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Solutions of two-point boundary value problems via phase-plane analysis

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Abstract. We consider period annuli (continua of periodic solutions) in equations of the type $x'' + g(x) = 0$ and $x'' + f(x)x'^2 + g(x) = 0$, where g and f are polynomials. The conditions are provided for existence of multiple nontrivial (encircling more than one critical point) period annuli. The conditions are obtained (by phase-plane analysis of period annuli) for existence of families of solutions to the Neumann boundary value problems.

Keywords: Neumann boundary conditions, phase portrait, period annulus, multiplicity of solutions.

2010 Mathematics Subject Classification: 34B15, 34C25, 34C37.

1 Introduction

We consider the equation

$$x'' + f(x)x'^2 + g(x) = 0, \tag{1.1}$$

where $f(x)$ and $g(x)$ are polynomials.

A number of papers deal with the equation

$$x'' + f(x)x' + g(x) = 0, \tag{1.2}$$

concerning oscillatory behavior of solutions and looking for isolated periodic solutions – limit cycles. We look for non-isolated periodic solutions. Our goal is to give conditions for existence of multiple period annuli and to study the relation between period annuli and solutions of boundary value problems.

Definition 1.1. Consider the equivalent two-dimensional differential system

$$x' = y, \qquad y' = -f(x)y^2 - g(x). \tag{1.3}$$

It has critical points at $(p_i, 0)$, where p_i are zeros of $g(x)$. Recall that a critical point O of (1.3) is a center if it has a punctured neighborhood covered with nontrivial cycles. Due to

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terminology in Sabatini [5], the largest connected region covered with cycles surrounding O is called *central region*. Every connected region covered with nontrivial concentric cycles is usually called *a period annulus*.

Definition 1.2. We will call a period annulus associated with a central region *a trivial period annulus*. Periodic trajectories of a trivial period annulus encircle exactly one critical point of the type center. Respectively period annuli enclosing several (more than one) critical points will be called *nontrivial period annuli*.

We are looking for multiple nontrivial period annuli.

First we consider equation (1.1) with $f \equiv 0$. We show that period annuli exist if the primitive (anti-derivative) $G(x) = \int^x g(s)\,ds$ has non-neighbouring points M_1 and M_2 of local maxima such that values of G at M_1 and M_2 are strictly greater than those at intermediate points of maxima.

In order to pass to equation (1.1) with non-zero functions f we use transformation by Sabatini [5] which allows the reduction of equation (1.1) to a conservative one of the form

$$u'' + h(u) = 0, \tag{1.4}$$

where periodic solutions $x(t)$ of (1.1) are in one-to-one correspondence with periodic solutions $u(t)$. The behaviour of the function $g(\xi)e^{2F(\xi)}$ is crucial for existence of period annuli in both equations (1.1) and (1.4).

2 Equation $x'' + g(x) = 0$

We assume that $g(x)$ is a polynomial with simple zeros, that is, if $g(z) = 0$ then $g'(z) \neq 0$.

The primitive $G(x) = \int_0^x g(s)\,ds$ may have multiple maxima. It is easy to observe that the equivalent differential system

$$x' = y, \qquad y' = -g(x) \tag{2.1}$$

has centers at the points $(m_i, 0)$ and saddle points at $(M_j, 0)$, where m_i and M_j are points of local minima and maxima respectively.

Proposition 2.1. *Critical points of the system (2.1) are "saddles" and "centers" which alternate.*

Theorem 2.2. *Let M_1 and M_2 ($M_1 < M_2$) be non-neighbouring points of maxima of the function $G(x)$. Suppose that $G(x) < \min\{G(M_1); G(M_2)\}$ for any $x \in (M_1; M_2)$. Then there exists at least one nontrivial period annulus.*

Proof. Consider the case $G(M_1) > G(M_2)$.

There are at least two critical points of the type "center" in the interval $(M_1; M_2)$. We will show that there exists a homoclinic solution emanating from the point $(M_2; 0)$ and enclosing all critical points in the interval $(M_1; M_2)$.

Consider the primitive $G_{M_2}(x) = \int_{M_2}^x g(s)\,ds$. Let r be the first zero of $G_{M_2}(x)$ to the left of M_2. One has that $G_{M_2}(x) < 0$ for $x \in (r, M_2)$. Consider the trajectory defined by the relation

$$x'^2(t) = -2G_{M_2}(x(t)) + 2G_{M_2}(M_2) = -2G_{M_2}(x(t)) \tag{2.2}$$

and passing through the points $(r; 0)$ and $(M_2, 0)$. Let T be the time needed for the point $(r; 0)$ to move to a position $(M_2; 0)$ along the trajectory. This time is given by the formula

$$T = \int_r^{M_2} \frac{ds}{\sqrt{-2G_{M_2}(s)}}. \tag{2.3}$$

Notice that $G'(s) = g(s),\ g(M_2) = 0$ and thus $G'_{M_2} = 0$. One obtains then the relation

$$\begin{aligned} G_{M_2}(s) = G_{M_2}(s) - G_{M_2}(M_2) &= \frac{1}{2}G''_2(M_2)(s-M_2)^2 + (s-M_2)^2\varepsilon(s-M_2) \\ &= \frac{1}{2}g'(M_2)(s-M_2)^2 + (s-M_2)^2\varepsilon(s-M_2), \end{aligned} \tag{2.4}$$

where $\varepsilon(\xi) \to 0$ as $\xi \to 0$. Then

$$\begin{aligned} &\int_r^{M_2} \frac{ds}{\sqrt{-g'(M_2)(s-M_2)^2 - 2(s-M_2)^2\varepsilon(s-M_2)}} \\ &= \frac{1}{\sqrt{-g'(M_2)}} \int_r^{M_2} \frac{d(s-M_2)}{\sqrt{(s-M_2)^2 + (s-M_2)^2\varepsilon_1(s-M_2)}} \\ &= \frac{1}{\sqrt{-g'(M_2)}} \int_{r-M_2}^{0} \frac{d(\xi)}{\sqrt{\xi^2 + \xi^2\varepsilon_1(\xi)}} \\ &= \frac{1}{\sqrt{-g'(M_2)}} \int_{r-M_2}^{0} \frac{d(\xi)}{\mid \xi \mid \sqrt{1+\varepsilon_1(\xi)}} \\ &> \frac{1}{\sqrt{-g'(M_2)}} \int_{-\delta}^{0} \frac{d(\xi)}{\mid \xi \mid \sqrt{1+\frac{1}{2}}} = \frac{\sqrt{2}}{\sqrt{-3g'(M_2)}} \int_{-\delta}^{0} \frac{d(\xi)}{\mid \xi \mid} = +\infty, \end{aligned} \tag{2.5}$$

where $\delta > 0$ is such that $\mid \varepsilon_1(\xi) \mid < \frac{1}{2}$ for $\xi \in [-\delta, 0],\ \varepsilon_1 = 2\varepsilon/g'(M_2)$.

Any trajectory of equation (2.2) is symmetric with respect to the x-axis. Therefore the existence of a homoclinic solution emanating from/entering the point $(M_2, 0)$ follows.

Let M_* be a point of greatest maximum of $G(x)$ in the interval (M_1, M_2). Recall that $G(M_2) > G(M_1) > G(M_*)$. Let $r_* > M_1$ be a nearest to M_1 point such that $G(r_*) = G(M_*)$. Any trajectory starting from a point $(s, 0)$, where $s \in (r, r_*)$, is described by the relation

$$x'^2(t) = -2G(x(t)) + 2G(s) \tag{2.6}$$

and therefore intersects the x-axis at some point $s_1 \in (M_{**}, M_2)$, where $M_{**} < M_2$ is a nearest to M_1 point such that $G(M_{**}) = G(M_*)$. A collection of these trajectories forms a period annulus which encloses all the critical points in the interval (M_1, M_2).

The case of $G(M_1) < G(M_2)$ can be considered similarly. In the case of $G(M_1) = G(M_2)$ we have $r = M_1$ and a heteroclinic trajectory exists that connects points $(M_1, 0)$ and $(M_2, 0)$. Otherwise the proof is the same. □

3 The Neumann problem

We consider the equation

$$x'' + g(x) = 0 \tag{3.1}$$

together with conditions

$$x'(a) = 0, \qquad x'(b) = 0. \tag{3.2}$$

Estimations of the number of solutions of the Neumann problem in trivial period annuli are easy. Since a trivial period annulus is bounded by a homoclinic or by heteroclinics, movement along trajectories near boundaries of an annulus is slow. Movement along trajectories in a neighbourhood of a unique critical point (center) depends entirely on linearization around this point.

Example 3.1. Consider the equation

$$x'' - (x+4)(x+2.5)(x+1.5)x(x-2)(x-3.6)(x-5) = 0 \tag{3.3}$$

together with conditions $x'(0) = 0,\ x'(1) = 0.$

The respective primitive function

$$G(x) = -\frac{x^8}{8} + \frac{13x^7}{35} + \frac{199x^6}{40} - \frac{41x^5}{4} - \frac{1241x^4}{20} + 61x^3 + 270x^2 \tag{3.4}$$

has four local maxima and three local minima. On phase plane there are four saddle points $(-4,0)$, $(-1.5,0)$, $(2,0)$, $(5,0)$ and three centers $(-2.5,0)$, $(0,0)$, $(3.6,0)$. There is a trivial period annulus around each center.

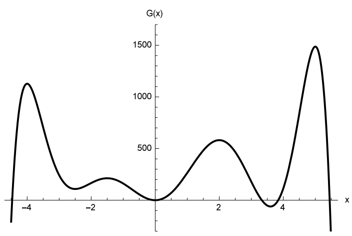


Figure 3.1: The function $G(x)$ (3.4).

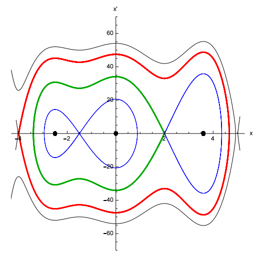


Figure 3.2: The respective phase-portrait.

Consider the trivial period annulus enclosing center $(0,0)$. By computing $|g_x(0)| = 540$ we get that the Neumann BVP has 14 solutions (see Figure 3.3).

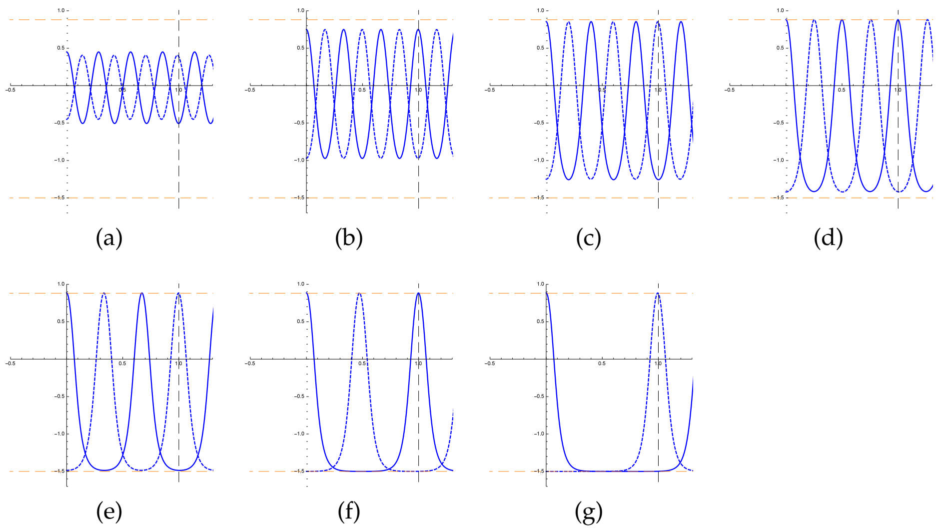


Figure 3.3: The solution for equation (3.3) around center $x = 0$ and which satisfy the initial conditions $x'(0) = 0$ and (solid line – positive conditions, dashed line – negative conditions) a) $x(0) = 0.45$ and $x(0) = -0.45$; b) $x(0) = 0.75$ and $x(0) = -0.97$; c) $x(0) = 0.855$ and $x(0) = -1.25$; d) $x(0) = 0.881$ and $x(0) = -1.42$; e) $x(0) = 0.8845$ and $x(0) = -1.485$; f) $x(0) = 0.88459908$ and $x(0) = -1.4992$; g) $x(0) = 0.8845991867$ and $x(0) = -1.499999985$.

Consider the trivial period annulus enclosing the center at $(-2.5,0)$. By computing $|g_x(-2.5)| = 772.031$ we get that the Neumann BVP has 16 solutions.

Consider the trivial period annulus enclosing center $(3.6, 0)$. By computing $|g_x(3.6)| = 1906.62$ we get that the Neumann BVP has 26 solutions.

Then the Neumann BVP (3.3) has at least 56 solutions (counting solutions in the trivial period annuli).

Theorem 3.2. *Suppose equation (3.1) has a period annulus C. Let C_x be intersection of a period annulus C with the x-axis (this intersection is a sum of two open intervals I_1 and I_2). Let $T(x)$ stand for a half-period of a periodic solution which belongs to C, $x \in (x_0^*, x_0^{**}) =: I_1$. Let k be the largest integer such that*

$$\begin{aligned} kT_{min} < b - a < (k+1)T_{min}, \\ T_{min} = \min\{T(x) : x \in (x_0^*, x_0^{**})\}. \end{aligned} \tag{3.5}$$

Then the problem (3.1), (3.2) has at least $4k$ solutions.

Proof. Consider trajectories which emanate from the points $(x, 0)$ of a phase plane (x, x'), where $x \in (x_0^*, x_0^{**})$. The half-period function $T(x)$ is continuous and tends to infinity as $x \to x_0^*$ or $x \to x_0^{**}$. This is because the initial values $(x_0^*, 0)$ and $(x_0^{**}, 0)$ correspond to homoclinic (in exceptional cases – to heteroclinic) solutions, which have "infinite" period. Let the minimal value $T_{min} := \min\{T(x) : x \in (x_0^*, x_0^{**})\}$ satisfy the relation $kT_{min} < b - a$ and let x_0 be a point of minimum. Consider $T(x)$ as x moves from x_0 to x_0^*. By continuity, there exist x_i such that $iT(x_i) = b - a$, $i = 1, \ldots, k$. Therefore at least k solutions of the boundary value problem, if $k > 0$. If $k = 0$, then there may be no solutions to the problem. Similar consideration for $x \in (x_0, x_0^{**})$ leads to conclusion that there exist another k solutions.

Consider now $T(x)$ on the second interval I_2. The respective trajectories for initial values of t are located in the lower part of the phase plane (x, x'). Due to symmetry of phase trajectories with respect to the x-axis a minimal value T_{min} over the interval I_2 is the same as that on I_1. Arguing as above we are led to conclusion that there exist at least $2k$ additional solutions.

Totally at least $4k$ solutions. □

Remark 3.3. The half-period $T(x)$ is given by the formula $T(x) = \frac{1}{\sqrt{2}} \int_x^{x_1} \frac{ds}{\sqrt{G(s) - G(x)}}$, where x_1 is the first larger than x root of the equation $G(s) - G(x) = 0$ (equation is with respect to s).

Example 3.4. Consider equation (3.3) together with conditions $x'(0) = 0,\ x'(1) = 0$.

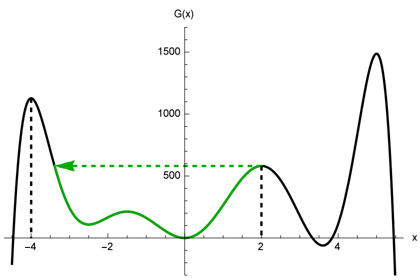


Figure 3.4: The function $G(x)$.

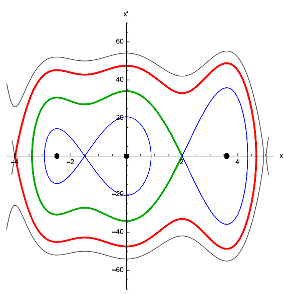


Figure 3.5: The respective phase-portrait with nontrivial period annulus (green).

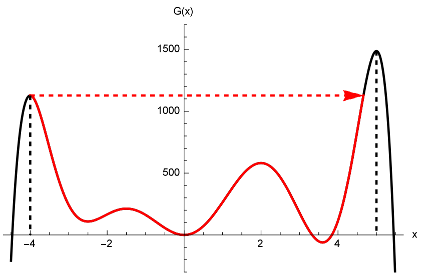


Figure 3.6: The function $G(x)$.

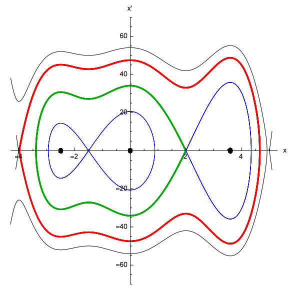


Figure 3.7: The respective phase-portrait with nontrivial period annulus (red).

First we consider nontrivial period annulus. The corresponding homoclinic trajectory on phase plane (see Figure 3.5) encircles two centers and one saddle point. Let us compute the time.

The U-shaped graphs of $T(x)$ on the interval I_1 and I_2 are depicted in Figure 3.8.

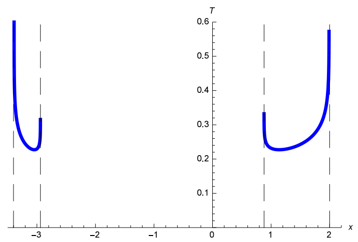
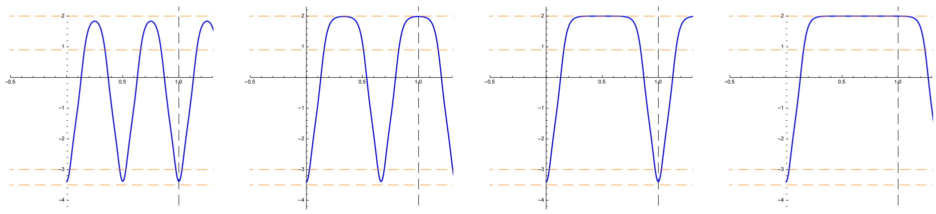


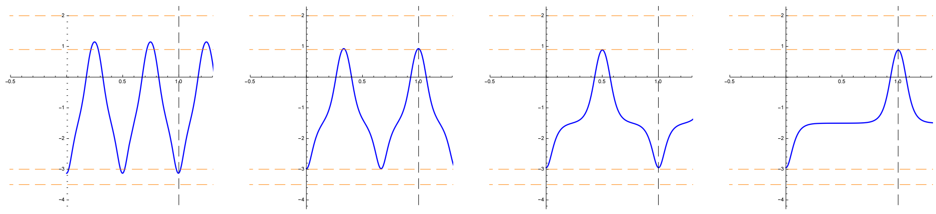
Figure 3.8: Time $T(x)$ graphs $x \in (-3.39097, -2.94664) \cup (0.88459, 2)$.

The minimal value T_{min} is approximately 0.24. Therefore the number k is 2. Then 8 solutions to the problem are expected. These solutions were computed and their graphs are depicted in Figures 3.9 to 3.12.



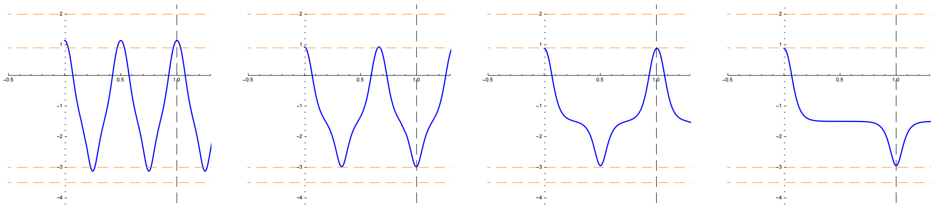
(a) (b) (c) (d)

Figure 3.9: The solution for equation (3.3) and which satisfy the initial conditions $x'(0) = 0$ and (a) $x(0) = -3.38$; (b) $x(0) = -3.3909$; (c) $x(0) = -3.39096801$; (d) $x(0) = -3.3909680129842$.



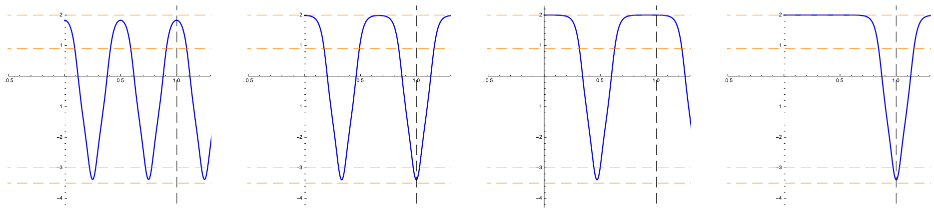
(a) (b) (c) (d)

Figure 3.10: The solution for equation (3.3) and which satisfy the initial conditions $x'(0) = 0$ and (a) $x(0) = -3.13$; (b) $x(0) = -2.983$; (c) $x(0) = -2.9479$; (d) $x(0) = -2.94669253$.



(a) (b) (c) (d)

Figure 3.11: The solution for equation (3.3) and which satisfy the initial conditions $x'(0) = 0$ and (a) $x(0) = 1.14$; (b) $x(0) = 0.93$; (c) $x(0) = 0.886$; (d) $x(0) = 0.88459923$.



(a) (b) (c) (d)

Figure 3.12: The solution for equation (3.3) and which satisfy the initial conditions $x'(0) = 0$ and (a) $x(0) = 1.83$; (b) $x(0) = 1.987$; (c) $x(0) = 1.9998$; (d) $x(0) = -1.99999999999$.

The first two figures (3.9 and 3.10) show eight solutions which emanate from the interval I_1. Figures 3.11 and 3.12 show eight solutions of the Neumann boundary value problem which emanate from points of the interval I_2

4 Equation $x'' + f(x)x'^2 + g(x) = 0$. Reduction to a conservative equation

It was shown by Sabatini [6] that equation (1.1) can be reduced to the form $u'' + h(u) = 0$ by the following transformation. Let $F(x) = \int_0^x f(s)\,ds$ and $G(x) = \int_0^x g(s)\,ds$. Introduce

$$u := \Phi(x) = \int_0^x e^{F(s)}\,ds. \tag{4.1}$$

Since $\frac{du}{dx} > 0$, this is one-to-one transformation and the inverse $x = x(u)$ is well defined.

Proposition 4.1 ([5, Lemma 1]). *The function $x(t)$ is a solution to (1.1) if and only if $u(t) = \Phi(x(t))$ is a solution to*

$$u'' + g(x(u))e^{F(x(u))} = 0. \tag{4.2}$$

Denote $H(u) = \int_0^u g(x(s))e^{F(x(s))}\,ds$. The existence of periodic solutions depends entirely on properties of the primitive H.

Let us state some easy assertions [1–4] about equation (1.1), the equivalent system

$$x' = y, \qquad y' = -f(x)y^2 - g(x) \tag{4.3}$$

and the system

$$x' = y, \qquad y' = -g(x). \tag{4.4}$$

Proposition 4.2. *Critical points and their character are the same for systems (4.4) and (4.3).*

Consider a system

$$u' = v, \qquad v' = -g(x(u))e^{F(x(u)}, \tag{4.5}$$

which is equivalent to equation (4.2).

Proposition 4.3. *Critical points $(x, 0)$ and $(u(x), 0)$ of systems (4.4) and (4.5) respectively are in 1-to-1 correspondence and their characters are the same.*

Proposition 4.4. *Periodic solutions $x(t)$ of equation (1.1) turn to periodic solutions $u(t) = \Phi(x(t))$ of equation (4.2) by transformation (4.1).*

Proposition 4.5. *Homoclinic solutions of equation (1.1) turn to homoclinic solutions of equation (4.2) by transformation (4.1).*

Proposition 4.6. *Let p_i be a zero of $g(x)$. The equality*

$$g_x(p_i) = g_u(x(u))e^{F(x(u))}|_{u=p_i} \tag{4.6}$$

is valid.

Proof. By calculation of the derivative. □

5 Comparison of Newtonian equation $x'' + g(x) = 0$ and the Liénard type equation

We will show that the following is possible: 1) the Newtonian equation has nontrivial period annuli; 2) these nontrivial period annuli disappeared in the Liénard type equation. Then the Neumann problem solutions associated with these nontrivial period annuli, also do not exist.

In what follows we consider the example of a Liénard type equation with quadratical term. This equation contains the parameter k, that may be changed. We demonstrate, that the Newtonian equation obtained using the Sabatini's transformation, has different properties for different values of k.

Example 5.1. Consider the Liénard type equation

$$x'' - k(x+2)x(x-3)x'^2 + g(x) = 0 \tag{5.1}$$

where k is the positive parameter, and $g(x) = -(x+4)(x+2.5)(x+1.5)x(x-2)(x-3.6)(x-5)$.

Then

$$F(x) = \int_0^x -k(s+2)s(s-3)\, ds = -\frac{x^4}{4} + \frac{x^3}{3} + 3x^2, \tag{5.2}$$

$$u = \int_0^x e^{F(s)}\, ds. \tag{5.3}$$

Evidently, the inverse function $x = x(u)$ exists. Equation (5.1) reduces to $u'' + h(u) = 0$ where

$$h(u) = g(x(u))e^{F(x(u))}\, ds \tag{5.4}$$

$$H(u) = \int_0^u h(s)\, ds = \int_0^{x(u)} g(\xi)e^{2F(x\xi)}\, d\xi \tag{5.5}$$

Equation

$$u'' + h(u) = 0 \tag{5.6}$$

depends on the parameter k through $F(u)$. We consider it at various values of k. We observe the process of decomposing of non-trivial period annuli in equation (5.1) under the increase of k. Indeed, look at Figure 5.1 to Figure 5.3.

The primitive $H(x(u))$ is visualized in Figure 5.1 (left) for $k = 0.01$. Due to Theorem 2.2 there exist two non-trivial period annuli (depicted in green (the bigger) and red). One might expect solutions of the Neumann BVP in both non-trivial period annuli and three trivial period annuli. Therefore 5 groups of solutions. The respective phase portrait is in Figure 5.1 (right).

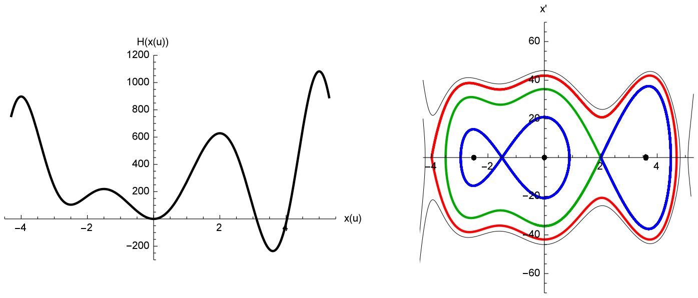


Figure 5.1: The function $H(x(u))$ (on the left) and the respective phase portrait for Liénard type equation (on the right) for value $k = 0.01$.

Figure 5.2 (left) shows the primitive $H(x(u))$ for $k = 0.03$. The respective phase portrait is on the right. The graph of H has four local maxima but only two of them (the first one and the third) indicate that there exists a non-trivial period annulus. The fourth maximum became less than two mentioned maxima. Therefore the largest non-trivial period annulus disappeared.

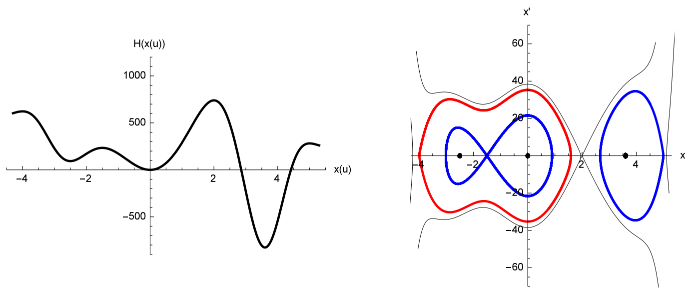


Figure 5.2: The function $H(x(u))$ (on the left) and the respective phase portrait for Liénard type equation (on the right) for value $k = 0.03$.

Figure 5.3 shows further changes in equation (5.1). The parameter $k = 0.1$. The graph of the primitive $H(x(u))$ has no two non-neighbouring maxima and therefore no more non-trivial period annuli exist in equation (5.1). There are only three trivial period annuli.

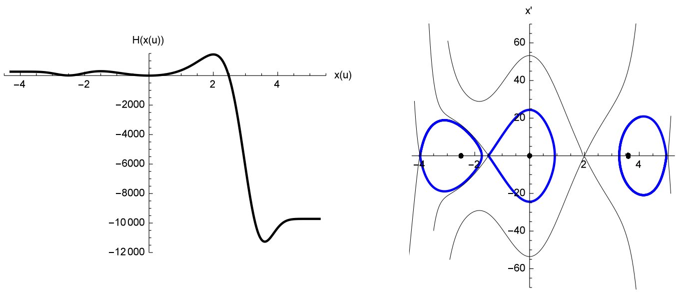


Figure 5.3: The function $H(x(u))$ (on the left) and the respective phase portrait for Liénard type equation (on the right) for value $k = 0.1$.

6 Conclusions

- The Liénard type equation (1.1) with quadratical term may be reduced to the Newtonian equation (4.2);
- Newtonian equations (3.1) may have trivial period annuli and non-trivial period annuli, this depends entirely on properties of the primitive $G(x)$;
- solutions of the Neumann boundary value problem may appear in period annuli of both types; in non-trivial period annuli graphs of these solutions may be of peculiar shapes due to the form of period annuli;
- Sabatini's transformation may be used to reduce equation of the form (1.1) to the Newtonian equation (4.2); it is useful to compare equation (1.1) (and the respective equation (4.2)) with the shortened Newtonian equation (3.1);
- it is possible that all essential properties of equation (3.1) remain in equation (4.2) and therefore in equation (1.1);
- it is possible that graphs of the primitives G and H differ significantly, the relative positions of maxima are different; consequently the phase portraits for the shortened equation (3.1) and equation (4.2) are essentially different (for instance, non-trivial period annuli disappear in (4.2)). In the latter case this means that addition of the middle term in equation (1.1) can change significantly properties of the shortened equation $x'' + g(x) = 0$.

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