doi: 10.14232/actasm-014-542-y

*Acta Sci. Math. (Szeged)*
**82** (2016), 221–234

# Spectra of some weighted composition operators on $H^2$

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*Communicated by L. Kérchy*

**Abstract.** We completely characterize the spectrum of a weighted composition operator $W_{\psi,\varphi}$ on $H^2(\mathbb{D})$ when $\varphi$ has Denjoy–Wolff point $a$ with $0 < |\varphi'(a)| < 1$, the iterates, $\varphi_n$, converge uniformly to $a$, and $\psi$ is in $H^\infty$ (the space of bounded analytic functions on $\mathbb{D}$) and continuous at $a$. We also give bounds and some computations when $|a| = 1$ and $\varphi'(a) = 1$ and, in addition, show that these symbols include all linear fractional $\varphi$ that are hyperbolic and parabolic non-automorphisms. Finally, we use these results to eliminate possible weights $\psi$ so that $W_{\psi,\varphi}$ is seminormal.

## 1. Introduction

The work of this paper is concerned with weighted composition operators on the space of functions $H^2(\mathbb{D})$, which we will denote $H^2$ for brevity. ($H^\infty(\mathbb{D})$, the space of bounded analytic functions on $\mathbb{D}$, will also appear and will likewise be denoted $H^\infty$.) If $\psi$ is in $H^\infty(\mathbb{D})$ and $\varphi$ is analytic map of the unit disk into itself, the *weighted composition operator on $H^2$ with symbols $\psi$ and $\varphi$* is the operator $W_{\psi,\varphi}$, where $T_\psi$ is the analytic Toeplitz operator given by $T_\psi(h) = \psi h$ for $h$ in $H^2$, and $C_\varphi$ is the composition operator on $H^2$ given by $C_\varphi(h) = h \circ \varphi$. Clearly, if $\psi$ is bounded on the disk, then $W_{\psi,\varphi}$ is bounded on $H^2$ and $\|W_{\psi,\varphi}\| \leq \|\psi\|_\infty \|C_\varphi\|$. Although it will have

Received May 21, 2014, and in revised form September 18, 2014.
*AMS Subject Classifications*: 47B33, 47B35, 47A10, 47B20, 47B38.
*Key words and phrases*: weighted composition operator, spectrum of an operator, hyponormal operator.
*Supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) grant funded by the Ministry of Education, Science and Technology (2012R1A2A2A02008590).

little impact on our work, it is not necessary for $\psi$ to be bounded for $W_{\psi,\varphi}$ to be bounded. To avoid trivialities and special cases, we will assume $\psi$ is not identically zero and $\varphi$ is not a constant mapping.

Weighted composition operators have been studied occasionally over the past few decades, but have usually arisen in answering other questions related to operators on spaces of analytic functions, such as questions about multiplication operators or composition operators. For example, Forelli [12] showed that the only isometries of $H^p$ for $p \neq 2$ are weighted composition operators and that the isometries for $H^p$ with $p \neq 2$ have analogues that are isometries of $H^2$ (but there are also many other isometries of $H^2$). Weighted composition operators also arise in the description of commutants of analytic Toeplitz operators (see for example [5,6] and in the adjoints of composition operators (see for example [7–9]).

Recently, work has begun on studying the spectrum of weighted composition operators on $H^2$ more carefully. Gunatillake [13] characterized the spectrum when $\varphi$ has an interior fixed point and $W_{\psi,\varphi}$ is compact. The first two authors [10] characterized the spectrum when $W_{\psi,\varphi}$ is a self-adjoint operator. Bourdon and Narayan extended their work [2] to characterize the spectrum when $W_{\psi,\varphi}$ is unitary and when $W_{\psi,\varphi}$ is normal with interior fixed point. Gunatillake [14] defined invertible weighted composition operators on $H^2$ and identified their spectrum. Very recently, Hyvärinen, Lindström, Nieminen, and Saukko [15] extended his work to when $\varphi$ is an automorphism but $W_{\psi,\varphi}$ is not necessarily invertible. They also improved his work when $\varphi$ is a hyperbolic automorphism and $W_{\psi,\varphi}$ is invertible on $H^2$.

Our work finds the spectra of $W_{\psi,\varphi}$ with relatively weak conditions on $\psi$, but a rather strong one on $\varphi$, which is that the iterates of $\varphi$ converge uniformly on *all* of $\mathbb{D}$ to the Denjoy–Wolff point $a$, rather than just on compact subsets of $\mathbb{D}$. In Section 2, we identify situations when $\varphi$ satisfies this uniformity condition on the convergence of its iterates, and show that this class of symbols is non-trivial. In Section 3, we give general bounds for $\sigma(W_{\psi,\varphi})$ that define the spectrum when $\sigma(C_\varphi)$ is given by the closure of $\sigma_p(C_\varphi)$. In Section 4, we are much more specific about $\sigma_p(W_{\psi,\varphi})$ when $\varphi'(a) < 1$ and give some examples. In Section 5, we eliminate some possibilities where $W_{\psi,\varphi}$ could be seminormal (that is, hyponormal or cohyponormal). Finally, we suggest further areas of study in Section 6.

## 2. When are the iterates of $\varphi$ uniformly convergent?

To accomplish the work of this paper, we make a rather strong assumption that $\varphi$ converges uniformly on all of $\mathbb{D}$ to the Denjoy–Wolff point $a$. Our work in this section will further explain when this phenomenon occurs. To facilitate reference to the property of uniform convergence of the iterates of $\varphi$, we make the following definition.

**Definition.** We say *UCI holds for* $\varphi$ or $\varphi$ *satisfies UCI* if $\varphi$ is an analytic map of the unit disk $\mathbb{D}$ into itself with Denjoy–Wolff point $a$ and the iterates $\varphi_n$ of $\varphi$ converge uniformly, on all of $\mathbb{D}$, to $a$.

We begin by showing that this condition is not particularly helpful when the Denjoy–Wolff point $a$ belongs to $\mathbb{D}$.

**Theorem 1.** *Suppose* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *is analytic and continuous on* $\partial\mathbb{D}$. *If the Denjoy–Wolff point* $a$ *of* $\varphi$ *is in* $\mathbb{D}$, *then* $\varphi_n \to a$ *uniformly if and only if there is* $N > 0$ *such that* $\varphi_N(\overline{\mathbb{D}}) \subseteq \mathbb{D}$.

**Proof.** Suppose there is $N > 0$ such that $\varphi_N(\overline{\mathbb{D}}) \subseteq \mathbb{D}$. Since $\varphi_n$ always converges uniformly on compact subsets of $\mathbb{D}$ to $a$ by the Denjoy–Wolff Theorem [8] and $\varphi_N(\overline{\mathbb{D}})$ is a compact subset of $\mathbb{D}$, we have that $\varphi_n \to a$ uniformly on $\mathbb{D}$.

To prove the other direction, let $M$ be the minimum distance between $a$ and the unit circle. Since $\varphi_n \to a$ uniformly on $\mathbb{D}$, for $\epsilon = M/2$, there exists $N > 0$ such that $|\varphi_N(z) - a| < \epsilon, \forall z \in \mathbb{D}$. Suppose $\varphi_N(b_1) = b_2, |b_1| = |b_2| = 1$. Then for our given $\epsilon$, since $\varphi_N$ is continuous on the unit circle, there exists $\delta > 0$ so that $|b_1 - z| < \delta \Rightarrow |b_2 - \varphi_N(z)| < \epsilon$. However, for $z$ such that $|b_1 - z| < \delta$,

$$\begin{aligned} M \leq |b_2 - a| &= |b_2 - \varphi_N(z) + \varphi_N(z) - a| \\ &\leq |b_2 - \varphi_N(z)| + |\varphi_N(z) - a| < 2\epsilon = M \end{aligned}$$

which is a contradiction, so $\varphi_N(\overline{\mathbb{D}}) \subseteq \mathbb{D}$. ■

The following corollary shows that this is of interest.

**Corollary 2.** *Suppose* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *is analytic and continuous on* $\partial\mathbb{D}$. *If the Denjoy–Wolff point* $a$ *of* $\varphi$ *is in* $\mathbb{D}$ *and* $\varphi_n \to a$ *uniformly, then* $C_\varphi$ *is power-compact. Furthermore, any associated weighted composition operator* $W_{\psi,\varphi}$ *with* $\psi \in H^\infty$ *is power-compact.*

**Proof.** Since $\varphi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$ is a sufficient condition for $C_\varphi$ to be compact [8], we see that by Theorem 1 $C_{\varphi_N} = C_\varphi^N$ is compact for some $N > 0$ and $C_\varphi$ is power-compact. Since compact operators are an ideal in $\mathcal{B}(H^2)$, $(W_{\psi,\varphi})^N = T_\zeta C_{\varphi_N}$ is compact, where $\zeta = \psi(\psi \circ \varphi) \ldots (\psi \circ \varphi_{N-1})$. ■

Since Gunatillake [13] and others have already characterized the spectrum of compact weighted composition operators (and therefore power compact weighted composition operators) when $\varphi$ has an interior fixed point, we will instead turn our work to when the Denjoy–Wolff point is on $\partial\mathbb{D}$, although our results will include the interior fixed point case. Next, we indicate some conditions on $\varphi$ when the Denjoy–Wolff point is on $\partial\mathbb{D}$, and give some examples.

**Theorem 3.** *If $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic in $\mathbb{D}$ and continuous on $\partial\mathbb{D}$, has Denjoy–Wolff point $a$ with $|a| = 1$ and $\varphi_n \to a$ uniformly, then $a$ is the only fixed point of $\varphi$ in the closed unit disk.*

**Proof.** Suppose $\varphi(b) = b$, $b \neq a$. Since the Denjoy–Wolff point is on the boundary, we must have $|b| = 1$, or else $b$ would be the Denjoy–Wolff point. Since $\varphi_n \to a$ uniformly, given $\epsilon > 0$, there is an $N$ such that $|\varphi_N(z) - a| < \epsilon, \forall z \in \mathbb{D}$. Note that $\varphi_N$ is continuous at $b$. For the same $\epsilon$, there is $\delta$ such that $|b - z| < \delta \Rightarrow |\varphi_N(b) - \varphi_N(z)| = |b - \varphi_N(z)| < \epsilon$. Let $z$ be such that $|b - z| < \delta$. Then

$$|b - a| = |b - \varphi_N(z) + \varphi_N(z) - a| \leq |b - \varphi_N(z)| + |\varphi_N(z) - a| < 2\epsilon$$

However, if we take $\epsilon < |b - a|/2$, we have a contradiction. ∎

Although our work so far indicates that the class of weighted composition operators where $\varphi$ satisfies UCI may be small, we now give sufficient conditions for $\varphi$ to satisfy UCI and follow with some examples. Much of the following proof is owed to [1].

**Theorem 4.** *Suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic in $\mathbb{D}$ and continuous on $\partial\mathbb{D}$ and has Denjoy–Wolff point $a$ with $|a| = 1, \varphi'(a) < 1$. If $\varphi_N(\overline{\mathbb{D}}) \subseteq \mathbb{D} \cup \{a\}$ for some $N$, then $\varphi_n \to a$ uniformly in $\mathbb{D}$.*

**Proof.** Without loss of generality, we will assume $\varphi(\overline{\mathbb{D}}) \subseteq \mathbb{D} \cup \{a\}$. Since $\varphi(\overline{\mathbb{D}}) \subseteq \mathbb{D} \cup \{a\}$ and $\varphi(\overline{\mathbb{D}})$ is connected, it fits within the disk

$$H(a, \lambda) := \{z \in \mathbb{C} : |a - z|^2 \leq \lambda(1 - |z|^2)\}$$

for some fixed $\lambda > 0$. Disks of this type are Euclidean subdiscs of $\mathbb{D}$ centered at $a/(1 + \lambda)$ with radius $\lambda/(1 + \lambda)$, and are tangent to the unit circle at $a$. Julia's Lemma [8] shows that $\varphi(H(a, \lambda)) \subseteq H(a, \varphi'(a)\lambda)$. Applying $\varphi$ iteratively, we see that for any $z$ in this set, we have

$$|a - \varphi_n(z)|^2 \leq \varphi'(a)^n \lambda(1 - |\varphi_n(z)|^2)$$

and therefore

$$|a - \varphi_n(z)| \leq \sqrt{\lambda}\varphi'(a)^{n/2}(1 - |\varphi_n(z)|) \leq \sqrt{\lambda}\varphi'(a)^{n/2}.$$

Thus, for any $\epsilon > 0$, there is $N > 0$ such that for $n > N$, $\sqrt{\lambda}\varphi'(a)^{n/2} < \epsilon$ (since $\varphi'(a) < 1$). Then $|\varphi_n(z) - a| \leq \sqrt{\lambda}\varphi'(a)^{n/2} < \epsilon$ for $n > N$. ∎

Although this does not completely characterize UCI holding for $\varphi$ when $|a| = 1$ and $\varphi'(a) < 1$, we see from this sufficient condition that this class includes, at least, $\varphi$ that are linear fractional non-automorphisms, such as $\varphi(z) = \frac{1}{2}z + \frac{1}{2}$. When $\varphi'(a) = 1$, the situation is even more delicate because the conditions above are not sufficient, as can be seen when $\varphi$ is a parabolic automorphism. However, if $\varphi$ is a linear fractional non-automorphism with $\varphi'(a) = 1$, we see that $\varphi$ actually satisfies UCI:

**Example 5.** Let $\varphi$ be a linear fractional map, not an automorphism, with Denjoy–Wolff point $a$ such that $|a| = 1$ and $\varphi'(a) = 1$. Without loss of generality, assume $a = 1$. Such symbols form a semigroup $\varphi_t(z) = \frac{t+(2-t)z}{(2+t)-tz}$. Then we have

$$\begin{aligned}|\varphi_t(z) - 1| &= \left|\frac{t + (2-t)z}{(2+t) - tz} - \frac{(2+t) - tz}{(2+t) - tz}\right| = \left|\frac{2z - 2}{(2+t) - tz}\right| = \left|\frac{2(z-1)}{2 + t(1-z)}\right| \\ &= \left|\frac{2(1-z)}{2 + t(1-z)}\right| = \left|\frac{2}{\frac{2}{1-z} + t}\right| \leq \left|\frac{2}{t}\right| = \frac{2}{t} \to 0, \text{ as } t \to \infty\end{aligned}$$

since $\text{Re}\left\{\frac{2}{1-z}\right\} > 1$ for $z \in \mathbb{D}$. Thus if $\varphi$ is a linear fractional non-automorphism with Denjoy–Wolff point $a$ and $\varphi'(a) = 1$, then $\varphi_n \to a$ uniformly in $\mathbb{D}$.

Now we see that UCI holding for $\varphi$ can arise when $\varphi'(a) < 1$ or $\varphi'(a) = 1$. Next, we show general bounds for the spectra of a weighted composition operator with UCI holding for the compositional symbol, and later we discuss the differences more specifically between the two cases.

## 3. Spectral bounds for $W_{\psi,\varphi}$

Throughout the remainder of the paper, we will assume that $\psi$ is in $H^\infty$, continuous at the Denjoy–Wolff point $a$ of $\varphi$, and that $\psi(a) \neq 0$.

We now offer some lemmas which will give us an inequality between the spectra of $W_{\psi,\varphi}$ and $\psi(a)C_\varphi$.

**Lemma 6.** *If $A$ and $B$ are bounded operators on a Hilbert space $\mathcal{H}$, then:*

(1) *If $ABv = \lambda v$ and $Bv \neq 0$, then $Bv$ is an eigenvector for $BA$ with eigenvalue $\lambda$.*

(2) $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.

**Proof.** (1) Trivial. (2) See [3, p. 199, exercise 7]. ∎

Although part (1) of Lemma 6 requires that $Bv \neq 0$, we will only be using analytic Toeplitz operators and composition operators with trivial kernels when we apply the lemma.

**Lemma 7.** *Suppose* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *is analytic with Denjoy–Wolff point* $a$, $\varphi_n \to a$ *uniformly in* $\mathbb{D}$, *and* $\psi \in H^\infty$ *is continuous at* $z = a$. *Then* $\|T_{\psi(a)}C_\varphi - T_{\psi\circ\varphi_n}C_\varphi\| \to 0$ *in* $\mathcal{B}(H^2)$ *as* $n \to \infty$.

**Proof.** If $\varphi_n \to a$ uniformly in $\mathbb{D}$, and $\psi$ is continuous at $a$, then $\psi \circ \varphi_n \to \psi(a)$ uniformly in $D$, which implies that $\|\psi(a) - \psi \circ \varphi_n\|_\infty \to 0$ as $n \to \infty$. Then

$$\left\|T_{\psi(a)}C_\varphi - T_{\psi\circ\varphi_n}C_\varphi\right\| \leq \left\|T_{\psi(a)-\psi\circ\varphi_n}\right\| \|C_\varphi\| = \|\psi(a) - \psi \circ \varphi_n\|_\infty \|C_\varphi\| \to 0$$

as $n \to \infty$, since $C_\varphi$ is bounded. ∎

**Theorem 8.** *If* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *is analytic with Denjoy–Wolff point* $a$, $\varphi_n \to a$ *uniformly in* $\mathbb{D}$, *and* $\psi \in H^\infty$ *is continuous at* $z = a$, *then* $\sigma(T_\psi C_\varphi) \subseteq \sigma(\psi(a)C_\varphi)$.

**Proof.** Note that $(T_\psi C_\varphi - \lambda I)$ is invertible if and only if $(C_\varphi T_\psi - \lambda I) = (T_{\psi\circ\varphi}C_\varphi - \lambda I)$ is invertible by Lemma 6. Applying this iteratively, we see that $(T_\psi C_\varphi - \lambda I)$ if and only if $(T_{\psi\circ\varphi_n}C_\varphi - \lambda I)$ is invertible for all $n$.

Let $\lambda \in \sigma(T_\psi C_\varphi)$. Then $\lambda \in \sigma(T_{\psi\circ\varphi_n}C_\varphi)$ for all $n$ by above. By Lemma 7, the operators $(T_{\psi\circ\varphi_n}C_\varphi - \lambda I)$ converge to $(T_{\psi(a)}C_\varphi - \lambda I)$ in $H^2$ norm. Since the invertible operators in $\mathcal{B}(H^2)$ are an open set and each operator in the sequence is not invertible, we know that $(T_{\psi(a)}C_\varphi - \lambda I)$ is also not invertible, so $\lambda \in \sigma(\psi(a)C_\varphi)$. ∎

Given the theorem above, it is seen that we assume $\psi(a) \neq 0$ simply to avoid trivial cases where $\sigma(W_{\psi,\varphi}) = \{0\}$. Our next goal is to find a lower bound on the spectra of $W_{\psi,\varphi}$ and use a squeeze-type argument. The following theorems will accomplish that.

**Theorem 9.** *Suppose* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *is analytic with Denjoy–Wolff point* $a$, $\varphi_n \to a$ *uniformly in* $\mathbb{D}$, *and* $\psi \in H^\infty$ *is continuous at* $z = a$ *with* $\psi(a) \neq 0$. *Then for any eigenvalue* $\lambda$ *of* $C_\varphi$, $\psi(a)\lambda \in \sigma_{ap}(T_\psi C_\varphi)$. *In particular,* $\sigma_p(\psi(a)C_\varphi) \subseteq \sigma_{ap}(T_\psi C_\varphi)$.

**Proof.** Let

$$h_m(z) = \prod_{n=0}^{m} \frac{\psi \circ \varphi_n}{\psi(a)}.$$

Note that $T_\psi C_\varphi h_m(z) = \psi(a)h_{m+1}(z)$. These vectors are finite products of $H^\infty$ functions, so they belong in $H^\infty$ and therefore to $H^2$ as well.

Let $g$ be an eigenvector for $C_\varphi$ with eigenvalue $\lambda$. Then since the vectors $h_m$ are all in $H^\infty$, $g_m = gh_m$ are all in $H^2$. Now we have

$$\begin{aligned} T_\psi C_\varphi g_m = T_\psi C_\varphi h_m g = (\psi h_m \circ \varphi)(g \circ \varphi) &= (\psi(a) h_{m+1})(\lambda g) \\ &= \psi(a)\lambda h_{m+1} g = \psi(a)\lambda g_{m+1}. \end{aligned}$$

Let $G_m = \frac{g_m}{\|g_m\|_2}$. Then we have

$$\begin{aligned} &\|(T_\psi C_\varphi - \psi(a)\lambda I)G_m\|_2 \\ &= \frac{1}{\|g_m\|_2}\|(T_\psi C_\varphi - \psi(a)\lambda I)g_m\|_2 = \frac{1}{\|g_m\|_2}\|T_\psi C_\varphi g_m - \psi(a)\lambda g_m\|_2 \\ &= \frac{1}{\|g_m\|_2}\|\psi(a)\lambda g_{m+1} - \psi(a)\lambda g_m\|_2 = \frac{1}{\|g_m\|_2}\left\|\psi(a)\lambda g_m\left(\frac{\psi \circ \varphi_{m+1}}{\psi(a)}\right) - \psi(a)\lambda g_m\right\|_2 \\ &= \frac{1}{\|g_m\|_2}\|\lambda g_m(\psi \circ \varphi_{m+1} - \psi(a))\|_2 \\ &\le \frac{1}{\|g_m\|_2}\|\lambda g_m\|_2\|\psi \circ \varphi_{m+1} - \psi(a)\|_\infty = |\lambda|\,\|\psi \circ \varphi_{m+1} - \psi(a)\|_\infty \to 0. \end{aligned}$$

The last line is by UCI holding for $\varphi$ and continuity of $\psi$. Since this is true for any eigenvalue $\lambda$ of $C_\varphi$, we have $\sigma_p(\psi(a)C_\varphi) \subseteq \sigma_{ap}(T_\psi C_\varphi)$. ∎

Taking this together with Theorem 8, we get the following string of inequalities:

**Corollary 10.** *Suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic with Denjoy–Wolff point $a$, $\varphi_n \to a$ uniformly in $\mathbb{D}$, and $\psi \in H^\infty$ is continuous at $z = a$ with $\psi(a) \neq 0$. Then we have*

$$\overline{\sigma_p(\psi(a)C_\varphi)} \subseteq \overline{\sigma_{ap}(T_\psi C_\varphi)} \subseteq \sigma(T_\psi C_\varphi) \subseteq \sigma(\psi(a)C_\varphi).$$

*In particular, if $\overline{\sigma_p(C_\varphi)} = \sigma(C_\varphi)$, then $\sigma(T_\psi C_\varphi) = \sigma(\psi(a)C_\varphi)$.*

**Proof.** The first containment is given by Theorem 9; the second containment is trivial; the third containment is given by Theorem 8. ∎

As a consequence of this corollary, we can define the spectrum in the case where $\varphi'(a) < 1$, and give some examples where $\varphi'(a) = 1$.

**Corollary 11.** *Suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic with Denjoy–Wolff point $a$, $\varphi_n \to a$ uniformly in $\mathbb{D}$, and $\varphi'(a) < 1$. Then for any $\psi \in H^\infty$ continuous at $z = a$ with $\psi(a) \neq 0$, $\sigma(W_{\psi,\varphi}) = \sigma(\psi(a)C_\varphi)$.*

**Proof.** When the Denjoy–Wolff point of $\varphi$ is on the boundary with $\varphi'(a) < 1$ and $a$ is the only fixed point of $\varphi$, then every point in the spectrum except for 0 and the peripheral spectrum is an eigenvalue of infinite multiplicity [8], p. 281. Thus $\overline{\sigma_p(C_\varphi)} = \sigma(C_\varphi)$ and $\sigma(W_{\psi,\varphi}) = \sigma(\psi(a)C_\varphi)$ by Corollary 10. ∎

**Example 12.** If $\varphi(z) = \frac{1}{2-z}$ so that $\varphi(1) = 1, \varphi'(1) = 1$, then it is known that $C_\varphi$ has spectrum $[0,1]$ and point spectrum $(0,1)$ [8]. Since $\overline{\sigma_p(C_\varphi)} = \sigma(C_\varphi)$, we have $\sigma(W_{\psi,\varphi}) = \sigma(\psi(a)C_\varphi)$ by Corollary 10, for any $\psi \in H^\infty$ continuous at $z = 1$ with $\psi(1) \neq 0$.

So far, our work in this section has not taken the value of $\varphi'(a)$ into account until the corollary above. When $\varphi'(a) < 1$, we can actually be much more specific about the point spectrum, which we will do in the next section.

## 4. Point spectra of $W_{\psi,\varphi}$ when $\varphi'(a) < 1$

For this section, our goal is to show that except for 0 and the peripheral spectrum, $\sigma(W_{\psi,\varphi})$ otherwise consists entirely of eigenvalues when $\varphi'(a) < 1$. We accomplish this by extending the vector in the proof of Theorem 9 to an infinite series bounded by $\varphi'(a)^n$.

**Theorem 13.** *Suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic with Denjoy–Wolff point $a$, $0 < |\varphi'(a)| < 1$, and $\varphi_n \to a$ uniformly in $\mathbb{D}$. Then for any $\psi$ in $H^\infty$ that is bounded away from zero on $\mathbb{D}$ and continuous at $a$, there is an eigenvector $h$ for $T_\psi C_\varphi$ with eigenvalue $\psi(a)$ and $h$ invertible in $H^\infty$.*

**Proof.** Since $\psi$ is a bounded, analytic, and non-vanishing map on $\mathbb{D}$, we may assume that there exists a bounded analytic map $\eta$ so that $\psi = e^\eta$. Since $\eta$ is analytic and bounded on $\mathbb{D}$, it has bounded derivative there, so $\eta$ is Lipschitz on $\mathbb{D}$, i.e. $|\eta(z_1) - \eta(z_2)| \leq K|z_1 - z_2|$ for $z_1, z_2 \in \mathbb{D}$ and some constant $\tilde{K}$ independent of $z_1, z_2$. Since $\eta$ is continuous at $a$, it can be seen that the above inequality holds on $\mathbb{D} \cup \{a\}$. Additionally, since $\varphi_n$ converges uniformly on $\mathbb{D}$, $|\varphi_n(z) - a| \leq K\varphi'(a)^n$ for some constant $K$ independent of $z$, as seen in the proof of Theorem 4 above. Since

$$\lim_{n\to\infty} \eta \circ \varphi_n = \eta(a),$$

we want to show that $\sum_{n=0}^{\infty}(\eta \circ \varphi_n - \eta(a))$ converges in $H^\infty$. Since

$$|\eta(\varphi_n(z)) - \eta(a)| \leq \tilde{K}|\varphi_n(z) - a| \leq K\tilde{K}\varphi'(a)^n$$

and $|\varphi'(a)| < 1$, the series converges. Set

$$g = \sum_{n=0}^{\infty}(\eta \circ \varphi_n - \eta(a)).$$

Then $h(z) = e^{g(z)}$ is an eigenfunction for $W_{\psi,\varphi}$ with eigenvalue $\psi(a)$. Since $\psi$ is bounded below, so is $\eta$, and now $g(z)$ is bounded above and below, so $\frac{1}{h} = e^{-g(z)}$ is also in $H^\infty$.

■

The next theorem shows that the special eigenvector above completely identifies the point spectrum with that of $\psi(a)C_\varphi$.

**Theorem 14.** *If $T_\psi C_\varphi$ has eigenvalue $\alpha$ with an eigenvector $g \in H^\infty$ for $\alpha$, and $\lambda$ is any eigenvalue of $C_\varphi$ with eigenvector $f$, then $\alpha\lambda$ is an eigenvalue of $T_\psi C_\varphi$ with eigenvector $gf$. Furthermore, if $\frac{1}{g} \in H^\infty$ as well, then $\sigma_p(\alpha C_\varphi) = \sigma_p(T_\psi C_\varphi)$. In particular, if $\alpha = \psi(a)$, then $\sigma_p(\psi(a)C_\varphi) = \sigma_p(T_\psi C_\varphi)$.*

**Proof.** We have $T_\psi C_\varphi g = \alpha g$ and $C_\varphi f = \lambda f$. Note since $g \in H^\infty, gf \in H^2$. Then

$$T_\psi C_\varphi(gf) = \psi(gf) \circ \varphi = (\psi g \circ \varphi)(f \circ \varphi) = (\alpha g)(\lambda f) = \alpha\lambda(gf)$$

so $gf$ is an eigenfunction for $T_\psi C_\varphi$ with eigenvalue $\alpha\lambda$. So $\sigma_p(\alpha C_\varphi) \subseteq \sigma_p(T_\psi C_\varphi)$.

Now, if $\frac{1}{g} \in H^\infty$ as well, then for any eigenvalue $\mu \in \sigma_p(T_\psi C_\varphi)$ with eigenvector $h$, we can write $v = \frac{h}{g}$ which is in $H^2$, so $gv = h$. Then

$$\mu gv = \mu h = T_\psi C_\varphi h = T_\psi C_\varphi gv = (\psi g \circ \varphi)(v \circ \varphi) = \alpha gv \circ \varphi$$

Dividing the far sides by $g$, we see that $\mu v = \alpha v \circ \varphi$, so $\mu \in \sigma_p(\alpha C_\varphi)$. Thus $\sigma_p(T_\psi C_\varphi) \subseteq \sigma_p(\alpha C_\varphi)$, so now $\sigma_p(\alpha C_\varphi) = \sigma_p(T_\psi C_\varphi)$. ∎

Putting these theorems together, we have the following corollary.

**Corollary 15.** *Suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic with Denjoy–Wolff point $a$, $0 < |\varphi'(a)| < 1$, and $\varphi_n \to a$ uniformly in $\mathbb{D}$. Then for any $\psi \in H^\infty$ that is bounded away from zero on $\mathbb{D}$ and continuous at $a$, $\sigma_p(W_{\psi,\varphi}) = \sigma_p(\psi(a)C_\varphi)$.*

Although we required stricter conditions on $\psi$ to achieve the above corollary, we can in fact use UCI holding for $\varphi$ to relax those conditions:

**Corollary 16.** *Suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic with Denjoy–Wolff point $a, 0 < |\varphi'(a)| < 1$, and $\varphi_n \to a$ uniformly in $\mathbb{D}$. Then for any $\psi \in H^\infty$ that is continuous at $a$ with $\psi(a) \neq 0$, $\sigma_p(W_{\psi,\varphi}) = \sigma_p(\psi(a)C_\varphi)$.*

**Proof.** By part (2) of Lemma 6, $\sigma_p(T_\psi C_\varphi) = \sigma_p(C_\phi T_\psi) = \sigma_p(T_{\psi\circ\phi} C_\phi)$, since 0 belongs to the spectrum of both operators. Applying the lemma repeatedly, we have $\sigma_p(T_\psi C_\varphi) = \sigma_p(T_{\psi\circ\varphi_n} C_\varphi)$ for all $n$. Since $\psi$ is continuous at $a$ and $\psi(a) \neq 0$, there is $\epsilon > 0$ so that $\psi(z)$ is bounded away from zero on the set $\{z : |z - a| < \epsilon\}$. Since $\varphi_n \to a$ uniformly, there is $N$ such that for $n \geq N$, $|\varphi_n(z) - a| < \epsilon$, for all $z \in \mathbb{D}$. Then $T_{\psi\circ\varphi_N} C_\varphi$ satisfies the conditions of Corollary 15, so $\sigma_p(T_\psi C_\varphi) = \sigma_p(T_{\psi\circ\varphi_N} C_\varphi) = \sigma_p(\psi(a)C_\varphi)$. ∎

Since we can now entirely characterize the spectrum and point spectrum when $\varphi'(a) < 1$, we illustrate this with an example below.

**Example 17.** Let $\varphi(z) = \frac{1}{2}z + \frac{1}{2}$ and $\psi(z) = e^{(2-z)}$. Note $\varphi_n(z) = \frac{1}{2^n}z + 1 - \frac{1}{2^n}$. Then for $\eta$ as in the proof of Theorem 13, we have $\eta(z) = 2 - z$ and we can compute $g(z)$ as in Theorem 13:

$$\sum_{n=0}^{\infty} \eta \circ \varphi_n - \eta(1) = \sum_{n=0}^{\infty} 2 - \left(\frac{1}{2^n}z + 1 - \frac{1}{2^n}\right) - 1 = \sum_{n=0}^{\infty} \frac{1}{2^n}(1-z) = 2 - 2z.$$

Then $h(z) = e^{(2-2z)}$ is an $H^\infty$ eigenvector for $W_{\psi,\varphi}$ with eigenvalue $\psi(1) = e$, as is seen below:

$$\psi h \circ \varphi = e^{(2-z)}e^{(2-2(\frac{1}{2}z+\frac{1}{2}))} = e^{(2-z)}e^{(1-z)} = e^{(1+2-2z)} = e^1 e^{(2-2z)} = eh.$$

Note that $\frac{1}{h} = e^{(2z-2)}$ is also in $H^\infty$. It is known that the functions $(1-z)^\lambda$ are eigenvectors of $C_\varphi$ with eigenvalue $(1/2)^\lambda$, that these belong to $H^2$ when $\text{Re}(\lambda) > -1/2$, and that $\sigma_p(C_\varphi) = \{\lambda : 0 < \lambda < \sqrt{2}\}$ [8]. Then $\sigma_p(W_{\psi,\varphi}) = \{\lambda : 0 < \lambda < \sqrt{2}e\}$ and $e^{(2-2z)}(1-z)^\lambda$ is an eigenvector for eigenvalue $\frac{e}{2^\lambda}$.

Our work here depended on the fact that $\varphi'(a) < 1$. The following two examples show that an analogous statement cannot be made when $\varphi'(a) = 1$.

**Example 18.** Let $\varphi(z) = \frac{1}{2-z}$ and $\psi(z) = 2 - z$. Then we see that $h(z) = 1 - z$ is an eigenvector for $W_{\psi,\varphi}$ with eigenvalue 1. It is known that $C_\varphi$ has spectrum $[0,1]$ with point spectrum $(0,1)$ [8]. Since $h$ is in $H^\infty$, any eigenvector $g$ for an eigenvalue $\lambda$ of $C_\varphi$ corresponds to an eigenvector $gh$ of $W_{\psi,\varphi}$ with eigenvalue $\lambda$. Thus $W_{\psi,\varphi}$ has spectrum $[0,1]$ and every element $0 < \lambda \leq 1$ is an eigenvalue.

**Example 19.** Let $\varphi(z) = \frac{1}{2-z}$ and $\psi(z) = \frac{1}{1-\frac{1}{2}z}$. The first two authors [10] showed that the operator $W_{\psi,\varphi}$ is self-adjoint and has no eigenvalues, but rather consists entirely of approximate point spectrum.

## 5. Seminormality of $W_{\psi,\varphi}$

In [10], the first two authors showed that the semigroup of parabolic non-automorphisms studied in this paper have a companion weight so that $W_{\psi,\varphi}$ is self-adjoint. The form of the companion weight associated with most known self-adjoint [10], normal [2], and cohyponormal [11] weighted composition operators is $\psi = pK_{\sigma(0)}$, where $p$ is a constant and $\sigma$ is the Cowen auxillary function of $\varphi$ (which is linear fractional in these situations). As a result of our work above, we eliminate possibilities for $\psi$ when $\varphi$ is a linear fractional non-automorphism with Denjoy–Wolff point on $\partial D$ and $W_{\psi,\varphi}$ is seminormal.

First, we show that when $\varphi$ is a parabolic non-automorphism, there are no other weight functions $\psi$ continuous at the Denjoy–Wolff point $a$ so that $W_{\psi,\varphi}$ is (co)hyponormal.

**Theorem 20.** *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a parabolic non-automorphism with Denjoy–Wolf point $a$ and let $\psi \in H^\infty$ be continuous at $z = a$. If $W_{\psi,\varphi}$ is (co)hyponormal, then it is normal and $\psi$ is a multiple of $K_{\sigma(0)}$, where $\sigma$ is the Cowen auxillary function of $\varphi$. Furthermore, if $\psi(a)$ is real, then $W_{\psi,\varphi}$ is self-adjoint.*

**Proof.** Without loss of generality, assume $a = 1$ since composition with a rotation is unitary. For now, assume $\psi(a)$ is real. Any (co)hyponormal operator whose spectrum has zero area is normal [16]. Since $C_\varphi$ has spectrum $[0, 1]$ and point spectrum $(0, 1)$, $W_{\psi,\varphi}$ (and therefore also $W^*_{\psi,\varphi}$) has spectrum equal to the line segment $[0, \psi(a)]$ by Corollary 10. Since line segments have zero area, $W_{\psi,\varphi}$ is normal. Since $W_{\psi,\varphi}$ is (sub)normal and $\sigma(W_{\psi,\varphi}) \subseteq \mathbb{R}$, it is self-adjoint [4]. The self-adjoint weighted composition operators on $H^2$ have been completely characterized in [10] and $\psi$ must therefore be a real multiple of $K_{\sigma(0)}$.

If $\psi(a)$ is not real, we get the same result for the weight $\lambda\psi$, where $\lambda$ is a non-zero constant so that $\lambda\psi(a)$ is real. Then we see that $\psi$ must be a (non-real) multiple of $K_{\sigma(0)}$ and that $W_{\psi,\varphi}$ is normal. ∎

Next, let $\varphi$ be a hyperbolic non-automorphism. Here, $W_{\psi,\varphi}$ can be cohyponormal and in fact cosubnormal. For example, if $\varphi(z) = sz + 1 - s, 0 < s < 1$, then $\sigma(0) = 0, K_0 = 1$ and $C_\varphi$ is a "weighted" composition operator which is cosubnormal. In [2], it is shown that if $\psi$ is in $C^1$ on $\overline{\mathbb{D}}$ then $W_{\psi,\varphi}$ cannot be essentially normal. Due to our understanding of the spectrum from Section 4 above, we can show that no weight $\psi$ in $H^\infty$ continuous at the Denjoy–Wolff point (but with no other conditions on $\psi$ at the boundary) creates a hyponormal weighted composition operator when $\varphi$ is a hyperbolic non-automorphism. However, first we need a lemma.

**Lemma 21.** *Let $g$ be a vector in $H^2$ such that $\langle g, z^n g\rangle = \langle g, g\rangle$ for all integers $n \geq 1$. Then $g$ is the zero vector.*

**Proof.** Suppose that $g$ is not the zero vector. Writing $g$ as $g = \sum_{k=0}^\infty a_k z^k$, since $g$ is not the zero vector, not all $a_k$ are zero. Therefore, there is an integer $n$ such that the vector $g_n = \sum_{k=n}^\infty a_k z^k$ satisfies $\|g_n\| < \|g\|/2$. Then

$$\|g\|^2 = |\langle g, g\rangle| = |\langle g, z^n g\rangle| = |\langle g_n, z^n g\rangle| \leq \|g_n\| \|z^n g\| = \|g_n\| \|g\| < \|g\|^2 /2$$

which is impossible. Therefore $g$ is the zero vector. ∎

**Theorem 22.** *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a hyperbolic non-automorphism. There is no $\psi \in H^\infty$ continuous at $z = a$ such that $W_{\psi,\varphi}$ is hyponormal.*

**Proof.** Without loss of generality, assume $\varphi(z) = sz + 1 - s$ for some $0 < s < 1$. (Otherwise, conjugate $W_{\psi,\varphi}$ by the unitary weighted composition operator $T_g C_\zeta$ where $g = K_{\zeta(0)}$ and $\zeta$ is an automorphism so that $\zeta \circ \varphi \circ \zeta$ is in this form. This will change the weight function $\psi$, but it will still be continuous at $a$ and it is otherwise arbitrary.) Now assume $W_{\psi,\varphi}$ is hyponormal.

It is known that $(1 - z)^n$ is an eigenvector for $C_\varphi$ with eigenvalue $s^n$. By Theorem 13, there is an eigenfunction $h \in H^\infty$ for $W_{\psi,\varphi}$ with eigenvalue $\psi(a)$, and thus $h(1 - z)^n$ is an eigenfunction for $W_{\psi,\varphi}$ with eigenvalue $\psi(a)s^n$ by Theorem 14. Since $W_{\psi,\varphi}$ is hyponormal, eigenvectors corresponding to different eigenvalues must be perpendicular [4]. Then

$$0 = \langle h, (1 - z)h \rangle = \langle h, h \rangle - \langle h, zh \rangle \Rightarrow \langle h, h \rangle = \langle h, zh \rangle .$$

Keeping this result in mind, we now consider the vectors $h$ and $(1 - z)^2 h$:

$$0 = \left\langle h, (1 - z)^2 h \right\rangle = \langle h, h \rangle - 2 \langle h, zh \rangle + \left\langle h, z^2 h \right\rangle \Rightarrow \langle h, h \rangle = \left\langle h, z^2 h \right\rangle .$$

Continuing inductively, we have $\langle h, h \rangle = \langle h, z^n h \rangle$ for all integers $n > 0$. Therefore, by Lemma 21, $h$ is the 0 vector, which is a contradiction since eigenvectors are non-zero. Therefore $W_{\psi,\varphi}$ cannot be hyponormal. ∎

## 6. Further questions

Below is a list of questions that would extend our work:

(1) Characterize exactly when the iterates $\varphi_n$ converge uniformly to $a$ on all of $\mathbb{D}$.

(2) Completely characterize the point spectrum of $W_{\psi,\varphi}$ when $|a| = 1$, $\varphi'(a) = 1$ and the iterates $\varphi_n$ converge uniformly to $a$ in all of $\mathbb{D}$.

(3) Completely characterize (co)(hypo)normal weighted composition operators on $H^2$. (For example, it has *not* been shown that if $W_{\psi,\varphi}$ is normal, $\varphi$ must be linear fractional.)

(4) In our work and many of our referenced papers, it seems that when $\varphi$ has exactly one fixed point $a$ in $\overline{\mathbb{D}}$, that $\sigma(W_{\psi,\varphi}) = \sigma(\psi(a)C_\varphi)$. How often is this true?

**Notes added in proof.** Theorem 22 can be stated in much greater capacity and much more simply. Since we've shown that whenever $\varphi$ satisfies UCI and $\varphi'(a) < 1$

with Denjoy–Wolff point $a$ on the boundary $T_\psi C_\varphi$ will have uncountably many eigenvectors just as $C_\varphi$ does, any such operator clearly cannot be hyponormal. (Hyponormal operators must have orthogonal eigenvectors when the eigenvectors correspond to different eigenvalues, and here we have uncountably many eigenvalues and a space with a countable basis.)

**Added in proof.** It was pointed out after submission that the results, with identical proof, extend to any Banach space $X$ of analytic functions on the disk with the following two properties. First, for any $f \in H^\infty$, $g \in X$, $fg \in X$. Second, for $f \in H^\infty$, $\|T_f\|_X \leq \|f\|_\infty$. This includes $H^p$ and $A^2_\alpha$, and the proofs could possibly extend to spaces with other multiplier algebras. The authors are indebted to Flavia Colonna for pointing this out.

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