



## Oscillation of second order neutral dynamic equations with distributed delay

Qiaoshun Yang <sup>1</sup>, Zhiting Xu<sup>2</sup> and Ping Long<sup>1</sup>

<sup>1</sup>Department of Mathematics and Computer Science, Normal College of Jishou University,  
Jishou, Hunan, 416000, P. R. China

<sup>2</sup>School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P. R. China

Received 16 November 2015, appeared 19 June 2016

Communicated by Zuzana Došlá

**Abstract.** In this paper, we establish new oscillation criteria for second order neutral dynamic equations with distributed delay by employing the generalized Riccati transformation. The obtained theorems essentially improve the oscillation results in the literature. And two examples are provided to illustrate to the versatility of our main results.

**Keywords:** oscillation, neutral dynamic equation, time scale, distributed delay.

**2010 Mathematics Subject Classification:** 34K11, 34N05, 39A10.


### 1 Introduction

In this paper, we are concerned with the oscillatory behavior of the following second order neutral dynamic equations with distributed delay

$$(r(t)(Z^\Delta(t))^\alpha)^\Delta + \int_c^d f(t, x(\theta(t, \xi)))\Delta\xi = 0 \quad (1.1)$$

on time scales  $[t_0, \infty)_{\mathbb{T}}$ , where  $\mathbb{T}$  is a time scale with  $\sup \mathbb{T} = \infty$ ;  $Z(t) = x(t) + p(t)x(\tau(t))$  and  $\alpha$  is a quotient of odd positive integers.

Since we are interested in oscillation of solutions near infinity, we assume that  $\sup \mathbb{T} = \infty$  and define the time scale interval  $[t_0, \infty)_{\mathbb{T}}$  by  $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$ . For completeness, we recall the following concepts related to the notion of time scales. A time scale  $\mathbb{T}$  is an arbitrary nonempty closed subset of the real numbers  $\mathbb{R}$ . On any time scale we define the forward and backward jump operators by  $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$  and  $\rho(t) := \sup\{s \in \mathbb{T}, s < t\}$ , where  $\inf \emptyset := \sup \mathbb{T}$  and  $\sup \emptyset := \inf \mathbb{T}$ ; here  $\emptyset$  denotes the empty set. A point  $t \in \mathbb{T}$  and  $t > \inf \mathbb{T}$ , is said to be left-dense if  $\rho(t) = t$ , right-dense if  $t < \sup \mathbb{T}$  and  $\sigma(t) = t$ , left-scattered if  $\rho(t) < t$  and right-scattered if  $\sigma(t) > t$ . The graininess function  $\mu$  for the time scale  $\mathbb{T}$  is defined by  $\mu(t) := \sigma(t) - t$ , and for any function  $f : \mathbb{T} \rightarrow \mathbb{R}$ , the notation  $f^\sigma(t)$

 Corresponding author. Email: [yqs244@163.com](mailto:yqs244@163.com)

denotes  $f(\sigma(t))$ . A function  $g : \mathbb{T} \rightarrow \mathbb{R}$  is said to be  $rd$ -continuous provided  $g$  is continuous at right-dense points and at left-dense points in  $\mathbb{T}$ , left-hand limits exist and are finite. The set of all such  $rd$ -continuous functions is denoted by  $C_{rd}(\mathbb{T})$ . The set of functions  $f : \mathbb{T} \rightarrow \mathbb{R}$  which are differentiable and whose derivative is  $rd$ -continuous function is denoted by  $C_{rd}^1(\mathbb{T}, \mathbb{R})$ . For more details, see the monograph [5].

Throughout this paper, we always assume that

- (A1)  $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$  with  $\int_{t_0}^{\infty} r^{-1/\alpha}(t)\Delta t = \infty$ ;
- (A2)  $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$  with  $0 \leq p(t) < 1$ ;
- (A3)  $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ ,  $\tau(t) \leq t$ , and  $\lim_{t \rightarrow \infty} \tau(t) = \infty$ ;
- (A4)  $c, d \in [t_0, \infty)_{\mathbb{T}}$ ,  $\theta(t, \xi) \in C_{rd}([t_0, \infty)_{\mathbb{T}} \times [c, d]_{\mathbb{T}}, \mathbb{T})$ ,  $[c, d]_{\mathbb{T}} = \{\xi \in \mathbb{T} : c \leq \xi \leq d\}$ ,  $\theta(t, c) \leq \theta(t, \xi)$  for  $(t, \xi) \in [t_0, \infty)_{\mathbb{T}} \times [c, d]_{\mathbb{T}}$ , and  $\lim_{t \rightarrow \infty} \theta(t, c) = \infty$ ;
- (A5)  $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that  $uf(t, u) > 0$  for all  $u \neq 0$  and there exists a function  $q(t, \xi) \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, +\infty))$  such that  $|f(t, u)| \geq q(t, \xi)|u^\alpha|$ .

By a solution of Eq. (1.1), we mean a nontrivial real-valued function  $x \in C_{rd}^1([T_x, \infty), \mathbb{R})$ ,  $T_x \geq t_0$  which has the property that  $r(t)(Z^\Delta(t))^\alpha \in C_{rd}^1([T_x, \infty), \mathbb{R})$  and satisfies Eq. (1.1) on  $[T_x, \infty)$ . The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution  $x(t)$  of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

In recent years, there has been an increasing interest in studying oscillatory behavior of solutions to various classes of dynamic equations on time scales. In particular, oscillation of second order neutral dynamic equations attracted significant attention of researchers due to the fact that such equations arise in many real life problems; see, for example, [1, 2, 7, 8, 10, 11, 13–24] and the references cited therein. Chen [10], Şahiner [21], Saker et al. [18], Saker and O'Regan [20] considered the second-order nonlinear neutral dynamic equation with variable delays

$$(r(t)[(x(t) + p(t)x(\tau(t)))^\Delta]^\gamma)^\Delta + f(t, x(\delta(t))) = 0, \quad (1.2)$$

where  $0 \leq p(t) < 1$ . Han et al. [13] and Saker et al. [16] examined the oscillation of Eq. (1.2) when  $\gamma = 1$ . In particular, Han et al. [13] investigated the case where  $\gamma = 1$  and  $p(t) \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, p_0])$ , where  $p_0$  is a constant.

Regarding the oscillation of dynamic equations with *distributed delay*, Candan [7] studied the oscillation of the second order neutral delay dynamic equation

$$(r(t)((x(t) + p(t)x(\tau(t)))^\Delta)^\gamma)^\Delta + \int_c^d f(t, x(\theta(t, \xi)))\Delta\xi = 0,$$

where  $f(t, u) \geq q(t)|u^\beta|$ ;  $\gamma > 0$  and  $\beta > 0$  are ratios of odd positive integers. He gave some oscillation results when  $\theta(t, d) > t$  and  $\theta(t, d) \leq t$ , respectively. For more related works on the oscillation of second order neutral dynamic equations with distributed delay, we refer the readers to [9, 11, 15, 22, 24].

In this paper, inspired by the works [6, 12, 24], we will study the oscillation of (1.1). Here we will employ the generalized Riccati transformation technique to establish new oscillation criteria for (1.1) when  $\delta(t) \leq \sigma(t)$  and  $\delta(t) > \sigma(t)$ , respectively, the obtained results improve the main results in [7, 15, 22]. Finally, we give two examples to illustrate the main results.

## 2 Main results

In what follows, we use the following notation for the convenience of the reader.

$$\delta(t) = \theta(t, c); \quad R(t) = \int_c^d q(t, \xi)(1 - p(\theta(t, \xi)))^\alpha \Delta \xi,$$

and

$$\eta_1(t, u) = \frac{r(\delta(t), u)}{r(\sigma(t), u)}, \quad \eta_2(t, u) = \frac{r(t, u)}{r(\sigma(t), u)},$$

where

$$r(t, u) = \int_u^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \Delta s, \quad t > u \geq t_0.$$

Further, for any given functions  $\eta(t), a(t) \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$  with  $\eta(t) > 0$  and  $a(t) > -1/[r(t)r^\alpha(t, T)]$ , we let

$$\begin{aligned} \phi_+(t) &:= \max\{0, \phi(t)\}, \\ \phi(t) &:= \eta^\sigma(t) [R(t)\eta_1^\alpha(t, T) + r(t)\eta_2^{1+\alpha}(t, T)a^{1+\frac{1}{\alpha}}(t) - (r(t)a(t))^\Delta], \\ \varphi_1(t) &:= \frac{\eta^\Delta(t)}{\eta(t)} + (\alpha + 1)a^{\frac{1}{\alpha}}(t) \frac{\eta^\sigma(t)\eta_2^{1+\alpha}(t)}{\eta(t)}, \\ \varphi_2(t) &:= \frac{\alpha \eta^\sigma(t)\eta_2^{1+\alpha}(t, T)}{r^{\frac{1}{\alpha}}(t)\eta^{1+\frac{1}{\alpha}}(t)}. \end{aligned}$$

### 2.1 Two lemmas

In order to prove the main results, we need the following two lemmas.

**Lemma 2.1.** *Let  $x(t)$  be an eventually positive solution of Eq. (1.1). Then there exists some  $T > t_0$  large enough such that for all  $t > T$ ,*

$$Z(t) > 0, \quad Z^\Delta(t) > 0, \quad Z(t) > r^{\frac{1}{\alpha}}(t)Z^\Delta(t)r(t, T), \quad Z(t) \geq \eta_2(t, T)Z(\sigma(t)). \quad (2.1)$$

*Proof.* Without loss of generality, we assume that there exists a  $T \in [t_0, \infty)_{\mathbb{T}}$  such that  $x(t), x(\tau(t)), x(\theta(t, \xi)) > 0$  on  $[T, \infty)_{\mathbb{T}}$ , then  $Z(t) \geq x(t) > 0$ . It follows from (1.1) and (A5) that

$$(r(t)(Z^\Delta(t))^\alpha)^\Delta \leq - \int_c^d q(t, \xi)x^\alpha(\theta(t, \xi))\Delta \xi \leq 0.$$

Hence,  $r(t)(Z^\Delta(t))^\alpha$  is decreasing on  $[T, \infty)_{\mathbb{T}}$ . We now claim that  $Z^\Delta(t) > 0$  eventually on  $t \in [T, \infty)_{\mathbb{T}}$ . If not, then there exists a  $t_1 \in [T, \infty)_{\mathbb{T}}$  such that  $Z^\Delta(t_1) < 0$ . Then

$$r(t)(Z^\Delta(t))^\alpha \leq r(t_1)(Z^\Delta(t_1))^\alpha := -c^* < 0, \quad t \geq t_1,$$

i.e.,

$$Z^\Delta(t) \leq - \left( \frac{c^*}{r(t)} \right)^{\frac{1}{\alpha}}. \quad (2.2)$$

Integrating (2.2) from  $t_1$  to  $t$ , we find from (A1) that

$$Z(t) \leq Z(t_1) - (c^*)^{\frac{1}{\alpha}} \int_{t_1}^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \Delta s \rightarrow -\infty \quad \text{as } t \rightarrow \infty,$$

which implies that  $Z(t)$  is eventually negative. This contradicts  $Z(t) > 0$ . Thus,  $Z^\Delta(t) > 0$  on  $[T, \infty)_{\mathbb{T}}$ .

Since  $r(t)(Z^\Delta(t))^\alpha$  is decreasing on  $[T, \infty)_{\mathbb{T}}$ , we have

$$\begin{aligned} Z(t) &> Z(t) - Z(T) = \int_T^t \frac{1}{r^{\frac{1}{\alpha}}(s)} (r(s)(Z^\Delta(s))^\alpha)^{\frac{1}{\alpha}} \Delta s \\ &\geq (r(t)(Z^\Delta(t))^\alpha)^{\frac{1}{\alpha}} \int_T^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \Delta s \\ &= (r(t))^{1/\alpha} Z^\Delta(t) r(t, T). \end{aligned}$$

Thus,  $(Z(t)/r(t, T))^\Delta \leq 0$ , which implies that

$$\frac{Z(t)}{r(t, T)} \geq \frac{Z^\sigma(t)}{r(\sigma(t), T)}.$$

This completes the proof.  $\square$

For the positive solution  $x(t)$  of Eq. (1.1), it follows from the definition of  $Z(t)$  and Lemma 2.1 that, for  $t \geq T$ ,

$$x(t) = Z(t) - p(t)x(\tau(t)) \geq Z(t) - p(t)Z(\tau(t)) \geq (1 - p(t))Z(t),$$

consequently,

$$x^\alpha(\theta(t, \xi)) \geq (1 - p(\theta(t, \xi)))^\alpha Z^\alpha(\theta(t, \xi)). \quad (2.3)$$

Multiplying (2.3) by  $q(t, \xi)$  and integrating both sides from  $c$  to  $d$ , we have

$$\int_c^d q(t, \xi) x^\alpha(\theta(t, \xi)) \Delta \xi \geq \int_c^d q(t, \xi) (1 - p(\theta(t, \xi)))^\alpha Z^\alpha(\theta(t, \xi)) \Delta \xi.$$

It follows from (1.1) that

$$(r(t)(Z^\Delta(t))^\alpha)^\Delta \leq - \int_c^d q(t, \xi) (1 - p(\theta(t, \xi)))^\alpha Z^\alpha(\theta(t, \xi)) \Delta \xi.$$

Since  $\theta(t, \xi) \geq \theta(t, c)$  and  $Z^\Delta(t) > 0$ , then  $Z(\theta(t, \xi)) \geq Z(\theta(t, c))$ . By the definition of  $R(t)$  and  $\delta(t)$ , we obtain

$$(r(t)(Z^\Delta(t))^\alpha)^\Delta \leq -R(t)Z^\alpha(\theta(t, c)) = -R(t)Z^\alpha(\delta(t)). \quad (2.4)$$

**Lemma 2.2.** *Let  $x(t)$  be an eventually positive solution of Eq. (1.1). Then there exists some  $T > t_0$  large enough such that for all  $t > T$ ,*

$$\frac{Z(\delta(t))}{Z(\sigma(t))} \geq \begin{cases} 1, & \delta(t) > \sigma(t), \\ \eta_1(t, T), & \delta(t) \leq \sigma(t). \end{cases} \quad (2.5)$$

The proof is similar to that of [24, Lemma 2.2], we omitted the details here.

## 2.2 Oscillation of (1.1) for the case $\delta(t) \leq \sigma(t)$

**Theorem 2.3.** Assume that there exist a function  $a(t)$  and a positive  $\Delta$ -differentiable function  $\eta(t)$ , such that for sufficiently large  $T \in [t_0, \infty)_{\mathbb{T}}$ ,

$$\limsup_{t \rightarrow \infty} \int_{T_1}^t \left( \varphi(s) - \frac{\alpha^\alpha ([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \varphi_2^\alpha(s)} \right) \Delta s > \eta(T_1) \left( \frac{1}{r^\alpha(T_1, T)} + r(T_1)a(T_1) \right), \quad (2.6)$$

where  $T_1 > T \geq t_0$ . Then

- (i) every solution  $x(t)$  of (1.1) is oscillatory for  $\alpha \geq 1$ ;
- (ii) every solution  $x(t)$  of (1.1) oscillates for  $0 < \alpha < 1$  and  $a(t) = 0$ .

*Proof.* Assume that  $x(t)$  is a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists a  $T \in [t_0, \infty)_{\mathbb{T}}$  (sufficiently large) such that  $x(t), x(\tau(t)), x(\delta(t)) > 0$  on  $[T, \infty)_{\mathbb{T}}$ . Then by Lemmas 2.1 and 2.2, (2.1) and (2.5) hold. Consider the generalized Riccati substitution

$$w(t) = \eta(t) \left( \frac{r(t)(Z^\Delta(t))^\alpha}{Z^\alpha(t)} + r(t)a(t) \right), \quad t \geq T.$$

Clearly,  $w(t) > 0$ . In view of [5, Theorem 1.20] and (2.4), we get

$$\begin{aligned} w^\Delta(t) &= \eta^\Delta(t) \left( \frac{r(t)(Z^\Delta(t))^\alpha}{Z^\alpha(t)} + r(t)a(t) \right) + \eta^\sigma(t) \left( \frac{r(t)(Z^\Delta(t))^\alpha}{Z^\alpha(t)} + r(t)a(t) \right)^\Delta \\ &= \frac{\eta^\Delta(t)}{\eta(t)} w(t) + \eta^\sigma(t) \left( \frac{r(t)(Z^\Delta(t))^\alpha}{Z^\alpha(t)} \right)^\Delta + \eta^\sigma(t) (r(t)a(t))^\Delta \\ &= \frac{\eta^\Delta(t)}{\eta(t)} w(t) + \eta^\sigma(t) (r(t)a(t))^\Delta \\ &\quad + \eta^\sigma(t) \frac{(r(t)(Z^\Delta(t))^\alpha)^\Delta Z^\alpha(t) - r(t)(Z^\Delta(t))^\alpha (Z^\alpha(t))^\Delta}{Z^\alpha(t)(Z^\sigma(t))^\alpha} \\ &\leq \frac{\eta^\Delta(t)}{\eta(t)} w(t) + \eta^\sigma(t) (r(t)a(t))^\Delta \\ &\quad - \eta^\sigma(t) \frac{R(t)Z^\alpha(\delta(t))}{(Z^\sigma(t))^\alpha} - \frac{\eta^\sigma(t)r(t)(Z^\Delta(t))^\alpha (Z^\alpha(t))^\Delta}{Z^\alpha(t)(Z^\sigma(t))^\alpha} \\ &= \frac{\eta^\Delta(t)}{\eta(t)} w(t) + \eta^\sigma(t) (r(t)a(t))^\Delta - \eta^\sigma(t) R(t) \left( \frac{Z(\delta(t))}{Z^\sigma(t)} \right)^\alpha \\ &\quad - \eta^\sigma(t) \left( \frac{w(t)}{\eta(t)} - r(t)a(t) \right) \frac{(Z^\alpha(t))^\Delta}{(Z^\sigma(t))^\alpha}. \end{aligned} \quad (2.7)$$

By the Pötzsche chain rule [5, Theorem 1.87], then

$$\begin{aligned} (Z^\alpha(t))^\Delta &= \alpha \left\{ \int_0^1 [(1-h)Z(t) + hZ(\sigma(t))]^{\alpha-1} dh \right\} Z^\Delta(t) \\ &\geq \begin{cases} \alpha (Z(t))^{\alpha-1} Z^\Delta(t), & \alpha > 1, \\ \alpha (Z^\sigma(t))^{\alpha-1} Z^\Delta(t), & 0 < \alpha \leq 1. \end{cases} \end{aligned}$$

Consequently,

$$\frac{(Z^\alpha(t))^\Delta}{Z^\alpha(t)} \geq \begin{cases} \alpha \frac{Z^\Delta(t)}{Z(t)}, & \alpha > 1, \\ \alpha \frac{(Z^\sigma(t))^{\alpha-1}}{Z^\alpha(t)} Z^\Delta(t), & 0 < \alpha \leq 1. \end{cases}$$

Note that  $Z(t)$  is increasing on  $[T, \infty)_{\mathbb{T}}$ . Then  $Z(t) \leq Z(\sigma(t))$  for  $t \in [T, \infty)_{\mathbb{T}}$ . Therefore

$$\frac{(Z^\alpha(t))^\Delta}{Z^\alpha(t)} \geq \alpha \frac{Z^\Delta(t)}{Z^\sigma(t)},$$

which implies

$$\begin{aligned} \frac{(Z^\alpha(t))^\Delta}{(Z^\sigma(t))^\alpha} &= \frac{(Z^\alpha(t))^\Delta}{Z^\alpha(t)} \frac{Z^\alpha(t)}{(Z^\sigma(t))^\alpha} \geq \alpha \frac{Z^\Delta(t)}{Z(t)} \left( \frac{Z(t)}{Z^\sigma(t)} \right)^{1+\alpha} \\ &\geq \frac{\alpha}{r^{\frac{1}{\alpha}}(t)} \left( \frac{w(t)}{\eta(t)} - r(t)a(t) \right)^{\frac{1}{\alpha}} \eta_2^{1+\alpha}(t, T). \end{aligned} \quad (2.8)$$

Substituting (2.8) into (2.7), and by (2.5), we obtain

$$\begin{aligned} w^\Delta(t) &< \frac{\eta^\Delta(t)}{\eta(t)} w(t) + \eta^\sigma(t) (r(t)a(t))^\Delta - \eta^\sigma(t) R(t) \eta_1^\alpha(t, T) \\ &\quad - \frac{\alpha \eta^\sigma(t) \eta_2^{1+\alpha}(t, T)}{r^{\frac{1}{\alpha}}(t)} \left( \frac{w(t)}{\eta(t)} - r(t)a(t) \right)^{1+\frac{1}{\alpha}}, \quad t > T. \end{aligned} \quad (2.9)$$

For the case  $\alpha \geq 1$ , using the inequality (see [3, (2.18)]),

$$A^{1+\frac{1}{\alpha}} - (A - B)^{1+\frac{1}{\alpha}} \leq B^{\frac{1}{\alpha}} \left( \left(1 + \frac{1}{\alpha}\right) A - \frac{1}{\alpha} B \right), \quad \alpha = \frac{\text{odd}}{\text{odd}},$$

with  $A := w(t)/\eta(t)$  and  $B := r(t)a(t)$ , we get

$$\left( \frac{w(t)}{\eta(t)} - r(t)a(t) \right)^{1+\frac{1}{\alpha}} \geq \left( \frac{w(t)}{\eta(t)} \right)^{1+\frac{1}{\alpha}} + \frac{1}{\alpha} (r(t)a(t))^{1+\frac{1}{\alpha}} - \frac{1+\alpha}{\alpha} (r(t)a(t))^{\frac{1}{\alpha}} \frac{w(t)}{\eta(t)}. \quad (2.10)$$

Substituting (2.10) into (2.9), we obtain

$$w^\Delta(t) < -\varphi(t) + \varphi_1(t)w(t) - \varphi_2(t)w^{1+\frac{1}{\alpha}}(t) \quad (2.11)$$

$$\leq -\varphi(t) + [\varphi_1(t)]_+ w(t) - \varphi_2(t)w^{1+\frac{1}{\alpha}}(t), \quad t > T. \quad (2.12)$$

For the case when  $0 < \alpha < 1$  and  $a(t) = 0$ , in view of the definitions of  $w(t)$ ,  $\varphi(t)$ ,  $\varphi_1(t)$ ,  $\varphi_2(t)$ , we find that (2.12) also holds by (2.9). Using the inequality (see [4, (2.8)]),

$$B_1 w - A_1 w^{1+\frac{1}{\alpha}} \leq \frac{\alpha^\alpha B_1^{1+\alpha}}{(1+\alpha)^{1+\alpha} A_1^\alpha}, \quad (2.13)$$

with  $B_1 = [\varphi_1(t)]_+$  and  $A_1 = \varphi_2(t)$ , we have

$$[\varphi_1(t)]_+ w(t) - \varphi_2(t)w^{1+\frac{1}{\alpha}}(t) \leq \frac{\alpha^\alpha ([\varphi_1(t)]_+)^{1+\alpha}}{(1+\alpha)^{1+\alpha} (\varphi_2(t))^\alpha}. \quad (2.14)$$

Substituting (2.14) into (2.12) we obtain

$$w^\Delta(t) < -\varphi(t) + \frac{\alpha^\alpha ([\varphi_1(t)]_+)^{1+\alpha}}{(1+\alpha)^{1+\alpha} (\varphi_2(t))^\alpha}, \quad t > T. \quad (2.15)$$

Integrating both sides of (2.15) from  $T_1$  to  $t (t > T_1 > T)$ , we have

$$\int_{T_1}^t \left( \varphi(s) - \frac{\alpha^\alpha ([\varphi_1(s)]_+)^{1+\alpha}}{(1+\alpha)^{1+\alpha} (\varphi_2(s))^\alpha} \right) \Delta s < w(T_1) < \eta(T_1) \left( \frac{1}{r^\alpha(T_1, T)} + r(T_1)a(T_1) \right),$$

which contradicts (2.6). This completes the proof.  $\square$

Let  $\mathbb{D}_0 \equiv \{t > s \geq t_0, t, s \in [t_0, \infty)_{\mathbb{T}}\}$  and  $\mathbb{D} \equiv \{t \geq s \geq t_0, t, s \in [t_0, \infty)_{\mathbb{T}}\}$ . The function  $K \in C_{rd}(\mathbb{D}, \mathbb{R})$  is said to belong to the class  $\mathfrak{K}$  (defined by  $K \in \mathfrak{K}$ , for short) if

$$K(t, t) = 0, \quad t \geq t_0, \quad K(t, s) > 0, \quad t > s \geq t_0,$$

and  $K$  has a nonpositive continuous  $\Delta$ -partial derivative  $K^{\Delta s}(t, s)$  on  $\mathbb{D}_0$  with respect to the second variable.

**Theorem 2.4.** Assume that  $K \in \mathfrak{K}$ ,  $k \in C_{rd}(\mathbb{D}_0, \mathbb{R})$  and there exist a function  $a(t)$  and a positive  $\Delta$ -differentiable function  $\eta(t)$ , such that for sufficiently large  $T \in [t_0, \infty)_{\mathbb{T}}$ ,

$$K^{\Delta s}(\sigma(t), s) + K(\sigma(t), \sigma(s))\varphi_1(s) = k(t, s),$$

and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{K(\sigma(t), T_1)} \int_{T_1}^t \left( K(\sigma(t), \sigma(s))\varphi(s) - \frac{\alpha^\alpha ([k(t, s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} (K(\sigma(t), \sigma(s))\varphi_2(s))^\alpha} \right) \Delta s \\ & > \eta(T_1) \left( \frac{1}{r^\alpha(T_1, T)} + r(T_1)a(T_1) \right), \end{aligned} \quad (2.16)$$

where  $T_1 > T \geq t_0$ . Then

(i) every solution  $x(t)$  of (1.1) is oscillatory for  $\alpha \geq 1$ ;

(ii) every solution  $x(t)$  of (1.1) oscillates for  $0 < \alpha < 1$  and  $a(t) = 0$ .

*Proof.* Assume that  $x(t)$  is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that  $x(t)$  is an eventually positive. Proceeding as the proof of Theorem 2.3, we get (2.11) holds, i.e.,

$$w^\Delta(t) < -\varphi(t) + \varphi_1(t)w(t) - \varphi_2(t)w^{1+\frac{1}{\alpha}}(t), \quad t > T_1 > T. \quad (2.11)$$

Multiplying both sides of (2.11), with  $t$  replaced by  $s$ , by  $K(\sigma(t), \sigma(s))$ , integrating with respect to  $s$  from  $T_1$  to  $\sigma(t)$ , we get

$$\begin{aligned} & \int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))\varphi(s)\Delta s \\ & < - \int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))w^\Delta(s)\Delta s + \int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))\varphi_1(s)w(s)\Delta s \\ & \quad - \int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s)\Delta s. \end{aligned} \quad (2.17)$$

Using integration by parts for the first part of the right-hand side of (2.17), we obtain

$$\int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))w^\Delta(s)\Delta s = -K(\sigma(t), T_1)w(T_1) - \int_{T_1}^{\sigma(t)} K^{\Delta s}(\sigma(t), s)w(s)\Delta s. \quad (2.18)$$

Substitution (2.18) into (2.17) implies that

$$\begin{aligned} & \int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))\varphi(s)\Delta s \\ & < K(\sigma(t), T_1)w(T_1) + \int_{T_1}^{\sigma(t)} K^{\Delta s}(\sigma(t), s)w(s)\Delta s \\ & \quad + \int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))\varphi_1(s)w(s)\Delta s - \int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s)\Delta s \\ & = K(\sigma(t), T_1)w(T_1) + \int_{T_1}^{\sigma(t)} k(t, s)w(s)\Delta s - \int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s)\Delta s. \end{aligned}$$

Using the definition of  $K$  and

$$\int_{T_1}^{\sigma(t)} \phi(s) \Delta s = \int_{T_1}^t \phi(s) \Delta s + \int_t^{\sigma(t)} \phi(s) \Delta s = \int_{T_1}^t \phi(s) \Delta s + \mu(t) \phi(t),$$

we derive from  $\mu(t)K^{\Delta_s}(\sigma(t), t)w(t) \leq 0$  that

$$\begin{aligned} & \int_{T_1}^{\sigma(t)} k(t, s)w(s) \Delta s - \int_{T_1}^{\sigma(t)} K(\sigma(t), \sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s) \\ &= \int_{T_1}^t k(t, s)w(s) \Delta s - \int_{T_1}^t K(\sigma(t), \sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s) \\ & \quad + \int_t^{\sigma(t)} k(t, s)w(s) \Delta s - \int_t^{\sigma(t)} K(\sigma(t), \sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s) \\ &= \int_{T_1}^t k(t, s)w(s) \Delta s - \int_{T_1}^t K(\sigma(t), \sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s) \\ & \quad + \mu(t) \left[ K^{\Delta_s}(\sigma(t), t) + K(\sigma(t), \sigma(t))\varphi_1(t) \right] w(t) - \mu(t)K(\sigma(t), \sigma(t))\varphi_2(t)w^{1+\frac{1}{\alpha}}(t) \\ &\leq \int_{T_1}^t [k(t, s)]_+ w(s) \Delta s - \int_{T_1}^t K(\sigma(t), \sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s). \end{aligned}$$

Now using the inequality (2.13) with  $B_1 = [k(t, s)]_+$  and  $A_1 = K(\sigma(t), \sigma(s))\varphi_2(s)$ , we get

$$[k(t, s)]_+ w(s) - K(\sigma(t), \sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s) \leq \frac{\alpha^\alpha [k(t, s)]_+^{1+\alpha}}{(1+\alpha)^{1+\alpha} (K(\sigma(t), \sigma(s))\varphi_2(s))^\alpha}.$$

Hence, noting that  $K(\sigma(t), \sigma(t)) = 0$ , we get

$$\int_{T_1}^t \left( K(\sigma(t), \sigma(s))\varphi(s) - \frac{\alpha^\alpha ([k(t, s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} (K(\sigma(t), \sigma(s))\varphi_2(s))^\alpha} \right) \Delta s \leq K(\sigma(t), T_1)w(T_1).$$

Thus,

$$\begin{aligned} & \frac{1}{K(\sigma(t), T_1)} \int_{T_1}^t \left( K(\sigma(t), \sigma(s))\varphi(s) - \frac{\alpha^\alpha ([k(t, s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} (K(\sigma(t), \sigma(s))\varphi_2(s))^\alpha} \right) \Delta s \\ & \leq w(T_1) < \eta(T_1) \left( \frac{1}{r^\alpha(T_1, T)} + r(T_1)a(T_1) \right), \end{aligned}$$

which contradicts (2.16), and then the proof is complete.  $\square$

**Remark 2.5.** If (2.6) and (2.16) are replaced respectively by

$$\limsup_{t \rightarrow \infty} \int_{T_1}^t \left( \varphi(s) - \frac{\alpha^\alpha ([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \varphi_2^\alpha(s)} \right) \Delta s = \infty,$$

$$\limsup_{t \rightarrow \infty} \frac{1}{K(\sigma(t), T_1)} \int_{T_1}^t \left( K(\sigma(t), \sigma(s))\varphi(s) - \frac{\alpha^\alpha ([k(t, s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} (K(\sigma(t), \sigma(s))\varphi_2(s))^\alpha} \right) \Delta s = \infty,$$

then the conclusions of Theorems 2.3, 2.4 are also true which are special cases of Theorems 2.3, 2.4. Thus, Theorems 2.3, 2.4 essentially improve the related results established by [7, 15, 22].

**Remark 2.6.** The assumption  $(H_6)$  in D. Chen [10] required  $b^\Delta(t) > 0$  and  $b(\sigma(t)) = \sigma(b(t))$ , the function  $\psi(t)$  in Theorem 3.1 and 3.2 required  $\psi(t) \geq 0$  which are stronger than that of  $(H4)$  and  $a(t)$  in our work, respectively. Therefore, Theorem 2.4 improves Theorem 3.2 in D. Chen [10]



### 2.3 Oscillation of (1.1) for the case $\delta(t) > \sigma(t)$

In this case when  $\delta(t) > \sigma(t)$ , by Lemma 2.2, we get  $Z(\delta(t))/Z(\sigma(t)) > 1$ . Now we replace  $Z(\delta(t)/Z(\sigma(t)))$  by 1 in (2.7), and similarly to the proof of Theorems 2.3–2.4, we can obtain following results.

**Theorem 2.7.** Assume that there exist a function  $a(t)$  and a positive  $\Delta$ -differentiable function  $\eta(t)$ , such that for sufficiently large  $T \in [t_0, \infty)_{\mathbb{T}}$ ,

$$\limsup_{t \rightarrow \infty} \int_{T_1}^t \left( \bar{\varphi}(s) - \frac{\alpha^\alpha ([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \varphi_2^\alpha(s)} \right) \Delta s > \eta(T_1) \left( \frac{1}{r^\alpha(T_1, T)} + r(T_1)a(T_1) \right).$$

where  $T_1 > T \geq t_0$ , and

$$\bar{\varphi}(t) = \eta^\sigma(t) [R(t) + r(t)\eta_2^{1+\alpha}(t, T)a^{1+\frac{1}{\alpha}}(t) - (r(t)a(t))^\Delta].$$

Then

- (i) every solution  $x(t)$  of (1.1) is oscillatory for  $\alpha \geq 1$ ;
- (ii) every solution  $x(t)$  of (1.1) oscillates for  $0 < \alpha < 1$  and  $a(t) = 0$ .

**Theorem 2.8.** Assume that  $K \in \mathfrak{K}$ ,  $k \in C_{rd}(\mathbb{D}_0, \mathbb{R})$  and there exist a function  $a(t)$  and a positive  $\Delta$ -differentiable function  $\eta(t)$ , such that for sufficiently large  $T \in [t_0, \infty)_{\mathbb{T}}$ ,

$$K^{\Delta s}(\sigma(t), s) + K(\sigma(t), \sigma(s))\varphi_1(s) = k(t, s),$$

and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{K(\sigma(t), T_1)} \int_{T_1}^t \left( K(\sigma(t), \sigma(s))\bar{\varphi}(s) - \frac{\alpha^\alpha ([k(t, s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} (K(\sigma(t), \sigma(s))\varphi_2(s))^\alpha} \right) \Delta s \\ & > \eta(T_1) \left( \frac{1}{r^\alpha(T_1, T)} + r(T_1)a(T_1) \right), \end{aligned}$$

where  $T_1 > T \geq t_0$ , and  $\bar{\varphi}(s)$  is defined as in Theorem 2.7. Then

- (i) every solution  $x(t)$  of (1.1) is oscillatory for  $\alpha \geq 1$ ;
- (ii) every solution  $x(t)$  of (1.1) oscillates for  $0 < \alpha < 1$  and  $a(t) = 0$ .

## 3 Two examples

In this section, we give two examples to illustrate our main results.

**Example 3.1.** Consider the neutral dynamic equation

$$\left( \frac{1}{(t + \sigma(t))^\alpha} (Z^\Delta(t))^\alpha \right)^\Delta + \int_c^d \frac{1 + \sin^2(t\xi)}{t\sigma(t)} x^\alpha(t - \xi) \Delta \xi = 0, \quad (3.1)$$

where  $1/2 \leq \alpha < 1$  with  $\alpha = \text{odd}/\text{odd}$ ,  $\tau(t)$  satisfies (A3), and  $Z(t) = x(t) + \frac{1}{2}x(\tau(t))$ .

For (1.1), we let

$$r(t) = \frac{1}{(t + \sigma(t))^\alpha}, \quad p(t) = \frac{1}{2}, \quad q(t, \xi) = \frac{1}{t\sigma(t)},$$

and

$$\theta(t, \xi) = t - \xi, \quad \delta(t) = \theta(t, c) = t - c.$$

Obviously, (A1) holds and we have

$$r(t, T) = t^2 - T^2, \quad \eta_1(t, T) = \frac{\delta^2(t) - T^2}{\sigma^2(t) - T^2} = \frac{(t - c)^2 - T^2}{\sigma^2(t) - T^2}.$$

Then the function  $\eta_1(t, T)$  is strictly increasing and we have  $1 \geq \eta_1(t, T) \geq 1/3$  for sufficiently large  $T \in [t_0, \infty)_{\mathbb{T}}$ . Then

$$\begin{aligned} R(t) &= \int_c^d q(t, \xi) (1 - p(\theta(t, \xi)))^\alpha \Delta \xi \\ &= \frac{1}{2^\alpha} \int_c^d \frac{1}{t\sigma(t)} \Delta \xi \\ &= \frac{d - c}{2^\alpha t\sigma(t)}. \end{aligned}$$

Let  $a(t) = 0$  and  $\eta(t) = 1 + 1/t$ , choosing  $T_1 = 2T$ , then

$$\begin{aligned} \frac{d - c}{2^{\alpha-1} t\sigma(t)} &> \varphi(t) = \eta^\sigma(t) R(t) \eta_1^\alpha(t, T) > \frac{d - c}{6^\alpha t\sigma(t)}, \\ [\varphi_1(s)]_+ &= 0, \quad \frac{\eta(T_1)}{r^\alpha(T_1, T)} = \left(1 + \frac{1}{T_1}\right) \frac{1}{(T_1^2 - T^2)^\alpha} < \frac{1}{3^\alpha T}. \end{aligned}$$

Hence,

$$\begin{aligned} \infty &> \limsup_{t \rightarrow \infty} \int_{T_1}^t \frac{d - c}{2^{\alpha-1} t\sigma(t)} \Delta s > \limsup_{t \rightarrow \infty} \int_{T_1}^t \left( \varphi(s) - \frac{\alpha^\alpha ([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha + 1)^{\alpha+1} \varphi_2^\alpha(s)} \right) \Delta s \\ &> \limsup_{t \rightarrow \infty} \int_{T_1}^t \frac{d - c}{6^\alpha t\sigma(t)} \Delta s = \limsup_{t \rightarrow \infty} \frac{d - c}{6^\alpha} \left( \frac{1}{2T} - \frac{1}{t} \right). \end{aligned}$$

We can choose  $c, d$  such that

$$\frac{d - c}{6^\alpha} > \frac{2}{3^\alpha} + 2.$$

Consequently,

$$\limsup_{t \rightarrow \infty} \int_{T_1}^t \left( \varphi(s) - \frac{\alpha^\alpha ([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha + 1)^{\alpha+1} \varphi_2^\alpha(s)} \right) \Delta s > \frac{\eta(T_1)}{r^\alpha(T_1, T)}.$$

Thus, by Theorem 2.3, Eq. (3.1) is oscillatory.

**Example 3.2.** Let  $\mathbb{T} = 2^{\mathbb{N}}$ ,  $\alpha > \gamma + 1 > 2$  and  $\alpha = \text{odd/odd}$ . Consider the neutral dynamic equation

$$\left( (Z^\Delta(t))^\alpha \right)^\Delta + \int_c^d \frac{(|\sin t| + 1)^\alpha}{t^\gamma (2 - \sin^2(t\xi))} x^\alpha(t) \Delta \xi = 0, \quad (3.2)$$

where  $\tau(t)$  satisfies (A3), and  $Z(t) = x(t) + [|\sin t| / (|\sin t| + 1)] x(\tau(t))$ . Here

$$r(t) = 1, \quad p(t) = \frac{|\sin t|}{|\sin t| + 1}, \quad q(t, \xi) = \frac{(|\sin t| + 1)^\alpha}{2 t^\gamma},$$

and

$$\theta(t, \xi) = t, \quad \delta(t) = \theta(t, c) = t.$$

Clearly (A1) holds. Noting that the function  $r(t, T)/r(\sigma(t), T) = (t - T)/(\sigma(t) - T)$  is strictly increasing. Hence,  $\eta_1(t, T) = \eta_2(t, T) \geq 1/3$  for sufficiently large  $T \in [t_0, \infty)_{\mathbb{T}}$  and  $t \geq \sigma(T) = 2T$ . Then

$$\begin{aligned} R(t) &= \int_c^d q(t, \xi)(1 - p(\theta(t, \xi)))^\alpha \Delta \xi \\ &= \int_c^d \frac{(|\sin t| + 1)^\alpha}{2 t^\gamma} \frac{1}{(|\sin t| + 1)^\alpha} \Delta \xi \\ &= \frac{d - c}{2 t^\gamma}, \end{aligned}$$

Let  $a(t) = 0$  and  $\eta(t) = 1$  in Theorem 2.3, we have  $\varphi_1(t) = 0$  for  $t > T$ . Choosing  $T_1 = 2T$ , then

$$\frac{d - c}{2 t^\gamma} > \varphi(t) = \eta^\sigma(t)R(t)\eta_1^\alpha(t, T) \geq \frac{d - c}{2(3^\alpha t^\gamma)},$$

and

$$[\varphi_1(t)]_+ = 0, \quad \frac{\eta(T_1)}{r^\alpha(T_1, T)} = \frac{1}{T^\alpha}.$$

Hence

$$\begin{aligned} \infty &> \limsup_{t \rightarrow \infty} \int_{T_1}^t \frac{d - c}{2 s^\gamma} \Delta s > \limsup_{t \rightarrow \infty} \int_{T_1}^t \left( \varphi(s) - \frac{\alpha^\alpha ([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha + 1)^{\alpha+1} \varphi_2^\alpha(s)} \right) \Delta s \\ &\geq \limsup_{t \rightarrow \infty} \int_{T_1}^t \frac{d - c}{2(3^\alpha s^\gamma)} \Delta s > \frac{\eta(T_1)}{r^\alpha(T_1, T)}, \end{aligned}$$

Thus, by Theorem 2.3, Eq. (3.2) is oscillatory.

## Acknowledgements

The first author was supported by The Foundation of Jishou University (No. 2015JSUJGB65) and was part supported by The Hunan province Natural Science Foundation (No. 2015JJ4041).

We are grateful to the anonymous referees for their careful reading and helpful comments which led to an improvement of our original manuscript.

## References

- [1] H. A. AGWO, Oscillation of nonlinear second order neutral delay dynamic equations on time scales, *Bull. Korean Math. Soc.* **45**(2008), 299–312. [MR2419078](#)
- [2] R. P. AGARWAL, D. O' REGAN, S. H. SAKER, Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales, *Appl. Anal.* **86**(2007), 1–17. [MR2295322](#)
- [3] R. P. AGARWAL, C. ZHANG, T. LI, New Kamenev-type oscillation criteria for second-order nonlinear advanced dynamic equations, *Appl. Math. Comput.* **225**(2013), 822–828. [MR3129695](#)
- [4] R. P. AGARWAL, M. BOHNER, T. LI, C. ZHANG, Oscillation criteria for second-order dynamic equations on time scales, *Appl. Math. Lett.* **31**(2014), 34–40. [MR3178426](#)

- [5] M. BOHNER, A. PETERSON, Dynamic equations on time scales: an introduction with applications, Birkhäuser, Boston, 2001. [MR1843232](#)
- [6] M. BOHNER, T. LI, Kamenev-type criteria for nonlinear damped dynamic equations, *Sci. China Math.* **58**(2015), 1445–1452. [MR3353981](#)
- [7] T. CANDAN, Oscillation of second-order nonlinear neutral dynamic equations on time scales with distributed deviating arguments, *Comput. Math. Appl.* **62**(2011), 4118–4125. [MR2859967](#)
- [8] T. CANDAN, Oscillation criteria for second-order nonlinear neutral dynamic equations with distributed deviating arguments on time scales, *Adv. Difference Equ.* **2013**, 2013:112, 8 pp. [MR3057259](#)
- [9] T. CANDAN, Oscillatory behavior of second order nonlinear neutral differential equations with distributed deviating arguments, *Appl. Math. Comput.* **262**(2015), 199–203. [MR3346536](#)
- [10] D. CHEN, Oscillation of second-order Emden–Fowler neutral delay dynamic equations on time scales, *Math. Comput. Modelling* **51**(2010), 1221–1229. [MR2608908](#)
- [11] D. CHEN, J. LIU, Oscillation theorems for second-order nonlinear neutral dynamic equations on time scales with distributed delay (in Chinese), *J. Systems Sci. Math. Sci.* **30**(2010), 1191–1205. [MR2785243](#)
- [12] X. DENG, Q. WANG, Z. ZHOU, Oscillation criteria for second order nonlinear delay dynamic equations on time scales, *Appl. Math. Comput.* **269**(2015), 834–840. [MR3396826](#)
- [13] Z. HAN, T. LI, S. SUN, C. ZHANG, On the oscillation of second-order neutral delay dynamic equations on time scales. *Afr. Diaspora J. Math. (N.S.)* **9**(2010), 76–86. [MR2565577](#)
- [14] Z. HAN, S. SUN, T. LI, C. ZHANG, Oscillatory behavior of quasilinear neutral delay dynamic equations on time scales, *Adv. Difference Equ.* **2010**, Art. ID 450264. [MR2601358](#)
- [15] T. LI, E. THANDAPANI, Oscillation of second-order quasilinear neutral functional dynamic equations with distributed deviating arguments, *J. Nonlinear Sci. Appl.* **4**(2011), 180–192. [url](#)
- [16] S. H. SAKER, D. O'REGAN, R. P. AGARWAL, Oscillation theorems for second-order nonlinear neutral delay dynamic equations on time scales. *Acta Math. Sin. (Engl. Ser.)* **24**(2008), 1409–1432. [MR2438311](#)
- [17] S. H. SAKER, Oscillation of superlinear and sublinear neutral delay dynamic equations, *Commun. Appl. Anal.* **12**(2008), 173–187. [MR2191492](#)
- [18] S. H. SAKER, Oscillation criteria for a certain class of second-order neutral delay dynamic equations. *Dynam. Cont. Discr. Impul. Syst. Ser B Appl. Algorithms* **16**(2009), 433–452. [MR2513020](#)
- [19] S. H. SAKER, Oscillation criteria for a second-order quasilinear neutral functional dynamic equation on time scales. *Nonlinear Oscil.* **13**(2011), 407–428. [MR2797503](#); [url](#)

- [20] S. H. SAKER, D. O'REGAN, New oscillation criteria for second-order neutral functional dynamic equations via the generalized Riccati substitution, *Commun. Nonlinear Sci. Numer. Simul.* **16**(2011), 423–434. [MR2679193](#)
- [21] Y. ŞAHINER, Oscillation of second-order neutral delay and mixed-type dynamic equations on time scales, *Adv. Difference Equ.* **2006**, Art. ID 65626, 9 pp. [MR2255172](#); [url](#)
- [22] E. THANDAPANI, V. PIRAMANANTHAM, Oscillation criteria of second order neutral delay dynamic equations with distributed deviating arguments, *Electron. J. Qual. Theory Differ. Equ.* **2010**, No. 61, 1–15. [MR2725004](#); [url](#)
- [23] Q. YANG, Z. XU, Oscillation criteria for second order quasilinear neutral delay differential equations on time scales, *Comput. Math. Appl.* **62**(2011), 3682–3691. [MR2852090](#); [url](#)
- [24] Q. YANG, B. JIA, Z. XU, Nonlinear oscillation of second-order neutral dynamic equations with distributed delay, *Math. Methods Appl. Sci.* **39**(2016), 202–213. [MR3453705](#); [url](#)