

# Hartman – Wintner type theorem for PDE with $p$ –Laplacian

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## Abstract

The well known Hartman–Wintner oscillation criterion is extended to the PDE

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0, \quad p > 1. \quad (\text{E})$$

The condition on the function  $c(x)$  under which (E) has no solution positive for large  $\|x\|$ , i.e.  $\infty$  belongs to the closure of the set of zeros of every solution defined on the domain  $\Omega = \{x \in \mathbb{R}^n : \|x\| > 1\}$ , is derived.

**Keywords.**  $p$ –Laplacian, positive solution, Riccati equation.

## 1 Introduction

Let us consider the following partial differential equation with  $p$ –Laplacian

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \quad (1.1)$$

where  $p > 1$ ,  $x = (x_1, x_2, \dots, x_n)$ ,  $\|\cdot\|$  is the usual Euclidean norm in  $\mathbb{R}^n$  and  $\nabla$  is the usual nabla operator. Define the sets  $\Omega(a) = \{x \in \mathbb{R}^n : a \leq \|x\|\}$ ,  $\Omega(a, b) = \{x \in \mathbb{R}^n : a \leq \|x\| \leq b\}$ . The function  $c(x)$  is assumed to be integrable on every compact subset of  $\Omega(1)$ . Under solution of the equation (1.1) we understand every absolutely continuous function  $u : \Omega(1) \rightarrow \mathbb{R}$  such that  $|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}$  is absolutely continuous with respect to  $x_i$  and  $u$  satisfies the equation (1.1) almost everywhere on  $\Omega(1)$ .

Equation (1.1) appears for example in the study of non-Newtonian fluids, nonlinear elasticity and in glaciology. Special cases of the equation (1.1) are the linear Schrödinger equation

$$\Delta u + c(x)u = 0 \quad (1.2)$$

if  $p = 2$ , the half-linear ordinary differential equation

$$\left(|u'|^{p-2}u'\right)' + c(x)|u|^{p-2}u = 0 \quad ' = \frac{d}{dx} \quad (1.3)$$

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This paper is in final form and no version of it will be submitted for publication elsewhere.

if  $n = 1$ , and the ordinary differential equation

$$u'' + c(x)u = 0 \tag{1.4}$$

if both  $n = 1$  and  $p = 2$  holds.

Remark that if  $c(x)$  is radial function, i.e.,  $c(x) = \tilde{c}(|x|)$ , then the equation for radial solution  $u(x) = \tilde{u}(|x|)$  of the equation (1.1) becomes

$$\left( r^{n-1} |\tilde{u}'|^{p-2} \tilde{u}' \right)' + r^{n-1} \tilde{c}(r) |\tilde{u}|^{p-2} \tilde{u} = 0 \quad ' = \frac{d}{dr}, \tag{1.5}$$

which can be transformed into the equation (1.3).

This paper is motivated by the papers [1, 4] and [5, 6], where the Riccati technique is used to establish oscillation criteria for the equation (1.3) and (1.2), respectively.

The well-known result from the theory of second order ODE is the following theorem.

**Theorem (Hartman–Wintner).** *If either*

$$-\infty < \liminf_{t \rightarrow \infty} \frac{1}{t} \int_1^t \int_1^s c(\xi) \, d\xi \, ds < \limsup_{t \rightarrow \infty} \frac{1}{t} \int_1^t \int_1^s c(\xi) \, d\xi \, ds \leq \infty, \tag{1.6}$$

or

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_1^t \int_1^s c(\xi) \, d\xi \, ds = \infty, \tag{1.7}$$

then the equation (1.4) is oscillatory.

This theorem is proved using Riccati technique in [3, Chap. XI]. The aim of this paper is to extend this statement to the equation (1.1). Another statement of this type was proved in [2, Theorem 3.4] under additional condition  $p \geq n + 1$ . Here we prove a similar criterion, without the restriction on  $p$ .

We use the following function  $C(t)$ :

$$C(t) = \frac{p-1}{t^{p-1}} \int_1^t s^{p-2} \int_{\Omega(1,s)} \|x\|^{1-n} c(x) \, dx \, ds. \tag{1.8}$$

## 2 Main results

First we introduce main ideas from the Riccati technique.

Suppose that there exists a number  $a \in \mathbb{R}^+$  and a solution  $u$  of (1.1) which is positive on  $\Omega(a)$ . The vector function  $\mathbf{w} = \frac{\|\nabla u\|^{p-2} \nabla u}{|u|^{p-2} u}$  is defined on  $\Omega(a)$  and solves the *Riccati type equation*

$$\operatorname{div} \mathbf{w} + c(x) + (p-1) \|\mathbf{w}\|^q = 0, \tag{2.1}$$

where  $q$  is the conjugate number to  $p$  (i.e.  $\frac{1}{p} + \frac{1}{q} = 1$  holds). Really, direct computation shows

$$\begin{aligned} \operatorname{div} \mathbf{w} &= \operatorname{div}(\|\nabla u\|^{p-2} \nabla u) u^{1-p} - (p-1) \|\nabla u\|^{p-2} u^{-p} \langle \nabla u, \nabla u \rangle \\ &= -c(x) - (p-1) \frac{\|\nabla u\|^p}{u^p} \\ &= -c(x) - (p-1) \|\mathbf{w}\|^q. \end{aligned}$$

The following Lemma plays a crucial role in our consideration. It is a straightforward generalization of [3, Lemma 7.1, Chap XI.]

**Lemma 2.1.** *Let  $\mathbf{w}$  be the solution of (2.1) defined on  $\Omega(a)$  for some  $a > 1$ . The following statements are equivalent:*

(i)

$$\int_{\Omega(a)} \|x\|^{1-n} \|\mathbf{w}\|^q dx < \infty; \quad (2.2)$$

(ii) *there exists a finite limit*

$$\lim_{t \rightarrow \infty} C(t) = C_0; \quad (2.3)$$

(iii)

$$\liminf_{t \rightarrow \infty} C(t) > -\infty, \quad (2.4)$$

where the function  $C(t)$  is defined by (1.8).

Our main theorem now follows from Lemma 2.1.

**Theorem 2.2 (Hartman–Wintner type oscillation criterion).** *If either*

$$-\infty < \liminf_{t \rightarrow \infty} C(t) < \limsup_{t \rightarrow \infty} C(t) \leq \infty$$

or

$$\lim_{t \rightarrow \infty} C(t) = \infty,$$

then the equation (1.1) has no positive solution positive on  $\Omega(a)$  for any  $a > 1$ .

*Proof.* It follows from the assumptions of the theorem that  $\liminf_{t \rightarrow \infty} C(t) > -\infty$ . If there would exist a number  $a > 1$  such that (1.1) has a solution positive on  $\Omega(a)$ , then Theorem 2.1 would imply that there exists a finite limit  $\lim_{t \rightarrow \infty} C(t)$ . This contradiction ends the proof.  $\square$

**Corollary 2.3 (Leighton–Wintner type criterion).** *If*

$$\lim_{t \rightarrow \infty} \int_{\Omega(1,t)} \|x\|^{1-n} c(x) dx = \infty, \quad (2.5)$$

then equation (1.1) has no positive solution on  $\Omega(a)$  for any  $a > 1$ .

*Proof.* If (2.5) holds, then  $\lim_{t \rightarrow \infty} C(t) = \infty$  and the statement follows from Theorem 2.2.  $\square$

**Remark.** For the equation (1.2) were the results from this paper proved in [5]. Criteria analogous to the second part of Theorem 2.2 and Corollary 2.3 were proved in [2] without the term  $\|x\|^{1-n}$  but under additional conditions  $p \geq n+1$  and  $p \geq n$ , respectively.

*Proof of Lemma 2.1.* First we multiply the Riccati equation (2.1) by  $\|x\|^{1-n}$  and integrate on  $\Omega(a, t)$ . Application of the identity

$$\|x\|^{1-n} \operatorname{div} \mathbf{w} = \operatorname{div}(\|x\|^{1-n} \mathbf{w}) - (1-n)\|x\|^{-n} \langle \mathbf{w}, \mathbf{j} \rangle,$$

and Gauss divergence theorem yields

$$\begin{aligned} & \int_{\|x\|=t} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma - \int_{\|x\|=a} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma \\ & - (1-n) \int_{\Omega(a,t)} \|x\|^{-n} \langle \mathbf{w}, \mathbf{j} \rangle dx + (p-1) \int_{\Omega(a,t)} \|x\|^{1-n} \|\mathbf{w}\|^q dx \\ & + \int_{\Omega(a,t)} \|x\|^{1-n} c(x) dx = 0, \quad (2.6) \end{aligned}$$

where  $\int \cdot d\sigma$  denotes the surface integral,  $\mathbf{j}$  is the unit outside normal vector to the sphere in  $\mathbb{R}^n$  and  $\langle \cdot, \cdot \rangle$  is the usual scalar product.

“(i) $\Rightarrow$ (ii)” Suppose that (2.2) holds. The Hölder inequality implies that

$$\begin{aligned} \int_{\Omega(a,t)} \|x\|^{1-n} |\langle \mathbf{w}, \mathbf{j} \rangle| dx & \leq \left( \int_{\Omega(a,t)} \|x\|^{1-n} \|\mathbf{w}\|^q dx \right)^{1/q} \left( \int_{\Omega(a,t)} \|x\|^{1-n-p} dx \right)^{1/p} \\ & \leq \left( \int_{\Omega(a)} \|x\|^{1-n} \|\mathbf{w}\|^q dx \right)^{1/q} \left( \omega_n \int_a^t s^{-p} ds \right)^{1/p}, \end{aligned}$$

where  $\omega_n$  is the measure of surface of the  $n$ -dimensional unit sphere in  $\mathbb{R}^n$ . Hence

$$\int_{\Omega(a)} \|x\|^{-n} \langle \mathbf{w}, \mathbf{j} \rangle dx \leq \infty. \quad (2.7)$$

Denote

$$\begin{aligned} \widehat{C} & = -(p-1) \int_{\Omega(a)} \|x\|^{1-n} \|\mathbf{w}\|^q dx + \int_{\|x\|=a} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma \\ & + (1-n) \int_{\Omega(a)} \|x\|^{-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma + \int_{\Omega(1,a)} \|x\|^{1-n} c(x) dx. \end{aligned}$$

Below we will show that  $\widehat{C} = C_0$ . The equation (2.6) can be written in the form

$$\begin{aligned} \widehat{C} - \int_{\Omega(1,t)} \|x\|^{1-n} c(x) dx &= \int_{\|x\|=t} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma \\ &- (p-1) \int_{\Omega(t)} \|x\|^{1-n} \|\mathbf{w}\|^q dx + (1-n) \int_{\Omega(t)} \|x\|^{-n} \langle \mathbf{w}, \mathbf{j} \rangle dx. \end{aligned} \quad (2.8)$$

Multiplying (2.8) by  $t^{p-2}$ , integrating over  $[a, t]$  and multiplying by  $\frac{p-1}{t^{p-1}}$  we obtain

$$\begin{aligned} \widehat{C} - \left(\frac{a}{t}\right)^{p-1} [\widehat{C} - C(a)] - C(t) &= \frac{p-1}{t^{p-1}} \int_a^t s^{p-2} \int_{\|x\|=s} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma ds \\ &- \frac{(p-1)^2}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(s)} \|x\|^{1-n} \|\mathbf{w}\|^q dx ds \\ &+ \frac{(1-n)(p-1)}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(s)} \|x\|^{-n} \langle \mathbf{w}, \mathbf{j} \rangle dx ds. \end{aligned} \quad (2.9)$$

The second and the third integral on the right hand side tend to zero as  $t$  tends to infinity in view of (2.2) and (2.7). The Hölder inequality implies

$$\begin{aligned} &\left| \frac{1}{t^{p-1}} \int_a^t s^{p-2} \int_{\|x\|=s} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma ds \right| \\ &\leq \frac{1}{t^{p-1}} \int_a^t s^{p-2} \left( \int_{\|x\|=s} \|x\|^{1-n} \|\mathbf{w}\|^q d\sigma \right)^{1/q} \omega_n^{1/p} ds \\ &\leq \frac{\omega_n^{1/p}}{t^{p-1}} \left( \int_{\Omega(a,t)} \|x\|^{1-n} \|\mathbf{w}\|^q dx \right)^{1/q} \left( \int_0^t s^{p^2-2p} ds \right)^{1/p} \\ &\leq \frac{\omega_n^{1/p}}{(p-1)^{2/p}} \left( \int_{\Omega(a)} \|x\|^{1-n} \|\mathbf{w}\|^q dx \right)^{1/q} t^{\frac{(p-1)^2}{p} - (p-1)} \end{aligned} \quad (2.10)$$

and the first integral in (2.9) tends to zero too. Hence

$$\lim_{t \rightarrow \infty} C(t) = \widehat{C} = C_0. \quad (2.11)$$

The implication “(ii) $\Rightarrow$ (iii)” is trivial.

“(iii) $\Rightarrow$ (i)” Suppose, by contradiction, that (2.4) holds and

$$\int_{\Omega(a)} \|x\|^{1-n} \|\mathbf{w}\|^q dx = +\infty. \quad (2.12)$$

Multiplication of (2.6) by  $t^{p-2}$ , integration over the interval  $[a, b]$  and multiplication by  $t^{1-p}$  gives

$$\begin{aligned} &\frac{1}{t^{p-1}} \int_a^t s^{p-2} \int_{\|x\|=s} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma ds \\ &+ \frac{p-1}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{1-n} \|\mathbf{w}\|^q dx ds \end{aligned}$$

$$\begin{aligned}
& - \frac{1-n}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{-n} \langle \mathbf{w}, \mathbf{j} \rangle dx ds \\
& = \frac{1}{t^{p-1}} \int_a^t s^{p-2} ds \int_{\|x\|=a} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma \\
& \quad - \frac{1}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{1-n} c(x) dx ds. \quad (2.13)
\end{aligned}$$

Define the function

$$v(t) := (p-1) \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{1-n} \|\mathbf{w}\|^q dx ds.$$

The function  $v$  satisfies

$$\frac{v(t)}{t^{p-1}} \rightarrow \infty \text{ for } t \rightarrow \infty. \quad (2.14)$$

Because of the right hand side of the equality (2.13) is bounded from above, there exists  $t_a$  such that the right hand side of (2.13) is less than  $\frac{v(t)}{3t^{p-1}}$  for  $t \geq t_a$ . Now we have from (2.13)

$$\begin{aligned}
\frac{2}{3}v(t) & < \left| \int_a^t s^{p-2} \int_{\|x|=s} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma ds \right| \\
& \quad + \left| (1-n) \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{-n} \langle \mathbf{w}, \mathbf{j} \rangle dx ds \right| \quad (2.15)
\end{aligned}$$

for  $t \geq t_a$ . The same way as in the inequality (2.10) gives

$$\begin{aligned}
& \left| \int_a^t s^{p-2} \int_{\|x|=s} \|x\|^{1-n} \langle \mathbf{w}, \mathbf{j} \rangle d\sigma ds \right| \\
& \leq \left( \int_{\Omega(a,t)} \|x\|^{1-n} \|\mathbf{w}\|^q dx \right)^{1/q} \frac{\omega_n^{1/p} t^{\frac{(p-1)^2}{p}}}{(p-1)^{2/p}} = K (tv'(t))^{1/q}, \quad (2.16)
\end{aligned}$$

where  $K = \omega_n^{1/p} (p-1)^{-\frac{2}{p} - \frac{1}{q}}$ . The Hölder inequality gives

$$\begin{aligned}
& \left| (1-n) \int_a^t s^{p-2} \int_{\Omega(a,s)} \|x\|^{-n} \langle \mathbf{w}, \mathbf{j} \rangle dx ds \right| \\
& \leq (n-1) \int_a^t s^{p-2} \left( \int_{\Omega(a,t)} \|x\|^{1-n} \|\mathbf{w}\|^q dx \right)^{1/q} \left( \int_1^\infty \omega_n \xi^{-p} d\xi \right)^{1/p} ds \\
& \leq (n-1) \left( \int_a^t s^{p-2} \int_{\Omega(a,t)} \|x\|^{1-n} \|\mathbf{w}\|^q dx ds \right)^{1/q} \\
& \quad \times \left( \int_1^\infty \omega_n s^{-p} ds \right)^{1/p} \left( \int_0^t s^{p-2} ds \right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&= (n-1) \left( \frac{v(t)}{p-1} \right)^{1/q} t^{\frac{p-1}{p}} \omega_n^{1/p} \\
&= \frac{(n-1)\omega_n^{1/p}}{(p-1)^{\frac{1}{q}+\frac{2}{p}}} v^{1/q}(t) t^{\frac{p-1}{p}}.
\end{aligned} \tag{2.17}$$

In view of the fact (2.14) there exists a number  $t_b \geq t_a$  such that

$$\frac{(n-1)\omega_n^{1/p}}{(p-1)^{\frac{1}{q}+\frac{2}{p}}} t^{\frac{p-1}{p}} \leq \frac{1}{3} v^{1/p}(t) \tag{2.18}$$

for  $t \geq t_b$ . Combining (2.15), (2.16), (2.17) and (2.18) we get

$$\frac{1}{3} v(t) \leq K (tv'(t))^{1/q}$$

for  $t \geq t_b$ . From here

$$\frac{v'(t)}{v^q(t)} \geq \frac{1}{t} \left( \frac{1}{3K} \right)^q$$

for  $t \geq t_b$ . Integration of this inequality from  $t_b$  to  $\infty$  gives a convergent integral on the left hand side and divergent integral on the right hand side. This contradiction ends the proof.  $\square$

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