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On conjugacy of second-order half-linear differential equations on the real axis

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Abstract. Some conjugacy criteria are given for the equation

$$\big(|u'|^\alpha \operatorname{sgn} u'\big)' + p(t)|u|^\alpha \operatorname{sgn} u = 0,$$

where $p\colon \mathbb{R} \to \mathbb{R}$ is a locally integrable function and $\alpha > 0$, which generalise and supplement results known in the existing literature. Illustrative examples justifying applicability of the main results are given, as well. The results obtained are new even for linear differential equations, i.e., if $\alpha = 1$.

Keywords: second-order half-linear equation, conjugacy, oscillation.

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1 Introduction

On the real axis, we consider the equation

$$\big(|u'|^\alpha \operatorname{sgn} u'\big)' + p(t)|u|^\alpha \operatorname{sgn} u = 0, \tag{1.1}$$

where $p\colon \mathbb{R} \to \mathbb{R}$ is a locally integrable function and $\alpha > 0$.

A function $u\colon I \to \mathbb{R}$ is said to be *a solution to equation* (1.1) *on the interval* $I \subseteq \mathbb{R}$, if it is continuously differentiable on I, $|u'|^\alpha \operatorname{sgn} u'$ is absolutely continuous on every compact subinterval of I, and u satisfies equality (1.1) almost everywhere on I. In [8, Lemma 2.1], Mirzov proved that every solution to equation (1.1) is extendable to the whole real axis. Therefore, speaking about a solution to equation (1.1), we assume that it is defined on $\mathbb{R}$. Moreover, for any $a \in \mathbb{R}$, the initial value problem

$$\big(|u'|^\alpha \operatorname{sgn} u'\big)' + p(t)|u|^\alpha \operatorname{sgn} u = 0; \quad u(a) = 0,\ u'(a) = 0$$

has only the solution $u \equiv 0$ (see [8, Lemma 1.1]). Hence, a solution u to equation (1.1) is said to be *non-trivial*, if $u \not\equiv 0$ on $\mathbb{R}$.

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Definition 1.1. We say that equation (1.1) is *conjugate on* $\mathbb{R}$ if it has a non-trivial solution with at least two zeros, and *disconjugate on* $\mathbb{R}$ otherwise.

It is clear that in the case $\alpha = 1$, equation (1.1) reduces to the linear equation

$$u'' + p(t)u = 0. \tag{1.2}$$

As it is mentioned in [2], a history of the problem of conjugacy of (1.2) began in the paper by Hawking and Penrose [6]. In [10], Tipler presented an interesting relevance of the study of conjugacy of (1.2) to the general relativity and improved Hawking–Penrose's criterion, showing that (1.2) is conjugate on $\mathbb{R}$ if the inequality

$$\liminf_{\substack{t\to+\infty \\ \tau\to-\infty}} \int_\tau^t p(s)\mathrm{d}s > 0 \tag{1.3}$$

holds. Later, Peña [9] proved that the same condition is sufficient also for the conjugacy of half-linear equation (1.1).

The study of conjugacy of (1.1) on $\mathbb{R}$ is closely related to the question of oscillation of (1.1) on the whole real axis. It is known that Sturm's separation theorem holds for equation (1.1) (see [8, Theorem 1.1]). Therefore, if equation (1.1) possesses a non-trivial solution with a sequence of zeros tending to $+\infty$ (resp. $-\infty$), then any other its non-trivial solution has also a sequence of zeros tending to $+\infty$ (resp. $-\infty$).

Definition 1.2. Equation (1.1) is said to be *oscillatory in the neighbourhood of* $+\infty$ (resp. *in the neighbourhood of* $-\infty$) if every its non-trivial solution has a sequence of zeros tending to $+\infty$ (resp. to $-\infty$). We say that equation (1.1) is *oscillatory on* $\mathbb{R}$ if it is oscillatory in the neighbourhood of either $+\infty$ or $-\infty$, and *non-oscillatory on* $\mathbb{R}$ otherwise.

Clearly, if equation (1.1) is oscillatory on $\mathbb{R}$, then it is conjugate on $\mathbb{R}$, as well. It is well known that oscillations of (1.1) in the neighbourhood of $+\infty$ (resp. $-\infty$) can be described by means of behaviour of the Hartman–Wintner type expression

$$\frac{1}{|t|}\int_0^t \left(\int_0^s p(\xi)\mathrm{d}\xi\right)\mathrm{d}s \tag{1.4}$$

in the neighbourhood of $+\infty$ (resp. $-\infty$), see [7, Theorem 12.3]. However, expression (1.4) is very useful also in the study of conjugacy of (1.1) on $\mathbb{R}$. In particular, efficient conjugacy and disconjugacy criteria for linear equation (1.2) formulated by means of expression (1.4) are given in [2]. Abd-Alla and Abu-Risha [1] observed that for the study of conjugacy on whole real axis, it is more convenient to consider a Hartman–Wintner type expression in a certain symmetric form, where all values of the function p are involved simultaneously. They proved in [1], among other things, that equation (1.1) with a continuous p is conjugate on $\mathbb{R}$ provided that $p \not\equiv 0$ and

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t \left(\int_{-s}^s p(\xi)\mathrm{d}\xi\right)\mathrm{d}s \geq 0, \tag{1.5}$$

which obviously improves Peña's criterion (1.3). In the present paper, we generalise and supplement criterion (1.5) (see Theorems 2.1 and 2.3 below), and we establish further statements, which can be applied in the cases not covered by Theorems 2.1 and 2.3 (see Subsections 2.1 and 2.2). Moreover, we provide illustrative examples justifying the meaningfulness of the results obtained (see Section 3). In Sections 4 and 5, we establish auxiliary statements and prove the main results in detail.

2 Main results

For any $\nu < 1$, we put

$$c(t;\nu) := \frac{1-\nu}{(1+t)^{1-\nu}} \int_0^t \frac{1}{(1+s)^{\nu}} \left(\int_{-s}^{s} p(\xi) \mathrm{d}\xi \right) \mathrm{d}s \quad \text{for } t \geq 0. \tag{2.1}$$

We start with a Hartman–Wintner type result, which guarantees that equation (1.1) is oscillatory on $\mathbb{R}$ (not only conjugate).

Theorem 2.1. *Let $\nu < 1$ be such that either*

$$\lim_{t \to +\infty} c(t;\nu) = +\infty, \tag{2.2}$$

or

$$-\infty < \liminf_{t \to +\infty} c(t;\nu) < \limsup_{t \to +\infty} c(t;\nu). \tag{2.3}$$

Then equation (1.1) is oscillatory on $\mathbb{R}$ and consequently, conjugate on $\mathbb{R}$.

Remark 2.2. Using integration by parts, it is easy to verify that for any $\nu_1, \nu_2 < 1$, we have

$$c(t;\nu_2) = \frac{1-\nu_2}{1-\nu_1} c(t;\nu_1) + \frac{\nu_2 - \nu_1}{1-\nu_1} \frac{1-\nu_2}{(1+t)^{1-\nu_2}} \int_0^t \frac{1}{(1+s)^{\nu_2}} c(s;\nu_1) \mathrm{d}s \quad \text{for } t \geq 0,$$

whence we get the following assertions.

(i) There exists a finite $\lim_{t \to +\infty} c(t;\nu_2)$ if and only if there exists a finite $\lim_{t \to +\infty} c(t;\nu_1)$, in which case both limits are equal.

(ii) If $\nu_2 > \nu_1$, then

$$\liminf_{t \to +\infty} c(t;\nu_2) \geq \liminf_{t \to +\infty} c(t;\nu_1), \qquad \limsup_{t \to +\infty} c(t;\nu_2) \leq \limsup_{t \to +\infty} c(t;\nu_1).$$

In view of Remark 2.2 (i), Theorem 2.1 cannot be applied, in particular, if the function $c(\cdot\,;1-\alpha)$ has a finite limit as $t \to +\infty$. The following statement provides a conjugacy criterion covering this case.

Theorem 2.3. *Let $p \not\equiv 0$ and*

$$0 \leq \lim_{t \to +\infty} c(t;1-\alpha) < +\infty. \tag{2.4}$$

Then equation (1.1) is conjugate on $\mathbb{R}$.

Theorems 2.1 and 2.3 yield

Corollary 2.4. *Let $p \not\equiv 0$ and $\nu < 1$ be such that*

$$\liminf_{t \to +\infty} c(t;\nu) > -\infty \tag{2.5}$$

and

$$\limsup_{t \to +\infty} c(t;\nu) \geq 0. \tag{2.6}$$

Then equation (1.1) is conjugate on $\mathbb{R}$.

Corollary 2.4 generalises several conjugacy criteria known in the existing literature. In particular, [4, Theorem 2.2] can be derived from Corollary 2.4. Moreover, conjugacy criterion (1.5) given in [1, Theorem 2.2] follows immediately from Corollary 2.4 with $\nu := 0$. Corollary 2.4 also yields the following half-linear extension of [2, Theorem 1].

Corollary 2.5. *Let $p \not\equiv 0$ and the function*

$$M\colon t \mapsto \frac{1}{|t|}\int_0^t \left(\int_0^s p(\xi)\mathrm{d}\xi\right)\mathrm{d}s$$

have finite limits as $t \to \pm\infty$. If

$$\lim_{t\to+\infty} M(t) + \lim_{t\to-\infty} M(t) \geq 0, \tag{2.7}$$

then equation (1.1) is conjugate on $\mathbb{R}$.

According to the above said, we conclude that neither of Theorems 2.1 and 2.3 can be applied in the following two cases:

$$\lim_{t\to+\infty} c(t;1-\alpha) =: c(+\infty) \in\,]-\infty, 0[\tag{2.8}$$

and

$$\liminf_{t\to+\infty} c(t;\nu) = -\infty \quad \text{for every } \nu < 1. \tag{2.9}$$

In Subsections 2.1 and 2.2 below, we provide some conjugacy criteria in both cases (2.8) and (2.9). It is worthwhile mentioning here that the results obtained therein are new even for linear equation (1.2), i.e., if $\alpha = 1$.

2.1 The case (2.8)

In the first statement, we require that the function $c(\cdot\,;1-\alpha)$ is at some point far enough from its limit $c(+\infty)$.

Theorem 2.6. *Let (2.8) hold and*

$$\sup\left\{\frac{(1+t)^{\alpha}}{\ln(1+t)}\big[c(+\infty) - c(t;1-\alpha)\big] : t > 0\right\} > 2\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}. \tag{2.10}$$

Then equation (1.1) is conjugate on $\mathbb{R}$.

Remark 2.7. It follows from the proof of Theorem 2.6, Proposition 4.8, and Lemma 4.12 that if (2.10) is replaced by

$$\limsup_{t\to+\infty}\frac{(1+t)^{\alpha}}{\ln(1+t)}\big[c(+\infty) - c(t;1-\alpha)\big] > 2\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}, \tag{2.11}$$

then we can claim in Theorem 2.6 that equation (1.1) is even oscillatory on $\mathbb{R}$.

Now we put

$$Q_\alpha(t) := \frac{(1+t)^{1+\alpha}}{t}\left[c(+\infty) - \int_{-t}^{t} p(s)\mathrm{d}s\right] \quad \text{for } t > 0 \tag{2.12}$$

and

$$H_\alpha(t) := \frac{1}{t}\int_{-t}^{t}(1+|s|)^{1+\alpha}p(s)\mathrm{d}s \quad \text{for } t > 0. \tag{2.13}$$

Theorem 2.8. *Let* (2.8) *hold and*

$$\sup\left\{Q_\alpha(t)+H_\alpha(t):t>0\right\}>2. \tag{2.14}$$

Then equation (1.1) *is conjugate on* $\mathbb{R}$.

Remark 2.9. It follows from the proof of Theorem 2.8, Proposition 4.8, and Lemma 4.13 that if

$$\limsup_{t\to+\infty}\Big(Q_\alpha(t)+H_\alpha(t)\Big)>2,$$

then we can claim in Theorem 2.8 that equation (1.1) is even oscillatory on $\mathbb{R}$.

2.2 The case (2.9)

First observe that, in condition (2.9), the assumption that $\liminf_{t\to+\infty}c(t;\nu)=-\infty$ for **every** $\nu<1$ is, in fact, not too restrictive. Indeed, let $\liminf_{t\to+\infty}c(t;\nu_1)=-\infty$ for some $\nu_1<1$. Then Remark 2.2(i) yields that for any $\nu<1$, the function $c(\cdot\,;\nu)$ does not possesses any finite limit. Consequently, if there exists $\nu_2<1$ such that $\liminf_{t\to+\infty}c(t;\nu_2)>-\infty$, then equation (1.1) is oscillatory on $\mathbb{R}$ as follows from Theorem 2.1.

Proposition 2.10. *Let condition* (2.9) *hold and there exist a number* $\kappa>\alpha$ *such that*

$$\limsup_{t\to+\infty}\frac{1}{t^\kappa}\int_{-t}^{t}(t-|s|)^\kappa p(s)\mathrm{d}s>-\infty. \tag{2.15}$$

Then equation (1.1) *is oscillatory on* $\mathbb{R}$ *and consequently, conjugate on* $\mathbb{R}$.

Finally, we provide a statement which can be applied in the case, when condition (2.9) holds, but (2.15) is violated for every $\kappa>\alpha$, i.e.,

$$\lim_{t\to+\infty}\frac{1}{t^\kappa}\int_{-t}^{t}(t-|s|)^\kappa p(s)\mathrm{d}s=-\infty\quad\text{for every }\kappa>\alpha$$

(it may happen as it is shown in Example 3.6).

Theorem 2.11. *Let there exist a number* $\kappa>\alpha$ *such that*

$$\sup\left\{\frac{1}{t^{\kappa-\alpha}}\int_{-t}^{t}(t-|s|)^\kappa p(s)\mathrm{d}s:t>0\right\}>\frac{2}{\kappa-\alpha}\left(\frac{\kappa}{1+\alpha}\right)^{1+\alpha}. \tag{2.16}$$

Then equation (1.1) *is conjugate on* $\mathbb{R}$.

Remark 2.12. Observe that Theorem 2.11 does not require assumption (2.9), it is a general statement applicable without regard to behaviour of the function $c(\cdot\,;\nu)$.

3 Illustrative examples

In this section, we give three illustrative examples justifying meaningfulness of Theorems 2.1 and 2.3, as well as results presented in Subsections 2.1 and 2.2. In Example 3.1, Theorem 2.1 is applied (see Proposition 3.2) to get oscillation of a given equation. For an equation constructed in Example 3.3 with $m:=0$, Theorem 2.1 cannot be applied but Theorem 2.3 yields its conjugacy (see Proposition 3.4). Further, Example 3.3 with $m:=1$ and Example 3.6 justify meaningfulness of results stated in Subsections 2.1 and 2.2. Namely, in Example 3.3 with $m:=1$, Theorem 2.6 is applied (see Proposition 3.5) and Example 3.6 gives an example of equation (1.1) for which Proposition 2.10 (resp. Theorem 2.11) can be used (see Proposition 3.7, resp. Proposition 3.8) in the case, when $\alpha<1$ (resp. $\alpha\geq1$).

Example 3.1. Let $p\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable function such that

$$p(t) + p(-t) = 2\sin(2t) + 2(1+t)\cos(2t) \quad \text{for a.e. } t \geq 0, \tag{3.1}$$

e.g.,

$$p(t) := \sin(2|t|) + (1+|t|)\cos(2t) \quad \text{for } t \in \mathbb{R}.$$

Then it is clear that

$$\int_{-t}^{t} p(s)\mathrm{d}s = \sin^2 t + (1+t)\sin(2t) \quad \text{for } t \geq 0$$

and

$$\int_0^t \left(\int_{-s}^{s} p(\xi)\mathrm{d}\xi \right) \mathrm{d}s = (1+t)\sin^2 t \quad \text{for } t \geq 0.$$

Therefore, we have $c(t;0) = \sin^2 t$ for $t \geq 0$, which leads to

$$\liminf_{t\to+\infty} c(t;0) = 0, \qquad \limsup_{t\to+\infty} c(t;0) = 1.$$

Consequently, Theorem 2.1 with $\nu := 0$ yields the following statement.

Proposition 3.2. *Equation* (1.1) *with p satisfying* (3.1) *is oscillatory on $\mathbb{R}$ and consequently, conjugate on $\mathbb{R}$.*

Example 3.3. Put

$$f(t) := \begin{cases} 2t^2 & \text{for } t \in [0,1[\,, \\ 6t^3 - 29t^2 + 44t - 19 & \text{for } t \in [1,2[\,, \\ 1 & \text{for } t \in [2,+\infty[\,. \end{cases}$$

Let $m \in \{0,1\}$ and $p\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable function such that

$$p(t) + p(-t) = (-1)^m \Big[2f'(t) + (1+t)f''(t) \Big] \quad \text{for a.e. } t \geq 0, \tag{3.2}$$

e.g.,

$$p(t) := (-1)^m \begin{cases} 6|t| + 2 & \text{for } t \in\,]-1,1[\,, \\ 36t^2 - 69|t| + 15 & \text{for } t \in\,]-2,-1] \cup [1,2[\,, \\ \left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \frac{\operatorname{sgn} t}{|t|^{1+\alpha}} & \text{for } t \in\,]-\infty,-2] \cup [2,+\infty[\,. \end{cases}$$

Then it is clear that

$$\int_{-t}^{t} p(s)\mathrm{d}s = (-1)^m \Big[f(t) + (1+t)f'(t) \Big] = (-1)^m \frac{\mathrm{d}}{\mathrm{d}t}(1+t)f(t) \quad \text{for } t \geq 0,$$

whence we get

$$c(t;1-\alpha) = (-1)^m \left[\alpha f(t) - \frac{\alpha(\alpha-1)}{(1+t)^\alpha} \int_0^t (1+s)^{\alpha-1} f(s)\mathrm{d}s \right] \quad \text{for } t \geq 0. \tag{3.3}$$

Consequently, we have

$$\lim_{t\to+\infty} c(t;1-\alpha) = (-1)^m \tag{3.4}$$

and thus, Theorem 2.1 cannot be applied.

However, if $m = 0$, Theorem 2.3 yields the following statement.

Proposition 3.4. *Let $m = 0$. Then equation (1.1) with p satisfying relation (3.2) is conjugate on $\mathbb{R}$.*

Now assume that $m = 1$. It follows from (3.4) that neither of Theorems 2.1 and 2.3 can be applied. Put

$$\ell(t) := \frac{(1+t)^\alpha}{\ln(1+t)}\left[-1+\alpha f(t) - \frac{\alpha(\alpha-1)}{(1+t)^\alpha}\int_0^t (1+s)^{\alpha-1} f(s)\mathrm{d}s\right] \quad \text{for } t > 0.$$

Observe that

$$\ell(t) = \frac{\alpha-1}{\ln(1+t)}\left[3^\alpha - \alpha\int_0^2 (1+s)^{\alpha-1} f(s)\mathrm{d}s\right] \quad \text{for } t \geq 2$$

and thus, we have

$$\lim_{t\to+\infty} \frac{(1+t)^\alpha}{\ln(1+t)}\Big[-1-c(t;1-\alpha)\Big] = 0,$$

i.e., condition (2.11) is violated. On the other hand, we have

$$\begin{aligned}\ell(1) &= \frac{2^\alpha}{\ln 2}\left[-1+2\alpha - \frac{\alpha(\alpha-1)}{2^{\alpha-1}}\int_0^1 (1+s)^{\alpha-1}s^2\mathrm{d}s\right] \\ &\geq \frac{2^\alpha}{\ln 2}\left[-1+2\alpha - \alpha(\alpha-1)\int_0^1 s^2\mathrm{d}s\right] \\ &= \frac{2^\alpha}{\ln 2}\left[-1+\frac{\alpha(7-\alpha)}{3}\right].\end{aligned}$$

Therefore, if $\alpha < 7$ and

$$\frac{3}{\alpha(7-\alpha)}\left[\frac{\ln 2}{2^{\alpha-1}}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} + 1\right] < 1 \tag{3.5}$$

(for example, if $\alpha \in \left[\frac{3}{5}, \frac{32}{5}\right]$), then

$$\sup\left\{\frac{(1+t)^\alpha}{\ln(1+t)}\big[-1-c(t;1-\alpha)\big] : t > 0\right\} > 2\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}.$$

Consequently, Theorem 2.6 yields the following statement.

Proposition 3.5. *Let $m = 1$ and $\alpha < 7$ be such that condition (3.5) holds. Then equation (1.1) with p satisfying (3.2) is conjugate on $\mathbb{R}$.*

We conclude this example by the following remark. As we have mentioned above, condition (2.11) is not fulfilled. Therefore, we cannot claim in Proposition 3.5 that equation (1.1) with p satisfying (3.2) is oscillatory on $\mathbb{R}$ (see Remark 2.7).

Example 3.6. Let $p\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable function such that

$$\begin{aligned}p(t) + p(-t) = &-12(t-\pi)\sin^2 t \\ &-12(t-\pi)^2\sin(2t) - 4(t-\pi)^3\cos(2t) \quad \text{for a.e. } t \geq 0,\end{aligned} \tag{3.6}$$

e.g.,

$$p(t) := -6(|t|-\pi)\sin^2 t - 6(|t|-\pi)^2\sin(2|t|) - 2(|t|-\pi)^3\cos(2t) \quad \text{for } t \in \mathbb{R}.$$

Then it is clear that

$$\int_{-t}^t p(s)\mathrm{d}s = -6(t-\pi)^2\sin^2 t - 2(t-\pi)^3\sin(2t) \quad \text{for } t \geq 0$$

and

$$\int_0^t \left(\int_{-s}^{s} p(\xi) \mathrm{d}\xi \right) \mathrm{d}s = -2(t-\pi)^3 \sin^2 t \quad \text{for } t \geq 0.$$

Therefore, for any $\nu < 1$ we get

$$\begin{aligned} \int_0^t \frac{1}{(1+s)^\nu} \left(\int_{-s}^{s} p(\xi) \mathrm{d}\xi \right) \mathrm{d}s &= \int_0^t \frac{1}{(1+s)^\nu} \left[\int_0^s \left(\int_{-\eta}^{\eta} p(\xi) \mathrm{d}\xi \right) \mathrm{d}\eta \right]' \mathrm{d}s \\ &= -\frac{2(t-\pi)^3}{(1+t)^\nu} \sin^2 t + 2\nu \int_0^\pi \frac{(\pi-s)^3}{(1+s)^{1+\nu}} \sin^2 s \, \mathrm{d}s \\ &\quad - 2\nu \int_\pi^t \frac{(s-\pi)^3}{(1+s)^{1+\nu}} \sin^2 s \, \mathrm{d}s \quad \text{for } t \geq \pi, \end{aligned}$$

which yields that

$$\begin{aligned} c(t;\nu) = &-\frac{2(1-\nu)(t-\pi)^3}{1+t} \sin^2 t + \frac{2\nu(1-\nu)}{(1+t)^{1-\nu}} \int_0^\pi \frac{(\pi-s)^3}{(1+s)^{1+\nu}} \sin^2 s \, \mathrm{d}s \\ &- \frac{2\nu(1-\nu)}{(1+t)^{1-\nu}} \int_\pi^t \frac{(s-\pi)^3}{(1+s)^{1+\nu}} \sin^2 s \, \mathrm{d}s \quad \text{for } t \geq \pi. \end{aligned}$$

We first show that

$$\liminf_{t \to +\infty} c(t;\nu) = -\infty \quad \text{for } \nu < 1. \tag{3.7}$$

(i) Let $\nu \in [0,1[$. Then

$$\begin{aligned} \liminf_{t \to +\infty} c(t;\nu) &\leq \liminf_{t \to +\infty} \left(-\frac{2(1-\nu)(t-\pi)^3}{1+t} \sin^2 t + \frac{2\nu(1-\nu)}{(1+t)^{1-\nu}} \int_0^\pi \frac{(\pi-s)^3}{(1+s)^{1+\nu}} \sin^2 s \, \mathrm{d}s \right) \\ &= -\limsup_{t \to +\infty} \frac{2(1-\nu)(t-\pi)^3}{1+t} \sin^2 t = -\infty. \end{aligned}$$

(ii) Let $\nu < 0$. Then it follows from Remark 2.2 (ii) that

$$\liminf_{t \to +\infty} c(t;\nu) \leq \liminf_{t \to +\infty} c(t;0) = -\infty.$$

Consequently, condition (3.7) holds and thus, neither of Theorems 2.1 and 2.3 can be applied. For any $\kappa > \alpha$ we put

$$\ell(t;\kappa) := \frac{1}{t^\kappa} \int_{-t}^{t} (t-|s|)^\kappa p(s) \mathrm{d}s \quad \text{for } t > 0.$$

Observe that

$$\ell(t;1) = \frac{1}{t} \int_0^t (t-s) \left(\int_{-s}^{s} p(\xi) \mathrm{d}\xi \right)' \mathrm{d}s = \frac{1+t}{t} c(t;0) \quad \text{for } t > 0$$

whence we get

$$\limsup_{t \to +\infty} \ell(t;1) = \limsup_{t \to +\infty} \left(-\frac{2(t-\pi)^3}{t} \sin^2 t \right) = 0.$$

Therefore, Proposition 2.10 with $\kappa := 1$ yields the following proposition.

Proposition 3.7. *Let $\alpha \in]0,1[$. Then equation (1.1) with p satisfying relation (3.6) is oscillatory on $\mathbb{R}$ and consequently, conjugate on $\mathbb{R}$.*

Now we show that

$$\lim_{t\to+\infty} \ell(t;\kappa) = -\infty \quad \text{for } \kappa > 1. \tag{3.8}$$

(a) Let $\kappa \in]1,2]$. Then

$$\begin{aligned}\ell(t;\kappa) &= \frac{1}{t^\kappa}\int_0^t (t-s)^\kappa \left(\int_{-s}^s p(\xi)\mathrm{d}\xi\right)' \mathrm{d}s \\ &= \frac{\kappa(\kappa-1)}{t^\kappa}\int_0^t (t-s)^{\kappa-2}\left[\int_0^s \left(\int_{-\eta}^{\eta} p(\xi)\mathrm{d}\xi\right)\mathrm{d}\eta\right]\mathrm{d}s \\ &\le -\frac{2\kappa(\kappa-1)}{t^2}\int_\pi^t (s-\pi)^3 \sin^2 s\mathrm{d}s + \frac{2\kappa\pi^3}{t} \quad \text{for } t \ge \pi.\end{aligned}$$

However, by direct calculation, one can verify that

$$\int_\pi^t (s-\pi)^3 \sin^2 s\mathrm{d}s \ge \frac{(t-\pi)^4}{8} - \frac{(t-\pi)^3}{2} \quad \text{for } t \ge \pi$$

and thus, we have

$$\ell(t;\kappa) \le -\frac{\kappa(\kappa-1)}{4}\frac{(t-\pi)^3}{t^2}(t-\pi-4) + \frac{2\kappa\pi^3}{t} \quad \text{for } t \ge \pi.$$

Consequently,

$$\lim_{t\to+\infty} \ell(t;\kappa) = -\infty \quad \text{for } \kappa \in]1,2]. \tag{3.9}$$

Observe that

$$\ell(t;2) = \frac{2}{t^2}\int_0^t \left[\int_0^s \left(\int_{-\eta}^{\eta} p(\xi)\mathrm{d}\xi\right)\mathrm{d}\eta\right]\mathrm{d}s \quad \text{for } t \ge 0.$$

Let $M > 0$ be arbitrary. In view of (3.9), there exists $t_0 > \pi$ such that

$$\ell(t;2) \le -M \quad \text{for } t \ge t_0. \tag{3.10}$$

(b) Let $\kappa \in]2,3]$. Then, using previous calculations and (3.10), one gets

$$\begin{aligned}\ell(t;\kappa) &= \frac{\kappa(\kappa-1)}{t^\kappa}\int_0^t (t-s)^{\kappa-2}\left(\frac{s^2}{2}\ell(s;2)\right)'\mathrm{d}s \\ &= \frac{\kappa(\kappa-1)(\kappa-2)}{2t^\kappa}\int_0^t (t-s)^{\kappa-3}s^2\ell(s;2)\mathrm{d}s \\ &\le -\frac{M\kappa(\kappa-1)(\kappa-2)}{2t^\kappa}\int_{t_0}^t (t-s)^{\kappa-3}s^2\mathrm{d}s \\ &\quad + \frac{\kappa(\kappa-1)(\kappa-2)}{2t^\kappa}(t-t_0)^{\kappa-3}\int_0^{t_0} s^2|\ell(s;2)|\mathrm{d}s \\ &\le -\frac{M\kappa(\kappa-1)(\kappa-2)}{3!} + \frac{M\kappa(\kappa-1)(\kappa-2)}{3!}\left(\frac{t_0}{t}\right)^3 \\ &\quad + \frac{\kappa(\kappa-1)(\kappa-2)}{2(t-t_0)^3}\int_0^{t_0} s^2|\ell(s;2)|\mathrm{d}s \quad \text{for } t > t_0,\end{aligned}$$

which yields that

$$\limsup_{t\to+\infty} \ell(t;\kappa) \le -M\,\frac{\kappa(\kappa-1)(\kappa-2)}{3!}\,.$$

Since $M>0$ was arbitrary, from the latter inequality we get

$$\lim_{t\to+\infty} \ell(t;\kappa) = -\infty \quad \text{for } \kappa\in\,]2,3]. \tag{3.11}$$

(c) Let $\kappa\in\,]n-1,n]$, where $n\in\{4,5,\dots\}$. Then, using previous calculations and (3.10), one gets

$$\begin{aligned}\ell(t;\kappa) \le &-\frac{M\kappa(\kappa-1)(\kappa-2)}{2t^{\kappa}}\int_{t_0}^{t}(t-s)^{\kappa-3}s^2\mathrm{d}s\\ &+\frac{\kappa(\kappa-1)(\kappa-2)}{2t^3}\int_0^{t_0} s^2|\ell(s;2)|\mathrm{d}s \quad \text{for } t\ge t_0.\end{aligned}$$

If we integrate by parts the first term on the right-hand side of the latter inequality, for any $t\ge t_0$ we obtain

$$\begin{aligned}\ell(t;\kappa) \le &-M\binom{\kappa}{n}\frac{n}{t^{\kappa}}\int_{t_0}^{t}(t-s)^{\kappa-n}s^{n-1}\mathrm{d}s\\ &+M\sum_{m=3}^{n-1}\binom{\kappa}{m}\left(\frac{t_0}{t}\right)^m+\frac{\kappa(\kappa-1)(\kappa-2)}{2t^3}\int_0^{t_0} s^2|\ell(s;2)|\mathrm{d}s\\ \le &-M\binom{\kappa}{n}\frac{n}{t^{n}}\int_{t_0}^{t}s^{n-1}\mathrm{d}s\\ &+M\sum_{m=3}^{n-1}\binom{\kappa}{m}\left(\frac{t_0}{t}\right)^m+\frac{\kappa(\kappa-1)(\kappa-2)}{2t^3}\int_0^{t_0} s^2|\ell(s;2)|\mathrm{d}s\\ = &-M\binom{\kappa}{n}+M\sum_{m=3}^{n}\binom{\kappa}{m}\left(\frac{t_0}{t}\right)^m+\frac{\kappa(\kappa-1)(\kappa-2)}{2t^3}\int_0^{t_0} s^2|\ell(s;2)|\mathrm{d}s\end{aligned}$$

whence we get

$$\limsup_{t\to+\infty} \ell(t;\kappa) \le -M\binom{\kappa}{n}.$$

Since $M>0$ was arbitrary, from the latter inequality we get

$$\lim_{t\to+\infty} \ell(t;\kappa) = -\infty \quad \text{for } \kappa>3. \tag{3.12}$$

Therefore, it follows from (3.9), (3.11), and (3.12) that condition (3.8) holds and thus, Proposition 2.10 cannot be applied if $\alpha\ge 1$.

On the other hand, we have

$$\begin{aligned}\ell(\pi;1+\alpha) &= \frac{\alpha(\alpha+1)}{\pi^{\alpha+1}}\int_0^{\pi}(\pi-s)^{\alpha-1}\left[\int_0^{s}\left(\int_{-\eta}^{\eta}p(\xi)\mathrm{d}\xi\right)\mathrm{d}\eta\right]\mathrm{d}s\\ &= \frac{2\alpha(\alpha+1)}{\pi^{\alpha+1}}\int_0^{\pi}(\pi-s)^{\alpha+2}\sin^2 s\mathrm{d}s.\end{aligned}$$

Assuming $\alpha \geq 1$, the latter integral can be estimated from below as follows

$$\begin{aligned}
2\int_0^{\pi}(\pi - s)^{\alpha+2}\sin^2 s\mathrm{d}s &= \int_0^{\pi}(\pi - s)^{\alpha+2}\big(1-\cos(2s)\big)\mathrm{d}s \\
&= \frac{\pi^{\alpha+3}}{\alpha+3} - \frac{(\alpha+2)\pi^{\alpha+1}}{4} \\
&\quad + \frac{\alpha(\alpha+1)(\alpha+2)}{8}\int_0^{\pi}(\pi - s)^{\alpha-1}\sin(2s)\mathrm{d}s \\
&> \frac{\pi^{\alpha+3}}{\alpha+3} - \frac{(\alpha+2)\pi^{\alpha+1}}{4} \\
&\quad - \frac{\alpha(\alpha+1)(\alpha+2)}{8}\int_0^{\pi}(\pi - s)^{\alpha-1}\mathrm{d}s \\
&= \frac{\pi^{\alpha}\left[8\pi^3 - 2(\alpha+2)(\alpha+3)\pi - (\alpha+1)(\alpha+2)(\alpha+3)\right]}{8(\alpha+3)}.
\end{aligned}$$

Hence, for any $\alpha \geq 1$, we have

$$\begin{aligned}
\sup\left\{\frac{1}{t}\int_{-t}^{t}\big(t-|s|\big)^{1+\alpha}p(s)\mathrm{d}s : t>0\right\} &\geq \pi^{\alpha}\ell(\pi;1+\alpha) \\
&> \frac{\alpha(\alpha+1)\pi^{\alpha-1}}{8(\alpha+3)}\Big[8\pi^3 - 2(\alpha+2)(\alpha+3)\pi - (\alpha+1)(\alpha+2)(\alpha+3)\Big].
\end{aligned}$$

Therefore, if $\alpha \geq 1$ and

$$\frac{16(\alpha+3)}{\alpha(\alpha+1)\pi^{\alpha-1}} + 2(\alpha+2)(\alpha+3)\pi + (\alpha+1)(\alpha+2)(\alpha+3) \leq 8\pi^3 \tag{3.13}$$

(for example, if $\alpha \in \left[1, \frac{5}{2}\right]$), then

$$\sup\left\{\frac{1}{t}\int_{-t}^{t}(t-|s|)^{1+\alpha}p(s)\mathrm{d}s : t>0\right\} > 2.$$

Consequently, Theorem 2.11 with $\kappa := 1+\alpha$ yields the following statement.

Proposition 3.8. *Let $\alpha \geq 1$ be such that condition (3.13) holds. Then equation (1.1) with p satisfying (3.6) is conjugate on $\mathbb{R}$.*

4 Auxiliary statements

Lemma 4.1 ([5, Theorem 16])**.** *If $r \geq 1$ then*

$$\left(\frac{a+b}{2}\right)^r \leq \frac{a^r+b^r}{2} \quad \textit{for } a,b \geq 0.$$

Lemma 4.2 ([3, Lemma 3.1])**.** *Let $\alpha > 0$ and $\omega \geq 0$. Then*

$$\omega|z| - \alpha|z|^{\frac{1+\alpha}{\alpha}} \leq \left(\frac{\omega}{1+\alpha}\right)^{1+\alpha} \quad \textit{for } z \in \mathbb{R}.$$

Lemma 4.3. *Let $a \in \mathbb{R}$, $\tau > a$, and u_1, u_2 be solutions to equation (1.1) satisfying the inequalities*

$$u_1(t) > 0, \quad u_2(t) > 0 \quad \text{for } t \in [a, \tau[\tag{4.1}$$

and

$$u_1(a) = u_2(a), \qquad u_1'(a) \geq u_2'(a). \tag{4.2}$$

Then

$$u_1(t) \geq u_2(t) \quad \text{for } t \in [a, \tau]. \tag{4.3}$$

Proof. Assume on the contrary that inequality (4.3) is violated. Then there exist $a_0 \in [a, \tau[$ and $a_1 \in]a_0, \tau[$ such that

$$u_1(t) < u_2(t) \quad \text{for } t \in]a_0, a_1[\tag{4.4}$$

and

$$u_1(a_0) = u_2(a_0). \tag{4.5}$$

It is clear that

$$u_1'(a_0) \leq u_2'(a_0). \tag{4.6}$$

Put

$$\sigma_k(t) := \frac{|u_k'(t)|^\alpha \operatorname{sgn} u_k'(t)}{u_k^\alpha(t)} \quad \text{for } t \in [a, \tau[\,,\ k \in \{1, 2\}.$$

Then the functions σ_1, σ_2 are absolutely continuous on every compact subinterval of $[a, \tau[$ and it follows from (1.1) that

$$\sigma_k'(t) = -p(t) - \alpha |\sigma_k(t)|^{\frac{1+\alpha}{\alpha}} \quad \text{for a.e. } t \in [a, \tau[\,,\ k = 1, 2. \tag{4.7}$$

Let

$$w(t) := \sigma_1(t) - \sigma_2(t) \quad \text{for } t \in [a, \tau[$$

and

$$\varphi(t) := f(\sigma_1(t), \sigma_2(t)) \quad \text{for } t \in [a, \tau[\,,$$

where

$$f(x, y) := \begin{cases} -\frac{\alpha}{x-y}\left(|x|^{\frac{1+\alpha}{\alpha}} - |y|^{\frac{1+\alpha}{\alpha}}\right) & \text{for } x, y \in \mathbb{R},\ x \neq y, \\ -(1+\alpha)|x|^{\frac{1}{\alpha}} \operatorname{sgn} x & \text{for } x, y \in \mathbb{R},\ x = y. \end{cases} \tag{4.8}$$

It is not difficult to verify that $f \colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and thus, the function φ is continuous on $[a, \tau[$. It follows from (4.7) that

$$w'(t) = \varphi(t) w(t) \quad \text{for a.e. } t \in [a, \tau]. \tag{4.9}$$

Moreover, in view of (4.2), (4.5), and (4.6), we get

$$w(a) \geq 0, \qquad w(a_0) \leq 0. \tag{4.10}$$

Therefore, conditions (4.9) and (4.10) yields that $w(t) = 0$ for $t \in [a, \tau[$. Consequently, we have $\sigma_1(t) = \sigma_2(t)$ for $t \in [a, \tau[$ which yields that

$$\frac{u_1'(t)}{u_1(t)} = \frac{u_2'(t)}{u_2(t)} \quad \text{for } t \in [a, \tau[\,.$$

Since $u_1(a) = u_2(a)$, the integration of the latter equality over the interval $[a, t]$ leads to the relation $u_1(t) = u_2(t)$ for $t \in [a, \tau[$ that contradicts (4.4). □

Analysis similar to that in the proof of Lemma 4.3 shows that the following statement holds.

Lemma 4.4. *Let $a \in \mathbb{R}$, $\tau < a$, and u_1, u_2 be solutions to equation (1.1) satisfying the inequalities*

$$u_1(t) > 0, \quad u_2(t) > 0 \quad \textit{for } t \in \,]\tau, a]$$

and

$$u_1(a) = u_2(a), \qquad u_1'(a) \le u_2'(a).$$

Then

$$u_1(t) \ge u_2(t) \quad \textit{for } t \in [\tau, a].$$

Lemma 4.5. *Let equation (1.1) be disconjugate on $\mathbb{R}$. Then for any $a \in \mathbb{R}$ and $b > a$, there exists a solution u to equation (1.1) such that*

$$\begin{gathered} u(t)\operatorname{sgn}(t-a) > 0 \quad \textit{for } t \in \mathbb{R} \setminus \{a\}, \quad u(a) = 0, \quad u(b) = 1 \\ \Big(\textit{resp.} \quad u(t)\operatorname{sgn}(t-b) < 0 \quad \textit{for } t \in \mathbb{R} \setminus \{b\}, \quad u(a) = 1, \quad u(b) = 0\Big). \end{gathered} \tag{4.11}$$

Proof. Let $a \in \mathbb{R}$ and $b > a$ be arbitrary and w be a solution to equation (1.1) satisfying the initial conditions

$$w(a) = 0, \quad w'(a) = 1 \qquad \Big(\text{resp.} \quad w(b) = 0, \quad w'(b) = -1\Big).$$

Since equation (1.1) is disconjugate on $\mathbb{R}$, we have

$$\begin{gathered} w(t)\operatorname{sgn}(t-a) > 0 \quad \text{for } t \in \mathbb{R} \setminus \{a\} \\ \Big(\text{resp.} \quad w(t)\operatorname{sgn}(t-b) < 0 \quad \text{for } t \in \mathbb{R} \setminus \{b\}\Big). \end{gathered}$$

Therefore, the function u defined by the formula

$$u(t) := \frac{w(t)}{w(b)} \quad \text{for } t \in \mathbb{R} \qquad \left(\text{resp.} \quad u(t) := \frac{w(t)}{w(a)} \quad \text{for } t \in \mathbb{R}\right)$$

is a solution to equation (1.1) satisfying desired conditions (4.11). □

Proposition 4.6. *If equation (1.1) is disconjugate on $\mathbb{R}$, then it has a solution u such that*

$$u(t) > 0 \quad \textit{for } t \in \mathbb{R}. \tag{4.12}$$

Proof. Assume that equation (1.1) is disconjugate on $\mathbb{R}$. By virtue of Lemma 4.5, for any $n \in \mathbb{N}$, there are solutions u_n, z_n to equation (1.1) such that

$$u_n(-n) = 0, \quad u_n(0) = 1, \quad z_n(0) = 1, \quad z_n(n) = 0, \tag{4.13}$$

$$u_n(t)\operatorname{sgn}(t+n) > 0 \quad \text{for } t \in \mathbb{R} \setminus \{-n\}, \tag{4.14}$$

$$z_n(t)\operatorname{sgn}(t-n) < 0 \quad \text{for } t \in \mathbb{R} \setminus \{n\}. \tag{4.15}$$

We first show that

$$u_n'(0) > z_k'(0) \quad \text{for } n, k \in \mathbb{N}. \tag{4.16}$$

Indeed, assuming $u_n'(0) \le z_k'(0)$ for some $n, k \in \mathbb{N}$, it follows from Lemma 4.3 that $u_n(t) \le z_k(t)$ for $t \in [0, k]$. Consequently, we get $u_n(k) \le z_k(k) = 0$, which contradicts (4.14). Similarly, using Lemma 4.4, one can show that

$$u_n'(0) < u_k'(0) \quad \text{for } n, k \in \mathbb{N},\ n > k. \tag{4.17}$$

Inequalities (4.16) and (4.17) immediately yield that

$$|u_n'(0)| \le c_0 \quad \text{for } n \in \mathbb{N}, \tag{4.18}$$

where $c_0 := |u_1'(0)| + |z_1'(0)|$. Moreover, taking inequalities (4.16) and (4.17) into account, it follows from Lemmas 4.3 and 4.4 that

$$u_n(t) \ge u_k(t) \quad \text{for } t \in [-k, 0],\ n, k \in \mathbb{N},\ n > k, \tag{4.19}$$
$$u_n(t) \le u_k(t) \quad \text{for } t \ge 0,\ n, k \in \mathbb{N},\ n > k, \tag{4.20}$$
$$u_n(t) \ge z_k(t) \quad \text{for } t \in [0, k],\ n, k \in \mathbb{N}, \tag{4.21}$$
$$u_n(t) \le z_k(t) \quad \text{for } t \in [-n, 0],\ n, k \in \mathbb{N}. \tag{4.22}$$

In particular, we have

$$0 \le u_n(t) \le w(t) \quad \text{for } t \ge -n,\ n \in \mathbb{N}, \tag{4.23}$$

where

$$w(t) := \begin{cases} u_1(t) & \text{for } t \ge 0, \\ z_1(t) & \text{for } t < 0. \end{cases}$$

Now we put

$$h_n(t) := |u_n'(t)|^\alpha \operatorname{sgn} u_n'(t) \quad \text{for } t \in \mathbb{R},\ n \in \mathbb{N}. \tag{4.24}$$

In view of (4.18) and (4.23), from (1.1) we get

$$|h_n(t) - h_n(s)| \le \left| \int_s^t |p(\xi)| w^\alpha(\xi) \mathrm{d}\xi \right| \quad \text{for } s, t \in [-n, +\infty[,\ n \in \mathbb{N} \tag{4.25}$$

and

$$|h_n(t)| \le |h_n(0)| + \left| \int_0^t |p(s)| w^\alpha(s) \mathrm{d}s \right| \le \varphi(t) \quad \text{for } t \ge -n,\ n \in \mathbb{N}, \tag{4.26}$$

where

$$\varphi(t) := c_0^\alpha + \left| \int_0^t |p(s)| w^\alpha(s) \mathrm{d}s \right| \quad \text{for } t \ge -n,\ n \in \mathbb{N}.$$

Moreover, by virtue of (4.26) equality (4.24) yields that

$$\begin{aligned} |u_n(t) - u_n(s)| &= \left| \int_s^t |h_n(\xi)|^{\frac{1}{\alpha}} \operatorname{sgn} h_n(\xi) \mathrm{d}\xi \right| \\ &\le \left| \int_s^t |\varphi(\xi)|^{\frac{1}{\alpha}} \mathrm{d}\xi \right| \quad \text{for } s, t \in [-n, +\infty[,\ n \in \mathbb{N}. \end{aligned} \tag{4.27}$$

Therefore, it follows from estimates (4.23), (4.25), (4.26), and (4.27) that the sequences $\{u_n\}_{n=1}^{+\infty}$ and $\{h_n\}_{n=1}^{+\infty}$ are uniformly bounded and equicontinuous on every compact subinterval of $\mathbb{R}$. Consequently, by virtue of the Arzelà–Ascoli lemma, we can assume without loss of generality that

$$\lim_{n \to +\infty} u_n(t) = u(t), \quad \lim_{n \to +\infty} h_n(t) = h(t) \quad \text{uniformly in}^* \ \mathbb{R}, \tag{4.28}$$

*It means uniformly on every compact subinterval of $\mathbb{R}$.

where $u, h\colon \mathbb{R} \to \mathbb{R}$ are continuous functions.

Now we show that u is a solution to equation (1.1). Indeed, (1.1) yields that

$$h_n(t) = h_n(0) - \int_0^t p(s)|u_n(s)|^\alpha \operatorname{sgn} u_n(s) \mathrm{d}s \quad \text{for } t \in \mathbb{R},\ n \in \mathbb{N}.$$

Letting $n \to +\infty$ in the latter equality and taking (4.28) into account, one gets

$$h(t) = h(0) - \int_0^t p(s)|u(s)|^\alpha \operatorname{sgn} u(s) \mathrm{d}s \quad \text{for } t \in \mathbb{R}.$$

Consequently, the function h is absolutely continuous on every compact subinterval of $\mathbb{R}$ and

$$h'(t) = -p(t)|u(t)|^\alpha \operatorname{sgn} u(t) \quad \text{for a.e. } t \in \mathbb{R}. \tag{4.29}$$

On the other hand, it follows from (4.24) that

$$u_n(t) = u_n(0) + \int_0^t |h_n(s)|^{\frac{1}{\alpha}} \operatorname{sgn} h_n(s) \mathrm{d}s \quad \text{for } t \in \mathbb{R},\ n \in \mathbb{N}.$$

Letting $n \to +\infty$ in the latter equality and taking (4.28) into account, one gets

$$u(t) = u(0) + \int_0^t |h(s)|^{\frac{1}{\alpha}} \operatorname{sgn} h(s) \mathrm{d}s \quad \text{for } t \in \mathbb{R}.$$

Therefore, the function u is continuously differentiable on $\mathbb{R}$ and

$$u'(t) = |h(t)|^{\frac{1}{\alpha}} \operatorname{sgn} h(t) \quad \text{for } t \in \mathbb{R},$$

which yields that

$$h(t) = |u'(t)|^\alpha \operatorname{sgn} u'(t) \quad \text{for } t \in \mathbb{R}. \tag{4.30}$$

However, it means that the function $|u'|^\alpha \operatorname{sgn} u'$ is absolutely continuous on every compact subinterval of $\mathbb{R}$ (because h has this property) and, in view of (4.29) and (4.30), u is a solution to equation (1.1).

It remains to show that the function u is positive. Letting $n \to +\infty$ in inequalities (4.19), (4.21) and taking (4.28) into account, we get

$$\begin{aligned} u(t) &\geq u_k(t) \quad \text{for } t \in [-k, 0],\ k \in \mathbb{N}, \\ u(t) &\geq z_k(t) \quad \text{for } t \in [0, k],\ k \in \mathbb{N}. \end{aligned} \tag{4.31}$$

Since the functions u_k and z_k satisfy (4.14) and (4.15), inequalities (4.31) guarantee that desired condition (4.12) is fulfilled. □

Proposition 4.7. *Let $p \not\equiv 0$ and equation (1.1) be disconjugate on $\mathbb{R}$. Then the equation*

$$\left(|v'|^\alpha \operatorname{sgn} v'\right)' + \frac{1}{2}\big(p(t) + p(-t)\big)|v|^\alpha \operatorname{sgn} v = 0 \tag{4.32}$$

possesses a solution v such that

$$v(0) = 1, \qquad v'(0) = 0, \tag{4.33}$$

$$v(t) > 0 \quad \text{for } t \geq 0, \tag{4.34}$$

$$v'(t_0) \neq 0 \quad \text{for some } t_0 > 0. \tag{4.35}$$

Proof. Let u be a positive solution to equation (1.1) whose existence follows from Proposition 4.6. Put

$$\varrho(t) := \frac{1}{2}\left[\frac{|u'(t)|^{\alpha}\operatorname{sgn} u'(t)}{u^{\alpha}(t)} - \frac{|u'(-t)|^{\alpha}\operatorname{sgn} u'(-t)}{u^{\alpha}(-t)}\right] \quad \text{for } t \in \mathbb{R}. \tag{4.36}$$

It is clear that the function ϱ is absolutely continuous on every compact subinterval of $\mathbb{R}$. Hence, in view of (1.1), we get

$$\begin{aligned}\varrho'(t) &= \frac{1}{2}\left[-p(t) - \alpha\left|\frac{u'(t)}{u(t)}\right|^{1+\alpha} - p(-t) - \alpha\left|\frac{u'(-t)}{u(-t)}\right|^{1+\alpha}\right] \\ &= -\frac{1}{2}\big(p(t)+p(-t)\big) - \frac{\alpha}{2}\left[\left(\left|\frac{u'(t)}{u(t)}\right|^{\alpha}\right)^{\frac{1+\alpha}{\alpha}} + \left(\left|\frac{u'(-t)}{u(-t)}\right|^{\alpha}\right)^{\frac{1+\alpha}{\alpha}}\right]\end{aligned} \tag{4.37}$$

for a. e. $t \in \mathbb{R}$. Therefore, Lemma 4.1 with $r := \frac{1+\alpha}{\alpha}$ yields that

$$\begin{aligned}\varrho'(t) &\le -\frac{1}{2}\big(p(t)+p(-t)\big) - \alpha\left[\frac{1}{2}\left(\left|\frac{u'(t)}{u(t)}\right|^{\alpha} + \left|\frac{u'(-t)}{u(-t)}\right|^{\alpha}\right)\right]^{\frac{1+\alpha}{\alpha}} \\ &\le -\frac{1}{2}\big(p(t)+p(-t)\big) - \alpha|\varrho(t)|^{\frac{1+\alpha}{\alpha}} \quad \text{for a.e. } t \in \mathbb{R}.\end{aligned} \tag{4.38}$$

On the other hand, problem (4.32), (4.33) has a solution v on the interval $[0,+\infty[$. Put

$$v(t) := v(-t) \quad \text{for } t < 0. \tag{4.39}$$

Then it is clear that the function v is a solution to equation (4.32) on $\mathbb{R}$ satisfying conditions (4.33).

Now we show that v satisfies also inequality (4.34). Indeed, assume on the contrary that (4.34) is violated. Then, by virtue of (4.33) and (4.39), there exists $t_0 > 0$ such that

$$v(t) > 0 \quad \text{for } t \in\,]-t_0, t_0[\,, \quad v(-t_0) = 0, \quad v(t_0) = 0. \tag{4.40}$$

Since for any $t^* \in \mathbb{R}$, the problem

$$\big(|v'|^{\alpha}\operatorname{sgn} v'\big)' + \frac{1}{2}\big(p(t)+p(-t)\big)|v|^{\alpha}\operatorname{sgn} v = 0; \quad v(t^*) = 0,\ v'(t^*) = 0 \tag{4.41}$$

has only the trivial solution (see [8, Lemma 1.1]), we have

$$v'(-t_0) > 0, \qquad v'(t_0) < 0. \tag{4.42}$$

Let

$$\sigma(t) := \frac{|v'(t)|^{\alpha}\operatorname{sgn} v'(t)}{v^{\alpha}(t)} \quad \text{for } t \in\,]-t_0, t_0[\,.$$

It is clear that the function σ is absolutely continuous on each compact subinterval of $]-t_0, t_0[$. Hence, in view of (4.32), we get

$$\sigma'(t) = -\frac{1}{2}\big(p(t)+p(-t)\big) - \alpha|\sigma(t)|^{\frac{1+\alpha}{\alpha}} \quad \text{for a.e. } t \in [-t_0, t_0]. \tag{4.43}$$

Moreover, (4.40) and (4.42) imply

$$\lim_{t\to -t_0+} \sigma(t) = +\infty, \qquad \lim_{t\to t_0-} \sigma(t) = -\infty. \tag{4.44}$$

Therefore, there exists $t_1 \in]-t_0, t_0[$ such that

$$\varrho(t) > \sigma(t) \quad \text{for } t \in]t_1, t_0[, \qquad \varrho(t_1) = \sigma(t_1) \tag{4.45}$$

Put

$$w(t) := \varrho(t) - \sigma(t) \quad \text{for } t \in [t_1, t_0[$$

and

$$\varphi(t) := f(\varrho(t), \sigma(t)) \quad \text{for } t \in [t_1, t_0[,$$

where the function f is defined by formula (4.8). It is not difficult to verify that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and thus, the function φ is continuous on $[t_1, t_0[$. In view of (4.45), it follows from (4.38) and (4.43) that

$$w'(t) \le \varphi(t) w(t) \quad \text{for a.e. } t \in [t_1, t_0], \qquad w(t_1) = 0.$$

Consequently, we get $w(t) \le 0$ for $t \in [t_1, t_0[$, which is in a contradiction with (4.45).

It remains to show that the solution v satisfies condition (4.35). Assume on the contrary that $v'(t) = 0$ for $t \ge 0$. Then equation (4.32) immediately yields that

$$p(t) + p(-t) = 0 \quad \text{for a.e. } t \in \mathbb{R} \tag{4.46}$$

and thus, from inequality (4.38) we get

$$\varrho'(t) \le -\alpha |\varrho(t)|^{\frac{1+\alpha}{\alpha}} \quad \text{for a.e. } t \in \mathbb{R}. \tag{4.47}$$

It is clear that $\varrho \not\equiv 0$ because otherwise it follows from (4.37) and (4.46) that $u' \equiv 0$ on $\mathbb{R}$, which together with (4.12) leads to the contradiction $p \equiv 0$. Since $\varrho(t) = -\varrho(-t)$ for $t \in \mathbb{R}$, inequality (4.47) yields that there exists $t_2 < 0$ such that

$$\varrho(t) > 0 \quad \text{for } t \le t_2. \tag{4.48}$$

Integrating inequality (4.47) over the interval $[t, t_2]$ and taking (4.48) into account, one gets

$$\varrho^{-\frac{1}{\alpha}}(t_2) > \varrho^{-\frac{1}{\alpha}}(t_2) - \varrho^{-\frac{1}{\alpha}}(t) \ge t_2 - t \quad \text{for } t \le t_2,$$

i.e.,

$$\varrho^{\frac{1}{\alpha}}(t_2) < \frac{1}{t_2 - t} \quad \text{for } t < t_2.$$

Passing to the limit $t \to -\infty$ in the latter inequality, we obtain $\varrho^{1/\alpha}(t_2) \le 0$ which is in a contradiction with inequality (4.48). □

Proposition 4.8. *If equation* (1.1) *is non-oscillatory on* $\mathbb{R}$*, then equation* (4.32) *is not oscillatory in the neighbourhood of* $+\infty$*.*

Proof. Assume that equation (1.1) is non-oscillatory on $\mathbb{R}$. Then there exists a solution u to equation (1.1) such that $u(t) \ne 0$ for $|t| \ge t_u$ with $t_u \ge 0$. Put

$$\varrho(t) := \frac{1}{2}\left[\frac{|u'(t)|^\alpha \operatorname{sgn} u'(t)}{u^\alpha(t)} - \frac{|u'(-t)|^\alpha \operatorname{sgn} u'(-t)}{u^\alpha(-t)}\right] \quad \text{for } t \ge t_u.$$

It is clear that the function ϱ is absolutely continuous on every compact subinterval of $[t_u, +\infty[$. Steps analogous to those in the proof of Proposition 4.7 shows that

$$\varrho'(t) \le -\frac{1}{2}\big(p(t) + p(-t)\big) - \alpha |\varrho(t)|^{\frac{1+\alpha}{\alpha}} \quad \text{for a.e. } t \ge t_u. \tag{4.49}$$

Assume on the contrary that equation (4.32) is oscillatory in the neighbourhood of $+\infty$. Then there exist a solution v to equation (4.32), $a > t_u$, and $t_0 > a$ such that

$$v(t) > 0 \quad \text{for } t \in]a, t_0[\,, \quad v(a) = 0, \quad v(t_0) = 0. \tag{4.50}$$

Since for any $t^* \in \mathbb{R}$, problem (4.41) has only the trivial solution (see [8, Lemma 1.1]), we have

$$v'(a) > 0, \qquad v'(t_0) < 0. \tag{4.51}$$

Let

$$\sigma(t) := \frac{|v'(t)|^{\alpha} \operatorname{sgn} v'(t)}{v^{\alpha}(t)} \quad \text{for } t \in]a, t_0[\,.$$

It is clear that the function σ is absolutely continuous on every compact subinterval of $]a, t_0[$ and, in view of (4.32), we get

$$\sigma'(t) = -\frac{1}{2}\big(p(t) + p(-t)\big) - \alpha|\sigma(t)|^{\frac{1+\alpha}{\alpha}} \quad \text{for a.e. } t \in [a, t_0]. \tag{4.52}$$

Moreover, (4.50) and (4.51) imply

$$\lim_{t \to a+} \sigma(t) = +\infty, \qquad \lim_{t \to t_0-} \sigma(t) = -\infty. \tag{4.53}$$

Therefore, there exists $t_1 \in]a, t_0[$ such that (4.45) holds. However, analogously to the proof of Proposition 4.7 we show that $\varrho(t) - \sigma(t) \le 0$ for $t \in [t_1, t_0[$, which is in a contradiction with (4.45). □

Lemma 4.9. *Let $\beta > 0$, $\kappa > \alpha$, and v be a solution to equation (4.32) such that*

$$v(t) \neq 0 \quad \text{for } t \ge t_v \tag{4.54}$$

with $t_v \ge 0$. If

$$\limsup_{t \to +\infty} \frac{1}{(1+t)^{\kappa\beta}} \int_{-t}^{t} \big[(1+t)^{\beta} - (1+|s|)^{\beta}\big]^{\kappa} p(s)\mathrm{d}s > -\infty, \tag{4.55}$$

then

$$\int_{t_v}^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}} \mathrm{d}s < +\infty, \tag{4.56}$$

where

$$\varrho(t) := \left|\frac{v'(t)}{v(t)}\right|^{\alpha} \operatorname{sgn} \frac{v'(t)}{v(t)} \quad \text{for } t \ge t_v. \tag{4.57}$$

Proof. The function ϱ is absolutely continuous on every compact subinterval of $[t_v, +\infty[$ and, in view of (4.32), relation (4.57) yields that

$$\varrho'(t) = -\frac{1}{2}\big(p(t) + p(-t)\big) - \alpha|\varrho(t)|^{\frac{1+\alpha}{\alpha}} \quad \text{for a.e. } t \ge t_v. \tag{4.58}$$

Put

$$f(t,s) := (1+t)^{\beta} - (1+s)^{\beta} \quad \text{for } t \ge s \ge t_v. \tag{4.59}$$

Then it follows from equality (4.58) that

$$\begin{aligned} \int_{t_v}^{t} f^{\kappa}(t,s)\varrho'(s)\mathrm{d}s = &-\frac{1}{2}\int_{t_v}^{t} f^{\kappa}(t,s)\big(p(s) + p(-s)\big)\mathrm{d}s \\ &- \alpha \int_{t_v}^{t} f^{\kappa}(t,s)|\varrho(s)|^{\frac{1+\alpha}{\alpha}} \mathrm{d}s \quad \text{for } t \ge t_v. \end{aligned} \tag{4.60}$$

Observe that for any $t > t_v$ and $\zeta > 0$,

$$\text{the function } s \mapsto f^{\zeta}(t,s) \text{ is absolutely continuous on } [t_v, t]$$

and thus, we obtain

$$\int_{t_v}^{t} f^{\kappa}(t,s)\varrho'(s)\mathrm{d}s = -f^{\kappa}(t,t_v)\varrho(t_v) + \kappa\beta \int_{t_v}^{t} f^{\kappa-1}(t,s)(1+s)^{\beta-1}\varrho(s)\mathrm{d}s$$

for $t \geq t_v$. Therefore, from equality (4.60) we get

$$\begin{aligned}
&\int_{t_v}^{t} f^{\kappa}(t,s)\big(p(s)+p(-s)\big)\mathrm{d}s \\
&\quad = 2f^{\kappa}(t,t_v)\varrho(t_v) - \alpha\int_{t_v}^{t} f^{\kappa}(t,s)|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \\
&\qquad + \int_{t_v}^{t} f^{\kappa-1-\alpha}(t,s)(1+s)^{(1+\alpha)(\beta-1)} \\
&\qquad \times \left[-2\kappa\beta(1+s)^{\alpha(1-\beta)}f^{\alpha}(t,s)\varrho(s) - \alpha\big|(1+s)^{\alpha(1-\beta)}f^{\alpha}(t,s)\varrho(s)\big|^{\frac{1+\alpha}{\alpha}}\right]\mathrm{d}s
\end{aligned}$$

for $t \geq t_v$ which, by virtue of Lemma 4.2, yields that

$$\begin{aligned}
\int_{-t}^{t} f^{\kappa}(t,|s|)p(s)\mathrm{d}s \leq 2f^{\kappa}(t,t_v)\varrho(t_v) &+ \int_{-t_v}^{t_v} f^{\kappa}(t,|s|)p(s)\mathrm{d}s \\
&- \alpha\int_{t_v}^{t} f^{\kappa}(t,s)|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \\
&+ \left(\frac{2\kappa\beta}{1+\alpha}\right)^{1+\alpha}\int_{t_v}^{t} f^{\kappa-1-\alpha}(t,s)(1+s)^{(1+\alpha)(\beta-1)}\mathrm{d}s
\end{aligned} \tag{4.61}$$

for $t \geq t_v$. Now observe that

$$\frac{1}{(1+t)^{\kappa\beta}} f^{\kappa}(t,t_v)\varrho(t_v) \leq |\varrho(t_v)| \quad \text{for } t \geq t_v, \tag{4.62}$$

$$\frac{1}{(1+t)^{\kappa\beta}}\int_{-t_v}^{t_v} f^{\kappa}(t,|s|)p(s)\mathrm{d}s \leq \int_{-t_v}^{t_v}|p(s)|\mathrm{d}s \quad \text{for } t \geq t_v, \tag{4.63}$$

and

$$\frac{1}{(1+t)^{\kappa\beta}}\int_{t_v}^{t} f^{\kappa-1-\alpha}(t,s)(1+s)^{(1+\alpha)(\beta-1)}\mathrm{d}s \leq \frac{(1+t_v)^{-\alpha}}{(\kappa-\alpha)\beta} \quad \text{for } t \geq t_v, \tag{4.64}$$

because:

- If $\beta \leq 1$, then

$$\begin{aligned}
&\frac{1}{(1+t)^{\kappa\beta}}\int_{t_v}^{t} f^{\kappa-1-\alpha}(t,s)(1+s)^{\beta-1}\frac{1}{(1+s)^{\alpha(1-\beta)}}\,\mathrm{d}s \\
&\quad \leq \frac{1}{(1+t)^{\kappa\beta}(1+t_v)^{\alpha(1-\beta)}}\int_{t_v}^{t}\mathrm{d}_s\left(\frac{-1}{(\kappa-\alpha)\beta}\left[(1+t)^{\beta}-(1+s)^{\beta}\right]^{\kappa-\alpha}\right) \\
&\quad \leq \frac{1}{(\kappa-\alpha)\beta}\,\frac{1}{(1+t_v)^{\alpha}} \quad \text{for } t \geq t_v.
\end{aligned}$$

- If $\beta > 1$, then

$$\begin{aligned}\frac{1}{(1+t)^{\kappa\beta}} \int_{t_v}^{t} f^{\kappa-1-\alpha}(t,s)(1+s)^{\beta-1}(1+s)^{\alpha(\beta-1)}\mathrm{d}s \\ \leq \frac{(1+t)^{\alpha(\beta-1)}}{(1+t)^{\kappa\beta}} \int_{t_v}^{t} \mathrm{d}_s \left(\frac{-1}{(\kappa-\alpha)\beta} \left[(1+t)^{\beta} - (1+s)^{\beta}\right]^{\kappa-\alpha} \right) \\ \leq \frac{1}{(\kappa-\alpha)\beta} \frac{1}{(1+t_v)^{\alpha}} \quad \text{for } t \geq t_v.\end{aligned}$$

Put

$$F(t,s) := \left[1 - \left(\frac{1+s}{1+t}\right)^{\beta}\right]^{\kappa} \quad \text{for } t \geq s \geq t_v.$$

Then, by using relations (4.62), (4.63), and (4.64), from inequality (4.61) we get

$$\frac{1}{(1+t)^{\kappa\beta}} \int_{-t}^{t} f^{\kappa}(t,|s|)p(s)\mathrm{d}s \leq -\alpha \int_{t_v}^{t} F(t,s)|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s + \delta(t_v) \tag{4.65}$$

for $t \geq t_v$, where

$$\delta(t_v) := 2|\varrho(t_v)| + \int_{-t_v}^{t_v} |p(s)|\mathrm{d}s + \left(\frac{2\kappa\beta}{1+\alpha}\right)^{1+\alpha} \frac{1}{(\kappa-\alpha)\beta} \frac{1}{(1+t_v)^{\alpha}}.$$

Assume on the contrary that inequality (4.56) is violated, i.e.,

$$\int_{t_v}^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s = +\infty. \tag{4.66}$$

It is clear that for any $\tau \geq t_v$, we have

$$\int_{t_v}^{t} F(t,s)|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \geq F(t,\tau) \int_{t_v}^{\tau} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } t \geq \tau$$

and thus,

$$\liminf_{t\to+\infty} \int_{t_v}^{t} F(t,s)|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \geq \int_{t_v}^{\tau} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } \tau \geq t_v.$$

Since we suppose that equality (4.66) is satisfied, the last relation guarantees that

$$\lim_{t\to+\infty} \int_{t_v}^{t} F(t,s)|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s = +\infty.$$

Consequently, inequality (4.65) yields

$$\lim_{t\to+\infty} \frac{1}{(1+t)^{\kappa\beta}} \int_{-t}^{t} f^{\kappa}(t,|s|)p(s)\mathrm{d}s = -\infty$$

which, in view of notation (4.59), contradicts assumption (4.55). □

Lemma 4.10. *Let v be a solution to equation (4.32) fulfilling relation (4.54) with $t_v \geq 0$ and inequality (4.56) hold, where the function ϱ is defined by formula (4.57). Then the function $c(\cdot\,;1-\alpha)$ has a finite limit*

$$c(+\infty) := \lim_{t\to+\infty} c(t;1-\alpha). \tag{4.67}$$

Moreover, the equalities

$$\varrho(t) = \frac{c(+\infty)}{2} - \frac{1}{2}\int_{-t}^{t} p(s)\mathrm{d}s + \alpha \int_{t}^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \textit{for } t \geq t_v \tag{4.68}$$

and

$$c(+\infty) = 2\varrho(t_v) + \int_{-t_v}^{t_v} p(s)\mathrm{d}s - 2\alpha \int_{t_v}^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \tag{4.69}$$

are satisfied.

Proof. The function ϱ is absolutely continuous on every compact subinterval of $[t_v, +\infty[$ and, in view of (4.32), relation (4.57) yields that equality (4.58) holds, whence we obtain

$$\varrho(t) = \varrho(t_v) - \frac{1}{2}\int_{t_v}^{t} (p(s) + p(-s))\mathrm{d}s - \alpha \int_{t_v}^{t} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } t \geq t_v.$$

Therefore,

$$\varrho(t) = \delta(t_v) - \frac{1}{2}\int_{-t}^{t} p(s)\mathrm{d}s + \alpha \int_{t}^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } t \geq t_v, \tag{4.70}$$

where

$$\delta(t_v) := \varrho(t_v) + \frac{1}{2}\int_{-t_v}^{t_v} p(s)\mathrm{d}s - \alpha \int_{t_v}^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s. \tag{4.71}$$

Now it follows from equality (4.70) that

$$\begin{aligned}\int_{t_v}^{t} \frac{1}{(1+s)^{1-\alpha}}\left(\int_{-s}^{s} p(\xi)\mathrm{d}\xi\right)\mathrm{d}s = {} & \frac{2\delta(t_v)}{\alpha}\left[(1+t)^{\alpha} - (1+t_v)^{\alpha}\right] \\ & + 2\alpha \int_{t_v}^{t} \frac{1}{(1+s)^{1-\alpha}}\left(\int_{s}^{+\infty} |\varrho(\xi)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}\xi\right)\mathrm{d}s \\ & - 2\int_{t_v}^{t} \frac{1}{(1+s)^{1-\alpha}}\varrho(s)\mathrm{d}s \quad \text{for } t \geq t_v.\end{aligned} \tag{4.72}$$

Observe that for any $\tau \geq t_v$, we have

$$\begin{aligned}\int_{t_v}^{t} \frac{1}{(1+s)^{1-\alpha}}\left(\int_{s}^{+\infty} |\varrho(\xi)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}\xi\right)\mathrm{d}s & \leq \int_{t_v}^{\tau} \frac{1}{(1+s)^{1-\alpha}}\left(\int_{s}^{+\infty} |\varrho(\xi)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}\xi\right)\mathrm{d}s \\ & \quad + \int_{\tau}^{t} \frac{1}{(1+s)^{1-\alpha}}\mathrm{d}s \int_{\tau}^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \\ & \leq \int_{t_v}^{\tau} \frac{1}{(1+s)^{1-\alpha}}\left(\int_{s}^{+\infty} |\varrho(\xi)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}\xi\right)\mathrm{d}s \\ & \quad + \frac{(1+t)^{\alpha}}{\alpha}\int_{\tau}^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\end{aligned}$$

for $t \geq \tau$. Therefore,

$$\limsup_{t\to+\infty} \frac{\alpha}{(1+t)^{\alpha}}\int_{t_v}^{t} \frac{1}{(1+s)^{1-\alpha}}\left(\int_{s}^{+\infty} |\varrho(\xi)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}\xi\right)\mathrm{d}s \leq \int_{\tau}^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s$$

for $\tau \geq t_v$ and thus, we have

$$\lim_{t\to+\infty} \frac{\alpha}{(1+t)^{\alpha}}\int_{t_v}^{t} \frac{1}{(1+s)^{1-\alpha}}\left(\int_{s}^{+\infty} |\varrho(\xi)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}\xi\right)\mathrm{d}s = 0. \tag{4.73}$$

Furthermore, by using Hölder's inequality, we get

$$\begin{aligned}\left|\int_{t_v}^{t}\frac{1}{(1+s)^{1-\alpha}}\varrho(s)\mathrm{d}s\right| &\le \int_{t_v}^{t}(1+s)^{\alpha-1}|\varrho(s)|\mathrm{d}s\\ &\le \left(\int_{t_v}^{t}(1+s)^{\alpha^2-1}\mathrm{d}s\right)^{\frac{1}{1+\alpha}}\left(\int_{t_v}^{t}|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\right)^{\frac{\alpha}{1+\alpha}}\\ &\le \left(\frac{1}{\alpha^2}\right)^{\frac{1}{1+\alpha}}(1+t)^{\frac{\alpha^2}{1+\alpha}}\left(\int_{t_v}^{+\infty}|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\right)^{\frac{\alpha}{1+\alpha}}\end{aligned}$$

for $t \ge t_v$ which yields that

$$\lim_{t\to+\infty}\frac{\alpha}{(1+t)^{\alpha}}\int_{t_v}^{t}\frac{1}{(1+s)^{1-\alpha}}\varrho(s)\mathrm{d}s = 0. \tag{4.74}$$

Consequently, by virtue of relations (4.73) and (4.74), from equality (4.72) we obtain

$$\lim_{t\to+\infty}\frac{\alpha}{(1+t)^{\alpha}}\int_{t_v}^{t}\frac{1}{(1+s)^{1-\alpha}}\left(\int_{-s}^{s}p(\xi)\mathrm{d}\xi\right)\mathrm{d}s = 2\delta(t_v).$$

Therefore, in view of (2.1) and (4.71), the function $c(\cdot\,;1-\alpha)$ has a finite limit (4.67) and $c(+\infty)$ satisfies (4.69). To finish the proof it is sufficient to mention that desired equality (4.68) now follows from (4.70), (4.71), and the above-proved equality (4.69). □

Lemma 4.11. *Let $\nu < 1$. If inequality (2.5) holds, then there exists $\kappa > \alpha$ such that*

$$\liminf_{t\to+\infty}\frac{1}{(1+t)^{\kappa(1-\nu)}}\int_{-t}^{t}\left[(1+t)^{1-\nu}-(1+|s|)^{1-\nu}\right]^{\kappa}p(s)\mathrm{d}s > -\infty. \tag{4.75}$$

Proof. Let $n \in \mathbb{N}$ be such that $n > \max\{1,\alpha\}$. Using integration by parts, one gets

$$\begin{aligned}&\frac{1}{(1+t)^{n(1-\nu)}}\int_{-t}^{t}\left[(1+t)^{1-\nu}-(1+|s|)^{1-\nu}\right]^{n}p(s)\mathrm{d}s\\ &\quad=\frac{1}{(1+t)^{n(1-\nu)}}\int_{0}^{t}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n}\big(p(s)+p(-s)\big)\mathrm{d}s\\ &\quad=\frac{n(1-\nu)}{(1+t)^{n(1-\nu)}}\int_{0}^{t}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n-1}\frac{1}{(1+s)^{\nu}}\left(\int_{-s}^{s}p(\xi)\mathrm{d}\xi\right)\mathrm{d}s\\ &\quad=\frac{n(n-1)(1-\nu)}{(1+t)^{n(1-\nu)}}\int_{0}^{t}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n-2}\frac{c(s;\nu)}{(1+s)^{2\nu-1}}\mathrm{d}s \quad \text{for } t \ge 0.\end{aligned}$$

Assume that inequality (2.5) holds. Then there exist $A \in \mathbb{R}$ and $t_0 \ge 0$ such that

$$c(t;\nu) \ge A \quad \text{for } t \ge t_0.$$

Then we have

$$\begin{aligned}&\frac{1}{(1+t)^{n(1-\nu)}}\int_{-t}^{t}\left[(1+t)^{1-\nu}-(1+|s|)^{1-\nu}\right]^{n}p(s)\mathrm{d}s\\ &\quad\ge\frac{n(n-1)(1-\nu)}{(1+t)^{n(1-\nu)}}\int_{0}^{t_0}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n-2}\frac{c(s;\nu)-A}{(1+s)^{2\nu-1}}\mathrm{d}s\\ &\qquad+A\frac{n(n-1)(1-\nu)}{(1+t)^{n(1-\nu)}}\int_{0}^{t}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n-2}\frac{\mathrm{d}s}{(1+s)^{2\nu-1}}\end{aligned} \tag{4.76}$$

for $t \geq t_0$. It is clear that

$$\left|\frac{1}{(1+t)^{n(1-\nu)}}\int_0^{t_0}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n-2}\frac{c(s;\nu)-A}{(1+s)^{2\nu-1}}\,\mathrm{d}s\right|$$
$$\leq \frac{1}{(1+t)^{2(1-\nu)}}\int_0^{t_0}\left[1-\left(\frac{1+s}{1+t}\right)^{1-\nu}\right]^{n-2}\frac{|c(s;\nu)-A|}{(1+s)^{2\nu-1}}\,\mathrm{d}s$$
$$\leq \frac{1}{(1+t)^{2(1-\nu)}}\int_0^{t_0}\frac{|c(s;\nu)-A|}{(1+s)^{2\nu-1}}\,\mathrm{d}s \quad \text{for } t\geq t_0$$

and thus,

$$\lim_{t\to+\infty}\frac{n(n-1)(1-\nu)}{(1+t)^{n(1-\nu)}}\int_0^{t_0}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n-2}\frac{c(s;\nu)-A}{(1+s)^{2\nu-1}}\,\mathrm{d}s=0.$$

On the other hand,

$$(n-1)\int_0^{t}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n-2}\frac{\mathrm{d}s}{(1+s)^{2\nu-1}}$$
$$=\frac{(n-1)\cdots(n-m+1)}{(m-1)!}\int_0^{t}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n-m}\frac{\mathrm{d}s}{(1+s)^{1-m(1-\nu)}}$$
$$-\sum_{\ell=2}^{m-1}\frac{(n-1)\cdots(n-\ell+1)}{\ell!(1-\nu)}\left[(1+t)^{1-\nu}-1\right]^{n-\ell}\quad \text{for } t\geq 0,\ m=2,\ldots,n,$$

which yields that

$$\frac{n(n-1)(1-\nu)}{(1+t)^{n(1-\nu)}}\int_0^{t}\left[(1+t)^{1-\nu}-(1+s)^{1-\nu}\right]^{n-2}\frac{\mathrm{d}s}{(1+s)^{2\nu-1}}$$
$$=1-\frac{1}{(1+t)^{2(1-\nu)}}\sum_{\ell=2}^{n}\frac{n(n-1)\cdots(n-\ell+1)}{\ell!(1+t)^{(\ell-2)(1-\nu)}}\left[1-\frac{1}{(1+t)^{1-\nu}}\right]^{n-\ell}$$

for $t>0$ (note that we set $\sum_{\ell=2}^{1}=0$). Consequently, from inequality (4.76) we get

$$\liminf_{t\to+\infty}\frac{1}{(1+t)^{n(1-\nu)}}\int_{-t}^{t}\left[(1+t)^{1-\nu}-(1+|s|)^{1-\nu}\right]^{n}p(s)\mathrm{d}s\geq A$$

and thus, desired condition (4.75) holds with $\kappa := n$. □

Lemma 4.12. *Let the function $c(\cdot\,;1-\alpha)$ have a finite limit (4.67) and v be a solution to equation (4.32) fulfilling relation (4.54) with $t_v \geq 0$. Then*

$$\begin{aligned}(1+t)^{\alpha}\left[c(+\infty)-c(t;1-\alpha)\right]\\ \leq 2\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}\ln\frac{1+t}{1+t_v}\\ +(1+t_v)^{\alpha}\left[2\left|\frac{v'(t_v)}{v(t_v)}\right|^{\alpha}\operatorname{sgn}\frac{v'(t_v)}{v(t_v)}+\int_{-t_v}^{t_v}p(s)\mathrm{d}s-c(t_v;1-\alpha)\right]\end{aligned}\tag{4.77}$$

for $t \geq t_v$.

Proof. Define the function ϱ by formula (4.57). It follows from Lemma 4.11 with $\nu := 1-\alpha$ that there exists $\kappa > \alpha$ such that condition (4.55) holds with $\beta := \alpha$. Therefore, Lemma 4.9 yields that inequality (4.56) is satisfied and thus, by virtue of Lemma 4.10, we obtain equalities (4.68) and (4.69).

Multiplying equality (4.68) by $(1+t)^{\alpha-1}$ and integrating it from t_v to t, one gets

$$\begin{aligned}\int_{t_v}^{t}(1+s)^{\alpha-1}\varrho(s)\,\mathrm{d}s &= \frac{(1+t)^\alpha}{2\alpha}\big[c(+\infty)-c(t;1-\alpha)\big]\\ &\quad-\frac{(1+t_v)^\alpha}{2\alpha}\big[c(+\infty)-c(t_v;1-\alpha)\big]\\ &\quad+\alpha\int_{t_v}^{t}(1+s)^{\alpha-1}\left(\int_s^{+\infty}|\varrho(\xi)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}\xi\right)\mathrm{d}s\end{aligned}\tag{4.78}$$

for $t \geq t_v$. Observe that

$$\begin{aligned}\int_{t_v}^{t}(1+s)^{\alpha-1}\left(\int_s^{+\infty}|\varrho(\xi)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}\xi\right)\mathrm{d}s &= \frac{(1+t)^\alpha}{\alpha}\int_t^{+\infty}|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\\ &\quad-\frac{(1+t_v)^\alpha}{\alpha}\int_{t_v}^{+\infty}|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\\ &\quad+\frac{1}{\alpha}\int_{t_v}^{t}(1+s)^\alpha|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } t\geq t_v.\end{aligned}$$

Therefore, by virtue of (4.69) and Lemma 4.2, from (4.78) we get

$$\begin{aligned}&(1+t)^\alpha\big[c(+\infty)-c(t;1-\alpha)\big]\\ &\quad=(1+t_v)^\alpha\left[c(+\infty)-c(t_v;1-\alpha)+2\alpha\int_{t_v}^{+\infty}|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\right]\\ &\qquad-2\alpha(1+t)^\alpha\int_t^{+\infty}|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\\ &\qquad+2\int_{t_v}^{t}\frac{1}{1+s}\left[\alpha(1+s)^\alpha\varrho(s)-\alpha|(1+s)^\alpha\varrho(s)|^{\frac{1+\alpha}{\alpha}}\right]\mathrm{d}s\\ &\quad\leq(1+t_v)^\alpha\left[2\varrho(t_v)+\int_{-t_v}^{t_v}p(s)\mathrm{d}s-c(t_v;1-\alpha)\right]\\ &\qquad+2\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}\ln\frac{1+t}{1+t_v}\quad \text{for } t\geq t_v\end{aligned}$$

and thus, in view of (4.57), inequality (4.77) holds for $t \geq t_v$. □

Lemma 4.13. *Let the function $c(\cdot\,;1-\alpha)$ have a finite limit (4.67) and v be a solution to equation (4.32) fulfilling relation (4.54) with $t_v \geq 0$. Then*

$$\begin{aligned}Q_\alpha(t)+H_\alpha(t)\leq 2+\frac{1}{t}\Bigg[&2(1+t_v)^{1+\alpha}\left|\frac{v'(t_v)}{v(t_v)}\right|^\alpha \operatorname{sgn}\frac{v'(t_v)}{v(t_v)}\\ &+\int_{-t_v}^{t_v}(1+|s|)^{1+\alpha}p(s)\mathrm{d}s-2t_v\Bigg]\quad \textit{for } t>t_v,\end{aligned}\tag{4.79}$$

where the functions Q_α and H_α are defined by formulae (2.12) and (2.13), respectively.

Proof. Define the function ϱ by formula (4.57). It is clear that the function ϱ is absolutely continuous on every compact subinterval of $[t_v, +\infty[$ and, in view of (4.32), relation (4.57) yields that equality (4.58) holds. It follows from Lemma 4.11 with $\nu := 1 - \alpha$ that there exists $\kappa > \alpha$ such that condition (4.55) holds with $\beta := \alpha$. Therefore, Lemma 4.9 yields that inequality (4.56) is satisfied and thus, by virtue of Lemma 4.10, we obtain equality (4.68).

Equality (4.68) yields that

$$2\varrho(t) \geq c(+\infty) - \int_{-t}^{t} p(s)\mathrm{d}s = \frac{t}{(1+t)^{1+\alpha}} Q_\alpha(t) \quad \text{for } t \geq t_v. \tag{4.80}$$

On the other hand, multiplying equality (4.58) by $(1+t)^{1+\alpha}$ and integrating it from t_v to t, one gets

$$\int_{t_v}^{t} (1+s)^{1+\alpha} \varrho'(s)\mathrm{d}s = -\frac{1}{2}\int_{t_v}^{t} (1+s)^{1+\alpha}\big(p(s)+p(-s)\big)\mathrm{d}s \\ - \alpha \int_{t_v}^{t} (1+s)^{1+\alpha} |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } t \geq t_v.$$

Consequently, we have

$$(1+t)^{1+\alpha}\varrho(t) - (1+t_v)^{1+\alpha}\varrho(t_v) - (1+\alpha)\int_{t_v}^{t} (1+s)^{\alpha}\varrho(s)\mathrm{d}s \\ = -\frac{1}{2}\int_{-t}^{t} (1+|s|)^{1+\alpha} p(s)\mathrm{d}s + \frac{1}{2}\int_{-t_v}^{t_v} (1+|s|)^{1+\alpha} p(s)\mathrm{d}s \\ - \alpha\int_{t_v}^{t} (1+s)^{1+\alpha}|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } t \geq t_v$$

which, by virtue of Lemma 4.2, yields that

$$\begin{aligned} H_\alpha(t) &= \frac{1}{t}\int_{-t}^{t} (1+|s|)^{1+\alpha} p(s)\mathrm{d}s \\ &= -\frac{2(1+t)^{1+\alpha}}{t}\varrho(t) \\ &\quad + \frac{2}{t}\int_{t_v}^{t} \Big[(1+\alpha)(1+s)^{\alpha}\varrho(s) - \alpha|(1+s)^{\alpha}\varrho(s)|^{\frac{1+\alpha}{\alpha}}\Big]\mathrm{d}s \\ &\quad + \frac{1}{t}\Big[2(1+t_v)^{1+\alpha}\varrho(t_v) + \int_{-t_v}^{t_v} (1+|s|)^{1+\alpha}p(s)\mathrm{d}s\Big] \\ &\leq -\frac{2(1+t)^{1+\alpha}}{t}\varrho(t) + 2\left(1 - \frac{t_v}{t}\right) \\ &\quad + \frac{1}{t}\Big[2(1+t_v)^{1+\alpha}\varrho(t_v) + \int_{-t_v}^{t_v} (1+|s|)^{1+\alpha}p(s)\mathrm{d}s\Big] \quad \text{for } t > t_v. \end{aligned}$$

Therefore, in view of (4.57) and (4.80), inequality (4.79) holds. □

5 Proofs of main results

Proof of Theorem 2.1. Assume on the contrary that equation (1.1) is non-oscillatory on $\mathbb{R}$. Then it follows from Proposition 4.8 that there exists a solution v to equation (4.32) satisfying condition (4.54) with $t_v \geq 0$. Define the function ϱ by formula (4.57). It is clear that in both cases (2.2) and (2.3), inequality (2.5) holds. Therefore, Lemma 4.11 yields that there exists $\kappa > \alpha$ such that condition (4.55) is fulfilled with $\beta := 1 - \nu$ and thus, from Lemma 4.9 we get inequality (4.56). Hence, by virtue of Lemma 4.10, the function $c(\cdot\,; 1-\alpha)$ has a finite limit (4.67). Consequently, it follows from Remark 2.2 (i) that the function $c(\cdot\,;\nu)$ has a finite limit as $t \to +\infty$, which contradicts both conditions (2.2) and (2.3). □

Proof of Theorem 2.3. Assume on the contrary that equation (1.1) is disconjugate on $\mathbb{R}$. Then it follows from Proposition 4.7 that there exists a solution v to equation (4.32) fulfilling conditions (4.33), (4.34), and (4.35). Put $t_v := 0$ and define the function ϱ by formula (4.57). In view of assumption (2.4), Lemma 4.11 with $\nu := 1 - \alpha$ yields that there exists $\kappa > \alpha$ such that condition (4.55) holds with $\beta := \alpha$ and thus, from Lemma 4.9 we get inequality (4.56). Therefore, by virtue of (4.33), (4.34), and Lemma 4.10, the function $c(\cdot\,; 1-\alpha)$ has a finite limit

$$\lim_{t\to+\infty} c(t; 1-\alpha) = -2\alpha \int_0^{+\infty} |\varrho(s)|^{\frac{1+\alpha}{\alpha}} \mathrm{d}s. \tag{5.1}$$

Since the solution v satisfies condition (4.35), we have $\varrho \not\equiv 0$ on $[0, +\infty[$. Consequently, it follows from (5.1) that $\lim_{t\to+\infty} c(t; 1-\alpha) < 0$, which contradicts assumption (2.4). □

Proof of Corollary 2.4. It is clear that either (2.2) holds, or (2.3) is satisfied, or the function $c(\cdot\,;\nu)$ has a finite non-negative limit as $t \to +\infty$. Therefore, the assertion of the corollary follows from Theorems 2.1, 2.3 and Remark 2.2 (i). □

Proof of Corollary 2.5. It is easy to see that

$$\begin{aligned} c(t;0) &= \frac{1}{1+t}\int_0^t \left(\int_0^s p(\xi)\mathrm{d}\xi\right)\mathrm{d}s + \frac{1}{1+t}\int_{-t}^0 \left(\int_s^0 p(\xi)\mathrm{d}\xi\right)\mathrm{d}s \\ &= \frac{t}{1+t}M(t) + \frac{t}{1+t}M(-t) \quad \text{for } t > 0. \end{aligned}$$

Therefore,

$$\lim_{t\to+\infty} c(t;0) = \lim_{t\to+\infty} M(t) + \lim_{t\to-\infty} M(t) \geq 0$$

and thus, the assertion of the corollary follows from Corollary 2.4 with $\nu := 0$. □

Proof of Theorem 2.6. Assume on the contrary that equation (1.1) is disconjugate on $\mathbb{R}$. Observe that, in view of assumption (2.8), we have $p \not\equiv 0$. It follows from Proposition 4.7 that there exists a solution v to equation (4.32) fulfilling conditions (4.33) and (4.54) with $t_v := 0$. Therefore, in view of (4.33), Lemma 4.12 yields that

$$(1+t)^{\alpha}\big[c(+\infty) - c(t;1-\alpha)\big] \leq 2\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \ln(1+t) \quad \text{for } t \geq 0,$$

which contradicts assumption (2.10). □

Proof of Theorem 2.8. Assume on the contrary that equation (1.1) is disconjugate on $\mathbb{R}$. Observe that, in view of assumption (2.8), we have $p \not\equiv 0$. It follows from Proposition 4.7 that there exists a solution v to equation (4.32) fulfilling conditions (4.33) and (4.54) with $t_v := 0$. Therefore, in view of (4.33), Lemma 4.13 yields that

$$Q_\alpha(t) + H_\alpha(t) \leq 2 \quad \text{for } t > 0,$$

which contradicts assumption (2.14). □

Proof of Proposition 2.10. Assume on the contrary that equation (1.1) is non-oscillatory on $\mathbb{R}$. Then it follows from Proposition 4.8 that there exists a solution v to equation (4.32) fulfilling condition (4.54) with $t_v \geq 0$. Assumption (2.15) yields that condition (4.55) is satisfied with $\beta := 1$ and thus, from Lemma 4.9 we get inequality (4.56), where the function ϱ is defined by formula (4.57). Therefore, Lemma 4.10 guarantees that the function $c(\cdot\,;1-\alpha)$ has a finite limit (4.67), which is in a contradiction with assumption (2.9). □

Proof of Theorem 2.11. Assume on the contrary that equation (1.1) is disconjugate on $\mathbb{R}$. Observe that, in view of assumption (2.16), we have $p \not\equiv 0$. It follows from Proposition 4.7 that there exists a solution v to equation (4.32) fulfilling conditions (4.33) and (4.34). Put

$$\varrho(t) := \frac{|v'(t)|^\alpha \operatorname{sgn} v'(t)}{v^\alpha(t)} \quad \text{for } t \geq 0.$$

It is clear that the function ϱ is absolutely continuous on every compact subinterval of $[0, +\infty[$ and, in view of (4.32), we get

$$\varrho'(t) = -\frac{1}{2}\big(p(t) + p(-t)\big) - \alpha|\varrho(t)|^{\frac{1+\alpha}{\alpha}} \quad \text{for a.e. } t \geq 0.$$

It follows from the latter inequality that

$$\begin{aligned} \int_0^t (t-s)^\kappa \varrho'(s)\mathrm{d}s = &-\frac{1}{2}\int_{-t}^t (t-|s|)^\kappa p(s)\mathrm{d}s \\ &- \alpha\int_0^t (t-s)^\kappa |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } t \geq 0. \end{aligned} \tag{5.2}$$

Moreover, in view of (4.33), we have

$$\int_0^t (t-s)^\kappa \varrho'(s)\mathrm{d}s = \kappa\int_0^t (t-s)^{\kappa-1}\varrho(s)\mathrm{d}s \quad \text{for } t \geq 0.$$

Therefore, by virtue of Lemma 4.2, equality (5.2) yields that

$$\begin{aligned} \int_{-t}^t (t-|s|)^\kappa p(s)\mathrm{d}s &= 2\int_0^t (t-s)^{\kappa-\alpha-1}\left[-\kappa(t-s)^\alpha\varrho(s) - \alpha\big|(t-s)^\alpha\varrho(s)\big|^{\frac{1+\alpha}{\alpha}}\right]\mathrm{d}s \\ &\leq 2\left(\frac{\kappa}{1+\alpha}\right)^{1+\alpha}\int_0^t (t-s)^{\kappa-\alpha-1}\mathrm{d}s \\ &= \frac{2}{\kappa-\alpha}\left(\frac{\kappa}{1+\alpha}\right)^{1+\alpha} t^{\kappa-\alpha} \quad \text{for } t \geq 0, \end{aligned}$$

which is in a contradiction with assumption (2.16). □

6 Concluding remark

All results presented in Section 2 can be easily generalised for the equation

$$\big(r(t)|u'|^\alpha \operatorname{sgn} u'\big)' + p(t)|u|^\alpha \operatorname{sgn} u = 0$$

with a positive function r, continuous on $\mathbb{R}$ and such that

$$\int_{-\infty}^0 \frac{\mathrm{d}s}{r^{1/\alpha}(s)} = +\infty, \qquad \int_0^{+\infty} \frac{\mathrm{d}s}{r^{1/\alpha}(s)} = +\infty. \tag{6.1}$$

However, the case, when condition (6.1) is violated, deserves a further investigation because it is related to the question of conjugacy of equation (1.1) either on a finite interval or on a half-line.

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