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Forced Oscillations of Beams on Elastic Bearings

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Abstract

We study the existence of weak periodic solutions for certain damped and forced linear beam equations resting on semi-linear elastic bearings. Conditions for the periodic forcing term and semi-linear elastic bearings are derived which ensure either the existence or nonexistence of periodic solutions of the beam equation. Topological degree arguments are used to achieve these results.

Keywords. beam equations, periodic solutions, topological degree
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1 Introduction

In this note, we consider a periodically forced and damped beam resting on two different bearings with purely elastic responses. The length of the beam is $\pi/4$. The equation of vibrations is as follows

$$\begin{aligned} &u_{tt} + u_{xxxx} + \delta u_t + h(x,t) = 0\,, \\ &u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ &u_{xxx}(0,\cdot) = -ku(0,\cdot) - f(u(0,\cdot))\,, \\ &u_{xxx}(\pi/4,\cdot) = ru(\pi/4,\cdot) + g(u(\pi/4,\cdot))\,, \end{aligned} \tag{1}$$

where $\delta > 0$, $r \geq 0$, $k \geq 0$ are constants, $h \in C([0,\pi/4] \times S^T)$, and $f, g \in C(\mathbb{R})$ have at most linear growth at infinity. Here S^T is the circle $S^T = \mathbb{R}/\{T\mathbb{Z}\}$.

The undamped and unforced case of the form

$$\begin{aligned} &u_{tt} + u_{xxxx} = 0\,, \\ &u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ &u_{xxx}(0,\cdot) = -f(u(0,\cdot))\,, \\ &u_{xxx}(\pi/4,\cdot) = f(u(\pi/4,\cdot)) \end{aligned} \tag{2}$$

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is studied in [7] and [9] by using variational methods, where among others the following results are proved.

Theorem. ([9]) *If the function $f(u)$ satisfies the following assumptions:*

(i) $f \in C^1(\mathbb{R})$, $f(-u) = -f(u)$ *for all* $u \in \mathbb{R}$.

(ii) *For any* $C > 0$ *there is a* $K(C)$ *such that* $f(u) \geq Cu - K(C)$ *for all* $u \geq 0$.

(iii) $\frac{1}{2} f(u)u - F(u) \geq c_1|f(u)| - c_2$ *for all* $u \in \mathbb{R}$, *where* $F(u) = \int\limits_0^u f(s)\,ds$ *and* c_1, c_2 *are positive constants.*

(iv) $f(0) = f'(0) = 0$.

Then there is a sufficiently large positive integer M such that equation (2) *possesses at least one nonzero time periodic solution with the period $2\pi M^2$.*

Theorem. ([7]) *If the continuous function $f(u)$ is odd on $\mathbb{R}$, C^1-smooth near $u = 0$ with $f'(0) > 0$ and $\lim\limits_{|u|\to\infty} f(u)/u = 0$. Then equation* (2) *possesses infinitely many odd time periodic solutions with periods densely distributed in an interval $(a_1, 2a_1)$ for a constant $a_1 > 0$.*

It is pointed out in [9] that equation (2) is a simple analogue of a more complicated shaft dynamics model introduced in the works [5] and [6].

When the nonlinearities and parameters are small, i.e. (1) is of the form

$$\begin{aligned} & u_{tt} + u_{xxxx} + \varepsilon\delta u_t + \varepsilon\mu h(x, \sqrt{\varepsilon}t) = 0\,, \\ & u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ & u_{xxx}(0,\cdot) = -\varepsilon f(u(0,\cdot)), \quad u_{xxx}(\pi/4,\cdot) = \varepsilon f(u(\pi/4,\cdot))\,, \end{aligned} \tag{3}$$

where $\varepsilon > 0$ and μ are small parameters, $\delta > 0$ is a constant, $f \in C^2(\mathbb{R})$, $h \in C^2([0,\pi/4] \times \mathbb{R})$ and $h(x,t)$ is 1-periodic in t. Then by using analytic methods, the following result is proved in [2] and [3].

Theorem. ([2], [3]) *Let the following assumptions hold:*

(I) $f(0) = 0$, $f'(0) < 0$ *and the equation* $\ddot{x} + f(x) = 0$ *has a homoclinic solution* $\gamma(t) \neq 0$ *that is a non trivial bounded solution such that* $\lim\limits_{t\to\pm\infty} \gamma(t) = 0$.

(II) *The homoclinic solution* $\gamma_1(t) := \frac{\sqrt{\pi}}{2}\gamma\Big(2\sqrt{\frac{2}{\pi}}t\Big)$ *is non-degenerate, that is the linear equation*

$$\ddot{v} + \frac{24}{\pi} f'\Big(\frac{2}{\sqrt{\pi}}\gamma_1(t)\Big)v = 0$$

has no nontrivial bounded solutions.

(III) $10.5705675493 \cdot |f'(0)| < \delta$.

If $\eta \neq 0$ can be chosen in such a way that the equation

$$\delta \int\limits_{-\infty}^{\infty} \dot{\gamma}_1(s)^2 \, ds + \frac{2}{\sqrt{\pi}} \eta \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\pi/4} \dot{\gamma}_1(s) h(x, s+\alpha) \, dx \, ds = 0$$

has a simple root α, then there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$ and $\mu = \sqrt{\varepsilon}\eta$, equation (3) *has a unique bounded solution on $\mathbb{R}$ near $\gamma\Big(2\sqrt{\frac{2}{\pi}}\big(\sqrt{\varepsilon}t - \alpha\big)\Big)$ which is exponentially homoclinic to a unique small periodic solution of* (3). *Moreover, the Smale horseshoe (see* [12]*) can be embedded into the dynamics of* (3).

Finally, a damped case is studied in [8] of the form

$$\begin{aligned} &u_{tt} + u_{xxxx} + \delta u_t + h(x,t) = 0\,, \\ &u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ &u_{xxx}(0,\cdot) = -f(u(0,\cdot))\,, \\ &u_{xxx}(\pi/4,\cdot) = g(u(\pi/4,\cdot))\,, \end{aligned} \tag{4}$$

where δ is a positive constant, f and g are analytic, the function $h \in C([0,\pi/4] \times S^T)$ is splitted as follows

$$h(x,t) = 8\frac{\theta_2 - 2\theta_1}{T\pi} + 96\frac{\theta_1 - \theta_2}{T\pi^2}x + p(x,t)$$

for $\theta_{1,2} \in \mathbb{R}$ and

$$\int_0^T \int_0^{\pi/4} p(x,t)\,dx\,dt = \int_0^T \int_0^{\pi/4} xp(x,t)\,dx\,dt = 0\,.$$

Conditions are found in [8] between the numbers $\theta_{1,2}$, the function $p(x,t)$ and the nonlinearities f, g under which (4) has a T-periodic solution. It is also shown that under certain assumptions, constants $\theta_{1,2}$ are functions of $p(x,t)$ in order to get a T-periodic solution of (4).

In this note, we are interested in T-periodic vibrations of (1) by using topological degree arguments. We show the existence of T-periodic vibrations of (1) for $r > 0$, $k > 0$ and f, g sublinear at infinity. Also a generic result is derived for this case when in addition $f, g \in C^1(\mathbb{R})$. If either $r = 0$ or $k = 0$, then we derive Landesman-Lazer type conditions on h, f, g for showing either existence or nonexistence results of T-periodic vibrations of (1).

2 Setting of the Problem

By a weak T-periodic solution of (1), we mean any $u(x,t) \in C([0,\pi/4] \times S^T)$ satisfying the identity

$$
\begin{aligned}
&\int_0^T \int_0^{\pi/4} \Big[u(x,t) \Big\{ v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t) \Big\} + h(x,t) v(x,t) \Big] \, dx \, dt \\
&+ \int_0^T \Big\{ \Big(k u(0,t) + f(u(0,t)) \Big) v(0,t) \\
&+ \Big(r u(\pi/4,t) + g(u(\pi/4,t)) \Big) v(\pi/4,t) \Big\} \, dt = 0
\end{aligned}
\tag{5}
$$

for any $v(x,t) \in C^\infty([0,\pi/4] \times S^T)$ such that the following boundary value conditions hold

$$
v_{xx}(0,\cdot) = v_{xx}(\pi/4,\cdot) = v_{xxx}(0,\cdot) = v_{xxx}(\pi/4,\cdot) = 0\,. \tag{6}
$$

The eigenvalue problem

$$
\begin{aligned}
&w_{xxxx}(x) = \mu^4 w(x)\,, \\
&w_{xx}(0) = w_{xx}(\pi/4) = 0, \quad w_{xxx}(0) = w_{xxx}(\pi/4) = 0
\end{aligned}
$$

is known [9] to possesses a sequence of eigenvalues μ_k, $k = -1, 0, 1, \cdots$ with

$$
\mu_{-1} = \mu_0 = 0
$$

and

$$
\cos(\mu_k \pi/4) \cosh(\mu_k \pi/4) = 1, \quad k = 1, 2, \cdots\,. \tag{7}
$$

The corresponding orthonormal in $L^2(0,\pi/4)$ system of eigenvectors is

$$
\begin{aligned}
w_{-1}(x) &= \tfrac{2}{\sqrt{\pi}}, \quad w_0(x) = \tfrac{16}{\pi} \left(x - \tfrac{\pi}{8} \right) \sqrt{\tfrac{3}{\pi}} \\
w_k(x) &= \tfrac{4}{\sqrt{\pi} W_k} \Big[\cosh(\mu_k x) + \cos(\mu_k x) \\
&\quad - \tfrac{\cosh \xi_k - \cos \xi_k}{\sinh \xi_k - \sin \xi_k} \big(\sinh(\mu_k x) + \sin(\mu_k x) \big) \Big]
\end{aligned}
$$

where the constants W_k are given by the formulas

$$
W_k = \cosh(\xi_k) + \cos \xi_k - \tfrac{\cosh \xi_k - \cos \xi_k}{\sinh \xi_k - \sin \xi_k} \big(\sinh \xi_k + \sin \xi_k \big)
$$

for $\xi_k = \mu_k \pi/4$. From (7) we get the asymptotic formulas

$$
1 < \mu_k = 2(2k+1) + r(k) \quad \forall k \geq 1
$$

along with

$$
|r(k)| \leq \bar{c}_1 e^{-\bar{c}_2 k} \quad \forall k \geq 1\,,
$$

where $\bar{c}_1$, $\bar{c}_2$ are positive constants. Moreover, the eigenfunctions $\{w_i\}_{i=-1}^{\infty}$ are uniformly bounded in $C([0,\pi/4])$.

3 Preliminary Results

Let $H_1(x,t) \in C([0,\pi/4] \times S^T)$, $H_2(t), H_3(t) \in C(S^T)$ be continuous T-periodic functions and consider the equation

$$\begin{aligned}&\int_0^T \int_0^{\pi/4} \Big[z(x,t) \Big\{ v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t) \Big\} + H_1(x,t) v(x,t) \Big] \, dx \, dt \\ &+ \int_0^T \Big\{ H_2(t) v(0,t) + H_3(t) v(\pi/4,t) \Big\} \, dt = 0\end{aligned} \tag{8}$$

for any $v(x,t) \in C^\infty([0,\pi/4] \times \mathbb{R})$ satisfying the boundary conditions (6) along with

$$\int_0^{\pi/4} v(x,t) dx = \int_0^{\pi/4} x v(x,t) dx = 0 \quad \forall t \in S^T. \tag{9}$$

Note that conditions (9) correspond to the orthogonality of $v(x,t)$ to $w_{-1}(x)$ and $w_0(x)$, for any $t \in S^T$. We look for $z(x,t)$ in the form

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t) w_i(x) \, . \tag{10}$$

We formally put (10) into (8) to get a system of ordinary differential equations

$$\ddot{z}_i(t) + \delta \dot{z}_i(t) + \mu_i^4 z_i(t) = h_i(t) \, , \tag{11}$$

where

$$h_i(t) = -\Big(\int_0^{\pi/4} H_1(x,t) w_i(x) \, dx + H_2(t) w_i(0) + H_3(t) w_i(\pi/4) \Big) \, . \tag{12}$$

Let us put

$$M_1 = \sup_{i \geq 1, x} |w_i(x)|, \quad M_2 = 4 M_1 \sum_{i=1}^{\infty} 1/\mu_i^2 < \infty \quad M_3 = \sup_{i \geq 1} i^2/\mu_i^2 < \infty \, . \tag{13}$$

Since $\mu_i > 0$ for $i \geq 1$, equation (11) has a unique T-periodic solution $z_i(t)$, for $2\mu_i^2 > \delta$ given by

$$z_i(t) = \frac{2}{\bar{\omega}_i} \int_{-\infty}^{t} e^{-\delta(t-s)/2} \sin \Big(\frac{\bar{\omega}_i}{2} (t-s) \Big) \times h_i(s) \, ds \, , \tag{14}$$

where $\bar{\omega}_i = \sqrt{4\mu_i^4 - \delta^2}$, for $2\mu_i^2 = \delta$ given by

$$z_i(t) = \int_{-\infty}^{t} e^{-\delta(t-s)/2} (t-s) \times h_i(s) \, ds \, , \tag{15}$$

and for $2\mu_i^2 < \delta$ given by

$$z_i(t) = \int\limits_{-\infty}^{t} \frac{1}{\tilde{\omega}_i}\Big(e^{(-\delta+\tilde{\omega}_i)(t-s)/2} - e^{(-\delta-\tilde{\omega}_i)(t-s)/2}\Big) \times h_i(s)\, ds\,, \tag{16}$$

where $\tilde{\omega}_i = \sqrt{\delta^2 - 4\mu_i^4}$. Let $\|\cdot\|_\infty$ denote the maximum norm on $[0, T]$.

From (14) for $3\mu_i^4 > \delta^2$ we get

$$\begin{aligned} &\|z_i\|_\infty \le \tfrac{4}{\bar{\omega}_i\delta}\|h_i\|_\infty \le \tfrac{4}{\mu_i^2\delta}\|h_i\|_\infty \\ &\|\dot{z}_i\|_\infty \le \tfrac{4}{\delta}\|h_i\|_\infty\,, \end{aligned} \tag{17}$$

and for $3\mu_i^4 < \delta^2 < 4\mu_i^4$ from (14) we get

$$\begin{aligned} &\|z_i\|_\infty \le \|h_i\|_\infty \int\limits_{-\infty}^{t} e^{-\delta(t-s)/2}(t-s)\, ds = \tfrac{4}{\delta^2}\|h_i\|_\infty \\ &\le \tfrac{4}{\sqrt{3}\mu_i^2\delta}\|h_i\|_\infty \le \tfrac{4}{\mu_i^2\delta}\|h_i\|_\infty\,, \\ &\|\dot{z}_i\|_\infty \le \tfrac{4}{\delta}\|h_i\|_\infty\,, \end{aligned} \tag{18}$$

where we use in derivation of (18) the inequality $|\sin x| \le |x|\ \forall x \in \mathbb{R}$. From (15) for $2\mu_i^2 = \delta$ we get

$$\begin{aligned} &\|z_i\|_\infty \le \|h_i\|_\infty \int\limits_{-\infty}^{t} e^{-\delta(t-s)/2}(t-s)\, ds = \tfrac{4}{\delta^2}\|h_i\|_\infty \\ &= \tfrac{2}{\mu_i^2\delta}\|h_i\|_\infty \le \tfrac{4}{\mu_i^2\delta}\|h_i\|_\infty\,, \\ &\|\dot{z}_i\|_\infty \le \tfrac{4}{\delta}\|h_i\|_\infty\,. \end{aligned} \tag{19}$$

From (16) for $2\mu_i^2 < \delta$ we get

$$\begin{aligned} &\|z_i\|_\infty \le \tfrac{1}{\mu_j^4}\|h_i\|_\infty \\ &\|\dot{z}_i\|_\infty \le \tfrac{\delta}{\mu_i^4}\|h_i\|_\infty \le \delta\|h_i\|_\infty\,. \end{aligned} \tag{20}$$

From (12) we get

$$\|h_i\|_\infty \le M_1\Big(\frac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\Big)\,. \tag{21}$$

We consider the Banach space $C([0, \pi/4] \times S^T)$ with the usual maximum norm $\|\cdot\|_\infty$. We need the following result.

Proposition 1. *A sequence* $\{z^n(x,t)\}_{n=1}^\infty \subset C([0, \pi/4] \times S^T)$ *is precompact if there is a constant* $M > 0$ *such that*

$$\sup_{i\ge1, n\ge1} \|z_i^n\|_\infty i^2 < M, \qquad \sup_{i\ge1, n\ge1} \|\dot{z}_i^n\|_\infty < M\,, \tag{22}$$

where $z^n(x,t) = \sum\limits_{i=1}^{\infty} z_i^n(t) w_i(x)$.

Proof. From (22) we get

$$|z_i^n(t)|i^2 \le M, \quad |\dot z_i^n(t)| \le M \quad \forall t \in S^T .$$

By the Arzela-Ascoli theorem, there is a subsequence $\{z_1^{n_k}\}_{k=1}^\infty$ of $\{z_1^n\}_{n=1}^\infty$ such that $z_1^{n_k}(t) \to z_1^0(t)$ uniformly on S^T. Similarly we have a subsequence $\{z_2^{n_{k_s}}\}_{s=1}^\infty$ of $\{z_2^{n_k}\}_{k=1}^\infty$ such that $z_2^{n_{k_s}}(t) \to z_2^0(t)$ uniformly on S^T. Then we follow this construction. By using the Cantor diagonal procedure, we find an increasing sequence $\{m_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $z_i^{m_k}(t) \to z_i^0(t)$ uniformly on S^T for any $i \ge 1$. Clearly we have

$$\sup_{i\ge 1} \|z_i^0\|_\infty i^2 \le M .$$

Hence $z^0(x,t) = \sum_{i=1}^\infty z_i^0(t)w_i(x) \in C([0,\pi/4]\times S^T)$. Let $\varepsilon > 0$ be given. Then we choose $i_1 \in \mathbb{N}$ so large that $2M_1M\sum_{i=i_1}^\infty i^{-2} < \varepsilon/2$. We estimate

$$\begin{aligned}
&\|z^{m_k} - z^0\|_\infty \le M_1 \sum_{i=1}^\infty \|z_i^{m_k} - z_i^0\|_\infty \\
&\le 2M_1M\sum_{i=i_1}^\infty i^{-2} + M_1\sum_{i=1}^{i_1} \|z_i^{m_k} - z_i^0\|_\infty < \tfrac{\varepsilon}{2} + M_1\sum_{i=1}^{i_1} \|z_i^{m_k} - z_i^0\|_\infty .
\end{aligned}$$

For $1 \le i < i_1$, we have

$$\|z_i^{m_k} - z_i^0\|_\infty \to 0$$

as $k \to \infty$. Hence $\|z^{m_k} - z^0\|_\infty < \varepsilon/2$ for k large . This implies $z^{m_k} \to z^0$ in $C([0,\pi/4]\times S^T)$. The proof is finished.

Now if $h_i(t)$, $i \ge 1$ is given by (12) and T-periodic $z_i(t)$ are defined by (11), then $z(x,t)$ given by (10) satisfies $z(x,t) \in C([0,\pi/4]\times S^T)$. Indeed, from $\sum_{i=1}^\infty \mu_i^{-2} < \infty$ and (17) - (20) we have that the series (10) is uniformly convergent. Hence $z(x,t) \in C([0,\pi/4]\times S^T)$ and (17) - (20) also imply

$$\begin{aligned}
&\|z\|_\infty \le M_1 \sum_{2\mu_i<\delta} \tfrac{1}{\mu_i^4}\|h_i\|_\infty + M_1 \sum_{2\mu_i\ge\delta} \tfrac{4}{\mu_i^2\delta}\|h_i\|_\infty \\
&\le M_1 \sum_{i=1}^\infty \Big(\tfrac{1}{\mu_i^2} + \tfrac{4}{\mu_i^2\delta}\Big)\|h_i\|_\infty \\
&\le M_2\Big(\tfrac{1}{\delta} + \tfrac{1}{4}\Big)\Big(\tfrac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\Big) .
\end{aligned}$$

Moreover, we derive

$$\sup_{i\ge1} \|z_i\|_\infty i^2 \le M_3\Big(\frac{4}{\delta} + 1\Big)\Big(\frac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\Big) \tag{23}$$

and

$$\sup_{i\ge1} \|\dot z_i\|_\infty \le \Big(\frac{4}{\delta} + \delta\Big)\Big(\frac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\Big) . \tag{24}$$

We also know from [3] that such $z(x,t)$ satisfies (8). On the other hand, if $z(x,t) \in C([0,\pi/4]\times S^T)$ satisfies (8), then $z(x,t) = \sum_{i=1}^{\infty} z_i(t)w_i(x)$ in $L^2(0,\pi/4)$ for a.e. $t \in S^T$. By inserting $v(x,t) = \phi(t)w_i(x)$ in (8) with $\phi \in C^\infty(S^T)$, we get (11) with (12). So $z(x,t)$ has the above properties.

Finally, let us define the following Banach space

$$C_0([0,\pi/4]\times S^T) := \Big\{ z(x,t) \in C([0,\pi/4]\times S^T) \mid \int_0^{\pi/4} z(x,t)\,dx = \int_0^{\pi/4} z(x,t)x\,dx = 0 \quad \forall\, t \in S^T \Big\}$$

with the maximum norm $\|\cdot\|_\infty$ on $[0,\pi/4]\times S^T$. Summarizing, we arrive at the following result.

Proposition 2. *For any given functions* $H_1(x,t) \in C([0,\pi/4]\times S^T)$, $H_2(t), H_3(t) \in C(S^T)$, *equation* (8) *has a unique solution* $z(x,t) \in C_0([0,\pi/4]\times S^T)$ *of the form*

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t)w_i(x)\,.$$

Such a solution satisfies the condition (9) *along with:*

(a) $z(x,t) \in X$ *for the Banach space*

$$X = \Big\{ z(x,t) \in C([0,\pi/4]\times S^T) \mid z(x,t) = \sum_{i=1}^{\infty} z_i(t)w_i(x), \sup_{i\geq 1} \|z_i\|_\infty i^2 < \infty \Big\}\,.$$

(b) $\|z\|_\infty \leq M_2\Big(\frac{1}{\delta} + \frac{1}{4}\Big)\big(\frac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\big)$.

(c) *The mapping* $L : C([0,\pi/4]\times S^T)\times C(S^T)\times C(S^T) \to C_0([0,\pi/4]\times S^T)$ *defined by* $L(H_1,H_2,H_3) := z(x,t)$ *is compact.*

Proof. Properties (a) and (b) are proved above. Property (c) follows from Lemma 1 and inequalities (23) and (24). The proof is finished.

4 Nonhomogeneous Linear Problems

In this section, we consider the linear problem

$$\begin{aligned}
&u_{tt} + u_{xxxx} + \delta u_t + h(x,t) = 0\,,\\
&u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,,\\
&u_{xxx}(0,\cdot) = -ku(0,\cdot) - f_1(t)\,,\\
&u_{xxx}(\pi/4,\cdot) = ru(u(\pi/4,\cdot) + f_2(t)\,,
\end{aligned} \tag{25}$$

where $h(x,t) \in C([0,\pi/4] \times S^T)$, $f_1(t), f_2(t) \in C(S^T)$. Of course, we consider (25) in the sense of (5). Now we split $u(x,t)$ as follows

$$u(x,t) = y_1(t) w_{-1}(x) + y_2(t) w_0(x) + z(x,t)$$

with $z(x,t) \in C_0([0,\pi/4] \times S^T)$. Then (25) is equivalent to the system

$$\begin{aligned} &\ddot{y}_1(t) + \delta \dot{y}_1(t) + \tfrac{2}{\sqrt{\pi}} \int_0^{\pi/4} h(x,t)\, dx \\ &+ \tfrac{4}{\pi}(k+r) y_1(t) + \tfrac{4\sqrt{3}}{\pi}(r-k) y_2(t) \\ &+ \tfrac{2}{\sqrt{\pi}} \left(k z(0,t) + r z(\pi/4,t) + f_1(t) + f_2(t) \right) = 0\,, \end{aligned} \tag{26}$$

$$\begin{aligned} &\ddot{y}_2(t) + \delta \dot{y}_2(t) + \tfrac{16}{\pi}\sqrt{\tfrac{3}{\pi}} \int_0^{\pi/4} h(x,t)\Big(x - \tfrac{\pi}{8}\Big)\, dx \\ &+ \tfrac{4\sqrt{3}}{\pi}(r-k) y_1(t) + \tfrac{12}{\pi}(k+r) y_2(t) \\ &+ 2\sqrt{\tfrac{3}{\pi}} \left(r z(\pi/4,t) - k z(0,t) + f_2(t) - f_1(t) \right) = 0\,, \end{aligned} \tag{27}$$

$$\begin{aligned} &\int_0^T \int_0^{\pi/4} \Big[z(x,t) \Big\{ v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t) \Big\} + h(x,t) v(x,t) \Big]\, dx\, dt \\ &+ \int_0^T \Big\{ \Big(k \Big(\tfrac{2}{\sqrt{\pi}} y_1(t) - 2\sqrt{\tfrac{3}{\pi}} y_2(t) + z(0,t) \Big) + f_1(t) \Big) v(0,t) \\ &+ \Big(r \Big(\tfrac{2}{\sqrt{\pi}} y_1(t) + 2\sqrt{\tfrac{3}{\pi}} y_2(t) + z(\pi/4,t) \Big) + f_2(t) \Big) v(\pi/4,t) \Big\}\, dt = 0 \end{aligned} \tag{28}$$

where $v(x,t) \in C^\infty([0,\frac{\pi}{4}] \times S^T)$ satisfies the conditions (6), (9). Let us define

$$\begin{aligned} &L_1 : C(S^T) \times C(S^T) \to C_0([0,\pi/4] \times S^T)\,, \\ &L_2 : C_0([0,\pi/4] \times S^T) \to C_0([0,\pi/4] \times S^T)\,, \\ &H_4 \in C_0([0,\pi/4] \times S^T) \end{aligned}$$

by

$$\begin{aligned} &L_1(y_1,y_2) := L\Big(0, k\Big(\tfrac{2}{\sqrt{\pi}} y_1(t) - 2\sqrt{\tfrac{3}{\pi}} y_2(t)\Big), r\Big(\tfrac{2}{\sqrt{\pi}} y_1(t) + 2\sqrt{\tfrac{3}{\pi}} y_2(t)\Big)\Big)\,, \\ &L_2(z) := L\Big(0, k z(0,t), r z(\pi/4,t)\Big)\,, \\ &H_4(t) := L\Big(h(x,t), f_1(t), f_2(t)\Big)\,. \end{aligned}$$

Then according to Proposition 2, equation (28) has the form

$$z = L_2(z) + L_1(y_1,y_2) + H_4\,. \tag{29}$$

Moreover, operators L_1 and L_2 are compact. Furthermore, since for $r > 0$, $k > 0$ the matrix

$$A = \frac{4}{\pi} \begin{pmatrix} (k+r) & \sqrt{3}(r-k) \\ \sqrt{3}(r-k) & 3(r+k) \end{pmatrix}$$

is invertible, the system

$$\ddot{y} + \delta \dot{y} + A y = \bar{h}(t) = (h_1(t), h_2(t)) \in C(S^T)^2 \tag{30}$$

has a unique T-periodic solution $y = (y_1, y_2) := L_3(h_1, h_2)$. Let us define

$$L_4 : C_0([0, \pi/4] \times S^T) \to C(S^T)^2, \quad H_5 \in C(S^T)$$

given by

$$\begin{aligned}
L_4(z) &:= L_3\Big(\tfrac{2}{\sqrt{\pi}}\left(kz(0,t) + rz(\pi/4,t)\right), 2\sqrt{\tfrac{3}{\pi}}\left(rz(\pi/4,t) - kz(0,t)\right)\Big)\,,\\
H_5(t) &:= L_3\Big(\tfrac{2}{\sqrt{\pi}}\int_0^{\pi/4} h(x,t)\,dx + \tfrac{2}{\sqrt{\pi}}(f_1(t) + f_2(t)),\\
&\qquad \tfrac{16}{\pi}\sqrt{\tfrac{3}{\pi}}\int_0^{\pi/4} h(x,t)\Big(x - \tfrac{\pi}{8}\Big)\,dx + 2\sqrt{\tfrac{3}{\pi}}(f_2(t) - f_1(t))\Big)\,.
\end{aligned}$$

Then (26) and (27) are equivalent to

$$y = L_4(z) + H_5(t) \tag{31}$$

and L_4 is compact. Consequently, in order to solve uniquely equations (29) and (31), we must show that if

$$\begin{array}{l} z = L_2(z) + L_1(y_1, y_2), \quad z \in C_0([0, \pi/4] \times S^T)\\ y = L_4(z), \quad y = (y_1, y_2) \in C(S^T)^2\,, \end{array} \tag{32}$$

then $z = 0$ and $y = 0$. Equation (32) is equivalent to the system

$$\begin{aligned}
&\ddot{z}_j(t) + \delta\dot{z}_j(t) + \mu_j^4 z_j(t)\\
&+ \sum_{s=-1}^{\infty}\Big(kz_s(t)w_s(0)w_j(0) + rz_s(t)w_s(\pi/4)w_j(\pi/4)\Big) = 0
\end{aligned} \tag{33}$$

for $z_j \in C^2(S^T)$, $j \geq -1$ with $\sup_{j\geq -1}\|z_j\|_\infty(j^2+1) < \infty$. Let us expand

$$z_j(t) = \sum_{m\in\mathbb{Z}} e^{\imath 2\pi mt/T} c_{mj}\,.$$

Note that $c_{mj} \sim j^{-2}$ as $j \to \infty$ uniformly for $m \in \mathbb{Z}$. Then by (33) we derive

$$\begin{aligned}
&c_{mj}\left(\mu_j^4 - \tfrac{4\pi^2 m^2}{T^2} + \imath\tfrac{2\delta\pi}{T}m\right)\\
&+ \sum_{s=-1}^{\infty}\Big(kc_{ms}w_s(0)w_j(0) + rc_{ms}w_s(\pi/4)w_j(\pi/4)\Big) = 0\,.
\end{aligned} \tag{34}$$

By taking $c_{mj} = a_{mj} + \imath b_{mj}$, from (34) we derive

$$\begin{aligned}
&a_{mj}\left(\mu_j^4 - \tfrac{4\pi^2 m^2}{T^2}\right) - \tfrac{2\delta\pi}{T}mb_{mj}\\
&+ \sum_{s=-1}^{\infty}\Big(kw_s(0)w_j(0) + rw_s(\pi/4)w_j(\pi/4)\Big)a_{ms} = 0\,,
\end{aligned}$$

$$\begin{aligned}
&a_{mj}\tfrac{2\delta\pi}{T}m + \left(\mu_j^4 - \tfrac{4\pi^2 m^2}{T^2}\right)b_{mj}\\
&+ \sum_{s=-1}^{\infty}\Big(kw_s(0)w_j(0) + rw_s(\pi/4)w_j(\pi/4)\Big)b_{ms} = 0\,.
\end{aligned}$$

Since $a_{mj}, b_{mj} \sim j^{-2}$ as $j \to \infty$, we get

$$\sum_{j=-1}^{\infty} \left(a_{mj}^2 + b_{mj}^2\right) \frac{2\pi\delta}{T} m = 0\,,$$

hence $a_{mj} = b_{mj} = 0$ for any $m \neq 0$ and j. For $m = 0$ we get

$$a_{0j}\mu_j^4 + \sum_{s=-1}^{\infty} \Big(kw_s(0)w_j(0) + rw_s(\pi/4)w_j(\pi/4)\Big) a_{0s} = 0\,, \tag{35}$$

$$b_{0j}\mu_j^4 + \sum_{s=-1}^{\infty} \Big(kw_s(0)w_j(0) + rw_s(\pi/4)w_j(\pi/4)\Big) b_{0s} = 0\,. \tag{36}$$

We put $a_{0j}(\mu_j^4 + 1) = e_j$ and from (35) we get

$$\sum_{j=-1}^{\infty} \frac{e_j^2}{\mu_j^4+1} \frac{\mu_j^4}{\mu_j^4+1} + k\Big(\sum_{s=-1}^{\infty} w_s(0) \frac{e_s}{\mu_s^4+1}\Big)^2 + r\Big(\sum_{s=-1}^{\infty} w_s(\pi/4) \frac{e_s}{\mu_s^4+1}\Big)^2 = 0\,. \tag{37}$$

From (37) for $r > 0$, $k > 0$ we immediately get $e_j = 0$ for $j \geq 1$ and

$$\frac{2}{\sqrt{\pi}} e_{-1} - 2\sqrt{\frac{3}{\pi}} e_0 = 0, \quad \frac{2}{\sqrt{\pi}} e_{-1} + 2\sqrt{\frac{3}{\pi}} e_0 = 0\,,$$

which imply also $e_{-1} = e_0 = 0$. Similar results hold for (36). Hence, (32) has the only solution $z = 0$ and $y = 0$. Consequently, (29) and (31) are uniquely solvable in z, y for $r > 0$, $k > 0$. Summarizing, we arrive at the following result.

Proposition 3. *If $r > 0$, $k > 0$ then for any given functions $h(x,t) \in C([0,\pi/4] \times S^T)$ and $f_1(t)$, $f_2(t) \in C(S^T)$, equation (25) has a unique solution $u(x,t) \in C([0,\pi/4] \times S^T)$ of the form*

$$u(x,t) = \sum_{i=-1}^{\infty} z_i(t) w_i(x)\,.$$

Such a solution satisfies:

(a) *$u(x,t) \in Y$ for the Banach space*

$$Y = \Big\{ u(x,t) \in C([0,\pi/4] \times S^T) \mid u(x,t) = \sum_{i=-1}^{\infty} z_i(t) w_i(x), \\ \|u\| := \sup_{i \geq -1} \|z_i\|_\infty (|i|+1)^2 < \infty \Big\}\,.$$

(b) *$\|u\|$, $\|u\|_\infty \leq c\,(\|h\|_\infty + \|f_1\|_\infty + \|f_2\|_\infty)$ for a constant $c > 0$.*

(c) *The mapping $\tilde{L} : C([0,\pi/4] \times S^T) \times C(S^T) \times C(S^T) \to C([0,\pi/4] \times S^T)$ defined by $\tilde{L}(h, f_1, f_2) := u(x,t)$ is compact.*

We also define a compact mapping $\bar{L} : C(S^T) \times C(S^T) \to C([0, \pi/4] \times S^T)$ by $\bar{L}(f_1, f_2) := \tilde{L}(0, f_1, f_2)$. We denote by $\|\bar{L}\|$ the norm of $\bar{L}$.

Now we study the case when $r = 0$ and $k > 0$ in (25). Then equation (29) remains, but the matrix A is no more invertible. Equation (30) has a T-periodic solution if and only if

$$\int_0^T (\sqrt{3} h_1(t) + h_2(t))\, dt = 0$$

and the linear equation

$$\ddot{y} + \delta \dot{y} + Ay = 0$$

has the only T-periodic solutions $y(t) = c(\sqrt{3}, 1)$, $c \in \mathbb{R}$. Consequently, we are still working with Fredholm operators of index 0 possessing forms of compact perturbations of identity operators [11]. Hence, in order to study (25) we consider like in (33) the equations

$$\ddot{z}_j(t) + \delta \dot{z}_j(t) + \mu_j^4 z_j(t) + k \sum_{s=-1}^{\infty} z_s(t) w_s(0) w_j(0) = 0 \tag{38}$$

$$\ddot{z}_j(t) + \delta \dot{z}_j(t) + \mu_j^4 z_j(t) + k \sum_{s=-1}^{\infty} z_s(t) w_s(0) w_j(0) + h_j(t) = 0 \tag{39}$$

for $h_j \in C(S^T)$, $z_j \in C^2(S^T)$, $j \geq -1$ with $\sup\limits_{j \geq -1} \|z_j\|_\infty (j^2 + 1) < \infty$. Like for (33), we get that $z_{-1}(t) = c\sqrt{3}$, $z_0(t) = c$, $c \in \mathbb{R}$, $z_j(t) = 0$, $j \geq 1$ for (38). According to the above comments, the set of all $\{h_j(t)\}_{j \geq -1}$ for which (39) is solvable must have a codimension 1. For this reason, we consider an adjoint equation to (38) of the form

$$\ddot{v}_j(t) - \delta \dot{v}_j(t) + \mu_j^4 v_j(t) + k \sum_{s=-1}^{\infty} v_s(t) w_s(0) w_j(0) = 0 \tag{40}$$

for $v_j \in C^2(S^T)$, $j \geq -1$ with $\sup\limits_{j \geq -1} \|v_j\|_\infty (j^2 + 1) < \infty$. Like above we get $v_{-1}(t) = c\sqrt{3}$, $v_0(t) = c$, $c \in \mathbb{R}$, $v_j(t) = 0$, $j \geq 1$ for (40). By multiplying (39) with $v_j(t)$ and using integration by parts, we get

$$\begin{aligned} &\int_0^T z_j(t)\Big(\ddot{v}_j(t) - \delta \dot{v}_j(t) + \mu_j^4 v_j(t)\Big)\, dt \\ &+ k\int_0^T \Big(\sum_{s=-1}^{\infty} z_s(t) w_s(0) w_j(0) v_j(t)\Big)\, dt + \int_0^T h_j(t) v_j(t)\, dt = 0\,. \end{aligned} \tag{41}$$

Inserting (40) to (41) we obtain

$$\begin{aligned} &k\int_0^T \sum_{s=-1}^{\infty} \Big(z_s(t) w_s(0) w_j(0) v_j(t) - z_j(t) w_s(0) w_j(0) v_s(t)\Big)\, dt \\ &+ \int_0^T h_j(t) v_j(t)\, dt = 0\,. \end{aligned} \tag{42}$$

Since $v_j(t) \sim j^{-2}$, $z_j(t) \sim j^{-2}$ uniformly on S^T, (42) implies

$$0 = \sum_{s=-1}^{\infty} \int_0^T h_j(t) v_j(t)\, dt = \int_0^T (\sqrt{3} h_{-1}(t) + h_0(t))\, dt \,. \qquad (43)$$

We recall that the set of all $\{h_j(t)\}_{j \geq -1}$ for which (39) is solvable has a codimension 1. Then condition (43) is necessary and also sufficient for solvability of (39). We note

$$\begin{aligned} h_{-1}(t) &= \tfrac{2}{\sqrt{\pi}} \int_0^{\pi/4} h(x,t)\, dx + \tfrac{2}{\sqrt{\pi}} \left(f_1(t) + f_2(t)\right) , \\ h_0(t) &= \tfrac{16}{\pi} \sqrt{\tfrac{3}{\pi}} \int_0^{\pi/4} h(x,t) \left(x - \tfrac{\pi}{8}\right) dx + 2\sqrt{\tfrac{3}{\pi}} \left(f_2(t) - f_1(t)\right) . \end{aligned}$$

Then condition (43) has the form

$$\frac{4}{\pi} \int_0^T \int_0^{\pi/4} h(x,t) x\, dx\, dt + \int_0^T f_2(t)\, dt = 0 \,. \qquad (44)$$

Finally, the corresponding kernel to (38) is spanned by the function

$$z_{-1}(t) w_{-1}(x) + z_0(t) w_0(x) = \frac{16}{\pi} \sqrt{\frac{3}{\pi}} x \,. \qquad (45)$$

Summarizing we get the next result.

Proposition 4. *If $r = 0$, $k > 0$ then for any given functions $h(x,t) \in C([0,\pi/4] \times S^T)$ and $f_1(t)$, $f_2(t) \in C(S^T)$, equation (25) has a solution $u(x,t) \in C([0,\pi/4] \times S^T)$ if and only if condition (44) holds. Such a solution is unique if*

$$\int_0^T \int_0^{\pi/4} u(x,t) x\, dx\, dt = 0 \,. \qquad (46)$$

Moreover, the mapping $K : C_1 \to C_2$ is compact where

$$\begin{aligned} C_1 &:= \left\{ (h, f_1, f_2) \in C([0,\pi/4] \times S^T) \times C(S^T)^2 \mid \quad \text{condition (44) holds} \right\} , \\ C_2 &:= \left\{ u \in C([0,\pi/4] \times S^T) \mid \quad \text{condition (46) holds} \right\} \end{aligned}$$

are Banach spaces endowed with the maximum norms and the mapping K is defined by $K(h, f_1, f_2) := u(x,t)$.

Similarly we derive the next results.

Proposition 5. *If $r > 0$, $k = 0$ then for any given functions $h(x,t) \in C([0,\pi/4]\times S^T)$ and $f_1(t)$, $f_2(t) \in C(S^T)$, equation* (25) *has a solution $u(x,t) \in C([0,\pi/4]\times S^T)$ if and only if*

$$\frac{4}{\pi}\int\limits_0^T\int\limits_0^{\pi/4} h(x,t)\Big(\frac{\pi}{4}-x\Big)\,dx\,dt + \int\limits_0^T f_1(t)\,dt = 0\,. \tag{47}$$

Such a solution is unique if

$$\int\limits_0^T\int\limits_0^{\pi/4} u(x,t)\Big(\frac{\pi}{4}-x\Big)\,dx\,dt = 0\,. \tag{48}$$

Moreover, the mapping $\tilde{K} : \tilde{C}_1 \to \tilde{C}_2$ is compact where

$$\begin{aligned}\tilde{C}_1 &:= \Big\{(h, f_1, f_2) \in C([0,\pi/4]\times S^T)\times C(S^T)^2 \mid \quad \text{condition (47) holds}\Big\}\,,\\ \tilde{C}_2 &:= \Big\{u \in C([0,\pi/4]\times S^T) \mid \quad \text{condition (48) holds}\Big\}\end{aligned}$$

are Banach spaces endowed with the maximum norms and the mapping K is defined by $\tilde{K}(h, f_1, f_2) := u(x,t)$.

Proposition 6. *If $r = k = 0$ then for any given functions $h(x,t) \in C([0,\pi/4]\times S^T)$ and $f_1(t)$, $f_2(t) \in C(S^T)$, equation* (25) *has a solution $u(x,t) \in C([0,\pi/4]\times S^T)$ if and only if the both conditions* (44) *and* (47) *hold. Such a solution is unique if the both conditions* (46) *and* (48) *hold.*

Moreover, the mapping $\bar{K} : \bar{C}_1 \to \bar{C}_2$ is compact where

$$\begin{aligned}\bar{C}_1 &:= \Big\{(h, f_1, f_2) \in C([0,\pi/4]\times S^T)\times C(S^T)^2 \mid \text{conditions (44), (47) hold}\Big\}\,,\\ \bar{C}_2 &:= \Big\{u \in C([0,\pi/4]\times S^T) \mid \quad \text{conditions (46), (48) hold}\Big\}\end{aligned}$$

are Banach spaces endowed with the maximum norms and the mapping $\bar{K}$ is defined by $\bar{K}(h, f_1, f_2) := u(x,t)$.

5 Nonlinear Problems

In this section, we present the main results concerning equation (1).

Theorem 1. *If $r > 0, k > 0$ and there are positive constants $c_{11}, c_{12}, c_{21}, c_{22}$ along with*

$$c_{12} + c_{22} < 1/\|\bar{L}\|$$

and such that

$$\begin{aligned}|f(u)| &\le c_{11} + c_{12}|u|, \quad \forall u \in \mathbb{R}\\ |g(u)| &\le c_{21} + c_{22}|u|, \quad \forall u \in \mathbb{R}\,,\end{aligned}$$

then for any given function $h(x,t) \in C([0,\pi/4] \times S^T)$, equation (1) possesses a weak T-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$.

Proof. By using the above results, the proof is standard. According to Proposition 3, equation (1) is equivalent to

$$u = F(u) := \tilde{L}(h,0,0) + \bar{L}\Big(f(u(0,\cdot)), g(u(\pi/4,\cdot))\Big)\,. \tag{49}$$

Proposition 3 also implies the compactness of the mapping

$$F : C([0,\pi/4] \times S^T) \to C([0,\pi/4] \times S^T)\,.$$

From the assumptions of Theorem 1 and (b) of Proposition 3, we get

$$\begin{aligned} \|F(u)\|_\infty &\le c\|h\|_\infty + \|\bar{L}\|\Big(\|f(u(0,\cdot))\|_\infty + \|g(u(\pi/4,\cdot))\|_\infty\Big) \\ &\le c\|h\|_\infty + \|\bar{L}\|\Big(c_{11} + c_{21} + (c_{12}+c_{22})\|u\|_\infty\Big)\,. \end{aligned} \tag{50}$$

Since $\|\bar{L}\|(c_{12}+c_{22}) < 1$, there is a unique $\tau > 0$ such that

$$\tau = c\|h\|_\infty + \|\bar{L}\|\big(c_{11} + c_{21} + (c_{12}+c_{22})\tau\big)\,.$$

Consequently, (50) implies that the ball

$$B_\tau = \Big\{u \in C([0,\pi/4] \times S^T) \mid \quad \|u\| \le \tau\Big\}$$

is mapped to itself by the mapping F. The Schauder fixed point theorem ensures the existence of a fixed point $u \in B_\tau$ of F. This gives a weak T-periodic solution of (1). The proof is finished.

Of course, when f, g have sublinear growth at infinity:

$$\lim_{|u|\to\infty} f(u)/u = 0, \qquad \lim_{|u|\to\infty} g(u)/u = 0$$

and $r > 0$, $k > 0$, then the assumptions of Theorem 1 hold and equation (1) possesses a weak T-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$ for any $h(x,t) \in C([0,\pi/4] \times S^T)$.

The implicit function theorem together with Proposition 3 gives the next result.

Theorem 2. *If $r > 0$, $k > 0$, $f(0) = f'(0) = g(0) = g'(0) = 0$ and $f, g \in C^1(S^T)$, then there are positive constants K_1, ε_0 such that for any given function $h(x,t) \in C([0,\pi/4] \times S^T)$ with $\|h\|_\infty < \varepsilon_0$, equation (1) possesses a unique small weak T-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$ satisfying $\|u\|_\infty \le K_1\|h\|_\infty$.*

Now we suppose that $r = 0$ and $k > 0$ in equation (1). Let

$$P : C([0, \pi/4] \times S^T) \times C(S^T)^2 \to C_1$$

be a continuous projection and let

$$C_2 \oplus \mathbb{R}x = C([0, \pi/4] \times S^T)$$

be a continuous splitting $u(x,t) = v(x,t) + c\frac{4}{\pi}x$ with $v(x,t) \in C_2$ and $c \in \mathbb{R}$. Then according to Proposition 4, equation (1) is equivalent to the system

$$v = \lambda K\Big(P\Big(h, f(v(0,\cdot)), g(c + v(\pi/4, \cdot))\Big)\Big)\,, \tag{51}$$

$$\frac{4}{\pi}\int_0^T\int_0^{\pi/4} h(x,t)x\,dx\,dt + \int_0^T g(c + \lambda v(\pi/4, t))\,dt = 0 \tag{52}$$

for $\lambda = 1$. Now we can prove the next result.

Theorem 3. *Let $r = 0$ and $k > 0$. If $\sup_{u\in\mathbb{R}} |f(u)| < \infty$, finite limits*

$$\lim_{u\to\pm\infty} g(u) := g_\pm$$

exist and it holds

$$\frac{4}{T\pi}\int_0^T\int_0^{\pi/4} h(x,t)x\,dx\,dt \in (-g_-, -g_+)\,. \tag{53}$$

Then equation (1) *possesses a weak T-periodic solution $u(x,t) \in C([0, \pi/4] \times S^T)$. On the other hand, if*

$$\Big|\frac{4}{T\pi}\int_0^T\int_0^{\pi/4} h(x,t)x\,dx\,dt\Big| > \sup_{u\in\mathbb{R}} |g(u)| \tag{54}$$

then equation (1) *has no weak T-periodic solutions.*

Proof. This is a Landesman-Lazer type result [4], [10]. We consider (51) and (52) for $0 \le \lambda \le 1$ on $C_2 \oplus \mathbb{R}$. Since h, f, g are bounded, Proposition 4 implies that any solution of (51) must satisfy $\|v\|_\infty \le K_1$, for a constant $K_1 > 0$. We take the set

$$B = \Big\{(v, c) \in C_2 \oplus \mathbb{R} \mid \ \|v\|_\infty < K_1 + 1, \quad |c| < K_2\Big\}$$

for a fixed large $K_2 > 0$. If $(v, c) \in \partial B$ then either $\|v\|_\infty = K_1 + 1$ and then (51) does not hold, or $\|v\|_\infty \le K_1 + 1$ and $c = \pm K_2$, and then (52) does not hold according to (53). Hence we can apply Leray-Schauder degree to (51) and (52) on B [4], [10]. For $\lambda = 0$ we get a function

$$c \to \frac{4}{T\pi} \int_0^T \int_0^{\pi/4} h(x,t) x \, dx \, dt + g(c) \,,$$

which according to (53) changes the sign on $[-K_2, K_2]$. Consequently, (51) and (52) are solvable on B. On the other hand, if (52) holds then clearly (54) can not be satisfied. The proof is finished.

Similarly we get the next result.

Theorem 4. *Let $r = 0$ and $k > 0$. If $\sup_{u \in \mathbb{R}} |f(u)| < \infty$, finite limits*

$$\lim_{u \to \pm\infty} g(u) := g_{\pm}$$

exist and g is monotonic on $\mathbb{R}$. Then equation (1) *possesses a weak T-periodic solution if* (53) *holds and it has no weak T-periodic solutions if*

$$\frac{4}{T\pi} \int_0^T \int_0^{\pi/4} h(x,t) x \, dx \, dt \notin [-g_-, -g_+] \,.$$

If in addition, g is strictly monotonic on $\mathbb{R}$, then equation (1) *possesses a weak T-periodic solution if and only if* (53) *holds.*

By using Proposition 5, similar arguments hold for the case $r > 0$ and $k = 0$. We state this result for the reader convenience.

Theorem 5. *Let $r > 0$ and $k = 0$. If $\sup_{u \in \mathbb{R}} |g(u)| < \infty$, finite limits*

$$\lim_{u \to \pm\infty} f(u) := f_{\pm}$$

exist and it holds

$$\frac{4}{T\pi} \int_0^T \int_0^{\pi/4} h(x,t) \Big(\frac{\pi}{4} - x \Big) \, dx \, dt \in (-f_-, -f_+) \,. \tag{55}$$

Then equation (1) *possesses a weak T-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$. On the other hand, if*

$$\Big| \frac{4}{T\pi} \int_0^T \int_0^{\pi/4} h(x,t) \Big(\frac{\pi}{4} - x \Big) \, dx \, dt \Big| > \sup_{u \in \mathbb{R}} |f(u)| \tag{56}$$

then equation (1) *has no weak T-periodic solutions.*

Theorem 4 can be also modified for the case $r > 0$ and $k = 0$. Now we study the case that $r = k = 0$ in equation (1). This is a codimension two problem. The above approach to Theorem 3 can be used with the following modifications. Let

$$\bar{P} : C([0, \pi/4] \times S^T) \times C(S^T)^2 \to \bar{C}_1$$

be a continuous projection and let

$$\bar{C}_2 \oplus \mathbb{R}\Big(1 - \frac{4}{\pi}x\Big) \oplus \mathbb{R}x = C([0, \pi/4] \times S^T)$$

be a continuous splitting $u(x,t) = v(x,t) + c_1\Big(1 - \frac{4}{\pi}x\Big) + c_2\frac{4}{\pi}x$ with $v(x,t) \in \bar{C}_2$ and $c_1, c_2 \in \mathbb{R}$. Then according to Proposition 6, equation (1) is equivalent to the system

$$v = \lambda\bar{K}\Big(\bar{P}\Big(h, f(c_1 + v(0,\cdot)), g(c_2 + v(\pi/4,\cdot))\Big)\Big), \tag{57}$$

$$\frac{4}{\pi}\int_0^T\int_0^{\pi/4} h(x,t)\Big(\frac{\pi}{4} - x\Big)\,dx\,dt + \int_0^T f(c_1 + \lambda v(0,t))\,dt = 0 \tag{58}$$

$$\frac{4}{\pi}\int_0^T\int_0^{\pi/4} h(x,t)x\,dx\,dt + \int_0^T g(c_2 + \lambda v(\pi/4,t))\,dt = 0 \tag{59}$$

for $\lambda = 1$. By repeating the proof of Theorem 3 to (57)-(59), we get the next result.

Theorem 6. *Let $r = k = 0$. If finite limits*

$$\lim_{u\to\pm\infty} f(u) := f_\pm, \quad \lim_{u\to\pm\infty} g(u) := g_\pm$$

exist and the both conditions (53) *and* (55) *hold, then equation* (1) *possesses a weak T-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$. On the other hand, if one of the conditions* (54) *and* (56) *is satisfied, then equation* (1) *has no weak T-periodic solutions.*

Now let us suppose that $f, g \in C^1(\mathbb{R})$. If we consider equation (8) for any $v(x,t) \in C^\infty([0,\pi/4] \times \mathbb{R})$ satisfying the boundary conditions (6) and also orthogonal to each $w_i(x)$, $i = -1, 0, \cdots, i_1$ for $i_1 \in \mathbb{N}$ large and fixed, like in (9). Then we look for $z(x,t)$ in the form

$$z(x,t) = \sum_{i=i_1+1}^{\infty} z_i(t)w_i(x),$$

and we get a result similar to Proposition 2 with an estimate as (b) for $M_2 \to 0$ as $i_1 \to \infty$. Consequently, we can locally reduce by means of the Ljapunov-Schmidt method the solvability of (1) to finite-dimensional mappings. In this way, we can repeat the proof of the Sard-Smale theorem [4], [11] for (1). Moreover, by following a method of [11], we can prove the next result.

Theorem 7. *Let the assumptions of Theorem* 1 *hold along with that* $f, g \in C^1(\mathbb{R})$*. Then there is an open and dense subset* $C_3 \subset C([0, \pi/4] \times S^T)$ *such that for any given* $h(x,t) \in C_3$*, equation* (1) *possesses a finite nonzero number of weak* T*-periodic solutions* $u(x,t) \in C([0, \pi/4] \times S^T)$*. This number of solutions is constant on each connected components of* C_3*.*

Finally, we note that the question on the existence of a global bounded weak solution of (1) remains open when $h(x,t)$ is only bounded on $[0, \pi/4] \times S^T$. A combination of methods of [1] and this paper would be hopeful.

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