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A CANTILEVER EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We prove new results on the existence of positive solutions for some cantilever equation subject to nonlocal and nonlinear boundary conditions. Our main ingredient is the classical fixed point index.

Key words and phrases: Fixed point index, cone, positive solution, cantilever equation.
AMS (MOS) Subject Classifications: Primary 34B18, secondary 34B10, 47H10, 47H30

1 Introduction

In this paper we establish new results on the existence of positive solutions for the fourth order differential equation

$$u^{(4)}(t) = g(t)f(t, u(t)),\ t \in (0, 1), \tag{1}$$

subject to the nonlocal boundary conditions (BCs)

$$u(0) = u'(0) = u''(1) = 0,\ u'''(1) + k_0 + B(\alpha[u]) = 0. \tag{2}$$

Here k_0 is a non-negative constant, B is a non-negative continuous function and $\alpha[u]$ is a positive functional given by

$$\alpha[u] = \int_0^1 u(s)\, dA(s),$$

involving a Stieltjes integral.

Equation (1) models the stationary states of the deflection of an elastic beam. Beam equations have been studied recently by several authors under different boundary conditions. For example, the BCs

$$u(0) = 0,\ u(1) = 0,\ u'(0) = 0,\ u'(1) = 0,$$

correspond to both ends of the beam being clamped, these have been studied recently in [32, 37, 42, 43, 48]; the BCs

$$u(0) = 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0,$$

model a bar with hinged ends and have been studied for example in [5, 7, 9, 12, 14, 19, 35, 36, 37, 42, 43, 46]. Also the BCs

$$u(0) = 0,\ u'(0) = 0,\ u(1) = 0,\ u''(1) = 0,$$

have been studied in [1, 42, 49] and correspond to the left end being clamped and the right end being hinged. The conditions

$$u(0) = 0,\ u'(0) = 0,\ u''(1) = 0,\ u'''(1) = 0$$

model the so-called cantilever bar, that is a bar clamped on the left end and where the right end is free to move with vanishing bending moment and shearing force. These types of BCs have been investigated in [4, 47] and, in particular, [4] provides a detailed insight on the physical motivation for this problem.

The BCs (2) describe a cantilever beam with forces acting on its right end, for example,

- $u'''(1) + k_0 = 0$ models a force acting in 1,
- $u'''(1) + k_1 u(1) = 0$ describes a spring in 1,
- $u'''(1)+B(u(1)) = 0$ models a spring with a strongly nonlinear rigidity (this could happen, for example, due to the type of material),
- $u'''(1) + B(u(\eta)) = 0$ describes a feedback mechanism, where the spring reacts to the displacement registered in a point η of the beam.

Thus the condition

$$u'''(1) + k_0 + B(\alpha[u]) = 0$$

covers a variety of cases and includes, as special cases when $k_0 = 0$ and $B(w) = w$, multi-point and integral boundary conditions, that are widely studied objects also in the case of fourth order BVPs, see for example [8, 11, 13, 15, 16, 18, 29, 30, 39]. BCs of nonlinear type are widely studied objects in the case of second and fourth order equations, see for example [3, 6, 10, 20, 21, 23, 25, 31, 45] and references therein. The study of positive solutions of BVPs that involve Stieltjes integrals has been done, in the case of positive measures, in [27, 28]. *Signed* measures were used in [41] and in the subsequent papers [40, 42]; here, as in [23, 25], we are forced, due to some inequalities involved in our theory, to restrict our attention to positive measures only.

Our approach is to rewrite the BVP (1)-(2) as a perturbed Hammerstein integral equation of the form

$$u(t) = \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds.$$

This type of integral equation has been investigated, when $k_0 = 0$ and $B(w) \equiv w$, in [38, 41, 43, 44], when $B(w) \equiv w$ in [22, 24, 27], and when $k_0 = 0$ in [21, 23, 25, 45]. In order to utilize classical fixed point index theory, we provide here a modification of the results of [27, 41] in order to prove the existence of multiple positive solutions of the BVP (1)-(2).

In a last remark we briefly illustrate how our approach may also be applied to the nonlinear BCs

$$u(0) = k_0 + B(\alpha[u]),\ u'(0) = 0,\ u''(1) = 0,\ u'''(1) = 0, \tag{3}$$

$$u(0) = 0,\ u'(0) = k_0 + B(\alpha[u]),\ u''(1) = 0,\ u'''(1) = 0, \tag{4}$$

$$u(0) = 0,\ u'(0) = 0,\ u''(1) = k_0 + B(\alpha[u]),\ u'''(1) = 0. \tag{5}$$

As in [42], where a different set of BCs were investigated, we point out that these nonlocal boundary conditions can be interpreted as feedback controls; in particular, the BCs (3) can be seen as a control on the displacement in the left end, the BCs (4) would be a device handling the angular attitude at the left end, whereas the BCs (5) describe a control on the bending moment at the right end.

2 Positive Solutions of the Fourth-Order BVPs

We firstly describe in details our approach for the BVP

$$u^{(4)}(t) = g(t)f(t,u(t)),\ t \in (0,1), \tag{6}$$

$$u(0) = u'(0) = u''(1) = 0,\ u'''(1) + k_0 + B(\alpha[u]) = 0. \tag{7}$$

Throughout the paper we assume that:

(C_1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ satisfies Carathéodory conditions, that is, for each u, $t \mapsto f(t,u)$ is measurable and for almost every t, $u \mapsto f(t,u)$ is continuous, and for every $r > 0$ there exists a L^∞-function $\phi_r : [0,1] \to [0,\infty)$ such that

$$f(t,u) \le \phi_r(t) \quad \text{for almost all } t \in [0,1] \text{ and all } u \in [0,r],$$

(C_2) $g \in L^1[0,1]$, $g \geq 0$ for almost every $t \in [0,1]$,

(C_3) A is a function of bounded variation and dA is a *positive* measure,

(C_4) $B : [0,\infty) \to [0,\infty)$ is continuous and there exist $\delta_1, \delta_2 \geq 0$ such that

$$\delta_1 w \leq B(w) \leq \delta_2 w \text{ for every } w \in [0,\infty).$$

The homogeneous BVP that corresponds to the BCs

$$u(0) = u'(0) = u''(1) = u'''(1) = 0,$$

has been studied in [4, 47], by means of an Hammerstein integral equation of the type

$$u(t) = \int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds, \tag{8}$$

where G_0 is the Green's function associated with these BCs, that is

$$G_0(t,s) = \begin{cases} \frac{1}{6}(3t^2 s - t^3), & s \geq t \\ \frac{1}{6}(3s^2 t - s^3), & s \leq t. \end{cases}$$

In order to rewrite the BVP (6)-(7) as a perturbation of the integral equation (8), we observe that the function

$$\gamma(t) = \frac{1}{6}(3t^2 - t^3) \text{ for all } t \in [0,1],$$

is the unique solution of the BVP

$$\gamma^{(4)}(t) = 0,\ \gamma(0) = \gamma'(0) = \gamma''(1) = 0,\ \gamma'''(1) + 1 = 0.$$

This allows us to associate with the BVP (6)-(7) the perturbed Hammerstein integral equation

$$u(t) = \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds, \tag{9}$$

where γ and G_0 are as above, and utilize the classical fixed point index for compact maps (see for example [2] or [17]).

Since the derivative of the function G_0 with respect to t is non-negative for all $t \in [0,1]$, G_0 is a non-decreasing function of t that attains its maximum when $t = 1$. Then we set

$$\Phi(s) := \max_{0 \leq t \leq 1} G_0(t,s) = G_0(1,s) = \frac{1}{2}s^2 - \frac{1}{2}s^3.$$

We now look for a suitable interval $[a,b] \subset [0,1]$ and for the constants $c_0, c_\gamma > 0$ that satisfy the inequalities

$$\begin{aligned} &G_0(t,s) \geq c_0 \Phi(s), \text{ for every } (t,s) \in [a,b] \times [0,1], \\ &\gamma(t) \geq c_\gamma \|\gamma\|, \text{ for every } t \in [a,b]. \end{aligned}$$

Since the derivative of the function $G_0(t,s)/\Phi(s)$ with respect to s is non-negative for all $s \in [0,1]$, the function $G_0(t,s)/\Phi(s)$ is a non-decreasing function of s.

If we set

$$c_0(t) = \min\Big\{\frac{2t}{3-t}, \frac{1}{2}t^2(3-t)\Big\} = \frac{1}{2}t^2(3-t),$$

we may take, for $[a,b] \subset (0,1]$,

$$c_0 := \min_{a \leq t \leq b} c_0(t) = \frac{1}{2}a^2(3-a).$$

We observe that $\|\gamma\| = \frac{1}{3}$, $\min_{t\in[a,b]} \gamma(t) = \gamma(a)$ and therefore we have

$$\gamma(t) \geq 3\gamma(a)\|\gamma\|, \text{ for every } t \in [a,b],$$

so that we can take

$$c_\gamma := 3\gamma(a).$$

Thus, for an arbitrary $[a,b] \subset (0,1]$, we may set

$$c := \min\{c_0, c_\gamma\} = \frac{1}{2}a^2(3-a). \tag{10}$$

By a solution of the BVP (6)-(7) we mean a solution of the corresponding integral equation (9) and we work in the Banach space $C[0,1]$ endowed with the usual supremum norm $\|u\| := \sup\{|u(t)| : \ t \in [0,1]\}$ and the above hypotheses enable us to utilize the cone

$$K = \big\{u \in C[0,1] : \min_{t\in[a,b]} u(t) \geq c\|u\|\big\},$$

where $[a,b] \subset (0,1]$ and c is as in (10).

If Ω is a bounded open subset of K (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. We write

$$K_r = \{u \in K : \|u\| < r\} \text{ and } \overline{K}_r = \{u \in K : \|u\| \leq r\}.$$

We consider now the map $T : C[0,1] \to C[0,1]$ defined by

$$Tu(t) := \gamma(t)(k_0 + B(\alpha[u])) + Fu(t),$$

where

$$Fu(t) := \int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds.$$

Theorem 2.1 *If the hypotheses* (C_1)-(C_4) *hold for some* $r > 0$, *then* T *maps* $\overline{K}_r$ *into* K. *When these hypotheses hold for each* $r > 0$, T *maps* K *into* K. *Moreover,* T *is a compact map.*

Proof. Take $u \in \overline{K}_r$. Then we have, for $t \in [0, 1]$,

$$\begin{aligned} Tu(t) &= \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds \\ &\leq \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 \Phi(s)g(s)f(s,u(s))\,ds, \end{aligned}$$

therefore

$$\|Tu\| \leq \|\gamma\|(k_0 + B(\alpha[u])) + \int_0^1 \Phi(s)g(s)f(s,u(s))\,ds.$$

Then we have

$$\min_{t\in[a,b]} Tu(t) \geq c\Big[\|\gamma\|(k_0 + B(\alpha[u])) + \int_0^1 \Phi(s)g(s)f(s,u(s))\,ds\Big] \geq c\|Tu\|.$$

Hence we have $Tu \in K$ for every $u \in \overline{K}_r$. Moreover, the map T is compact, since it is a sum of two compact maps. In fact, the compactness of F is well-known and the perturbation $\gamma(t)(k_0 + B(\alpha[u]))$ is compact since γ and B are continuous and it maps a bounded set into a bounded subset of a 1-dimensional space. □

We make use of the following numbers

$$f^{0,\rho} := \sup_{0\leq u\leq\rho,\ 0\leq t\leq 1} \frac{f(t,u)}{\rho}, \quad f_{\rho,\rho/c} := \inf_{\rho\leq u\leq\rho/c,\ a\leq t\leq b} \frac{f(t,u)}{\rho},$$

$$\frac{1}{m} := \sup_{t\in[0,1]} \int_0^1 G_0(t,s)g(s)\,ds, \quad \frac{1}{M(a,b)} := \inf_{t\in[a,b]} \int_a^b G_0(t,s)g(s)\,ds,$$

use the notation

$$\mathcal{G}(s) := \int_0^1 G_0(t,s)\,dA(t),$$

and assume

(C_5) $1 - \delta_2\alpha[\gamma] > 0.$

First, we give a condition that implies the index is 1 on the set K_ρ.

Lemma 2.1 *Assume that there exists* $\rho > 0$ *such that*

(I_ρ^1) *the following inequality holds:*

$$\frac{k_0}{3\rho(1 - \delta_2\alpha[\gamma])} + f^{0,\rho}\Big(\frac{\delta_2}{3(1 - \delta_2\alpha[\gamma])}\int_0^1 \mathcal{G}(s)g(s)\,ds + \frac{1}{m}\Big) < 1. \tag{11}$$

Then the fixed point index, $i_K(T, K_\rho)$, is 1.

Proof. We show that

$$\lambda u \neq Tu \text{ for every } u \in \partial K_\rho \text{ and for every } \lambda \geq 1,$$

which implies that the index is 1 on the set K_ρ. In fact, if there exists $\lambda \geq 1$ and $u \in \partial K_\rho$ such that

$$\lambda u(t) = Tu(t) = \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds$$

then we have

$$\lambda u(t) \leq \gamma(t)(k_0 + \delta_2\alpha[u]) + \int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds \tag{12}$$

and

$$\lambda\alpha[u] \leq \alpha[\gamma](k_0 + \delta_2\alpha[u]) + \int_0^1 \mathcal{G}(s)g(s)f(s,u(s))\,ds.$$

Hence we obtain

$$(\lambda - \delta_2\alpha[\gamma])\alpha[u] \leq k_0\alpha[\gamma] + \int_0^1 \mathcal{G}(s)g(s)f(s,u(s))\,ds.$$

Substituting into (12) gives

$$\lambda u(t) \leq \frac{\lambda k_0\gamma(t)}{\lambda - \delta_2\alpha[\gamma]} + \frac{\gamma(t)\delta_2}{\lambda - \delta_2\alpha[\gamma]}\int_0^1 \mathcal{G}(s)g(s)f(s,u(s))\,ds + \int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds.$$

Taking the supremum for $t \in [0,1]$ gives

$$\begin{aligned}\lambda\rho &\leq \frac{\lambda k_0\|\gamma\|}{\lambda - \delta_2\alpha[\gamma]} + \frac{\|\gamma\|\delta_2}{\lambda - \delta_2\alpha[\gamma]}\int_0^1 \mathcal{G}(s)g(s)\rho f^{0,\rho}\,ds + \sup_{t\in[0,1]}\int_0^1 G_0(t,s)g(s)\rho f^{0,\rho}\,ds \\ &\leq \frac{k_0\|\gamma\|}{1 - \delta_2\alpha[\gamma]} + \frac{\|\gamma\|\delta_2}{1 - \delta_2\alpha[\gamma]}\int_0^1 \mathcal{G}(s)g(s)\rho f^{0,\rho}\,ds + \sup_{t\in[0,1]}\int_0^1 G_0(t,s)g(s)\rho f^{0,\rho}\,ds.\end{aligned}$$

Thus we have,

$$\lambda \leq \frac{k_0}{3\rho(1 - \delta_2\alpha[\gamma])} + f^{0,\rho}\Big(\frac{\delta_2}{3(1 - \delta_2\alpha[\gamma])}\int_0^1 \mathcal{G}(s)g(s)\,ds + \frac{1}{m}\Big) < 1.$$

This contradicts the fact that $\lambda \geq 1$ and proves the result. □

In order to give a condition that implies the fixed point index is equal to 0, we make use of the open set

$$V_\rho = \big\{u \in K : \min_{t\in[a,b]} u(t) < \rho\big\}.$$

V_ρ is equal to the set called $\Omega_{\rho/c}$ in [33]. Note that $K_\rho \subset V_\rho \subset K_{\rho/c}$.

Lemma 2.2 *Assume that there exists $\rho > 0$ such that*

(I_{ρ}^{0}) *the following inequality holds:*

$$\frac{k_0 c}{3\rho(1-\delta_1\alpha[\gamma])} + \Big(\frac{c\delta_1}{3(1-\delta_1\alpha[\gamma])}\int_a^b \mathcal{G}(s)g(s)\,ds + \frac{1}{M(a,b)}\Big) f_{\rho,\rho/c} > 1. \tag{13}$$

Then we have $i_K(T,V_\rho)=0$.

Proof. Let $e(t)\equiv 1$ for $t\in[0,1]$. Then $e\in K$. We prove that

$$u \neq T(u)+\lambda e \quad \text{for all } u\in\partial V_\rho \quad \text{and } \lambda\geq 0,$$

which ensures that the index is 0 on the set V_ρ. In fact, if this does not happen, there exist $u\in\partial V_\rho$ and $\lambda\geq 0$ such that $u=Tu+\lambda e$, that is

$$u(t)=\gamma(t)(k_0+B(\alpha[u]))+\int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds+\lambda.$$

Then we have

$$u(t)\geq \gamma(t)(k_0+\delta_1\alpha[u])+\int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds+\lambda, \tag{14}$$

and

$$\begin{aligned}\alpha[u] &\geq \alpha[\gamma](k_0+\delta_1\alpha[u])+\int_0^1 \mathcal{G}(s)g(s)f(s,u(s))\,ds+\alpha[\lambda]\\ &\geq \alpha[\gamma](k_0+\delta_1\alpha[u])+\int_0^1 \mathcal{G}(s)g(s)f(s,u(s))\,ds.\end{aligned}$$

Hence we obtain

$$(1-\delta_1\alpha[\gamma])\alpha[u]\geq\ k_0\alpha[\gamma]+\int_0^1 \mathcal{G}(s)g(s)f(s,u(s))\,ds.$$

Substituting into (14) gives

$$u(t)\geq \frac{k_0\gamma(t)}{1-\delta_1\alpha[\gamma]}+\frac{\gamma(t)\delta_1}{1-\delta_1\alpha[\gamma]}\int_0^1 \mathcal{G}(s)g(s)f(s,u(s))\,ds+\int_0^1 G_0(t,s)g(s)f(s,u(s))\,ds+\lambda.$$

Then we have, for $t\in[a,b]$,

$$u(t)\geq \frac{k_0 c_\gamma\|\gamma\|}{1-\delta_1\alpha[\gamma]}+\frac{c_\gamma\|\gamma\|\delta_1}{1-\delta_1\alpha[\gamma]}\int_a^b \mathcal{G}(s)g(s)\rho f_{\rho,\rho/c}\,ds+\int_a^b G_0(t,s)g(s)\rho f_{\rho,\rho/c}\,ds+\lambda,$$

and therefore

$$\min_{t\in[a,b]} u(t) \geq \frac{k_0 c}{3\rho(1-\delta_1\alpha[\gamma])} + f_{\rho,\rho/c}\Big(\frac{c\delta_1}{3(1-\delta_1\alpha[\gamma])}\int_a^b \mathcal{G}(s)g(s)\,ds + \frac{1}{M(a,b)}\Big) + \lambda.$$

By (I^0_ρ) we have that $\min_{t\in[a,b]} u(t) > \rho + \lambda \geq \rho$. This contradict the fact that $u \in \partial V_\rho$ and proves the result. □

We can now state a result for the existence of one or two positive solutions for the integral equation (9).

Theorem 2.2 *Equation* (9) *has a positive solution in* K *if either of the following conditions hold.*

(S_1) *There exist* $\rho_1, \rho_2 \in (0,\infty)$*, with* $\rho_1 < \rho_2$*, such that* $(\mathrm{I}^1_{\rho_1})$ *and* $(\mathrm{I}^0_{\rho_2})$ *hold.*

(S_2) *There exist* $\rho_1, \rho_2 \in (0,\infty)$*, with* $\rho_1 < c\rho_2$*, such that* $(\mathrm{I}^0_{\rho_1})$ *and* $(\mathrm{I}^1_{\rho_2})$ *hold.*

Equation (9) *has two positive solutions in* K *if either of the following conditions hold.*

(D_1) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$*, with* $\rho_1 < \rho_2 < c\rho_3$*, such that* $(\mathrm{I}^1_{\rho_1})$*,* $(\mathrm{I}^0_{\rho_2})$ *and* $(\mathrm{I}^1_{\rho_3})$ *hold.*

(D_2) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$*, with* $\rho_1 < c\rho_2$ *and* $\rho_2 < \rho_3$*, such that* $(\mathrm{I}^0_{\rho_1})$*,* $(\mathrm{I}^1_{\rho_2})$ *and* $(\mathrm{I}^0_{\rho_3})$ *hold.*

We omit the proof which follows simply from properties of fixed point index, for details of similar proofs see [26, 34]. Note that, if the nonlinearity f has a suitable oscillatory behavior, one may establish, by the same method, the existence of more than two positive solutions (we refer the reader to [27, 33] to see the type of results that may be stated).

Example 2.1 *We now assume that* $g \equiv 1$*,* $\alpha[u] = u(\xi)$*,* $\xi \in (0,1)$ *and we make, as in [25], the choice*

$$B(w) = \begin{cases} \frac{1}{4}w, & 0 \leq w \leq 1, \\ \frac{1}{8}w + \frac{1}{8}, & w \geq 1. \end{cases}$$

In this case we have $\delta_1 = \frac{1}{8}$ *and* $\delta_2 = \frac{1}{4}$*,*

$$\alpha[\gamma] = \int_0^1 \gamma(t)\,dA(t) = \frac{1}{6}(3\xi^2 - \xi^3)$$

and

$$\int_0^1 \mathcal{G}(s)\,ds = \xi^2\Big(-\frac{1}{24}\xi^2 - \frac{1}{6}\xi + \frac{1}{4}\Big).$$

By direct calculation one gets $m = 8$*. The 'optimal'* $[a,b]$*, the interval for which*

$$M := M(a,b) = \frac{12}{3a^2b^2 - 2a^3b - a^4}$$

is a minimum, is given by the interval

$$\Big[\frac{-3+\sqrt{33}}{4},1\Big], \tag{15}$$

this gives $M = 22.032$ *and* $c = 0.545$.

Remark 2.1 *Until now we have discussed in details the case of the BCs* (2). *We point out that the same approach may be applied to the BCs* (3), (4), (5).

In all these cases the growth assumptions on the nonlinearities, that are the conditions equivalent to (11) *and* (13), *can be written in the form*

$$\frac{k_0\|\gamma\|}{(1-\delta_2\alpha[\gamma])\rho}+f^{0,\rho}\Big(\frac{\delta_2\|\gamma\|}{(1-\delta_2\alpha[\gamma])}\int_0^1 \mathcal{G}(s)g(s)\,ds+\frac{1}{m}\Big)<1,$$

and

$$\frac{k_0c_\gamma\|\gamma\|}{(1-\delta_1\alpha[\gamma])\rho}+\Big(\frac{c_\gamma\delta_1\|\gamma\|}{(1-\delta_1\alpha[\gamma])}\int_a^b \mathcal{G}(s)g(s)\,ds+\frac{1}{M(a,b)}\Big)f_{\rho,\rho/c}>1,$$

where

	BCs (2)	*BCs* (3)	*BCs* (4)	*BCs* (5)
$\gamma(t)$	$(3t^2-t^3)/6$	1	t	$t^2/2$
c_γ	$a^2(3-a)/2$	1	a	$a^2/2$
c	$a^2(3-a)/2$	$a^2(3-a)/2$	$a^2(3-a)/2$	$a^2/2$

This also illustrates that the cone K, *given by the constant* c, *varies according to the non-homogeneous BCs considered.*

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References

[1] R. P. Agarwal, On fourth order boundary value problems arising in beam analysis, *Differential Integral Equations* **2** (1989), 91-110.

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM. Rev.* **18** (1976), 620-709.

[3] P. Amster and P. P. Cárdenas Alzate, A shooting method for a nonlinear beam equation, *Nonlinear Anal.* **68** (2008), 2072-2078.

[4] D. R. Anderson and J. Hoffacker, Existence of solutions for a cantilever beam problem, *J. Math. Anal. Appl.* **323** (2006), 958-973.

[5] G. Bonanno and B. Di Bella, A boundary value problem for fourth-order elastic beam equations, *J. Math. Anal. Appl.* **343** (2008), 1166-1176.

[6] A. Cabada and F. Minhós, Fully nonlinear fourth-order equations with functional boundary conditions, *J. Math. Anal. Appl.* **340** (2008), 239-251.

[7] C. J. Chyan and J. Henderson, Positive solutions of $2m$th-order boundary value problems, *Appl. Math. Lett.* **15** (2002), 767-774.

[8] M. Eggensperger and N. Kosmatov, Positive solutions of a fourth-order multi-point boundary value problem, *Commun. Math. Anal.* **6** (2009), 22-30.

[9] J. Fialho and F. Minhós, Existence and location results for hinged beams equations with unbounded nonlinearities, *Nonlinear Anal.*, to appear.

[10] D. Franco and D. O'Regan, Existence of solutions to second order problems with nonlinear boundary conditions, *Discrete Contin. Dyn. Syst.*, **suppl.** (2005), 273-280.

[11] J. R. Graef, J. Henderson, P. J. Y. Wong and B. Yang, Three solutions of an nth order three-point focal type boundary value problem, *Nonlinear Anal.* **69** (2008), 3386-3404.

[12] J. R. Graef, C. Qian and B. Yang, Multiple symmetric positive solutions of a class of boundary value problems for higher order ordinary differential equations, *Proc. Amer. Math. Soc.* **131** (2003), 577-585.

[13] J. R. Graef, C. Qian and B. Yang, A three point boundary value problem for nonlinear fourth order differential equations, *J. Math. Anal. Appl.* **287** (2003), 217-233.

[14] J. R. Graef and B. Yang, On a nonlinear boundary value problem for fourth order equations, *Appl. Anal.* **72** (1999), 439-448.

[15] J. R. Graef and B. Yang, Positive solutions to a multi-point higher order boundary value problem, *J. Math. Anal. Appl.* **316** (2006), 409-421.

[16] J. R. Graef and J. R. L. Webb, Third order boundary value problems with nonlocal boundary conditions, *Nonlinear Anal.*, to appear.

[17] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[18] J. Henderson and D. Ma, Uniqueness of solutions for fourth-order nonlocal boundary value problems, *Bound. Value Probl.*, **2006**, 12 pp.

[19] G. Han and Z. Xu, Multiple solutions of some nonlinear fourth-order beam equations, *Nonlinear Anal.* **68** (2008), 3646-3656.

[20] G. Infante, Positive solutions of differential equations with nonlinear boundary conditions, *Discrete Contin. Dyn. Syst.*, **suppl.** (2003), 432-438.

[21] G. Infante, Nonzero solutions of second order problems subject to nonlinear BCs, *Dynamic systems and applications Vol. 5*, 222-226, Dynamic, Atlanta, GA, 2008.

[22] G. Infante, Positive solutions of some nonlinear BVPs involving singularities and integral BCs, *Discrete Contin. Dyn. Syst. Ser. S* **1** (2008), 99-106.

[23] G. Infante, Nonlocal boundary value problems with two nonlinear boundary conditions, *Commun. Appl. Anal.* **12** (2008), 279–288.

[24] G. Infante and P. Pietramala, Eigenvalues and non-negative solutions of a system with nonlocal BCs, *Nonlinear Stud.*, to appear.

[25] G. Infante and P. Pietramala, Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations, *Nonlinear Anal.*, to appear.

[26] G. Infante and J. R. L. Webb, Nonzero solutions of Hammerstein integral equations with discontinuous kernels, *J. Math. Anal. Appl.* **272** (2002), 30-42.

[27] G. Infante and J. R. L. Webb, Nonlinear nonlocal boundary value problems and perturbed Hammerstein integral equations, *Proc. Edinb. Math. Soc.* **49** (2006), 637-656.

[28] G. L. Karakostas and P. Ch. Tsamatos, Existence of multiple positive solutions for a nonlocal boundary value problem, *Topol. Methods Nonlinear Anal.* **19** (2002), 109-121.

[29] B. K. Karna, E. R. Kaufmann and J. Nobles, Comparison of eigenvalues for a fourth-order four-point boundary value problem, *Electron. J. Qual. Theory Differ. Equ.*, **2005**, 9 pp.

[30] P. S. Kelevedjiev, P. K. Palamides and N. I. Popivanov, Another understanding of fourth-order four-point boundary-value problems, *Electron. J. Differential Equations*, **2008**, 15 pp.

[31] L. Kong and J. Wang, Multiple positive solutions for the one-dimensional p-Laplacian, *Nonlinear Anal.* **42** (2000), 1327-1333.

[32] N. Kosmatov, Countably many solutions of a fourth order boundary value problem, *Electron. J. Qual. Theory Differ. Equ.*, **2004**, 15 pp.

[33] K. Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, *J. London Math. Soc* **63** (2001), 690-704.

[34] K. Q. Lan, Multiple positive solutions of Hammerstein integral equations with singularities, *Differential Equations and Dynamical Systems* **8** (2000), 175-195.

[35] Y. Li, On the existence of positive solutions for the bending elastic beam equations, *Appl. Math. Comput.* **189** (2007), 821-827.

[36] F. Minhós, T. Gyulov and A. I. Santos, Existence and location result for a fourth order boundary value problem, *Discrete Contin. Dyn. Syst.*, **suppl.** (2005), 662-671.

[37] B. P. Rynne, Infinitely many solutions of superlinear fourth order boundary value problems, *Topol. Methods Nonlinear Anal.* **19** (2002), 303-312.

[38] J. R. L. Webb, Fixed point index and its application to positive solutions of nonlocal boundary value problems, *Seminar of Mathematical Analysis, Univ. Sevilla Secr. Publ., Seville*, (2006), 181-205.

[39] J. R. L. Webb, Nonlocal conjugate type boundary value problems of higher order, *Nonlinear Anal.*, to appear.

[40] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach *J. London Math. Soc.* **74** (2006), 673-693.

[41] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions, *NoDEA Nonlinear Differential Equations Appl.* **15** (2008), 45-67.

[42] J. R. L. Webb and G. Infante, Nonlocal boundary value problems of arbitrary order, *J. London Math. Soc.* **79** (2009), 238-258.

[43] J. R. L. Webb, G. Infante and D. Franco, Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A* **138** (2008), 427-446.

[44] J. R. L. Webb and M. Zima, Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems, *Nonlinear Anal.*, to appear.

[45] Z. Yang, Positive solutions of a second-order integral boundary value problem, *J. Math. Anal. Appl.* **321** (2006), 751-765.

[46] Q. Yao, Existence of n solutions and/or positive solutions to a semipositone elastic beam equation, *Nonlinear Anal.* **66** (2007), 138-150.

[47] Q. Yao, Monotonically iterative method of nonlinear cantilever beam equations, *Appl. Math. Comput.* **205** (2008), 432-437.

[48] Q. Yao, Existence and multiplicity of positive solutions to a singular elastic beam equation rigidly fixed at both ends, *Nonlinear Anal.* **69** (2008), 2683-2694.

[49] Q. Yao, Positive solutions of a nonlinear elastic beam equation rigidly fastened on the left and simply supported on the right, *Nonlinear Anal.* **69** (2008), 1570-1580.