Periodic Orbits and the Global Attractor for Delayed Monotone Negative Feedback

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Abstract

We study the delay differential equation $\dot{x}(t) = -\mu x(t) + f(x(t-1))$ with $\mu \geq 0$ and C^1-smooth real functions f satisfying $f(0) = 0$ and $f' < 0$. For a set of μ and f, we determine the number of periodic orbits, and describe the structure of the global attractor as the union of the strong unstable sets of the periodic orbits and of the stationary point 0.

The delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1}$$

with parameter $\mu \geq 0$ and C^1-smooth nonlinearities $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$ models a system governed by delayed feedback and instantaneous damping. The negative feedback case, that is, $\xi f(\xi) < 0$ for all $\xi \neq 0$, arises e.g. in physiological processes or diseases [MG]. The positive feedback case, that is, $\xi f(\xi) > 0$ for all $\xi \neq 0$, occurs e.g. in neural network theory [MW].

A tremendous number of results are known about Eq. (1), mostly concerning existence and qualitative properties of periodic solutions. The introduction of a discrete Lyapunov functional by Mallet-Paret and Sell in [MPS1] (see also Mallet-Paret [MP] for an earlier version) opened the door to a general inquiry into the structure of the attractor of Eq. (1) and more general cyclic feedback systems with delay.

For the negative feedback case Mallet-Paret [MP] obtained a Morse decomposition of the global attractor of Eq. (1). An analogous result for the positive feedback case was shown by Polner [P]. Connecting orbits between some of the Morse sets were obtained by Fiedler and Mallet-Paret [FMP] and by McCord and Mischaikow [MCM]. Although the existence of a Morse decomposition means a gradient-like structure of the attractor, the dynamics under the above negative or positive feedback conditions can be complicated. We refer to Lani-Wayda [LW] and references in it for chaotic behavior of solutions of Eq. (1).

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In the monotone feedback case, i.e.,

$$\text{either } f'(\xi) < 0 \quad \text{for all } \xi \in \mathbb{R}, \qquad \text{or } f'(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R},$$

much more is known about the asymptotic behavior of the solutions of Eq. (1). Mallet-Paret and Sell [MPS2] proved that a Poincaré–Bendixson type theorem holds. In the monotone negative feedback case, i.e., $f' < 0$, Walther [Wa1,Wa2] and Walther and Yebdri [WY] described the attractor of slowly oscillating solutions. Mallet-Paret and Walther [MPW] obtained that the domain of attraction of the attractor of the slowly oscillating solutions is an open and dense subset of the phase space. In [Wa1], for a set of μ and f, Walther showed that the global attractor of Eq. (1) is a 2-dimensional disk with a slowly oscillating periodic orbit on the boundary. In the monotone positive feedback case, i.e., $f' > 0$, Krisztin, Walther and Wu [KWW] and Krisztin and Walther [KWa] proved that, for certain μ and f, the global attractor is a 3-dimensional smooth submanifold of the phase space, and gave a description of it. Higher dimensional partial analogues of these results can be found in [KWu] and [K]. In spite of the above mentioned nice results, there are still several open problems for the monotone feedback cases. For example, the dynamics of the famous Wright's equation $\dot{x}(t) = -\alpha\left(e^{x(t-1)} - 1\right)$ is still not completely understood.

In this note we consider Eq. (1) under a monotone negative feedback condition, that is, we assume

(H1) $\mu \geq 0$, $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $f'(\xi) < 0$ for all $\xi \in \mathbb{R}$, $f(0) = 0$, and f is bounded from above or from below.

Under hypothesis (H1) and an additional condition we state a result describing the global attractor of the semiflow generated by Eq. (1) as the finite union of strong unstable sets of periodic orbits and the strong unstable set of the stationary point 0. We sketch the main ideas and main steps of the proof, and indicate how existing results for the monotone positive feedback case can be modified for this situation.

Let C denote the Banach space of continuous functions $\phi : [-1, 0] \to \mathbb{R}$ with the norm given by $||\phi|| = \max_{-1 \leq t \leq 0} |\phi(t)|$. C^1 is the Banach space of all C^1-maps $\phi : [-1, 0] \to \mathbb{R}$, with the norm $||\phi||_1 = ||\phi|| + ||\dot{\phi}||$. If $I \subset \mathbb{R}$ is an interval, $x : I \to \mathbb{R}$ is a continuous function, $t \in \mathbb{R}$ so that $[t-1, t] \subset I$, then the segment $x_t \in C$ is defined by $x_t(s) = x(t+s)$, $-1 \leq s \leq 0$.

Every $\phi \in C$ uniquely determines a solution $x^\phi : [-1, \infty) \to \mathbb{R}$ with $x_0^\phi = \phi$, i.e., a continuous function $x : [-1, \infty) \to \mathbb{R}$ such that x is differentiable on $(0, \infty)$, $x_0 = \phi$, and x satisfies Eq. (1) for all $t > 0$. The map

$$F : \mathbb{R}^+ \times C \ni (t, \phi) \mapsto x_t^\phi \in C$$

is a continuous semiflow. 0 is the only stationary points of F. All maps $F(t, \cdot) : C \to C$, $t \geq 0$, are injective. It follows that for every $\phi \in C$ there is at most one solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1) with $x_0 = \phi$. We denote also by x^ϕ such a solution on $\mathbb{R}$ whenever it exists.

The global attractor of the semiflow F is a nonempty compact set $A \subset C$ which is invariant in the sense that

$$F(t, A) = A \qquad \text{for all } t \geq 0,$$

and which attracts bounded sets in the sense that for every bounded set $B \subset C$ and for every open set $U \supset A$ there exists $t \geq 0$ with

$$F([t, \infty) \times B) \subset U.$$

Under hypothesis (H1) the semiflow F has a global attractor, see e.g. [Wa2].

It is not difficult to show that

$$A = \{\phi \in C : \text{There are a bounded solution } x : \mathbb{R} \to \mathbb{R} \text{ of Eq. (1) and } t \in \mathbb{R} \text{ so that } \phi = x_t\}.$$

The compactness of A, its invariance property and the injectivity of the maps $F(t, \cdot)$, $t \geq 0$, combined give that the map

$$\mathbb{R}^+ \times A \ni (t, \phi) \mapsto F(t, \phi) \in A$$

extends to a continuous flow

$$F_A : \mathbb{R} \times A \to A;$$

for every $\phi \in A$ and for all $t \in \mathbb{R}$ we have

$$F_A(t, \phi) = x_t$$

with the uniquely determined solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1) satisfying $x_0 = \phi$.

Now we linearize the semiflow F at its stationary point 0. The smoothness of f implies that each map $F(t, \cdot)$, $t \geq 0$, is continuously differentiable. The operators $D_2F(t, 0)$, $t \geq 0$, form a strongly continuous semigroup. The spectrum of the generator of the semigroup $(D_2F(t, 0))_{t \geq 0}$ consists of the solutions $\lambda \in \mathbb{C}$ of the characteristic equation

$$\lambda + \mu - f'(0)e^{-\lambda} = 0. \tag{2}$$

In case $f'(0) < -e^{-\mu-1}$, all points in the spectrum form a sequence of complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_0^\infty$ with

$$\operatorname{Re} \lambda_0 > \operatorname{Re} \lambda_1 > \operatorname{Re} \lambda_2 > \dots, \quad 2j\pi < \operatorname{Im} \lambda_j < (2j + 1)\pi$$

for all $j \in \mathbb{N}$, and $\operatorname{Re} \lambda_j \to -\infty$ as $j \to \infty$. In particular, if the stationary point 0 is linearly unstable, then all points in the spectrum occur in complex conjugate pairs.

The following explicit condition in terms of μ and $f'(0)$ for the location of the solutions of (2) can be obtained e.g. from [DGVW].

Proposition 1. *Let $j \in \mathbb{N}$, and let θ_j denote the unique solution of the equation $-\theta \cot \theta = \mu$ in $(2j\pi, (2j + 1)\pi)$. Then*

$$\operatorname{Re} \lambda_j > 0 \ (= 0)$$

if and only if

$$f'(0) < -\frac{\theta_j}{\sin\theta_j} \ \left(= -\frac{\theta_j}{\sin\theta_j}\right).$$

Assume that there exists $N \in \mathbb{N}$ so that

$$\operatorname{Re}\lambda_{N+1} < 0 < \operatorname{Re}\lambda_N.$$

Let P denote the realified generalized eigenspace of the generator associated with the spectral set $\{\lambda_0, \overline{\lambda_0}, \dots, \lambda_N, \overline{\lambda_N}\}$. Let Q denote the realified generalized eigenspace given by the spectral set of all $\lambda_k, \overline{\lambda_k}$ with $k \geq N+1$. Then $C = P \oplus Q$. The spaces P and Q are also realified generalized eigenspaces of $D_2F(1,0)$ given by the spectral sets $\{e^{\lambda_0}, e^{\overline{\lambda_0}}, \dots, e^{\lambda_N}, e^{\overline{\lambda_N}}\}$ and $\{e^{\lambda_k} : k \geq N+1\} \cup \{e^{\overline{\lambda_k}} : k \geq N+1\}$, respectively.

Choose $\beta > 1$ with $\beta < e^{\operatorname{Re}\lambda_N}$. According to Theorem I.3 in [KWW] there exist convex open neighbourhoods N_Q, N_P of Q, P, respectively, and a C^1-map $w_u : N_P \to Q$ with $W_u(N_P) \subset N_Q$, $w_u(0) = 0$, $Dw_u(0) = 0$ so that the strong unstable manifold of the fixed point 0 of $F(1,\cdot)$ in $N_Q + N_P$, namely

$$\begin{aligned} W^u(0, F(1,\cdot), N_Q + N_P) = \{\phi \in N_Q + N_P :& \text{ There is a trajectory } (\phi_n)_{-\infty}^0 \\ & \text{of } F(1,\cdot) \text{ with } \phi_0 = \phi,\ \phi_n\beta^{-n} \in N_Q + N_P \text{ for all } n \in -\mathbb{N}, \\ & \text{and } \phi_n\beta^{-n} \to 0 \text{ as } n \to -\infty\} \end{aligned}$$

coincides with the graph $\{\chi + w_u(\chi) : \chi \in N_P\}$. It is easy to show that every $\phi \in W^u(0, F(1,\cdot), N_Q + N_P)$ uniquely determines a solution $x^\phi : \mathbb{R} \to \mathbb{R}$ of Eq. (1), and for this solution $x^\phi(t) \to 0$ as $t \to -\infty$ holds, moreover there exists $t \in \mathbb{R}$ with $x_s^\phi \in W^u(0, F(1,\cdot), N_Q + N_P)$ for all $s \leq t$.

We call the forward extension

$$W^u_{str}(0) = F(\mathbb{R}^+ \times W^u(0, F(1,\cdot), N_Q + N_P))$$

the strong unstable set of 0. The unstable set of 0 is defined by

$$\begin{aligned} W^u(0) = \{\phi \in C :& \text{There is a solution } x : \mathbb{R} \to \mathbb{R} \text{ of Eq. (1)} \\ & \text{with } x_0 = \phi \text{ and } x_t \to 0 \text{ as } t \to -\infty\}. \end{aligned}$$

In case $\operatorname{Re}\lambda_{N+1} < 0 < \operatorname{Re}\lambda_N$, 0 is hyperbolic and

$$W^u(0) = W^u_{str}(0).$$

We recall the definition and some properties of a discrete Lyapunov functional

$$V : C \setminus \{0\} \to \mathbb{N} \cup \{\infty\}$$

which goes back to the work of Mallet-Paret [MP]. The version which we use was introduced in Mallet-Paret and Sell [MPS1].

The definition is as follows. First, set $\mathrm{sc}(\phi)=0$ whenever $\phi\in C\setminus\{0\}$ is nonnegative or nonpositive, otherwise, for nonzero elements of C, let

$$\begin{aligned}\mathrm{sc}(\phi)=\sup\big\{k\in\mathbb{N}\setminus\{0\}:\ &\text{There is a strictly increasing finite sequence}\\ &(s^i)_0^k \text{ in } [-1,0] \text{ with } \phi(s^{i-1})\phi(s^i)<0 \text{ for all } i\in\{1,2,\dots,k\}\big\}\le\infty.\end{aligned}$$

Then define

$$V(\phi)=\begin{cases}\mathrm{sc}(\phi) & \text{if } \mathrm{sc}(\phi) \text{ is odd or } \infty,\\ \mathrm{sc}(\phi)+1 & \text{if } \mathrm{sc}(\phi) \text{ is even.}\end{cases}$$

Set

$$\begin{aligned}R=\{\phi\in C^1:\ &\phi(0)\neq 0 \text{ or } \dot\phi(0)\phi(-1)<0,\\ &\phi(-1)\neq 0 \text{ or } \dot\phi(-1)\phi(0)>0,\\ &\text{all zeros of } \phi \text{ in } (-1,0) \text{ are simple}\}.\end{aligned}$$

The next lemma lists basic properties of V [MPS1,MPS2].

Proposition 2.

(i) *For every $\phi\in C\setminus\{0\}$ and for every sequence $(\phi_n)_0^\infty$ in $C\setminus\{0\}$ with $\phi_n\to\phi$ as $n\to\infty$,*

$$V(\phi)\le\liminf_{n\to\infty}V(\phi_n).$$

(ii) *For every $\phi\in R$ and for every sequence $(\phi_n)_0^\infty$ in $C^1\setminus\{0\}$ with $||\phi_n-\phi||_1\to 0$ as $n\to\infty$,*

$$V(\phi)=\lim_{n\to\infty}V(\phi_n)<\infty.$$

(iii) *Let an interval $I\subset\mathbb{R}$, a real $\nu\ge 0$, and continuous functions $b:I\to(-\infty,0)$ and $z:I+[-1,0]\to\mathbb{R}$ be given so that $z|_I$ is differentiable with*

$$\dot z(t)=-\nu z(t)+b(t)z(t-1) \tag{3}$$

for $\inf I<t\in I$, and $z(t)\neq 0$ for some $t\in I+[-1,0]$. Then the map $I\ni t\mapsto V(z_t)\in\mathbb{N}\cup\{\infty\}$ is monotone nonincreasing. If $t\in I$, $t-3\in I$ and $z(t)=0=z(t-1)$, then $V(z_t)=\infty$ or $V(z_{t-3})>V(z_t)$. For all $t\in I$ with $t-4\in I$ and $V(z_{t-4})=V(z_t)<\infty$, we have $z_t\in R$.

(iv) *If $\nu\ge 0$, $b:\mathbb{R}\to(-\infty,0)$ is continuous and bounded, $z:\mathbb{R}\to\mathbb{R}$ is differentiable and bounded, z satisfies (3) for all $t\in\mathbb{R}$, and $z(t)\neq 0$ for some $t\in\mathbb{R}$, then $V(z_t)<\infty$ for all $t\in\mathbb{R}$.*

We need the following corollary of a general Poincaré–Bendixson type theorem for monotone cyclic feedback systems due to Mallet-Paret and Sell [MPS2].

Proposition 3. *Let $x:\mathbb{R}\to\mathbb{R}$ be a bounded solution of Eq. (1). Then $\alpha(x)$ is either the orbit of a nonconstant periodic solution of Eq. (1), or for every solution $y:\mathbb{R}\to\mathbb{R}$ of Eq. (1) with $y_0\in\alpha(x)$ the sets $\alpha(y)$ and $\omega(y_0)$ consist of stationary points of F. An analogous statement holds for ω-limit sets.*

We introduce an additional hypothesis on f:

(H2) $f(\xi)=-f(-\xi)$ for all $\xi\in\mathbb{R}$, and the function $(0,\infty)\ni\xi\mapsto\frac{\xi f'(\xi)}{f(\xi)}\in\mathbb{R}$ is strictly decreasing.

From Lemma 2(iii) and (iv) it follows that for any nonconstant periodic solution $x:\mathbb{R}\to\mathbb{R}$ of Eq. (1) there exists $k\in\mathbb{N}$ so that $V(x_t)=2k+1$ and $x_t\in R$ for all $t\in\mathbb{R}$. For $k\in\mathbb{N}$, we say that Eq. (1) has a periodic orbit in $V^{-1}(2k+1)$ if it has a nonconstant periodic solution $x:\mathbb{R}\to\mathbb{R}$ with $V(x_t)=2k+1$ for all $t\in\mathbb{R}$.

The following result considers uniqueness and absence of periodic orbits.

Proposition 4. *Assume that hypotheses (H1) and (H2) are satisfied.*

(i) For every $k\in\mathbb{N}$, Eq. (1) has at most one periodic orbit in $V^{-1}(2k+1)$.

(ii) Eq. (1) has no periodic orbit in $V^{-1}(2k+1)$ if $\operatorname{Re}\lambda_k\le 0$.

In [KWa] we proved an analogous result for the monotone positive feedback. That proof can be easily modified to obtain Proposition 4. The approach uses the technique of Cao [Ca2] who studied slowly oscillating periodic orbits, i.e., periodic orbits in $V^{-1}(1)$, for Eq. (1). For periodic orbits in $V^{-1}(2k+1)$ with $0<k\in\mathbb{N}$, not all arguments from [Ca2] seem to work. The oddness condition in (H2) is applied to overcome the difficulties. By a results of Mallet-Paret and Sell [MPS2] the oddness of f implies a special symmetry of the periodic solutions of Eq. (1).

The next result of [KWu] guarantees the existence of a periodic orbit with a given oscillation frequency.

Proposition 5. *Assume that hypothesis (H1) holds. If $k\in\mathbb{N}$ and $\operatorname{Re}\lambda_k>0$, then Eq. (1) has a periodic orbit $\mathcal{O}_k$ in $V^{-1}(2k+1)$.*

For a given $k\in\mathbb{N}$, let $p:\mathbb{R}\to\mathbb{R}$ denote the periodic solution guaranteed by Proposition 5 and normalized so that $p(0)=0$ and $p(-1)<0$. Then $\mathcal{O}_k=\{p_t:t\in\mathbb{R}\}$. It is also true that three consecutive zeros of p determine the minimal period ω of p [MPS2]. All zeros of p are simple since $p_t\in R$ for all $t\in\mathbb{R}$ by Proposition 2(iii). Then the definition of V and the fact $V(p_t)=2k+1$ for all $t\in\mathbb{R}$ combined yield $(k+1)\omega>1$. Define the monodromy operator

$$M=D_2F(\omega,p_0).$$

The operator M^{k+1} is compact since $\omega>1/(k+1)$. We then have that the spectrum σ of M contains 0, and that every point $\lambda\in\sigma\setminus\{0\}$ is an eigenvalue of M of finite multiplicity, and is isolated in σ. These eigenvalues in $\sigma\setminus\{0\}$ are called Floquet multipliers.

For $0\neq\lambda\in\sigma$ with $\operatorname{Im}\lambda\ge 0$, let $G_{\mathbb{R}}(\lambda)$ stand for the realified generalized eigenspace of the eigenvalue λ of M. If $r>0$ and $\{\lambda\in\sigma:r<|\lambda|\}\neq\emptyset$, then we use $C_{\le r}$ and $C_{r<}$ to denote the realified generalized eigenspaces of M associated with the nonempty disjoint spectral sets $\{\lambda \in \sigma : |\lambda| \le r\}$ and $\{\lambda \in \sigma : r < |\lambda|\}$, respectively. Then

$$C = C_{\le r} \oplus C_{r<}, \qquad C_{r<} = \bigoplus_{\lambda\in\sigma, r<|\lambda|, \operatorname{Im}\lambda\ge 0} G_{\mathbb{R}}(\lambda).$$

In [KWu] the following result can be found on the Floquet multipliers of the periodic orbit $\mathcal{O}_k$:

There exists $r_M \in (0,1)$ such that

$$C_{\le r_M} \cap V^{-1}(\{1,3,\dots,2k+1\}) = \emptyset, \quad C_{r_M<} \cap C_{\le 1} \subset V^{-1}(2k+1) \cup \{0\},$$
$$\dim C_{r_M<} \cap C_{\le 1} = 2, \quad 0 \le \dim C_{1<} \le 2k.$$

Choose $\lambda \in (0,1)$ so that

$$\lambda > \max\left\{ \max_{\zeta\in\sigma,|\zeta|>1} \frac{1}{|\zeta|}, \max_{\zeta\in\sigma,|\zeta|<1} |\zeta| \right\}.$$

Theorem I.3 in [KWW] guarantees the existence of a local strong unstable manifold of the period-ω map $F(\omega,\cdot)$ at its fixed point p_0; namely, there are convex open neighbourhoods $N_{1<}$ of 0 in $C_{1<}$ and $N_{\le 1}$ of 0 in $C_{\le 1}$, a C^1-map $w^u : N_{1<} \to C_{\le 1}$ so that $w^u(0) = 0$, $Dw^u(0) = 0$, $w^u(N_{1<}) \subset N_{\le 1}$, and with $N^u = N_{\le 1} + N_{1<}$ the shifted graph

$$W^u(p_0, F(\omega,\cdot), N^u) = \{p_0 + \chi + w^u(\chi) : \chi \in N_{1<}\}$$

is equal to the set

$$\{\chi \in p_0 + N^u : \text{There is a trajectory } (\chi^n)_{-\infty}^0 \text{ of } F(\omega,\cdot) \text{ with } \chi^0 = \chi,$$
$$\lambda^n(\chi^n - p_0) \in N^u \text{ for all } n \in -\mathbb{N}, \text{ and } \lambda^n(\chi^n - p_0) \to 0 \text{ as } n \to -\infty\}.$$

The C^1-submanifold $W^u(p_0, F(\omega,\cdot), N^u)$ of C is called a local strong unstable manifold of $F(\omega,\cdot)$ at p_0.

The strong unstable set $W^u_{str}(\mathcal{O}_k)$ of the periodic orbit $\mathcal{O}_k$ is defined by

$$W^u_{str}(\mathcal{O}_k) = F(\mathbb{R}^+ \times W^u(p_0, F(\omega,\cdot), N^u)).$$

The unstable set $W^u(\mathcal{O}_k)$ of the periodic orbit $\mathcal{O}_k$ is given by

$$W^u(\mathcal{O}_k) = \{\phi \in C : \text{There exists a solution } x : \mathbb{R} \to \mathbb{R}$$
$$\text{so that } x_0 = \phi \text{ and } \operatorname{dist}(x_t, \mathcal{O}_k) \to 0 \text{ as } t \to -\infty\}.$$

It is not difficult to show that

$$W^u_{str}(\mathcal{O}_k) \subset W^u(\mathcal{O}_k).$$

If $\mathcal{O}_k$ is hyperbolic, i.e., $\sigma \cap S^1_{\mathbb{C}} = \{1\}$ and the generalized eigenspace of M associated with 1 is 1-dimensional, then the equality $W^u_{str}(\mathcal{O}_k) = W^u(\mathcal{O}_k)$ holds. For a nonhyperbolic $\mathcal{O}_k$, in general, we do not have equality.

In [Kr] in the case of monotone positive delayed feedback we proved the equality $W^u_{str}(\mathcal{O}) = W^u(\mathcal{O})$ for a periodic orbit $\mathcal{O}$ without assuming hyperbolicity. We can use essentially the same ideas even for the negative feedback case to obtain

Proposition 6. *Under hypotheses (H1) and (H2), for each periodic orbit $\mathcal{O}$, $W^u_{str}(\mathcal{O}) = W^u(\mathcal{O})$ holds.*

The basic idea of the proof of the above equality is simple. Propositions 4 and 5 guarantee existence and uniqueness of periodic orbits with a given oscillation frequency. Let $p : \mathbb{R} \to \mathbb{R}$ be a periodic solution of Eq. (1) with minimal period $\omega > 0$ so that $\mathcal{O} = \{p_t : t \in [0, \omega]\}$. We construct two solutions $x : [-1, \infty) \to \mathbb{R}$ and $y : [-1, \infty) \to \mathbb{R}$ of Eq. (1) such that in the plane $\mathbb{R}^2$ the curve

$$X : [0, \infty) \ni t \mapsto \begin{pmatrix} x(t) \\ x(t-1) \end{pmatrix} \in \mathbb{R}^2$$

spirals toward the trace $|P|$ of the simple closed curve

$$P : [0, \omega] \ni t \mapsto \begin{pmatrix} p(t) \\ p(t-1) \end{pmatrix} \in \mathbb{R}^2$$

in the interior of P as $t \to \infty$, while the curve

$$Y : [0, \infty) \ni t \mapsto \begin{pmatrix} y(t) \\ y(t-1) \end{pmatrix} \in \mathbb{R}^2$$

spirals toward $|P|$ in the exterior of P as $t \to \infty$. If $W^u(\mathcal{O}) \neq W^u_{str}(\mathcal{O})$ then there is a solution $z : \mathbb{R} \to \mathbb{R}$ of Eq. (1) such that the curve

$$Z : (-\infty, 0] \ni t \mapsto \begin{pmatrix} z(t) \\ z(t-1) \end{pmatrix} \in \mathbb{R}^2$$

does not intersect the curves P, X, Y, and $Z(t)$ spirals toward $|P|$ as $t \to -\infty$. A planar argument applying the Jordan curve theorem leads to a contradiction. A solution x with the above property is given in Krisztin and Wu [KWu]. The existence of the solution y is shown by using homotopy methods and the Brouwer degree. The construction of z requires some information about the Floquet multipliers of the periodic orbit $\mathcal{O}$.

Now we can state the main result of this note.

Theorem 7. *Assume that hypotheses (H1) and (H2) hold, and $N \geq 0$ is an integer such that*

$$-\frac{\theta_{N+1}}{\sin \theta_{N+1}} < f'(0) < -\frac{\theta_N}{\sin \theta_N}$$

is satisfied where θ_N, θ_{N+1} denote the unique solution of $-\theta \cot \theta = \mu$ in $(2N\pi, (2N+1)\pi)$, $(2(N+2)\pi, (2N+3)\pi)$, respectively. Then the semiflow F has exactly $N+1$ periodic orbits $\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_N$, and, for the global attractor A of F, we have

$$A = W^u_{str}(0) \cup \left(\bigcup_{k=1}^{N} W^u_{str}(\mathcal{O}_k) \right). \tag{4}$$

Sketch of the proof. Hypothesis (H1) implies that the semiflow F has a global attractor A. 0 is the only stationary point, and it is hyperbolic and unstable. In particular, $W^u(0) = W^u_{str}(0)$.

Propositions 1, 4 and 5 imply that F has exactly $N+1$ periodic orbits $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_N$, and $\mathcal{O}_k \subset V^{-1}(2k+1)$, $k \in \{0, 1, \ldots, N\}$. Proposition 6 shows $W^u(\mathcal{O}_k) = W^u_{str}(\mathcal{O}_k)$ for all $k \in \{0, 1, \ldots, N\}$.

Let $\phi \in A$. By the invariance of A, there exists a solution $x : \mathbb{R} \to \mathbb{R}$ so that $x_0 = \phi$ and $x_t \in A$ for all $t \in \mathbb{R}$. Proposition 3 gives that either $\alpha(x) = \mathcal{O}_k$ for some $k \in \{0, 1, \ldots, N\}$ or, for every solution $y : \mathbb{R} \to \mathbb{R}$ of Eq. (1) with $y_0 \in \alpha(x)$, the sets $\alpha(y)$ and $\omega(y_0)$ consist of stationary points of F, i.e., 0. In order to show (4) it suffices to verify that in case $\alpha(x)$ is not a periodic orbit we have $\alpha(x) = \{0\}$. Suppose

$$\alpha(x) \neq \mathcal{O}_k \qquad \text{for all } k \in \{0, 1, \ldots, N\}.$$

Then $0 \in \alpha(x)$ since 0 is the only stationary point of F. Assume $\alpha(x) \neq \{0\}$. Then there exist $\psi \in \alpha(x) \setminus \{0\}$ and a nonzero solution $y : \mathbb{R} \to \mathbb{R}$ with $y_0 = \psi$ and $\alpha(y) \cup \omega(y_0) \subset \alpha(x) \cap \{0\} = \{0\}$. Thus, $\alpha(y) = \omega(y_0) = \{0\}$.

From $y(t) \to 0$ as $t \to -\infty$, it follows that there is a sequence $(t_n)_0^\infty$ with $t_n \to -\infty$ so that

$$|y(t_n)| = \sup_{t \leq 0} |y(t + t_n)|.$$

The functions

$$z^n : \mathbb{R} \ni t \mapsto \frac{y(t + t_n)}{|y(t_n)|} \in \mathbb{R}$$

satisfy $|z^n(t)| \leq 1$ for all $t \leq 0$, and

$$\dot{z}^n(t) = -\mu z^n(t) + b^n(t) z^n(t-1) \qquad \text{for all } t \in \mathbb{R}$$

with

$$b^n(t) = \int_0^1 f'(sy(t - 1 + t_n))\, ds \to f'(0) \qquad \text{as } n \to \infty \text{ uniformly in } (-\infty, 0].$$

The Arzela–Ascoli theorem can be applied to find a subsequence $(z^{n_k})_{k=0}^\infty$ of $(z^n)_0^\infty$ and a continuously differentiable function $z : (-\infty, 0] \to \mathbb{R}$ with

$$z^{n_k} \to z, \quad \dot{z}^{n_k} \to \dot{z} \qquad \text{as } k \to \infty$$

uniformly on compact subsets of $(-\infty, 0]$, and

$$\dot{z}(t) = -\mu z(t) + f'(0) z(t-1) \qquad \text{for all } t \leq 0,$$

$||z_0|| = 1$, $|z(t)| \leq 1$ for all $t \leq 0$. It is not difficult to show that $z_t \in P$ for all $t \leq 0$. Indeed, this follows using the definition of (z^n), the facts that $y_t \in W^u(0) = W^u_{str}(0)$ for all

$t \in \mathbb{R}$, that $W^u_{str}(0)$ is the forward extension of $W^u(0, F(1,\cdot), N_Q + N_P) = \{\chi + w_u(\chi) : \chi \in N_P\}$, and $Dw_u(0) = 0$. The information on the imaginary parts of $\lambda_0, \lambda_1, \ldots, \lambda_N$ yields $V(z_t) \le 2N+1$ for all $t \le 0$. By Proposition 2(iii), there exists $T < 0$ with $z_T \in R$. Using Proposition 2(ii) and $||z_T^{n_k} - z_T||_1 \to 0$ as $k \to \infty$, we find $V(z_T^{n_k}) = V(z_T) \le 2N+1$ for all sufficiently large k. This fact, the definition of (z^n) and the monotonicity of V combined imply

$$V(y_t) \le 2N+1 \qquad \text{for all } t \in \mathbb{R}.$$

By the above upper bound on the number of sign changes of y and the boundedness of y, a result of Cao [Ca1] or Arino [Ar] (see also Mallet-Paret [MP]) can be used to show that $y(t)$ can not decay too fast as $t \to \infty$. More precisely, there exist $a > 0$, $b > 0$ so that

$$||y_t|| \ge ae^{-bt} \qquad \text{for all } t \ge 0.$$

Then, as $y(t) \to 0$ $(t \to \infty)$, an asymptotic expansion holds for $y(t)$ as $t \to \infty$. Namely, there are an integer $j > N$ and $(c, d) \in \mathbb{R}^2 \setminus \{(0,0)\}$ so that

$$y(t) = e^{\operatorname{Re}\lambda_j t}\Big(c\cos(\operatorname{Im}\lambda_j t) + d\sin(\operatorname{Im}\lambda_j t) + o(1)\Big) \qquad \text{as } t \to \infty.$$

A consequence of this fact is that

$$V(y_t) \ge 2j + 1 > 2N + 1$$

for all sufficiently large t. This is a contradiction. Therefore $\alpha(x) = \{0\}$, and the proof is complete.

Remarks 1. We emphasize that no hyperbolicity condition on the periodic orbits is assumed in Theorem 7.

2. As the maps $F(t,\cdot)$ and $D_2F(t,\cdot)$ are injective for all $t \ge 0$, Theorem 6.1.9 in Henry [He] can be used to show that the strong unstable sets

$$W^u_{str}(\mathcal{O}_0), \ldots, W^u_{str}(\mathcal{O}_N)$$

in (4) are C^1 immersed submanifolds of C. We suspect that these strong unstable sets are also C^1-submanifolds of C.

3. Hypotheses (H1) and (H2) hold, for example, for the functions

$$f(\xi) = -\alpha\tanh(\beta\xi), \qquad f(\xi) = -\alpha\tan^{-1}(\beta\xi)$$

with parameters $\alpha > 0$ and $\beta > 0$.

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