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Intersection theory and the Horn inequalities for invariant subspaces

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Abstract. We provide a direct, intersection theoretic, argument that the Jordan models of an operator of class C_0, of its restriction to an invariant subspace, and of its compression to the orthogonal complement, satisfy a multiplicative form of the Horn inequalities, where 'inequality' is replaced by 'divisibility'. When one of these inequalities is saturated, we show that there exists a splitting of the operator into quasidirect summands which induces similar splittings for the restriction of the operator to the given invariant subspace and its compression to the orthogonal complement. The result is true even for operators acting on nonseparable Hilbert spaces. For such operators the usual Horn inequalities are supplemented so as to apply to all the Jordan blocks in the model.

1. Introduction

Consider a complex Hilbert space $\mathcal{H}$ and an operator T of class C_0 acting on it. It is known (see [3,23]) that T is quasisimilar to a uniquely determined Jordan model, that is, to an operator of the form

$$\bigoplus_{1\le n<\aleph} S(\theta_n),$$

where the sum is indexed by ordinal numbers n less than some cardinal $\aleph$ and each θ_n is an inner function in the unit disk such that θ_n divides θ_m if $\mathrm{card}(m) \le \mathrm{card}(n)$. If $\mathcal{H}'$ is an invariant subspace for T, the restriction $T|\mathcal{H}'$ and the compression

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$P_{\mathcal{H}''}T|\mathcal{H}''$, $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$, are of class C_0 as well, and therefore they have Jordan models, say

$$\bigoplus_{1\le n<\aleph} S(\theta_n') \text{ and } \bigoplus_{1\le n<\aleph} S(\theta_n'').$$

(Note that the first ordinal in the sum is 1 rather than 0 as in [3]. This way of labeling the Jordan blocks is more convenient for stating the Horn inequalities.) It has been known for some time [5, 6, 18] that the functions $\{\theta_n, \theta_n', \theta_n'' : 1 \le n < \aleph_0\}$ satisfy a version of the Littlewood–Richardson rule provided that

$$\bigwedge_{n=1}^{\infty} \theta_n \equiv 1$$

and, due to results of [15, 16], this is equivalent (in the case of finite multiplicity N) to saying that $\prod_{n=1}^{N} \theta_n = \prod_{n=1}^{N} (\theta_n'\theta_n'')$, and that these functions satisfy a collection of divisibility relations, analogous to the Horn inequalities. The collection of these divisibility relations is indexed by triples of Schubert cells in a Grassmann variety whose intersection has dimension zero. It is natural to ask whether a direct connection between these divisibility relations and intersection theory can be made. Indeed, in the context of finitely generated torsion modules over a principal ideal domain, such a connection was made in [10], and it is our purpose to extend that approach to the context of arbitrary C_0 operators. The result is that a sufficient number of these divisibility relations can be obtained from special invariant subspaces $\mathcal{M}$ of T. More precisely, assume that $T|\mathcal{M}$ has cyclic multiplicity $r < \infty$, set $\mathcal{M}' = \mathcal{M} \cap \mathcal{H}'$, $\mathcal{M}'' = \overline{P_{\mathcal{H}''}\mathcal{M}}$, and let

$$\bigoplus_{n=1}^{r} S(\alpha_n), \quad \bigoplus_{n=1}^{r} S(\alpha_n'), \quad \bigoplus_{n=1}^{r} S(\alpha_n'')$$

be the Jordan models of $T|\mathcal{M}$, $T|\mathcal{M}'$, and $P_{\mathcal{M}''}T|\mathcal{M}''$, respectively. It is known [3] that

$$\prod_{n=1}^{r} \alpha_n \equiv \prod_{n=1}^{r} (\alpha_n'\alpha_n''),$$

where we use the notation $\varphi \equiv \psi$ to indicate that the quotient of the inner functions φ and ψ is a constant (necessarily of absolute value equal to 1). We show that $\mathcal{M}$ can be chosen in such a way that $\prod_{n=1}^{r} \alpha_n$ is divisible by a product of the form $\prod_{n=1}^{r} \theta_{i_n}$, while the products $\prod_{n=1}^{r} \alpha_n'$, $\prod_{n=1}^{r} \alpha_n''$ divide $\prod_{n=1}^{r} \theta_{j_n}'$, $\prod_{n=1}^{r} \theta_{k_n}''$ respectively, thus establishing that $\prod_{n=1}^{r} \theta_{i_n}$ divides $\prod_{n=1}^{r} (\theta_{j_n}'\theta_{k_n}'')$ for certain indices $\{k_i, \ell_i, m_i : i = 1, 2, \ldots, r\}$. Moreover, in case this last divisibility is an equality, we show that $\mathcal{M}$ is, in a weak sense, a reducing subspace for T, with similar statements about $\mathcal{M}'$

and $\mathcal{M}''$. Two special cases of this result were proved in [7], but the methods of that paper do not seem to extend beyond the two classes of inequalities considered there. The existence of such (almost) reducing spaces is analogous to the existence of common reducing spaces for selfadjoint matrices whose sum saturates one of the Horn inequalities (see [12]).

When the space $\mathcal{H}$ is separable, the summands $S(\theta_n)$ in the model of T are constant for $n \geq \aleph_0$, which means that $S(\theta_n)$ acts on a space of dimension zero which can then be omitted from the sum. One may ask what form the divisibility relations take in the nonseparable case. The only additional relations state that θ_n divides $\theta'_n\theta''_n$ when $n \geq \aleph_0$. Just as in the case of the Horn relations, these divisibility relations can be obtained by exhibiting an invariant subspace $\mathcal{M}$ for T which, in this case, is reducing in the usual sense for T and for $P_{\mathcal{H}'}$.

The divisibility relations we consider were studied earlier when $\mathcal{H}$ is finite dimensional. We refer to [24] for a survey of these results. The Littlewood–Richardson rule in this context and, indeed, in the case of finitely generated modules over a discrete valuation ring, was first proved in [13] (see also [20] for a different argument.) The basic ideas in this paper originated in the study of singular numbers for products of operators [8] and in the study of torsion modules over principal ideal domains [10]. The techniques we use are necessarily different, and they may obscure to some extent the essential simplicity of the arguments.

The remainder of this paper is organized as follows. Section 2 contains some preliminaries about the class C_0 and intersection theory. In Section 3 we consider special invariant subspaces for operators of class C_0 with finite defect indices. In Section 4 we prove the Horn divisibility relations, first for contractions with finite multiplicity and then in general. The relations pertaining to θ_n for $n \geq \aleph_0$ are established in Section 5. We conclude in Section 6 with a discussion of the 'inverse' problem: given Jordan operators

$$J = \bigoplus_{1\leq n<\aleph} S(\theta_n), \quad J' = \bigoplus_{1\leq n<\aleph} S(\theta'_n), \quad J'' = \bigoplus_{1\leq n<\aleph} S(\theta''_n),$$

do there exist a C_0 operator T and an invariant subspace $\mathcal{H}'$ for T such that T, $T|\mathcal{H}'$, and $P_{\mathcal{H}'^\perp}T|\mathcal{H}'^\perp$ are quasisimilar to J, J', and J'', respectively?

2. Preliminaries

Recall [23] that an operator T acting on a complex Hilbert space $\mathcal{H}$ is a *contraction* if $\|T\| \leq 1$, and it is a *completely nonunitary contraction* if, in addition, T has no unitary restriction to any invariant subspace. Given a completely nonunitary contraction T on $\mathcal{H}$; the usual polynomial calculus $p \mapsto p(T)$ extends to a functional calculus (discovered by Sz.-Nagy and Foias) defined on the algebra H^∞ of bounded analytic functions in the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. The operator T is of *class* C_0 if the ideal $\mathcal{J}_T = \{u \in H^\infty : u(T) = 0\}$ is not zero, in which case $\mathcal{J}_T$ is a principal ideal generated by an inner function m_T, uniquely determined up to a scalar factor and called the *minimal function* of T.

The simplest operators of class C_0 are the *Jordan blocks*. Given an inner function $\theta \in H^\infty$, the Jordan block $S(\theta)$ is obtained by compressing the unilateral shift S on the Hardy space H^2 to its co-invariant subspace

$$\mathcal{H}(\theta) = H^2 \ominus \theta H^2.$$

A *Jordan operator* is, as already indicated in the introduction, an operator of the form

$$J = \bigoplus_{1 \le n < \aleph} S(\theta_n),$$

where $\aleph$ is a cardinal number (that is, the smallest ordinal of some cardinality) and, for each ordinal n, θ_n is an inner function such that $\theta_n | \theta_m$ whenever $\operatorname{card}(n) \ge \operatorname{card}(m)$. (We write $\varphi | \psi$ to indicate that φ is a divisor of ψ in H^∞.) In particular, every θ_n divides θ_1 which is in fact the minimal function of J. Note that, when θ is a constant inner function, we have $\mathcal{H}(\theta) = \{0\}$, so summands with $\theta_n \equiv 1$ do not contribute to the sum defining a Jordan operator. One is tempted to write J as a direct sum extended over the entire class of ordinal numbers (with $\theta_n = 1$ for sufficiently large n), but this does not seem wise if set theoretical decorum is to be maintained.

The class C_0 is completely classified by the relation of quasisimilarity. Two operators T, T' acting on $\mathcal{H}, \mathcal{H}'$, respectively, are said to be *quasisimilar* if there exist continuous linear operators $X : \mathcal{H} \to \mathcal{H}'$ and $Y : \mathcal{H}' \to \mathcal{H}$ which are one-to-one, have dense ranges, and satisfy the following intertwining relations

$$XT = T'X, \quad TY = YT'.$$

We write $T \sim T'$ to indicate that T is quasisimilar to T'. Quasisimliarity is a weaker relation than similarity, but it is just right for the class C_0. The following result is [3, Theorem III.5.23].

Theorem 2.1. *The quasisimilarity equivalence class of every operator of class C_0 contains a unique Jordan operator, called the Jordan model of T.*

Given an operator T of class C_0, we may occasionally write θ_n^T for the inner functions in its Jordan model. In order to characterize these functions, we need the concept of *cyclic multiplicity* for an operator T. This is a cardinal number μ_T defined as the smallest cardinality of a subset $C \subset \mathcal{H}$ with the property that the invariant subspace for T generated by C is the entire space $\mathcal{H}$:

$$\mathcal{H} = \bigvee\{T^k h : h \in C, k \geq 0\}.$$

Proposition 2.2. ([3, Corollary III.3.25]) *Given an operator T of class C_0 on $\mathcal{H}$, an inner function φ, and an ordinal number $n \geq 1$, the following statements are equivalent:*

(1) $\theta^T_{n+1}|\varphi$.
(2) *The cyclic multiplicity of the restriction* $T|\overline{\varphi(T)\mathcal{H}}$ *is at most* $\operatorname{card}(n)$.

Note again the shift $n \mapsto n+1$ compared to the corresponding statement in [3]. This is immaterial for transfinite n for which we have $\theta^T_{n+1} \equiv \theta^T_n$ as well as $\operatorname{card}(n+1) = \operatorname{card}(n)$. One way to use this result is as follows. If $\theta^T_n \not\equiv 1$ for some finite n, then T has a restriction to some invariant subspace which is quasisimilar to the direct sum of n copies of $S(\varphi)$ for any nonconstant inner divisor φ of θ^T_n. This follows from the fact that $S(\varphi)$ is unitarily equivalent to the restriction of $S(\psi)$ to some invariant subspace provided that $\varphi|\psi$. The (unique) invariant subspace in question is $(\psi/\varphi)H^2 \ominus \psi H^2$. If $\theta^T_n \not\equiv 1$ for some transfinite n, then T has a restriction which is quasisimilar to the direct sum of m copies of $S(\varphi)$, $\varphi|\theta^T_n$, where m is the smallest cardinal *greater than* n. Indeed m is precisely the cardinality of the set of ordinals $\{n' : \operatorname{card}(n') = \operatorname{card}(n)\}$.

Another way to describe the structure of an operator in terms of its Jordan model is to use *quasidirect* decompositions of the Hilbert space. Let $\{\mathcal{H}_i\}_{i\in I}$ be a collection of closed subspaces of the Hilbert space $\mathcal{H}$. We say that $\mathcal{H}$ is the *quasidirect sum* of the spaces $\{\mathcal{H}_i\}_{i\in I}$ if

$$\mathcal{H} = \bigvee_{i\in I} \mathcal{H}_i$$

and, for each $j \in I$, we have

$$\mathcal{H}_j \cap \Big[\bigvee_{i\neq j} \mathcal{H}_i\Big] = \{0\}.$$

Given a quasidirect decomposition $\mathcal{H} = \mathcal{M} \vee \mathcal{N}$, we say that $\mathcal{N}$ is a *quasidirect complement* of $\mathcal{M}$.

Theorem 2.3. ([3, Theorem III.6.10]) *Let T be an operator of class C_0 on $\mathcal{H}$. For each ordinal $n \geq 1$, there exists an invariant subspace $\mathcal{H}_n \subset \mathcal{H}$ such that $T|\mathcal{H}_n$ is quasisimilar to $S(\theta^T_n)$, and $\mathcal{H}$ is the quasidirect sum of the spaces $\{\mathcal{H}_n\}$. In addition, $\mathcal{H}_{n+i} \perp \mathcal{H}_{m+j}$ if n and m are distinct limit ordinals and $i, j < \aleph_0$.*

We note again that we do not run into any set theory difficulties because $\mathcal{H}_n = \{0\}$ for sufficiently large n. Since $T|\mathcal{H}_n$ has cyclic multiplicity equal to one when $\mathcal{H}_n \neq \{0\}$, we can select for each n a vector $x_n \in \mathcal{H}_n$ which is cyclic for $T|\mathcal{H}_n$. A collection $\{x_n\}$ obtained this way will be called a C_0-*basis* for the operator T. The vector x_1 is also known as a *maximal vector* for T. Just like a linearly independent set in a vector space can be completed to a basis, a partial C_0-basis can be completed to a C_0-basis. We formulate the result for the case of finite multiplicity, in an essentially equivalent form. The proof is contained in [3, Theorem III.6.10]. The last statement follows from the main result of [2].

Proposition 2.4. *Let T be an operator of class C_0 on $\mathcal{H}$. Assume that k is a positive integer, and that $\mathcal{M} \subset \mathcal{H}$ is an invariant subspace for T such that $T|\mathcal{M}$ is quasisimilar to*

$$\bigoplus_{n=1}^{k} S(\theta_n^T).$$

Then $\mathcal{M}$ has a T-invariant quasidirect complement. If $\mathcal{N}$ is such a complement of $\mathcal{M}$, the Jordan model of $T|\mathcal{N}$ is

$$\bigoplus_{n\geq 1} S(\theta_{n+k}^T).$$

If two T-invariant subspaces $\mathcal{M}, \mathcal{N}$ are quasidirect complements of each other and if $T|\mathcal{M}$ is quasisimilar to

$$\bigoplus_{j=1}^{k} S(\theta_{n_j}^T),$$

where $1 \leq n_1 < n_2 < \cdots < n_k < \aleph_0$, then $T|\mathcal{N}$ is quasisimilar to

$$\bigoplus_{n\notin\{n_1,n_2,\ldots,n_k\}} S(\theta_n^T).$$

In the case of operators of finite multiplicity it is somewhat easier to verify that a system of vectors is a C_0-basis.

Proposition 2.5. *Let T be an operator of class C_0 on $\mathcal{H}$ and let $\bigoplus_{n=1}^{N} S(\theta_n)$ be the Jordan model of T. Assume that the vectors $\{h_n\}_{n=1}^{N} \subset \mathcal{H}$ satisfy the following properties:*

(1) *The smallest invariant subspace for T containing $\{h_n\}_{n=1}^{N}$ is $\mathcal{H}$.*

(2) $\theta_n(T)h_n = 0$.

Then $\{h_n\}_{n=1}^{N}$ is a C_0-basis for T.

Proof. Denote by $\mathcal{H}_n$ the cyclic space for T generated by h_n. Condition (2) implies that $T|\mathcal{H}_n \sim S(\varphi_n)$ for some inner divisor φ_n of θ_n. Choose injective operators with dense ranges $X_n\colon \mathcal{H}(\varphi_n) \to \mathcal{H}_n$ such that

$$(T|\mathcal{H}_n)X_n = X_n S(\varphi_n), \quad n = 1, 2, \ldots, N,$$

and define $X\colon \bigoplus_{n=1}^N \mathcal{H}(\varphi_n) \to \mathcal{H}$ by

$$X(u_1 \oplus u_2 \oplus \cdots \oplus u_N) = \sum_{n=1}^N X_n u_n, \quad u_1 \oplus u_2 \oplus \cdots \oplus u_N \in \bigoplus_{n=1}^N \mathcal{H}(\varphi_n).$$

Then X has dense range, and [3, Theorem VI.3.16] implies that $\prod_{n=1}^N \theta_n$ divides $\prod_{n=1}^N \varphi_n$. It follows that $\varphi_n \equiv \theta_n$, $n = 1, 2, \ldots, N$, and a second application of [3, Theorem VI.3.16] implies that X is one-to-one as well. In fact, X implements an isomorphism between the lattices of invariant subspaces of $\bigoplus_{n=1}^N S(\varphi_n) = \bigoplus_{n=1}^N S(\theta_n)$ and T via the map $\mathcal{M} \mapsto \overline{X\mathcal{M}}$, and this implies immediately the conclusion of the proposition. ∎

There is a natural way to transform a C_0-basis into another. We recall that the algebraic adjoint (or cofactor matrix) of an $N \times N$ matrix A is a matrix A^{Ad} such that

$$AA^{\mathrm{Ad}} = A^{\mathrm{Ad}}A = \det(A)I_N,$$

where I_N denotes the identity matrix of size N. Given functions $u, v \in H^\infty$, at least one of which is nonzero, we denote by $u \wedge v$ their greatest common inner divisor.

Corollary 2.6. *Let T be an operator of class C_0 on $\mathcal{H}$ with Jordan model $\bigoplus_{n=1}^N S(\theta_n)$, and let $\{h_n\}_{n=1}^N \subset \mathcal{H}$ be a C_0-basis for T. Consider functions $\{u_{nm}\}_{n,m=1}^N \subset H^\infty$ with the following properties:*

(1) $\theta_1 \wedge \det[u_{nm}]_{n,m=1}^N \equiv 1$.

(2) *θ_m/θ_n divides u_{nm} if $n > m$.*

Then the vectors

$$h'_n = \sum_{m=1}^N u_{nm}(T)h_m, \quad n = 1, 2, \ldots, N,$$

also form a C_0-basis for T.

Proof. Denote by $[v_{nm}]_{n,m=1}^N$ the algebraic adjoint of the matrix $[u_{nm}]_{n,m=1}^N$, and set $g = \det[u_{nm}]_{n,m=1}^N$ so that

$$\sum_{j=1}^N v_{nm}u_{mk} = g\delta_{ik}, \quad i, k = 1, 2, \ldots, N.$$

We have

$$\sum_{i=1}^{N} v_{nm}(T)h'_m = g(T)h_n, \quad n = 1, 2, \ldots, N,$$

so that the invariant subspace for T generated by $\{h'_n\}_{n=1}^N$ contains $g(T)\mathcal{H}$. Condition (1) implies that $g(T)$ has dense range, and therefore this invariant subspace is $\mathcal{H}$. The corollary follows from Proposition 2.5 once we show that $\theta_m(T)h'_m = 0$. Indeed, since $\theta_k|\theta_m$ for $k > m$,

$$\theta_m(T)h'_m = \sum_{n=1}^{m-1} u_{mn}(T)\theta_m(T)h_n.$$

Condition (2) implies that $\theta_n|u_{mn}\theta_m$ and therefore all the terms in the sum above vanish.

∎

There is yet another quasidirect decomposition which serves as a substitute for the primary decomposition of torsion modules over a principal ideal domain. The following result is easily obtained from [3, Theorem II.4.6].

Proposition 2.7. *Consider an operator T of class C_0 on a Hilbert space $\mathcal{H}$. Assume that the minimal function m_T is factored as a product*

$$m_T = \gamma_1\gamma_2\cdots\gamma_n$$

of inner functions such that $\gamma_i \wedge \gamma_j \equiv 1$ for $i \neq j$, and set $\Gamma_j = m_T/\gamma_j$ for $j = 1, 2, \ldots, n$. Then $\mathcal{H}$ is the quasidirect sum of the invariant subspaces

$$\mathcal{H}_j = \overline{\Gamma_j(T)\mathcal{H}}, \quad j = 1, 2, \ldots, n.$$

If $\mathcal{M}$ is an invariant subspace for T, we have

$$\overline{\Gamma_j(T)\mathcal{M}} = \mathcal{M} \cap \overline{\Gamma_j(T)\mathcal{H}}, \quad j = 1, 2, \ldots, n,$$

and $\mathcal{M}$ is the quasidirect sum of the spaces $\mathcal{M}_j = \overline{\Gamma_j(T)\mathcal{M}}$. Moreover, $\mathcal{M}$ has an invariant quasidirect complement in $\mathcal{H}$ if and only if each $\mathcal{M}_j$ has an invariant quasidirect complement in $\mathcal{H}_j$.

The decomposition of m_T to which this proposition is applied arises as follows.

Lemma 2.8. (1) *Consider functions $m, f_1, f_2, \ldots, f_k \in H^\infty$ such that m is inner. There exist pairwise relative prime inner functions $\gamma_1, \gamma_2, \ldots, \gamma_n$ in H^∞ with the property that $m = \gamma_1\gamma_2\cdots\gamma_n$ and the set of inner functions*

$$\{f_1 \wedge \gamma_i, f_2 \wedge \gamma_i, \ldots, f_k \wedge \gamma_i\}$$

is totally ordered by divisibility for $i = 1, 2, \ldots, n$.

(2) *Consider inner functions* $\theta_1, \theta_2, \ldots, \theta_N \in H^\infty$ *such that* $\theta_{j+1} | \theta_j$ *for* $j = 1, 2, \ldots, N-1$. *There exists a factorization* $\theta_1 = \gamma_1 \gamma_2 \cdots \gamma_n$ *into pairwise relatively prime inner factors with the following property: if* ω *is a nonconstant inner factor of* γ_k *for some* $k = 1, 2, \ldots, n$ *and* $\theta_j \wedge \gamma_k$ *is not constant for some* $j = 2, \ldots, N$, *then*

$$\omega \wedge \theta_j \wedge \gamma_k \not\equiv 1.$$

Proof. For the proof, it suffices to consider the case when m is either a Blaschke product or a singular inner function. We only treat the case when

$$m(\lambda) = \exp\Big[\int_0^{2\pi} \frac{\lambda + e^{it}}{\lambda - e^{it}}\, d\mu(t)\Big], \quad \lambda \in \mathbb{D},$$

where μ is a singular measure on the interval $[0, 2\pi)$. We have then

$$(f_j \wedge m)(\lambda) = \exp\Big[\int_0^{2\pi} \frac{\lambda + e^{it}}{\lambda - e^{it}} h_j(t)\, d\mu(t)\Big], \quad \lambda \in \mathbb{D}, j = 1, 2 \ldots, k,$$

where the functions $h_j \colon [0, 2\pi) \to [0, 1]$ are Borel measurable. There is a Borel partition $[0, 2\pi) = \bigcup_\sigma A_\sigma$ indexed by the permutations σ of $\{1, 2, \ldots, k\}$ such that

$$h_{\sigma(1)}(t) \le h_{\sigma(2)}(t) \le \cdots \le h_{\sigma(k)}(t), \quad t \in A_\sigma,$$

for each σ. We write the function m as the product of the inner functions

$$\gamma_\sigma(\lambda) = \exp\Big[\int_{A_\sigma} \frac{\lambda + e^{it}}{\lambda - e^{it}}\, d\mu(t)\Big], \quad \lambda \in \mathbb{D}.$$

This decomposition satisfies the conclusion of (1). For (2), we assume again that the functions θ_j are singular, so that

$$\theta_j(\lambda) = \exp\Big[\int_0^{2\pi} \frac{\lambda + e^{it}}{\lambda - e^{it}} h_j(t)\, d\mu(t)\Big], \quad \lambda \in \mathbb{D}, j = 1, 2 \ldots, N,$$

where the Borel functions h_j are such that

$$1 = h_1 \ge h_2 \ge \cdots \ge h_N \ge 0.$$

Define now Borel sets $B_k = \{t \in [0, 2\pi) : h_k(t) > h_{k+1}(t) = 0\}$, $k = 1, 2, \ldots, N$, using the convention $h_{N+1} = 0$ in the definition of B_N. The functions

$$\gamma_k(\lambda) = \exp\Big[\int_{B_k} \frac{\lambda + e^{it}}{\lambda - e^{it}}\, d\mu(t)\Big], \quad \lambda \in \mathbb{D}, k = 1, 2 \ldots, N,$$

satisfy the requirements of (2) with $n = N$. The case of Blaschke products is treated similarly, with the functions h_j being replaced by the functions $\nu_j(\lambda)$ representing the order of $\lambda \in \mathbb{D}$ as a zero of θ_j. ∎

We need more precise information about the construction of the Jordan model for operators T of class C_0 for which $I - T^*T$ has finite rank N. Such operators are said to be of class $C_0(N)$; they are constructed as follows. Consider an inner function Θ on the unit disk whose values are complex $N \times N$ matrices. Thus, $\|\Theta(\lambda)\| \leq 1$ for every $\lambda \in \mathbb{D}$, and the boundary values $\Theta(\zeta)$ are unitary matrices for almost every $\zeta \in \mathbb{T} = \partial\mathbb{D}$. Such a function determines a multiplication operator M_Θ on the space $H^2 \otimes \mathbb{C}^N$, and the space $M_\Theta[H^2 \otimes \mathbb{C}^N]$ is invariant for the shift $S \otimes I_{\mathbb{C}^N}$ of multiplicity N. As in the case $N = 1$, we write

$$\mathcal{H}(\Theta) = [H^2 \otimes \mathbb{C}^N] \ominus M_\Theta[H^2 \otimes \mathbb{C}^N],$$

and denote by $S(\Theta)$ the compression of $S \otimes I_{\mathbb{C}^N}$ to $\mathcal{H}(\Theta)$. The operator $S(\Theta)$ constructed this way is of class $C_0(N')$ for some $N' \leq N$, and every operator of class $C_0(N)$ is unitarily equivalent to $S(\Theta)$ for some function Θ with the above properties. The Jordan model of $S(\Theta)$ can be obtained directly by finding an analogue of the Smith normal form for the matrix Θ [21, 22]. We recall the basic definitions. Assume that A and B are two $p \times q$ matrices with elements from H^∞. We say that A is *quasiequivalent* to B if, for any inner function ω, there exist matrices X, Y over H^∞ of sizes $p \times p, q \times q$, respectively, such that

$$XA = BY \tag{2.1}$$

and the functions $\det(X), \det(Y)$ are relatively prime to ω, that is, neither of them has any nonconstant common inner factors with ω. Despite the asymmetry in the definition, quasiequivalence is an equivalence relation. Indeed, equation (2.1) implies the relation

$$X_1 B = AY_1,$$

where $X_1 = \det(Y)X^{\mathrm{Ad}}$ and $Y_1 = \det(X)Y^{\mathrm{Ad}}$. According to [22], every $p \times q$ matrix A over H^∞ is quasiequivalent to a matrix of the form

$$\begin{bmatrix} \theta_1 & 0 & 0 & \cdots & 0 \\ 0 & \theta_2 & 0 & \cdots & 0 \\ 0 & 0 & \theta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \end{bmatrix},$$

where the functions $\theta_1, \theta_2, \ldots, \theta_{\min\{p,q\}}$ are inner or zero and satisfy $\theta_{n+1} | \theta_n$. These functions are uniquely determined except for scalar factors of absolute value 1. None of the functions θ_n is zero if A has a nonzero minor of order $\min\{p, q\}$. This result can be applied to an inner $N \times N$ matrix Θ to yield inner functions $\theta_1, \theta_2, \ldots, \theta_N$ such that $\theta_{n+1} | \theta_n$ for $n = 1, 2, \ldots, N-1$, and $N \times N$ matrices X, Y over H^∞ such that

$$\Theta X = Y \Theta'$$

and $\det(X), \det(Y)$ are relatively prime to θ_1, where Θ' is the diagonal matrix with diagonal entries $\theta_1, \theta_2, \ldots, \theta_N$. The conditions on the determinants above can be written as

$$\det(X) \wedge \theta_1 \equiv \det(Y) \wedge \theta_1 \equiv 1.$$

Denote by $y_1, y_2, \ldots, y_N$ the columns of the matrix Y, which can be viewed as vectors in $H^2 \otimes \mathbb{C}^N$. In other words, $y_n = Y(1 \otimes e_n)$, where $\{e_n\}_{n=1}^N$ is the standard basis in $\mathbb{C}^N$. Then the results of [21] say that $\theta_n^{S(\Theta)} \equiv \theta_n$ for $n = 1, 2, \ldots, N$, and the vectors $\{P_{\mathcal{H}(\Theta)} y_n\}_{n=1}^N$ form a C_0-basis for $S(\Theta)$. Note incidentally that $S(\Theta)$ need not have multiplicity equal to N. Indeed, the last few of the functions θ_n could be constant, and the corresponding vectors in the C_0-basis would be 0. The following lemma provides a formulation of the C_0-basis property in terms of the vectors y_n.

Lemma 2.9. *With the above notation, assume that $\{u_n\}_{n=1}^N \subset H^\infty$ and*

$$\sum_{n=1}^N u_n y_n \in M_\Theta [H^2 \otimes \mathbb{C}^N]. \tag{2.2}$$

Then $\theta_n | u_n$ for $n = 1, 2, \ldots, N$.

Proof. Observe that

$$\sum_{n=1}^N u_n y_n = \sum_{n=1}^N Y(u_n \otimes e_n) = Y\Theta' \sum_{n=1}^N \frac{u_n}{\theta_n} \otimes e_j = \Theta X \sum_{n=1}^N \frac{u_n}{\theta_n} \otimes e_n,$$

and (2.2) implies that

$$X \sum_{n=1}^N \frac{u_n}{\theta_n} \otimes e_n \in H^2 \otimes \mathbb{C}^N$$

because M_Θ is one-to-one. Therefore

$$\det(X) \sum_{n=1}^N \frac{u_n}{\theta_n} \otimes e_n = X^{\mathrm{Ad}} X \sum_{n=1}^N \frac{u_n}{\theta_n} \otimes e_n \in H^2 \otimes \mathbb{C}^N$$

as well, so that $\theta_n | u_n \det(X)$ for all n. Since $\theta_n \wedge \det(X) | \theta_1 \wedge \det(X) \equiv 1$, we conclude that $\theta_n | u_n$, as claimed. ∎

We use repeatedly the following result about operators of class C_0 with finite multiplicity. The proof follows from [3, Proposition VII.6.9].

Proposition 2.10. *Let T be an operator of class C_0 on $\mathcal{H}$ with $\mu_T < +\infty$, let T' be a completely nonunitary contraction on $\mathcal{H}'$, and let $X : \mathcal{H}' \to \mathcal{H}$ be a linear operator with dense range such that $XT' = TX$. Then every invariant subspace $\mathcal{M}$ for T is of the form $\overline{X\mathcal{M}'}$, where $\mathcal{M}'$ is an invariant subspace for T'. If $T|\mathcal{M}$ has a cyclic vector, then it also has a cyclic vector of the form Xh' with $h' \in \mathcal{H}'$.*

It is useful to consider more general invariant subspaces of $S \otimes I_{\mathbb{C}^N}$. These are characterized by the Beurling–Lax–Halmos theorem [23, Theorem V.3.3].

Theorem 2.11. *Consider an invariant subspace $\mathcal{K} \subset H^2 \otimes \mathbb{C}^N$ for $S \otimes I_{\mathbb{C}^N}$. There exist an integer $r \leq N$ and an inner function Ψ with values $N \times r$ complex matrices so that $\mathcal{K} = M_\Psi[H^2 \otimes \mathbb{C}^r]$.*

The fact that Ψ is inner implies that it has nonzero minors of order r, and therefore quasidiagonalization produces a matrix with r inner functions $\psi_1, \psi_2, \ldots, \psi_r$ on the main diagonal. We call the number r the rank of the invariant subspace $\mathcal{K}$, and observe that r is simply the multiplicity of the unilateral shift $S \otimes I_{\mathbb{C}^N}|\mathcal{K}$. We also use the notation

$$d(\mathcal{K}) = \psi_1\psi_2 \cdots \psi_r$$

for the product of these functions. The function $d(\mathcal{K})$ is inner, and it is uniquely determined up to a scalar factor. In the special case when $\mathcal{K}$ is of maximum rank $r = N$, we have

$$d(\mathcal{K}) \equiv \det(\Psi).$$

More generally, if V is a unilateral shift of finite multiplicity on a space $\mathcal{M}$ and $\mathcal{K} \subset \mathcal{M}$ is an invariant subspace for V, we can define an inner function

$$d_\mathcal{M}(\mathcal{K})$$

by noting that V is unitarily equivalent to $S \otimes I_{\mathbb{C}^N}$ for some N, and identifying $\mathcal{K}$ with the range of an inner function as above. The multiplicative property of determinants implies that

$$d_\mathcal{M}(\mathcal{L}) \equiv d_\mathcal{M}(\mathcal{K})d_\mathcal{K}(\mathcal{L})$$

if $\mathcal{L} \subset \mathcal{K}$ are invariant subspaces of rank N of V.

Lemma 2.12. *Let V be a unilateral shift of finite multiplicity on a space $\mathcal{M}$, and let $\mathcal{K} \subset \mathcal{M}$ be an invariant subspace. The following conditions are equivalent:*

(1) $d_{\mathcal{M}}(\mathcal{K}) \not\equiv 1$.
(2) *There exist an inner function* $\varphi \in H^\infty$ *and a vector* $f \in \mathcal{M} \setminus \mathcal{K}$ *such that* $\varphi(V)f \in \mathcal{K}$.

Proof. We assume without loss of generality that $V = S \otimes I_{\mathbb{C}^N}$ for some $N \in \mathbb{N}$, and that $\mathcal{K} = M_\Psi[H^2 \otimes \mathbb{C}^r]$ for some $r \in \{1, 2, \ldots, N\}$ and some inner $N \times r$ matrix Ψ. Choose square matrices X, Y over H^∞ such that $\det(X) \wedge d_{\mathcal{M}}(\mathcal{K}) \equiv \det(Y) \wedge d_{\mathcal{M}}(\mathcal{K}) \equiv 1$ and $\Psi X = Y\Psi'$, where Ψ' has inner entries $\psi_1, \psi_2, \ldots, \psi_r$ on the main diagonal, zero entries elsewhere, and $\psi_{j+1}|\psi_j$ for $j = 1, 2, \ldots, r-1$. Denote by $y_1, y_2, \ldots, y_r$ the columns of the matrix Y, and denote by $e_1, e_2, \ldots, e_r$ the standard basis in $\mathbb{C}^r$, which we also view as a subspace of $\mathbb{C}^N$. Condition (1) is equivalent to $\psi_1 \not\equiv 1$. Assume first that $\psi_1 \not\equiv 1$, and observe that

$$\psi_1(V)y_1 = \psi_1 y_1 = Y(\psi_1 \otimes e_1) = Y\Psi'(1 \otimes e_1) = \Psi X(1 \otimes e_1)$$

belongs to the space $\mathcal{K}$, but $y_1 \notin \mathcal{K}$, as can be seen by repeating the proof of Lemma 2.9. Thus (1) implies (2). Conversely, assume that $\psi_1 \equiv 1$ and a vector $f \in H^2 \otimes \mathbb{C}^N$ satisfies $\varphi f \in \mathcal{K}$ for some inner function φ, say $\varphi f = \Psi h$ for some $h \in H^2 \otimes \mathbb{C}^r$. Note that the matrices X, Y above can now be chosen so that their determinants are relatively prime to φ, and Ψ' can be chosen to simply be the matrix representing the inclusion $\mathbb{C}^r \subset \mathbb{C}^N$. We have

$$\varphi \det(X) f = \Psi \det(X) h = \Psi X X^{\mathrm{Ad}} h = Y\Psi' X^{\mathrm{Ad}} h,$$

so that multiplying by Y^{Ad} yields

$$\varphi \det(X) Y^{\mathrm{Ad}} f = \det(Y)\Psi' X^{\mathrm{Ad}} h.$$

Apply now the matrix $X \oplus I_{\mathbb{C}^{N-r}}$ to both sides to obtain

$$(X \oplus I_{\mathbb{C}^{N-r}}) \det(X) Y^{\mathrm{Ad}} \varphi f = \det(X) \det(Y) \Psi' h.$$

In other words, since $\varphi \wedge (\det(X)\det(Y)) \equiv 1$, φ divides all the components of the vector $\Psi' h$, and this simply means that $h/\varphi \in H^2 \otimes \mathbb{C}^r$. We conclude that $f = \Psi(h/\varphi)$ does belong to $\mathcal{K}$, thus showing that property (2) does not hold. Thus (2) implies (1). ∎

Corollary 2.13. *Let* V *be a shift of finite multiplicity on a Hilbert space* $\mathcal{M}$, *and let* $\mathcal{K}$ *be an invariant subspace for* V *such that* $d_{\mathcal{M}}(\mathcal{K}) \equiv 1$ *and* $V|\mathcal{K}$ *has multiplicity* r. *Fix an inner function* $\omega \in H^\infty$. *There exist vectors* $h_1, h_2, \ldots, h_r \in \mathcal{K}$ *with the following property: if the functions* $u_1, u_2, \ldots, u_r \in H^\infty$ *are such that* $\sum_{j=1}^r u_j(V)h_j \in \omega(V)\mathcal{M}$, *then* $\omega | u_j$ *for all* $j = 1, 2, \ldots, r$.

Proof. We assume with no loss of generality that $V = S \otimes I_{\mathbb{C}^N}$ on $\mathcal{M} = H^2 \otimes \mathbb{C}^N$, and $\mathcal{Q} = M_\Psi[H^2 \otimes \mathbb{C}^r]$ for some inner function Ψ. We apply quasiequivalence to obtain square matrices X, Y over H^∞ such that $\Psi X = Y\Psi'$ and $\det(X) \wedge \omega \equiv \det(Y) \wedge \omega \equiv 1$, where Ψ' is the matrix representing the inclusion $\mathbb{C}^r \subset \mathbb{C}^N$. We claim that the columns $h_1, h_2, \dots, h_r$ of the matrix $Y\Psi'$ satisfy the required property. Indeed, the equation $h_j = Y\Psi'(1 \otimes e_j) = \Psi X(1 \otimes e_j)$ shows that $h_j \in \mathcal{K}$. Assume now that $\sum_{j=1}^r u_j h_j = \omega h$ for some $h \in H^2 \otimes \mathbb{C}^N$, where the coefficients u_j belong to H^∞. We have

$$\det(Y) \sum_{j=1}^r u_j \otimes e_j = Y^{\mathrm{Ad}} Y \sum_{j=1}^r u_j \otimes e_j = Y^{\mathrm{Ad}} \sum_{j=1}^r u_j h_j = \omega Y^{\mathrm{Ad}} h,$$

and the divisibility $\omega | u_j$ follows because $\det(Y) \wedge \omega \equiv 1$. ∎

Subspaces $\mathcal{K}$ with $d_\mathcal{M}(\mathcal{K}) \equiv 1$ are obtained as follows. Consider the field of fractions $\mathfrak{D}$ of H^∞. That is $\mathfrak{D} = \{\varphi/\psi : \varphi \in H^\infty, \psi \in H^\infty \setminus \{0\}\}$; recall that $H^2 \subset \mathfrak{D}$. A unilateral shift V of finite multiplicity on $\mathcal{M}$ turns $\mathcal{M}$ into a module over H^∞, and this module is contained in the finite dimensional vector space $\mathfrak{D}\mathcal{M}$ over $\mathfrak{D}$. Indeed, $\mathfrak{D}(H^2 \otimes \mathbb{C}^N) = \mathfrak{D} \otimes \mathbb{C}^N = \mathfrak{D}^N$, thus showing that $\dim_\mathfrak{D}(\mathfrak{D}\mathcal{M})$ equals the multiplicity of V.

Lemma 2.14. *Let V be a shift of finite multiplicity on a Hilbert space $\mathcal{M}$, and let $\mathcal{Q} \subset \mathfrak{D}\mathcal{M}$ be a $\mathfrak{D}$-vector subspace. Then $\mathcal{K} = \mathcal{Q} \cap \mathcal{M}$ is closed in $\mathcal{M}$, it is invariant for V, and $d_\mathcal{M}(\mathcal{K}) \equiv 1$. Conversely, every invariant subspace $\mathcal{K} \subset \mathcal{M}$ such that $d_\mathcal{M}(\mathcal{K}) \equiv 1$ satisfies the relation $\mathcal{K} = (\mathfrak{D}\mathcal{K}) \cap \mathcal{M}$.*

Proof. We assume without loss of generality that $V = S \otimes I_{\mathbb{C}^N}$ so that

$$\mathfrak{D}\mathcal{M} = \mathfrak{D} \otimes \mathbb{C}^N = \left\{ \sum_{j=1}^N u_j \otimes e_j : u_1, u_2, \dots, u_N \in \mathfrak{D} \right\}.$$

The vector space $\mathcal{Q}$ is defined by a finite number of linear equations of the form

$$\sum_{j=1}^N \alpha_j u_j = 0, \tag{2.3}$$

with coefficients $\alpha_j \in \mathfrak{D}$. The solution set of such an equation is not modified if we multiply all the coefficients by the same function in $H^\infty \setminus \{0\}$. We can thus assume that $\alpha_j \in H^\infty$ for all j. It follows that $\mathcal{K} = \mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N)$ consists of those vectors $\sum_{j=1}^N u_j \otimes e_j$ for which the functions $u_j \in H^2$ satisfy all the equations (2.3) defining $\mathcal{Q}$, an therefore $\mathcal{K}$ is indeed closed. Finally, $\mathcal{K}$ is easily seen not to satisfy property (2) of Lemma 2.12.

Assume now that $d_{\mathcal{M}}(\mathcal{K}) \equiv 1$ for some invariant space $\mathcal{K} \subset H^2 \otimes \mathbb{C}^N$. Clearly, $\mathcal{K} \subset (\mathfrak{D}\mathcal{K}) \cap \mathcal{M}$, so it suffices to prove the opposite inclusion. Consider a vector $h \in (\mathfrak{D}\mathcal{K}) \cap \mathcal{M}$, so there exist $\varphi \in H^\infty$, $\psi \in H^\infty \setminus \{0\}$, and $k \in \mathcal{K}$ such that $h = (\varphi/\psi)k$. Factor $\psi = \psi_1\psi_2$, where ψ_1 is inner and ψ_2 is outer. We have

$$\psi_1\psi_2 h = \psi h = \varphi k \in \mathcal{K},$$

and therefore $\psi_2 h \in \mathcal{K}$ by Lemma 2.12. Since ψ_2 is outer, there exists a sequence $\{u_n\}_{n=1}^\infty \subset H^\infty$ such that $\lim_{n\to\infty} u_n\psi_2 = 1$ in the weak* topology of H^∞. This implies that $\lim_{n\to\infty} u_n\psi_2 h = h$ in $H^2 \otimes \mathbb{C}^N$, and therefore $h \in \mathcal{K}$. ∎

The following result is useful when we want to replace a linear combination with coefficients in $\mathfrak{D}$ by another one with coefficients in H^∞. It is a stronger property which cyclic vectors for an operator of class C_0 have.

Lemma 2.15. *Consider an operator T of class C_0 on a Hilbert space $\mathcal{H}$, a cyclic vector $h \in \mathcal{H}$ for T, and an inner function ω. For every vector $k \in \mathcal{H}$ there exist functions $\alpha, \beta \in H^\infty$ such that $\beta \wedge \omega \equiv 1$ and $\alpha(T)h = \beta(T)k$.*

Proof. Let $S(\theta)$ be the Jordan model of T, and let $X\colon \mathcal{H} \to \mathcal{H}(\theta)$ be an injective operator with dense range such that $XT = S(\theta)X$. If $\alpha, \beta \in H^\infty$, we have

$$\alpha(S(\theta))Xh = X\alpha(T)h, \quad \beta(S(\theta))Xk = X\beta(T)k.$$

Th equality $\alpha(T)h = \beta(T)k$ is equivalent to

$$\alpha(S(\theta))Xh = \beta(S(\theta))Xk \tag{2.4}$$

because X is one-to-one. Since Xh is a cyclic vector for $S(\theta)$, this observation shows that it suffices to prove the lemma when $T = S(\theta)$. In this case the functions $h, k \in H^2$ can be written as $h = \beta/\gamma$ and $k = \alpha/\gamma$ for some functions $\alpha, \beta, \gamma \in H^\infty$ such that γ is outer. The equation (2.4) is satisfied with this choice of α and β. Moreover, the fact that h is a cyclic vector for $S(\theta)$ amounts to the equality $\beta \wedge \theta \equiv 1$. Equation (2.4) remains valid if β is replaced by $\beta + t\theta$ for some scalar $t \in \mathbb{C}$. It is known [3, Theorem III.1.14] that for t in a dense G_δ set, we have $(\beta + t\theta) \wedge \omega \equiv 1$. The lemma follows. ∎

We continue with one useful result from intersection theory. Consider an arbitrary field $\mathfrak{D}$ and a vector space L over $\mathfrak{D}$ of dimension $N < +\infty$. Given an integer $r \in \{1, 2, \ldots, N\}$, the *Grassmann variety* $G(L, r)$ consists of all vector subspaces $M \subset L$ of dimension r. A *complete flag* in L is a collection

$$\mathcal{E} = \{\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_N\}$$

of subspaces of L such that $\dim_{\mathfrak{D}}(\mathcal{E}_i) = i$ for $i = 0, 1, \ldots, N$. Assume that $\mathcal{E}$ is a complete flag and

$$I = \{i_1 < i_2 < \cdots < i_r\}$$

is a subset of $\{1, 2, \ldots, N\}$. The *Schubert variety* $\mathfrak{S}(\mathcal{E}, I)$ is the subset of $G(L, r)$ consisting of those subspaces M for which

$$\dim_{\mathfrak{D}}(M \cap \mathcal{E}_{i_x}) \geq x, \quad x = 1, 2, \ldots, r.$$

The Littlewood–Richardson rule provides a test to determine whether the intersection of three Schubert varieties $\mathfrak{S}(\mathcal{E}, I)$, $\mathfrak{S}(\mathcal{F}, J)$, and $\mathfrak{S}(\mathcal{G}, K)$ is nonempty. More precisely, assume that the three sets $I, J, K \subset \{1, 2, \ldots, N\}$ satisfy the equation

$$\sum_{x=1}^{r} (i_x + j_x + k_x - 3x) = 2r(N - r).$$

Then one associates to the triple (I, J, K) a nonnegative integer c_{IJK} with the property that

$$\mathfrak{S}(\mathcal{E}, I) \cap \mathfrak{S}(\mathcal{F}, J) \cap \mathfrak{S}(\mathcal{G}, K) \tag{2.5}$$

contains generically c_{IJK} elements in case $\mathfrak{D}$ is algebraically closed. For general $\mathfrak{D}$, one can still state that the intersection (2.5) contains at least one element when $c_{IJK} = 1$; see [9].

3. Invariant subspaces related to Schubert varieties

In this section we fix an operator T of class C_0 with finite defect numbers. As seen above, we can assume that $T = S(\Theta)$, where Θ is an inner function with values $N \times N$ matrices. The Jordan model of T has the form $\bigoplus_{n=1}^{N} S(\theta_n)$, where it may happen that the last few functions θ_n are constant. Denote by $h_1, h_2, \ldots, h_N$ a C_0-basis for T, where $h_n = 0$ if θ_n is constant. We study invariant subspaces for T of the form $[P_{\mathcal{H}(\Theta)}\mathcal{R}]^-$, where $\mathcal{R} \subset H^2 \otimes \mathbb{C}^N$ is an invariant subspace for $S \otimes I_{\mathbb{C}^N}$ such that $d(\mathcal{R}) = 1$. For the remainder of the paper, $\mathfrak{D}$ denotes the field of fractions of H^∞.

Lemma 3.1. *Fix $m, r \in \{1, 2, \ldots, N\}$ such that $r \leq m$. Assume that $\mathcal{Q} \subset \mathfrak{D} \otimes \mathbb{C}^N$ is a $\mathfrak{D}$-vector space of dimension r contained in the $\mathfrak{D}$-linear span of $\{h_1, h_2, \ldots, h_m\}$,* *set* $\mathcal{R} = \mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N)$, *and* $\mathcal{K} = [P_{\mathcal{H}(\Theta)}\mathcal{R}]^-$. *If the Jordan model of* $T|\mathcal{K}$ *is* $\bigoplus_{n=1}^r S(\alpha_n)$ *then* $\theta_m|\alpha_r$.

Proof. Observe that $(S \otimes I_{\mathbb{C}^N})|\mathcal{R}$ has cyclic multiplicity r. Projecting onto $\mathcal{K}$ a cyclic set for $(S \otimes I_{\mathbb{C}^N})|\mathcal{R}$ yields a cyclic set for $T|\mathcal{K}$, and therefore $\mu_{T|\mathcal{R}} \leq r$. Thus, indeed, the Jordan model of $T|\mathcal{R}$ consists of at most r summands. The lemma is vacuously satisfied if $\theta_m \equiv 1$, so we assume that is not the case. We show next that it suffices to prove the lemma in the special case $m = N$. Denote by $\mathcal{E}$ the $\mathfrak{D}$-linear span of $\{h_1, h_2, \ldots, h_m\}$, and set

$$\mathcal{E}_+ = \mathcal{E} \cap (H^2 \otimes \mathbb{C}^N), \quad V = (S \otimes I_{\mathbb{C}^N})|\mathcal{E}_+.$$

The operator $A = P_{\mathcal{H}(\Theta)}|\mathcal{E}_+$ satisfies the intertwining relation

$$AV = TA,$$

and therefore the restriction $B = A|\mathcal{H}'$, where $\mathcal{H}' = \mathcal{E}_+ \ominus \ker(A)$, intertwines the compression T' of V to $\mathcal{H}'$ and T, that is

$$BT' = TB.$$

Moreover, B is one-to-one and its range contains $h_1, h_2, \ldots, h_m$. It is easy to see now that $T' \sim \bigoplus_{n=1}^m S(\theta_n)$. Indeed, T' is an operator of class $C_0(m)$ so its Jordan model contains at most m summands $S(\varphi_n)$, and $\varphi_n|\theta_n$ because T' is quasisimilar to a restriction of T. On the other hand, $\theta_n|\varphi_n$, $n = 1, 2, \ldots, m$, because the restriction of T to its invariant subspace generated by $\{h_1, h_2, \ldots, h_m\}$ is quasisimilar to a restriction of T'. Set now $\mathcal{R}' = \mathcal{Q} \cap \mathcal{E}_+$ and $\mathcal{K}' = [P_{\mathcal{H}'}\mathcal{R}']^-$. We have $B\mathcal{K}' \subset \mathcal{K}$ and therefore the Jordan model $\bigoplus_{n=1}^r S(\alpha_n')$ of $T'|\mathcal{K}'$ satisfies $\alpha_n'|\alpha_n$ for $n = 1, 2, \ldots, r$. It suffices therefore to show that $\theta_m|\alpha_r'$. This concludes our reduction to the case $m = N$.

Represent the space $\mathcal{R}$ as

$$\mathcal{R} = M_\Psi[H^2 \otimes \mathbb{C}^r]$$

for some inner $N \times r$ matrix Ψ. As noted in Lemma 2.14, Ψ is quasiequivalent to the constant matrix J representing the inclusion $\mathbb{C}^r \subset \mathbb{C}^N$. Fix square matrices X, Y over H^∞ such that

$$\Psi X = YJ$$

and $\det(X) \wedge \theta_N \equiv \det(Y) \wedge \theta_N \equiv 1$, and denote by $y_n = Y(1 \otimes e_n) \in \mathcal{R}$, $n = 1, 2, \ldots, r$, the columns of Y. We claim that, given functions $u_1, u_2, \ldots, u_r \in H^\infty$ such that

$$\sum_{n=1}^r u_n(T)P_{\mathcal{H}(\Theta)}y_n = 0,$$

it follows that $\theta_N | u_n$ for $n = 1, 2, \ldots, r$. Indeed, the relation above is equivalent to $\sum_{n=1}^{r} u_n y_n \in M_\Theta[H^2 \otimes \mathbb{C}^N]$. We use now the fact that θ_N divides all the entries of Θ to conclude that

$$\frac{1}{\theta_N} YJ\Big(\sum_{n=1}^{r} u_n \otimes e_n\Big) = \frac{1}{\theta_N}\sum_{n=1}^{r} u_n y_n \in H^2 \otimes \mathbb{C}^N.$$

Multiplying by Y^{Ad}, we see that

$$\frac{\det(Y)}{\theta_N}\sum_{n=1}^{r} u_n \otimes e_n \in H^2 \otimes \mathbb{C}^r.$$

The relation $\theta_N | u_n$ follows because $\det(Y) \wedge \theta_N \equiv 1$. The conclusion of the lemma follows by showing that $T|\mathcal{K}$ has a restriction quasisimilar to

$$\underbrace{S(\theta_N) \oplus S(\theta_N) \oplus \cdots \oplus S(\theta_N)}_{r \text{ times}}.$$

Indeed, denote by ψ_n an inner function satisfying

$$\psi_n H^\infty = \{\psi \in H^\infty : \psi(T)P_{\mathcal{H}(\Theta)} y_n = 0\}, \quad n = 1, 2, \ldots, r.$$

The invariant subspace $\mathcal{K}' \subset \mathcal{K}$ for T generated by the vectors

$$k_n = (\psi_n/\theta_N)(T)P_{\mathcal{H}(\Theta)} y_n, \quad n = 1, 2 \ldots, r,$$

is such that

$$T|\mathcal{K}' \sim \underbrace{S(\theta_N) \oplus S(\theta_N) \oplus \cdots \oplus S(\theta_N)}_{r \text{ times}}.$$

This is seen by noting that $k_1, k_2, \ldots, k_r$ form a C_0-basis for $T|\mathcal{K}'$. ■

Lemma 3.2. *Fix $m, p, r \in \{1, 2, \ldots, N\}$ with $r \le p \le N - m$, let $\mathcal{Q} \subset \mathfrak{D} \otimes \mathbb{C}^N$ be a subspace of dimension r, and let $z_1, z_2, \ldots, z_r \in \mathcal{Q}$ be given by*

$$z_\ell = \sum_{n=1}^{m+p} u_{\ell,n} h_n, \quad \ell = 1, 2, \ldots, r,$$

where all the coefficients $u_{\ell,n}$ are in H^∞. Set $\mathcal{R} = \mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N)$, and $\mathcal{K} = [P_{\mathcal{H}(\Theta)}\mathcal{R}]^-$. Assume that:

(1) *$\theta_{m+1} \equiv \theta_{m+2} \equiv \cdots \equiv \theta_{m+p}$ and θ_m/θ_{m+1} is not constant.*
(2) *Every nonconstant inner factor ω of θ_{m+1} satisfies $\omega \wedge (\theta_m/\theta_{m+1}) \not\equiv 1$.*

(3) *The Jordan model of* $T|\mathcal{K}$ *is*

$$\underbrace{S(\theta_{m+1}) \oplus S(\theta_{m+1}) \oplus \cdots \oplus S(\theta_{m+1})}_{r \text{ times}}.$$

(4) *The vectors* $w_\ell = P_{\mathcal{H}(\Theta)} z_\ell$, $\ell = 1, 2, \ldots, r$, *form a* C_0*-basis for* $T|\mathcal{K}$.

(5) *The set consisting of all inner functions of the form*

$$\theta_n \wedge \det \begin{bmatrix} u_{1,j_1} & u_{1,j_2} & \cdots & u_{1,j_s} \\ u_{2,j_1} & u_{2,j_2} & \cdots & u_{2,j_s} \\ \vdots & \vdots & \ddots & \vdots \\ u_{s,j_1} & u_{s,j_2} & \cdots & u_{s,j_s} \end{bmatrix},$$

where $s \in \{1, 2, \ldots, r\}$ *and* $n, j_1, j_2, \ldots, j_s \in \{1, 2 \ldots, N\}$, *is totally ordered by divisibility.*

Then there exist integers $m + 1 \leq j_1 < j_2 < \cdots < j_r \leq m + p$ *such that*

$$\det[u_{\ell,j_k}]_{\ell,k=1}^{r} \wedge \theta_{m+1} \equiv 1.$$

Proof. The hypothesis implies that $\theta_{m+1}(T) w_\ell = 0$ or, equivalently,

$$\sum_{n=1}^{m+p} \theta_{m+1}(T) u_{\ell,n}(T) h_n = 0, \quad \ell = 1, 2, \ldots, r.$$

Since $\{h_n\}_{n=1}^{N}$ form a C_0-basis for T, we conclude that $\theta_n | \theta_{m+1} u_{\ell,n}$ for all n, in particular $u_{\ell,n}$ is divisible by θ_m / θ_{m+1} for $n = 1, 2, \ldots, m$ and $\ell = 1, 2, \ldots, r$. Let $s \in \{0, 1, \ldots, r\}$ be largest with the property that there exist

$$m + 1 \leq j_1 < j_2 < \cdots < j_s \leq m + p$$

such that

$$\det[u_{\ell,j_m}]_{\ell,m=1}^{s} \wedge \theta_{m+1} \equiv 1,$$

and assume to get a contradiction that $s < r$. Consider indeterminates $\xi_1, \xi_2, \ldots, \xi_{s+1}$ and write the determinant

$$\det \begin{bmatrix} u_{1,j_1} & u_{1,j_2} & \cdots & u_{1,j_s} & \xi_1 \\ u_{2,j_1} & u_{2,j_2} & \cdots & u_{2,j_s} & \xi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{s,j_1} & u_{s,j_2} & \cdots & u_{s,j_s} & \xi_s \\ u_{s+1,j_1} & u_{s+1,j_2} & \cdots & u_{s+1,j_s} & \xi_{s+1} \end{bmatrix} = \sum_{\ell=1}^{s+1} \alpha_\ell \xi_\ell.$$

The coefficients $\alpha_\ell \in H^\infty$ are determinants of size s, and

$$\alpha_{s+1} \wedge \theta_{m+1} \equiv 1. \tag{3.1}$$

The vector

$$z = \sum_{\ell=1}^{s+1} \alpha_\ell z_\ell = \sum_{n=1}^{m+p} \beta_n h_n$$

has coefficients

$$\beta_n = \sum_{\ell=1}^{s+1} \alpha_\ell u_{\ell,n}, \quad n = 1, 2, \ldots, N,$$

which are determinants of order $s+1$. Note that $\beta_n = 0$ if $n \in \{j_1, j_2, \ldots, j_s\}$ and θ_m/θ_{m+1} divides β_n for $n \in \{1, 2, \ldots, m\}$. For other values of n, the function $\beta_n \wedge \theta_{m+p}$ is not constant by the definition of s. Conditions (2) and (5) imply now that there exists a nonconstant inner function ω such that $\omega | \beta_n \wedge \theta_{m+1}$ for all $n = 1, 2, \ldots, m+p$. Thus the vector z/ω belongs to $\mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N)$, and therefore $P_{\mathcal{H}(\Theta)}(z/\omega) \in \mathcal{K}$. This in turn implies that $P_{\mathcal{H}(\Theta)}[(\theta_{m+1}/\omega)z] = \theta_{m+1}(T) P_{\mathcal{H}(\Theta)}(z/\omega) = 0$, so

$$\frac{\theta_{m+1}}{\omega} z = \sum_{\ell=1}^{s+1} \frac{\theta_{m+1}}{\omega} \alpha_\ell z_\ell \in M_\Theta[H^2 \otimes \mathbb{C}^N],$$

or

$$\sum_{\ell=1}^{s+1} (\theta_{m+1}/\omega)(T) \alpha_\ell(T) w_\ell = 0.$$

Assumption (4) implies that $\theta_{m+1} | (\theta_{m+1}/\omega)\alpha_\ell$, that is, $\omega | \alpha_\ell$ for $\ell = 1, 2, \ldots, s+1$. This however contradicts (3.1) when $\ell = s+1$, thus concluding the proof. ∎

Define $\mathcal{E}_n$ to be the $\mathfrak{D}$-linear span of $\{h_1, h_2, \ldots, h_n\}$ when $1 \leq n \leq N$ and $h_n \neq 0$. The nonzero vectors h_n are linearly independent over $\mathfrak{D}$ because they from a C_0 basis for T, and therefore $\dim_{\mathfrak{D}} \mathcal{E}_n = n$. This sequence of spaces can be supplemented to form a complete flag $\mathcal{E}$ by defining appropriate spaces $\mathcal{E}_n$ of dimension n when $h_n = 0$.

Lemma 3.3. *Assume that $n \in \{1, 2, \ldots, N\}$ is such that $\theta_n \not\equiv 1$. Then*

$$[P_{\mathcal{H}(\Theta)}(\mathcal{E}_n \cap (H^2 \otimes \mathbb{C}^N))]^-$$

is equal to the smallest invariant subspace for T containing the vectors $h_1, h_2, \ldots, h_n$.

Proof. Clearly the space $\mathcal{M} = [P_{\mathcal{H}(\Theta)}(\mathcal{E}_n \cap (H^2 \otimes \mathbb{C}^N))]^-$ contains the vectors $h_1, h_2, \ldots, h_n$. To conclude the proof, it suffices to show that these vectors form a C_0-basis for $T|\mathcal{M}$. The operator $T|\mathcal{M}$ has multiplicity at most n, as seen at the beginning of the proof of Lemma 3.1. Thus the Jordan model of $T|\mathcal{M}$ is of the form $\bigoplus_{j=1}^n S(\varphi_j)$ with $\varphi_j|\theta_j$ for $j = 1, 2, \ldots, n$. On the other hand, $\theta_j|\varphi_j$ for $j = 1, 2, \ldots, n$ because $\mathcal{M}$ contains the subset $\{h_1, h_2, \ldots, h_n\}$ of the C_0-basis of T. The conclusion follows immediately from Proposition 2.5. ∎

Proposition 3.4. *Fix a positive integer $r \leq N$, a subset*

$$I = \{i_1 < i_2 < \cdots < i_r\} \subset \{1, 2, \ldots, N\},$$

and a subspace $\mathcal{Q} \in \mathfrak{S}(\mathcal{E}, I)$. Set $\mathcal{R} = \mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N)$ and $\mathcal{K} = [P_{\mathcal{H}(\Theta)}\mathcal{R}]^-$. Then:

1. *The space $\mathcal{K}$ is invariant for $T = S(\Theta)$, and $T|\mathcal{K}$ has cyclic multiplicity less than or equal to r.*
2. *If the Jordan model of $T|\mathcal{K}$ is $S(\beta_1) \oplus S(\beta_2) \oplus \cdots \oplus S(\beta_r)$, then θ_{i_x} divides β_x for $x = 1, 2, \ldots, r$.*
3. *If $\beta_x \equiv \theta_{i_x}$ for all $x = 1, 2, \ldots, r$, then $\mathcal{K}$ has a T-invariant quasidirect complement $\mathcal{L}$ in $\mathcal{H}(\Theta)$ such that $T|\mathcal{L} \sim \bigoplus_{i \notin I} S(\theta_i)$.*

Proof. We recall that $S \otimes I_{\mathbb{C}^N}|\mathcal{R}$ is a unilateral shift of multiplicity r, and therefore it has cyclic multiplicity r as well. Projecting onto $\mathcal{H}(\Theta)$ a cyclic set of r elements for $S \otimes I_{\mathbb{C}^N}|(\mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N))$ we obtain a cyclic set for $S(\Theta)|\mathcal{R}$, and this yields (1).

For the proofs of (2) and (3) we need some preparation. For each $x \in \{1, 2, \ldots, r\}$, choose a subspace $\mathcal{Q}_x \subset \mathcal{Q} \cap \mathcal{E}_{i_x}$ such that $\dim_{\mathfrak{D}} \mathcal{Q}_x = x$. The subspace $\mathcal{R}_x = \mathcal{Q}_x \cap (H^2 \otimes \mathbb{C}^N)$ is invariant for $S \otimes I_{\mathbb{C}^N}$, and $(S \otimes I_{\mathbb{C}^N})|\mathcal{R}_x$ has multiplicity equal to x. Also set $\mathcal{K}_x = [P_{\mathcal{H}(\Theta)}\mathcal{R}_x]^-$. When $r > 1$, we may assume that

$$\mathcal{Q}_x \subset \mathcal{Q}_{x+1}, \text{ thus } \mathcal{R}_x \subset \mathcal{R}_{x+1} \text{ and } \mathcal{K}_x \subset \mathcal{K}_{x+1}, \quad x = 1, 2, \ldots, r-1. \tag{3.2}$$

To prove (2), it suffices to consider the case when $\theta_{i_x} \not\equiv 1$. Denote by

$$S(\beta_1^x) \oplus S(\beta_2^x) \oplus \cdots \oplus S(\beta_x^x)$$

the Jordan model of $T|\mathcal{K}_x$, $x = 1, 2, \ldots, r$. An application of Lemma 3.1 to the space $\mathcal{Q}_x$ shows that $\theta_{i_x}|\beta_x^x$, and (2) follows because $\mathcal{K}_x \subset \mathcal{K}_r = \mathcal{K}$ and thus $\beta_x^x|\beta_x^r = \beta_x$.

As we observed above, we have

$$\theta_{i_1}|\beta_1^x|\beta_1, \; \theta_{i_2}|\beta_2^x|\beta_2, \ldots, \; \theta_{i_x}|\beta_x^x|\beta_x, \quad x = 1, 2, \ldots, r-1.$$

If the hypothesis of (3) is satisfied, that is, $\beta_x \equiv \theta_{i_x}$ for $x = 1, 2, \ldots, r$, we must have

$$\beta_j^x \equiv \theta_{i_x}, \quad j = 1, 2, \ldots, x,$$

for $x = 1, 2, \dots, r$. Thus the Jordan model of $T|\mathcal{K}_x$ is precisely

$$S(\theta_{i_1}) \oplus S(\theta_{i_2}) \oplus \cdots \oplus S(\theta_{i_x})$$

for $x \in \{1, 2, \dots, r\}$. The case in which $\theta_{i_r} \equiv 1$ reduces to the same statement with r replaced by $r - 1$, and therefore an inductive argument allows us to assume that $\theta_{i_r} \not\equiv 1$ for the remainder of the proof. A repeated application of Proposition 2.4 shows that we can find vectors

$$w_1 \in \mathcal{K}_1, w_2 \in \mathcal{K}_2, \dots, w_r \in \mathcal{K}_r = \mathcal{K},$$

such that $\{w_1, w_2, \dots, w_x\}$ is a C_0-basis for $T|\mathcal{K}_x$, $x = 1, 2, \dots, r$. Moreover, we may assume that these vectors are of the form

$$w_x = \sum_{j=1}^{i_x} u_{x,j}(T)h_j, \quad x = 1, 2, \dots, r, \tag{3.3}$$

where all the $u_{x,j} \in H^\infty$. This can be seen as follows. We have

$$\mathcal{K}_x \subset [P_{\mathcal{H}(\Theta)}(\mathcal{E}_{i_x} \cap (H^2 \otimes \mathbb{C}^N))]^-,$$

and according to Lemma 3.3, the space on the right is the invariant subspace for T generated by $\{h_1, h_2, \dots, h_{i_x}\}$. Denote by $\mathcal{H}_i$ the cyclic subspace for T generated by h_i, and let

$$X\colon \bigoplus_{j=1}^{i_x} \mathcal{H}_j \to [P_{\mathcal{H}(\Theta)}(\mathcal{E}_{i_x} \cap (H^2 \otimes \mathbb{C}^N))]^-$$

be defined by

$$X\Big[\bigoplus_{j=1}^{i_x} v_j\Big] = \sum_{j=1}^{i_x} v_j, \quad v_j \in \mathcal{H}_j, j = 1, 2, \dots, i_x.$$

An application of Proposition 2.10 shows that the vector w_x can be chosen to belong to the range of X, thus $w_x = \sum_{j=1}^{i_x} v_j$ for some vectors $v_j \in \mathcal{H}_j$, $j = 1, 2, \dots, i_x$. Lemma 2.15 provides functions $\{\alpha_j, \beta_j\}_{j=1}^{i_x} \subset H^\infty$ such that $\alpha_j \wedge \theta_1 \equiv 1$ and $\alpha_j(T)v_j = \beta_j(T)h_j$ for $j = 1, 2, \dots, i_x$. The vector w_x can then be replaced by $\alpha(T)w_x$, where $\alpha = \alpha_1\alpha_2 \cdots \alpha_{i_x}$, and this vector satisfies (3.3) with $u_{x,j} = \alpha\beta_j$, $j = 1, 2, \dots, i_x$. For convenience, we write $u_{x,j} = 0$ if $j = i_x + 1, \dots, N$. Observe also that the condition $\theta_{i_x}(T)w_x = 0$ is equivalent to

$$\sum_{j=1}^{i_x} \theta_{i_x}(T)u_{x,j}(T)h_j = 0$$

and the fact that $\{h_j\}_{j=1}^N$ is a C_0-basis for T implies that $\theta_j|\theta_{i_x}u_{x,j}$ for $j = 1, 2, \ldots, i_x$. In other words,

$$\frac{\theta_j}{\theta_{i_x}}|u_{x,j}, \quad j = 1, 2, \ldots, i_x, x = 1, 2, \ldots, r.$$

At this point we need to make a reduction to a special case, namely, we can assume that the collection consisting of all inner functions of the form

$$\theta_j \wedge \det \begin{bmatrix} u_{1,j_1} & u_{1,j_2} & \cdots & u_{1,j_s} \\ u_{2,j_1} & u_{2,j_2} & \cdots & u_{2,j_s} \\ \vdots & \vdots & \ddots & \vdots \\ u_{s,j_1} & u_{s,j_2} & \cdots & u_{s,j_s} \end{bmatrix},$$

where $1 \le s \le r$, $1 \le j \le N$, and $1 \le j_1 < j_2 < \cdots < j_s \le N$ and of the inner functions

$$\theta_j/\theta_{j+1}, \quad n = 1, 2, \ldots, N-1,$$

is totally ordered by divisibility and, for every nonconstant inner factor ω of θ_1 we have $\omega \wedge \theta_j \not\equiv 1$, $j = 1, 2, \ldots, N-1$, unless θ_j itself is constant. This is accomplished as follows. Use Lemma 2.8 to find a decomposition of $\theta_1 = m_T$ into a product

$$\theta_1 = \gamma_1\gamma_2\cdots\gamma_n$$

of relatively prime inner factors with the property that the collection consisting of all inner functions of the form

$$\gamma_\ell \wedge \theta_j \wedge \det \begin{bmatrix} u_{1,j_1} & u_{1,j_2} & \cdots & u_{1,j_s} \\ u_{2,j_1} & u_{2,j_2} & \cdots & u_{2,j_s} \\ \vdots & \vdots & \ddots & \vdots \\ u_{s,j_1} & u_{s,j_2} & \cdots & u_{s,j_s} \end{bmatrix},$$

where $1 \le s \le r$, $1 \le j \le N$, and $1 \le j_1 < j_2 < \cdots < j_s \le N$ and of the inner functions

$$\gamma_\ell \wedge (\theta_j/\theta_{j+1}) \quad , j = 1, 2, \ldots, N-1,$$

is totally ordered by divisibility, and such that condition (2) of that lemma is satisfied as well. Setting $\Gamma_\ell = \theta_1/\gamma_\ell$, Proposition 2.7 shows that it suffices to show that $[\Gamma_\ell(T)\mathcal{R}]^-$ has an invariant quasidirect complement in $[\Gamma_\ell(T)\mathcal{H}(\Theta)]^-$ for $\ell = 1, 2, \ldots, n$. In order to do this, we replace $T|[\Gamma_\ell(T)\mathcal{H}(\Theta)]^-$ by a quasisimilar operator as follows. Define $A_\ell\colon H^2 \otimes \mathbb{C}^N \to [\Gamma_\ell(T)\mathcal{H}(\Theta)]^-$ by setting

$$A_\ell h = P_{\mathcal{H}(\Theta)}(\Gamma_\ell h).$$

We have $TA_\ell = A_\ell(S\otimes I_{\mathbb{C}^N})$, so that the injective operator $B_\ell|[(H^2\otimes\mathbb{C}^N)\ominus\ker(A_\ell)]$ satisfies

$$TB_\ell = B_\ell T_\ell,$$

where T_ℓ is the compression of $S\otimes I_{\mathbb{C}^N}$ to $\mathcal{H}_\ell = (H^2\otimes\mathbb{C}^N)\ominus\ker(A_\ell)$. It follows immediately that T_ℓ is an operator of class C_0 and, since the range of B_ℓ is dense in $[\Gamma_\ell(T)\mathcal{H}(\Theta)]^-$, T_ℓ is quasisimilar to $T|[\Gamma_\ell(T)\mathcal{H}(\Theta)]^-$. The vectors $P_{\mathcal{H}_\ell}h_1, P_{\mathcal{H}_\ell}h_2, \ldots, P_{\mathcal{H}_\ell}h_N$ form a C_0-basis for T_ℓ because the vectors

$$B_\ell P_{\mathcal{H}_\ell}h_j = \Gamma_\ell(T)h_j$$

form a C_0-basis for $T|[\Gamma_\ell(T)\mathcal{H}(\Theta)]^-$. Define next a subspace $\mathcal{K}^\ell\subset\mathcal{H}_\ell$ as the invariant subspace for T_ℓ generated by $P_{\mathcal{H}_\ell}w_x$, $x=1,2,\ldots,r$. Then $[B_\ell\mathcal{K}^\ell]^- = [\Gamma_\ell(T)\mathcal{K}]^-$ and the vectors $P_{\mathcal{H}_\ell}w_x$, $x=1,2,\ldots,r$, form a C_0-basis for $T_\ell|\mathcal{K}^\ell$. We see that it suffices to show that $\mathcal{K}^\ell$ has an invariant quasi-complement in $\mathcal{H}_\ell$. The spaces $\mathcal{K}^\ell$ and $\mathcal{H}_\ell$ are entirely analogous to the original spaces $\mathcal{K}$ and $\mathcal{H}$, and they have the additional divisibility properties outlined above. Of course, $\theta_i\wedge\gamma_\ell\equiv\theta_i^{T_\ell}$. Thus it suffices to prove (3) under these additional divisibility conditions. This is accomplished by showing that the vectors $\{w_x\}_{x=1}^r$ and a set $F\subset\{h_j\}_{j=1}^N$ of cardinality $N-r$ form a C_0-basis for T. The quasidirect complement of $\mathcal{K}$ is then the invariant subspace for T generated by F. Divide the functions $\{\theta_{i_x}\}_{x=1}^r$ into equivalence classes, that is, find integers $0=r_0<r_1<r_2<\cdots<r_p=r$ such that $\theta_{i_x}\equiv\theta_{i_{x+1}}$ when $r_{k-1}<x\le r_k$ and $1\le i\le p$, but $\theta_{i_{x+1}}/\theta_{i_x}$ is not constant if $x=r_k$, $1\le k<p$. We apply Lemma 3.2 to the $\mathfrak{D}$-vector space generated by $\{w_x\}_{x=r_{k-1}+1}^{r_k}$ to obtain indices $\{j_m\}_{m=r_{k-1}+1}^{r_k}$ with the property that $\theta_{j_m}\equiv\theta_{i_{r_k}}$ for $r_{k-1}+1\le m\le r_k$, and the function

$$\det[u_{\ell,j_k}]_{\ell,k=r_{i-1}+1}^{r_i}$$

is relatively prime to $\theta_{i_{r_k}}$, and therefore to θ_1 because of the divisibility assumptions we made about $\theta_1,\theta_2,\ldots,\theta_N$. The proof of (3) is now completed by showing that the set

$$A=\{w_\ell\}_{\ell=1}^r\cup\{h_j : j\notin\{j_1,j_2,\ldots,j_r\}\}$$

is a C_0-basis for T. In other words, this basis is obtained by replacing each h_{j_ℓ} by the corresponding vector w_ℓ. It is clear however that the set A is obtained from the C_0-basis $\{h_j\}_{j=1}^N$ by the process described in Corollary 2.6. This concludes the proof of (3) and of the proposition. ■

The preceding proposition shows that, for certain flags $\mathcal{E}$ and for spaces $\mathcal{Q}\in\mathfrak{S}(\mathcal{E},I)$, the invariant subspace $\mathcal{M}=[P_{\mathcal{H}(\Theta)}(\mathcal{Q}\cap(H^2\otimes\mathbb{C}^N))]^-$ has the property that $T|\mathcal{M}$ has a 'large' Jordan model. Next we produce flags $\mathcal{E}$ with the property that the Jordan models of operators of the form $T|\mathcal{M}$ are 'small' if $\mathcal{Q} \in \mathfrak{S}(\mathcal{E}, I)$. Some preparation is needed first.

Lemma 3.5. *Consider a factorization*

$$\Theta(\lambda) = \Theta''(\lambda)\Theta'(\lambda), \quad \lambda \in \mathbb{D},$$

where Θ' and Θ'' are inner functions. Write

$$\mathcal{H}(\Theta) = \mathcal{H}' \oplus \mathcal{H}'',$$

where

$$\mathcal{H}' = M_{\Theta''}[H^2 \otimes \mathbb{C}^N] \ominus M_\Theta[H^2 \otimes \mathbb{C}^N] = M_{\Theta''}\mathcal{H}(\Theta')$$

is T-invariant and $\mathcal{H}'' = \mathcal{H}(\Theta'')$. Let $\mathcal{Q} \subset \mathfrak{D} \otimes \mathbb{C}^N$ be a $\mathfrak{D}$-vector space, and define invariant subspaces

$$\mathcal{M} = [P_{\mathcal{H}(\Theta)}(\mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N))]^-,$$
$$\mathcal{M}' = [P_{\mathcal{H}'}(\mathcal{Q} \cap M_{\Theta''}[H^2 \otimes \mathbb{C}^N])]^-,$$
$$\mathcal{M}'' = [P_{\mathcal{H}''}(\mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N))]^-,$$

for T, $T' = T|\mathcal{H}'$, and $T'' = P_{\mathcal{H}''}T|\mathcal{H}''$, respectively. Then we have

$$\mathcal{M}'' = [P_{\mathcal{H}''}\mathcal{M}]^- \text{ and } \mathcal{M}' = \mathcal{M} \cap \mathcal{H}'.$$

Proof. The first equality and the inclusion

$$\mathcal{M}' \subset \mathcal{M} \cap \mathcal{H}'$$

follow directly from the definitions of $\mathcal{M}, \mathcal{M}'$, and $\mathcal{M}''$. For the opposite inclusion, note that the set of vectors $h' \in \mathcal{M} \cap \mathcal{H}'$ of the form $P_{\mathcal{H}(\Theta)}q$ for some $q \in \mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N)$ is dense in $\mathcal{M} \cap \mathcal{H}'$ by Proposition 2.10. Consider such a vector $h' = P_{\mathcal{H}(\Theta)}q$, and observe that $q - h' \in M_\Theta[H^2 \otimes \mathbb{C}^N]$ so that

$$q \in \mathcal{H}' + M_\Theta[H^2 \otimes \mathbb{C}^N] \subset M_{\Theta''}[H^2 \otimes \mathbb{C}^N].$$

It follows that $h' \in \mathcal{M}'$, and this concludes the proof. ∎

The preceding lemma is applied in the proofs of Proposition 3.7 and Theorem 4.1. For the first application, we recall that any inner multiple ω of θ_1 is a scalar multiple of Θ, that is, there exists an inner function Ω such that

$$\Theta(\lambda)\Omega(\lambda) = \omega(\lambda)I_{\mathbb{C}^N}, \quad \lambda \in \mathbb{D}. \tag{3.4}$$

Of course, the operator $S(\omega I_{\mathbb{C}^N})$ is unitarily equivalent to the orthogonal sum of N copies of $S(\omega)$.

Lemma 3.6. *Let $\omega \in H^\infty$ be an inner multiple of θ_1. Then there exist bounded vectors $y_1, y_2, \ldots, y_N \in H^2 \otimes \mathbb{C}^N$ such that:*

(1) $P_{\mathcal{H}(\omega I_{\mathbb{C}^N})}y_1, P_{\mathcal{H}(\omega I_{\mathbb{C}^N})}y_2, \ldots, P_{\mathcal{H}(\omega I_{\mathbb{C}^N})}y_N$ *form a C_0-basis for $S(\omega) \otimes I_{\mathbb{C}^N}$,*

(2) $P_{\mathcal{H}(\Theta)}y_1, P_{\mathcal{H}(\Theta)}y_2, \ldots, P_{\mathcal{H}(\Theta)}y_N$ *form a C_0-basis for $T = S(\Theta)$, and*

(3) $P_{\mathcal{H}(\omega I_{\mathbb{C}^N})}(\theta_N y_N), P_{\mathcal{H}(\omega I_{\mathbb{C}^N})}(\theta_{N-1}y_{N-1}), \ldots, P_{\mathcal{H}(\omega I_{\mathbb{C}^N})}(\theta_1 y_1)$ *form a C_0-basis for the restriction of $S \otimes I_{\mathbb{C}^N}$ to $M_\Theta[H^2 \otimes \mathbb{C}^N] \ominus [\omega H^2 \otimes \mathbb{C}^N]$.*

Proof. Denote by Θ' the diagonal matrix with diagonal entries $\theta_1, \ldots, \theta_N$. As noted earlier, Θ and Θ' are quasiequivalent. Choose $N \times N$ matrices X, Y over H^∞ such that $\Theta X = Y\Theta'$ and $\det(X) \wedge \omega \equiv \det(Y) \wedge \omega \equiv 1$, and let $y_j = Y(1 \otimes e_j)$, $j = 1, \ldots, N$, be the columns of Y. We claim that these vectors satisfy the conclusion of the lemma. Indeed, (2) follows from Lemma 2.9 while (1) follows from the same lemma because $(\omega \otimes I_{\mathbb{C}^N})Y = Y(\omega \otimes I_{\mathbb{C}^N})$. Finally, we observe that the vectors $P_{\mathcal{H}(\omega I_{\mathbb{C}^N})}(\theta_j y_j)$ belong to $M_\Theta[H^2 \otimes \mathbb{C}^N] \ominus [\omega H^2 \otimes \mathbb{C}^N]$ because $\theta_j y_j = \Theta X(1 \otimes e_j)$ for $j = 1, \ldots, N$. Moreover, these vectors form a C_0-basis for the $S(\omega) \otimes I_{\mathbb{C}^N}$-invariant subspace $\mathcal{M}$ they generate. To conclude the proof of (3) we need to show that $\mathcal{M} = M_\Theta[H^2 \otimes \mathbb{C}^N] \ominus [\omega H^2 \otimes \mathbb{C}^N]$ and for this purpose it suffices to verify that the restrictions of $S(\omega) \otimes I_{\mathbb{C}^N}$ to these two subspaces have the same Jordan model. This follows from the main result of [4]. ∎

For our result on 'small' invariant subspaces, we fix an inner multiple $\omega \in H^\infty$ of θ_1^2, a sequence $\{y_n\}_{n=1}^N$ satisfying the conclusion of Lemma 3.6, and denote by $\mathcal{F}$ the complete flag in $\mathfrak{D} \otimes \mathbb{C}^N$ defined by letting $\mathcal{F}_n$ be the space generated by $\{y_N, y_{N-1}, \ldots, y_{N-n+1}\}$ for $n = 1, 2, \ldots, N$. The choice of ω insures that all the functions ω/θ_n are divisible by θ_1.

Proposition 3.7. *Fix a positive integer $r \le N$, a subset*

$$I = \{i_1 < i_2 < \cdots < i_r\} \subset \{1, 2, \ldots, N\},$$

and a subspace $\mathcal{Q} \in \mathfrak{S}(\mathcal{F}, I)$. Set $\mathcal{R} = \mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N)$ and $\mathcal{K} = [P_{\mathcal{H}(\Theta)}\mathcal{R}]^-$. Then:

(1) *The space $\mathcal{K}$ is invariant for $T = S(\Theta)$, and $T|\mathcal{K}$ has cyclic multiplicity less than or equal to r.*

(2) *If the Jordan model of $T|\mathcal{K}$ is $S(\alpha_1) \oplus S(\alpha_2) \oplus \cdots \oplus S(\alpha_r)$, then α_x divides $\theta_{N+1-i_{r+1-x}}$ for $x = 1, 2, \ldots, r$.*

(3) *If $\alpha_x \equiv \theta_{N+1-i_{r+1-x}}$ for all $x = 1, 2, \ldots, r$, then $\mathcal{K}$ has a T-invariant quasidirect complement $\mathcal{L}$ in $\mathcal{H}(\Theta)$ such that $T|\mathcal{L} \sim \bigoplus_{N+1-i \notin I} S(\theta_i)$.*

Proof. Using (3.4) and the notation in Lemma 3.5, with $\omega I_{\mathbb{C}^N}, \Omega, \Theta$ playing the roles of $\Theta, \Theta', \Theta''$, respectively, we have $\mathcal{K} = \mathcal{M}''$. Continuing with that notation, we claim that the compression of $S \otimes I_{\mathbb{C}^N}$ to the space $\mathcal{M}$ has Jordan model

$$\underbrace{S(\omega) \oplus S(\omega) \oplus \cdots \oplus S(\omega)}_{r \text{ times}}.$$

Indeed, apply Proposition 3.4 with $S(\omega) \otimes I_{\mathbb{C}^N}$ in place of T to deduce that this Jordan model has the form $S(\gamma_1) \oplus S(\gamma_2) \oplus \cdots \oplus S(\gamma_r)$ where all the functions γ_j are divisible by ω. However, these functions must also divide the minimal function of $S(\omega) \otimes I_{\mathbb{C}^N}$ which is ω, and therefore $\gamma_j \equiv \omega$ for $j = 1, 2, \ldots, r$.

Next we observe that

$$T' = S(\omega) \otimes I_{\mathbb{C}^N} | [\mathcal{H}(\omega) \otimes \mathbb{C}^N] \ominus \mathcal{H}(\Theta)$$

is an operator of class C_0 with Jordan model

$$S(\omega/\theta_N) \oplus S(\omega/\theta_{N-1}) \oplus \cdots \oplus S(\omega/\theta_1),$$

where the summands are written in this order so that this is a Jordan operator. Indeed, this follows from [4]. We can now apply Proposition 3.4 with T' and $(S \otimes I_{\mathbb{C}^N})|M_\Theta[H^2 \otimes \mathbb{C}^N]$ in place of T and $S \otimes I_{\mathbb{C}^N}$ to deduce that the Jordan model of $T'|\mathcal{M}'$ is of the form $S(\beta_1) \oplus S(\beta_2) \oplus \cdots \oplus S(\beta_r)$ with the property that ω/θ_{N+1-i_x} divides β_x for $x = 1, 2, \ldots, r$. It follows from [4] that the Jordan model of the compression of T to $\mathcal{M} \ominus \mathcal{M}'$ has Jordan model $S(\alpha_1) \oplus S(\alpha_2) \oplus \cdots \oplus S(\alpha_r)$, where $\alpha_{r+1-x}\beta_x = \omega$ for $x = 1, 2, \ldots, r$. We deduce that $\alpha_{r+1-x} = \omega/\beta_x$ divides

$$\frac{\omega}{\omega/\theta_{N+1-i_x}} = \theta_{N+1-i_x}$$

or, equivalently, $\alpha_x | \theta_{N+1-i_{r+1-x}}$ for $x = 1, 2, \ldots, r$. Parts (1) and (2) of the statement follow now once we prove that $T|\mathcal{M}'' = T|\mathcal{K}$ is quasisimilar to the compression of T to $\mathcal{M} \ominus \mathcal{M}'$. In fact, an operator X which is one-to-one with dense range and intertwines these two operators is obtained by setting $X = P_{\mathcal{H}(\Theta)}|\mathcal{M} \ominus \mathcal{M}'$. The claimed properties of X follow readily from Lemma 3.5.

Assume finally that $\alpha_x \equiv \theta_{N+1-i_{r+1-x}}$ for $x = 1, 2, \ldots, r$. We have then $\beta_x \equiv \omega/\theta_{N+1-x}$ for $x = 1, 2, \ldots, r$. The proof of Proposition 3.4(3) can now be applied with T' in place of T and $M_\Theta[H^2 \otimes \mathbb{C}^N]$ in place of $H^2 \otimes \mathbb{C}^N$. Following that argument, and recalling that $P_{\mathcal{H}'}(\theta_{N+1-j} y_{N+1-j})$ form a C_0-basis for T', we first produce a C_0-basis $w_1, w_2, \ldots, w_r$ for $T'|\mathcal{M}'$ such that

$$w_x = P_{\mathcal{H}'} \sum_{j=1}^{N} u_{x,j} \theta_j y_j, \quad x = 1, 2, \ldots, r,$$

with $\sum_{j=1}^{N} u_{x,j}\theta_j z_j \in \mathcal{Q}$, coefficients $u_{x,j} \in H^\infty$ such that $u_{x,j} = 0$ for $j = 1, 2, \ldots, N+1-i_x$ and $\theta_{N+1-i_x}/\theta_j$ divides $u_{x,j}$ for $j \geq N - i_x$. We now apply a reduction which allows us to assume that a family of inner functions is totally ordered by divisibility. We partition I into subsets $I_1, I_2, \ldots, I_p$ with the property that $\theta_{N+1-i} \equiv \theta_{N+1-i'}$ if and only if i and i' belong to the same I_k, and apply the arguments in the proof of Proposition 3.4 to find minors of the form

$$\det[u_{x,j}]_{N+1-x\in I_k, N+1-j\in J_k}$$

which are relatively prime to θ_1. Here J_k and I_k have the same cardinality, and $\theta_{N+1-j} \equiv \theta_{N+1-i}$ for $i \in I_k$ and $j \in J_k$. The functions

$$v_{x,j} = \frac{u_{x,j}\theta_j}{\theta_{N+1-i_x}}$$

belong again to H^∞, and thus we can define vectors $\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_r \in \mathcal{H}(\Theta)$ by setting

$$\widetilde{w}_x = P_{\mathcal{H}(\Theta)} \sum_{j=1}^{N} v_{x,j} y_j, \quad x = 1, 2, \ldots, r.$$

Since $\sum_{j=1}^{N} v_{x,j} y_j \in \mathcal{Q}$, these vectors actually belong to $\mathcal{M}'' = \mathcal{K}$, and they form a C_0-basis for $T|\mathcal{K}$. The argument is now concluded by observing that the set

$$\{\widetilde{w}_x\}_{x=1}^{r} \cup \left\{P_{\mathcal{H}(\Theta)} y_j : j \in \{1, 2, \ldots, N\} \setminus \bigcup_{q=1}^{p} J_q\right\}$$

is a C_0-basis for T, so that an invariant quasidirect complement for $\mathcal{K}$ is generated by the vectors

$$\left\{P_{\mathcal{H}(\Theta)} y_j : j \in \{1, 2, \ldots, N\} \setminus \bigcup_{q=1}^{p} J_q\right\}. \qquad \blacksquare$$

4. The Horn inequalities

In this section we consider an integer N and three sets $I, J, K \subset \{1, 2, \ldots, N\}$, each containing $r \leq N$ elements, such that $c_{I\widetilde{J}\widetilde{K}} = 1$, where

$$\widetilde{J} = \{N+1-j : j \in J\}, \quad \widetilde{K} = \{N+1-k : k \in K\}.$$

The Horn inequalities associated with such triples of sets are sufficient to imply all the Horn inequalities associated with sets I, J, K such that $c_{I\widetilde{J}\widetilde{K}} > 0$ (see [1] or [17]). Our main result is as follows.

Theorem 4.1. *Assume that T is an operator of class C_0 on $\mathcal{H}$, $\mathcal{H}'$ is an invariant subspace for T, $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$, and*

$$T = \begin{bmatrix} T' & * \\ 0 & T'' \end{bmatrix}$$

is the matrix of T corresponding to the orthogonal decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$. Let

$$\bigoplus_{1 \le n < \aleph} S(\theta_n), \quad \bigoplus_{1 \le n < \aleph} S(\theta'_n), \quad \bigoplus_{1 \le n < \aleph} S(\theta''_n)$$

be the Jordan models of T, T', and T'', respectively. Let $I, J, K \subset \{1, 2, \ldots, N\}$ be sets of cardinality $r \le N$ such that $c_{I\tilde{J}\tilde{K}} = 1$. Then there exists an invariant subspace $\mathcal{M}$ for T with the following properties.

(1) *The cyclic mutiplicity of $T|\mathcal{M}$ is at most r, and its Jordan model $\bigoplus_{x=1}^r S(\beta_x)$ is such that $\theta_{i_x} | \beta_x$ for $x = 1, 2, \ldots, r$.*
(2) *The Jordan model $\bigoplus_{x=1}^r S(\beta'_x)$ of $T|\mathcal{M}'$, where $\mathcal{M}' = \mathcal{M} \cap \mathcal{H}'$ is such that $\beta'_x | \theta'_{j_x}$ for $x = 1, 2, \ldots, r$.*
(3) *The Jordan model $\bigoplus_{x=1}^r S(\beta''_x)$ of $T|\mathcal{M}''$, where $\mathcal{M}'' = \overline{P_{\mathcal{H}''}\mathcal{M}}$, is such that $\beta''_x | \theta''_{k_x}$ for $x = 1, 2, \ldots, r$.*
(4) $\prod_{x=1}^r \beta_x = \prod_{x=1}^r (\beta'_x \beta''_x)$.

We conclude that

$$\prod_{i \in I} \theta_i \Big| \prod_{j \in J} \theta'_j \prod_{k \in K} \theta''_k.$$

If $\prod_{i \in I} \theta_i \equiv \prod_{j \in J} \theta'_j \prod_{k \in K} \theta''_k$ then $\mathcal{M}$ (respectively, $\mathcal{M}', \mathcal{M}''$) has a T-invariant (respectively, T'-invariant, T''-invariant) quasidirect complement in $\mathcal{H}$ (respectively, $\mathcal{H}, \mathcal{H}''$).

Proof. The key case to consider is that in which T has cyclic multiplicity at most equal to N. In this case we can replace T by any operator of class C_0 which is quasisimilar to it. Indeed, quasisimilarity between operators of class C_0 with finite multiplicity allows one to identify their lattices of invariant subspaces (see [3, Proposition VII.1.21]). We can then assume that $T = S(\Theta)$, where Θ is an $N \times N$ inner function, $\mathcal{H} = \mathcal{H}(\Theta)$, $\mathcal{H}'' = \mathcal{H}(\Theta'')$, and $\mathcal{H}' = M_{\Theta''}\mathcal{H}(\Theta')$ for some inner factorization $\Theta = \Theta''\Theta'$. Propositions 3.4 and 3.7 allow us to choose three complete flags $\mathcal{E}, \mathcal{F}, \mathcal{G}$ in $\mathfrak{D}^N = \mathfrak{D} \otimes \mathbb{C}^N$ such that, given a space $\mathcal{Q} \in \mathfrak{S}(\mathcal{E}, I)$ (respectively, $\mathfrak{S}(\mathcal{F}, \tilde{J}), \mathfrak{S}(\mathcal{G}, \tilde{K})$) the restriction of T (respectively, T', T'') to $[P_{\mathcal{H}}(\mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N))]^-$ (respectively, $[P_{\mathcal{H}'}(\mathcal{Q} \cap M_{\Theta''}(H^2 \otimes \mathbb{C}^N))]^-, [P_{\mathcal{H}''}(\mathcal{Q} \cap (H^2 \otimes \mathbb{C}^N))]^-$) has Jordan model $\bigoplus_{x=1}^r S(\beta_x)$ (respectively, $\bigoplus_{x=1}^r S(\beta'_x), \bigoplus_{x=1}^r S(\beta''_x)$) satisfying $\theta_{i_x} | \beta_x$ (respectively, $\beta'_x | \theta'_{j_x}, \beta''_x | \theta''_{k_x}$) for $x = 1, 2, \ldots, r$. The assumption that $c_{I\tilde{J}\tilde{K}} = 1$ implies now that we can find a space $\mathcal{Q}$ in the intersection

$$\mathfrak{S}(\mathcal{E},I)\cap\mathfrak{S}(\mathcal{F},\widetilde{J})\cap S(\mathcal{G},\widetilde{K}),$$

and this yields immediately statements (1), (2), and (3). Statement (4) follows from [3, Theorem VI.3.16] and the fact that $T''|\mathcal{M}''$ is quasisimilar to the compression of $T|\mathcal{M}$ to the space $\mathcal{M}\ominus\mathcal{M}'$. If $\prod_{i\in I}\theta_i\equiv\prod_{j\in J}\theta'_j\prod_{k\in K}\theta''_k$ then, of course, $\beta_x\equiv\theta_{i_x}$, $\beta'_x\equiv\theta'_{j_x}$, and $\beta''_x\equiv\theta''_x$ for $x=1,2,\ldots,r$, and the existence of invariant quasidirect complements follows from part (3) of Propositions 3.4 and 3.7.

We consider next an operator T of class C_0 with finite multiplicity $N'>N$. This case reduces to the previous one as follows. The sets I,J,K are also contained in $\{1,2,\ldots,N'\}$, and setting

$$\overline{J}=\{N'+1-j:j\in J\},\quad \overline{K}=\{N'+1-k:k\in K\},$$

we still have $c_{I\overline{J}\,\overline{K}}=1$. (This fact is verified using the Littlewood–Richardson rule for partitions. More generally, $c_{I\overline{J}\,\overline{K}}=c_{I\widetilde{J}\widetilde{K}}$, see [11,12].) Therefore the preceding argument works simply replacing N by N'.

Finally, assume that T has infinite multiplicity, and consider quasidirect decompositions

$$\mathcal{H}=\bigvee_{1\le n<\aleph}\mathcal{H}_n,\quad \mathcal{H}'=\bigvee_{1\le n<\aleph}\mathcal{H}'_n,\quad \mathcal{H}''=\bigvee_{1\le n<\aleph}\mathcal{H}''_n,$$

into invariant spaces for T,T',T'', respectively, such that $T|\mathcal{H}_n\sim S(\theta_n)$, $T'|\mathcal{H}'_n\sim S(\theta'_n)$, and $T''|\mathcal{H}''_n\sim S(\theta''_n)$ for all $n<\aleph$. The invariant subspace $\widetilde{\mathcal{H}}$ for T generated by the spaces $\mathcal{H}_n,\mathcal{H}'_n,\mathcal{H}''_n$ for $1\le n\le N$ has the property that $T|\widetilde{\mathcal{H}}$ has finite multiplicity. We can therefore apply the theorem, already proved for the case of operators with finite multiplicity, with $T|\widetilde{\mathcal{H}}$ in place of T, $\widetilde{\mathcal{H}}'=\mathcal{H}'\cap\widetilde{\mathcal{H}}$ in place of $\mathcal{H}'$, and $\widetilde{\mathcal{H}}''=\widetilde{\mathcal{H}}\ominus\widetilde{\mathcal{H}}'$ in place of $\mathcal{H}''$. We obtain a $T|\widetilde{\mathcal{H}}$-invariant subspace $\mathcal{M}\subset\widetilde{\mathcal{H}}$ which satisfies requirements (1–4) of the theorem. Indeed, the first N functions in the Jordan model of $T|\widetilde{\mathcal{H}}$ are still $\theta_1,\theta_2,\ldots,\theta_N$, and similar observations hold for $T|\widetilde{\mathcal{H}}'$ and the compression of T to $\widetilde{\mathcal{H}}''$. It remains to verify the final assertion of the theorem. Assume therefore that $\prod_{i\in I}\theta_i\equiv\prod_{j\in J}\theta'_j\prod_{k\in K}\theta''_k$. We already know that $\mathcal{M}$ has a $T|\widetilde{\mathcal{H}}$-invariant quasidirect complement in $\widetilde{\mathcal{H}}$. Setting $N'=\mu_{T|\widetilde{\mathcal{H}}}$, there exists a quasidirect decomposition

$$\widetilde{\mathcal{H}}=\bigvee_{n=1}^{N'}\widetilde{\mathcal{H}}_n$$

into invariant subspaces for T such that $T|\widetilde{\mathcal{H}}_n\sim S(\theta_n)$ for $n=1,2,\ldots,N$, and

$$\mathcal{M}=\bigvee_{i\in I}\widetilde{\mathcal{H}}_i.$$

Proposition 2.4 allows us to construct an invariant subspace $\mathcal{L}$ for T such that

$$\mathcal{H} = \mathcal{L} \vee \bigvee_{n=1}^{N} \widetilde{\mathcal{H}}_n, \quad \mathcal{L} \cap \Big[\bigvee_{n=1}^{N} \widetilde{\mathcal{H}}_n \Big] = \{0\}.$$

An invariant quasidirect complement for $\mathcal{M}$ is then given by

$$\mathcal{L} \vee \bigvee_{n \notin I} \widetilde{\mathcal{H}}_n.$$

Similar arguments show the existence of invariant quasidirect complements for $\mathcal{M}'$ and $\mathcal{M}''$. ∎

5. Operators on nonseparable spaces

With the notation of Theorem 4.1, we show that $\theta_\beta | \theta'_\beta \theta''_\beta$ for $\beta \geq \aleph_0$. This relation can also be established by exhibiting an appropriate invariant subspace for T, but in this case the subspace can be chosen to be reducing for T, as well as for the invariant subspace $\mathcal{H}'$.

Theorem 5.1. *Assume that T is an operator of class C_0 on $\mathcal{H}$, $\mathcal{H}'$ is an invariant subspace for T, $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$, and*

$$T = \begin{bmatrix} T' & * \\ 0 & T'' \end{bmatrix}$$

is the matrix of T corresponding to the orthogonal decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$. Let

$$\bigoplus_{1 \leq n < \aleph} S(\theta_n), \quad \bigoplus_{1 \leq n < \aleph} S(\theta'_n), \quad \bigoplus_{1 \leq n < \aleph} S(\theta''_n)$$

be the Jordan models of T, T', and T'', respectively. Fix an ordinal $\beta \geq \aleph_0$. There exist separable reducing spaces $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ for T, T', T'', respectively, such that $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ and the Jordan models of $T|\mathcal{M}$, $T'|\mathcal{M}'$, and $T''|\mathcal{M}''$ are orthogonal sums of countably many copies of $S(\theta_\beta)$, $S(\theta'_\beta)$, and $S(\theta''_\beta)$, respectively. In particular, $\theta_\beta | \theta'_\beta \theta''_\beta$.

Proof. We may assume without loss of generality that $\theta_\beta \not\equiv 1$. Consider quasidirect decompositions

$$\mathcal{H} = \bigvee_{1 \leq n < \aleph} \mathcal{H}_n, \quad \mathcal{H}' = \bigvee_{1 \leq n < \aleph} \mathcal{H}'_n, \quad \mathcal{H}'' = \bigvee_{1 \leq n < \aleph} \mathcal{H}''_n,$$

into invariant spaces for T, T', T'', respectively, such that $T|\mathcal{H}_n \sim S(\theta_n)$, $T'|\mathcal{H}'_n \sim S(\theta'_n)$, and $T''|\mathcal{H}''_n \sim S(\theta''_n)$ for all $n < \aleph$. Denote by $\beth$ the cardinality of β, and let $\mathcal{K}$ be the smallest subspace of $\mathcal{H}$ which reduces both T and the orthogonal projection $P_{\mathcal{H}'}$ and contains all the spaces $\mathcal{H}_n, \mathcal{H}'_n, \mathcal{H}''_n$ for $n < \beth$. Since $\beth$ is transfinite and each $\mathcal{H}_n$ is separable, it follows that the space $\mathcal{K}$ has dimension at most $\beth$. Moreover, the first $\aleph_0$ inner functions in the Jordan model of $T|\mathcal{K}^\perp$ are equal to θ_β, and similar statements hold for $T'|\mathcal{H}' \cap \mathcal{K}^\perp$ and $T''|\mathcal{H}'' \cap \mathcal{K}^\perp$. Indeed, the minimal function of $T|\mathcal{K}^\perp$ divides θ_β because $P_{\mathcal{K}^\perp}[\bigvee_{\mathrm{card}(n)\geq\beth} \mathcal{H}_n]$ is dense in $\mathcal{K}^\perp$. On the other hand, the multiplicity of $T|[\varphi(T)\mathcal{K}^\perp]^-$ is at least $\beth$ if φ is an inner function which does not divide θ_β. Thus the Jordan model of $T|\mathcal{K}^\perp$ contains at least $\beth$ summands equal to $S(\theta_\beta)$. Choose invariant subspaces $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ for $T|\mathcal{K}^\perp$, $T'|\mathcal{H}' \cap \mathcal{K}^\perp$, $T''|\mathcal{H}'' \cap \mathcal{K}^\perp$, respectively, such that the Jordan models of the three restrictions are orthogonal sums of countably many copies of $S(\theta_\beta), S(\theta'_\beta)$, and $S(\theta''_\beta)$, respectively. The desired space $\mathcal{M}$ is obtained as the smallest subspace containing $\mathcal{L} \cup \mathcal{L}' \cup \mathcal{L}''$ which reduces both T and $P_{\mathcal{H}'}$. We then define $\mathcal{M}' = \mathcal{M} \cap \mathcal{H}'$ and $\mathcal{M}'' = \mathcal{M} \cap \mathcal{H}''$. The space $\mathcal{M}$ is separable, and the Jordan model $\bigoplus_{1\leq n<\aleph_0} S(\varphi_n)$ satisfies $\varphi_n|\theta_\beta$ (because θ_β is the minimal function of $T|\mathcal{K}^\perp$ and $\mathcal{K}^\perp \supset \mathcal{M}$) and $\theta_\beta|\varphi_n$ (because $\mathcal{M} \supset \mathcal{L}$). Thus $T|\mathcal{M}$ has the desired Jordan model. Similar arguments determine the Jordan models of $T'|\mathcal{M}'$ and $T''|\mathcal{M}''$. ■

6. Comments on the inverse problem

Let T be an operator of class C_0 on $\mathcal{H}$, let $\mathcal{H}'$ be an invariant subspace for T, set $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$, and let

$$T = \begin{bmatrix} T' & * \\ 0 & T'' \end{bmatrix}$$

be the matrix of T corresponding to the orthogonal decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$. Assume further that

$$J = \bigoplus_{1\leq n<\aleph} S(\theta_n), \quad J' = \bigoplus_{1\leq n<\aleph} S(\theta'_n), \quad J'' = \bigoplus_{1\leq n<\aleph} S(\theta''_n) \tag{6.1}$$

are the Jordan models of T, T', and T'', respectively. We have seen that the functions $\{\theta_n, \theta'_n, \theta''_n\}_{1\leq n<\aleph}$ are subject to a collection of 'inequalities' of the form $\prod_{i\in I}\theta_i | \prod_{j\in J}\theta'_j \prod_{k\in K}\theta''_k$ for some finite equipotent sets $I, J, K \subset \{1, 2, \dots\}$, and also that $\theta_n|\theta'_n\theta''_n$ if $n \geq \aleph_0$. Other necessary conditions are that $\theta'_n|\theta_n$ and $\theta''_n|\theta_n$ for $1 \leq n < \aleph$. A natural question is whether these conditions on J, J', J'' are sufficient for the existence of an operator T of class C_0 and of an invariant subspace

$\mathcal{H}'$ for T such that $T \sim J$, $T|\mathcal{H}' \sim J'$, and $P_{\mathcal{H}'^\perp}T|\mathcal{H}'^\perp \sim J''$. The answer is in the negative: if $\mu_T \leq N < \infty$, we have $\theta_n \equiv 1$ for $n > N$ and the models also satisfy the 'determinant' condition

$$\prod_{n=1}^{N} \theta_n \equiv \prod_{n=1}^{N} (\theta'_n \theta''_n).$$

In this special case (with $\theta_n \equiv 1$ if $n > N$), the relations outlined above are in fact sufficient for the existence of T and $\mathcal{H}'$ (see [18,19] and, for the algebraic case, [14] or [20]). An appropriate substitute for the determinant condition has not been found in the case of infinite multiplicity. We argue that, at least, the problem reduces to the separable case.

Proposition 6.1. *Let J, J', and J'' be Jordan operators given by* (6.1). *Assume that:*
(1) $\theta_n \equiv \theta_1$, $\theta'_n \equiv \theta'_1$, *and* $\theta''_n \equiv \theta''_1$ *for all* $n \leq \aleph_0$,
(2) $\theta'_n|\theta_n$ *and* $\theta''_n|\theta_n$ *for all* $n < \aleph$, *and*
(3) $\theta_n|\theta'_n\theta''_n$ *for all* $n < \aleph$.

Then there exists an operator T of class C_0 and an invariant subspace $\mathcal{H}'$ for T such that $T \sim J$, $T|\mathcal{H}' \sim J'$, and $P_{\mathcal{H}'^\perp}T|\mathcal{H}'^\perp \sim J''$.

Proof. Define $T = J$ on $\mathcal{H} = \bigoplus_{1 \leq n < \aleph} S(\theta_n)$ and $\mathcal{H}' = \bigoplus_{1 \leq n < \aleph} \mathcal{H}'_n$, where for every ordinal number written as $n = m + k$ with m a limit ordinal and $k < \aleph_0$ we set

$$\mathcal{H}'_n = \begin{cases} (\theta_n/\theta'_n)H^2 \ominus \theta_n H^2, & \text{if } k \text{ is even,} \\ (\theta_n/\theta''_n)H^2 \ominus \theta_n H^2, & \text{if } k \text{ is odd.} \end{cases}$$

It is easy to verify that these objects satisfy the requirements of the proposition. ∎

This proposition shows that there is no need for more elaborate conditions on the functions θ_n for $n \geq \aleph_0$. It also reduces the inverse problem, which remains open, to the separable case.

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