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The study of higher-order resonant and non-resonant boundary value problems

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Abstract. The existence of at least one solution to a nonlinear n^{th} order differential equation $x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$, $0 < t < 1$, under both non-resonant and resonant boundary conditions, is proved. The methods involve the characterization of the R_δ-set and an application of a new generalization for a multi-valued version of the Miranda Theorem.

Keywords: nonlinear boundary value problem, decomposable maps, R_δ-set, set-valued map.

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1 Introduction

Higher order differential equations have been extensively studied in recent years. A variety of results ranging from the theoretical aspects of existence and uniqueness of solutions to analytic and numerical methods for finding solutions have appeared in the literature [1, 6, 11–13, 18]. Such differential equations can be written in the form $Lx = Nx$, where L is a linear differential operator defined in appropriate Banach spaces and N is a nonlinear operator. When L is a linear Fredholm operator of index 0 under certain boundary conditions, then the kernel of the linear part of the above equation is trivial, and in this case, the corresponding BVP is called non-resonant. This means that there exists an integral operator; then, topological methods can be applied to prove existence theorems. If the kernel of L is nontrivial, then the problem is said to be at resonance, and then the problem can be managed by using coincidence degree theory. Such boundary value problems for higher order differential equations have been studied by standard methods in many papers; see, for instance, the papers [2, 4, 7–10, 15–17].

Motivated in this paper by the above research, by applying a generalized Miranda Theorem [14] and a technique completely different from the methods mentioned above, we obtain results devoted to the study of the following higher-order nonlinear differential equation

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}), \qquad 0 < t < 1, \tag{1.1}$$

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under the non-resonant boundary conditions

$$x^{(i)}(0) = 0, \qquad x(1) = 0, \qquad i = 1, 2, \ldots, n-1, \tag{1.2}$$

and under the resonant conditions

$$x^{(i)}(0) = 0, \qquad x'(1) = 0, \qquad i = 1, 2, \ldots, n-1, \tag{1.3}$$

respectively, where

$$f\colon [0,1] \times \underbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}_{n} \to \mathbb{R}^k$$

is a continuous vector function and satisfies appropriate growth conditions; in particular, we assume:

(H1) $|f(t, X_1, X_2, \ldots, X_n)| \le a_1(t)|X_1| + a_2(t)|X_2| + \cdots + a_n(t)|X_n| + a_{n+1}(t)$, where $a_1, a_2, \ldots, a_{n+1} \in C([0,1], \mathbb{R}_+)$;

(H2) there exists $m_i > 0$ such that $x_{1i} \cdot f_i(t, X_1, X_2, \ldots, X_n) \ge 0$ for $t \in [0,1]$, $X_j = (x_{j1}, x_{j2}, \ldots, x_{jk}) \in \mathbb{R}^k$, and $|x_{1i}| \ge m_i$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$.

The rest of this paper is organized as follows. In Section 2, we give some preliminary definitions and theorems on the topological structure of certain sets in metric spaces, which will be employed to obtain the main results. In Section 3, we study the non-resonant BVP (1.1)–(1.2). By introducing an auxiliary initial value problem and characterizing the upper semicontinuous set-valued R_δ-map, we show that this problem has at least one solution. Finally, in Section 4, we deal with the resonant BVP (1.1)–(1.3). By a differential transformation, we get the desired results by adopting the techniques used in Section 3.

2 Preliminaries

First, we present some notations and terminologies.

Definition 2.1. A metric space X is an absolute neighborhood retract (written *ANR*) if, given a space Y and a homeomorphic embedding $i\colon X \to Y$ of X onto a closed subset $i(X) \subset Y$, $i(X)$ is a neighborhood retract of Y, i.e. there is an open neighborhood U of $i(X)$ in Y and a retraction $r\colon U \to i(X)$. A map $r\colon U \to i(X)$ is a retraction provided that $r(y) = y$ for $y \in i(X)$.

Definition 2.2. A nonempty space X is contractible provided there exist $x_0 \in X$ and a homotopy $h\colon X \times [0,1] \to X$ such that $h(x,0) = x$ and $h(x,1) = x_0$ for every $x \in X$.

Definition 2.3. A compact (nonempty) space X is an R_δ-set (we write $X \in R_\delta$), if there is a decreasing sequence X_n of compact contractible spaces such that $X = \cap_{n\ge1} X_n$.

Definition 2.4. A set-valued map $\Phi\colon X \multimap Y$ is upper semicontinuous (written *USC*) if, given an open $V \subset Y$, the set $\{x \in X : \Phi(x) \subset V\}$ is open. We say Φ is an R_δ-map if it is *USC* and, for each $x \in X$, $\Phi(x) \in R_\delta$.

Definition 2.5. By a decomposable map we mean a pair (D, F) consisting of a set-valued map $F\colon X \multimap Y$ and a diagram $D\colon X \overset{\Phi}{\multimap} Z \xrightarrow{\varphi} Y$, where $Z \in ANR$, $\Phi\colon X \multimap Z$ is an R_δ-map, and $\varphi\colon Z \to Y$ a single-valued continuous map, such that $F = \varphi \circ \Phi$.

Remark 2.6. In our case, Z will be a Banach space, which is *ANR*. Moreover, notice that a decomposable map (D, F) is an admissible map in the sense of Górniewicz (see [5]).

Remark 2.7. A superposition of a set-valued map with compact values and continuous function is an *USC* map, so any decomposable map is *USC*.

Definition 2.8. We say the two decomposable maps (D_0, F_0), (D_1, F_1) where $D_k\colon X \overset{\Phi_k}{\multimap} Z_k \xrightarrow{\varphi_k} Y$, $k = 0, 1$, are homotopic (we write $(D_0, F_0) \simeq (D_1, F_1)$), if there is a decomposable map $(\check{D}, \check{F})$ with $\check{D}\colon X \times [0, 1] \overset{\check{\Phi}}{\multimap} Z \xrightarrow{\check{\varphi}} Y$ and maps $j_k\colon Z_k \to Z$, $k = 0, 1$, such that the diagram

$$\begin{array}{ccccc}
X & \overset{\Phi_0}{\longrightarrow} & Z_0 & & \\
\downarrow^{i_0} & & \downarrow^{j_0} & \searrow^{\varphi_0} & \\
X \times [0,1] & \overset{\check{\Phi}}{\longrightarrow} & Z & \overset{\check{\varphi}}{\longrightarrow} & Y \\
\uparrow_{i_1} & & \uparrow_{j_1} & \nearrow_{\varphi_1} & \\
X & \overset{\Phi_1}{\longrightarrow} & Z_1 & &
\end{array}$$

where $i_k(x) = (x, k)$ for $x \in X$, $k = 0, 1$, is commutative.

Next, we present a result from [3] about the topological structure of the set of solutions for some nonlinear functional equations.

Theorem 2.9. *Let X be a space, $(B, \|\cdot\|)$ a Banach space and $h\colon X \to B$ a proper map, i.e. h is continuous and for every compact $E \subset B$ the set $h^{-1}(E)$ is compact. Assume further that for each $\varepsilon > 0$ a proper map $h_\varepsilon\colon X \to B$ is given and the following two conditions are satisfied:*

(a) $\|h_\varepsilon(x) - h(x)\| < \varepsilon$, *for every* $x \in X$;

(b) *for any* $\varepsilon > 0$ *and* $u \in B$ *such that* $\|u\| \le \varepsilon$, *the equation* $h_\varepsilon(x) = u$ *has exactly one solution.*

Then the set $S = h^{-1}(0)$ is R_δ.

Next, we present a generalization of the Miranda Theorem proven in [14], which will be of crucial importance.

Theorem 2.10. *Let $M_i > 0$, $i = 1, \dots, k$, and F be an admissible map from $\prod_{i=1}^k [-M_i, M_i]$ to $\mathbb{R}^k$, i.e. there exist a Banach space E, $\dim E \ge k$, a linear, bounded and surjective map $\varphi\colon E \to \mathbb{R}^k$ and an R_δ-map Φ from $\prod_{i=1}^k [-M_i, M_i]$ to E such that $F = \varphi \circ \Phi$. If for any $i = 1, \dots, k$ and every $y \in F(x)$, where $|x_i| = M_i$, we have*

$$x_i \cdot y_i \ge 0, \tag{2.1}$$

then there exists $x \in \prod_{i=1}^k [-M_i, M_i]$ such that $0 \in F(x)$.

3 Solutions for BVP (1.1)–(1.2)

In this section, we discuss the BVP (1.1)–(1.2).

First, in setting the stage for the application of Theorem 2.10, let the Banach space $(B, \|\cdot\|_B)$ be defined by

$$\begin{aligned}
B &:= \{x \in C^{n-1}([0,1], \mathbb{R}^k) : x^{(i)}(0) = 0,\ i = 1, 2, \dots, n-1\}, \\
\|x\|_B &:= \max\{|x|_0,\ |x^{(n-1)}|_0\},
\end{aligned}$$

where $|x|_0 = \max\{x(t) : t \in [0,1]\}$.

Next, we consider the equation (1.1) under the following initial conditions:

$$x(0) = c, \qquad x^{(i)}(0) = 0, \qquad i = 1, 2, \ldots, n-1, \tag{3.1}$$

where $c \in \mathbb{R}^k$ is fixed. Notice that the IVP (1.1)–(3.1) is equivalent to

$$x(t) = c + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f\big(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\big) ds, \qquad t \in [0,1]. \tag{3.2}$$

Moreover, we have

$$x^{(n-1)}(t) = \int_0^t f\big(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\big) ds, \qquad t \in [0,1]. \tag{3.3}$$

Note that

$$\begin{aligned} x(t) &= c + \frac{1}{(n-2)!} \int_0^t (t-s)^{n-2} x^{(n-1)}(s) ds, \\ x^{(i)}(t) &= \frac{1}{(n-2-i)!} \int_0^t (t-s)^{n-2-i} x^{(n-1)}(s) ds, \qquad i = 1, 2, \ldots, n-2. \end{aligned}$$

Then, by applying (3.3) and (H1), for $t \in [0,1]$, we get

$$\begin{aligned} |x^{(n-1)}(t)| \le{}& \int_0^t \Big(a_1(s)|x(s)| + a_2(s)|x'(s)| + \cdots + a_n(s)|x^{(n-1)}(s)| + a_{n+1}(s) \Big) ds \\ \le{}& \int_0^t \Bigg\{ a_1(s) \bigg(|c| + \frac{1}{(n-2)!} \int_0^s (s-\tau)^{n-2} |x^{(n-1)}(\tau)| d\tau \bigg) \\ & \quad + a_2(s) \frac{1}{(n-3)!} \int_0^s (s-\tau)^{n-3} |x^{(n-1)}(\tau)| d\tau \\ & \quad + \cdots \\ & \quad + a_{n-1}(s) \int_0^s |x^{(n-1)}(\tau)| d\tau + a_n(s)|x^{(n-1)}(s)| + a_{n+1}(s) \Bigg\} ds \\ \le{}& \int_0^t \Bigg\{ a_1(s)|c| + a_{n+1}(s) \\ & \quad + \left[\frac{a_1(s)s^{n-1}}{(n-1)!} + \frac{a_2(s)s^{n-2}}{(n-2)!} + \cdots + a_{n-1}(s)s + a_n(s) \right] \cdot \max_{\tau \in [0,s]} |x^{(n-1)}(\tau)| \Bigg\} ds \end{aligned}$$

Set $w(t) = \max_{s \in [0,t]} |x^{(n-1)}(s)|$. We obtain

$$\begin{aligned} |w(t)| \le{}& \int_0^t (a_1(s)|c| + a_{n+1}(s)) ds \\ & + \int_0^t \left[\frac{a_1(s)s^{n-1}}{(n-1)!} + \frac{a_2(s)s^{n-2}}{(n-2)!} + \cdots + a_{n-1}(s)s + a_n(s) \right] w(s) ds \\ \le{}& K_c + \int_0^t \left[\frac{a_1(s)s^{n-1}}{(n-1)!} + \frac{a_2(s)s^{n-2}}{(n-2)!} + \cdots + a_{n-1}(s)s + a_n(s) \right] w(s) ds, \end{aligned}$$

where

$$K_c = \int_0^1 (a_1(s)|c| + a_{n+1}(s)) ds.$$

Now, in view of Gronwall's Lemma, we have

$$w(t) \le K_c \exp \int_0^t \left[\frac{a_1(s)s^{n-1}}{(n-1)!} + \frac{a_2(s)s^{n-2}}{(n-2)!} + \cdots + a_{n-1}(s)s + a_n(s) \right] ds.$$

Hence

$$|x^{(n-1)}(t)| \le K_c \cdot e^K, \qquad t \in [0,1], \tag{3.4}$$

where

$$K = \int_0^1 \left[\frac{a_1(s)s^{n-1}}{(n-1)!} + \frac{a_2(s)s^{n-2}}{(n-2)!} + \cdots + a_{n-1}(s)s + a_n(s) \right] ds.$$

By (3.4), we get the following estimate

$$|x(t)| = \left| c + \frac{1}{(n-2)!} \int_0^t (t-s)^{n-2} x^{(n-1)}(s) ds \right| \le |c| + \frac{K_c \cdot e^K}{(n-1)!} < \infty. \tag{3.5}$$

From above, the Leray–Schauder Alternative implies that the IVP (1.1)–(3.1) has a bounded global solution for every $t \in [0,1]$ and fixed $c \in \mathbb{R}^k$.

Now, given $c \in \mathbb{R}^k$, consider the nonlinear operator $T\colon \mathbb{R}^k \times B \to B$, $(c,x) \mapsto T_c(x)$, defined as

$$T_c(x)(t) = c + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, x(s), x'(s), \ldots, x^{(n-1)}(s)) ds, \qquad t \in [0,1]. \tag{3.6}$$

It is clear that $T_c(x)\colon [0,1] \to \mathbb{R}^k$ is continuous. Moreover, by applying (H1), (3.4) and (3.5) one can easily show that the image of

$$\{(c,x) \in \mathbb{R}^k \times B : \|(c,x)\|_{\mathbb{R}^k \times B} \le L\}$$

under T is relatively compact. we obtain the following results.

Lemma 3.1. *Let assumption* (H1) *hold. Then the operator T is completely continuous.*

Notice that the solutions of the IVP (1.1)–(3.1) are fixed points of the operator T defined by (3.6). Let $\operatorname{Fix} T_c(\cdot)$ denote the set of fixed points of operator T_c, where $c \in \mathbb{R}^k$ is given.

Lemma 3.2. *Let assumption* (H1) *hold and $\Phi\colon \mathbb{R}^k \ni c \multimap \operatorname{Fix} T_c(\cdot) \subseteq C^{n-1}([0,1], \mathbb{R}^k)$. Then the set-valued map Φ is USC with compact values.*

Proof. The set-valued map Φ is *USC* with compact values, if given a sequence $\{c_n\}$ in $\mathbb{R}^k$, $c_n \to c_0$ and $\{x_n\} \subseteq \Phi(c_n)$, $\{x_n\}$ has a converging subsequence to some $x_0 \in \Phi(c_0)$.

Taking any sequence $\{c_n\}$, $c_n \to c_0$ and $\{x_n\} \subseteq \Phi(c_n)$, we have

$$x_n = T_{c_n}(x_n). \tag{3.7}$$

Since $\{c_n\}$ is bounded, by (3.4) and (3.5), we see that $\{x_n(t)\} \subset B$, $t \in [0,1]$ is equibounded. So $\{x_n\}$ is bounded in B. Lemma 3.1 yields that the operator T is completely continuous. Then, by (3.7), $\{x_n\}$ is relatively compact in B. Passing to a subsequence if necessary, we may assume that $x_n \to x_0$ in B. The continuity of T implies that

$$x_0 = T_{c_0}(x_0).$$

Thus, $x_0 \in \Phi(c_0)$ and the proof is complete. □

Lemma 3.3. *Let assumption* (H1) *hold. Then* Φ *is an* R_δ*-map.*

Proof. By Lemma 3.2, the map Φ is *USC*. It remains to be shown that for any $c \in \mathbb{R}^k$, $\Phi(c) \in R_\delta$, i.e. the set $\operatorname{Fix} T_c(\cdot)$ is an R_δ-set.

Let $E = \{x \in B : \|x\| \le R\}$, where $R := \max\{K_c \cdot e^K,\ |c| + \frac{K_c \cdot e^K}{(n-1)!}\}$. Define an operator $h\colon E \to B$ as $h(x) = x - T_c(x)$. Since $T_c\colon E \to B$ is a compact map from Lemma 3.1, h is a compact vector field associated with $T_c(\cdot)$. Then we shall show that there exists a sequence $h_n\colon E \to B$ of continuous proper mappings satisfying conditions (a) and (b) of Theorem 2.9 with respect to h.

For the proof it is sufficient to define a sequence $T_c^n(\cdot)\colon E \to B$ of compact maps such that $T_c(x) = \lim\limits_{n\to\infty} T_c^n(x)$ uniformly in E and show that $h_n(x) = x - T_c^n(x)$ is a one-to-one map. To do this, we define auxiliary mappings $r_n\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$r_n(t) := \begin{cases} 0, & t \in [0, \frac{1}{n}], \\ t - \frac{1}{n}, & t \in (\frac{1}{n}, 1]. \end{cases}$$

Now, we can define the sequence $\{T_c^n(\cdot)\}$ as follows:

$$T_c^n(x)(t) = T_c(x)(r_n(t)), \qquad x \in E,\ n \in \mathbb{N}_+. \tag{3.8}$$

We see that $T_c^n(\cdot)$ are continuous and compact. Since $|r_n(t) - t| \le \frac{1}{n}$, we deduce from the compactness of $T_c^n(\cdot)$ and (3.8) that $T_c^n(x) \to T_c(x)$ uniformly in E.

Next, we shall prove that h_n is a one-to-one map. Assume that for some $u, v \in E$, we have $h_n(u) = h_n(v)$. This implies that

$$u - v = T_c^n(u) - T_c^n(v).$$

If $t \in [0, \frac{1}{n}]$, then we have

$$u(t) - v(t) = T_c(u)(r_n(t)) - T_c(v)(r_n(t)) = T_c(u)(0) - T_c(v)(0) = 0.$$

Thus, we obtain $u(t) = v(t)$ for $t \in [0, \frac{1}{n}]$.

If $t \in [\frac{1}{n}, \frac{2}{n}]$, then $r_n(t) \in [0, \frac{1}{n}]$, $r_n(r_n(t)) = 0$. Hence, by the property of operator $T_c(\cdot)$ mentioned above, we have

$$T_c^n(u)(r_n(t)) = T_c(u)(r_n(r_n(t))) = 0 \quad \text{and} \quad T_c(u)(t) = \lim_{n\to\infty} T_c^n(u)(r_n(t)) = 0$$

for $t \in [\frac{1}{n}, \frac{2}{n}]$. So, we can get $T_c(v)(t) = 0$ for $t \in [\frac{1}{n}, \frac{2}{n}]$. Thus, we have $u(t) = v(t)$ for $t \in [0, \frac{2}{n}]$. By repeating this procedure n times we infer that $u(t) = v(t)$ for $t \in [0,1]$. Therefore, h_n is a one-to-one map. Hence the assumptions of Theorem 2.9 hold and $h^{-1}(0) = \operatorname{Fix} T_c(\cdot)$ is an R_δ-set. ☐

Consider a multifunction $F\colon \mathbb{R}^k \multimap \mathbb{R}^k$ given by

$$F(c) := \{x(1) : x \in \operatorname{Fix} T_c\}. \tag{3.9}$$

Now, let $\varphi\colon B \to \mathbb{R}^k$ be such that

$$\varphi(x) = x(1).$$

It is easy to see that φ is continuous, linear and surjective. Hence the map $F = \varphi \circ \Phi$ is decomposable with a decomposition

$$\mathbb{R}^k \overset{\Phi}{\multimap} B \overset{\varphi}{\to} \mathbb{R}^k.$$

Theorem 3.4. *If assumptions* (H1) *and* (H2) *are satisfied, then the BVP* (1.1)–(1.2) *has at least one solution.*

Proof. Let $x \in \operatorname{Fix} T_c(\cdot)$ be a bounded global solution of the IVP (1.1)–(3.1). Observe that $x(t)$ is a solution of the BVP (1.1)–(1.2) if there exists a $c \in \mathbb{R}^k$ such that $0 \in F(c)$. So we will show that all the assumptions in Theorem 2.10 are satisfied.

By Lemma 3.2 and 3.3, φ, Φ and F satisfy the assumptions of Theorem 2.10. Now, we shall show that the condition (2.1) holds.

Let $c_i = m_i + 1$, where m_i is as in (H2) for $i = 1, 2, \ldots, k$. First, we shall prove that $x_i'(t) \geq 0$ for $t \in [0, 1]$. From (3.1), we have $x_i'(0) = 0$. Assume that for some $t \in [0, 1]$, we have $x_i'(t) < 0$. Then there exists $t_* := \inf\{t \in [0, 1] : x_i'(t) < 0\}$ such that $x_i'(t_*) = 0$ and $x_i'(t) \geq 0$ for $t < t_*$. Since $x_i'(t)$ is continuous, there exists $t_1 > t_*$ such that $\int_{t_*}^{t_1} |x_i'(t)| dt \leq 1$. Hence, in view of $x_i(0) = c_i$, we get

$$x_i(t) \geq c_i + \int_{t_*}^{t} x_i'(s) ds \geq m_i + 1 - \int_{t_*}^{t_1} |x_i'(t)| dt \geq m_i > 0, \qquad t \in [t_*, t_1].$$

Now, by condition (H2), we obtain

$$x_i(t) \cdot f_i(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) = x_i(t) \cdot x_i^{(n)}(t) \geq 0, \qquad t \in [t_*, t_1].$$

So, $x_i^{(n)}(t) \geq 0$ for $t \in [t_*, t_1]$. This means that $x_i^{(n-1)}(t)$ is nondecreasing on $[t_*, t_1]$. Since $x_i^{(n-1)}(0) = 0$, we have $x_i^{(n-1)}(t) \geq 0$ for $t \in [t_*, t_1]$, which means that $x_i^{(n-2)}(t)$ is nondecreasing on $[t_*, t_1]$. By repeating this procedure, we can see that $x_i''(t) \geq 0$ for $t \in [t_*, t_1]$, which means that $x_i'(t)$ is nondecreasing on $[t_*, t_1]$, and $x_i'(t) \geq x_i'(t_*) = 0$ for $t \in [t_*, t_1]$, which is a contradiction. Hence $x_i'(t) \geq 0$ for $t \in [0, 1]$.

Since $x_i(0) = c_i$, we have $x_i(t) \geq c_i$ for $t \in [0, 1]$. Then $x_i(1) \geq c_i$. Therefore, by the definition of F, $c_i \cdot x_i(1) \geq 0$. Hence, the condition (2.1) in Theorem 2.10 is satisfied for $M_i := m_i + 1$, $i = 1, 2, \ldots, k$. We can proceed analogously to prove (2.1) in the case $c_i = -(m_i + 1)$.

By Theorem 2.10, there exists $c \in \mathbb{R}^k$ such that $0 \in F(c)$. This completes the proof. □

Remark 3.5. In view of (3.4) and (3.5), one can see that the solutions of the BVP (1.1)–(1.2) are globally bounded.

4 Solutions for BVP (1.1)–(1.3)

In this section, based on the technique and results in the previous section, we search for solutions for the resonant BVP (1.1)–(1.3).

First, we define the Banach space $(\hat{B}, \|\cdot\|_{\hat{B}})$ as

$$\hat{B} := \{y \in C^{n-2}([0, 1], \mathbb{R}^k) : y^{(i)}(0) = 0,\ i = 0, 1, 2, \ldots, n-2\}$$

with the norm $\|y\|_{\hat{B}} := |y^{(n-2)}|_0$. Let $y(t) = x'(t)$, then the equation (1.1) can be written as

$$y^{(n-1)} = f\Big(t, c + \int_0^t y(s) ds, y, y', \ldots, y^{(n-2)}\Big), \qquad 0 < t < 1, \tag{4.1}$$

where $c \in \mathbb{R}^k$. Now, we consider the equation (4.1) under the following initial conditions:

$$y^{(i)}(0) = 0, \qquad i = 0, 1, 2, \ldots, n-2. \tag{4.2}$$

We can see that the IVP (1.1)–(3.1) is equivalent to the IVP (4.1)–(4.2). Also, we can write the IVP (4.1)–(4.2) in the following form

$$y(t) = \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2} f\Big(s, c + \int_0^s y(\tau)d\tau, y(s), y'(s), \dots, y^{n-2}(s)\Big)ds \tag{4.3}$$

for $t \in [0,1]$. Moreover, we have

$$y^{(n-2)}(t) = \int_0^t f\Big(s, c + \int_0^s y(\tau)d\tau, y(s), y'(s), \dots, y^{n-2}(s)\Big)ds, \qquad t \in [0,1]. \tag{4.4}$$

Then, by (H1) and (4.4) and applying Gronwall's Lemma as in Section 3, we can show that

$$|y^{(n-2)}(t)| \le \hat{K}_c \cdot e^{\hat{K}} < \infty, \qquad t \in [0,1], \tag{4.5}$$

where

$$\hat{K}_c = \int_0^1 (a_1(s)|c| + a_{n+1}(s))ds,$$

$$\hat{K} = \int_0^1 \Big[\frac{1}{(n-1)!}a_1(s)s^{n-1} + \frac{1}{(n-2)!}a_2(s)s^{n-2} + \dots + a_{n-1}(s)s + a_n(s)\Big]ds.$$

So, the IVP (4.1) has a bounded global solution for every fixed $c \in \mathbb{R}^k$ and $t \in [0,1]$. Now, we also consider a nonlinear operator $\hat{T} \colon \mathbb{R}^k \times \hat{B} \to \hat{B}$, $(c, y) \mapsto \hat{T}_c(y)$, given by

$$\hat{T}_c(y)(t) = \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2} f\Big(s, c + \int_0^s y(\tau)d\tau, y(s), y'(s), \dots, y^{n-2}(s)\Big)ds, \tag{4.6}$$

which is completely continuous.

Notice that the solutions of the IVP (4.1) are fixed points of the operator $\hat{T}$ defined by (4.6). Let Fix $\hat{T}_c(\cdot)$ denote the set of fixed points of operator $\hat{T}_c$, where $c \in \mathbb{R}^k$ is given. We consider the map:

$$F \colon \mathbb{R}^k \multimap \mathbb{R}^k, \qquad F(c) := \{y(1) : y \in \operatorname{Fix} \hat{T}_c\}. \tag{4.7}$$

If $\hat{\varphi} \colon \hat{B} \to \mathbb{R}^k$ is a map such that

$$\hat{\varphi}(y) = y(1),$$

then $\hat{\varphi}$ is continuous, linear and surjective. Next, we define a map $\hat{\Phi}$ by

$$\hat{\Phi} \colon \mathbb{R}^k \ni c \multimap \operatorname{Fix} \hat{T}_c(\cdot) \subset \hat{B}$$

and notice that $F = \hat{\varphi} \circ \hat{\Phi}$.

The proof of the lemma below is similar to the proof of Lemma 3.2, so we omit it here.

Lemma 4.1. *Let assumption* (H1) *hold. Then the set-valued map $\hat{\Phi}$ is USC with compact values.*

Lemma 4.2. *Let assumption* (H1) *hold. Then $\hat{\Phi}$ is an R_δ-map.*

Proof. By Lemma 4.1, the map $\hat{\Phi}$ is *USC*. We shall show that for any $c \in \mathbb{R}^k$, the set $\hat{\Phi}(c) \in R_\delta$, i.e. the set Fix $\hat{T}_c(\cdot)$ is R_δ-set.

Let $\hat{E} = \{x \in B : \|x\| \le \hat{R}\}$, where $\hat{R} := \hat{K}_c \cdot e^{\hat{K}}$ is taken from (4.5). It is sufficient to observe for the operator $\hat{T}_c \colon \hat{E} \to \hat{B}$, analogous to the proof of Lemma 3.3, the conditions of Theorem 2.9 are satisfied. Hence, $\hat{\Phi}$ is an R_δ-map. □

The following theorem holds true.

Theorem 4.3. *Under assumptions* (H1) *and* (H2), *the BVP* (1.1)–(1.3) *has at least one solution.*

Proof. Let $y \in \operatorname{Fix} \hat{T}_c(\cdot)$ be a bounded global solution of the IVP (4.1). Observe that $x(t) = c + \int_0^t y(s)ds$ is a solution of the BVP (1.1)–(1.3) if there exists a $c \in \mathbb{R}^k$ such that $0 \in \hat{F}(c)$. Notice that functions $\hat{\varphi}$, $\hat{\Phi}$ and $\hat{F}$ satisfy the assumptions of Theorem 2.10. Now, we shall show that the condition (2.1) holds.

Let $c_i = m_i + 1$, where m_i is as in (H2). First, we shall prove that $y_i'(t) \geq 0$ for $t \in [0,1]$. To prove this fact we proceed similarly as in the proof of Theorem 3.4. Now, by the definition of $\hat{F}$, $c_i \cdot y_i(1) \geq 0$. Hence the condition (2.1) in Theorem 2.10 is satisfied for $M_i := m_i + 1$. We can proceed analogously to prove (2.1) in the case that $c_i = -(m_i + 1)$.

Hence, by Theorem 2.10, there exists $c \in \mathbb{R}^k$ such that $0 \in \hat{F}(c)$ and the proof is complete. $\square$

Remark 4.4. Consider the following boundary conditions

$$x^{(i)}(0) = 0, \qquad x^{(j)}(1) = 0, \qquad i = 1, 2, \ldots, n-1, \tag{4.8}$$

where j is a fixed integer such that $1 \leq j \leq n-1$. Then for each j, the BVP (1.1)–(4.8) is resonant. Let $y(t) = x^{(j)}(t)$, then the IVP (1.1)–(3.1) can be written as

$$\begin{cases} y^{(n-j)} = f\Big(t, c + \dfrac{1}{(j-1)!}\displaystyle\int_0^t (t-s)^{j-1} y(s)ds, \dfrac{1}{(j-2)!}\displaystyle\int_0^t (t-s)^{j-2} y(s)ds, \ldots, \\ \qquad\qquad \displaystyle\int_0^t (t-s)y(s)ds, \int_0^t y(s)ds, y, y', \ldots, y^{(n-j-1)}\Big), \quad 0 < t < 1, \\ y^{(i)}(0) = 0, \quad i = 0, 1, 2, \ldots, n-j-1, \end{cases} \tag{4.9}$$

where $c \in \mathbb{R}^k$. Define the Banach space $(\hat{\hat{B}}, \|\cdot\|_{\hat{\hat{B}}})$ as

$$\hat{\hat{B}} := \Big\{ y \in C^{n-j-1}([0,1], \mathbb{R}^k) : y^{(i)}(0) = 0,\ i = 0, 1, 2, \ldots, n-j-1 \Big\}, \qquad \|y\|_{\hat{\hat{B}}} := |y^{(n-j-1)}|_0.$$

Adopting the similar techniques used for the BVP (1.1)–(1.3) in this section, we also can obtain the corresponding results for the BVP (1.1)–(4.8).

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