



Oscillation and non-oscillation criterion for Riemann–Weber type half-linear differential equations

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Abstract. By the combination of the modified half-linear Prüfer method and the Riccati technique, we study oscillatory properties of half-linear differential equations. Taking into account the transformation theory of half-linear equations and using some known results, we show that the analysed equations in the Riemann–Weber form with perturbations in both terms are conditionally oscillatory. Within the process, we identify the critical oscillation values of their coefficients and, consequently, we decide when the considered equations are oscillatory and when they are non-oscillatory. As a direct corollary of our main result, we solve the so-called critical case for a certain type of half-linear non-perturbed equations.

Keywords: half-linear equations, Prüfer angle, Riccati equation, oscillation theory, conditional oscillation, oscillation constant, oscillation criterion.

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1 Introduction

This paper is devoted to the study of the half-linear differential equations


$$\left[r(t)t^{p-1}\Phi(x') \right]' + \frac{s(t)}{t \log^p t} \Phi(x) = 0, \quad \Phi(x) = |x|^{p-1} \operatorname{sgn} x \quad (1.1)$$

and

$$\left[\left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{1}{t \log^p t} \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2} \right) \Phi(x) = 0 \quad (1.2)$$

with continuous coefficients $r > 0, s, r_1 > 0, r_2, s_1, s_2$, where \log denotes the natural logarithm, $p > 1$ is a given real constant, and q stands for the number conjugated with p , i.e., $p + q = pq$.

The main interest in our investigation is the so-called conditional oscillation. Therefore, we begin with recalling this notion. At first, we should mention that the Sturmian theory is extendable to half-linear differential equations. Especially, the separation theorem is extendable

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to half-linear equations. This fact enables us to categorize the studied equations as oscillatory (zeros of every solution tend to infinity) and non-oscillatory (every non-zero solution has the biggest zero). An important role is played by the so-called conditionally oscillatory equations. They are special types of equations, whose oscillatory properties are determined by “measuring” their coefficients and the oscillation and non-oscillation can be changed using the multiplication of at least one coefficient by positive constants. More precisely, we say that the second order half-linear differential equation

$$[R(t)\Phi(x')] + \gamma S(t)\Phi(x) = 0 \quad (1.3)$$

is conditionally oscillatory if there exists a positive constant Γ (called the critical oscillation constant) such that Eq. (1.3) is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$. In general, it is difficult to solve the critical case given by $\gamma = \Gamma$. Many half-linear equations are non-oscillatory in the critical case. But, there are known cases when it is not possible to decide whether the studied equations are oscillatory or non-oscillatory. Equations with general coefficients may be both oscillatory and non-oscillatory in the critical case, i.e., while one equation is oscillatory, another one is non-oscillatory. For more details, see, e.g., [2, 6, 10, 18, 19, 31, 35]. For the basic theory background on half-linear equations, we refer to books [1, 9].

In this paper, we fully solve the critical case of Eq. (1.1) with periodic coefficients r and s , i.e., we analyse the oscillation of this equation in full. At the same time, we turn our attention to the perturbed equation (1.2), where the coefficients r_1, s_1 are periodic and the coefficients r_2, s_2 in the perturbations are very general and they can change their signs.

Let us briefly mention the current state of the conditional oscillation theory and give some historical remarks. As far as we know, the first attempt to the conditional oscillation comes from [22], where the linear differential equation

$$x'' + \frac{\gamma}{t^2}x = 0 \quad (1.4)$$

was studied and its oscillation constant $\gamma_0 = 1/4$ was obtained. Note that it was also shown in [22] that Eq. (1.4) is non-oscillatory in the critical case. The non-constant coefficients were treated in [15, 28], where the equation

$$[r(t)x'] + \frac{\gamma s(t)}{t^2}x = 0 \quad (1.5)$$

with positive periodic coefficients r, s was analysed. The critical case of Eq. (1.5) was solved as non-oscillatory in [29] as a consequence of the study of the perturbed equation

$$[r(t)x'] + \frac{1}{t^2} \left[\gamma s_1(t) + \frac{\mu s_2(t)}{\log^2 t} \right] x = 0$$

with positive periodic coefficients r, s_1, s_2 .

In the theory of half-linear equations, the first attempt was made in [11, 12], from where it follows that the half-linear Euler equation

$$[\Phi(x')] + \frac{\gamma}{t^p}\Phi(x) = 0 \quad (1.6)$$

is conditionally oscillatory with the oscillation constant $\gamma_p := q^{-p}$. From [13], it is known that the half-linear Riemann–Weber equation

$$[\Phi(x')] + \frac{1}{t^p} \left[\gamma_p + \frac{\mu}{\log^2 t} \right] \Phi(x) = 0 \quad (1.7)$$

is also conditionally oscillatory with respect to the oscillation constant $\mu_p := q^{1-p}/2$.

As a natural continuation of the research of Eq. (1.6) and (1.7), positive constant coefficients were replaced by positive periodic functions in [8]. The main result of [8] deals with the Euler type equation

$$[r(t)\Phi(x')] + \frac{\gamma c(t)}{t^p} \Phi(x) = 0 \quad (1.8)$$

and with the Riemann–Weber type equation

$$[r(t)\Phi(x')] + \frac{1}{t^p} \left[\gamma c(t) + \frac{\mu d(t)}{\log^2 t} \right] \Phi(x) = 0, \quad (1.9)$$

where r, c , and d are periodic positive functions with the same period. Since [8] is one of the main motivations for our research, we reformulate its main result in full. We should recall that the mean value of a periodic function f over its period, say $T > 0$, is the number

$$M(f) = \frac{1}{T} \int_a^{a+T} f(x) dx,$$

where $a \in \mathbb{R}$ is arbitrary. We can also refer to Definition 4.8 below.

Theorem 1.1 ([8]). *Eq. (1.8) is non-oscillatory if and only if*

$$\gamma \leq \gamma_{rc} := \gamma_p \left[M(r^{1-q}) \right]^{1-p} [M(c)]^{-1}.$$

In the limiting case $\gamma = \gamma_{rc}$, Eq. (1.9) is non-oscillatory if

$$\mu < \mu_{rd} := \mu_p \left[M(r^{1-q}) \right] [M(d)]^{-1},$$

and it is oscillatory if $\mu > \mu_{rd}$.

The next motivation comes from papers [4–7,27]. At this place, we state a result concerning the equation

$$\left[\left(\alpha_1 + \frac{\alpha_2}{\log^2 t} \right)^{-\frac{p}{q}} \Phi(x') \right]' + \frac{1}{t^p} \left(\beta_1 + \frac{\beta_2}{\log^2 t} \right) \Phi(x) = 0, \quad (1.10)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants and $\alpha_1 > 0$. Note that, due to the exponent in the first term of Eq. (1.10), the formulations of results are technically easier and the exponent does not mean any restriction and can be removed. The following theorem can be obtained, e.g., as a direct corollary of the main result of [6] (or deduced from [4,5,7]). We will also use this theorem in the proof of Lemma 3.1 below which is essential to prove our main result.

Theorem 1.2 ([6]). *The following statements hold.*

- (i) Eq. (1.10) is oscillatory if $\beta_1 \alpha_1^{p-1} > \gamma_p$, and non-oscillatory if $\beta_1 \alpha_1^{p-1} < \gamma_p$.
- (ii) Let $\beta_1 \alpha_1^{p-1} = \gamma_p$. Eq. (1.10) is oscillatory if $\beta_2 \alpha_1^{p-1} + (p-1)\gamma_p \alpha_2 \alpha_1^{-1} > \mu_p$, and non-oscillatory if $\beta_2 \alpha_1^{p-1} + (p-1)\gamma_p \alpha_2 \alpha_1^{-1} < \mu_p$.

As the third result which is strongly connected to the presented one, we mention a result from [33]. This result is focused on an equation of the same type as Eq. (1.1). More precisely, it deals with the equation

$$\left[r^{-\frac{p}{q}}(t) t^{p-1} \Phi(x') \right]' + \frac{s(t)}{t \log^p t} \Phi(x) = 0, \quad (1.11)$$

where $r > 0$ and s are periodic functions with the same period.

Theorem 1.3 ([33]). *Eq. (1.11) is oscillatory if $[M(r)]^{p-1} M(s) > \gamma_p$. Eq. (1.11) is non-oscillatory if $[M(r)]^{p-1} M(s) < \gamma_p$.*

Besides the above given references, we should mention at least papers [16, 21, 23–26, 32]. Note that the treated topic is also studied in the field of difference equations (see, e.g., [17, 30, 34]) and in the field of dynamic equations on time scales (see, e.g., [20, 36]).

In this paper, we generalize Theorem 1.3 into a very general situation. Especially, we solve the critical case $[M(r)]^{p-1} M(s) = \gamma_p$. Our aim is to obtain a result similar to Theorem 1.1 which covers also non-periodic coefficients. To make this, we apply Theorem 1.2 and the method based on the combination of the modified half-linear Prüfer angle and the Riccati equation. To the best of our knowledge, the used method and the announced result are new even in the linear case (see also Corollary 4.3 and Example 4.4 below).

This paper is organized as follows. In the following section, we derive the equation for the modified Prüfer angle, which will be an important tool in the rest of this paper. Then, we study the behaviour of the Prüfer angle. This leads to the proof of the main result in Section 3. The paper is finished by corollaries and examples in Section 4.

2 Modified Prüfer angle and average function

At this place, we provide some background calculations which lead to auxiliary equations that are necessary for our approach. Throughout this paper, we will consider an arbitrarily given number $p > 1$ and the conjugated number $q := p/(p-1)$ and we will use the notation $\mathbb{R}_a := (a, \infty)$ for $a \in \mathbb{R}$. In our main result (see Theorem 3.3 below), we will consider the equation

$$\left[\left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{1}{t \log^p t} \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2} \right) \Phi(x) = 0, \quad (2.1)$$

where $r_1 : \mathbb{R} \rightarrow \mathbb{R}_0$ and $s_1 : \mathbb{R} \rightarrow \mathbb{R}$ are α -periodic continuous functions for some $\alpha \in \mathbb{R}_0$ and where $r_2, s_2 : \mathbb{R}_e \rightarrow \mathbb{R}$ are continuous functions such that

$$r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} > 0, \quad t \in \mathbb{R}_e, \quad (2.2)$$

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t \log t}} \int_t^{t+\alpha} |r_2(u)| \, du = 0, \quad (2.3)$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t \log t}} \int_t^{t+\alpha} |s_2(u)| \, du = 0. \quad (2.4)$$

For future use, we put

$$r_1^+ := \max_{t \in [0, \alpha]} r_1(t), \quad s_1^+ := \max_{t \in [0, \alpha]} |s_1(t)|. \quad (2.5)$$

For our investigation of Eq. (2.1), we need to express the half-linear Prüfer angle in a very special form. Let us briefly describe its derivation. At first, we apply the Riccati transformation

$$w(t) = \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi \left(\frac{x'(t)}{x(t)} \right), \quad (2.6)$$

where x is a non-zero solution of Eq. (2.1). The obtained function w satisfies the Riccati equation

$$w'(t) + \frac{1}{t \log^p t} \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2} \right) + \frac{p-1}{t} \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right) |w(t)|^q = 0 \quad (2.7)$$

associated to Eq. (2.1) whenever $x(t) \neq 0$. For details about the Riccati transformation and equation, we refer to [9, Section 1.1.4].

Now we use the transformation

$$v(t) = (\log t)^{\frac{p}{q}} w(t), \quad t \in \mathbb{R}_e, \quad (2.8)$$

in Eq. (2.7) which gets us to the adapted (or weighted) Riccati type equation

$$\begin{aligned} v'(t) &= \frac{p}{q} (\log t)^{\frac{p}{q}-1} \frac{w(t)}{t} + (\log t)^{\frac{p}{q}} w'(t) \\ &= \frac{p}{q} \frac{v(t)}{t \log t} - \frac{1}{t \log t} \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2} \right) \\ &\quad - \frac{p-1}{t} \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right) \frac{|v(t)|^q}{\log t}. \end{aligned} \quad (2.9)$$

Thus, in one hand, we keep the adapted Riccati equation (2.9). In the other hand, we have the modified half-linear Prüfer transformation

$$x(t) = \rho(t) \sin_p \varphi(t), \quad \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right)^{-1} t x'(t) = \frac{\rho(t)}{\log t} \cos_p \varphi(t), \quad (2.10)$$

where \sin_p and \cos_p denote the half-linear sine and cosine functions. For fundamental properties of the half-linear trigonometric functions \sin_p and \cos_p , see [9, Section 1.1.2]. In this paper, we have to mention only that the half-linear sine and cosine functions are periodic and that they satisfy the half-linear Pythagorean identity

$$|\sin_p x|^p + |\cos_p x|^p = 1, \quad x \in \mathbb{R}. \quad (2.11)$$

Especially,

$$|\sin_p x| \leq 1, \quad |\cos_p x| \leq 1, \quad |\Phi(\cos_p x)| \leq 1, \quad x \in \mathbb{R}. \quad (2.12)$$

We combine the adapted Riccati equation (2.9) with the Prüfer transformation (2.10). We begin with the observations that the function

$$y(t) = \Phi\left(\frac{\cos_p t}{\sin_p t}\right)$$

solves the equation

$$y'(t) + p - 1 + (p - 1)|y(t)|^q = 0$$

and that (see (2.6), (2.8), and (2.10))

$$v(t) = (\log t)^{\frac{p}{q}} \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi\left(\frac{x'(t)}{x(t)}\right) = \Phi\left(\frac{\cos_p \varphi(t)}{\sin_p \varphi(t)}\right). \quad (2.13)$$

Using (2.11), these two observations lead to the second expression (the first one is the adapted Riccati equation (2.9) itself)

$$\begin{aligned} v'(t) &= [y(\varphi(t))]' = [1 - p + (1 - p)|y(\varphi(t))|^q] \varphi'(t) \\ &= (1 - p) \left[1 + \left| \Phi\left(\frac{\cos_p \varphi(t)}{\sin_p \varphi(t)}\right) \right|^q \right] \varphi'(t) \\ &= (1 - p) \left[1 + \left| \frac{\cos_p \varphi(t)}{\sin_p \varphi(t)} \right|^p \right] \varphi'(t) = \frac{1 - p}{|\sin_p \varphi(t)|^p} \varphi'(t). \end{aligned} \quad (2.14)$$

Finally, we compare both of the expressions for $v'(t)$, namely (2.9) and (2.14). Hence, we have

$$\begin{aligned} \frac{1 - p}{|\sin_p \varphi(t)|^p} \varphi'(t) &= \frac{p}{q} \frac{v(t)}{t \log t} - \frac{1}{t \log t} \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2} \right) \\ &\quad - \frac{p - 1}{t} \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right) \frac{|v(t)|^q}{\log t}, \end{aligned}$$

from where we immediately express the derivative of the modified Prüfer angle (we are aware of (2.13))

$$\begin{aligned} \varphi'(t) &= \frac{1}{t \log t} \left[\left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right) |\cos_p \varphi(t)|^p - \Phi(\cos_p \varphi(t)) \sin_p \varphi(t) \right. \\ &\quad \left. + \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2} \right) \frac{|\sin_p \varphi(t)|^p}{p - 1} \right]. \end{aligned} \quad (2.15)$$

We will use Eq. (2.15) to the study of oscillatory properties of Eq. (2.1).

For the period α of the functions r_1, s_1 , we define the function φ_{ave} which determines the average value of an arbitrarily given solution $\varphi : \mathbb{R}_e \rightarrow \mathbb{R}$ of Eq. (2.15) over intervals of the length α ; i.e., we put

$$\varphi_{ave}(t) := \frac{1}{\alpha} \int_t^{t+\alpha} \varphi(u) \, du, \quad t \in \mathbb{R}_e, \quad (2.16)$$

where φ is a solution of Eq. (2.15) on \mathbb{R}_e .

We prove an auxiliary result concerning the function φ_{ave} .

Lemma 2.1. *The limit*

$$\lim_{t \rightarrow \infty} \sqrt{t \log t} |\varphi(s) - \varphi_{ave}(t)| = 0 \quad (2.17)$$

exists uniformly with respect to $s \in [t, t + \alpha]$.

Proof. For $s \in [t, t + \alpha]$, we have

$$\begin{aligned} 0 &\leq \liminf_{t \rightarrow \infty} \sqrt{t \log t} |\varphi(s) - \varphi_{ave}(t)| \leq \limsup_{t \rightarrow \infty} \sqrt{t \log t} |\varphi(s) - \varphi_{ave}(t)| \\ &= \limsup_{t \rightarrow \infty} \sqrt{t \log t} \left| \varphi(s) - \frac{1}{\alpha} \int_t^{t+\alpha} \varphi(u) \, du \right| = \limsup_{t \rightarrow \infty} \sqrt{t \log t} \left| \frac{1}{\alpha} \int_t^{t+\alpha} \varphi(s) - \varphi(u) \, du \right| \\ &\leq \limsup_{t \rightarrow \infty} \sqrt{t \log t} \max_{s_1, s_2 \in [t, t+\alpha]} |\varphi(s_1) - \varphi(s_2)| = \limsup_{t \rightarrow \infty} \sqrt{t \log t} \max_{s_1, s_2 \in [t, t+\alpha]} \left| \int_{s_1}^{s_2} \varphi'(u) \, du \right| \\ &= \limsup_{t \rightarrow \infty} \sqrt{t \log t} \max_{s_1, s_2 \in [t, t+\alpha]} \left| \int_{s_1}^{s_2} \frac{1}{u \log u} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right. \right. \\ &\quad \left. \left. - \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \right. \right. \\ &\quad \left. \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \right] du \right| \\ &= \limsup_{t \rightarrow \infty} \sqrt{t \log t} \left\{ \max_{s_1, s_2 \in [t, t+\alpha]} \left| \frac{1}{s_1 \log s_1} \int_{s_1}^{s_3} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right. \right. \right. \\ &\quad \left. \left. - \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \right. \right. \\ &\quad \left. \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \right] du \right. \\ &\quad \left. + \frac{1}{s_2 \log s_2} \int_{s_3}^{s_2} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right. \right. \\ &\quad \left. \left. - \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \right. \right. \\ &\quad \left. \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \right] du \right\} \\ &\leq \limsup_{t \rightarrow \infty} \sqrt{t \log t} \left\{ \max_{s_1 \in [t, t+\alpha]} \left| \frac{1}{s_1 \log s_1} \right| \cdot \left| \int_{s_1}^{s_3} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right. \right. \right. \\ &\quad \left. \left. - \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \right. \right. \\ &\quad \left. \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \right] du \right| \end{aligned}$$

$$\begin{aligned}
& + \max_{s_2 \in [t, t+\alpha]} \left| \frac{1}{s_2 \log s_2} \right| \cdot \left| \int_{s_3}^{s_2} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right. \right. \\
& \quad \left. \left. - \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \right. \right. \\
& \quad \left. \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \right] du \right| \Bigg\} \\
\leq & \limsup_{t \rightarrow \infty} \sqrt{t \log t} \left\{ \frac{1}{t \log t} \max_{s_1, s_2 \in [t, t+\alpha]} \left| \int_{s_1}^{s_2} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right. \right. \right. \\
& \quad \left. \left. - \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \right. \right. \\
& \quad \left. \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \right] du \right| \\
& + \frac{1}{t \log t} \max_{s_1, s_2 \in [t, t+\alpha]} \left| \int_{s_1}^{s_2} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right. \right. \\
& \quad \left. \left. - \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \right. \right. \\
& \quad \left. \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \right] du \right| \Bigg\} \\
\leq & \limsup_{t \rightarrow \infty} \frac{2}{\sqrt{t \log t}} \max_{s_1, s_2 \in [t, t+\alpha]} \int_{s_1}^{s_2} \left| \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right. \\
& \quad \left. - \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \right. \\
& \quad \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \right| du \\
\leq & 2 \limsup_{t \rightarrow \infty} \frac{1}{\sqrt{t \log t}} \max_{s_1, s_2 \in [t, t+\alpha]} \int_{s_1}^{s_2} \left[r_1^+ + \frac{|r_2(u)|}{[\log(\log u)]^2} + 1 + \frac{1}{p-1} \left(s_1^+ + \frac{|s_2(u)|}{[\log(\log u)]^2} \right) \right] du \\
\leq & 2 \limsup_{t \rightarrow \infty} \frac{1}{\sqrt{t \log t}} \int_t^{t+\alpha} \left[r_1^+ + \frac{|r_2(u)|}{[\log(\log t)]^2} + 1 + \frac{1}{p-1} \left(s_1^+ + \frac{|s_2(u)|}{[\log(\log t)]^2} \right) \right] du = 0,
\end{aligned}$$

where (2.3), (2.4), (2.5), and (2.12) are used. \square

3 Results

At first, we discuss the oscillatory behaviour of the equation

$$\left[\left(\alpha_1 + \frac{\alpha_2}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{1}{t \log^p t} \left(\beta_1 + \frac{\beta_2}{[\log(\log t)]^2} \right) \Phi(x) = 0 \quad (3.1)$$

with constant coefficients $\alpha_1 \in \mathbb{R}_0$, $\alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Applying a simple transformation, one can get the following lemma.

Lemma 3.1. Eq. (3.1) is oscillatory for $\alpha_1^{p-1}\beta_1 > q^{-p}$ and non-oscillatory for $\alpha_1^{p-1}\beta_1 < q^{-p}$. In the limiting case $\alpha_1^{p-1}\beta_1 = q^{-p}$, Eq. (3.1) is oscillatory if

$$\beta_2\alpha_1^{p-1} + \frac{p-1}{q^p} \frac{\alpha_2}{\alpha_1} > \frac{q^{1-p}}{2}, \quad (3.2)$$

and non-oscillatory if

$$\beta_2\alpha_1^{p-1} + \frac{p-1}{q^p} \frac{\alpha_2}{\alpha_1} < \frac{q^{1-p}}{2}. \quad (3.3)$$

Proof. In Eq. (3.1), we have $x = x(t)$ and $(\cdot)' = d/dt$. Using the transformation of the independent variable

$$s = \log t, \quad \text{i.e.,} \quad \frac{d}{dt} = \frac{1}{t} \frac{d}{ds},$$

we obtain (we put $x(t) = y(s)$)

$$\frac{1}{t} \frac{d}{ds} \left[\left(\alpha_1 + \frac{\alpha_2}{\log^2 s} \right)^{-\frac{p}{q}} t^{p-1} \Phi \left(\frac{1}{t} \frac{dy}{ds} \right) \right] + \frac{1}{ts^p} \left(\beta_1 + \frac{\beta_2}{\log^2 s} \right) \Phi(y) = 0.$$

This leads to the equation

$$\left[\left(\alpha_1 + \frac{\alpha_2}{\log^2 s} \right)^{-\frac{p}{q}} \Phi(y') \right]' + \frac{1}{s^p} \left(\beta_1 + \frac{\beta_2}{\log^2 s} \right) \Phi(y) = 0,$$

where $y = y(s)$ and $(\cdot)' = d/ds$. Now it suffices to use Theorem 1.2. \square

From Lemma 3.1, we get the lemma below which closes the preliminary results.

Lemma 3.2. Let $M(r_1), M(s_1) \in \mathbb{R}_0$ be such that $[M(r_1)]^{p-1}M(s_1) = q^{-p}$.

(i) If $X, Y \in \mathbb{R}$ satisfy

$$[M(r_1)]^{p-1}Y + \frac{p-1}{q^p} \frac{X}{M(r_1)} > \frac{q^{1-p}}{2}, \quad (3.4)$$

then any solution $\theta : \mathbb{R}_e \rightarrow \mathbb{R}$ of the equation

$$\theta'(t) = \frac{1}{t \log t} \left[\left(M(r_1) + \frac{X}{[\log(\log t)]^2} \right) |\cos_p \theta(t)|^p - \Phi(\cos_p \theta(t)) \sin_p \theta(t) + \left(M(s_1) + \frac{Y}{[\log(\log t)]^2} \right) \frac{|\sin_p \theta(t)|^p}{p-1} \right] \quad (3.5)$$

is unbounded from above.

(ii) If $V, W \in \mathbb{R}$ satisfy

$$[M(r_1)]^{p-1}W + \frac{p-1}{q^p} \frac{V}{M(r_1)} < \frac{q^{1-p}}{2}, \quad (3.6)$$

then any solution $\xi : \mathbb{R}_e \rightarrow \mathbb{R}$ of the equation

$$\xi'(t) = \frac{1}{t \log t} \left[\left(M(r_1) + \frac{V}{[\log(\log t)]^2} \right) |\cos_p \xi(t)|^p - \Phi(\cos_p \xi(t)) \sin_p \xi(t) + \left(M(s_1) + \frac{W}{[\log(\log t)]^2} \right) \frac{|\sin_p \xi(t)|^p}{p-1} \right] \quad (3.7)$$

is bounded from above.

Proof. Comparing Eq. (3.5) and Eq. (3.7) with Eq. (2.15), one can see that Eq. (3.5) and Eq. (3.7) is the equation of the Prüfer angle for

$$\left[\left(M(r_1) + \frac{X}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{1}{t \log^p t} \left(M(s_1) + \frac{Y}{[\log(\log t)]^2} \right) \Phi(x) = 0 \quad (3.8)$$

and

$$\left[\left(M(r_1) + \frac{V}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{1}{t \log^p t} \left(M(s_1) + \frac{W}{[\log(\log t)]^2} \right) \Phi(x) = 0, \quad (3.9)$$

respectively.

Let us focus on the first case. The assumption $[M(r_1)]^{p-1}M(s_1) = q^{-p}$ and (3.4) give that Eq. (3.8) is oscillatory (see (3.2) in Lemma 3.1). Now it suffices to consider directly the Prüfer transformation (2.10) and take into account the form of Eq. (3.5), where $\sin_p \theta(t) = 0$ implies $\theta'(t) > 0$ for all large t . Therefore, Eq. (3.8) is oscillatory if and only if its Prüfer angle θ is unbounded from above. Part (i) is proved.

Considering (3.3) and (3.6), the case (ii) is analogous (Eq. (3.9) is non-oscillatory if and only if the Prüfer angle ζ is bounded from above). \square

Now we are ready to formulate and to prove the main result of our paper.

Theorem 3.3. *Let $[M(r_1)]^{p-1}M(s_1) = q^{-p}$.*

(i) *If there exist $R, S \in \mathbb{R}$ such that*

$$\frac{1}{\alpha} \int_t^{t+\alpha} r_2(u) \, du \geq R, \quad \frac{1}{\alpha} \int_t^{t+\alpha} s_2(u) \, du \geq S \quad (3.10)$$

for all sufficiently large t and that

$$[M(r_1)]^{p-1}S + \frac{p-1}{q^p} \frac{R}{M(r_1)} > \frac{q^{1-p}}{2}, \quad (3.11)$$

then Eq. (2.1) is oscillatory.

(ii) *If there exist $R, S \in \mathbb{R}$ such that*

$$\frac{1}{\alpha} \int_t^{t+\alpha} r_2(u) \, du \leq R, \quad \frac{1}{\alpha} \int_t^{t+\alpha} s_2(u) \, du \leq S \quad (3.12)$$

for all sufficiently large t and that

$$[M(r_1)]^{p-1}S + \frac{p-1}{q^p} \frac{R}{M(r_1)} < \frac{q^{1-p}}{2}, \quad (3.13)$$

then Eq. (2.1) is non-oscillatory.

Proof. Let us consider the function φ_{ave} given by (2.16), where φ is an arbitrary solution of Eq. (2.15) on \mathbb{R}_e . It holds

$$\begin{aligned}
\varphi'_{ave}(t) &= \frac{1}{\alpha} [\varphi(t+\alpha) - \varphi(t)] = \frac{1}{\alpha} \int_t^{t+\alpha} \varphi'(u) \, du \\
&= \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{u \log u} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right. \\
&\quad \left. - \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \right. \\
&\quad \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \right] du \quad (3.14) \\
&= \frac{1}{\alpha} \left[\int_t^{t+\alpha} \frac{1}{u \log u} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \, du \right. \\
&\quad \left. - \int_t^{t+\alpha} \frac{1}{u \log u} \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \, du \right. \\
&\quad \left. + \int_t^{t+\alpha} \frac{1}{u \log u} \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p \varphi(u)|^p}{p-1} \, du \right]
\end{aligned}$$

for any $t \in \mathbb{R}_e$. Let $\varepsilon \in \mathbb{R}_0$ be arbitrarily given.

We have

$$\begin{aligned}
&\left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{u \log u} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \, du \right. \\
&\quad \left. - \frac{1}{\alpha t \log t} \int_t^{t+\alpha} \left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) du |\cos_p \varphi_{ave}(t)|^p \right| \\
&\leq \left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{u \log u} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \, du \right. \\
&\quad \left. - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \, du \right| \\
&\quad + \left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \, du \right. \\
&\quad \left. - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi_{ave}(t)|^p \, du \right| \\
&\quad + \left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi_{ave}(t)|^p \, du \right. \\
&\quad \left. - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) |\cos_p \varphi_{ave}(t)|^p \, du \right|
\end{aligned}$$

for all $t \in \mathbb{R}_e$. Since

$$\lim_{t \rightarrow \infty} t(t+\alpha) \log t \left(\frac{1}{t \log t} - \frac{1}{(t+\alpha) \log(t+\alpha)} \right) = \alpha, \quad (3.15)$$

we obtain (see (2.3), (2.5), and (2.12))

$$\begin{aligned} & \left| \frac{1}{\alpha} \int_t^{t+\alpha} \left[\frac{1}{u \log u} - \frac{1}{t \log t} \right] \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \, du \right| \\ & \leq \frac{1}{\alpha} \int_t^{t+\alpha} \left[\frac{1}{t \log t} - \frac{1}{(t+\alpha) \log(t+\alpha)} \right] \left(r_1(u) + \frac{|r_2(u)|}{[\log(\log t)]^2} \right) |\cos_p \varphi(u)|^p \, du \\ & \leq 2 \int_t^{t+\alpha} \frac{1}{t^2 \log t} \left(r_1^+ + \frac{|r_2(u)|}{[\log(\log t)]^2} \right) \, du < \frac{1}{t^{\frac{3}{2}}} \int_t^{t+\alpha} \frac{1}{\sqrt{t \log t}} (r_1^+ + |r_2(u)|) \, du < \frac{1}{t^{\frac{3}{2}}} \end{aligned}$$

for all large t . Especially, we can assume that

$$\left| \frac{1}{\alpha} \int_t^{t+\alpha} \left[\frac{1}{u \log u} - \frac{1}{t \log t} \right] \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \, du \right| < \frac{\varepsilon}{t \log t [\log(\log t)]^2}.$$

We recall that the considered half-linear functions \sin_p and $\Phi(\cos_p)$ are periodic and continuously differentiable. In particular, these facts imply the existence of a positive number L such that

$$|\sin_p x| - |\sin_p y| \leq L|x - y|, \quad |\Phi(\cos_p x) - \Phi(\cos_p y)| \leq L|x - y|, \quad (3.16)$$

and

$$|\sin_p x|^p - |\sin_p y|^p \leq L|x - y|, \quad |\cos_p x|^p - |\cos_p y|^p \leq L|x - y| \quad (3.17)$$

for any $x, y \in \mathbb{R}$. Applying the second inequality in (3.17), we have (see (2.17) in Lemma 2.1 and again (2.3) and (2.5))

$$\begin{aligned} & \left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) [|\cos_p \varphi(u)|^p - |\cos_p \varphi_{ave}(t)|^p] \, du \right| \\ & \leq \frac{L}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1^+ + \frac{|r_2(u)|}{[\log(\log t)]^2} \right) |\varphi(u) - \varphi_{ave}(t)| \, du \\ & \leq \frac{L}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1^+ + \frac{|r_2(u)|}{[\log(\log t)]^2} \right) \frac{1}{\sqrt{t \log t}} \, du < \frac{\varepsilon}{t \log t [\log(\log t)]^2} \end{aligned} \quad (3.18)$$

for sufficiently large t .

Using

$$\lim_{t \rightarrow \infty} t \log t \left(\frac{1}{[\log(\log t)]^2} - \frac{1}{[\log(\log[t + \alpha])]^2} \right) = 0$$

and (2.3), we obtain the estimation

$$\begin{aligned} & \left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{r_2(u)}{[\log(\log t)]^2} \, du - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{r_2(u)}{[\log(\log u)]^2} \, du \right| \\ & \leq \frac{1}{\alpha} \int_t^{t+\alpha} |r_2(u)| \left(\frac{1}{[\log(\log t)]^2} - \frac{1}{[\log(\log[t + \alpha])]^2} \right) \, du \\ & \leq \frac{1}{t \log t} \int_t^{t+\alpha} |r_2(u)| \, du \leq \frac{1}{\sqrt{t \log t}} \end{aligned}$$

for every large t , which gives (consider also (2.12))

$$\begin{aligned}
& \left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi_{ave}(t)|^p du \right. \\
& \quad \left. - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) |\cos_p \varphi_{ave}(t)|^p du \right| \\
& \leq \frac{1}{\alpha t \log t} \int_t^{t+\alpha} \left| \frac{r_2(u)}{[\log(\log u)]^2} - \frac{r_2(u)}{[\log(\log t)]^2} \right| |\cos_p \varphi_{ave}(t)|^p du \\
& \leq \left(\frac{1}{t \log t} \right)^{\frac{3}{2}} < \frac{\varepsilon}{t \log t [\log(\log t)]^2}
\end{aligned} \tag{3.19}$$

for all large t .

Thus (see (3.18) and (3.19)), we have

$$\begin{aligned}
& \left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{u \log u} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p du \right. \\
& \quad \left. - \frac{1}{\alpha t \log t} \int_t^{t+\alpha} \left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) du |\cos_p \varphi_{ave}(t)|^p \right| < \frac{3\varepsilon}{t \log t [\log(\log t)]^2}
\end{aligned} \tag{3.20}$$

for all large t .

Analogously (cf. (2.3) and (2.4)), one can show that

$$\begin{aligned}
& \left| \frac{1}{\alpha(p-1)} \int_t^{t+\alpha} \frac{1}{u \log u} \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) |\sin_p \varphi(u)|^p du \right. \\
& \quad \left. - \frac{1}{\alpha(p-1)t \log t} \int_t^{t+\alpha} \left(s_1(u) + \frac{s_2(u)}{[\log(\log t)]^2} \right) du |\sin_p \varphi_{ave}(t)|^p \right| \\
& < \frac{3\varepsilon}{t \log t [\log(\log t)]^2}
\end{aligned} \tag{3.21}$$

for all large t .

For large t , we have (see (2.12) and (3.15))

$$\begin{aligned}
& \left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) du - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{u \log u} \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) du \right| \\
& \leq \frac{1}{\alpha} \int_t^{t+\alpha} \left| \frac{1}{t \log t} - \frac{1}{u \log u} \right| |\Phi(\cos_p \varphi(u)) \sin_p \varphi(u)| du \\
& \leq \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} - \frac{1}{(t+\alpha) \log(t+\alpha)} du = \frac{1}{t \log t} - \frac{1}{(t+\alpha) \log(t+\alpha)} \leq \frac{2\alpha}{t^2 \log t}
\end{aligned} \tag{3.22}$$

and (see (2.12), (2.17) in Lemma 2.1, and (3.16))

$$\begin{aligned}
& \left| \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) - \frac{1}{\alpha} \int_t^{t+\alpha} \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \, du \right| \\
& \leq \left| \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) - \frac{1}{\alpha} \int_t^{t+\alpha} \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi(u) \, du \right| \\
& \quad + \left| \frac{1}{\alpha} \int_t^{t+\alpha} \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi(u) \, du - \frac{1}{\alpha} \int_t^{t+\alpha} \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \, du \right| \\
& \leq \frac{1}{\alpha} \int_t^{t+\alpha} |\sin_p \varphi_{ave}(t) - \sin_p \varphi(u)| \, du \\
& \quad + \frac{1}{\alpha} \int_t^{t+\alpha} |\Phi(\cos_p \varphi_{ave}(t)) - \Phi(\cos_p \varphi(u))| \, du \\
& \leq \frac{L}{\alpha} \int_t^{t+\alpha} |\varphi_{ave}(t) - \varphi(u)| \, du + \frac{L}{\alpha} \int_t^{t+\alpha} |\varphi_{ave}(t) - \varphi(u)| \, du \leq \frac{1}{\sqrt{t}}.
\end{aligned} \tag{3.23}$$

Hence (see (3.22) and (3.23)), it holds

$$\begin{aligned}
& \left| \frac{1}{t \log t} \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{u \log u} \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \, du \right| \\
& \leq \frac{1}{t \log t} \left| \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) - \frac{1}{\alpha} \int_t^{t+\alpha} \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \, du \right| \\
& \quad + \left| \frac{1}{\alpha t \log t} \int_t^{t+\alpha} \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \, du \right. \\
& \quad \quad \left. - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{u \log u} \Phi(\cos_p \varphi(u)) \sin_p \varphi(u) \, du \right| \\
& \leq \frac{1}{t \log t} \cdot \frac{1}{\sqrt{t}} + \frac{2\alpha}{t^2 \log t} < \frac{\varepsilon}{t \log t [\log(\log t)]^2}
\end{aligned} \tag{3.24}$$

for all large t .

Finally (see (3.14), (3.20), (3.21), and (3.24)), we have

$$\begin{aligned}
& \left| \varphi'_{ave}(t) - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) |\cos_p \varphi_{ave}(t)|^p \right. \right. \\
& \quad \quad \left. \left. - \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) \right. \right. \\
& \quad \quad \left. \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log t)]^2} \right) \frac{|\sin_p \varphi_{ave}(t)|^p}{p-1} \right] \, du \right| \\
& < \frac{7\varepsilon}{t \log t [\log(\log t)]^2}
\end{aligned} \tag{3.25}$$

for any sufficiently large t .

Part (i). Let $\vartheta \in \mathbb{R}_0$ be such that (see (3.11))

$$[M(r_1)]^{p-1} (S - \vartheta) + \frac{p-1}{q^p} \frac{R - \vartheta}{M(r_1)} > \frac{q^{1-p}}{2}. \tag{3.26}$$

We consider $\varepsilon \in \mathbb{R}_0$ such that

$$7\varepsilon < \vartheta, \quad 7\varepsilon(p-1) < \vartheta. \tag{3.27}$$

For large t , we have (see (2.11), (3.10), (3.25), and (3.27))

$$\begin{aligned}
\varphi'_{ave}(t) &> \frac{1}{\alpha t \log t} \int_t^{t+\alpha} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) |\cos_p \varphi_{ave}(t)|^p - \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) \right. \\
&\quad \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log t)]^2} \right) \frac{|\sin_p \varphi_{ave}(t)|^p}{p-1} \right] du - \frac{7\varepsilon}{t \log t [\log(\log t)]^2} \\
&= \frac{1}{\alpha t \log t} \left[\int_t^{t+\alpha} \left(r_1(u) + \frac{r_2(u) - 7\varepsilon}{[\log(\log t)]^2} \right) |\cos_p \varphi_{ave}(t)|^p - \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) \right. \\
&\quad \left. + \left(s_1(u) + \frac{s_2(u) - 7\varepsilon(p-1)}{[\log(\log t)]^2} \right) \frac{|\sin_p \varphi_{ave}(t)|^p}{p-1} du \right] \\
&= \frac{1}{t \log t} \left[\left(M(r_1) + \frac{\frac{1}{\alpha} \int_t^{t+\alpha} r_2(u) du - 7\varepsilon}{[\log(\log t)]^2} \right) |\cos_p \varphi_{ave}(t)|^p - \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) \right. \\
&\quad \left. + \left(M(s_1) + \frac{\frac{1}{\alpha} \int_t^{t+\alpha} s_2(u) du - 7\varepsilon(p-1)}{[\log(\log t)]^2} \right) \frac{|\sin_p \varphi_{ave}(t)|^p}{p-1} \right] \\
&> \frac{1}{t \log t} \left[\left(M(r_1) + \frac{R - \vartheta}{[\log(\log t)]^2} \right) |\cos_p \varphi_{ave}(t)|^p - \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) \right. \\
&\quad \left. + \left(M(s_1) + \frac{S - \vartheta}{[\log(\log t)]^2} \right) \frac{|\sin_p \varphi_{ave}(t)|^p}{p-1} \right].
\end{aligned}$$

It suffices to use Lemma 3.2, (i) (compare (3.4) with (3.26) and Eq. (3.5) with the last estimation for $X = R - \vartheta$, $Y = S - \vartheta$). Since the Prüfer angle φ is unbounded from above (consider (2.17) in Lemma 2.1), Eq. (2.1) is oscillatory. Therefore, the first part of the theorem is proved.

Part (ii). We consider $\vartheta \in \mathbb{R}_0$ such that (see (3.13))

$$[M(r_1)]^{p-1}(S + \vartheta) + \frac{p-1}{q^p} \frac{R + \vartheta}{M(r_1)} < \frac{q^{1-p}}{2} \quad (3.28)$$

and $\varepsilon \in \mathbb{R}_0$ satisfying (3.27). We can proceed analogously as in the first case.

For large t , we have (see (2.11), (3.12), (3.25), and (3.27))

$$\begin{aligned}
\varphi'_{ave}(t) &< \frac{1}{\alpha t \log t} \int_t^{t+\alpha} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) |\cos_p \varphi_{ave}(t)|^p \right. \\
&\quad \left. - \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) \right. \\
&\quad \left. + \left(s_1(u) + \frac{s_2(u)}{[\log(\log t)]^2} \right) \frac{|\sin_p \varphi_{ave}(t)|^p}{p-1} \right] du \\
&\quad + \frac{7\varepsilon}{t \log t [\log(\log t)]^2} \\
&< \frac{1}{t \log t} \left[\left(M(r_1) + \frac{R + \vartheta}{[\log(\log t)]^2} \right) |\cos_p \varphi_{ave}(t)|^p \right. \\
&\quad \left. - \Phi(\cos_p \varphi_{ave}(t)) \sin_p \varphi_{ave}(t) \right. \\
&\quad \left. + \left(M(s_1) + \frac{S + \vartheta}{[\log(\log t)]^2} \right) \frac{|\sin_p \varphi_{ave}(t)|^p}{p-1} \right].
\end{aligned} \quad (3.29)$$

Using Lemma 2.1 and Lemma 3.2, (ii) (cf. (3.6), (3.7) and (3.28), (3.29) for $V = R + \vartheta$, $W = S + \vartheta$), we know that the Prüfer angle is bounded from above, which implies the non-oscillation of Eq. (2.1). The proof is complete. \square

4 Corollaries and examples

In this section, we illustrate the novelty of Theorem 3.3 on corollaries and examples which are not covered by any previously known criteria. As a corollary of Theorem 3.3, we obtain the following new result which enables us to detect the oscillatory behaviour of the non-perturbed equation in the critical case (cf. Theorem 1.3).

Corollary 4.1. *If $r : \mathbb{R} \rightarrow \mathbb{R}_0$ and $s : \mathbb{R} \rightarrow \mathbb{R}$ are continuous α -periodic functions such that $[M(r)]^{p-1} M(s) = q^{-p}$, then the equation*

$$\left[r^{-\frac{p}{q}}(t) t^{p-1} \Phi(x') \right]' + \frac{s(t)}{t \log^p t} \Phi(x) = 0 \quad (4.1)$$

is non-oscillatory.

Proof. It suffices to consider $r_1(t) = r(t)$, $r_2(t) \equiv 0$, $s_1(t) = s(t)$, and $s_2(t) \equiv 0$ in Eq (2.1) and to put $R := 0$ and $S := 0$ in Theorem 3.3, (ii). \square

Example 4.2. We can apply Corollary 4.1, e.g., to the equation

$$\left[\left(\frac{2}{1 + 2 \sin^2 t} \right)^{\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{q^{-p} + p \sin t - q \cos t}{t \log^p t} \Phi(x) = 0 \quad (4.2)$$

which is in the form of Eq. (4.1), where

$$\begin{aligned} M(r) &= M\left(\frac{1 + 2 \sin^2 t}{2}\right) = 1, \\ M(s) &= M(q^{-p} + p \sin t - q \cos t) = q^{-p}. \end{aligned}$$

Since $[M(r)]^{p-1} M(s) = q^{-p}$, Eq. (4.2) is in the critical case which means that it is non-oscillatory.

Now we formulate a direct consequence of Theorems 1.3 and 3.3 and Corollary 4.1 for linear equations.

Corollary 4.3. *Consider the equations*

$$\left[\frac{tx'}{r_1(t)} \right]' + \frac{s_1(t)x}{t \log^2 t} = 0, \quad (4.3)$$

$$\left(\frac{[\log(\log t)]^2 tx'}{r_1(t) [\log(\log t)]^2 + r_2(t)} \right)' + \frac{1}{t \log^2 t} \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2} \right) x = 0 \quad (4.4)$$

with continuous α -periodic coefficients $r_1 : \mathbb{R} \rightarrow \mathbb{R}_0$, $s_1 : \mathbb{R} \rightarrow \mathbb{R}$ and with continuous coefficients $r_2, s_2 : \mathbb{R}_e \rightarrow \mathbb{R}$ satisfying (2.2), (2.3), and (2.4).

(i) *If $4M(r_1)M(s_1) > 1$, then Eq. (4.3) is oscillatory.*

(ii) If $4M(r_1)M(s_1) \leq 1$, then Eq. (4.3) is non-oscillatory.

(iii) If $4M(r_1)M(s_1) = 1$ and if there exist $R, S \in \mathbb{R}$ satisfying

$$\frac{S}{M(s_1)} + \frac{R}{M(r_1)} > 1 \quad \text{and} \quad \frac{1}{\alpha} \int_t^{t+\alpha} r_2(u) \, du \geq R, \quad \frac{1}{\alpha} \int_t^{t+\alpha} s_2(u) \, du \geq S, \quad t \in \mathbb{R}_e,$$

then Eq. (4.4) is oscillatory.

(iv) If $4M(r_1)M(s_1) = 1$ and if there exist $R, S \in \mathbb{R}$ satisfying

$$\frac{S}{M(s_1)} + \frac{R}{M(r_1)} < 1 \quad \text{and} \quad \frac{1}{\alpha} \int_t^{t+\alpha} r_2(u) \, du \leq R, \quad \frac{1}{\alpha} \int_t^{t+\alpha} s_2(u) \, du \leq S, \quad t \in \mathbb{R}_e,$$

then Eq. (4.4) is non-oscillatory.

Example 4.4. Let $a \in \mathbb{R}_1$ and $b, c, d \in \mathbb{R}_0$. From Corollary 4.3, we know that the equation

$$\left[\frac{tx'}{a + \sin(cx)} \right]' + \frac{b + \cos(cx)}{t \log^2 t} x = 0$$

is oscillatory if and only if $4ab > 1$. Note that the case $4ab = 1$ is covered by Corollary 4.1 and the case $4ab \neq 1$ by Theorem 1.3. In addition, applying Corollary 4.3, we know that the equation

$$\left(\frac{tx'}{a + \sin(cx)} \right)' + \frac{1}{t \log^2 t} \left(\frac{1}{4a} + \cos(cx) + \frac{d + \sin(cx) \cos(cx)}{[\log(\log t)]^2} \right) x = 0$$

is oscillatory for $4ad > 1$ and non-oscillatory for $4ad < 1$.

To formulate the next corollary, we recall the definitions of almost periodicity and asymptotic almost periodicity. For more details, we refer to books [3, 14].

Definition 4.5. A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called almost periodic if, for any $\varepsilon \in \mathbb{R}_0$, there exists a number $p(\varepsilon) \in \mathbb{R}_0$ with the property that any real interval of length $p(\varepsilon)$ contains at least one point s for which

$$|f(t+s) - f(t)| < \varepsilon, \quad t \in \mathbb{R}.$$

It is well-known that there exist different (equivalent) ways to define almost periodic functions. The above given definition is the so-called Bohr definition. Another way is given by the Bochner definition which follows.

Definition 4.6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. We say that f is almost periodic if, from any sequence of the form $\{f(t + s_n)\}_{n \in \mathbb{N}}$, where s_n are real numbers, one can extract a subsequence which converges uniformly with respect to $t \in \mathbb{R}$.

We remark that the equivalence of Definitions 4.5 and 4.6 is shown, e.g., in [14, Theorem 1.14]. The notion of asymptotic almost periodicity is a direct generalization of almost periodicity.

Definition 4.7. We say that a continuous function $f : \mathbb{R}_0 \cup \{0\} \rightarrow \mathbb{R}$ is asymptotically almost periodic if it can be represented in the form $f(t) = f_1(t) + f_2(t)$, $t \in \mathbb{R}_0 \cup \{0\}$, where f_1 is almost periodic and f_2 has the property that $\lim_{t \rightarrow \infty} f_2(t) = 0$.

From Definition 4.7, it is seen that (2.3) and (2.4) hold for all asymptotically almost periodic functions r_2, s_2 . At the same time, (2.2) is valid for all large t if r_2 is asymptotically almost periodic. Therefore, we can use Theorem 3.3 for any equation of the form (2.1) with α -periodic coefficients r_1, s_1 and asymptotically almost periodic coefficients r_2, s_2 . To be as clear as possible, we use in Corollary 4.9 and Example 4.10 below the fact, that any asymptotically almost periodic function has its mean value in the sense of the following definition. Note that Theorem 3.3 can be applied also for equations with coefficients which have mean values and which are not asymptotically almost periodic. This situation is closely described and examples of such coefficients are presented, e.g., in [21, 35].

Definition 4.8. Let a continuous function $f : \mathbb{R}_0 \cup \{0\} \rightarrow \mathbb{R}$ be such that the limit

$$M(f) := \lim_{t \rightarrow \infty} \frac{1}{t} \int_a^{a+t} f(s) ds \quad (4.5)$$

is finite and exists uniformly with respect to $a \in \mathbb{R}_0 \cup \{0\}$. The number $M(f)$ is called the mean value of f .

Corollary 4.9. Let $R_1 : \mathbb{R} \rightarrow \mathbb{R}_0$, $S_1 : \mathbb{R} \rightarrow \mathbb{R}$ be continuous α -periodic functions such that $[M(R_1)]^{p-1} M(S_1) = q^{-p}$ and let $R_2, S_2 : \mathbb{R}_0 \cup \{0\} \rightarrow \mathbb{R}$ be asymptotically almost periodic functions.

(i) If

$$[M(R_1)]^{p-1} M(S_2) + \frac{p-1}{q^p} \frac{M(R_2)}{M(R_1)} > \frac{q^{1-p}}{2},$$

then the equation

$$\begin{aligned} & \left[\left(R_1(t) + \frac{R_2(t)}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' \\ & + \frac{1}{t \log^p t} \left(S_1(t) + \frac{S_2(t)}{[\log(\log t)]^2} \right) \Phi(x) = 0 \end{aligned} \quad (4.6)$$

is oscillatory.

(ii) If

$$[M(R_1)]^{p-1} M(S_2) + \frac{p-1}{q^p} \frac{M(R_2)}{M(R_1)} < \frac{q^{1-p}}{2},$$

then Eq. (4.6) is non-oscillatory.

Proof. The corollary follows from Theorem 3.3 as well. It suffices to replace α by $n\alpha$ for a sufficiently large number $n \in \mathbb{N}$ and to use the definition of the mean value given in (4.5) and the existence of $\delta \in \mathbb{R}_0$ with the property that

$$[M(R_1)]^{p-1} \delta + \frac{p-1}{q^p} \frac{\delta}{M(R_1)} < \left| [M(R_1)]^{p-1} M(S_2) + \frac{p-1}{q^p} \frac{M(R_2)}{M(R_1)} - \frac{q^{1-p}}{2} \right|$$

if

$$[M(R_1)]^{p-1} M(S_2) + \frac{p-1}{q^p} \frac{M(R_2)}{M(R_1)} \neq \frac{q^{1-p}}{2}.$$

□

Example 4.10. Let $a, b, c \in \mathbb{R}$ and $u, v \in \mathbb{R} \setminus \{0\}$ determine the coefficients of the equation

$$\left[\left(\frac{3 + \Phi(\sin t)}{3} + \frac{a + \sin(bt) + \sin(ct)}{[\log(\log[t+1])]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{1}{t \log^p t} \left(\frac{2 \sin^2 t}{q^p} + \left[\frac{\sin(ut) \cos(ut) + v + t^{-2}}{\log(\log t)} \right]^2 \right) \Phi(x) = 0 \quad (4.7)$$

which has the form of Eq. (4.6) for

$$R_1(t) = \frac{3 + \Phi(\sin t)}{3}, \quad S_1(t) = \frac{2 \sin^2 t}{q^p},$$

$$R_2(t) = [a + \sin(bt) + \sin(ct)] \left[\frac{\log(\log t)}{\log(\log[t+1])} \right]^2, \quad S_2(t) = \left[\sin(ut) \cos(ut) + v + \frac{1}{t^2} \right]^2.$$

One can verify that R_2, S_2 are asymptotically almost periodic functions and that

$$M(R_1) = 1, \quad M(S_1) = q^{-p}, \quad M(R_2) = a, \quad M(S_2) = \frac{8v^2 + 1}{8}.$$

Especially, $[M(R_1)]^{p-1} M(S_1) = q^{-p}$. Hence, we can apply Corollary 4.9 which gives the oscillation of Eq. (4.7) for

$$\frac{8v^2 + 1}{8} + \frac{a(p-1)}{q^p} > \frac{q^{1-p}}{2}$$

and its non-oscillation for

$$\frac{8v^2 + 1}{8} + \frac{a(p-1)}{q^p} < \frac{q^{1-p}}{2}.$$

In the final corollary and example, we consider Eq. (2.1) with constant coefficients r_1, s_1 and periodic coefficients r_2, s_2 , which do not need to have any common period. We point out that we get a new result even in the case when the periods of r_2, s_2 are same.

Corollary 4.11. Let $a, b \in \mathbb{R}_0$ satisfy $a^{p-1}b = q^{-p}$. Let $R, S : \mathbb{R} \rightarrow \mathbb{R}$ be periodic continuous functions.

(i) If

$$a^{p-1}M(S) + \frac{p-1}{aq^p}M(R) > \frac{q^{1-p}}{2},$$

then the equation

$$\left[\left(a + \frac{R(t)}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{1}{t \log^p t} \left(b + \frac{S(t)}{[\log(\log t)]^2} \right) \Phi(x) = 0 \quad (4.8)$$

is oscillatory.

(ii) If

$$a^{p-1}M(S) + \frac{p-1}{aq^p}M(R) < \frac{q^{1-p}}{2},$$

then Eq. (4.8) is non-oscillatory.

Proof. The corollary is a special case of Corollary 4.9. □

Example 4.12. We illustrate Corollary 4.11 by the equation

$$\left[\left(1 + \frac{c + d \sin t}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{1}{t \log^p t} \left(q^{-p} + \frac{C + D \cos(\sqrt{2}t)}{[\log(\log t)]^2} \right) \Phi(x) = 0, \quad (4.9)$$

where $c, d, C, D \in \mathbb{R}$ are arbitrary constants. For $a := 1$, $b := q^{-p}$, $R(t) := c + d \sin t$, and $S(t) := C + D \cos(\sqrt{2}t)$, we have $a^{p-1}b = q^{-p}$ and $M(R) = c$, $M(S) = C$. Hence, Eq. (4.9) is oscillatory for

$$C + \frac{p-1}{q^p}c > \frac{q^{1-p}}{2}, \quad \text{i.e.,} \quad Cq^p + (p-1)c > \frac{q}{2},$$

and non-oscillatory for

$$C + \frac{p-1}{q^p}c < \frac{q^{1-p}}{2}, \quad \text{i.e.,} \quad Cq^p + (p-1)c < \frac{q}{2}.$$

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