Electronic Journal of Qualitative Theory of Differential Equations
Spec. Ed. I, 2009 No. 5, 1–11; **http://www.math.u-szeged.hu/ejqtde/**

THE FRIEDRICHS EXTENSION OF CERTAIN SINGULAR DIFFERENTIAL OPERATORS, II

Grey M. Ballard and John V. Baxley

Department of Mathematics, Wake Forest University
Winston-Salem, NC 27109 USA
email: grey.ballard@gmail.com, baxley@wfu.edu

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We study the Friedrichs extension for a class of $2n$th order ordinary differential operators. These selfadjoint operators have compact inverses and the central problem is to describe their domains in terms of boundary conditions.

Key words and phrases: Selfadjoint ordinary differential operators, Friedrichs extensions, boundary conditions.
AMS (MOS) Subject Classifications: 47B25, 47E05, 34B05, 34L05

1 Introduction

An unbounded symmetric operator L with domain $D(L)$ dense in a Hilbert space $\mathcal{H}$ with inner product (u, v) is called semibounded below if $(Lu, u) \geq c(u, u)$ for some constant c and every $u \in D(L)$. By adding a multiple of the identity operator to L, without loss of generality it may be assumed that $c > 0$. The largest such c is called the lower bound. Such an operator has a particular selfadjoint extension F, called the Friedrichs extension [9]. A characteristic of the Friedrichs extension is that it preserves the lower bound: $(Fu, u) \geq c(u, u)$ for every $u \in D(F)$. Even a cursory examination of the development of the Friedrichs extension in [8, Section XII.5] shows that this extension depends critically on the domain $D(L)$ of the operator. In fact, the domain of the Friedrichs extension of L is obtained by intersecting $D(L^*)$, the domain of the adjoint operator L^*, with a subset of the Hilbert space $\mathcal{H}$ which is obtained by completing $D(L)$, considered as an incomplete metric space, with the "new" inner product (Lu, u). The proof of the existence of the Friedrichs extension leaves open the possibility of the existence of additional selfadjoint extensions which preserve the lower bound.

For selfadjoint ordinary differential operators, domains are usually specified by the imposition of boundary conditions. Ever since [10], it has been of interest to find a boundary condition description of the domain of the Friedrichs extension of specific operators.

Let τ be the formal differential operator defined by

$$\tau u(x) = -\frac{1}{m(x)}(p(x)u')',$$

where $m(x) > 0, p(x) > 0$, m, p are infinitely differentiable on $(0,1]$ and

$$M = \int_0^1 m(x)\left[\int_x^1 \frac{1}{p(t)}\,dt\right]\,dx < \infty. \tag{1}$$

By varying the initial domain, Baxley [2, 3] considered several different semibounded symmetric operators defined by the nth iterate τ^n in the Hilbert space $L^2(0,1;m)$ of functions u satisfying $\int_0^1 |u(x)|^2 m(x)dx < \infty$ and with inner product

$$(u,v) = \int_0^1 u(x)\overline{v(x)}m(x)\,dx.$$

In each case, the Friedrichs extension was a selfadjoint operator with compact inverse. The main goal was to describe the domains of these Friedrichs extensions in the usual way in terms of boundary conditions. The motivation for this earlier work was an application to the theory of Toeplitz matrices [4].

We return again to this problem, with a similar motivation. Now the application is to the theory of Toeplitz integral operators associated with the Hankel transform (see [6, 7] for related earlier results), work which will appear elsewhere. The Friedrichs extension obtained here is most closely related to the one which was designated G_n in [3]. It is possible that the Friedrichs extension obtained here is the same as that G_n, but we have been unable to verify this conjecture. In any case, the initial domain here is different and is more convenient for our application; the development is also simpler and more self-contained.

In recent years, much attention has been given to characterizing the domains of Friedrichs extensions of ordinary differential operators (see [11, 13, 14, 15]), as well as partial differential operators (see [5], which contains further references).

2 Preliminary Results

For a given positive integer n, let $C^{2n}(0,1)$ denote the collection of all functions on $(0,1)$ with continuous derivatives up to and including $2n$, and let

$$\mathcal{C}_n = \Big\{u \in C^{2n}(0,1) : u^{(j)}(1) = 0 \text{ for } j = 0,1,\cdots,n-1,\ \ (\tau^{n-1}u)'(x) = 0 \text{ near } x = 0\ ,$$
$$\text{and for } n \geq 2,\ \ p(x)(\tau^j u)'(x) \to 0 \text{ as } x \to 0^+,\ \text{for } 0 \leq j \leq n-2\Big\}\,.$$

It is clear that if $u \in \mathcal{C}_n$, then $\tau u \in \mathcal{C}_{n-2}$. The conditions $u^{(j)}(1) = 0$, $j = 0, 1, \cdots, n-1$ are equivalent to the conditions

$$\tau^j u(1) = 0, (\tau^j u)'(1) = 0, \quad j = 0, 1, \cdots, \frac{n}{2} - 1, \text{ if } n \text{ is even}$$

and, if n is odd, $u(1) = 0$ together with

$$(\tau^{j-1}u)'(1) = 0, (\tau^j u)(1) = 0 \text{ for } j = 1, 2, \cdots, \frac{n-1}{2}, \text{ if } n \geq 3.$$

Lemma 2.1 *If $u \in \mathcal{C}_n$, then $\tau^j u(x)$ is bounded as $x \to 0^+$, for $j = 0, 1, \cdots, n-1$.*

Proof: For $u \in \mathcal{C}_n$, since $(\tau^{n-1}u)'(x) = 0$ for x near 0, $\tau^{n-1}u(x)$ is constant near $x = 0$ and therefore bounded as $x \to 0^+$. We complete the proof by showing that if $g(x) = \tau^j u(x)$ $(1 \leq j \leq n-1)$ is bounded as $x \to 0^+$, then so is $\tau^{j-1}u(x)$. Since g is bounded on $(0, 1]$ and

$$\tau^{j-1}u(x) = -\int_x^1 \frac{1}{p(y)} \left[\int_0^y m(t)g(t)\, dt\right] dy,$$

then

$$\left|\tau^{j-1}u(x)\right| \leq K \int_x^1 \frac{1}{p(y)} \left[\int_0^y m(t)\, dt\right] dy.$$

Since the integral in (1) is finite, applying Fubini's theorem, we see that

$$\int_0^1 \frac{1}{p(y)} \left[\int_0^y m(t)\, dt\right] dy$$

is also finite and we are done.

Lemma 2.2 *If $u \in \mathcal{C}_n$, the following conditions hold:*

1. *For $i, j \geq 1$, $i + j \leq n$, then $(\tau^i u, \tau^j u) = (\tau^{i+j}u, u)$.*
2. *For $0 < x_1 < x_2 \leq 1$, $|u(x_2) - u(x_1)|^2 \leq (\tau u, u) \int_{x_1}^{x_2} \frac{1}{p(t)}\, dt$.*
3. *For $n \geq 2$ and $0 < x_1 < x_2 \leq 1$, $|p(x_2)u'(x_2) - p(x_1)u'(x_1)|^2 \leq (\tau^2 u, u) \int_{x_1}^{x_2} m(t)dt$.*

Proof: Statement *1* follows by integrating by parts using Lemma 2.1 and the definition of $\mathcal{C}_n$ to see that all boundary terms vanish.

For *2*, we use the Schwarz inequality and integration by parts to get, for $0 < x_1 < x_2 \leq 1$,

$$|u(x_2) - u(x_1)|^2 = \left|\int_{x_1}^{x_2} u'(t)dt\right|^2 \leq \int_{x_1}^{x_2} \frac{1}{p(t)}dt \int_0^1 p(t)|u'(t)|^2 dt = (\tau u, u) \int_{x_1}^{x_2} \frac{1}{p(t)}\, dt.$$

For *3*, we similarly write

$$\begin{aligned} |p(x_2)u(x_2) - p(x_1)u(x_1)|^2 &= \left| \int_{x_1}^{x_2} (p(t)u'(t))' \, dt \right|^2 \\ &\le \int_{x_1}^{x_2} m(t) \, dt \int_0^1 \frac{1}{m(t)} \left|(p(t)u'(t))'\right|^2 \, dt \\ &= (\tau^2 u, u) \int_{x_1}^{x_2} m(t) \, dt. \end{aligned}$$

Lemma 2.3 *Let $u \in \mathcal{C}_n$ and $0 < x_1 < x_2 \le 1$. Then for k odd, $k = 2j+1 \le n$,*

$$\left|\tau^j u(x_2) - \tau^j u(x_1)\right|^2 \le (\tau^k u, u) \int_{x_1}^{x_2} \frac{1}{p(t)} \, dt$$

and for k even, $k = 2j \le n$,

$$\left|p(x_2)(\tau^{j-1}u)'(x_2) - p(x_1)(\tau^{j-1}u)'(x_1)\right|^2 \le (\tau^k u, u) \int_{x_1}^{x_2} m(t) \, dt.$$

Proof: In the first case, $\tau^j u \in C_{n-2j}$ and $n - 2j \ge 1$. In the second case, $\tau^{j-1}u \in C_{n-2(j-1)}$ and $n - 2(j-1) \ge 2$, so these statements follow quickly from Lemma 2.2.

Lemma 2.4 *If $u \in \mathcal{C}_n$, then $(\tau^i u, u) \le M(\tau^{i+1}u, u)$ for $0 \le i \le n-1$.*

Proof: Consider the $i = 0$ case. Since $u \in \mathcal{C}_n$, from Lemma 2.2 (*2*) we have

$$|u(x_2) - u(x_1)|^2 \le (\tau u, u) \int_{x_1}^{x_2} \frac{1}{p(t)} \, dt.$$

Letting $x_1 = x$ and $x_2 = 1$ gives

$$|u(x)|^2 \le (\tau u, u) \int_x^1 \frac{1}{p(t)} \, dt,$$

and integrating we have

$$(u, u) = \int_0^1 |u(x)|^2 m(x) \, dx \le (\tau u, u) \int_0^1 m(x) \left[\int_x^1 \frac{1}{p(t)} \, dt \right] dx = M(\tau u, u).$$

If $i = 2j \le n-1$ is even, since $u \in \mathcal{C}_n$ implies $\tau^j u \in \mathcal{C}_{n-2j}$ with $n - 2j \ge 1$, then Lemma 2.2 (*1*) and the above argument give

$$(\tau^i u, u) = (\tau^j u, \tau^j u) \le M(\tau^{j+1}u, \tau^j u) = M(\tau^{i+1}u, u).$$

If $i = 2j+1 \le n-1$ is odd,

$$\|\tau^j u\|^2 = (\tau^j u, \tau^j u) \le M(\tau^{j+1}u, \tau^j u) \le M\|\tau^{j+1}u\| \, \|\tau^j u\|$$

which implies $\|\tau^j u\| \le \|\tau^{j+1}u\|$, so

$$(\tau^i u, u) = (\tau^{j+1}u, \tau^j u) \le M\|\tau^{j+1}u\|^2 = M(\tau^{j+1}u, \tau^{j+1}u) = M(\tau^{i+1}u, u).$$

3 The Friedrichs Extension

Let $C^\infty(0,1)$ be the collection of all infinitely differentiable functions on $(0,1)$ and let $L_n u = \tau^n u$ for $u \in D(L_n)$ where

$$D(L_n) = \big\{u \in C^\infty(0,1) : u(x) = 0 \text{ near } x = 1, \;\; (\tau^{n-1}u)'(x) = 0 \text{ near } x = 0\,,$$

$$\text{and for } n \geq 2, \;\; p(x)(\tau^j u)'(x) \to 0 \text{ as } x \to 0^+, \text{ for } 0 \leq j \leq n-2\big\}\,.$$

Note that $D(L_n) \subset \mathcal{C}_n$. Let $C_0^\infty(0,1)$ consist of all functions in $C^\infty(0,1)$ with compact support in $(0,1)$. The next lemma is obvious.

Lemma 3.1 $C_0^\infty(0,1) \subset D(L_n) \subset D(L_{n+1})$ *for all* $n \geq 1$.

We view L_n as an operator in the weighted Hilbert space $L^2(0,1;m)$ defined earlier. From Lemma 3.1, $D(L_n)$ is dense in $L^2(0,1;m)$ for each n.

Theorem 3.1 *The operator* L_n *is symmetric and nonnegative for all* $n \geq 1$.

Proof: Symmetry follows directly from Lemma 2.2 (*1*). We note that for $n = 2j$ even,

$$(L_n u, u) = (\tau^n u, u) = (\tau^j u, \tau^j u) = \int_0^1 |\tau^j u(x)|^2 m(x)\, dx \geq 0,$$

and for $n = 2j+1$ odd,

$$(L_n u, u) = (\tau^n u, u) = (\tau^{j+1} u, \tau^j u) = \int_0^1 p(x)|(\tau^j u)'(x)|^2\, dx \geq 0,$$

and so L_n is nonnegative for all $k \geq 1$.

Iterating Lemma 2.4, we see that $(u,u) \leq M^n (L_n u, u)$ for any $u \in D(L_n)$. Thus M^{-n} is a positive lower bound for L_n.

For each n, Theorem 3.1 shows that L_n has a Friedrichs extension, which we denote by $\tilde{L}_n$.

Lemma 3.2 *Let* $u \in D(\tilde{L}_n)$. *Then*

1. $u^{(i)}(1) = 0$ *for* $0 \leq i \leq n-1$,
2. $|u(x_2) - u(x_1)|^2 \leq M^{n-1}(\tilde{L}_n u, u) \int_{x_1}^{x_2} \frac{1}{p(t)}\, dt$, *for* $0 < x_1 < x_2 \leq 1$,
3. $(u,u) \leq M^n(\tilde{L}_n u, u)$.

Proof: If $u \in D(\tilde{L}_n)$, then from [8, p. 1242], there exists $u_i \in D(L_n)$ such that $\|u_i - u\| \to 0$ and $(L_n u_i, u_i) \to (\tilde{L}_n u, u)$ as $i \to \infty$.

Since $u_i \in D(L_n)$ implies $\tau^k u_i \in D(L_n)$ for any integer $k \geq 0$, we can use Lemma 2.3 and Lemma 2.4 to conclude,

$$\left|\tau^j u_i(x_2) - \tau^j u_i(x_1)\right|^2 \leq M^{n-k}(L_n u_i, u_i) \int_{x_1}^{x_2} \frac{1}{p(t)}\, dt$$

for $k = 2j + 1 \leq n$ and

$$\left|p(x_2)(\tau^{j-1}u_i)'(x_2) - p(x_1)(\tau^{j-1}u_i)'(x_1)\right|^2 \leq M^{n-k}(L_n u_i, u_i) \int_{x_1}^{x_2} m(t)\, dt.$$

for $k = 2j \leq n$, whenever $0 < x_1 < x_2 \leq 1$. Since $\tau^j u_i$ and $(\tau^j u_i)'$ vanish for $x = 1$, it follows that all the sequences $\{\tau^j u_i\}$, for $2j + 1 \leq n$, and $\{p(\tau^j u_i)'\}$, for $2j \leq n$, are uniformly bounded and equicontinuous on any compact subset of $(0, 1]$. Using Ascoli's theorem and a familiar diagonalization argument, by passing to a subsequence we can assume without loss of generality that all of these sequences converge uniformly on any compact subset of $(0, 1]$. Since $\|u_i - u\| \to 0$, then $u_i \to u$ uniformly on any such compact subset of $(0, 1]$. Thus $u(1) = 0$. Letting $v = \lim_{i\to\infty} u_i'$, we write $-u_i(x) = \int_x^1 u_i'(t)dt$ and take limits using the bounded convergence theorem to get $-u(x) = \int_x^1 v(t)dt$. Differentiating gives $u'(x) = v(x)$ for $0 < x \leq 1$. So $u' = \lim_{i\to\infty} u_i'$. Continuing in this way, we find that each of these convergent sequences converge to the appropriate derivative of the limit function u, and it is easy to see that $u^{(j)}(1) = 0$ for $0 \leq j \leq n - 1$.

From Lemma 2.4, we have $(\tau u_i, u_i) \leq M^{n-1}(\tau^n u_i, u_i) = (L_n u_i, u_i)$. Then from Lemma 2.2 (*2*),

$$|u_i(x_2) - u_i(x_1)|^2 \leq M^{n-1}(L_n u_i, u_i) \int_{x_1}^{x_2} \frac{1}{p(t)}\, dt$$

for $0 < x_1 < x_2 \leq 1$. Taking limits, we have

$$|u(x_2) - u(x_1)|^2 \leq M^{n-1}(\tilde{L}_n u, u) \int_{x_1}^{x_2} \frac{1}{p(t)}\, dt.$$

Letting $x_1 = x$ and $x_2 = 1$ and integrating, we have $(u, u) \leq M^n(\tilde{L}_n u, u)$.

Theorem 3.2 *All eigenvalues of $\tilde{L}_n$ are strictly positive and $\tilde{L}_n$ has a compact inverse.*

Proof: Let u be an eigenfunction of $\tilde{L}_n$. Then from Lemma 3.2, $(u, u) \leq M^n(\tilde{L}_n u, u) = M^n\lambda(u, u)$ where λ is the corresponding eigenvalue. Hence all eigenvalues of $\tilde{L}_n$ are strictly positive. Thus $\tilde{L}_n^{-1}$ exists. To show $\tilde{L}_n^{-1}$ is compact, suppose there is a sequence $\{\tilde{L}_n u_i\}$ such that $\|\tilde{L}_n u_i\| \leq K$ for $n = 1, 2, \cdots$. We will show that $\{\tilde{L}_n^{-1}(\tilde{L}_n u_i)\} = \{u_i\}$

has a convergent subsequence. From Lemma 3.2 (*3*), $\|u_i\| \leq M^n \|\tilde{L}_n u_i\|$, and so we have

$$(\tilde{L}_n u_i, u_i) \leq M^n K^2 \text{ for all } n = 1, 2, \cdots .$$

Then from Lemma 3.2 (*2*),

$$|u_i(x_2) - u_i(x_1)|^2 \leq M^{2n-1} K^2 \int_{x_1}^{x_2} \frac{1}{p(t)}\, dt, \text{ for } 0 < x_1 < x_2 \leq 1$$

and so with $x_2 = 1$, $x_1 = x$

$$|u_i(x)|^2 \leq M^{2n-1} K^2 \int_x^1 \frac{1}{p(t)}\, dt, \text{ for } 0 < x \leq 1.$$

Thus the functions $\{u_i\}$ are equicontinuous and uniformly bounded on compact subintervals of $(0, 1]$. From Ascoli's theorem and a diagonalization argument we may assume the sequence converges uniformly to a limit function u on each compact subinterval of $(0, 1]$. Since the last inequality also holds for the limit function u, the dominated convergence theorem guarantees u_i converges to u in $L^2(0, 1; m)$, so $\tilde{L}_n^{-1}$ is compact.

The following lemma is immediate since 0 is not in the spectrum of $\tilde{L}_n$.

Lemma 3.3 *The range of $\tilde{L}_n$ is all of $L^2(0, 1; m)$.*

Because of the previous three lemmas, it follows from the theory of compact self-adjoint operators that all eigenvalues of the operator $\tilde{L}_n$ are positive and have finite multiplicity, and the eigenfunctions span $L^2(0, 1; m)$.

To prepare for the next lemma, for $0 \leq x \leq 1$, we define $Q_0(x) = 1$ and

$$Q_{j+1}(x) = \int_0^x \frac{1}{p(y)} \left[\int_0^y m(t) Q_j(t)\, dt \right] dy \text{ for } j = 0, 1, \cdots, n-2,$$

and we define $R_0(x) = \int_x^1 \frac{1}{p(t)}\, dt$ and

$$R_{j+1}(x) = \int_x^1 \frac{1}{p(y)} \left[\int_y^1 m(t) R_i(t)\, dt \right] dy, \text{ for } j = 0, 1, \cdots, n-2.$$

Just as in the proof of Lemma 2.1, it follows by induction that each Q_j is well-defined and bounded.

Lemma 3.4 *If $w \in N({L_n}^*)$, then for some constants a_j, $w = \sum_{j=0}^{n-1} a_j Q_j(x)$.*

Proof: It is easy to verify that

$$\tau^i Q_j = (-1)^i Q_{j-i}, \quad \text{for} \quad i \leq j, \quad \text{and} \quad \tau^i Q_j = 0 \quad \text{for} \quad i > j \tag{2}$$

and

$$p(\tau^i R_j)' = (-1)^i p R'_{j-i} \quad \text{for} \quad i < j \quad \text{and} \quad p(\tau^j R_j)' = (-1)^{j+1}. \tag{3}$$

Thus, τ^n maps all of the functions $Q_0, R_0, \cdots, Q_{k-1}, R_{k-1}$ to 0 and it is easy to show that the functions Q_j and R_j are linearly independent. Since $w \in N(L_n{}^*)$, it follows as in [8, pp. 1291-1294], that $w \in C^\infty(0,1)$ and $\tau^n u = 0$. Thus the $2n$ functions $\{Q_0, R_0, \cdots, Q_{k-1}, R_{k-1}\}$ must span $N(L_n{}^*)$. Therefore for $w \in N(L_n{}^*)$,

$$w = \sum_{j=0}^{n-1} (a_j Q_j + b_j R_j)$$

for some constants a_j and b_j. We need to show that $b_j = 0$ for all $j = 0, 1, \cdots, k-1$. Suppose $b_\gamma \neq 0$ where $b_j = 0$ for $\gamma + 1 \leq j \leq k-1$. Choose $u \in C^\infty(0,1)$ such that $u(x) = Q_{n-\gamma-1}(x)$ for $0 \leq x \leq 1/4$ and $u(x) = 0$ for $3/4 \leq x \leq 1$. Then $u \in D(L_n)$ and

$$(\tau^n u, w) = (L_n u, w) = (u, L_n{}^* w) = (u, \tau^n w).$$

Repeated integration by parts gives

$$(\tau^n u, w) = \sum_{i=0}^{n-1} (\tau^i u) p(x) (\tau^{n-i-1} w)' \Big|_0^1 - \sum_{i=0}^{n-1} p(x) (\tau^i u)' (\tau^{n-i-1} w) \Big|_0^1 + (u, \tau^n w)$$

and thus

$$\sum_{i=0}^{n-1} (\tau^i u) p(x) (\tau^{n-i-1} w)' \Big|_0^1 - \sum_{i=0}^{n-1} p(x) (\tau^i u)' (\tau^{n-i-1} w) \Big|_0^1 = 0.$$

Since $u(x) \equiv 0$ for $3/4 \leq x \leq 1$ our equation simplifies to

$$\lim_{x \to 0} \left[\sum_{i=0}^{n-1} (\tau^i u) p(x) (\tau^{n-i-1} w)' - \sum_{i=0}^{n-1} p(x) (\tau^i u)' (\tau^{n-i-1} w) \right] = 0.$$

For $0 \leq x \leq 1/4$, $u(x) = Q_{n-\gamma-1}(x)$ so from (2), $\tau^{n-\gamma-1} u = (-1)^{n-\gamma-1} Q_0 \equiv (-1)^{n-\gamma-1}$. Thus we have

$$\lim_{x \to 0} \left[\sum_{i=0}^{n-\gamma-1} (\tau^i Q_{n-\gamma-1}) p(x) (\tau^{n-i-1} w)' - \sum_{i=0}^{n-\gamma-2} p(x) (\tau^i Q_{n-\gamma-1})' (\tau^{n-i-1} w) \right] = 0. \tag{4}$$

Substituting for w, we see that any term with a product of the form $(\tau^i Q_j) p(x) (\tau^k Q_m)'$ tends to 0, so all terms in w involving the Q_j's vanish and we may as well assume that

$$w = \sum_{j=0}^{\gamma} b_j R_j.$$

In the second sum in (4), we encounter only factors of the form $\tau^k R_j$ where $k \geq \gamma + 1$ and $j \leq \gamma$. So by (3), all these terms tend to zero. In the first sum in (4), we encounter only factors of the form $p(x)(\tau^k R_j)'$ where $k \geq \gamma$ and $j \leq \gamma$. By (3), the only term which does not tend to zero is the one for which $k = j = \gamma$ so $i = n - \gamma - 1$ and (2), (3) give

$$\lim_{x \to 0} (\tau^{n-\gamma-1} Q_{n-\gamma-1}) p(x) (\tau^\gamma b_\gamma R_\gamma)' = (-1)^n b_\gamma = 0.$$

Thus $b_\gamma = 0$, a contradiction.

Since 0 is a point of regular type of L_n [1, pp. 91-94], the deficiency indices of L_n^* are equal to the dimension of the null space of L_n^*. By Lemma 3.4, these deficiency indices are each n. The following theorem almost follows from the general theory [8] which states that every selfadjoint extension of L_n comes by the imposition of n boundary conditions, all at the regular endpoint $x = 1$. The problem is that the general theory in [8] is all developed by starting with the initial domain $C_0^\infty(0, 1)$ and one would have to re-do that theory in this new context. It is likely that this approach would work, but it is not hard to give a direct proof.

Theorem 3.3 $D(\tilde{L}_n) = \{u \in D({L_n}^*) : u^{(i)}(1) = 0$ *for* $i = 0, 1, \cdots, n - 1\}$.

Proof: Since $D(\tilde{L}_n) \subset D({L_n}^*)$ and Lemma 3.2 (*1*) implies the boundary conditions are satisfied for $u \in D(\tilde{L}_n)$, we need only show the set described is a subset of $D(\tilde{L}_n)$.

Let $u \in D({L_n}^*)$ such that $u^{(i)}(1) = 0$ for $i = 0, 1, \cdots, n - 1$. From Lemma 3.3, the range of $\tilde{L}_n$ is all of $L^2(0, 1; m)$, so there exists $v \in D(\tilde{L}_n)$ such that $\tilde{L}_n v = {L_n}^* u$. We let $w = u - v$ and we will show that $w = 0$. Since $D(\tilde{L}_n) \subset D({L_n}^*)$, $w \in N({L_n}^*)$, and from Lemma 3.4, w has the form

$$w = \sum_{j=0}^{k-1} a_j Q_j(x). \tag{5}$$

It is easy to verify that w satisfies the conditions required for membership in $\mathcal{C}_n$ as $x \to 0^+$. Since u satisfies the boundary conditions by hypothesis and v satisfies the boundary conditions by Lemma 3.2 (*1*), we have $w^{(i)}(1) = 0$ for $i = 0, 1, \cdots, n - 1$, and so $w \in \mathcal{C}_n$. Iterating Lemma 2.4, we find

$$(w, w) \leq M^n (\tau^n w, w),$$

so the Schwarz inequality gives $\|w\| \leq M^n \|\tau^n w\|$, implying $w = 0$ or $u = v \in D(\tilde{L}_n)$.

A reader for whom the theory in [8] is not well-known might be worried that some functions in $D({L_n}^*)$ would lack sufficient smoothness for the boundary conditions in Theorem 3.3 to make sense. Although it follows from [8, Theorem 10, p. 1294] that all such functions have $2n - 1$ continuous derivatives, it is easy to see without this general theory that any $u \in D({L_n}^*)$ has at least $n - 1$ continuous derivatives. Returning to the proof of Theorem 3.3, one sees that $u = w + v$, where $w \in C^\infty(0,1)$ (as noted in the proof of Lemma 3.4), and $v \in D(\tilde{L}_n)$, which (as shown in the proof of Lemma 3.2) has at least $n-1$ continuous derivatives.

We now identify the eigenfunctions and eigenvalues of $\tilde{L}_1$ in one special case that arises in the case of Toeplitz integral operators associated with the Hankel transform.

Theorem 3.4 *Suppose $m(x) = p(x) = x^{2\nu}$ where $\nu > 0$ is a constant. Then the eigenvalues Λ_k of $\tilde{L}_1$ have multiplicity one and $\Lambda_k = z_k^2$, where z_k is the k^{th} positive zero of the Bessel function $J_{\nu-1/2}$. An eigenfunction corresponding to Λ_k is $x^{1/2-\nu}J_{\nu-1/2}(z_k x)$.*

Proof: If Λ is an eigenvalue of $\tilde{L}_1$ and u is a corresponding eigenfunction, then $\Lambda > 0$, u is in the null space of $\tilde{L}_1 - \Lambda I$ and we have the equation

$$L_1^* u - \Lambda u = \tilde{L}_1 u - \Lambda u = 0.$$

Again, using [8, pp. 1291-1294], u is infinitely differentiable and $\tau u = \Lambda u$. This equation reduces to

$$u'' + \frac{2\nu}{x}u' + \beta^2 u = 0$$

where $\Lambda = \beta^2$. It is easy to verify that the general solution of this equation is

$$u = c_1 x^{1/2-\nu}u_1(\beta x) + c_2 x^{1/2-\nu}u_2(\beta x),$$

where u_1, u_2 are any two linearly independent solutions of Bessel's equation. We may choose $u_1(x) = J_{\nu-1/2}(x)$ and $u_2(x) = Y_{\nu-1/2}(x)$. Let $w_2(x) = x^{1/2-\nu}u_2(\beta x)$. Known behavior (see [12]) of $u_2(x)$ as $x \to 0$ shows that

$$\lim_{x\to 0} x^{2\nu}w_2'(x) = d \neq 0$$

for some constant d. Choosing v in the domain of L_1 so that $v(x) = 1$ in a right neighborhood of $x = 0$, we can integrate by parts to get

$$(L_1 v, w_2) = (\tau v, w_2) = -d + (v, \tau w_2),$$

so that w_2 is not in the domain of L_1^*. Thus $c_2 = 0$. A similar calculation shows that $x^{1/2-\nu}u_2(\beta x)$ is in the domain of L_1^*. Then u is in the domain of $\tilde{L}_1$ if and only if $u(1) = 0$, which occurs if and only if $u_1(\beta) = J_{\nu-1/2}(\beta) = 0$. Thus β must be a zero z_k of this Bessel function.

References

[1] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space, Vol. I*, Ungar, New York, 1961.

[2] J. V. Baxley, The Friedrichs extension of certain singular differential operators, *Duke Math. J.* **35** (1968), 455-462.

[3] J. V. Baxley, Eigenvalues of singular differential operators by finite difference methods, II, *J. Math. Anal. Appl.* **38** (1972), 257-275.

[4] J. V. Baxley, Extreme eigenvalues of Toeplitz matrices associated with certain orthogonal polynomials, *SIAM J. Math. Anal.* **2** (1971), 470-482.

[5] J. F. Brasche and M. Melgaard, The Friedrichs extension of the Aharonov-Bohm Hamiltonian on a disc, *Integr. Equ. Oper. Theory* **52** (2005), 419436.

[6] J. R. Davis, Extreme eigenvalues of Toeplitz operators of the Hankel type I, *J. of Math. and Mech.* **14** (2) (1965), 245-275.

[7] J. R. Davis, On the extreme eigenvalues of Toeplitz operators of the Hankel type II, *Trans. Amer. Math. Soc.* **116** (1965), 267-299.

[8] N. Dunford and J. T. Schwartz, *Linear Operators, Part II: Spectral Theory*, Interscience, New York, 1963.

[9] K. O. Friedrichs, Spektraltheorie halbbeschrankter Operatoren, I, *Math. Ann.* **109** (1934), 465-487.

[10] K. O. Friedrichs, Uber die ausgezeichnete Randbedingung in der Spektraltheorie der halbbeschrankten gewohnlichen Differential-operatoren zweiter Ordnung, *Math. Ann.* **112** (1935), 1-23 .

[11] H. G. Kaper, M. K. Kwong, and A. Zettl, Characterizations of the Friedrichs extensions of singular Sturm-Liouville expressions, *SIAM J. Math. Anal.* **17** (1986), 772-777.

[12] N. N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall, New Jersey, 1965.

[13] M. Marletta and A. Zettl, The Friedrichs extension of singular differential operators, *J. Differential Equations* **160** (2000), 404-421.

[14] H.-D. Niessen and A. Zettl, Singular Sturm-Liouville problems: the Friedrichs extension and comparison of eigenvalues, *Proc. London Math. Soc.* **64** (1992), 545-578.

[15] A. Zettl, On the Friedrichs extension of singular differential operators, *Commun. Appl. Anal.* **2** (1998), 3136.