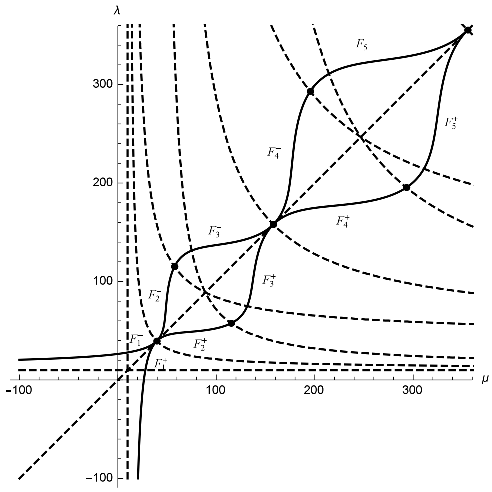


Figure 1.2: Some branches of spectrum of the problem with integral condition (1.2).

the type $x'' + g(t,x) = f(t,x,x'),\ x(0) = 0,\ x(1) = 0$, where nonlinearity g is asymptotically linear but asymmetric (f may be bounded). It was clear later that there are important practical problems [6] involving asymptotically asymmetric equations.

To get analytical expressions for the Fučík spectrum is a relatively simple task: one should just combine and glue solutions of two linear equations $x'' + \mu x = 0$ and $x'' + \lambda x = 0$ with positive parameters (negative does not fit since then the Dirichlet boundary conditions cannot be satisfied). When considering more general boundary conditions one finds that in many cases also negative values of parameters and combinations positive/negative and vice versa are possible. The resulting solution may consist of fragments of trigonometric functions and exponential functions as well.

A completely different spectrum was obtained in the case when the Fučík equation was considered with nonlocal integral condition

$$x(0) = 0, \qquad \int_0^1 x(s)ds = 0. \tag{1.2}$$

This spectrum was obtained in the author's work [8] only in the first quadrant, but entirely the spectrum was described in the work [9]. Several branches of the spectrum are shown in Figure 1.2. The spectrum branches are located in the regions between the branches of the classical Fučík spectrum.

The importance of integral boundary conditions stems from the fact that they generalize multi-point and nonlocal boundary conditions. More about the integral conditions and related references can be found in the paper [5].

In this article we consider the problem

$$x'' = -\mu x^+ + \lambda x^-, \tag{1.3}$$

$$x(0) = \gamma \int_0^1 x(s)ds = x(1), \qquad \gamma \in \mathbb{R}. \tag{1.4}$$

As a motivation for our work let us mention the work [2], where the problem (1.3), (1.4) is treated for the case $\mu = \lambda$. We wish also to generalize the results in [10], where the conditions $x(0) = 0$, $x(1) = \gamma \int_0^1 x(s)ds$ were considered.

2 The spectrum of the problem (1.3), (1.4)

2.1 Some features of the branches $F_0^{\pm}$

Consider the solutions of the problem (1.3), (1.4) without zeros in the interval $(0,1)$. The following results hold.

Lemma 2.1. *The branches $F_0^{\pm}$ of the spectrum for the problem (1.3), (1.4) do not exist for negative γ values.*

Proof. Let $\gamma < 0$. It is clear that in this case $\operatorname{sign} x(0) = \operatorname{sign} x(1) = \operatorname{sign} \int_0^1 x(s)ds \neq \operatorname{sign} \gamma \int_0^1 x(s)ds$, which proves the lemma. □

Lemma 2.2. *The branch F_0^+ (resp. F_0^-), which is a straight line parallel to the λ (resp. μ) axis*

- *is located in the first and the fourth (resp. the first and the second) quadrants of the (μ,λ)-plane for $\gamma \in [0,1)$;*
- *coincides with the λ (resp. μ) axis for $\gamma = 1$;*
- *is located in the second and the third (resp. the third and the fourth) quadrants of the (μ,λ)-plane for $\gamma \in (1,+\infty)$.*

Proof. If $\gamma = 0$ then we obtained the classical Fučík problem (1.1). The branch $F_0^+ = \{(\mu,\lambda) \mid \mu = \pi^2,\ \lambda \in \mathbb{R}\}$ in this case is located in the first and the fourth quadrant. Similarly the branch $F_0^- = \{(\mu,\lambda) \mid \mu \in \mathbb{R},\ \lambda = \pi^2\}$ is located in the first and the second quadrant.

Let us consider the case of $\gamma > 0$ and $x'(0) > 0$. We obtain the equation $x'' = -\mu x$. Solving this equation for $0 < \mu < \pi^2$ (it guarantees that the solution without zeros in the interval $(0,1)$ exists) we obtain the solution $x(t) = C_1 \cos\sqrt{\mu}t + C_2 \sin\sqrt{\mu}t$. In view of $x(0) = x(1)$ we obtain that $C_2 = C_1 \tan\frac{\sqrt{\mu}}{2}$. Taking into account the condition $x(0) = \gamma\int_0^1 x(s)ds$, we arrive to the following expression

$$\frac{\sqrt{\mu}}{2\tan\frac{\sqrt{\mu}}{2}} = \gamma. \tag{2.1}$$

Consider the left side of the expression (2.1) as a function of μ. The range of values of this function is the interval $(0,1)$. It proves the first assertion of the lemma for F_0^+.

Solving the equation $x'' = -\mu x$ for negative values of μ we obtain $x(t) = C_1\exp(\sqrt{-\mu}t) + C_2\exp(-\sqrt{-\mu}t)$. It follows that $C_2 = C_1\exp(\sqrt{-\mu})$ in view of the condition $x(0) = x(1)$. Taking into account the condition $x(0) = \gamma\int_0^1 x(s)ds$, we obtain the next expression

$$\frac{\sqrt{-\mu}}{2\tanh\frac{\sqrt{-\mu}}{2}} = \gamma. \tag{2.2}$$

The study of the equation (2.2) shows that it is solvable only for $\gamma > 1$. The last statement of the lemma for F_0^+ follows. The proof for F_0^- is similar.

If $\gamma = 1$ and $\mu = 0$ (or $\lambda = 0$) we obtain the next solution of the problem (1.3), (1.4) $x(t) = A$, where $A \in \mathbb{R}$. It follows that the axes are included in the spectrum. The proof of the second statement of the lemma is completed. □

2.2 Some features of the branches $F_{2i-1}^{\pm}$

Consider the solution of the problem (1.3), (1.4) with odd number of zeros in the interval $(0, 1)$. The following lemma is true.

Lemma 2.3. *The odd branches $F_{2i-1}^{\pm}$ of the spectrum for the problem* (1.3), (1.4) *represent the points on the (μ, λ)-plane bisectrix, $F_{2i-1}^{\pm} = \{(\mu, \lambda) \mid \mu = \lambda = (2\pi i)^2\}$.*

Proof. It is clear that the solution of the problem must have even number of zeros in the interval $(0, 1)$ in order to have the same values of the solution at the interval endpoints. That is why the odd number of zeros in the interval $(0, 1)$ is possible only in the case when $x(0) = 0 = x(1)$ and the value of integral $\int_0^1 x(s)ds = 0$ also. In this case $\mu = \lambda = (2\pi i)^2$. $\square$

Remark 2.4. The points $\mu = \lambda = (2\pi i)^2$ are the eigenvalues for the problem $x'' = -\mu x$, $x(0) = 0$, $\int_0^1 x(s)ds = 0$.

2.3 Some features of the branches $F_2^{\pm}$

Now consider the solution of the problem (1.3), (1.4) with two zeros in the interval $(0, 1)$ and $x'(0) > 0$. Let us recall that the corresponding (μ, λ) values belong to the branch F_2^+.

In this case the solutions of problem (1.3), (1.4) with two zeros τ_1 and τ_2 in the interval $(0, 1)$ may be of six types (described in Figure 2.1) for some values of γ. The hyperbolic sine function, linear function or sine function in the interval $(0, \tau_1)$ are to be continued by sine function in the interval (τ_1, τ_2) and then with hyperbolic sine function, linear function or sine function in the interval $(\tau_2, 1)$. The solution, which starts as sine function, may be of four different types. All types of solutions of problem (1.3), (1.4) with two zeros in the interval $(0, 1)$ and $x'(0) > 0$ are shown in Figure 2.1.

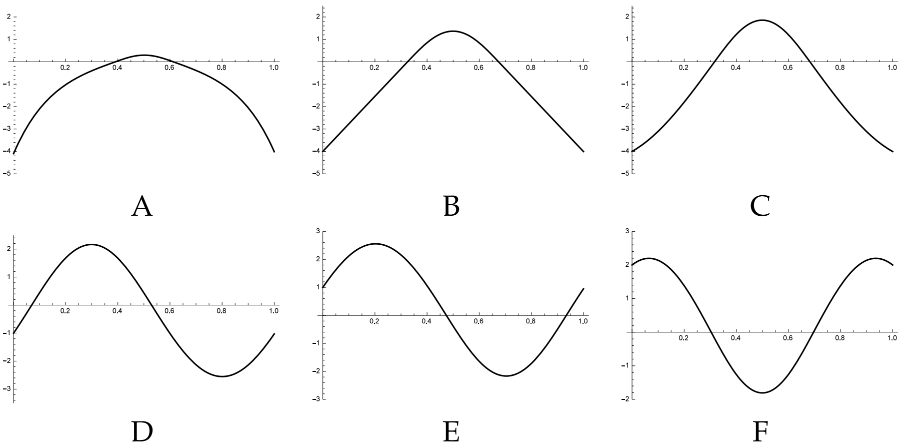


Figure 2.1: The solutions of the problem (1.3), (1.4) with two zeros in the interval $(0, 1)$ and $x'(0) > 0$ (for $\gamma = 4$).

Let us consider all of these types of solutions separately.

Case A. The solution which is depicted in the case A corresponds to the negative λ and positive μ values, so such point of the branch F_2^+ are located in the fourth (μ, λ)-plane quadrant. In view of the structure of a solution we obtain that

$$\tau_2 - \tau_1 = \frac{\pi}{\sqrt{\mu}}, \qquad \tau_1 = \frac{1}{2}\left(1 - \frac{\pi}{\sqrt{\mu}}\right), \qquad \tau_2 = \frac{1}{2}\left(1 + \frac{\pi}{\sqrt{\mu}}\right). \tag{2.3}$$

It follows that $\mu > \pi^2$.

Consider a solution of the problem (1.3), (1.4) in the interval $(0, \tau_1)$. The corresponding solution is $x(t) = A \sinh \sqrt{-\lambda}(t - \tau_1)$, it follows that $x(0) = -A \sinh \sqrt{-\lambda}\tau_1$.

In the interval (τ_1, τ_2) we obtain $x(t) = A\sqrt{-\lambda/\mu} \sin \sqrt{\mu}(t - \tau_1)$.

In the last interval $(\tau_2, 1)$ the solution of problem (1.3), (1.4) is $x(t) = -A \sinh \sqrt{-\lambda}(t - \tau_2)$, taking into account (2.3) we obtain $x(1) = -A \sinh \sqrt{-\lambda}\tau_1$.

It follows that

$$\int_0^{\tau_1} x(s)ds = \frac{A}{\sqrt{-\lambda}}(1 - \cosh \sqrt{-\lambda}\tau_1),$$

$$\int_{\tau_1}^{\tau_2} x(s)ds = \frac{2A\sqrt{-\lambda}}{\mu},$$

$$\int_{\tau_2}^{1} x(s)ds = \frac{A}{\sqrt{-\lambda}}(1 - \cosh \sqrt{-\lambda}\tau_1).$$

We obtain from the last relations and (1.4) the equation

$$\gamma \left(\frac{A}{\sqrt{-\lambda}}(1 - \cosh \sqrt{-\lambda}\tau_1 + \frac{2A\sqrt{-\lambda}}{\mu} + \frac{A}{\sqrt{-\lambda}}(1 - \cosh \sqrt{-\lambda}\tau_1) \right) = -A \sinh \sqrt{-\lambda}\tau_1$$

or

$$\gamma \left(\frac{2}{\lambda} - \frac{2}{\mu} - \frac{2}{\lambda} \cosh \frac{\sqrt{-\lambda}}{2} \left(1 - \frac{\pi}{\sqrt{\mu}} \right) \right) = \frac{1}{\sqrt{-\lambda}} \sinh \frac{\sqrt{-\lambda}}{2} \left(1 - \frac{\pi}{\sqrt{\mu}} \right). \tag{2.4}$$

Case B. The solution shown in the case B corresponds to $\lambda = 0$ and $\mu > 0$, so the point (μ, λ) is located on the μ axis.

Similarly as above, the expressions (2.3) hold.

Solutions of the problem (1.3), (1.4) in the intervals $(0, \tau_1)$, (τ_1, τ_2) and $(\tau_2, 1)$ are

$$x(t) = -A(t - \tau_1), \qquad x(t) = -\frac{A}{\sqrt{\mu}} \sin \sqrt{\mu}(t - \tau_1), \qquad x(t) = A(t - \tau_2)$$

respectively and

$$\int_0^{\tau_1} x(s)ds = \frac{A\tau_1^2}{2}, \qquad \int_{\tau_1}^{\tau_2} x(s)ds = -\frac{2A}{\mu}, \qquad \int_{\tau_2}^{1} x(s)ds = \frac{A\tau_1^2}{2}.$$

In a similar way as above, we obtain the equation

$$\gamma \left(\left(\frac{1}{2}(1 - \frac{\pi}{\sqrt{\mu}}) \right)^2 - \frac{2}{\mu} \right) = \frac{1}{2} \left(1 - \frac{\pi}{\sqrt{\mu}} \right)$$

that allows to calculate the corresponding values of μ.

Multiplying the last equation by 4μ we obtain

$$(\gamma - 2)\mu + 2(\pi - \pi\gamma)\sqrt{\mu} + \pi^2\gamma - 8\gamma = 0. \tag{2.5}$$

The investigation of (2.5) as quadratic equation with respect to $k = \sqrt{\mu}$ with parameter γ shows that solutions of this equation exist for any real values of γ, but these solutions satisfy the condition $\mu > \pi^2$ only for $\gamma < 0$ and $\gamma > 2$. This proves the next lemma.

Lemma 2.5. *The branch F_2^+ (resp. F_2^-) is located in the first and the fourth (resp. the first and the second) (μ, λ)-plane quadrant for $\gamma < 0$ and $\gamma > 2$.*

Let us mention that the equation (2.5) may be obtained from the equation (2.4) if $\lambda \to 0$ also.

Case C. The points of the spectrum corresponding the solutions which are depicted in the cases C, D, E and F belong to the first (μ, λ)-plane quadrant, so in this cases $\mu > 0$ and $\lambda > 0$.

In view of the structure of a solution in the case C we obtain the same zeros (2.3) as above. Analogously as in the case A consider the corresponding eigenvalue problems on each of the intervals $(0, \tau_1)$, (τ_1, τ_2) and $(\tau_2, 1)$, then calculate the integral values. So, in this case we obtain

$$\gamma\left(\frac{2}{\lambda} - \frac{2}{\mu} - \frac{2}{\lambda}\cos\frac{\sqrt{\lambda}}{2}\left(1 - \frac{\pi}{\sqrt{\mu}}\right)\right) = \frac{1}{\sqrt{\lambda}}\sin\frac{\sqrt{\lambda}}{2}\left(1 - \frac{\pi}{\sqrt{\mu}}\right). \tag{2.6}$$

It follows from geometrical considerations that branch F_2^+ given by the equation (2.6) is bounded by two restrictions $\frac{\pi}{\sqrt{\mu}} < 1$ and $\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} \geq 1$. It means that the corresponding (μ, λ) values are bounded by two consecutive branches F_0^+ and F_1^+ of classical Fučík spectrum.

Case F. In view of the structure of a solution it follows that

$$\tau_2 - \tau_1 = \frac{\pi}{\sqrt{\lambda}}, \qquad \tau_1 = \frac{1}{2}\left(1 - \frac{\pi}{\sqrt{\lambda}}\right), \qquad \tau_2 = \frac{1}{2}\left(1 + \frac{\pi}{\sqrt{\lambda}}\right). \tag{2.7}$$

In a similar way as above we obtain the next equation for F_2^+ in this case

$$\gamma\left(\frac{2}{\mu} - \frac{2}{\lambda} - \frac{2}{\mu}\cos\frac{\sqrt{\mu}}{2}\left(1 - \frac{\pi}{\sqrt{\lambda}}\right)\right) = \frac{1}{\sqrt{\mu}}\sin\frac{\sqrt{\mu}}{2}\left(1 - \frac{\pi}{\sqrt{\lambda}}\right). \tag{2.8}$$

This part of branch F_2^+ is bounded by two consecutive branches $F_1^+ = \{(\mu, \lambda) \mid \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} < 1\}$ and $F_2^+ = \{(\mu, \lambda) \mid \frac{2\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} \geq 1\}$ of classical Fučík spectrum.

Cases D, E. It follows from geometrical considerations that these parts of branch F_2^+ are given by a similar equation as that for F_1^+ in the classical Fučík spectrum, that is

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} = 1. \tag{2.9}$$

Let us recall that the point $\mu = \lambda = 4\pi^2$ does not belong to this branch, because this point is the degenerate branch F_1^+ of the spectrum for the problem (1.3), (1.4).

The values of (μ, λ) which correspond to solution shown in the case D are located in a part of (2.9) where $\mu > \lambda$, but the values of (μ, λ) which correspond to solution given in the case E are located in a part where $\mu < \lambda$.

Lemma 2.6. *The relation (2.9) describes two parts of $F_2^\pm$ for $\gamma \in (0, \frac{\pi}{2}]$. For $\gamma < 0$ and $\gamma > \frac{\pi}{2}$ these two components of branch are bounded by the points $\left(\frac{\pi^2(\pi-4\gamma)^2}{4\gamma^2}, \frac{\pi^2(\pi-4\gamma)^2}{(\pi-2\gamma)^2}\right)$ and $(4\pi^2, 4\pi^2)$ (in case E) and $(4\pi^2, 4\pi^2)$ and $\left(\frac{\pi^2(\pi-4\gamma)^2}{(\pi-2\gamma)^2}, \frac{\pi^2(\pi-4\gamma)^2}{4\gamma^2}\right)$ (in case D).*

Proof. Consider the solution of intermediate type between cases C and D. The solution with $x(0) < 0$ and $x'(0) = 0$ may exist. This type of solution is shown in Figure 2.2 in the first graph. In the second graph of Figure 2.2 a similar solution of intermediate type between cases E and F is depicted.

For definiteness let us consider the second case.

The solution of the problem (1.3), (1.4) in the interval $(0, \tau_1)$, where $\tau_1 = \frac{\pi}{2\sqrt{\mu}}$, is $x(t) = A \sin \sqrt{\mu}(t - \tau_1) = A \cos \sqrt{\mu} t$. In view of this, $x(0) = x(1) = A$.

Similarly as above we obtain that

$$\int_0^1 x(s)ds = \frac{2A}{\sqrt{\mu}} - \frac{2A\sqrt{\mu}}{\lambda}. \tag{2.10}$$

It follows from the conditions (1.4) and (2.10) that

$$\gamma \left(\frac{2}{\sqrt{\mu}} - \frac{2\sqrt{\mu}}{\lambda} \right) = 1. \tag{2.11}$$

By solving the system of equations (2.9) and (2.11) we obtain $\mu = \frac{\pi^2(\pi-4\gamma)^2}{4\gamma^2}$, $\lambda = \frac{\pi^2(\pi-4\gamma)^2}{(\pi-2\gamma)^2}$. The above mentioned values of μ and λ are greater then π^2 only for $\gamma < 0$ and $\gamma > \frac{\pi}{2}$, so only for such values of γ the solutions of the problem (1.3), (1.4) shown in Figure 2.2 exist.

The proof for branches F_2^- is similar. $\square$

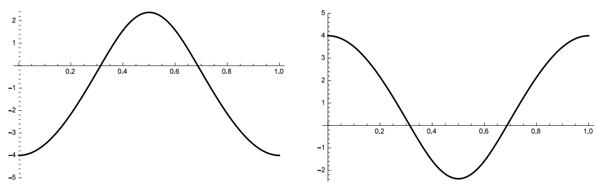


Figure 2.2: The solutions of the problem (1.3), (1.4) with two zeros in the interval $(0, 1)$ and $x'(0) = 0$ (for $\gamma = 4$).

2.4 Some features of the branches $F_{2i}^{\pm}$

Consider the solutions of the problem (1.3), (1.4) with even number of zeros $2i$ (where $i = 2, 3, \ldots$) in the interval $(0, 1)$. In this case the corresponding (μ, λ) values belong to the branches $F_{2i}^{\pm}$. The solutions of problem (1.3), (1.4) with $2i$ zeros in the interval $(0, 1)$ similarly as in above considered case may be of four types (see Figure 2.5 where all types of solutions corresponding to different components of F_4^+ are shown).

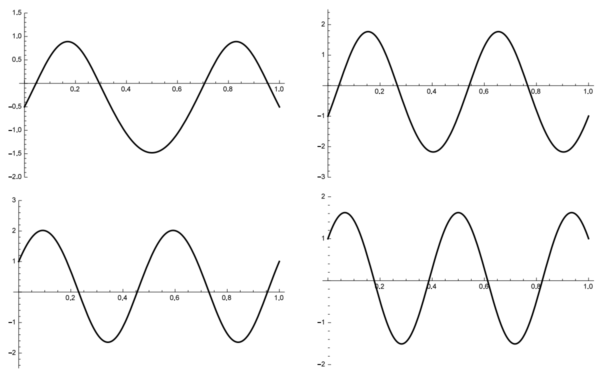


Figure 2.3: The solutions of the problem (1.3), (1.4) with four zeros in the interval $(0, 1)$ and $x'(0) > 0$ (for $\gamma = 4$).

Lemma 2.7. *The relation* $\frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} = 1$ *describes two parts of* $F_{2i}^{\pm}$ *for* $\gamma \in (0, \frac{\pi}{2}]$*. For* $\gamma < 0$ *and* $\gamma > \frac{\pi}{2}$ *these two components of a branch are bounded by the points* $\left(\frac{i^2\pi^2(\pi-4\gamma)^2}{4\gamma^2}, \frac{i^2\pi^2(\pi-4\gamma)^2}{(\pi-2\gamma)^2}\right)$ *and* $\left((2\pi i)^2, (2\pi i)^2\right)$ *for one component and* $\left((2\pi i)^2, (2\pi i)^2\right)$ *and* $\left(\frac{i^2\pi^2(\pi-4\gamma)^2}{(\pi-2\gamma)^2}, \frac{i^2\pi^2(\pi-4\gamma)^2}{4\gamma^2}\right)$ *for other one.*

Proof. The proof is similar to the proof of Lemma 2.6. □

Lemma 2.8. *The equation*

$$\gamma\left(\frac{2i}{\lambda} - \frac{2i}{\mu} - \frac{2}{\lambda}\cos\frac{\sqrt{\lambda}}{2}\left(1 - \frac{i\pi}{\sqrt{\mu}} - \frac{(i-1)\pi}{\sqrt{\lambda}}\right)\right) = \frac{1}{\sqrt{\lambda}}\sin\frac{\sqrt{\lambda}}{2}\left(1 - \frac{i\pi}{\sqrt{\mu}} - \frac{(i-1)\pi}{\sqrt{\lambda}}\right) \quad (2.12)$$

describes the component of F_{2i}^{+} *bounded by two restrictions* $\frac{i\pi}{\sqrt{\mu}} + \frac{(i-1)\pi}{\sqrt{\lambda}} < 1$ *and* $\frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} \geq 1$.

Proof. The derivation of the expression (2.12) is similar to the previously described derivation of the formula (2.6) in the above mentioned case C. □

Lemma 2.9. *The equation*

$$\gamma\left(\frac{2i}{\mu} - \frac{2i}{\lambda} - \frac{2}{\mu}\cos\frac{\sqrt{\mu}}{2}\left(1 - \frac{(i-1)\pi}{\sqrt{\mu}} - \frac{i\pi}{\sqrt{\lambda}}\right)\right) = \frac{1}{\sqrt{\mu}}\sin\frac{\sqrt{\mu}}{2}\left(1 - \frac{(i-1)\pi}{\sqrt{\mu}} - \frac{i\pi}{\sqrt{\lambda}}\right) \quad (2.13)$$

describes the component of F_{2i}^{+} *bounded by two restrictions* $\frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\mu}} < 1$ *and* $\frac{(i+1)\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} \geq 1$.

Proof. The derivation of the expression (2.13) is similar to the previously described derivation of the formula (2.8) in the above mentioned case F. □

2.5 Analytical and graphical description of the spectrum

The next theorem follows from the above lemmas.

Theorem 2.10. *The spectrum of the problem* (1.3), (1.4) *consists of the branches (if these branches exist for corresponding value of* γ*) given by*

$$F_0^{+} = \Big\{(\mu,\lambda) \,\Big|\, \gamma\tan\tfrac{\sqrt{\mu}}{2} = \tfrac{\sqrt{\mu}}{2},\ 0 < \mu < \pi^2,\ \lambda \in R \text{ or } \mu = 0,\ \lambda \in R \text{ or}$$
$$\gamma\tanh\tfrac{\sqrt{-\mu}}{2} = \tfrac{\sqrt{-\mu}}{2},\ \mu < 0,\ \lambda \in R\Big\},$$

$$F_{2i-3}^{+} = \Big\{(\mu,\lambda) \,\Big|\, \mu = \lambda = (2\pi(i-1))^2\Big\},$$

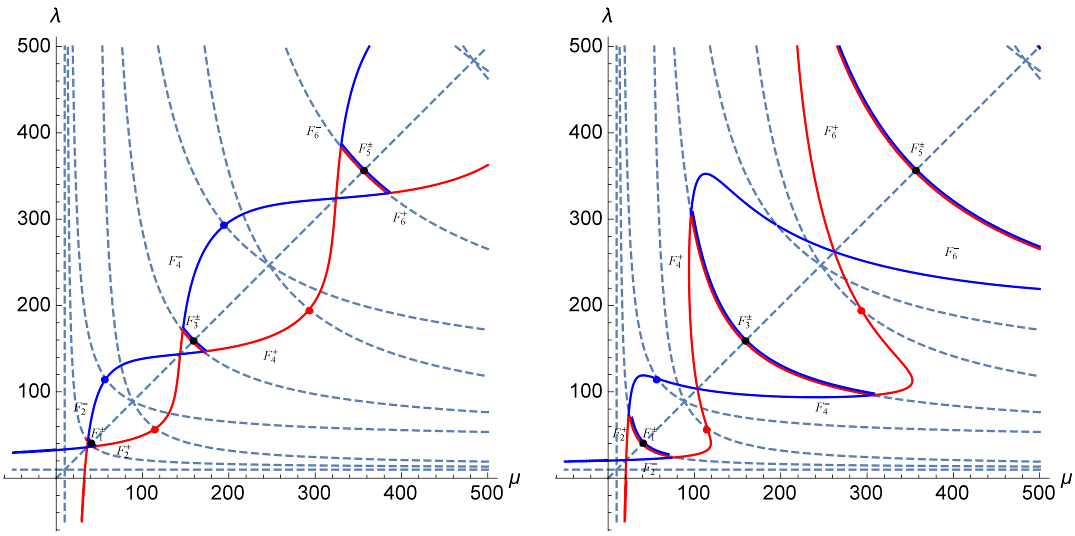
$$F_2^{+} = \Big\{(\mu,\lambda) \,\Big|\, \gamma\Big(\tfrac{2}{\lambda} - \tfrac{2}{\mu} - \tfrac{2}{\lambda}\cosh\tfrac{\sqrt{-\lambda}}{2}\big(1 - \tfrac{\pi}{\sqrt{\mu}}\big)\Big) = \tfrac{1}{\sqrt{-\lambda}}\sinh\tfrac{\sqrt{-\lambda}}{2}\big(1 - \tfrac{\pi}{\sqrt{\mu}}\big),\ \mu > \pi^2, \lambda < 0;$$
$$(\gamma-2)\mu + 2(\pi - \pi\gamma)\sqrt{\mu} + \pi^2\gamma - 8\gamma = 0,\ \mu > \pi^2,\ \lambda = 0;$$
$$\gamma\Big(\tfrac{2}{\lambda} - \tfrac{2}{\mu} - \tfrac{2}{\lambda}\cos\tfrac{\sqrt{\lambda}}{2}\big(1 - \tfrac{\pi}{\sqrt{\mu}}\big)\Big) = \tfrac{1}{\sqrt{\lambda}}\sin\tfrac{\sqrt{\lambda}}{2}\big(1 - \tfrac{\pi}{\sqrt{\mu}}\big),\ \mu > \pi^2,\ \tfrac{\pi}{\sqrt{\mu}} + \tfrac{\pi}{\sqrt{\lambda}} \geq 1;$$
$$\gamma\Big(\tfrac{2}{\mu} - \tfrac{2}{\lambda} - \tfrac{2}{\mu}\cos\tfrac{\sqrt{\mu}}{2}\big(1 - \tfrac{\pi}{\sqrt{\lambda}}\big)\Big) = \tfrac{1}{\sqrt{\mu}}\sin\tfrac{\sqrt{\mu}}{2}\big(1 - \tfrac{\pi}{\sqrt{\lambda}}\big),\ \tfrac{\pi}{\sqrt{\mu}} + \tfrac{\pi}{\sqrt{\lambda}} < 1,\ \tfrac{2\pi}{\sqrt{\mu}} + \tfrac{\pi}{\sqrt{\lambda}} \geq 1;$$
$$\tfrac{\pi}{\sqrt{\mu}} + \tfrac{\pi}{\sqrt{\lambda}} = 1,\ \mu \in \big(\tfrac{\pi^2(\pi-4\gamma)^2}{4\gamma^2},\ 4\pi^2\big) \cup \big(4\pi^2,\ \tfrac{\pi^2(\pi-4\gamma)^2}{(\pi-2\gamma)^2}\big),$$
$$\lambda \in \big(\tfrac{\pi^2(\pi-4\gamma)^2}{4\gamma^2}, 4\pi^2\big) \cup \big(4\pi^2,\ \tfrac{\pi^2(\pi-4\gamma)^2}{(\pi-2\gamma)^2}\big)\Big\},$$

$$
\begin{aligned}
F_{2i}^{+} = \Big\{(\mu,\lambda)\,\Big|\,&\gamma\Big(\tfrac{2i}{\lambda}-\tfrac{2i}{\mu}-\tfrac{2}{\lambda}\cos\tfrac{\sqrt{\lambda}}{2}\big(1-\tfrac{i\pi}{\sqrt{\mu}}-\tfrac{(i-1)\pi}{\sqrt{\lambda}}\big)\Big)=\tfrac{1}{\sqrt{\lambda}}\sin\tfrac{\sqrt{\lambda}}{2}\big(1-\tfrac{i\pi}{\sqrt{\mu}}-\tfrac{(i-1)\pi}{\sqrt{\lambda}}\big),\\
&\tfrac{i\pi}{\sqrt{\mu}}+\tfrac{(i-1)\pi}{\sqrt{\mu}}<1,\ \tfrac{i\pi}{\sqrt{\mu}}+\tfrac{i\pi}{\sqrt{\lambda}}\geq 1;\\
&\gamma\Big(\tfrac{2i}{\mu}-\tfrac{2i}{\lambda}-\tfrac{2}{\mu}\cos\tfrac{\sqrt{\mu}}{2}\big(1-\tfrac{(i-1)\pi}{\sqrt{\mu}}-\tfrac{i\pi}{\sqrt{\lambda}}\big)\Big)=\tfrac{1}{\sqrt{\mu}}\sin\tfrac{\sqrt{\mu}}{2}\big(1-\tfrac{(i-1)\pi}{\sqrt{\mu}}-\tfrac{i\pi}{\sqrt{\lambda}}\big),\\
&\tfrac{i\pi}{\sqrt{\mu}}+\tfrac{i\pi}{\sqrt{\mu}}<1,\ \tfrac{(i+1)\pi}{\sqrt{\mu}}+\tfrac{i\pi}{\sqrt{\lambda}}\geq 1;\ \tfrac{i\pi}{\sqrt{\mu}}+\tfrac{i\pi}{\sqrt{\lambda}}=1,\\
&\mu\in\Big(\tfrac{i^2\pi^2(\pi-4\gamma)^2}{4\gamma^2},(2\pi i)^2\Big)\cup\Big((2\pi i)^2,\tfrac{i^2\pi^2(\pi-4\gamma)^2}{(\pi-2\gamma)^2}\Big),\\
&\lambda\in\Big(\tfrac{i^2\pi^2(\pi-4\gamma)^2}{4\gamma^2},(2\pi i)^2\Big)\cup\Big((2\pi i)^2,\tfrac{i^2\pi^2(\pi-4\gamma)^2}{(\pi-2\gamma)^2}\Big)\Big\},
\end{aligned}
$$

$$F_i^{-}=\Big\{(\mu,\lambda)\,\Big|\,(\lambda,\mu)\in F_i^{+}\Big\},$$

where $i = 2, 3, \ldots$

Some branches of the spectrum for the problem (1.3), (1.4) are shown in Figures 2.4 and 2.5 for selected values of γ. The dashed curves form the classical Fučík spectrum (the spectrum of the problem (1.1)), the red ones are for F_i^{+} branches and the blue curves are for F_i^{-} branches of the spectrum for the problem (1.3), (1.4). The marked points on the bisectrix are degenerate odd branches $F_{2i-1}^{\pm}$ of the spectrum, the points which separate even branches from each other are shown with red (for F_{2i}^{+}) and blue (for F_{2i}^{-}) colours. Some points are specially marked with letters A, B and so on in the third graph in Figure 2.5, these points correspond to the solutions which are shown in Figure 2.1.



$\gamma = -20$ $\gamma = -2$

Figure 2.4: The spectrum for the problem (1.3), (1.4) for some negative values of γ.

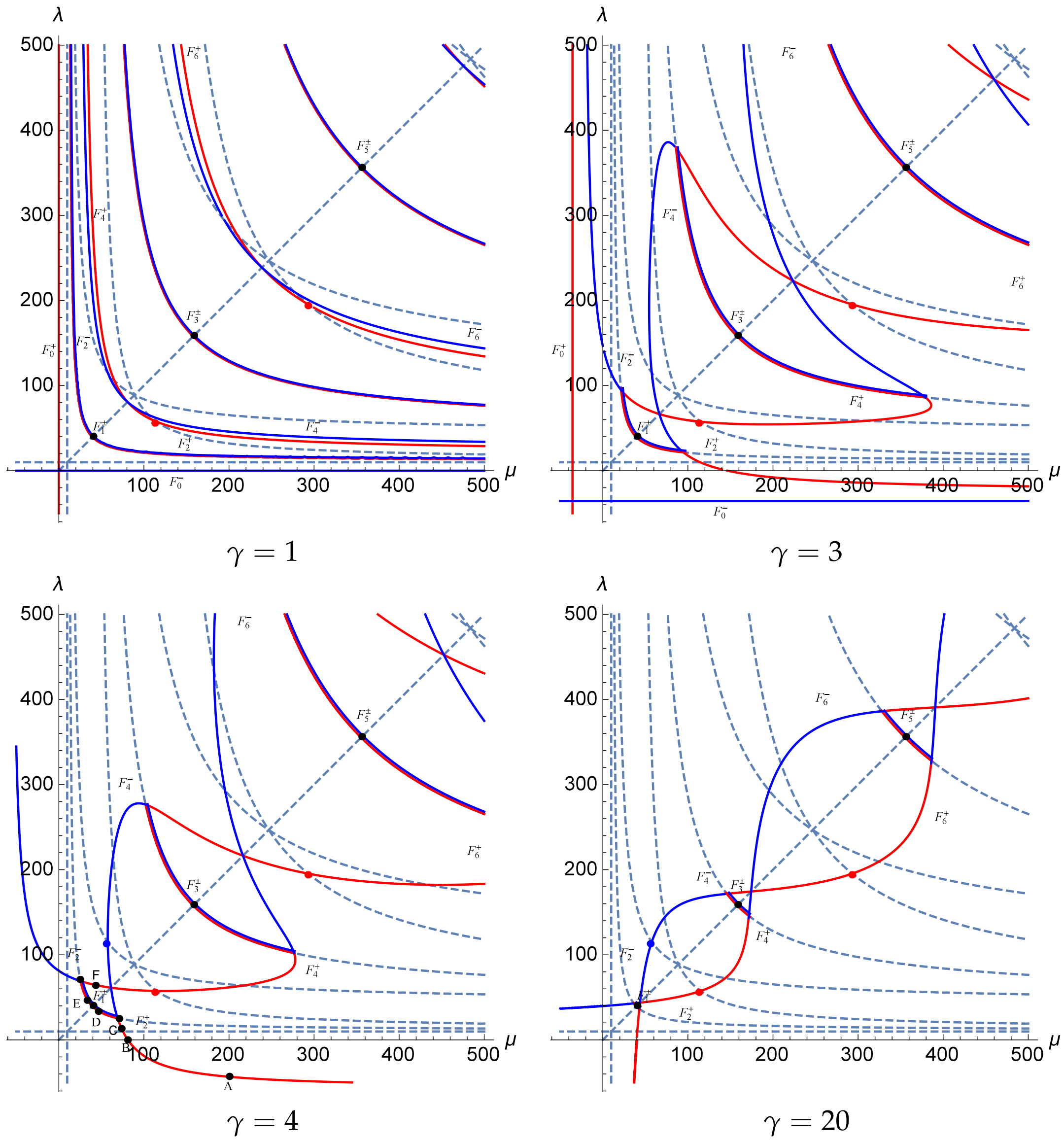


Figure 2.5: The spectrum for the problem (1.3), (1.4) for some positive values of γ.

3 Conclusions

- The new Fučík spectra were obtained for nonlocal integral boundary conditions (1.4).
- The full analytical description of the spectrum for the problem (1.3), (1.4) was obtained. It would be natural to consider in the future the more general boundary conditions of the form $x(0) = \gamma_1 \int_0^1 x(s)ds$, $x(1) = \gamma_2 \int_0^1 x(s)ds$, where $\gamma_1 \neq \gamma_2$.
- The following new feature of spectra was observed: positive (F_i^+) and negative (F_i^-) parts of the spectrum may contain common segments and entire branches for selected values of γ.
- The visualization of the spectrum for the problem (1.3), (1.4) was obtained for some selected values of γ.

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References

[1] C. De Coster, P. Habets, A two parameters Ambrosetti–Prodi problem, *Portugal. Math.* **53**(1996), No. 3, 279–303. MR1414868

[2] R. Čiupaila, Ž. Jesevičiutė, M. Sapagovas, On the eigenvalue problem for one-dimensional differential operator with nonlocal integral condition, *Nonlinear Anal. Model. Control* **9**(2004), No. 2, 109–116. MR2191914

[3] S. Fučík, A. Kufner, *Nonlinear differential equations*, Studies in Applied Mechanics, Vol. 2, Elsevier Scientific Publishing Co., Amsterdam–New York, 1980. MR558764

[4] A. Gritsans, F. Sadyrbaev, Two-parameter nonlinear oscillations: the Neumann problem, *Math. Model. Anal.* **16**(2011), No. 1, 23–38. MR2800669; url

[5] G. L. Karakostas, P. Ch. Tsamatos, Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems, *Electron. J. Differential Equations* **2002**, No. 30, 1–17. MR1907706

[6] A. C. Lazer, P. J. McKenna, Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis, *SIAM Rev.* **32**(1990), No. 4, 537–578. MR1084570; url

[7] F. Sadyrbaev, Multiplicity in parameter-dependent problems for ordinary differential equations, *Math. Model. Anal.* **14**(2009), No. 4, 503–514. MR2597076; url

[8] N. Sergejeva, On the unusual Fučík spectrum, *Discrete Contin. Dyn. Syst.* **2007**, Dynamical systems and differential equations. Proceedings of the 6th AIMS International Conference, suppl., 920–926. MR2409929

[9] N. Sergejeva, On nonlinear spectra, *Stud. Univ. Žilina Math. Ser.* **23**(2009), 101–110. MR2742003

[10] N. Sergejeva, On some problems with nonlocal integral condition, *Math. Model. Anal.* **15**(2010), No. 1, 113–126. MR2641930; url

[11] T. Shibata, Two-parameter nonlinear Sturm–Liouville problems, *Proc. Edinburgh Math. Soc. (2)* **41**(1998), No. 2, 225–245. MR1626480; url