# The second term of the asymptotics of the monodromy map in case of two even edges of Newton diagram

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**Abstract**

The second term of the asymptotics of the monodromy map of monodromic singular point for some class of vector fields, Newton diagram of which consists of two even edges is computed; in that case the principal term of the asymptotics is an identity mapping. The obtained result allows to formulate the sufficient condition of focus for the singular point from the class under consideration.

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*Key words and phrazes: monodromic singular point, center, focus, monodromy map, resolution of singularity, Newton diagram* [1]

## Introduction.

In this paper we calculate the second term of the asymptotics of the monodromy map of monodromic singular point in case when the principal term of the asymptotics is an identity mapping.

It is known ([1],[2]), that the monodromy map (return map) of the monodromic singular point of the analytic vector field on the plane has a linear principal term of the asymptotics

$$\Delta(\rho) = C\rho + o(\rho).$$

The logarithm of the coefficient of this principal term is computed in ([3]) for a so called $\Gamma$-nondegenerate vector field. It is expressed via the Taylor coefficients of the principal part of the vector field defined by Newton diagram $\Gamma$. If $\Delta(\rho) \equiv \rho$, then the the singular point is a center. The inequality $\ln C \neq 0$ is the suffucient condition for the singular point to be a focus.

It was found ([3]) if all the edges of the Newton diagram $\Gamma$ are even, then $\ln C$ is identically equal to zero in the class of all the $\Gamma$ -nondegenerate vector fields having a

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monodromic singular point. That is impossible to obtain the sufficient condition of focus with help of the principal term of asymptotics.

In this work we consider $\Gamma$-nondegenerate vector fields with a monodromic singular point Newton diagram of which consists of two even edges. Under some additional conditions we calculate the second term of asymptotics of the monodromy map.

Let us recall some notions connected with the Newton diagram.

We write an analytic vector field (germ) in the neighbourhood of a singular point $(x, y) = (0, 0)$ in the form

$$\frac{X(x,y)}{y}\frac{\partial}{\partial x} + \frac{Y(x,y)}{x}\frac{\partial}{\partial y}. \tag{0.1}$$

Here the functions $X$ and $Y$ are divisible by $y$ and $x$ respectively. The vector field (0.1) defines a dynamical system that it will be convenient to write in the form

$$y\dot{x} = X(x,y)\ ,\ x\dot{y} = Y(x,y). \tag{0.2}$$

**Definitions** 1. Let

$$(\sum a_{ij}x^i y^j, \sum b_{ij}x^i y^j)$$

be a Taylor expansion of the right hand-side of the system (0.2). The *support* of the vector field (0.1) and of the system (0.2) is the set $\{(i,j) : (a_{ij}, b_{ij}) \neq (0,0)\}$. The pair $(a_{ij}, b_{ij})$ is called the *vector coefficient* of the point $(i, j)$ of the support. *The index of the point* $(i, j)$ *of the support* is the quantity

$$\begin{cases} \frac{b_{ij}}{a_{ij}}, & if\ a_{ij} \neq 0 \\ \infty, & if\ a_{ij} = 0. \end{cases}$$

The vector coefficient of any other integer-valued point we define as $(0, 0)$.

2. Consider the set

$$\bigcup_{(i,j)} \{(i,j) + R_+^2\},$$

where $R_+^2$ is the positive quadrant, the points $(i, j)$ belong to the support. The boundary of the convex hull of this set consists of two open rays and one broken line, which can consist of one point. This broken line is called *the Newton diagram* of the vector field (0.1). The links of this broken line are called the *edges* of a Newton diagram, and their end-points are called the *vertices* of a Newton diagram.

3. The *index* of an edge of a Newton Diagram is the rational number that is equal to the tangent of the angle between the negative direction of the $j-$ axes, and the edge.

Consider the edge of the Newton diagram of the system (0.2) with index $\alpha = \frac{m}{n}$, where $\frac{m}{n}$ is irreducible fraction. We can group the terms of the Taylor series of the system (0.2) so that

$$y\dot{x} = \sum_{d=0}^{\infty} X_d(x,y), \qquad x\dot{y} = \sum_{d=0}^{\infty} Y_d(x,y), \tag{0.3}$$

where

$$X_d(x,y)=\sum_{ni+mj=d+d_0} a_{ij}x^iy^j,\qquad Y_d(x,y)=\sum_{ni+mj=d+d_0} b_{ij}x^iy^j \tag{0.4}$$

are quasihomogeneous polynomials of degree $d+d_0$ with weights $n$ and $m$ in the variables $x$ and $y$ respectively, $d_0\in\mathbf{N}$.

We set $F_d(x,y)=Y_d(x,y)-\alpha X_d(x,y)$.

**Proposition 0.1** *([3]) Let $\frac{m}{n}$ be an irreducible fraction. For any quasihomogeneous polynomial with weights $n$ and $m$ in the variables $x$ and $y$ respectively the decomposition*

$$R(x,y)=Ax^{s_1}y^{s_2}\prod(y^n-b_ix^m)^{k_i}$$

*holds, where $b_i$ are distinct nonzero complex numbers and $k_i\geq 0$.*

**Definition.** A factor of the form $y^n-b_ix^m$, $b_i\neq 0$, is called the *prime factor* of the polynomial $R(x,y)$, the number $k_i$ is called the *multiplicity* of this prime factor.

**Definition.** A vector field (germ) with Newton diagram $\Gamma$ is $\Gamma$-nondegenerate, if 1) none of the polynomials $F_0(x,y)$ corresponding to edges of the Newton diagram $\Gamma$ has a prime factor of multiplicity larger than one; 2) the index of any vertex not lying on a coordinate axis is different from the indices of the edges adjacent to it.

The set of $\Gamma$-nondegenerate vector fields having zero as a monodromic singular point will be denoted by $M_\Gamma$.

**Definition.** We call the Newton diagram $\Gamma$ *monodromic*, if the set $M_\Gamma$ is nonempty. A Newton diagram is *monodromic*, if and only if it has one vertex on each coordinate axis and the lengths of the projections of the edges on the coordinate axes are all even numbers ([3]).

**Definition.** Let $\alpha=\frac{m}{n}$ be an irreducible fraction. The edge of the Newton diagram with index $\alpha$ will be called *even*, if one of the numbers $m$ and $n$ and *odd* otherwise.

**Theorem 1** *([3]) Let $\Gamma$ be a monodromic Newton diagram.* ln $c=0$ *is identically zero on $M_\Gamma$ if and only if all the edges of the Newton diagram $\Gamma$ are even.*

Let the Newton diagram of the vector field $V$ consist of two edges with indices $\alpha=\frac{m}{n}$, $\tilde{\alpha}=\frac{\tilde{m}}{\tilde{n}}$, where $\tilde{\alpha}>\alpha$. For each edge we can consider the expansion (0.3) - (0.4). The polynomials analogous to $X_d,Y_d,F_d$ for the edge with index $\tilde{\alpha}$ we denote $\tilde{X}_d,\tilde{Y}_d,\tilde{F}_d$ respectively.

In this paper we prove the following theorem.

**Theorem 2** *Let $\Gamma$ be a monodromic Newton diagram consisting of two even edges with indices $\alpha=\frac{m}{n}$ and $\tilde{\alpha}=\frac{\tilde{m}}{\tilde{n}}$ $(\tilde{\alpha}>\alpha)$ and $V$ be a $\Gamma$-nondegenerate vector field having $(0,0)$ as a monodromic singular point, $(A_0,B_0)$ be a vector coefficient of the vertex of the Newton diagram between its edges. Let in addition the following conditions hold*

*1)* $\lambda = \frac{\tilde{n}(B_0 - \tilde{\alpha} A_0)}{B_0 - \alpha A_0} > 1$; *$\lambda$ is irrational number;*
*2)* $\frac{A_0}{(B_0 - \tilde{\alpha} A_0)} < 0$.

*Then the monodromy map associated to the origin (taking the axis of abscissa as transversal with a suitable chosen parameter) has possibly after a time reversal the form*

$$\Delta(\rho) = \rho(1 + F_2 \rho^{\frac{1}{n}} + o(\rho^{\frac{1}{n}})), \quad \rho \to 0,$$

*where in case $n > 1$ $F_2 = 0$,*

*in case $n = 1$, $\tilde{m}$ – even number, $r = \tilde{m}n - \tilde{n}m > 1$,*

$$F_2 = 2 \int\limits_{-\infty}^{+\infty} \tilde{\Phi}_1(1,\xi) e^{\int\limits_0^{\xi} \tilde{\Phi}_0(1,\tau) d\tau} d\xi$$

$$\tilde{\Phi}_0(x,y) = \frac{\tilde{X}_0(x,y)}{\tilde{n} y \tilde{F}_0(x,y)}, \quad \tilde{\Phi}_1(x,y) = \frac{\tilde{Y}_0(x,y)\tilde{X}_1(x,y) - \tilde{Y}_1(x,y)\tilde{X}_0(x,y)}{\tilde{n} y \tilde{F}_0^2(x,y)}.$$

If $F_2 \neq 0$, then the origin is a focus.

# 1 Resolution of the singularity.

Let the Newton diagram of the vector field $V$ consist of two edges with indices $\alpha = \frac{m}{n}$, $\tilde{\alpha} = \frac{\tilde{m}}{\tilde{n}}$, where $\tilde{\alpha} > \alpha$. In according to [3] in such a case the resolution of the singularity connected with a Newton diagram consists of the following: the first quadrant of the plain $(x,y)$ is broken up into sectors $S_\alpha, S_{\tilde{\alpha}}$ and $S_{\alpha\tilde{\alpha}}$, corresponding to the edges and the vertex between them of the Newton diagram.

Let $\varepsilon > 0$, $\delta > 0$ be small enough. The change of coordinates

$$x = w z^n, \; y = z^m. \tag{1.5}$$

turns the sector $S_\alpha = \{\varepsilon x^\alpha \le y \le \delta\}$ into the rectangle

$$P_\alpha = \{0 \le w \le \varepsilon^{-\frac{1}{\alpha}}, 0 \le z \le \delta^{\frac{1}{m}}\}.$$

The change of coordinates

$$x = u^{n\tilde{n}} v^n, \; y = u^{n\tilde{m}} v^m, \tag{1.6}$$

turns the sector $S_{\alpha\tilde{\alpha}} = \{\frac{1}{\varepsilon} x^{\tilde{\alpha}} \le y \le \varepsilon x^\alpha\}$ into the rectangle

$$P_{\alpha\tilde{\alpha}} = \{0 \le u \le \varepsilon_1, \; 0 \le v \le \varepsilon_2\},$$

where $\varepsilon_1 = \varepsilon^{\frac{1}{n\tilde{n}(\tilde{\alpha}-\alpha)}}$, $\varepsilon_2 = \varepsilon^{\frac{1}{n(\tilde{\alpha}-\alpha)}}$.

Finally the change of coordinates

$$x = \tilde{z}^{\tilde{n}}, \; y = \tilde{z}^{\tilde{m}} \tilde{w}, \tag{1.7}$$

turns the sector $S_{\tilde{\alpha}} = \{0 \le y \le \frac{1}{\varepsilon} x^{\tilde{\alpha}}\}$ into the rectangle

$$P_{\tilde{\alpha}} = \{0 \le \tilde{w} \le \frac{1}{\epsilon}, \; 0 \le \tilde{z} \le \delta^{\frac{1}{\tilde{n}}}\}.$$

(See fig.1.)

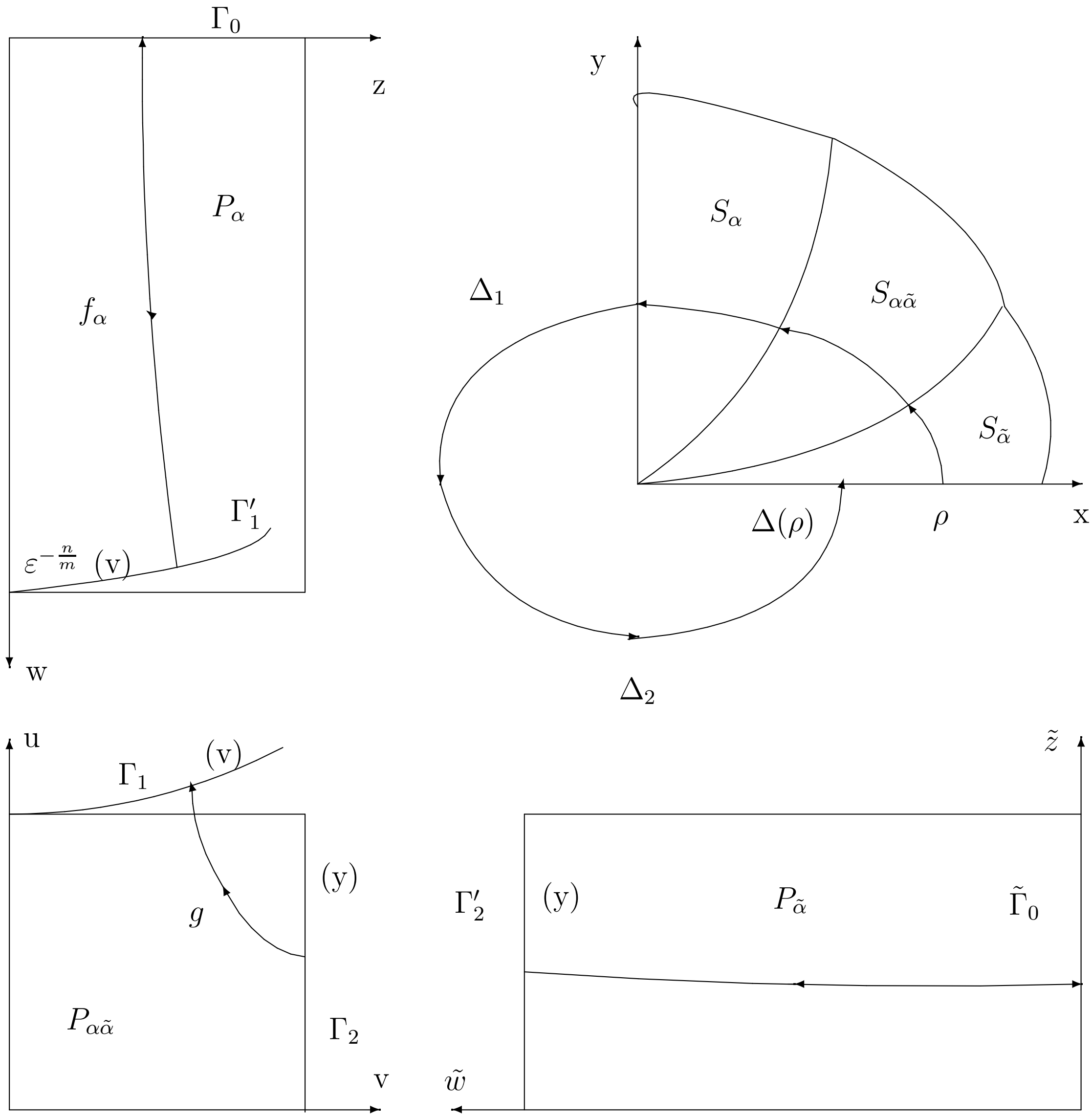


Figure 1: *Resolution of singularity.*

# 2 Change of coordinates in the sector corresponding to the edge of a Newton diagram.

Take the change (1.5) in the system (0.2), we obtain

$$\frac{dz}{dw} = z(\Phi_0(w,1) + z\Phi_1(w,1) + ...), \tag{2.8}$$

where

$$\Phi_0(x,y) = -\frac{Y_0(x,y)}{nxF_0(x,y)},$$

$$\Phi_1(x,y) = -\frac{m(Y_1(x,y)X_0(x,y) - X_1(x,y)Y_0(x,y))}{n^2 x F_0^2(x,y)}.$$

Analogously take the change (1.7), we obtain

$$\frac{d\tilde{z}}{d\tilde{w}} = \tilde{z}(\tilde{\Phi}_0(1,\tilde{w}) + \tilde{z}\tilde{\Phi}_1(1,\tilde{w}) + ...), \tag{2.9}$$

where $\tilde{\Phi}_0(x,y)$,*** $\tilde{\Phi}_1(x,y)$ are defined at the statement of the theorem 2.

# 3 Reflected vector fields.

Let $S^x$ and $S^y$ be the reflections of the $(x,y)$-plane about the $x$- and $y$-axes respectively, and let $S^{xy} = S^x \circ S^y$. The images of the vector field $V$ after reflections $S^x, S^y, S^{xy}$ we denote $V^x, V^y, V^{xy}$ respectively. Consider in the first quadrant four vector fields $V, V^x, V^y, V^{xy}$ and apply the described resolution of singularity to them. Corresponding formulas for the vector field $V$ are given at the previous section. The polinomials analogous to $X_d, Y_d, F_d, \Phi_d$ for the reflected vector fields we denote by the same letters with the corresponding index above.

From ([3]) we obtain

**Lemma 3.1**

$$\Phi_d^x(x,y) = \Phi_d(x,-y), \quad \tilde{\Phi}_d^x(x,y) = -\tilde{\Phi}_d(x,-y),$$
$$\Phi_d^y(x,y) = -\Phi_d(-x,y), \quad \tilde{\Phi}_d^y(x,y) = \tilde{\Phi}_d(-x,y),$$
$$\Phi_d^{xy}(x,y) = -\Phi_d(-x,-y), \quad \tilde{\Phi_d^{xy}}(x,y) = -\tilde{\Phi}_d(-x,-y).$$

*where* $d = 0,1$.

# 4 Transition map in the rectangle $P_{\alpha\tilde{\alpha}}$.

In this section we denote $F(u,v)$ be any analytic at the point $(0,0)$ function of variables $u,v$.

**Lemma 4.1** *Let $(A_0,B_0)$ be the vector coefficient of the vertex of the Newton diagram $\Gamma$ joining its edges, $(A_1,B_1)$ be the vector coefficient of the nearest to this vertex integer-valued point on the edge with the index $\tilde{\alpha}$; $(A_2,B_2)$ be the vector coefficient of the nearest to this vertex integer-valued point on the edge with the index $\alpha$. The change of variables (1.6) in the sector $S_{\alpha\tilde{\alpha}}$ followed by division by a power function converts fector field $V$ into a vector field*

$$\begin{aligned} \dot{u} &= u(\tilde{A}_0 + v^r(\tilde{A}_1 + f_1(v^r)) + u^{nr}(\tilde{A}_2 + f_2(u^{nr})) + uvF(u,v)), \\ \dot{v} &= v(\tilde{B}_0 + v^r(\tilde{B}_1 + g_1(v^r)) + u^{nr}(\tilde{A}_2 + g_2(u^{nr})) + uvF(u,v)), \end{aligned} \tag{4.10}$$

*where* $(\tilde{A}_i, \tilde{B}_i) = (\frac{1}{r}(B_i - \alpha A_i); -\frac{\tilde{n}}{r}(B_i - \tilde{\alpha} A_i)), \quad i = 0, 1, 2, \quad r = \tilde{m}n - \tilde{n}m, \ f_j, \ g_j-$ *polynomials,* $f_j(0) = g_j(0) = 0, \ j = 1, 2.$

**Proof.** The matrix of exponents corresponding to the change (1.6) has the form

$$C = \begin{pmatrix} n\tilde{n} & n \\ n\tilde{m} & m \end{pmatrix}.$$

In according to ([3]) the support of the new vector field is the image of the support of the vector field $V$ by means of the map $C^T$; vector coefficients are transformed by means of matrix $C^{-1}$:

$$C^T = \begin{pmatrix} n\tilde{n} & n\tilde{m} \\ n & m \end{pmatrix}, \quad C^{-1} = -\frac{1}{nr} \begin{pmatrix} m & -n \\ -n\tilde{m} & n\tilde{n} \end{pmatrix}.$$

The transformation of the support of the vector field $V$ is shown on the fig.2. From the form of the support of the transformed vector field and from the equality

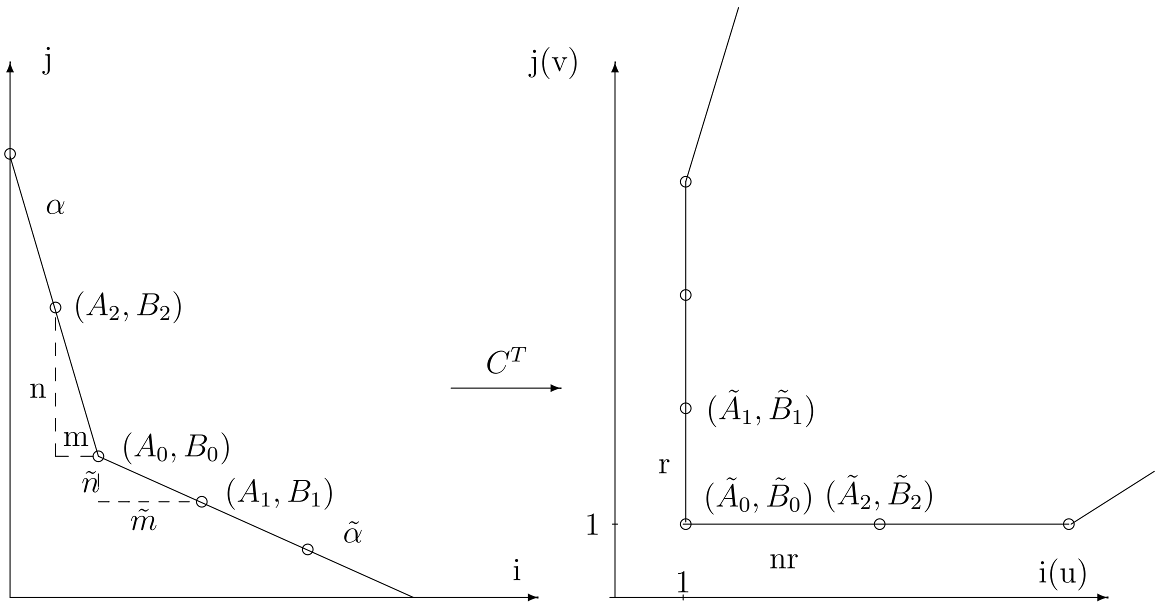


Figure 2: *Transformation of the support.*

$$\begin{pmatrix} \tilde{A}_i \\ \tilde{B}_i \end{pmatrix} = C^{-1} \begin{pmatrix} A_i \\ B_i \end{pmatrix}$$

the conclusion of lemma follows.

From conditions of $\Gamma$ - nondegegeneracy $\tilde{A}_0 \neq 0, \ \tilde{B}_0 \neq 0$. After division the system (4.10) on the expression in brackets from the second equation we obtain the system

$$\dot{u} = u(-\tfrac{1}{\lambda} + v^r(a + h_1(v^r)) + u^{nr}(b + h_2(u^{nr})) + uvF(u, v)), \quad \dot{v} = v, \tag{4.11}$$

where

$$\lambda = \frac{\tilde{n}(B_0 - \tilde{\alpha}A_0)}{B_0 - \alpha A_0}, \ a = \frac{A_0B_1 - B_0A_1}{\tilde{n}(B_0 - \tilde{\alpha}A_0)^2}(\tilde{\alpha} - \alpha), \ b = \frac{A_0B_2 - B_0A_2}{\tilde{n}(B_0 - \tilde{\alpha}A_0)^2}(\tilde{\alpha} - \alpha),$$

$h_1, \ h_2-$ polynomials, $h_1(0) = h_2(0) = 0$.

**Lemma 4.2** *The system (4.11) is reduced to the linear normal form*

$$\dot{v} = v, \quad \dot{y} = -\frac{1}{\lambda}y$$

*with help of the $C^\infty$-change of variables of the form*

$$u = y(1 + v^r(\tilde{a} + \tilde{h}_1(v^r)) + u^{nr}(\tilde{b} + \tilde{h}_2(u^{nr})) + uvF(u,v)), \tag{4.12}$$

*where* $\tilde{a} = -\frac{a}{r}, \quad \tilde{b} = \frac{b\lambda}{nr}, \ \tilde{h}_1, \ \tilde{h}_2-$ *polynomials,* $\tilde{h}_1(0) = \tilde{h}_2(0) = 0$.

**Proof.** In according to ([4],[5]) there exists $C^\infty$-change of variabels which linearized the system (4.11). We shall look for it in the form (4.12).

From (4.12) and (4.11) we obtain

$$\dot{y} = y(-\frac{1}{\lambda} + (a + \tilde{a}r)v^r + (b - \frac{\tilde{b}nr}{\lambda})y^{nr} + \ldots).$$

Setting equal $\dot{y}$ to $-\frac{1}{\lambda}y)$ we obtain: $\tilde{a} = -\frac{a}{r}, \quad \tilde{b} = \frac{b\lambda}{nr}$. Lemma is proved.

Because the vector field is $\Gamma$ - nondegenerate and the singular point is monodromic we have $\lambda > 0$.

On the plain $(y,v)$ we consider the rectangle $0 \leq y \leq \varepsilon_1, \ \ 0 \leq v \leq \varepsilon_2$. Let $L_1$ and $L_2$ be the sides of the rectangle not lying on the coordinate axes: $L_1 = \{y = \varepsilon_1\}, \ Ł_2 = \{v = \varepsilon_2\}$. Let $g : L_2 \to L_1$ be a transition map along the trajectories of the linear system

$$\dot{y} = y, \quad \dot{v} = -\lambda v. \tag{4.13}$$

On $L_1$ we consider the parameter $v$, on $L_2$ - parameter $y$.

**Lemma 4.3** $v = g(y) = \varepsilon_2 \left(\frac{y}{\varepsilon_1}\right)^\lambda$.

**Proof.** The trajectory of the system (4.13) goes from the point $(y, \varepsilon_2)$ to the point $(\varepsilon_1, v)$ during the time

$$t = \int\limits_y^{\varepsilon_1} \frac{dy}{y} = -\int\limits_{\varepsilon_2}^{v} \frac{dv}{\lambda v}.$$

From here $\ln \frac{\varepsilon_1}{y} = -\frac{1}{\lambda} \ln \frac{v}{\varepsilon_2}, \quad v = \varepsilon_2 \left(\frac{y}{\varepsilon_1}\right)^\lambda$.

Lemma is proved.

## 5 Parametrisation of transversals.

The change (4.12) converts the segments $L_1$ and $L_2$ into the curves $\Gamma_1$ and $\Gamma_2$. On $\Gamma_1$ we consider the parameter $v$, on $\Gamma_2 - y$. Then $\Gamma_1$ and $\Gamma_2$ accoding to (4.12) have the following form ($r > 1$):

$$\Gamma_1 : \ u = \varepsilon_1(1 + o(\varepsilon_1))(1 + O(\varepsilon_1)v + o(v)), \quad v = v, \tag{5.14}$$

$$\Gamma_2 : \begin{array}{l} u = y(1 + o(\varepsilon_2))(1 + O(\varepsilon_2)y + o(y)), \\ v = \varepsilon_2. \end{array} \tag{5.15}$$

From (1.5) and (1.6) we find that the connection between coordinates $(z, w)$ in the rectangle $P_\alpha$ and $(u, v)$ in the rectangle $P_{\alpha\tilde{\alpha}}$ is the following

$$w = u^{\frac{n\tilde{n}(\alpha-\tilde{\alpha})}{\alpha}}, \ z = u^{\frac{\tilde{m}}{\alpha}} v. \tag{5.16}$$

Analogously from (1.6) and (1.7) we obtain that the coordinates $(\tilde{z}, \tilde{w})$ in the rectangle $P_{\tilde{\alpha}}$ and $(u, v)$ in the rectangle $P_{\alpha\tilde{\alpha}}$ are connected by following formulas

$$\tilde{w} = v^{n(\alpha-\tilde{\alpha})}, \ \tilde{z} = u^n v^{\frac{n}{\tilde{n}}}. \tag{5.17}$$

Substituting (5.14) in (5.16) we obtain that $\Gamma_1$ in the coordinates $(z, w)$ has the form

$$\Gamma_1' : \begin{array}{l} w = \varepsilon_1^*(1 + o(v)) \\ z = \tilde{\varepsilon}_1 v(1 + O(\varepsilon_1)v + o(v)), \end{array} \tag{5.18}$$

where $\varepsilon_1^* = \varepsilon^{-\frac{n}{m}}(1 + o(\varepsilon_1))$, $\tilde{\varepsilon}_1 = \varepsilon_1^{\frac{\tilde{m}}{\alpha}}(1 + o(\varepsilon_1))$.

Analogously substituting (5.15) in (5.17) we obtain that $\Gamma_2$ in coordinates $(\tilde{z}, \tilde{w})$ has the form

$$\Gamma_2' : \begin{array}{l} \tilde{w} = \frac{1}{\varepsilon} \\ \tilde{z} = \tilde{\varepsilon}_2 y^n(1 + O(\varepsilon_2)y + o(y)), \end{array} \tag{5.19}$$

where $\tilde{\varepsilon}_2 = \varepsilon_2^{\frac{n}{\tilde{n}}}(1 + o(\varepsilon_2))$.

## 6 Transition maps in the rectangels corresponding to edges.

Consider in coordinates $(\tilde{z}, \tilde{w})$ two transversals $\tilde{\Gamma}_0 = \{\tilde{w} = 0\}$ with parameter $\rho = \tilde{z}$ and $\Gamma_2'$ (see (5.19)) with parameter $y$. Calculate the coefficients of the transition map $f_{\tilde{\alpha}} : \Gamma_2' \to \tilde{\Gamma}_0$.

**Lemma 6.1** *The map $f_{\tilde{\alpha}}$ has the asymptotics*

$$\rho = \tilde{d} y^n(1 + \tilde{d}_1 y + o(y)), \tag{6.20}$$

*where*

$$\tilde{d} = \tilde{\varepsilon}_2 e^{-\int_0^{\frac{1}{\varepsilon}} \tilde{\Phi}_0(1,\xi)d\xi}, \tag{6.21}$$

$$\frac{\tilde{d}_1}{\tilde{d}} = -\int_0^{\frac{1}{\varepsilon}} \tilde{\Phi}_1(1,\xi) e^{\int_0^{\xi} \tilde{\Phi}_0(1,\tau)d\tau} d\xi + \frac{O(\varepsilon_2)}{\tilde{d}}, \quad \textit{if} \;\; n=1, \quad \tilde{d}_1 = O(\varepsilon_2) \;\textit{if}\;\; n>1. \tag{6.22}$$

**Proof.** We look for the solution $\tilde{z}(\tilde{w},\rho)$ of the equation (2.9) with the initial condition $\tilde{z}(0,\rho)$ in the form

$$\tilde{z}(\tilde{w},\rho) = \rho(\tilde{C}_0(\tilde{w}) + \tilde{C}_1(\tilde{w})\rho + ...), \tag{6.23}$$

where $\tilde{C}_0(0)=1,\ \tilde{C}_i(0)=0 \quad \text{as} \quad i \geq 0.$

Solving corresponding equations in variations we obtain

$$\tilde{C}_0(\tilde{w}) = e^{\int_0^{\tilde{w}} \tilde{\Phi}_0(1,\xi)d\xi}, \quad \tilde{C}_1(\tilde{w}) = \tilde{C}_0(\tilde{w}) \int_0^{\tilde{w}} \tilde{C}_0(\xi)\tilde{\Phi}_1(1,\xi)d\xi. \tag{6.24}$$

For the map $\rho = f_{\tilde{\alpha}}(y)$ from (6.23) and (5.19) we obtain the equation

$$\tilde{C}_0\rho + \tilde{C}_1\rho^2 + ... = \tilde{\varepsilon}_2 y^n(1 + O(\varepsilon_2)y + o(y)), \tag{6.25}$$

where $\tilde{C}_i = \tilde{C}_i(\frac{1}{\varepsilon}),\ i=0,1.$

We shall look for $\rho = f_{\tilde{\alpha}}$ in the form (6.20). Substituting (6.20) in (6.25) we obtain

$$\tilde{C}_0\tilde{d}y^n(1+\tilde{d}_1y+o(y)) + \tilde{C}_1\tilde{d}^2y^{2n}(1+\tilde{d}_1y+o(y))^2 + o(y^2) = \tilde{\varepsilon}_2y^n(1+O(\varepsilon_2)y+o(y)).$$

Let $n=1$. Then

$$\tilde{C}_0\tilde{d}y(1+\tilde{d}_1y+o(y)) + \tilde{C}_1\tilde{d}^2y^2(1+2\tilde{d}_1y+o(y)) + o(y^2) = \tilde{\varepsilon}_2y(1+O(\varepsilon_2)y+o(y))$$

or $\tilde{C}_0\tilde{d}y + (\tilde{C}_0\tilde{d}\tilde{d}_1 + \tilde{C}_1\tilde{d}^2)y^2 + o(y^2) = \tilde{\varepsilon}_2y + \tilde{\varepsilon}_2O(\varepsilon_2)y^2 + o(y^2)$

Setting equal coefficients in the equal powers of $y$, we obtain the conclusion of lemma.

In the case that $n>1$ we obtain the same expression for $\tilde{d}$, and $\tilde{d}_1 = O(\varepsilon_2)$. Lemma is proved.

Consider in coordinates $(z,w)$ two curves: $\Gamma_0 = \{w=0\}$ with parameter $\rho = z$ and $\Gamma_1'$ (see (5.18)) with parameter $v$. It is evident, that the transition map $f_\alpha : \Gamma_1' \to \Gamma_0$ has the asymptotics

$$\rho = dv(1 + d_1v + ...) \tag{6.26}$$

As in the proof of Lemma 6.1 we obtain that

$$d = \tilde{\varepsilon}_1 e^{-\int_0^{\varepsilon_1^*} \Phi_0(\xi,1)d\xi}. \tag{6.27}$$

# 7 Coefficients of the composition of the transition maps.

The monodromy map $\Delta$ is the composition of the following maps:

$$\Delta = \Delta_2 \circ \Delta_1,$$

where $\Delta_1$ is a transition map for the upper half-plane which transforms the positive $x$-half-axis near the origin into the negative $x$-half-axis in the positive direction along the trajectories of the vector field, and $\Delta_2$ is the analogous map for the lower half-plane (see fig.1).

The maps analogous to $f_\alpha, f_{\tilde{\alpha}}, g$ for the reflected vector fields we denote by the same letters with corresponding index above. Then

$$\Delta_1 = f^y_{\tilde{\alpha}} \circ (g^y)^{-1} \circ (f^y_\alpha)^{-1} \circ f_\alpha \circ g \circ (f_{\tilde{\alpha}})^{-1}, \quad \Delta_2 = (\Delta^x_1)^{-1},$$

where

$$\Delta^x_1 = f^{xy}_{\tilde{\alpha}} \circ (g^{xy})^{-1} \circ (f^{xy}_\alpha)^{-1} \circ f^x_\alpha \circ g^x \circ (f^x_{\tilde{\alpha}})^{-1}.$$

According to Lemma 4.3 $v = g(y) = \varepsilon^\gamma y^\lambda$, where $\varepsilon^\gamma = \varepsilon_2 \varepsilon_1^{-\lambda}$. Observe that all the analogous to $g$ transition maps for the reflected vector fields have the same formula, because the number $\lambda$ is the same for all of them ([3]).

The maps $f_\alpha$ and $f_{\tilde{\alpha}}$ are defined by formulas (6.26) and (6.20) respectively. Inverse maps have respectively forms

$$v = (f^y_\alpha)^{-1}(\rho) = \frac{1}{d^y}\rho(1 - \frac{d^y_1}{d^y}\rho + ...),$$

$$y = f^{-1}_{\tilde{\alpha}}(\rho) = \left(\frac{1}{\tilde{d}}\right)^{\frac{1}{n}} \rho^{\frac{1}{n}}(1 - \frac{\tilde{d}_1}{n\tilde{d}^{\frac{1}{n}}}\rho^{\frac{1}{n}} + o(\rho^{\frac{1}{n}})).$$

It is easy to compute, that

$$y = g^{-1}(v) = \varepsilon^{\frac{-\gamma}{\lambda}} v^{\frac{1}{\lambda}}.$$

Taking into account that $\lambda > 1$ we obtain in consecutive order:

$$(f^y_\alpha)^{-1} \circ f_\alpha = \frac{d}{d^y} v(1 + \frac{d^y d_1 - d d^y_1}{d^y} v + o(v)),$$

$$(g^y)^{-1} \circ (f^y_\alpha)^{-1} \circ f_\alpha) \circ g = \left(\frac{d}{d^y}\right)^{\frac{1}{\lambda}} y(1 + o(y)).$$

Finally

$$\Delta_1 = f^y_{\tilde{\alpha}} \circ (g^y)^{-1} \circ (f^y_\alpha)^{-1} \circ f_\alpha \circ g \circ f^{-1}_{\tilde{\alpha}} = c\rho(1 + c_1 \rho^{\frac{1}{n}} + o(\rho^{\frac{1}{n}})),$$

where

$$* * * = \frac{\tilde{d}^y}{\tilde{d}} \left(\frac{d}{d^y}\right)^{\frac{n}{\lambda}}, \quad c_1 = \frac{\tilde{d}^y_1}{\tilde{d}^{\frac{1}{n}}} \left(\frac{d}{d^y}\right)^{\frac{1}{\lambda}} - \frac{\tilde{d}_1}{\tilde{d}^{\frac{1}{n}}}, \tag{7.28}$$

Analogously $\Delta_1^x = c^x\rho(1 + c_1^x\rho^{\frac{1}{n}} + o(\rho^{\frac{1}{n}}))$, where

$$c^x = \frac{\tilde{d}^{xy}}{\tilde{d}^x}\left(\frac{d^x}{d^{xy}}\right)^{\frac{n}{\lambda}}, \quad c_1^x \frac{\tilde{d}_1^{xy}}{(\tilde{d}^x)^{\frac{1}{n}}}\left(\frac{d^x}{d^{xy}}\right)^{\frac{1}{\lambda}} - \frac{\tilde{d}_1^x}{(\tilde{d}^x)^{\frac{1}{n}}}, \tag{7.29}$$

From here $\Delta_2 = (\Delta_1^x)^{-1} = \frac{1}{c^x}\rho(1 - \frac{c_1^x}{(c^x)^{\frac{1}{n}}}\rho^{\frac{1}{n}} + o(\rho^{\frac{1}{n}}))$. Finally

$$\Delta = \Delta_2 \circ \Delta_1 = \frac{c}{c^x}\rho(1 + F_2\rho^{\frac{1}{n}} + o(\rho^{\frac{1}{n}})),$$

where

$$F_2 = \frac{c_1(c^x)^{\frac{1}{n}} - c_1^x c^{\frac{1}{n}}}{(c^x)^{\frac{1}{n}}}.$$

Since $\frac{c}{c^x}$ is a coefficient of the principal term of the asymptotics of the monodromy map $\Delta$, then in our case of even edges it is equal to 1, hence

$$F_2 = c_1 - c_1^x. \tag{7.30}$$

The formula (6.22) implies that in case $n > 1$ $\tilde{d}_1 = O(\varepsilon_2)$, $\tilde{d}_1^y = O(\varepsilon_2)$. Consider the coefficient $c_1$. Notice that as $\varepsilon \to 0$

$$e^{\frac{1}{n}\int\limits_0^{\xi}\Phi_0(\xi,1)d\xi} \approx ***\xi^{b_0}, c > 0. \tag{7.31}$$

where $b_0 = \frac{A_0}{n\tilde{n}(B_0 - \tilde{\alpha}A_0)} < 0$.

From here

$$e^{\frac{1}{n}\int\limits_0^{\frac{1}{\varepsilon}}\Phi_0(\xi,1)d\xi} \approx ***\varepsilon^{-b_0}.$$

Hence

$$\frac{\tilde{d}_1}{\tilde{d}^{1/n}} = \frac{O(\varepsilon_2)}{\tilde{\varepsilon_2}^{1/n}} e^{\frac{1}{n}\int\limits_0^{\xi}\Phi_0(-\xi,1)d\xi} = \frac{O(\varepsilon_2)}{\varepsilon_2^{1/\tilde{n}}(1 + o(\varepsilon_2))}\varepsilon^{-b_0} = \frac{O(\varepsilon_2)}{\varepsilon_2^{1/\tilde{n}}}\varepsilon^{-b_0} \to 0$$

as $\varepsilon \to 0$. Analogously $\frac{\tilde{d}_1^y}{\tilde{d}^{1/n}} \to 0$ as $\varepsilon \to 0$.

We show, that the value $\frac{d}{d^y}$ is bounded as $\varepsilon \to 0$. Really from the formula (6.27) and lemma 3.1

$$d^y = \tilde{\varepsilon}_1 e^{-\int\limits_0^{\varepsilon_1^*}\Phi_0^y(\xi,1)d\xi} = \tilde{\varepsilon}_1 e^{\int\limits_0^{\varepsilon_1^*}\Phi_0(-\xi,1)d\xi} = \tilde{\varepsilon}_1 e^{\int\limits_{-\varepsilon_1^*}^{0}\Phi_0(\xi,1)d\xi}.$$

From here

$$\frac{d}{d^y} = (1 + o(\varepsilon_1)e^{-\int\limits_{-\varepsilon_1^*}^{\varepsilon_1^*}\Phi_0(\xi,1)d\xi}$$

Limits in the integral in the exponent are not symmetric, because the quantity

$$\varepsilon_1^* = \varepsilon^{-\frac{n}{m}}(1 + o(\varepsilon_1)) \tag{7.32}$$

contains $o$ - small, which for the reflected vector field can be differ from the analogous quantity for the initial vector field.

Therefore

$$\frac{d}{d^y} = (1 + o(\varepsilon_1))e^{-\int\limits_{-\varepsilon_1^*}^{\varepsilon_1^*} \Phi_0(\xi,1)d\xi} e^{\int\limits_{I(\varepsilon)} \Phi_0(\xi,1)d\xi}, \tag{7.33}$$

where the limits in the integral in the first exponent are symmetric, and both of the ends of the segment $I(\varepsilon)$ have the asymptotics (7.32). Because from $\Gamma$ - nondegeneracy conditions

$$\Phi_0(\xi, 1) = O(\frac{1}{\xi^k}), \ k \geq 1, \tag{7.34}$$

as $\xi \to \infty$, then the integraal in the first exponent turns to the finite limit as $\varepsilon \to 0$. From (7.34) we also obtain, that the integral in the second exponent in the formula (7.33) is $O(\varepsilon^{\frac{n}{m}k})\varepsilon_1^{-\frac{n}{m}} o(\varepsilon_1) = o(\varepsilon_1)$. Hence the second exponent turns to 1 as $\varepsilon \to 0$. From here the ratio $\frac{d}{d^y}$ is bounded. Notice that if $m$ is even the integrand in the first exponent (7.33) is odd and so

$$\frac{d}{d^y} = 1 + o(\varepsilon_1). \tag{7.35}$$

Analogously

$$\frac{d^x}{d^{xy}} = 1 + o(\varepsilon_1). \tag{7.36}$$

Because the ratio $\frac{d}{d^y}$ is bounded we have $c_1 \to 0$ as $\varepsilon \to 0$. Analogously $c_1^x \to 0$ as $\varepsilon \to 0$. From here and because $F_2$ is undependent on $\varepsilon$ we obtain that $F_2 = 0$.

Let $n = 1$.

Because in this case $m$ is even, that from (7.30),(7.28), (7.29), (7.35), (7.36) we obtain that

$$F_2 = \frac{\tilde{d}_1^y}{\tilde{d}} - \frac{\tilde{d}_1}{\tilde{d}} + o(\varepsilon_1)\frac{\tilde{d}_1{}^y}{\tilde{d}} - \left(\frac{\tilde{d}_1^{xy}}{\tilde{d}^x} - \frac{\tilde{d}_1^x}{\tilde{d}^x} + o(\varepsilon_1)\frac{\tilde{d}_1^{xy}}{\tilde{d}^x}\right). \tag{7.37}$$

Because $\tilde{m}$ is even, that from [3]

$$\tilde{\Phi}_0(-x, y) = \tilde{\Phi}_0(x, y), \ \ \tilde{\Phi}_1(-x, y) = -\tilde{\Phi}_1(x, y). \tag{7.38}$$

From here and from lemma 3.1 we obtain, that

$$\tilde{d}^y = \tilde{\varepsilon}_2 e^{-\int\limits_0^{\frac{1}{\varepsilon}} \tilde{\Phi}_0^y(1,\xi)d\xi} = \tilde{\varepsilon}_2 e^{-\int\limits_0^{\frac{1}{\varepsilon}} \tilde{\Phi}_0(-1,\xi)d\xi} = \tilde{\varepsilon}_2 e^{-\int\limits_0^{\frac{1}{\varepsilon}} \tilde{\Phi}_0(1,\xi)d\xi} = \tilde{d}(1 + o(\varepsilon_2)).$$

Analogously

$$\frac{\tilde{d}_1^y}{\tilde{d}} = \frac{\tilde{d}_1^y}{\tilde{d}^y}\frac{\tilde{d}^y}{\tilde{d}} = -\frac{\tilde{d}_1}{\tilde{d}}(1 + o(\varepsilon_2)) + \frac{O(\varepsilon_2)}{\tilde{d}}. \tag{7.39}$$

Analogously

$$\frac{\tilde{d}_1^{xy}}{\tilde{d}^x} = -\frac{\tilde{d}_1^x}{\tilde{d}^x}(1 + o(\varepsilon_2)) + \frac{O(\varepsilon_2)}{\tilde{d}^x}.$$

From here, from (7.39) and (7.37) we obtain, that

$$F_2 = c_1 - c_1^x = 2(\frac{\tilde{d}_1^x}{\tilde{d}^x} - \frac{\tilde{d}_1}{\tilde{d}}) + (o(\varepsilon_1) + o(\varepsilon_2))\frac{\tilde{d}_1}{\tilde{d}} + (o(\varepsilon_1) + o(\varepsilon_2))\frac{\tilde{d}_1^x}{\tilde{d}} + \frac{O(\varepsilon_2)}{\tilde{d}} + \frac{O(\varepsilon_2)}{\tilde{d}^x}. \tag{7.40}$$

Investigate the asymptotics of the integral in the formula (6.22) for the quantity $\frac{\tilde{d}_1}{\tilde{d}}$. The edge with index $\tilde{\alpha}$ is situated on the line $l:\ \tilde{n}i + \tilde{m}j = \tilde{d}_0$, the edge with index $\alpha$ is situated on the line $ni + mj = d_0$.

Let $(i_0, j_0)$ be the coordinates of the vertex, joining these edges. Then

$$ni_0 + mj_0 = d_0,\ \tilde{n}i_0 + \tilde{m}j_0 = \tilde{d}_0.$$

Consider the right line $l_1$, passing throw the point $(i_0 - m, j_0 + n)$ in parallel to $l$. Then $l_1$ has the equation $\tilde{n}i + \tilde{m}j = d$ and $\tilde{n}(i_0 - m) + \tilde{m}(j_0 + n) = d$. From here $d - \tilde{d}_0 = r > 1$. Therefore the supports of the functions $\tilde{X}_1$ and $\tilde{Y}_1$ do not lie on the right line $l_1$, hence they lie lower. Because $n = 1$, then the upper points of these supports lie not above the horizontal line $j = j_0$. So the powers of the polynomials $\tilde{X}_1(1, \xi)$ and $\tilde{Y}_1(1, \xi)$ are not greater than the power $j_0$ of polynomials $\tilde{X}_0,\ \tilde{Y}_0,\ \tilde{F}_0$. Therefore $\tilde{\Phi}_1(1, \xi) = O(\frac{1}{\xi^m}),\ m \geq 1$ as $\xi \to \infty$.

From here and from the formula (7.36) we obtain, that the integrand in (6.22) is $O(\frac{1}{\xi^{l-b_0}})$. So from the condition $b_0 < 0$ the integral in the formula (6.22) turns to the finite limit as $\varepsilon \to 0$.

We proved early, that two last terms in (7.40) turn to 0 as $\varepsilon \to 0$. From here and because the quantity $F_2$ does not depend on $\varepsilon$ we conclude that

$$F_2 = \lim_{\varepsilon \to 0} 2(\frac{\tilde{d}_1^x}{\tilde{d}^x} - \frac{\tilde{d}_1}{\tilde{d}}). \tag{7.41}$$

Continue the proof of the theorem. Twice changing the variable on the opposite one we obtain

$$\begin{aligned} &\frac{\tilde{d}_1^x}{\tilde{d}^x} = -(1 + o(\varepsilon_2)) \int_0^{\frac{1}{\varepsilon}} \tilde{\Phi}_1^x(1, \xi) e^{\int_0^{\xi} \tilde{\Phi}_0^x(1,\tau)d\tau} d\xi + \frac{O(\varepsilon_2)}{\tilde{d}^x} = \\ &(1 + o(\varepsilon_2)) \int_0^{\frac{1}{\varepsilon}} \tilde{\Phi}_1(1, -\xi) e^{-\int_0^{\xi} \tilde{\Phi}_0(1,-\tau)d\tau} d\xi + \frac{O(\varepsilon_2)}{\tilde{d}^x} = \\ &(1 + o(\varepsilon_2)) \int_{-\frac{1}{\varepsilon}}^{0} \tilde{\Phi}_1(1, \xi) e^{\int_0^{\xi} \tilde{\Phi}_0(1,\tau)d\tau} d\xi + \frac{O(\varepsilon_2)}{\tilde{d}^x}. \end{aligned} \tag{7.42}$$

From here, from (6.22) and (7.41) we obtain

$$F_2 = 2 \lim_{\varepsilon \to 0} \left( \frac{\tilde{d}_1^x}{\tilde{d}^x} - \frac{\tilde{d}_1}{\tilde{d}} \right) =$$

$$2 \lim_{\varepsilon \to 0} \left( \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \tilde{\Phi}_1(1, \xi) e^{\int_0^{\xi} \tilde{\Phi}_0(1,\tau)d\tau} d\xi + \frac{O(\varepsilon_2)}{\tilde{d}} + \frac{O(\varepsilon_2)}{\tilde{d}^x} \right) = 2 \int_{-\infty}^{\infty} \tilde{\Phi}_1(1, \xi) e^{\int_0^{\xi} \tilde{\Phi}_0(1,\tau)d\tau} d\xi.$$

Theorem 2 is proved.

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