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# On a nonlinear system containing nonlocal terms related to a fluid flow model

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**Abstract**

We consider a nonlinear system of differential equations where the main parts may contain nonlocal dependence on the unknowns. This system is a generalization of a model describing fluid flow in porous medium. Existence of weak solutions, boundedness and stabilization of solutions as $t \to \infty$ is shown by using the theory of monotone operators, and some examples are given.

## 1 Introduction

This paper was motivated by the work [21]. There the authors investigated fluid flow in porous media. A porous medium is a solid medium with lots of tiny holes (e.g., limestone). The flow of a fluid through the medium is determined by the large surface of the solid matrix and the closeness of the holes. For a detailed introduction to this topic, see [3]. If the fluid carries dissolved chemical species, chemical reactions can occur, see [16]. Among these include reactions that can change the porosity. In the cited paper the following model was derived for such flow in one dimension:

$$\omega(t,x)u_t(t,x) = \alpha \cdot (|v(t,x)|u_x(t,x))_x + K(\omega(t,x))p_x(t,x)u_x(t,x) - ku(t,x)g(\omega(t,x)) \tag{1}$$
$$\omega_t(t,x) = bu(t,x)g(\omega(t,x)) \tag{2}$$
$$(K(\omega(t,x))p_x(t,x))_x = bu(t,x)g(\omega(t,x)), \tag{3}$$
$$v(t,x) = -K(\omega(t,x))p_x(t,x), \qquad t>0,\ x\in(0,1), \tag{4}$$

with some initial and boundary conditions where $\omega$ is the porosity, $u$ is the concentration of the dissolved chemical solute carried by the fluid, $p$ is the pressure, $v$ is the velocity, further, $\alpha$, $k$, $b$ are given constants, $K$ and $g$ are given real functions. Observe that after elimintaion of the fourth equation one obtains a system that contains three different types of differential equations: an ordinary, a parabolic and an elliptic one, see [11, 21]. Similar model was studied in [11] by using the method of Rothe. Some numerical experiments were done in [21] concering the above system, however, correct proof on existence of solutions was not made (and one can hardly find papers dealing with such kind of systems in rigorous mathematical way). In the following, we consider a generalization of the above system. Namely, we admit also nonlocal dependence on the unknowns. Such nonlocality is especially reasonable for diffusion processes (heat or population) where the diffusion coefficient may depend on terms which depend on the unknowns in a nonlocal way (e.g., on the integral of the density). Furthermore, nonlocal models arise also in climatology, see the papers [2, 12, 13, 14] where a climatology model containing functional differential equations was studied. For some other problems involving nonlocal differential equations, such as transmission problems, see [17, 18, 19], and as nonlocal boundary conditions, see [26, 25, 23].

In the following we show existence and properties of weak solutions (such as boundedness and stabilization as $t \to \infty$) to a nonlocal generalization of the above system by using the theory of operators of monotone type. Our assumptions will be the generalizations of the classical conditions. However these are strict assumptions, the examples given after each statement will show that the results apply in a large class of problems.

### 1.1 Notation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the uniform $C^1$ regularity property (see [1]), further, let $0 < T < \infty$, $2 \le p_1, p_2 < \infty$ be real numbers. In the following, $Q_T := (0,T)\times\Omega$, $Q_\infty := (0,\infty)\times\Omega$. Denote by $W^{1,p_i}(\Omega)$ the usual Sobolev space with the norm

$$\|v\|_{W^{1,p_i}(\Omega)} = \Big(\int_\Omega (|v|^{p_i} + \sum_{j=1}^n |D_j v|^{p_i})\Big)^{1/p_i}$$

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where $D_j$ denotes the distributional derivative with respect to the $j$-th variable (briefly $D = (D_1, \dots, D_n)$). In addition, let $V_i$ be a closed linear subspace of $W^{1,p_i}(\Omega)$ which contains $W_0^{1,p_i}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p_i}(\Omega)$), and let $L^{p_i}(0,T;V_i)$ be the Banach space of measurable functions $u\colon (0,T) \to V_i$ such that $\|u\|_{V_i}^{p_i}$ is integrable and the norm is given by

$$\|u\|_{L^{p_i}(0,T;V_i)} = \left( \int_0^T \|u(t)\|_{V_i}^{p_i} dt \right)^{1/p_i}.$$

The dual space of $L^{p_i}(0,T;V_i)$ is $L^{q_i}(0,T;V_i^*)$ where $\frac{1}{p_i} + \frac{1}{q_i} = 1$ and $V_i^*$ is the dual of $V_i$. We write briefly $X_i := L^{p_i}(0,T;V_i)$. The pairing between $V_i^*$, $V_i$ and $X_i^*$, $X_i$ is denoted by $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$, respectively, further, $D_t u$ stands for the derivative of a function $u \in L^{p_i}(0,T;V_i)$. It is well known (see [27]) that if $u \in X_i$, $D_t u \in X_i^*$ then $u \in C([0,T], L^2(\Omega))$ so that $u(0)$ makes sense.

## 1.2 Formulation of the problem

Let us consider the following system of equations:

$$D_t\omega(t,x) = f(t,x,\omega(t,x),u(t,x);u), \qquad \omega(0,x) = \omega_0(x), \tag{5}$$

$$\begin{aligned} D_t u(t,x) - \sum_{i=1}^n D_i \left[a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})\right] + \\ + a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p}) = g(t,x), \qquad u(0,x) = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} -\sum_{i=1}^n D_i[b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})] + \\ + b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p}) = h(t,x) \end{aligned} \tag{7}$$

with some boundary conditions. This system is a generalization of the model (1)–(4), functions $f, a_i, b_i$ may contain nonlocal dependence on the unknown functions $\omega, u, \mathbf{p}$ which are written after the symbol ";". In the next section we formulate some assumptions on these functions then we may define the weak form of the above system and prove existence of weak solutions.

## 1.3 Assumptions

In what follows, $\xi$, $(\zeta_0, \zeta)$, $(\eta_0, \eta)$ refer for the variables $\omega$, $(u, Du)$ and $(\mathbf{p}, D\mathbf{p})$, respectively, further, $w$, $v_1$ and $v_2$ for the nonlocal dependence on $\omega$, $u$ and $\mathbf{p}$.

(A1) For fixed $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ functions $a_i\colon Q_T \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}$ $(i = 0, \dots, n)$ have the Carathéodory property, i.e., they are measurable in $(t,x) \in Q_T$ for every $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ for a.a. $(t,x) \in Q_T$.

(A2) There exist a continuous function $c_1\colon \mathbb{R} \to \mathbb{R}^+$ and bounded operators $\mathfrak{c}_1\colon L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}^+$, $k_1\colon L^\infty(Q_T) \times X_1 \times X_2 \to L^{q_1}(Q_T)$ such that

$$\begin{aligned} &|a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)| \le \\ &\le \mathfrak{c}_1(w,v_1,v_2)c_1(\xi)\left(|\zeta_0|^{p_1-1} + |\zeta|^{p_1-1} + |\eta_0|^{\frac{p_2}{q_1}} + |\eta|^{\frac{p_2}{q_1}} + [k_1(w,v_1,v_2)](t,x)\right), \end{aligned}$$

for a.a. $(t,x) \in Q_T$, every $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ $(i = 0, \dots, n)$.

(A3) There exists a constant $C > 0$ such that for a.a. $(t,x) \in Q_T$, every $(\xi, \zeta_0, \zeta, \eta_0, \eta)$, $(\xi, \zeta_0, \tilde{\zeta}, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$

$$\sum_{i=1}^n \left(a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) - a_i(t,x,\xi,\zeta_0,\tilde{\zeta},\eta_0,\eta;w,v_1,v_2)\right)(\zeta_i - \tilde{\zeta}_i) \ge C \cdot |\zeta - \tilde{\zeta}|^{p_1}.$$

(A4) There exist a constant $c_2 > 0$, a continuous function $\gamma\colon \mathbb{R} \to \mathbb{R}$ and bounded operators $\Gamma\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $k_2\colon X_1 \to L^1(Q_T)$ such that

$$\sum_{i=0}^n a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)\zeta_i \ge c_2\left(|\zeta_0|^{p_1} + |\zeta|^{p_1}\right) - \gamma(\xi)[\Gamma(w)](t,x)[k_2(v_1)](t,x)$$

for a.a. $(t,x) \in Q_T$ and every $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$. Further,

$$\lim_{\|v_1\|_{X_1} \to +\infty} \frac{\|k_2(v_1)\|_{L^1(Q_T)}}{\|v_1\|_{X_1}^{p_1}} = 0. \tag{8}$$

(A5) If $(\omega_k)$ is bounded in $L^\infty(Q_T)$, $\omega_k \to \omega$ a.e. in $Q_T$ and $u_k \to u$ weakly in $X_1$, strongly in $L^{p_1}(Q_T)$, further, $\mathbf{p}_k \to \mathbf{p}$ strongly in $X_2$ then

$$\lim_{k\to\infty} \|a_i(\cdot, \omega_k, u_k, Du_k, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_k, \mathbf{p}_k) - a_i(\cdot, \omega_k, u_k, Du_k, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p})\|_{L^{q_1}(Q_T)} = 0.$$

(B1) For fixed $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ functions $b_i\colon Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}$ $(i = 0, \dots, n)$ have the Carathéodory property, i.e., they are measurable in $(t,x) \in Q_T$ for every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ for a.a. $(t,x) \in Q_T$.

(B2) There exist a continuous function $\hat{c}_1\colon \mathbb{R} \to \mathbb{R}^+$ and bounded operators $\hat{\mathfrak{c}}_1\colon L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}^+$, $\hat{k}_1\colon L^\infty(Q_T) \times X_1 \times X_2 \to L^{q_2}(Q_T)$ such that

$$|b_i(t,x,\xi,\zeta_0,\eta_0,\eta; w, v_1, v_2)| \le \hat{\mathfrak{c}}_1(w, v_1, v_2)\hat{c}_1(\xi)\left(|\eta_0|^{p_2-1} + |\eta|^{p_2-1} + |\zeta_0|^{\frac{p_1}{q_2}} + [\hat{k}_1(w, v_1, v_2)](t,x)\right)$$

for a.a. $(t,x) \in Q_T$ and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ $(i = 0, \dots, n)$.

(B3) There exists a constant $\hat{C} > 0$ such that for a.a. $(t,x) \in Q_T$, every $(\xi, \zeta_0, \eta_0, \eta)$, $(\xi, \zeta_0, \tilde{\eta}_0, \tilde{\eta}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$

$$\sum_{i=0}^n \left(b_i(t,x,\xi,\zeta_0,\eta_0,\eta; w, v_1, v_2) - b_i(t,x,\xi,\zeta_0,\tilde{\eta}_0,\tilde{\eta}; w, v_1, v_2)\right)(\eta_i - \tilde{\eta}_i) \ge \hat{C} \cdot \left(|\eta_0 - \tilde{\eta}_0|^{p_2} + |\eta - \tilde{\eta}|^{p_2}\right).$$

(B4) There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma}\colon \mathbb{R} \to \mathbb{R}$ and bounded operators $\hat{\Gamma}\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\hat{k}_2\colon X_2 \to L^1(Q_T)$ such that

$$\sum_{i=0}^n b_i(t,x,\xi,\zeta_0,\eta_0,\eta; w, v_1, v_2)\eta_i \ge \hat{c}_2\left(|\eta_0|^{p_2} + |\eta|^{p_2}\right) - \hat{\gamma}(\xi)[\hat{\Gamma}(w)](t,x)\left(|\zeta_0|^{p_1} + [\hat{k}_2(v_2)](t,x)\right)$$

for a.a. $(t,x) \in Q_T$, and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$. Further,

$$\lim_{\|v_2\|_{X_2} \to \infty} \frac{\|\hat{k}_2(v_2)\|_{L^1(Q_T)}}{\|v_2\|_{X_2}^{p_2}} = 0. \tag{9}$$

(B5) If $(\omega_k)$ is bounded in $L^\infty(Q_T)$, $\omega_k \to \omega$ a.e. in $Q_T$ and $u_k \to u$ weakly in $X_1$, strongly in $L^{p_1}(Q_T)$, further, $\mathbf{p}_k \to \mathbf{p}$ weakly in $X_2$ then

$$\lim_{k\to\infty} \|b_i(\cdot, \omega_k, u_k, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_k, \mathbf{p}_k) - b_i(\cdot, \omega_k, u_k, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p})\|_{L^{q_2}(Q_T)} = 0.$$

(F1) For fixed $v \in X_1$ function $f\colon Q_T \times \mathbb{R}^2 \times L^\infty(Q_T) \times X_1 \to \mathbb{R}$ is a Carathéodory function, i.e., it is measurable in $(t,x) \in Q_T$ for every $(\xi, \zeta_0) \in \mathbb{R}^2$ and continuous in $(\xi, \zeta_0) \in \mathbb{R}^2$ for a.a. $(t,x) \in Q_T$. Further, there exists a bounded operator $\mathcal{K}_1\colon X_1 \to \mathbb{R}^+$ such that

- (i) for every bounded set $I \subset \mathbb{R}$ there is a continuous function $K_1\colon \mathbb{R} \to \mathbb{R}^+$ satisfying $|K_1(\zeta_0)| \le d_1|\zeta_0|^{\frac{p_1}{q_2}} + d_2$ for every $\zeta_0 \in \mathbb{R}$, with some nonnegative constants $d_1$, $d_2$ (depending on $I$),
- (ii) for a.a. $(t,x) \in Q_T$, every $(\xi, \zeta_0), (\tilde{\xi}, \zeta_0) \in I \times \mathbb{R}$ and every $v \in X_1$,

$$|f(t,x,\xi,\zeta_0; v) - f(t,x,\tilde{\xi},\zeta_0; v)| \le \mathcal{K}_1(v)K_1(\zeta_0) \cdot |\xi - \tilde{\xi}|.$$

(F2) There exist a bounded operator $\mathcal{K}_2\colon X_1 \to \mathbb{R}^+$ and a continuous function $K_2\colon \mathbb{R} \to \mathbb{R}^+$ such that for a.a. $(t,x) \in Q_T$, every $(\xi, \zeta_0)$, $(\xi, \tilde{\zeta}_0) \in \mathbb{R}^2$ and $v \in X_1$

$$|f(t,x,\xi,\zeta_0; v) - f(t,x,\xi,\tilde{\zeta}_0; v)| \le \mathcal{K}_2(v)K_2(\xi) \cdot |\zeta_0 - \tilde{\zeta}_0|.$$

(F3) There exists $\omega^* \in L^\infty(\Omega)$ such that for a.a. $(t,x) \in Q_T$, every $(\xi, \zeta_0) \in \mathbb{R}^2$ and $v \in X_1$,

$$(\xi - \omega^*(x)) \cdot f(t,x,\xi,\zeta_0; v) \le 0.$$

(F4) If $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $u_k \to u$ strongly in $L^{p_1}(Q_T)$ then

$$\lim_{k\to\infty} \|f(\cdot, \omega_k, u_k; u_k) - f(\cdot, \omega_k, u_k; u)\|_{L^1(Q_T)} = 0.$$

### 1.4 Weak form

If the above assumptions are satisfied we may define operators $A\colon L^\infty(Q_T)\times X_1\times X_2\to X_1^*$, $B\colon L^\infty(Q_T)\times X_1\times X_2\to X_2^*$ by:

$$[A(\omega,u,\mathbf{p}),v_1]:=\int_{Q_T}\sum_{i=1}^{n}a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})D_iv_1(t,x)\,dt\,dx+$$
$$+\int_{Q_T}a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})v_1(t,x)\,dt\,dx,\quad(10)$$

$$[B(\omega,u,\mathbf{p}),v_2]:=\int_{Q_T}\sum_{i=1}^{n}b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})D_iv_2(t,x)\,dt\,dx+$$
$$+\int_{Q_T}b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})v_2(t,x)\,dt\,dx,\quad(11)$$

for $v_1\in X_1$ and $v_2\in X_2$. In addition, let us introduce the linear operator $L:D(L)\to X_1^*$ by

$$D(L)=\{u\in X_1\colon D_tu\in X_1^*,\ u(0)=0\},\qquad Lu=D_tu.\quad(12)$$

By the operators above we may define the weak form of system (5)–(7) as

$$\omega(t,x)=\omega_0(x)+\int_0^t f(s,x,\omega(s,x),u(s,x);u)\,ds\quad\text{for a.a. }(t,x)\in Q_T\quad(13)$$
$$Lu+A(\omega,u,\mathbf{p})=G\quad(14)$$
$$B(\omega,u,\mathbf{p})=H\quad(15)$$

where $G\in X_1^*$ and $H\in X_2^*$ are given by

$$[G,v_1]=\int_{Q_T}g(t,x)v_1(t,x)\,dt\,dx,\quad[H,v_2]=\int_{Q_T}h(t,x)v_2(t,x)\,dt\,dx$$

where $v_i\in X_i$ $(i=1,2)$. It is well-known (see, e.g., [20]) that one obtains the above weak form by taking sufficiently smooth solutions, using Green's theorem and finally considering the whole system in the space $L^p(0,T;V)$. Clearly, if the boundary condition is homogeneous Neumann then $V=W^{1,p}(\Omega)$ (since the boundary term vanishes in Green's theorem) and if we have homogeneous Dirichlet boundary condition then $V=W_0^{1,p}(\Omega)$ (in order to eliminate the boundary term in Green's theorem). Further, if we have a partition, for example in one dimension with homogenous Dirichlet and Neumann boundary conditions then $V=\{v\in W^{1,p_1}(0,1):v(0)=0,D_xv(1)=0\}$.

## 2 Existence of solutions

In this section we prove

**Theorem 1.** *Suppose that conditions (A1)–(A5), (B1)–(B5), (F1)–(F4) are fulfilled. Then for every* $\omega_0\in L^\infty(\Omega)$, $G\in X_1^*$ *and* $H\in X_2^*$ *there exists a solution* $\omega\in L^\infty(Q_T)$, $u\in D(L)$, $\mathbf{p}\in L^{p_2}(0,T;V_2)$ *of problem* (13)–(15).

First we formulate some statements related to the solvability of the above equations (13)–(15).

**Proposition 2.** *Assume that conditions (F1), (F3) are satisfied. Then for every fixed* $u\in L^{p_1}(Q_T)$ *and* $\omega_0\in L^\infty(Q_T)$ *there exists a unique solution* $\omega\in L^\infty(Q_T)$ *of the integral equation* (13)*, further, for the solution* $\omega$*, estimate* $\|\omega\|_{L^\infty(Q_T)}\le\|\omega_0\|_{L^\infty(\Omega)}+\|\omega^*\|_{L^\infty(\Omega)}$ *holds.*

*Proof.* Immediately follows from Proposition 2.3 in [6] since for fixed nonlocal variable $u$, condition (F1) is the same as in the cited paper. □

**Proposition 3.** *Assume (F1)–(F4) and let* $(u_k)\subset L^{p_1}(Q_T)$*, further, let* $\omega_k$ *be the solution of* (13) *corresponding to* $u_k$*. If* $u_k\to u$ *in* $L^{p_1}(Q_T)$ *then* $\omega_k\to\omega$ *a.e. in* $Q_T$ *where* $\omega$ *is the solution of* (13) *corresponding to* $u$*.*

*Proof.* We may assume that for a.a. $x \in \Omega$, $u_k(\cdot,x) \to u(\cdot,x)$. Fix such a point $x \in \Omega$. Consider the following estimate:

$$|\omega_k(t,x) - \omega(t,x)| \le \int_0^t |f(s,x,\omega_k(s,x),u_k(s,x);u_k) - f(s,x,\omega_k(s,x),u_k(s,x);u)|\,ds +$$
$$\int_0^t |f(s,x,\omega_k(s,x),u_k(s,x);u) - f(s,x,\omega(s,x),u(s,x);u)|\,ds.$$

The first integral converges to 0 for a.a. $x \in \Omega$ by condition (F4), further, by (F1), (F2) it is easy to show that the second integral is less then

$$\text{const} \cdot \left( \int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2}\,ds \right)^{1/p_2} + \text{const} \cdot \int_0^T |u_k(s,x) - u(s,x)|\,ds.$$

Hence

$$|\omega_k(t,x) - \omega(t,x)|^{p_2} \le \text{const} \cdot \int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2}\,ds + r(u_k,\omega_k)$$

where the remainder term $r(u_k,\omega_k)$ tends to 0 as $k \to \infty$. Thus Gronwall's lemma yields $|\omega_k(t,x) - \omega(t,x)| \le \text{const} \cdot r(u_k) \to 0$ which implies the desired a.e. convergence of $(\omega_k)$. □

**Proposition 4.** *Assume (A1)–(A5). Then for every fixed $\omega \in L^\infty(Q_T)$, $\mathbf{p} \in X_2$ and $G \in X_1^*$ there exists a solution $u \in D(L)$ of problem $Lu + A(\omega,u,\mathbf{p}) = G$.*

*Proof.* The proof follows from Theorem 1.1 in [22] (based on the theory of monotone type operators, see [4]) since for fixed $\omega \in L^\infty(Q_T)$ and $\mathbf{p} \in X_2$ conditions (A1)–(A5) imply that operator $A(\omega,\cdot,\mathbf{p})\colon X_1 \to X_1^*$ fulfils conditions I–V of the mentioned theorem. □

**Proposition 5.** *Suppose that (B1)–(B5) hold. Then for every fixed $\omega \in L^\infty(Q_T)$, $u \in X_1$ and $H \in X_2^*$ there exists a solution $\mathbf{p} \in X_2$ of problem $B(\omega,u,\mathbf{p}) = H$.*

*Proof.* The statement follows from the theory of monotone operators (see [27]) since conditions (B1)–(B5) imply the boundedness, demicontinuity, pseudomonotonicity and coerciveness of operator $B(\omega,u,\cdot)\colon X_2 \to X_2^*$ for fixed $\omega \in L^\infty(Q_T), u \in X_1$. □

*Proof of Theorem 1.* The idea is similar as in [6]. We define sequences of approximate solutions of problem (13)–(15) and we show the boundedness of these sequences. By using the diagonal method we will choose wealky convergent subsequences and we verify that the weak limits of the subsequences are solutions of the problem. For simplicity, in the proof we omit the variable $(t,x)$ of functions $a_i$, $b_i$ if it is not confusing.

*Step 1: approximation.* Define the sequences $(\omega_k),(u_k),(\mathbf{p}_k)$ as follows. Let $\omega_0(t,x) \equiv u_0(t,x) \equiv \mathbf{p}_0(t,x) \equiv 0$ $((t,x) \in Q_T)$ and for $k = 0,1,\ldots$ let $\omega_{k+1}, u_{k+1}, \mathbf{p}_{k+1}$ be a solution of the system:

$$\omega_{k+1}(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega_{k+1}(s,x),u_k(s,x);u_k)\,ds \tag{16}$$
$$Lu_{k+1} + A(\omega_k,u_{k+1},\mathbf{p}_k) = G \tag{17}$$
$$B(\omega_k,u_k,\mathbf{p}_{k+1}) = H. \tag{18}$$

By Propositions 2, 4, 5 we have solutions $\omega_{k+1} \in L^\infty(Q_T)$, $u_{k+1} \in X_1$, $\mathbf{p}_{k+1} \in X_2$ so the above recurrence yields the sequences $(\omega_k) \subset L^\infty(Q_T), (u_k) \subset X_1, (\mathbf{p}_k) \subset X_2$.

*Step 2: boundedness.* We show that the above defined sequences are bounded. By Proposition 2 for fixed $\omega_0 \in L^\infty(\Omega)$ for the solution of equation (16) estimate $\|\omega_{k+1}\|_{L^\infty(Q_T)} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$ holds thus $(\omega_k)$ is bounded in $L^\infty(Q_T)$

Now by choosing the test function $v = u_{k+1}$ in (17) and by using condition (A4) and the monotonicity of operator $L$ we obtain

$$[G,u_{k+1}] = [Lu_{k+1},u_{k+1}] + [A(\omega_k,u_{k+1},\mathbf{p}_k),u_{k+1}] \ge c_2 \int_{Q_T} (|u_{k+1}|^{p_1} + |Du_{k+1}|^{p_1} - \gamma(\omega_k)\Gamma(\omega_k)k_2(u_{k+1})) \ge$$
$$\ge c_2\|u_{k+1}\|_{X_1} \left( \|u_{k+1}\|_{X_1}^{p_1-1} - \|\gamma(\omega_k)\Gamma(\omega_k)\|_{L^\infty(Q_T)} \cdot \frac{\|k_2(u_{k+1})\|_{L^1(Q_T)}}{\|u_{k+1}\|_{X_1}} \right).$$

Whence by the boundedness of $(\omega_k)$ we conclude for some $K > 0$

$$\|u_{k+1}\|_{X_1}^{p_1-1} \left( 1 - K \cdot \frac{\|k_2(u_{k+1})\|_{L^1(Q_T)}}{\|u_{k+1}\|_{X_1}^{p_1}} \right) \le \text{const}.$$

Now (8) implies that $(u_k)$ is bounded in $X_1$.

The boundedness of $(\mathbf{p}_k)$ in $X_2$ follows by similar arguments as above by using condition (B4) and the boundedness of the sequences $(\omega_k)$, $(u_k)$.

We need also the boundedness of the sequence $(Lu_k)$ in $X_1^*$. By Hölder's inequality

$$|[A(\omega_k, u_{k+1}, \mathbf{p}_k), v]| \le \left( \sum_{i=0}^{n} \|a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)} \right) \cdot \|v\|_{X_1}.$$

and from condition (A2) it follows that for all $i$

$$\|a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)} \le$$
$$\le \text{const} \cdot c_1(\omega_k)\mathfrak{c}_1(\omega_k, u_{k+1}, \mathbf{p}_k) \left( \|u_{k+1}\|_{X_1}^{p_1} + \|\mathbf{p}_k\|_{X_2}^{p_2} + \|k_1(\omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)} \right).$$

Therefore by the boundedness of the sequences $(\omega_k), (u_k), (\mathbf{p}_k)$ and the boundedness of operators $c_1, \mathfrak{c}_1, k_2$ we conclude $|[Lu_{k+1}, v]| = |[A(\omega_k, u_{k+1}, \mathbf{p}_k) + G, v]| \le \text{const} \cdot \|v\|_{X_1}$ so $(Lu_k)$ is a bounded sequence in $X_1^*$.

*Step 3: convergence.* Due to the boundedness of the sequences $(u_k), (Lu_k), (\mathbf{p}_k)$ (in reflexive Banach spaces) each has a weakly convergent subsequence, further, by applying a well known embedding theorem (see [20]) it follows that there exist subsequences (which will be denoted same as the original sequences) and functions $\omega \in L^\infty(Q_T)$, $u \in X_1$, $\mathbf{p} \in X_2$ such that

$$u_k \to u \text{ weakly in } X_1, \text{ strongly in } L^{p_1}(Q_T), \text{ a.e. in } Q_T;$$
$$Lu_k \to Lu \text{ weakly in } X_1^*;$$
$$\mathbf{p}_k \to \mathbf{p} \text{ weakly in } X_2.$$

In what follows, we show that $\omega, u, \mathbf{p}$ are solutions of problem (13)–(15).

Since $u_k \to u$ in $L^{p_1}(Q_T)$, further, $\omega_{k+1}$ is the solution of equation (16), by Proposition 3 it follows that $\omega_k \to \omega$ a.e. in $Q_T$ and functions $\omega, u$ satisfy the integral equation (13).

Now let us consider equation (18). First we show that $\mathbf{p}_k \to \mathbf{p}$ in $X_2$. To this end, let us introduce operator $\tilde{B}\colon L^\infty(Q_T) \times X_1 \times X_2 \times L^\infty(Q_T) \times X_1 \times X_2 \to X_2^*$ by

$$[\tilde{B}(\omega, u, \mathbf{p}; w, v_1, v_2), z_2] := \int_{Q_T} \sum_{i=1}^{n} b_i(t, x, \omega(t, x), u(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x); w, v_1, v_2) D_i z_2(t, x)\, dt\, dx +$$
$$+ \int_{Q_T} b_0(t, x, \omega(t, x), u(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x); w, v_1, v_2) z_2(t, x)\, dt\, dx \quad (19)$$

for $z_2 \in X_2$. Observe $B(\omega, u, \mathbf{p}) = \tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p})$. By condition (B3) we have

$$[\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] \ge \hat{C} \cdot \|\mathbf{p}_{k+1} - \mathbf{p}\|_{X_2}^{p_2}. \quad (20)$$

On the left hand side of the above inequality we have the following decomposition:

$$[\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] = [\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}), \mathbf{p}_{k+1} - \mathbf{p}] +$$
$$+ [\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}), \mathbf{p}_{k+1} - \mathbf{p}] +$$
$$+ [\tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] - [\tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}]. \quad (21)$$

We show that each term on the right hand side tends to 0. By recurrence (18), $\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}) = H$, further, $\mathbf{p}_{k+1} \to \mathbf{p}$ weakly in $X_2$ which implies the convergence of the first and the last term. The convergence of the second term follows from condition (B5). In order to verify the convergence of the third term, observe that

$$|[\tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}) - \tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}]| \le$$
$$\le \sum_{i=0}^{n} \|b_i(\omega_k, u_k, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p})\|_{L^{q_2}(Q_T)} \cdot \|\mathbf{p}_{k+1} - \mathbf{p}\|_{X_2} \quad (22)$$

and by condition (B2)

$$|b_i(\omega_k, u_k, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p})|^{q_2} \le$$
$$\le \text{const} \cdot \hat{\mathfrak{c}}_1(\omega, u, \mathbf{p}) \cdot (|\hat{c}_1(\omega_k)|^{q_2} + |\hat{c}_1(\omega)|^{q_2}) \left( |\mathbf{p}|^{p_2} + |D\mathbf{p}|^{p_2} + |u_k|^{p_1} + |u|^{p_1} + |\hat{k}_1(\omega, u, \mathbf{p})|^{q_2} \right).$$

Due to the boundedness of $(\omega_k)$ in $L^\infty(Q_T)$ and the convergence of $(u_k)$ in $L^{p_1}(Q_T)$ the left hand side of the above inequality is equi-integrable (see [10]), in addition, it a.e. converges to 0, therefore by Vitali's theorem the left hand side converges in $L^1(Q_T)$ to the zero function. Thus (because of the boundedness of $(\mathbf{p}_k)$) the right hand side of (22) tends to 0. Hence all terms on the right hand side of equation (21) converges to 0 thus (20) implies $\mathbf{p}_{k+1} \to \mathbf{p}$ in $X_2$.

Now by using the same arguments as in [6] one obtains that $\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) \to \tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}) = B(\omega, u, \mathbf{p})$ weakly in $X_2^*$. Further, by condition (B5) it is not difficult to see that $\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}) \to 0$ strongly in $X_2^*$ thus $\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}) \to B(\omega, u, \mathbf{p})$. Then from recurrence (18) we conclude $B(\omega, u, \mathbf{p}) = H$, i.e., $\omega, u, \mathbf{p}$ are solutions of problem (15).

Finally, $A(\omega, u, \mathbf{p}) = G$ can be shown by similar arguments as above. The proof of the theorem is complete. $\square$

# 3 Examples

We show some examples for functions satisfying conditions (A1)–(A5), (B1)–(B5). Let functions $a_i, b_i$ have the form

$$a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) = [\pi(w)](t,x)[\varphi(v_1)](t,x)[\psi(v_2)](t,x)P(\xi)Q(\eta_0,\eta)\zeta_i|\zeta|^{p_1-2} + \\ + [\tilde{\pi}(w)](t,x)[\tilde{\varphi}(v_1)](t,x)\tilde{P}(\xi)\zeta_i|\zeta|^{r_1-1}, \text{ if } i \neq 0, \quad (23)$$

$$a_0(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) = [\pi(w)](t,x)[\varphi(v_1)](t,x)[\psi(v_2)](t,x)P(\xi)Q(\eta_0,\eta)\zeta_0|\zeta_0|^{p_1-2} + \\ + [\tilde{\pi}_0(w)](t,x)[\tilde{\varphi}_0(v_1)](t,x)\tilde{P}_0(\xi)\zeta_0|\zeta_0|^{r_1-1}, \quad (24)$$

$$b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2) = [\kappa(w)](t,x)[\lambda(v_1)](t,x)[\vartheta(v_2)](t,x)R(\xi)S(\zeta_0)\eta_i|(\eta_0,\eta)|^{p_2-2} + \\ + [\tilde{\kappa}(w)](t,x)[\tilde{\vartheta}(v_2)](t,x)\tilde{R}(\xi)\eta_i|(\eta_0,\eta)|^{r_2-1}, \ i = 0,\dots,n, \quad (25)$$

where $1 \le r_i < p_i - 1$ $(i = 1, 2)$ and the following hold.

E1. a) Operators $\pi\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\varphi\colon L^{p_1}(Q_T) \to L^\infty(Q_T)$, $\psi\colon X_2 \to L^\infty(Q_T)$ are bounded, $\varphi$ and $\psi$ are continuous, further, if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\pi(\omega_k) \to \pi(\omega)$ in $L^\infty(Q_T)$. In addition, $P \in C(\mathbb{R})$, $Q \in C(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1})$, and there exists a positive lower bound for the values of $\pi, \varphi, \psi, P, Q$.

b) Operators $\tilde{\pi}, \tilde{\pi}_0\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\tilde{\varphi}, \tilde{\varphi}_0\colon L^{p_1}(Q_T) \to L^{\frac{p_1-1}{p_1-r_1-1}}(Q_T)$ are bounded, $\tilde{\varphi}$ and $\tilde{\varphi}_0$ are continuous, further, if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\tilde{\pi}(\omega_k) \to \tilde{\pi}(\omega)$ and $\tilde{\pi}_0(\omega_k) \to \tilde{\pi}_0(\omega)$ in $L^\infty(Q_T)$. In addition, $\tilde{P}, \tilde{P}_0 \in C(\mathbb{R})$, operators $\tilde{\pi}, \tilde{\varphi}$ and function $\tilde{P}$ are nonnegative and

$$\lim_{\|v_1\|_{X_1} \to +\infty} \frac{\int_{Q_T} |\tilde{\varphi}_0(v_1)|^{\frac{p_1-1}{p_1-r_1-1}}}{\|v_1\|_{X_1}^{p_1}} = 0.$$

E2. a) Operators $\kappa\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\lambda\colon L^{p_1}(Q_T) \to L^\infty(Q_T)$, $\vartheta\colon L^{p_2}(Q_T) \to L^\infty(Q_T)$ are bounded, $\lambda$ and $\vartheta$ are continuous, further, if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\kappa(\omega_k) \to \kappa(\omega)$ in $L^\infty(Q_T)$. In addition, $R \in C(\mathbb{R})$, $S \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and there exists a positive lower bound for the values of $\kappa, \lambda, \vartheta, R, S$.

b) Operators $\tilde{\kappa}\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\tilde{\vartheta}\colon L^{p_2}(Q_T) \to L^{\frac{p_2-1}{p_2-r_2-1}}(Q_T)$ are bounded, $\tilde{\vartheta}$ is continuous, function $\tilde{R} \in C(\mathbb{R})$, further, if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\tilde{\kappa}(\omega_k) \to \tilde{\kappa}(\omega)$ in $L^\infty(Q_T)$. In addition, operators $\tilde{\kappa}, \tilde{\vartheta}$ and function $\tilde{R} \in C(\mathbb{R})$ are nonnegative and

$$\lim_{\|v_2\|_{X_2} \to +\infty} \frac{\int_{Q_T} |\tilde{\vartheta}(v_2)|^{\frac{p_2-1}{p_2-r_2-1}}}{\|v_2\|_{X_2}^{p_2}} = 0.$$

**Proposition 6.** *Assume that E1-E2 hold, then functions* (23)–(25) *fulfil conditions (A1)–(A5), (B1)–(B5).*

By using Young's and Hölder's inequalities it is not difficult to prove the above statement, a detailed proof can be found in [5].

Operators $\pi, \tilde{\pi}, \tilde{\pi}_0, \kappa, \tilde{\kappa}$ may have the form $[\pi(w)](t,x) = \int_{Q_t} |w|^\beta$, where $1 \le \beta$. Further, operators $\varphi, \lambda$ may have one of the forms

$$[\varphi(v)](t,x) = \Phi\left(\int_{Q_t} |v|^\beta\right) \quad \text{or} \quad \Phi\left(\int_{Q_t} dv\right),$$

where $1 \le \beta \le p_1$, $d \in L^{q_1}(Q_T)$, $\Phi \in C(\mathbb{R})$ and $\Phi \ge \text{const} > 0$. Similarly, $\psi$ may have in the form

$$[\psi(v)](t,x) = \Psi\left(\int_{Q_t} |v|^\beta + |Dv|^\beta\right) \quad \text{or} \quad \Psi\left(\int_{Q_t} d_1 v + d_2|Dv|\right),$$

where $1 \le \beta \le p_2$, $d_1, d_2 \in L^{q_2}(Q_T)$, $\Psi \in C(\mathbb{R})$ and $\Psi \ge \text{const} > 0$ For $\tilde{\varphi}$ consider, e.g.,

$$[\tilde{\varphi}(v)](t,x) = \tilde{\Phi}\left(\int_0^t d(s,x)v(s,x)\,ds\right), \quad \tilde{\Phi}\left(\int_\Omega d(t,x)v(t,x)\,dx\right) \quad \text{or} \quad \tilde{\Phi}\left(\left[\int_0^t |v(s,x)|^\beta\,ds\right]^{\frac{1}{\beta}}\right),$$

where $d \in L^\infty(Q_T)$, $1 \le \beta \le p_1$, $\tilde{\Phi} \in C(\mathbb{R})$, $\tilde{\Phi} \ge 0$ and $|\tilde{\Phi}(\tau)| \le \text{const} \cdot |\tau|^{p_1 - r_1 - 1}$. In the case of $\tilde{\varphi}_0$ one has similar examples as for $\tilde{\varphi}$ above, except $\tilde{\Phi}$ does not have to be nonnegative.

For operators $\vartheta, \tilde{\vartheta}$ we may consider similar examples as for $\varphi, \tilde{\varphi}$ above, by replacing exponents $p_1$ with $p_2$ and $r_1$ with $r_2$.

It is not difficult to show that the above operators fulfil conditions E1–E2, for similar arguments see, e.g., [5].

As an example for function $f$ consider, e.g., $f(t,x,\xi,\zeta_0;v) = -[\varphi(v)](t,x)f_1(t,x)f_2(\zeta_0)(\xi - \omega^*(x))$, where $\varphi\colon L^{p_1}(Q_T) \to L^\infty(Q_T)$ is bounded and nonnegative, further, $f_1 \in L^\infty(Q_T)$, $f_2\colon \mathbb{R} \to \mathbb{R}$ is nonnegative, Lipschitz continuous and $|f_2(\zeta_0)| \le \text{const} \cdot |\zeta_0|^{\frac{p_1}{q_2}}$.

# 4 Solutions in $(0,\infty)$

In the previous section we have proved existence of solutions for all finite time interval $(0,T)$. In what follows we shall show existence of weak solutions in $(0,\infty)$. Denote by $X_i^\infty = L^{p_i}_{\text{loc}}(0,\infty;V_i)$ the space of measurable functions $u\colon (0,\infty) \to V_i$ such that $u|_{(0,T)} \in L^{p_i}(0,T;V_i)$ for every $0 < T < \infty$, further, let $L^\infty_{\text{loc}}(Q_\infty)$ be the space of functions $\omega\colon Q_\infty \to \mathbb{R}$ such that $\omega|_{Q_T} \in L^\infty(Q_T)$ for every $0 < T < \infty$. In the following we suppose

(Vol) Functions $a_i\colon Q_\infty \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times L^\infty(Q_\infty) \times X_1^\infty \times X_2^\infty \to \mathbb{R}$, $b_i\colon Q_\infty \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times L^\infty(Q_\infty) \times X_1^\infty \times X_2^\infty \to \mathbb{R}$ $(i = 0,\dots,n)$, $f\colon Q_\infty \times \mathbb{R}^2 \times L^\infty_{\text{loc}}(Q_\infty) \times X_1^\infty \to \mathbb{R}$ have the Volterra property, i.e., $a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)|_{Q_T}$, $b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2)|_{Q_T}$, $f(t,x,\xi,\zeta_0;w)|_{Q_T}$ depend only on $(w|_{Q_T}, v_1|_{Q_T}, v_2|_{Q_T})$ for every $0 < T < \infty$.

Now we may define the weak form of (5)–(7) in $Q_\infty$. For fixed $0 < T < \infty$ introduce operators $A_T\colon L^\infty(Q_T) \times L^{p_1}(0,T;V_1) \times L^{p_2}(0,T;V_2) \to L^{q_1}(0,T;V_1^*)$, $B_T\colon L^\infty(Q_T) \times L^{p_1}(0,T;V_1) \times L^{p_2}(0,T;V_2) \to L^{q_2}(0,T;V_2^*)$, $L_T\colon D(L_T) \to L^{q_i}_{\text{loc}}(0,T;V_i^*)$ by formulae (10)–(12). We say that $\omega \in L^\infty_{\text{loc}}(Q_\infty), u \in X_1^\infty, \mathbf{p} \in X_2^\infty$ is a solution of (5)–(7) in $(0,\infty)$ if for all $0 < T < \infty$ (for the restrictions of the functions to $Q_T$)

$$\omega(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega(s,x),u(s,x);u)\,ds \quad (t,x) \in Q_T \tag{26}$$

$$L_T u + A_T(\omega,u,\mathbf{p}) = G_T \tag{27}$$

$$B_T(\omega,u,\mathbf{p}) = H_T \tag{28}$$

where $G_T = G|_{(0,T)}, H_T = H|_{(0,T)}$ with $G \in L^{q_1}_{\text{loc}}(0,\infty;V_1^*), H \in L^{q_2}_{\text{loc}}(0,\infty;V_2^*)$. Observe that the Volterra property ensures that if $\omega, u, \mathbf{p}$ is a solution in $(0,T)$ for some $T$ then these functions are solutions in $(0,\tilde{T})$ for all $\tilde{T} < T$.

**Theorem 7.** *Suppose that (Vol), (A1)–(A5), (B1)–(B5), (F1)–(F4) hold (in the sense that they are satisfied by the restrictions of functions $a_i, b_i, f$ to $Q_T$ for all $0 < T < \infty$). Then there exists $\omega \in L^\infty(Q_\infty), u \in X_1^\infty, \mathbf{p} \in X_2^\infty$ such that $\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}$ is a solution of problem (13)–(15) for all $0 < T < \infty$ .*

*Idea of the proof.* One may apply the arguments of the proof of Theorem 1 in [7] word for word. The idea is the following. Due to Theorem 1 we have solutions in $(0,T_k)$ where $T_k \to \infty$. By showing the boundedness of these solutions and using a diagonal process one may choose weakly convergent sequences of solutions. After taking $k \to \infty$ we obtain a solution in $(0,\infty)$. ☐

**Remark 8.** *Examples (23)–(25) fulfil the conditions of the above theorem if $\pi, \tilde{\pi}, \tilde{\pi}_0, \kappa, \tilde{\kappa}\colon L^\infty_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\varphi, \lambda\colon L^{p_1}_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\psi, \vartheta\colon L^{p_2}_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\tilde{\varphi}, \tilde{\varphi}_0\colon L^{p_1}_{\text{loc}}(Q_\infty) \to L^{\frac{p_1-1}{p_1-r_1-1}}_{\text{loc}}(Q_\infty)$, $\tilde{\vartheta}\colon L^{p_2}_{\text{loc}}(Q_\infty) \to L^{\frac{p_2-1}{p_2-r_2-1}}_{\text{loc}}(Q_\infty)$ are of Volterra type and conditions E1–E2 are satisfied for all finite $T > 0$. E.g., the operators given after Proposition 6 serve as an example for the above.*

## 4.1 Boundedness

Now we show that under some further assumptions, the solutions, formulated in the previous theorem are bounded in appropriate norms in the time interval $(0,\infty)$. First suppose

A4$^*$. There exist a constant $c_2>0$, a continuous function $\gamma\colon \mathbb{R}\to\mathbb{R}$ and Volterra operators $\Gamma\colon L^\infty_{\mathrm{loc}}(Q_\infty)\to L^\infty_{\mathrm{loc}}(Q_\infty)$, $k_2\colon X_1^\infty\to L^1_{\mathrm{loc}}(Q_\infty)$ such that

$$\sum_{i=0}^{n} a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)\zeta_i \ge c_2\left(|\zeta_0|^{p_2}+|\zeta|^{p_2}\right)-\gamma(\xi)[\Gamma(w)](t,x)[k_2(v_1)](t,x)$$

for a.a. $(t,x)\in Q_\infty$, every $(\xi,\zeta_0,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n+1}$ and $(w,v_1,v_2)\in L^\infty(Q_\infty)\times X_1^\infty\times X_2^\infty$. Further, (8) holds and for all $v\in X_1^\infty$ and

$$\int_\Omega |[k_2(v_1)](t,x)|\,dx \le \alpha_1\left[\sup_{\tau\in[0,t]}\|v_1(\tau)\|^{\rho_1}_{L^2(\Omega)}+\chi_1(t)\sup_{\tau\in[0,t]}\|v_1(\tau)\|^{p_1}_{L^2(\Omega)}+1\right]$$

with some $\alpha_1>0,\rho_1<p_1$ and $\chi_1\colon\mathbb{R}\to\mathbb{R}$ such that $\lim_{t\to\infty}\chi_1(t)=0$.

B4$^*$. There exist a constant $\hat c_2>0$, a continuous function $\hat\gamma\colon\mathbb{R}\to\mathbb{R}$ and Volterra operators $\hat\Gamma\colon L^\infty(Q_\infty)\to L^\infty(Q_\infty)$, $k_2\colon X_2^\infty\to L^1_{\mathrm{loc}}(Q_\infty)$ such that

$$\sum_{i=0}^{n} b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2)\eta_i \ge \hat c_2\left(|\eta_0|^{p_2}+|\eta|^{p_2}\right)-\hat\gamma(\xi)[\hat\Gamma(w)](t,x)\left(|\zeta_0|^2+[\hat k_2(v_2)](t,x)\right)$$

for a.a. $(t,x)\in Q_\infty$, every $(\xi,\zeta_0,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n+1}$ and $(w,v_1,v_2)\in L^\infty(Q_\infty)\times X_1^\infty\times X_2^\infty$. Further, (9) holds and for all $v\in X_2^\infty$ and

$$\int_\Omega |[\hat k_2(v_2)](t,x)|\,dx\le\alpha_2\left[\operatorname*{ess\,sup}_{\tau\in[0,t]}\|v_2(\tau)\|^{\rho_2}_{V_2}+\chi_2(t)\operatorname*{ess\,sup}_{\tau\in[0,t]}\|v_2(\tau)\|^{p_2}_{V_2}+1\right]$$

with some $\alpha_2>0,\rho_2<p_2$ and $\chi_2\colon\mathbb{R}\to\mathbb{R}$ such that $\lim_{t\to\infty}\chi_2(t)=0$.

**Theorem 9.** *Suppose that $p_1,p_2>2$ and conditions (Vol), (A1)–A3, (A4$^*$), (A5), (B1)–(B3), (B4$^*$), (B5), (F1)–(F4) are satisfied, further, $\|G(\cdot)\|_{V_1^*},\|H(\cdot)\|_{V_2^*}\in L^\infty(0,\infty)$. Then for the solutions $\omega,u,\mathbf{p}$ formulated in Theorem 9, $\omega\in L^\infty(Q_\infty)$, $u\in L^\infty(0,\infty;L^2(\Omega)),\mathbf{p}\in L^\infty(0,\infty;V_2)$.*

*Idea of the proof.* We may apply the arguments of the proof of Theorem 2 in [7]. Introduce the notation $y(t)=\|u(t,\cdot)\|^2_{L^2(\Omega)}$ (then $y$ is continuous see, e.g., [27]). By choosing arbitrary $0<T_1<T_2<\infty$, (13), conditions (A4$^*$), Young's inequality and the continuous embedding $V_1\hookrightarrow L^2(\Omega)$ imply

$$\frac12\left(y(T_1)-y(T_2)\right)+\frac12 c_2\int_{T_1}^{T_2} y(t)^{\frac{p_1}{2}}\,dt\le \mathrm{const}\cdot\int_{T_1}^{T_2}\left(\sup_{\tau\in[0,t]}y(\tau)^{\frac{\varrho_1}{2}}+\chi_1(t)\sup_{\tau\in[0,t]}y(\tau)^{\frac{p_1}{2}}+1\right)dt.$$

It is not difficult to see that the above inequality implies the boundedness of $y$ in $(0,\infty)$, a detailed argument can be found in [24]. The boundedness of $\mathbf{p}$ follows from the boundedness of $y$ similarly as above, by using conditions (B4$^*$), see [7]. $\square$

**Remark 10.** *Example* (23)–(25) *fulfil the conditions of Theorem 9 if assumptions of Remark 8 are satisfied, in addition*

$$\int_\Omega |[\tilde\varphi_0(v_1)](t,x)|^{\frac{p_1-1}{p_1-r_1-1}}\,dx\le\alpha_1\left[\sup_{\tau\in[0,t]}\|v_1(\tau)\|^{\rho_1}_{L^2(\Omega)}+\chi_1(t)\sup_{\tau\in[0,t]}\|v_1(\tau)\|^{p_1}_{L^2(\Omega)}+1\right]$$

*for all $v_1\in L^{p_1}_{\mathrm{loc}}(Q_\infty)$ with some constants $\alpha_1>0,\varrho_1<p_1$ and function $\chi_1\colon\mathbb{R}\to\mathbb{R}$ such that $\lim_{t\to\infty}\chi_1(t)=0$, further, similar condition holds for $\tilde\vartheta$ (by changing the indeces from 1 to 2, and $L^2(\Omega)$ to $V_2$). For example, operator $\tilde\varphi_0$ may have the form*

$$[\tilde\varphi(v)](t,x)=\tilde\Phi\left(\int_\Omega d(t,x)v(t,x)\,dx\right),\ \tilde\Phi\left(\int_\Omega |d(t,x)||v(t,x)|^\beta\,dx\right)\ \textit{or}\ \chi_1(t)\tilde\Phi_0\left(\left[\int_\Omega|d(t,x)||v(t,x)|^2\,dx\right]^{\frac1\beta}\right),$$

*where $d\in L^\infty(Q_\infty)$, $1\le\beta\le2$, $\tilde\Phi,\tilde\Phi_0,\chi_1\in C(\mathbb{R})$ and $|\tilde\Phi(\tau)|\le\mathrm{const}\cdot|\tau|^{p_1-\varrho_1-1}$, $|\tilde\Phi_0(\tau)|\le\mathrm{const}\cdot|\tau|^{p_1-r_1-1}$, $\lim_{\tau\to\infty}\chi_1(\tau)=0$.*

## 4.2 Stabilization

In this section we consider a special case of problem (26)–(28), namely, let $p_1 = p_2 = p$ (thus $q_1 = q_2 = q$, $V_1 = V_2 = V$ and $X_1 = X_2 = X$). In what follows, we prove stabilization of the solutions of the system, that is, we show the convergence (in some sense) of solutions as $t \to \infty$ to the solutions of a stationary system. We need some further assumptions:

A6. There exist Carathéodory functions $a_{i,\infty}\colon \Omega \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 0, \dots, n)$ such that for a.a. $x \in \Omega$ and every $(\zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $\xi^* \in \mathbb{R}$, $w \in L^\infty(Q_\infty)$, $v_1 \in X^\infty \cap L^\infty(0, \infty; L^2(\Omega))$, $v_2 \in X^\infty \cap L^\infty(0, \infty; V)$,

$$\lim_{\substack{\xi\to\xi^*\\ t\to\infty}} a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2) = a_{i,\infty}(x, \xi^*, \zeta_0, \zeta, \eta_0, \eta).$$

B6. There exist Carathéodory functions $b_{i,\infty}\colon \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 0, \dots, n)$ such that for a.a. $x \in \Omega$ and every $(\zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1}$, $\xi^* \in \mathbb{R}$, $w \in L^\infty(Q_\infty)$, $v_1 \in X^\infty \cap L^\infty(0, \infty; L^2(\Omega))$, $v_2 \in X^\infty \cap L^\infty(0, \infty; V)$,

$$\lim_{\substack{\xi\to\xi^*\\ t\to\infty}} b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2) = b_{i,\infty}(x, \xi^*, \zeta_0, \eta_0, \eta).$$

AB There exists a positive constant $\mathcal{C}$ such that for a.a. $(t, x) \in Q_\infty$ and every $\xi \in \mathbb{R}$, $(\zeta_0, \zeta, \eta_0, \eta)$, $(\tilde\zeta_0, \tilde\zeta, \tilde\eta_0, \tilde\eta) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $(w, v_1, v_2) \in L^\infty(Q_\infty) \times X_1^\infty \times X_2^\infty$,

$$\sum_{i=0}^{n}\Big(a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2) - a_i(t, x, \xi, \tilde\zeta_0, \tilde\zeta, \tilde\eta_0, \tilde\eta; w, v_1, v_2)\Big)(\zeta_i - \tilde\zeta_i)+$$
$$+\sum_{i=0}^{n}\Big(b_i(t, x, \xi, \zeta_0, \eta_0, \eta) - b_i(t, x, \xi, \tilde\zeta_0, \tilde\eta_0, \tilde\eta)\Big)(\eta_i-\tilde\eta_i) \ge \mathcal{C}\cdot\Big(|\zeta_0 - \tilde\zeta_0|^p + |\zeta - \tilde\zeta|^p + |\eta_0 - \tilde\eta_0|^p + |\eta - \tilde\eta|^p\Big).$$

F5. For every fixed $v \in X^\infty \cap L^\infty(0, \infty; L^2(\Omega))$ there is a constant $m > 0$ such that $(\xi - \omega^*(x))f(t, x, \xi, \zeta_0; v) \le -m(\xi - \omega^*(x))^2$ for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0) \in \mathbb{R}^2$.

Now introduce operators $A_\infty\colon L^\infty(\Omega) \times V \times V \to V^*$, $B_\infty\colon L^\infty(\Omega) \times V \times V \to V^*$ by

$$\langle A_\infty(\omega, u, \mathbf{p}), v\rangle := \int_\Omega \sum_{i=1}^{n} a_{i,\infty}(x, \omega(x), u(x), Du(x), \mathbf{p}(x), D\mathbf{p}(x))D_i v(x)\, dx+$$
$$+\int_\Omega a_{0,\infty}(x, \omega(x), u(x), Du(x), \mathbf{p}(x), D\mathbf{p}(x))v(x)\, dx,$$

$$\langle B_\infty(\omega, u, \mathbf{p}), v\rangle := \int_\Omega \sum_{i=1}^{n} b_{i,\infty}(x, \omega(x), u(x), \mathbf{p}(x), D\mathbf{p}(x))D_i v(x)\, dx+$$
$$+\int_\Omega b_{0,\infty}(x, \omega(x), u(x), \mathbf{p}(x), D\mathbf{p}(x))v(x)\, dx,$$

**Theorem 11.** *Assume conditions (A1)–(A3), (A4*), (A5)–(A6), (B1)–(B3), (B4*), (B5)–(B6), (AB), (F1)–(F5) are satisfied (with $p = p_1 = p_2$), further, there exist $F_\infty, G_\infty \in V^*$ such that*

$$\lim_{t\to\infty} \|F(t) - F_\infty\|_{V^*} = 0, \quad \lim_{t\to\infty} \|G(t) - G_\infty\|_{V^*} = 0.$$

*Then there exist $u_\infty \in V, \mathbf{p}_\infty \in V$ such that for the solutions $\omega, u, \mathbf{p}$ of problem* (26)–(28), *$\omega(t, \cdot) \to \omega^*$ in $L^\infty(\Omega)$, $u(t) \to u_\infty$ in $L^2(\Omega)$, $\int_{t-1}^{t+1} \|u(s) - u_\infty\|_V^p\, ds \to 0$ , $\int_{t-1}^{t+1} \|\mathbf{p}(s) - \mathbf{p}_\infty\|_V^p\, ds \to 0$, further,*

$$A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = G_\infty \tag{29}$$
$$B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = H_\infty. \tag{30}$$

*Sketch of the proof.* We follow the proof of Theorem 3 in [7]. Let $\omega, u, \mathbf{p}$ be solutions of (26)–(28) then by Theorem 9, $\omega \in L^\infty(Q_\infty)$, $u \in L^\infty(0, \infty; L^2(\Omega))$, $\mathbf{p} \in L^\infty(0, \infty; V_2)$. By using the same arguments as in the above mentioned paper, conditions (F3), (F5) imply estimate $\|\omega(t, \cdot) - \omega^*(\cdot)\|_{L^\infty(\Omega)} \le \|\omega_0\|_{L^\infty(\Omega)}e^{-mt}$ which yields the convergence $\omega(t, \cdot) \to \omega^*(\cdot)$ in $L^\infty(\Omega)$.

Now by the using the idea of Proposition 5 and condition AB it is easy to see that for fixed $\omega^*$ there exist a unique solution $u_\infty \in V, \mathbf{p}_\infty \in V$ of problem (29)–(30) see, e.g., [27].

In order to show the desired convergences we prove a differential inequality for $u$ and $\mathbf{p}$. From equations (26)–(28) and (29)–(30) we obtain

$$\langle D_t(u(t)-u_\infty), u(t)-u_\infty\rangle + \langle [A(\omega,u,\mathbf{p})](t) - A_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), u(t)-u_\infty\rangle + \\ + \langle [B(\omega,u,\mathbf{p})](t) - B_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), \mathbf{p}(t)-\mathbf{p}_\infty\rangle = \langle G(t)-G_\infty, u(t)-u_\infty\rangle + \langle F(t)-F_\infty, \mathbf{p}(t)-\mathbf{p}_\infty\rangle. \quad (31)$$

Observe that the first term equals to $\frac{1}{2}y'(t)$ where $y(t) = \int_\Omega (u(t)-u_\infty)^2$. Further, for the second and third terms of the above equation we have by condition AB and Young's inequality

$$\langle [A(\omega,u,\mathbf{p})](t) - A_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), u(t)-u_\infty\rangle + \langle [B(\omega,u,\mathbf{p})](t) - B_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), \mathbf{p}(t)-\mathbf{p}_\infty\rangle \geq \\ \geq \mathcal{C}\cdot(\|u(t)-u_\infty\|_V^p + \|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p) - \frac{\varepsilon^p}{p}\|u(t)-u_\infty\|_V^p - \frac{\varepsilon^p}{p}\|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p - \\ - \frac{1}{q\varepsilon^q}\|[\tilde{A}(\omega,u_\infty,\mathbf{p}_\infty;\omega,u,\mathbf{p})](t) - A_\infty(\omega^*,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q - \\ - \frac{1}{q\varepsilon^q}\|[\tilde{B}(\omega,u_\infty,\mathbf{p}_\infty;\omega,u,\mathbf{p})](t) - B(\omega^*,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q \quad (32)$$

with some $\varepsilon > 0$ (and operator $\tilde{A}$ is defined similarly as $\tilde{B}$, see (19)). We show that last two terms on the right hand side of the above inequality converges to 0 as $t \to \infty$. Clearly,

$$\|[\tilde{A}(\omega,u_\infty,\mathbf{p}_\infty;\omega,u,\mathbf{p})](t) - A_\infty(\omega^*,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q \leq \\ \leq \sum_{i=0}^n \int_\Omega |a_i(t,\cdot,\omega(t,\cdot),u_\infty,Du_\infty,\mathbf{p}_\infty,D\mathbf{p}_\infty;\omega,u,\mathbf{p}) - a_{i,\infty}(\omega^*,u_\infty,Du_\infty,\mathbf{p}_\infty,D\mathbf{p}_\infty)|^q.$$

The integrand on the right hand side is a.e. convergent in $\Omega$ as $t \to \infty$ by condition A6 and since $\omega(t,x) \to \omega^*(x)$ for a.a. $x \in \Omega$. Further, it is integrable in $\Omega$ by conditions (A2), (A6) and estimate

$$|a_i(t,\cdot,\omega(t,\cdot),u_\infty,Du_\infty,\mathbf{p}_\infty,D\mathbf{p}_\infty) - a_{i,\infty}(\omega^*,u_\infty,Du_\infty,\mathbf{p}_\infty,D\mathbf{p}_\infty)|^q \leq \\ \leq \text{const}\cdot\left(\|c_1(\omega)\|_{L^\infty(Q_\infty)} + \|c_1(\omega^*)\|_{L^\infty(Q_\infty)}\right)\left(|u_\infty|^p + |Du_\infty|^p + |\mathbf{p}_\infty|^p + |D\mathbf{p}_\infty|^p + \|k_1\|_{L^q(\Omega)}\right)$$

thus by Lebesgue's theorem we obtain $\|[\tilde{A}(\omega,u_\infty,\mathbf{p}_\infty;\omega,u,\mathbf{p})](t) - A_\infty(\omega,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q \to 0$ as $t \to \infty$. The convergence of the last term in (32) can be proved similarly.

On the right hand side of (31) by Young's inequality we obtain

$$|\langle G(t)-G_\infty, u(t)-u_\infty\rangle + \langle F(t)-F_\infty, \mathbf{p}(t)-\mathbf{p}_\infty\rangle| \leq \\ \frac{\varepsilon^p}{p}\|u(t)-u_\infty\|_V^p + \frac{\varepsilon^p}{p}\|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p + \frac{1}{q\varepsilon^q}\|G(t)-G_\infty\|_{V^*}^q + \frac{1}{q\varepsilon^q}\|F(t)-F_\infty\|_{V^*}^q \quad (33)$$

where the last two terms tend to 0 as $t \to \infty$.

Now, by choosing sufficiently small $\varepsilon$ in (31) and by using (32), (33), the above convergences and the continuous embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$ we obtain

$$y'(t) + \text{const}\cdot y(t)^{\frac{p}{2}} + \text{const}\cdot\|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p \leq \varphi(t)$$

where $\varphi(t) \to 0$ as $t \to \infty$ and the constants are positive. It is not difficult to show that this inequality implies $\lim_{t\to\infty} y(t) = 0$ (see the proof of Theorem 2 in [24]), furthermore, by integrating (31) over $(t-1,t+1)$ one can deduce the convergences $\int_{t-1}^{t+1}\|u(s)-u_\infty\|_V^p\,ds \to 0$, $\int_{t-1}^{t+1}\|\mathbf{p}(s)-\mathbf{p}_\infty\|_V^p\,ds \to 0$, too. The proof of stabilization is complete. □

Consider the following functions for $i = 0,\ldots,n$

$$a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) = [\pi(w)](t,x)[\varphi(v_1)](t,x)[\psi(v_2)](t,x)P(\xi)\zeta_i|(\zeta_0,\zeta,\eta_0,\eta)|^{p-2}, \quad (34)$$

$$b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2) = [\kappa(w)](t,x)[\lambda(v_1)](t,x)[\vartheta(v_2)](t,x)R(\xi)\eta_i|(\zeta_0,\eta_0,\eta)|^{p-2}. \quad (35)$$

Suppose

E3. a) Operators $\pi\colon L^\infty_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\varphi, \psi\colon L^p_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$ are of Volterra type, further, for every $0 < T < \infty$, $\pi\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\varphi, \psi\colon L^p(Q_T) \to L^\infty(Q_T)$ are bounded, $\varphi$ and $\psi$ are continuous, and if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\pi(\omega_k) \to \pi(\omega)$ in $L^\infty(Q_T)$. In addition, $P \in C(\mathbb{R})$, and there exists a positive lower bound for the values of $\pi, \varphi, \psi, P$.

b) There exist $\pi_\infty, \varphi_\infty, \psi_\infty \in L^\infty(\Omega)$ such that for every $w \in L^\infty(Q_\infty)$, $v_1 \in X^\infty \cap L^\infty(0,\infty; L^2(\Omega))$, $v_2 \in X^\infty \cap L^\infty(0,\infty; V)$,

$$\lim_{t\to\infty} \left(\|[\pi(w)](t,\cdot) - \pi_\infty\|_{L^\infty(\Omega)} + \|[\varphi(v_1)](t,\cdot) - \varphi_\infty\|_{L^\infty(\Omega)} + \|[\psi(v_2)](t,\cdot) - \psi_\infty\|_{L^\infty(\Omega)}\right) = 0.$$

E4. a) Operators $\kappa\colon L^\infty_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\lambda, \vartheta\colon L^p_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$ are of Volterra type, further, for every $0 < T < \infty$, $\kappa\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\lambda, \vartheta\colon L^p(Q_T) \to L^\infty(Q_T)$ are bounded, $\lambda$ and $\vartheta$ are continuous, and if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\kappa(\omega_k) \to \kappa(\omega)$ in $L^\infty(Q_T)$. In addition, $R \in C(\mathbb{R})$, and there exists a positive lower bound for the values of $\pi, \varphi, \psi, R$.

b) There exist $\kappa_\infty, \lambda_\infty, \vartheta_\infty \in L^\infty(\Omega)$ such that for every $w \in L^\infty(Q_\infty)$, $v_1 \in X^\infty \cap L^\infty(0,\infty; L^2(\Omega))$, $v_2 \in X^\infty \cap L^\infty(0,\infty; V)$

$$\lim_{t\to\infty} \left(\|[\kappa(w)](t,\cdot) - \kappa_\infty\|_{L^\infty(\Omega)} + \|[\lambda(v_1)](t,\cdot) - \lambda_\infty\|_{L^\infty(\Omega)} + \|[\vartheta(v_2)](t,\cdot) - \vartheta_\infty\|_{L^\infty(\Omega)}\right) = 0.$$

It is not difficult to prove (for some arguments see, e.g., [9, 5, 22])

**Proposition 12.** *Suppose* $2 \le p \le 4$ *and E3–E4, then the above* (34)–(35) *functions satisfy conditions (A1)–(A3), (A4*), (A5)–(A6), (B1)–(B3), (B4*), (B5)–(B6), (AB) with* $p_1 = p_2 = p$.

If we consider

$$\begin{aligned} a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) &= \zeta_i|(\zeta_0,\zeta)|^{p-2} + [\pi(w)](t,x)[\phi(v_1)](t,x)P(\xi)\zeta_i|(\zeta_0,\zeta,\eta_0,\eta)|^{r-2},\\ b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2) &= \zeta_i|(\eta_0,\eta)|^{p-2} + [\kappa(w)](t,x)(t,x)[\vartheta(v_2)](t,x)R(\xi)\eta_i|(\zeta_0,\eta_0,\eta)|^{r-2} \end{aligned}$$

where $1 \le r \le 4$ and E3–E4 hold then it is easy to see that these functions satisfy conditions (A1)–(A3), (A4*), (A5)–(A6), (B1)–(B6), (AB) with $p_1 = p_2 = p \ge \max\{2, r\}$. E.g. operators $\pi, \varphi$ may have the form

$$[\pi(w)](t,x) = \chi(t)\int_{Q_t} |w|^\beta + \pi_\infty(x), \quad [\varphi(v)](t,x) = \tilde\chi(t)\int_\Omega |d(t,x)||v(t,x)|^\beta dx + \varphi_\infty(x),$$

where $\lim_{t\to\infty}\chi(t) = 0$, $\lim_{t\to\infty}\tilde\chi(t) = 0$ and $d, \pi_\infty, \varphi_\infty \in L^\infty(\Omega)$, $1 \le \alpha$, $1 \le \beta \le 2$. The other operators may have similar form.

As an example for function $f$ consider, e.g., $f(t,x,\xi,\zeta_0;v) = -(\xi - \omega^*(x))\int_\Omega |v(t,x)|^\beta\, dx$ where $1 \le \beta \le 2$.

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