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FILIPPOV'S THEOREM FOR IMPULSIVE DIFFERENTIAL INCLUSIONS WITH FRACTIONAL ORDER

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

In this paper, we present an impulsive version of Filippov's Theorem for fractional differential inclusions of the form:

$$\begin{array}{rcll} D_*^\alpha y(t) & \in & F(t,y(t)), & \text{a.e. } t\in J\backslash\{t_1,\dots,t_m\},\ \alpha\in(1,2],\\ y(t_k^+)-y(t_k^-) & = & I_k(y(t_k^-)), & k=1,\dots,m,\\ y'(t_k^+)-y'(t_k^-) & = & \overline{I}_k(y'(t_k^-)), & k=1,\dots,m,\\ y(0) & = & a,\ y'(0)=c, & \end{array}$$

where $J=[0,b]$, D_*^α denotes the Caputo fractional derivative and F is a set-valued map. The functions I_k, $\overline{I}_k$ characterize the jump of the solutions at impulse points t_k $(k=1,\dots,m)$.

Key words and phrases: Fractional differential inclusions, fractional derivative, fractional integral.
AMS (MOS) Subject Classifications: 34A60, 34A37.

1 Introduction

Differential equations with impulses were considered for the first time in the 1960's by Milman and Myshkis [47, 46]. A period of active research, primarily in Eastern Europe from 1960-1970, culminated with the monograph by Halanay and Wexler [32].

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of "impulses". As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biotechnology, industrial robotics, pharmcokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include [7, 42, 53, 57]. There are also many different studies in biology and medicine for which impulsive differential equation are a good model (see, for example, [2, 39, 40] and the references therein).

In recent years, many examples of differential equations with impulses with fixed moments have flourished in several contexts. In the periodic treatment of some diseases, impulses correspond to administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs.

During the last ten years, impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied by many mathematicians. At present the foundations of the general theory are already laid, and many of them are investigated in detail in the books of Aubin [3] and Benchohra *et al* [9], and in the papers of Graef *et al* [26, 27, 31], Graef and Ouahab [28, 29, 30] and the references therein.

Differential equations with fractional order have recently proved valuable tools in the modeling of many physical phenomena [19, 23, 24, 43, 44]. There has been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas *et al* [36], Miller and Ross [45], Podlubny [54], Samko *et al* [56], and the papers of Bai and Lu [6], Diethelm *et al* [19, 18, 20], El-Sayed and Ibrahim [21], Kilbas and Trujillo [37], Mainardi [43], Momani and Hadid, [48], Momani *et al* [49], Nakhushev [50], Podlubny *et al* [55], and Yu and Gao [59].

Very recently, some basic theory for initial value problems for fractional differential equations and inclusions involving the Riemann-Liouville differential operator was discussed by Benchohra *et al* [10] and Lakshmikantham [41]. El-Sayed and Ibrahim [21] initiated the study of fractional multivalued differential inclusions.

Applied problems require definitions of fractional derivatives allowing a utilization that is physically interpretable for initial conditions containing $y(0)$, $y'(0)$, etc. The same requirements are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types, see Podlubny [54].

Recently fractional functional differential equations and inclusions with standard Riemann-Liouville and Caputo derivatives with difference conditions were studied by

Benchohra *et al* [8, 10, 11], Henderson and Ouahab [34], and Ouahab [52].

When $\alpha \in (1, 2]$, the impulsive differential equations with Captuo fractional derivatives were studied by Agarwal *et al* [1].

In this paper, we shall be concerned with Filippov's theorem and global existence of solutions for impulsive fractional differential inclusions with fractional order. More precisely, we will consider the following problem,

$$D_*^\alpha y(t) \in F(t, y(t)),\ a.e.\ t \in J = [0, b], \quad 1 < \alpha \leq 2, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3}$$

$$y(0) = a, \quad y'(0) = c, \tag{4}$$

where D_*^α is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_k,\ \bar{I}_k \in C(\mathbb{R}, \mathbb{R})\,(k = 1, \dots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, $y(t_k^+) = \lim\limits_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h \to 0^+} y(t_k - h)$ stand for the right and the left limits of $y(t)$ at $t = t_k$, respectively.

The paper is organized as follows. We first collect some background material and basic results from multi-valued analysis and fractional calculus in Sections 2 and 3, respectively. Then, we shall be concerned with Filippov's theorem for impulsive differential inclusions with fractional order in Section 4.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $AC^i([0, b], \mathbb{R}^n)$ be the space of functions $y : [0, b] \to \mathbb{R}^n$, i-differentiable, and whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.

We take $C(J, \mathbb{R})$ to be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \leq t \leq b\}.$$

$L^1(J, \mathbb{R})$ refers to the Banach space of measurable functions $y : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable; it is normed by

$$|y|_1 = \int_0^b |y(s)| ds.$$

Let $(X, \|\cdot\|)$ be a Banach space, and denote:

$$\begin{array}{rcl}\mathcal{P}(X) & = & \{Y \subset X : Y \neq \emptyset\}, \\ \mathcal{P}_{cl}(X) & = & \{Y \in \mathcal{P}(X) : Y \text{closed}\}, \\ \mathcal{P}_b(X) & = & \{Y \in \mathcal{P}(X) : Y \text{bounded}\}, \\ \mathcal{P}_{cp}(X) & = & \{Y \in \mathcal{P}(X) : Y \text{compact}\}.\end{array}$$

We say that a multivalued mapping $G : X \to \mathcal{P}(X)$ has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

A multi-valued map $G : J \longrightarrow \mathcal{P}(\mathbb{R})$ is said to be *measurable* if for each $x \in \mathbb{R}$ the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\},$$

is measurable.

Lemma 2.1 *(see [25], Theorem 19.7) Let E be a separable metric space and G a multi-valued map with nonempty closed values. Then G has a measurable selection.*

Lemma 2.2 *(see [60], Lemma 3.2) Let $G : [0, b] \to \mathcal{P}(E)$ be a measurable multifunction and $u : [0, b] \to E$ a measurable function. Then for any measurable $v : [0, b] \to \mathbb{R}^+$ there exists a measurable selection g of G such that for a.e. $t \in [0, b]$,*

$$|u(t) - g(t)| \leq d(u(t), G(t)) + v(t).$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b),\ d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space; see [38].

Definition 2.1 *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \textit{for each } x,\ y \in X;$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

For more details on multi-valued maps we refer to the books by Aubin *et al* [5, 4], Deimling [16], Gorniewicz [25], Hu and Papageorgiou [35], Kisielewicz [38] and Tolstonogov [58].

3 Fractional Calculus

According to the Riemann-Liouville approach to fractional calculus, the notation of fractional integral of order α ($\alpha > 0$) is a natural consequence of the well known formula (usually attributed to Cauchy), that reduces the calculation of the $n-$fold primitive of a function $f(t)$ to a single integral of convolution type. In our notation, the Cauchy formula reads

$$J^n f(t) := \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)ds,\ t > 0 \quad,\ n \in \mathbb{N}.$$

Definition 3.1 *The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b],\mathbb{R})$ is defined by*

$$J^{\alpha}_{a^+} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $J^{\alpha} f(t) = f(t) * \phi_{\alpha}(t)$, where $\phi_{\alpha}(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\phi_{\alpha}(t) = 0$ for $t \leq 0$, and $\phi_{\alpha} \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function and Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt,\ \alpha > 0.$$

Also $J^0 = I$ (Identity operator), i.e. $J^0 f(t) = f(t)$. Furtheremore, by $J^{\alpha} f(0^+)$ we mean the limit (if it exists) of $J^{\alpha} f(t)$ for $t \to 0^+$; this limit may be infinite.)

After the notion of fractional integral, that of fractional derivative of order α ($\alpha > 0$) becomes a natural requirement and one is attempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integral and preserve the well known properties of the ordinary derivative of integer order. Denoting by D^n with $n \in \mathbb{N}$, the operator of the derivative of order n, we first note that

$$D^n J^n = I, \quad J^n D^n \neq I, \quad n \in \mathbb{N},$$

i.e., D^n is the left-inverse (and not the right-inverse) to the corresponding integral operator J^n. We can easily prove that

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a^+)\frac{(t-a)^k}{k!},\ t > 0.$$

As consequence, we expect that D^{α} is defined as the left-inverse to J^{α}. For this purpose, introducing the positive integer n such that $n-1 < \alpha \leq n$, one defines the fractional derivative of order $\alpha > 0$:

Definition 3.2 *For a function f given on interval $[a,b]$, the αth Riemann-Liouville fractional-order derivative of f is defined by*

$$D^{\alpha}f(t) \quad = \quad \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-s)^{-\alpha+n-1}f(s)ds,$$

where $n=[\alpha]+1$ and $[\alpha]$ is the integer part of α.

Also, we define $D^{0}=J^{0}=I$. Then we easily recognize that

$$D^{\alpha}J^{\alpha}=I, \qquad \alpha\geq 0, \tag{5}$$

and

$$D^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha},\ \alpha>0,\ \gamma-1,\ t>0. \tag{6}$$

Of course, the properties (5) and (6) are a natural generalization of those known when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^{\alpha}f$ is not zero for the constant function $f(t)=1$ if $\alpha\notin\mathbb{N}$. In fact, (6) with $\gamma=0$ teaches us that

$$D^{\alpha}1=\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)},\ \alpha>0,\quad t>0. \tag{7}$$

It is clear that $D^{\alpha}1=0$ for $\alpha\in\mathbb{N}$, due to the poles of the gamma function at the points $0,-1,-2,\dots$.

We now observe an alternative definition of fractional derivative, originally introduced by Caputo [12, 13] in the late sixties and adopted by Caputo and Mainardi [14] in the framework of the theory of "linear viscoelasticity" (see a review in [43]).

Definition 3.3 *Let $f\in AC^{n}([a,b])$. The Caputo fractional-order derivative of f is defined by*

$$(D_{*}^{\alpha}f)(t):=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}f^{n}(s)ds.$$

This definition is of course more restrictive than Riemann-Liouville definition, in that it requires the absolute integrability of the derivative of order m. Whenever we use the operator D_{*}^{α} we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^{\alpha}f(t):=D^{m}J^{m-\alpha}f(t)\neq J^{m-\alpha}D^{m}f(t):=D_{*}^{\alpha}f(t), \tag{8}$$

unless the function $f(t)$, along with its first $m-1$ derivatives, vanishes at $t=a^{+}$. In fact, assuming that the passage of the $m-$derivative under the integral is legitimate, one recognizes that, $m-1<\alpha<m$ and $t>0$,

$$D^{\alpha}f(t)=D_{*}^{\alpha}f(t)+\sum_{k=0}^{m-1}\frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(a^{+}), \tag{9}$$

and therefore, recalling the fractional derivative of the power function (6),

$$D^\alpha\left(f(t)-\sum_{k=0}^{m-1}\frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(a^+)\right)=D_*^\alpha f(t). \tag{10}$$

The alternative definition, that is, Definition 3.3, for the fractional derivative thus incorporates the initial values of the function and of order lower than α. The subtraction of the Taylor polynomial of degree $m-1$ at $t=a^+$ from $f(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, a relevant property is that the fractional derivative of a constant is sill zero, i.e.,

$$D_*^\alpha 1=0, \quad \alpha>0. \tag{11}$$

We now explore the most relevant differences between Definition 3.2 and Definition 3.3 for the two fractional derivatives. From the Riemann-Liouville fractional derivative, we have

$$D^\alpha(t-a)^{\alpha-j}=0, \quad \text{for } j=1,2,\dots,[\alpha]+1. \tag{12}$$

From (11) and (12) we thus recognize the following statements about functions, which for $t>0$ admit the same fractional derivative of order α, with $n-1<\alpha\le n$, $n\in\mathbb{N}$,

$$D^\alpha f(t)=D^\alpha g(t)\Leftrightarrow f(t)=g(t)+\sum_{j=1}^{m}c_j(t-a)^{\alpha-j}, \tag{13}$$

and

$$D_*^\alpha f(t)=D_*^\alpha g(t)\Leftrightarrow f(t)=g(t)+\sum_{j=1}^{m}c_j(t-a)^{n-j}. \tag{14}$$

In these formulas the coefficients c_j are arbitrary constants. For proving all mains results we present the following auxiliary lemmas.

Lemma 3.1 *[36] Let $\alpha>0$ and let $y\in L^\infty(a,b)$ or $C([a,b])$. Then*

$$(D_*^\alpha J^\alpha y)(t)=y(t).$$

Lemma 3.2 *[36] Let $\alpha>0$ and $n=[\alpha]+1$. If $y\in AC^n[a,b]$ or $y\in C^n[a,b]$, then*

$$(J^\alpha D_*^\alpha y)(t)=y(t)-\sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k.$$

For further readings and details on fractional calculus we refer to the books and papers by Kilbas [36], Podlubny [54], Samko [56], Captuo [12, 13, 14].

4 Filippov's Theorem

Let $J_k = (t_k, t_{k+1}]$, $k = 0, \dots, m$, and let y_k be the restriction of a function y to J_k. In order to define mild solutions for problem (1)-(4), consider the space

$$\begin{aligned} PC = \{y\colon J \to \mathbb{R} \mid y_k \in C(J_k, \mathbb{R}),\ k = 0, \dots, m, \text{ and } y(t_k^-) \\ \text{and } y(t_k^+) \text{ exist and satisfy } y(t_k^-) = y(t_k) \text{ for } k = 1, \dots, m\}. \end{aligned}$$

Endowed with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_\infty : \ k = 0, \dots, m\},$$

this is a Banach space.

Definition 4.1 *A function $y \in PC$ is said to be a solution of (1)-(4) if there exists $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$ for a.e. $t \in J$ such that y satisfies the fractional differential equation $D_*^\alpha y(t) = v(t)$ a.e. on J, and the conditions (2)-(4).*

Let $\overline{a}, \overline{c} \in \mathbb{R}$, $g \in L^1(J, \mathbb{R})$ and let $x \in PC$ be a solution of the impulsive differential problem with fractional order:

$$\left\{ \begin{array}{rcll} D_*^\alpha x(t) & = & g(t), & \text{a.e. } t \in J \backslash \{t_1, \dots, t_m\},\ \alpha \in (1, 2], \\ \Delta x_{t=t_k} & = & I_k(x(t_k^-)), & k = 1, \dots, m, \\ \Delta x'_{t=t_k} & = & \overline{I}_k(x(t_k^-)), & k = 1, \dots, m, \\ x(0) & = & \overline{a},\ x'(0) = \overline{c}. & \end{array} \right. \tag{15}$$

We will need the following two assumptions:

$(\mathcal{H}_1)$ The function $F\colon J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is such that
(a) for all $y \in \mathbb{R}$, the map $t \mapsto F(t, y)$ is measurable,
(b) the map $\gamma : t \mapsto d(g(t), F(t, x(t))$ is integrable.

$(\mathcal{H}_2)$ There exists a function $p \in L^1(J, \mathbb{R}^+)$ such that

$$H_d(F(t, z_1), F(t, z_2)) \le p(t)|z_1 - z_2|, \quad \text{for all } z_1,\ z_2 \in \mathbb{R}.$$

Remark 4.1 *From Assumptions $(\mathcal{H}_1(a))$ and $(\mathcal{H}_2)$, it follows that the multi-function $t \mapsto F(t, x_t)$ is measurable, and by Lemmas 1.4 and 1.5 from [22], we deduce that $\gamma(t) = d(g(t), F(t, x(t))$ is measurable (see also the Remark on p. 400 in [4]).*

Let $P(t) = \int_0^t p(s)ds$. Define the functions η_0 and H_0 by

$$\eta_0(t) = M\delta_0 + M \int_0^t [H_0(s)p(s) + \gamma(s)]\, ds, \quad t \in [0, t_1],$$

$$\text{where } H_0(t) = \delta_0 M \exp\left(M e^{P(t)}\right) + M \int_0^t \gamma(s) \exp\left(M e^{P(t) - P(s)}\right) ds,$$

where $M = \max\left(1, b, \dfrac{b^{\alpha-1}}{\Gamma(\alpha)}\right)$ and $\delta_0 = |a - \overline{a}| + |c - \overline{c}|$.

Theorem 4.1 *Suppose that hypotheses* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *are satisfied. Problem (1)-(4) has at least one solution* y *satisfying, for a.e.* $t \in [0,b]$, *the estimates*

$$|y(t)-x(t)| \;\; \leq \;\; M \sum_{0\leq k<i} \delta_k + M \sum_{0\leq t_k<t} \eta_k(t),$$

and

$$|D_*^{\alpha} y(t) - g(t)| \leq Mp(t) \sum_{0<t_k<t} H_k(t) + \sum_{0<t_k<t} \gamma_k(t),$$

and for $k=1,\ldots,m$, *where*

$$\eta_k(t) = M \int_{t_k}^{t} [H_k(s)p(s)+\gamma(s)]\,ds, \quad t \in (t_k, t_{k+1}],$$

and

$$\text{where } H_k(t) = \delta_k \exp\left(Me^{P_k(t)}\right) + \int_{t_k}^{t} \gamma(s) \exp\left(Me^{P_k(t)-P_k(s)}\right) ds,$$

where

$$\delta_k := |x(t_k)-y(t_k)| + |I_1(y(t_k)) - I_k(x(t_k))| + |x'(t_k)-y'(t_k)| + |I_1(y'(t_k)) - I_1(x'(t_k))|.$$

Before proving the theorem, we present a lemma.

Lemma 4.1 *Let* $G : [0,b] \to \mathcal{P}_{cl}(\mathbb{R})$ *be a measurable multifunction and* $u : [0,b] \to \mathbb{R}$ *a measurable function. Assume that there exist* $p \in L^1(J,\mathbb{R})$ *such that* $G(t) \subseteq p(t)B(0,1)$, *where* $B(0,1)$ *denotes the closed ball in* $\mathbb{R}$. *Then there exists a measurable selection* g *of* G *such that for a.e.* $t \in [0,b]$,

$$|u(t)-g(t)| \leq d(u(t), G(t)).$$

Proof. Let $v_\epsilon : [0,b] \to \mathbb{R}_+$ define by $v_\epsilon(t) = \epsilon > 0$. Then from Lemma 2.2, there exists a measurable selection g_ϵ of G such that

$$|u(t) - g_\epsilon(t)| \leq d(u(t), G(t)) + \epsilon.$$

Let $\epsilon = 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots,\; n \in \mathbb{N}$, hence

$$|u(t) - g_n(t)| \leq d(u(t), G(t)) + \frac{1}{n}.$$

Using the fact that G is integrable bounded, then

$$g_n(t) \in p(t)B(0,1),\;\; t \in J,$$

and we may pass to a subsequence if necessary to get that g_n converges to a measurable function g. Then
$$|u(t)-g(t)|\leq d(u(t),G(t)).$$

Proof of Theorem 4.1. We are going to study Problem (1)-(4) in the respective intervals $[0,t_1]$, $(t_1,t_2]$, $\ldots,(t_m,b]$. The proof will be given in three steps and then continued by induction, and then summarized in a fourth step.
Step 1. In this first step, we construct a sequence of functions $(y_n)_{n\in\mathbb{N}}$ which will be shown to converge to some solution of Problem (1)-(4) on the interval $[0,t_1]$, namely to
$$\left\{\begin{array}{rcl} D^\alpha_* y(t) & \in & F(t,y(t)),\ \ t\in J_0=[0,t_1],\ \alpha\in(1,2],\\ y(0) & = & a, \qquad\qquad\ \ y'(0)=c.\end{array}\right. \tag{16}$$
Let $f_0=g$ on $[0,t_1]$ and $y_0(t)=x(t),\ t\in[0,t_1]$, i.e.,
$$y_0(t)=\overline{a}+t\overline{c}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_0(s)ds,\qquad t\in[0,t_1].$$
Then define the multi-valued map U_1: $[0,t_1]\to\mathcal{P}(\mathbb{R})$ by $U_1(t)=F(t,y_0(t))\cap(g(t)+\gamma(t)B(0,1))$. Since g and γ are measurable, Theorem III.4.1 in [15] tells us that the ball $(g(t)+\gamma(t)B(0,1))$ is measurable. Moreover $F(t,y_0(t))$ is measurable (see Remark 4.1) and U_1 is nonempty. It is clear that
$$\begin{array}{rcl} d(0,F(t,0)) & \leq & d(0,g(t))+d(g(t),F(t,y_0(t)))+H_d(F(t,y_0(t)),F(t,0))\\ & \leq & |g(t)|+\gamma(t)+p(t)|y_0(t)|,\ \text{a.e. } t\in[0,t_1].\end{array}$$
Hence for all $w\in F(t,y_0(t))$ we have
$$\begin{array}{rcl} |w| & \leq & d(0,F(t,0))+H_d(F(t,0),F(t,y_0(t)))\\ & \leq & |g(t)|+\gamma(t)+2p(t)|y_0(t)|:=M(t),\ \text{a.e.}t\in[0,t_1].\end{array}$$
This implies that
$$F(t,y_0(t))\subseteq M(t)B(0,1),\ t\in[0,t_1].$$
From Lemma 4.1, there exists a function u which is a measurable selection of $F(t,y_0(t))$ such that
$$|u(t)-g(t)|\leq d(g(t),F(t,y_0(t)))=\gamma(t).$$
Then $u\in U_1(t)$, proving our claim. We deduce that the intersection multivalued operator $U_1(t)$ is measurable (see [4, 15, 25]). By Lemma 2.1 (Kuratowski-Ryll-Nardzewski selection theorem), there exists a function $t\to f_1(t)$ which is a measurable selection for U_1. Consider
$$y_1(t)=a+tc+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_1(s)ds,\qquad t\in[0,t_1].$$

For each $t \in [0, t_1]$, we have

$$\begin{array}{rcl} |y_1(t) - y_0(t)| & \leq & |a - \overline{a}| + t|c - \overline{c}| \\ & + & \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} |f_0(s) - f_1(s)|\, ds \\ & \leq & |a - \overline{a}| + b|c - \overline{c}| \\ & + & \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t |f_0(s) - f_1(s)| ds. \end{array} \tag{17}$$

Hence

$$|y_1(t) - y_0(t)| \ \leq \ \delta + \frac{t_1^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \gamma(s) ds, \quad t \in [0, t_1].$$

Now Lemma 1.4 in [22] tells us that $F(t, y_1(t))$ is measurable. The ball $(f_1(t) + p(t)|y_1(t) - y_0(t)|B(0, 1))$ is also measurable by Theorem III.4.1 in [15]. The set $U_2(t) = F(t, y_1(t)) \cap (f_1(t) + p(t)|y_1(t) - y_0(t)|B(0, 1))$ is nonempty. Indeed, since f_1 is a measurable function, Lemma 4.1 yields a measurable selection u of $F(t, y_1(t))$ such that

$$|u(t) - f_1(t)| \ \leq \ d(f_1(t), F(t, y_1(t))).$$

Then using $(\mathcal{H}_2)$, we get

$$\begin{array}{rcl} |u(t) - f_1(t)| & \leq & d(f_1(t), F(t, y_1(t))) \\ & \leq & H_d(F(t, y_0(t)), F(t, y_1(t))) \\ & \leq & p(t)|y_0(t) - y_1(t)|, \end{array}$$

i.e. $u \in U_2(t)$, proving our claim. Now, since the intersection multi-valued operator U_2 defined above is measurable (see [4, 15, 25]), there exists a measurable selection $f_2(t) \in U_2(t)$. Hence

$$|f_1(t) - f_2(t)| \leq p(t)|y_1(t) - y_0(t)|. \tag{18}$$

Define

$$y_2(t) = a + ct + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s) ds, \qquad t \in (0, t_1].$$

Using (17) and (18), a simple integration by parts yields the following estimates, valid for every $t \in [0, t_1]$,

$$\begin{array}{rcl} |y_2(t) - y_1(t)| & \leq & \dfrac{t_1^{\alpha-1}}{\Gamma(\alpha)} \displaystyle\int_0^t |f_2(s) - f_1(s)| ds \\ & \leq & \displaystyle\int_0^t p(s) \left(\delta + \int_0^s \gamma(u) du \right) ds \end{array}$$

$$\begin{aligned} &= M^2\left(\delta\int_0^t p(s)ds+\int_0^t p(s)ds\int_0^s\gamma(u)du\right)\\ &\le M^2\left(\delta\int_0^t p(s)e^{P(s)}ds+\int_0^t p(s)e^{P(s)}ds\int_0^s e^{-P(u)}\gamma(u)du\right)\\ &\le M^2\left(\delta e^{P(t)}+\int_0^t\gamma(s)e^{P(t)-P(s)}ds\right),\ t\in[0,t_1]. \end{aligned}$$

Let $U_3(t)=F(t,y_2(t))\cap(f_2(t)+p(t)|y_2(t)-y_1(t)|B(0,1))$. Arguing as for U_2, we can prove that U_3 is a measurable multi-valued map with nonempty values; so there exists a measurable selection $f_3(t)\in U_3(t)$. This allows us to define

$$y_3(t)=a+tc+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_3(s)ds,\qquad t\in[0,t_1].$$

For $t\in[0,t_1]$, we have

$$\begin{aligned} |y_3(t)-y_2(t)| &\le M\int_0^t|f_2(s)-f_3(s)|\,ds\\ &\le M\int_0^t p(s)|y_2(s)-y_1(s)|\,ds. \end{aligned}$$

Then

$$\begin{aligned} |y_3(s)-y_2(s)| &\le M^3\left(\delta e^{P(s)}+\int_0^s\gamma(u)e^{P(s)-P(u)}du\right)\\ &\le M^3\left(\delta e^{P(s)}+\int_0^s\gamma(u)e^{P(s)-P(u)}du\right). \end{aligned}$$

Performing an integration by parts, we obtain, since P is a nondecreasing function, the following estimates

$$\begin{aligned} |y_3(t)-y_2(t)| &\le \frac{M^3}{2}\int_0^t 2p(s)\left(\delta e^{2P(s)}+\int_0^s\gamma(u)e^{P(s)-P(u)}du\right)ds\\ &\le \frac{M^3}{2}\left(\delta e^{2P(t)}+\int_0^t 2p(s)ds\int_0^s\gamma(u)e^{2(P(s)-P(u))}du\right)\\ &\le \frac{M^3}{2}\left(\delta e^{2P(t)}+\int_0^t\left(e^{2P(s)}\right)'ds\int_0^s\gamma(u)e^{-2P(u))}du\right)\\ &\le \frac{M^3}{2}\left(\delta e^{2P(t)}+e^{2P(t)}\int_0^t\gamma(s)e^{-2P(s)}ds-\int_0^t\gamma(s)ds\right)\\ &\le \frac{M^3}{2}\left(\delta e^{2P(t)}+\int_0^t\gamma(s)e^{2(P(t)-P(s))}ds\right),\ t\in[0,t_1]. \end{aligned}$$

Repeating the process for $n = 1, 2, 3, \ldots$, we arrive at the following bound

$$\begin{aligned} |y_n(t) - y_{n-1}(t)| &\leq \frac{M^n}{(n-1)!} \int_0^t \gamma(s) e^{(n-1)(P(t)-P(s))} ds \\ &+ \frac{M^n}{(n-1)!} \delta e^{(n-1)P(t)}, \quad t \in [0, t_1]. \end{aligned} \tag{19}$$

By induction, suppose that (19) holds for some n and check (19) for $n+1$. Let $U_{n+1}(t) = F(t, y^n(t)) \cap (f_n + p(t)|y_n(t) - y_{n-1}(t)|B(0,1))$. Since U_{n+1} is a nonempty measurable set, there exists a measurable selection $f_{n+1}(t) \in U_{n+1}(t)$, which allows us to define for $n \in \mathbb{N}$

$$y_{n+1}(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_{n+1}(s) ds, \quad t \in [0, t_1]. \tag{20}$$

Therefore, for a.e. $t \in [0, t_1]$, we have

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq M \int_0^t |f_{n+1}(s) - f_n(s)|\, ds \\ &\leq \frac{M^{n+1}}{(n-1)!} \int_0^t p(s)|y_n(s) - y_{n-1}(s)|\, ds \\ &\leq \frac{M^{n+1}}{(n-1)!} \int_0^t p(s) ds \left(\delta e^{(n-1)P(s)} + \int_0^s \gamma(u) e^{(n-1)(P(s)-P(u))} du \right) \\ &\leq \frac{M^{n+1}}{n!} \int_0^t \delta n p(s) e^{nP(s)} ds \\ &+ \frac{M^{n+1}}{n!} \int_0^t n p(s) e^{nP(s)} ds \int_0^s \gamma(u) e^{-nP(u)} du. \end{aligned}$$

Again, an integration by parts leads to

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq \frac{M^{(n+1)}}{n!} \int_0^t \gamma(s) e^{n(P(t)-P(s))} ds \\ &+ \frac{M^{(n+1)}}{n!} \delta e^{nP(t)}. \end{aligned}$$

Consequently, (19) holds true for all $n \in \mathbb{N}$. We infer that $\{y_n\}$ is a Cauchy sequence in PC_1, converging uniformly to a limit function $y \in PC_1$, where

$$PC_1 = C([0, t_1], \mathbb{R}).$$

Moreover, from the definition of $\{U_n\}$, we have

$$|f_{n+1}(t) - f_n(t)| \leq p(t)|y_n(t) - y_{n-1}(t)|, \quad \text{a.e } t \in [0, t_1].$$

Hence, for almost every $t \in [0, t_1]$, $\{f_n(t)\}$ is also a Cauchy sequence in $\mathbb{R}$ and then converges almost everywhere to some measurable function $f(\cdot)$ in $\mathbb{R}$. In addition, since $f_0 = g$, we have for a.e. $t \in [0, t_1]$

$$\begin{aligned} |f_n(t)| &\leq \sum_{i=1}^{n} p(t)|f_i(t) - f_{i-1}(t)| + |f_0(t)| \\ &\leq \sum_{i=2}^{n} p(t)|y_{i-1}(t) - y_{i-2}(t)| + |g(t)| \\ &\leq p(t)\sum_{i=2}^{\infty} |y_i(t) - y_{i-1}(t)| + \gamma(t) + |g(t)|. \end{aligned}$$

Hence

$$|f_n(t)| \leq H_0(t)p(t) + \gamma(t) + |g(t)|,$$

where

$$H_0(t) := \delta M \exp\left(e^{P(t)}\right) + M\int_0^t \gamma(s)\exp\left(e^{P(t)-P(s)}\right) ds. \tag{21}$$

From the Lebesgue dominated convergence theorem, we deduce that $\{f_n\}$ converges to f in $L^1([0, t_1], \mathbb{R})$. Passing to the limit in (20), we find that the function

$$y(t) = a + tc + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, \ \ t \in (0, t_1]$$

is solution to Problem (1)-(4) on $[0, t_1]$; thus $y \in S_{[0,t_1]}(a, c)$. Moreover, for a.e. $t \in [0, t_1]$, we have

$$\begin{aligned} |x(t) - y(t)| &= \Bigg| a + tc + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds \\ &\quad -\overline{a} - t\overline{c} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds \Bigg| \\ &\leq M\delta + M\int_0^t |f(s) - f_0(s)|ds \\ &\leq M\delta + M\int_0^t |f(s) - f_n(s)|ds \\ &+ M\int_0^t |f_n(s) - f_0(s)|ds \\ &\leq M\delta + M\int_0^t |f(s) - f_n(s)|ds \\ &+ M\int_0^t (H(s)p(s) + \gamma(s))\, ds. \end{aligned}$$

Passing to the limit as $n \to \infty$, we get

$$|x(t) - y(t)| \leq \eta_0(t), \quad \text{a.e. } t \in [0, t_1], \tag{22}$$

with

$$\eta_0(t) := M\delta + M \int_0^t (H_0(s)p(s) + \gamma(s))\, ds.$$

Next, we give an estimate for $|D^\alpha y(t) - g(t)|$ for $t \in [0, t_1]$. We have

$$\begin{aligned} |D_*^\alpha y(t) - g(t)| &= |f(t) - f_0(t)| \\ &\leq |f_n(t) - f_0(t)| + |f_n(t) - f(t)| \\ &\leq p(t) \sum_{i=1}^{\infty} |y_{i+1}(t) - y_i(t)| + \gamma(t) + |f_n(t) - f(t)|. \end{aligned}$$

Arguing as in (21) and passing to the limit as $n \to +\infty$, we deduce that

$$|D_*^\alpha y(t) - g(t)| \leq H_0(t)p(t) + \gamma(t), \quad t \in [0, t_1].$$

The obtained solution is denoted by $y_1 := y_{|[0,t_1]}$.

Step 2. Consider now Problem (1)-(4) on the second interval $(t_1, t_2]$, i.e.,

$$\left\{ \begin{array}{rcll} D_*^\alpha y(t) & \in & F(t, y(t)), & \text{a.e. } t \in (t_1, t_2], \\ y(t_1^+) & = & y_1(t_1) + I_1(y_1(t_1)), & \\ y'(t_1^+) & = & y_1'(t_1) + \overline{I}_1(y_1(t_1)). & \end{array} \right. \tag{23}$$

Let $f_0 = g$ and set

$$\begin{aligned} y^0(t) &= x(t_1) + I_1(x(t_1)) + (t - t_1)[x(t_1) + \overline{I}_1(x(t_1))] \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f_0(s) ds, \quad t \in (t_1, t_2]. \end{aligned}$$

Notice that (22) allows us to use Assumption $(\mathcal{H}_2)$, apply again Lemma 1.4 in [22] and argue as in Step 1 to prove that the multi-valued map U_1: $[t_1, t_2] \to \mathcal{P}(\mathbb{R})$ defined by $U_1(t) = F(t, y^0(t)) \cap (g(t) + \gamma(t)B(0, 1))$ is $U_1(t)$ is measurable. Hence, there exists a function $t \mapsto f_1(t)$ which is a measurable selection for U_1. Define

$$\begin{aligned} y^1(t) &= y_1(t_1) + I_1(y_1(t_1)) + (t - t_1)[y_1(t_1) + \overline{I}_1(y_1(t_1))] \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f_1(s) ds, \quad t \in (t_1, t_2]. \end{aligned}$$

Next define the measurable multi-valued map $U_2(t) = F(t, y^1(t)) \cap (f_1(t) + p(t)|y^1(t) - y^0(t)|B(0, 1))$. It has a measurable selection $f_2(t) \in U_2(t)$ by the Kuratowski-Ryll-Nardzewski selection theorem. Repeating the process of selection as in Step 1, we can define by induction a sequence of multi-valued maps $U_n(t) = F(t, y^{n-1}(t)) \cap (f_{n-1}(t) + p(t)|y^{n-1}(t) - y^{n-2}(t)|B(0,1))$ where $\{f_n\} \in U_n$ and $(y^n)_{n\in\mathbb{N}}$ is as defined by

$$\begin{aligned} y^n(t) &= y_1(t_1) + I_1(y_1(t_1)) + (t-t_1)[y_1(t_1) + \overline{I}_1(y_1(t_1))] \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f_n(s)ds, t \in (t_1, t_2], \end{aligned}$$

and we can easily prove that

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq \frac{M^{(n+1)}}{n!} \int_{t_1}^t \gamma(s) e^{n(P_1(t)-P_1(s))} ds \\ &+ \frac{M^{(n+1)}}{n!} \delta_1 e^{nP_1(t)},\ t \in (t_1, t_2]. \end{aligned}$$

Let

$$PC_2 = \{y\colon y \in C(t_1, t_2] \text{ and } y(t_1^+) \ \text{ exists}\}.$$

As in Step 1, we can prove that the sequence $\{y^n\}$ converges to some $y \in PC_2$ solution to Problem (23) such that, for a.e. $t \in (t_1, t_2]$, we have

$$\begin{aligned} |x(t) - y(t)| &\leq M|x_1(t_1) - y_1(t_1)| + M|x'(t_1) - y_1'(t_1)| \\ &+ M|I_1(x(t_1)) - I_1(y_1(t_1))| + M|\overline{I}_1(x(t_1)) - \overline{I}_1(y_1(t_1))| \\ &+ M \int_{t_1}^t (H_1(s)p(s) + \gamma(s))\, ds \\ &\leq M\delta_1 + M \int_{t_1}^t (H_1(s)p(s) + \gamma(s))\, ds, \end{aligned}$$

and

$$|D_*^\alpha y(t) - g(t)| := |f(t) - f_0(t)| \leq H_1(t)p(t) + \gamma(t).$$

Denote the restriction $y_{|(t_1,t_2]}$ by y_2.

Step 3. We continue this process until we arrive at the function $y_{m+1} := y\Big|_{(t_m,b]}$ as a solution of the problem

$$\left\{ \begin{array}{rcll} D_*^\alpha y(t) & \in & F(t, y(t)), & \text{a.e. } t \in (t_m, b], \\ y(t_m^+) & = & y_{m-1}(t_m) + I_m(y_{m-1}(t_m)), & \\ y'(t_m^+) & = & y_{m-1}'(t_m) + \overline{I}_m(y(t_m)). & \end{array} \right.$$

Then, for a.e. $t \in (t_m, b]$, the following estimates are easily derived:

$$\begin{aligned} |x(t) - y(t)| &\leq |y_m(t_m) - x(t_m)| + |y_m'(t_m) - x'(t_m)| \\ &+ M[|I_m(x(t_m)) - I_m(y(t_m))| + |\overline{I}_m(x(t_m)) - \overline{I}_m(y(t_m))| \\ &+ M \textstyle\int_{t_m}^t (H_m(s)p(s) + \gamma(s))\, ds \\ &\leq M\delta_m + M \textstyle\int_{t_m}^t (H_m(s)p(s) + \gamma(s))\, ds \end{aligned}$$

and

$$|D_*^\alpha y(t) - g(t)| \leq H_m(t)p(t) + \gamma(t)).$$

Step 4. Summarizing, a solution y of Problem (1)-(4) can be defined as follows

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, t_2], \\ \dots & \dots \\ y_{m+1}(t), & \text{if } t \in (t_m, b]. \end{cases}$$

From Steps 1 to 3, we have that, for a.e. $t \in [0, t_1]$,

$$|x(t) - y(t)| \leq \eta_0(t), \quad \text{and} \quad |D_*^\alpha y(t) - g(t)| \leq H_0(t)p(t) + \gamma(t),$$

as well as the following estimates, valid for $t \in (t_1, b]$

$$|x(t) - y(t)| \quad \leq \quad M \sum_{k=0}^{m} \delta_k + \sum_{k=0}^{m} \eta_k(t).$$

Similarly

$$|D_*^\alpha y(t) - g(t)| \leq p(t) \sum_{0<t_k<t} H_k(t) + \sum_{0<t_k<t} \gamma_k(t),$$

where $\gamma_k := \gamma_{|_{J_k}}$. The proof of Theorem 4.1 is complete.

4.1 Filippov's Theorem on the Half-Line

We may consider Filippov's Problem on the half-line as given by,

$$\begin{cases} D_*^\alpha y(t) & \in & F(t, y(t)), & \text{a.e. } t \in \widetilde{J} \backslash \{t_1, \dots\}, \\ \Delta y_{t=t_k} & = & I_k(y(t_k^-)), & k = 1, \dots, \\ \Delta y'_{t=t_k} & = & \overline{I}_k(y(t_k^-)), & k = 1, \dots, \\ y(t) & = & a, & y'(0) = c, \end{cases} \tag{24}$$

where $\widetilde{J} = [0, \infty)$, $0 = t_0 < t_1 < \cdots < t_m < \cdots$, $\lim_{m \to \infty} t_m = +\infty$, F: $\widetilde{J} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multifunction, and $a, c \in \mathbb{R}$. Let x be the solution of Problem (15) on the half-line. We will need the following assumptions:

$(\widetilde{\mathcal{H}_1})$ The function F: $\widetilde{J} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is such that
(a) for all $y \in \mathbb{R}$, the map $t \mapsto F(t, y)$ is measurable,
(b) the map $t \mapsto \gamma(t) = d(g(t), F(t, x(t)) \in L^1([0, \infty), \mathbb{R}_+)$

$(\widetilde{\mathcal{H}_2})$ There exist a function $p \in L^1([0, \infty), \mathbb{R}^+)$ such that

$$H_d(F(t, z_1), F(t, z_2)) \leq p(t)|z_1 - z_2|, \quad \text{for all} \quad z_1, \, z_2 \in \mathbb{R},$$

$(\widetilde{\mathcal{H}_3})$ For every $x \in \mathbb{R}$, we have

$$\sum_{k=1}^{\infty} |I_k(x)| < \infty, \qquad \sum_{k=1}^{\infty} |\overline{I}_k(x)| < \infty.$$

Then we can extend Filippov's Theorem to the half-line.

Theorem 4.2 *Let $\gamma_k := \gamma_{|_{J_k}}$ and assume $(\widetilde{\mathcal{H}_1}) - (\widetilde{\mathcal{H}_3})$ hold. Then, Problem (24) has at least one solution y satisfying, for $t \in [0,\infty)$, the estimates*

$$|y(t) - x(t)| \;\leq\; M \sum_{0<t_k<t} \delta_k + M \sum_{0<t_k<t} \eta_k(t),$$

and

$$|D_*^{\alpha} y(t) - g(t)| \leq p(t) \sum_{0<t_k<t} H_k(t) + \sum_{0<t_k<t} \gamma_k(t).$$

Proof. The solution will be sought in the space

$$\begin{aligned}\widetilde{PC} = \{&y\colon [0,\infty) \to \mathbb{R},\ y_k \in C(J_k,\mathbb{R}),\ k = 0,\ldots,\ \text{ such that}\\ &y(t_k^-)\ \text{ and }\ y(t_k^+)\ \text{exist and satisfy }\ y(t_k^-) = y(t_k)\ \text{ for }\ k = 1,\ldots\},\end{aligned}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k \geq 0$. Theorem 4.1 yields estimates of y_k on each one of the bounded intervals $J_0 = [0, t_1]$, and $J_k = (t_{k-1}, t_k],\ k = 2, \ldots$. Let y_0 be solution of Problem (1)-(4) on J_0.

Then, consider the following problem

$$\left\{\begin{array}{rcll} D_*^{\alpha} y(t) & \in & F(t, y(t)), & \text{a.e. } \ t \in (t_1, t_2],\\ y(t_1^+) & = & y_0(t_1) + I_1(y_0(t_1)), & \\ y'(t_1^+) & = & y_0'(t_1) + \overline{I_1}(y_0(t_1)). & \end{array}\right.$$

From Theorem 4.1, this problem has a solution y_1. We continue this process taking into account that $y_m := y\Big|_{(t_m,t_{m+1}]}$ is a solution to the problem

$$\left\{\begin{array}{rcll} D_*^{\alpha} y(t) & \in & F(t, y(t)), & \text{a.e. } \ t \in (t_m, t_{m+1}],\\ y(t_m^+) & = & y_{m-1}(t_m) + I_m(y_{m-1}(t_m^-)), & \\ y'(t_m^+) & = & y_{m-1}'(t_m) + \overline{I}_m(y_{m-1}(t_m^-)). & \end{array}\right.$$

Then a solution y of Problem (24) may be rewritten as

$$y(t) = \left\{\begin{array}{ll} y_1(t), & \text{if } t \in [0, t_1],\\[2mm] y_2(t), & \text{if } t \in (t_1, t_2],\\[2mm] \ldots & \ldots\\[2mm] y_m(t), & \text{if } t \in (t_m, t_{m+1}],\\[2mm] \ldots & \ldots \end{array}\right.$$

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