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# DICHOTOMY AND ALMOST AUTOMORPHIC SOLUTION OF DIFFERENCE SYSTEM

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ABSTRACT. We study almost automorphic solutions of recurrence relations with values in a Banach space $V$ for quasilinear almost automorphic difference systems. Its linear part is a constant bounded linear operator $\Lambda$ defined on $V$ satisfying an exponential dichotomy. We study the existence of almost automorphic solutions of the non-homogeneous linear difference equation and to quasilinear difference equation. Assuming global Lipschitz type conditions, we obtain Massera type results for these abstract systems. The case where the eigenvalues $\lambda$ verify $|\lambda| = 1$ is also treated. An application to differential equations with piecewise constant argument is given.

## 1. Introduction

Almost automorphic sequences are natural extensions of almost periodic sequences. Almost periodic sequence was first introduced by Walther [43, 44] and then by Halanay [21] and Corduneanu [14]. See [4, 17]. Recently, several papers [5, 23, 25, 38, 39, 40] are devoted to study existence of almost periodic solutions of difference equations, see also [18, 27, 28]. However in very few papers [1, 2, 6], the concept of almost automorphic type sequence has been treated in the theory of difference equations. Abbas [1, 2] introduced pseudo almost periodic and weighted pseudo almost automorphic sequence and Araya et al. [6] almost automorphic ones.

The theory of difference equations:

$$y(n+1) = A(n)\,y(n)\,,\ n \in \mathbb{Z}, \tag{1.1}$$

$$y(n+1) = A(n)\,y(n) + f(n)\,,\ n \in \mathbb{Z}, \tag{1.2}$$

$$y(n+1) = A(n)\,y(n) + f(n) + g(n, y(n))\,,\ n \in \mathbb{Z}, \tag{1.3}$$

has gained a lot of attention from researchers. Difference equations play an important role in numerical analysis, dynamical system, control theory, etc. See [1, 2, 5, 6, 14, 16, 21, 25, 28], [31]-[40], [45]-[55].

One more time the convolution operator

$$C(f)(n) = \sum_{k=-\infty}^{\infty} e^{-\alpha|k|} f(n-k)\,,\ \ n \in \mathbb{Z}, \alpha > 0, \tag{1.4}$$

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defined for bounded sequences, is fundamental. The spaces $l_\infty$, $c_0$, $l_1$ of bounded, convergent to zero at $\pm\infty$ and summable sequences on $\mathbb{Z}$, respectively, will be used in this work.

When system (1.2) has a summable dichotomy (see [35, 36]) with Green function $G$, then:

$$y(n) = \sum_{k=-\infty}^{\infty} G(n-1,k) f(k) \tag{1.5}$$

is the unique bounded solution of (1.2). Thus (1.5) could be the unique almost automorphic solutions of (1.2). We would like to exploit this point.

For $f : \mathbb{Z} \to V$ almost automorphic sequence, perhaps the more simple equation (1.2), that is, with $A = I$ identity:

$$y(n+1) - y(n) = f(n), \tag{1.6}$$

can have no solution $y : \mathbb{Z} \to V$ almost automorphic sequence. If $f : \mathbb{Z} \to V$ is an almost automorphic sequence, the solution of (1.6) $F(n) = \sum_{k=0}^{n} f(k) : \mathbb{Z} \to V$ is an almost automorphic sequence, by the following result of Basit ([7, Theorem 1]) (see also [27, Lemma 2.8]).

**Theorem 1.** *(Basit [7]) Let $V$ be a Banach space that does not contain any subspace isomorphic to $c_0$. If $f : \mathbb{Z} \to V$ is an almost automorphic sequence, then every bounded solution $y : \mathbb{Z} \to V$ of equation (1.6) is an almost automorphic sequence.*

As it is well known a uniformly convex Banach space, every finite-dimensional normed space and a Hilbert space does not contain any subspace isomorphic to $c_0$.

About introduction of theory of continuous almost authomorphic functions can be found in [8, 11]. Contributions on this theory can be found, for example in [6, 20], [43]-[51], [19, 42], [29, Chapter 4]. Those contributions include topics like almost automorphic functions with values in Banach spaces, with values in fuzzy-number-type and on groups. Applications cover, studies in linear and nonlinear evolution equations, integro-differential, functional-differential equations and dynamical systems.

There are several types of differential equations, as those with impulsive effect, which connect sequences and functions, see Perestyuk-Samoilenko [32], Halanay-Wexler [22]. An other important class is the differential equations with piecewise constant argument as:

$$y'(t) = Ay(t) + g([t], y([t])),$$

where $[\cdot]$ is the integer part function. For these equations it holds that $y : \mathbb{R} \to V$ is almost automorphic if and only if the sequence $y : \mathbb{Z} \to V$ is almost automorphic, see section 5 and Huang et al. [24]. Recently this has been established for an abstract situation by Ming-Dat [28].

In this paper, we first review some important properties of almost automorphic sequences, and then we study the existence of almost automorphic solutions of linear difference equations (1.2) and (1.3). In section 2, we expose some basic and related properties about the theory of almost automorphic functions. In section 3, we establish the existence of almost automorphic solutions of non-homogeneous linear difference equation. In section 4, we discuss the existence of almost automorphic solutions of nonlinear difference equations (1.3), where $A$ is a bounded operator defined on a Banach space $V$. In Section 5, we show an application to

$$y'(t) = Ay(t) + By([t]) + h([t]), \tag{1.7}$$

where $A$ and $B$ are constant $p \times p$ complex matrices and $h : \mathbb{R} \to V^p$ is an almost automorphic function.

## 2. Preliminaries

Let $V$ be a real or complex Banach space. We recall that function $f : \mathbb{Z} \to V$ is said to be Bochner almost periodic sequence if and only if for any integer sequence $(k'_n)$, there exists a subsequence $(k_n)$ such that $f(k + k_n)$ converges uniformly on $\mathbb{Z}$ as $n \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence. We denote by $\mathrm{AP}(\mathbb{Z}, V)$ the set of almost periodic sequences. See [4, 15].

The pointwise convergence motivates the following definition.

**Definition 1.** *Let $V$ be a (real or complex) Banach space. A function $f : \mathbb{Z} \to V$ is said to be almost automorphic sequence if for every integer sequence $(k'_n)$, there exists a subsequence $(k_n)$ such that*

$$\lim_{n\to\infty} f(k + k_n) =: \tilde{f}(k) \text{ and } \lim_{n\to\infty} \tilde{f}(k - k_n) = f(k) \tag{2.1}$$

*are well defined for each $k \in \mathbb{Z}$.*

As in the continuous case we have that $f \in \mathrm{AA}(\mathbb{Z}, V)$ implies that $f$ is a bounded function and $\sup_{k\in\mathbb{Z}} \left\| \tilde{f}(k) \right\| = \sup_{k\in\mathbb{Z}} \| f(k) \|$ and for fixed $l_i$ $(i = 1, 2)$ in $\mathbb{Z}$, the function $u : \mathbb{Z} \to V$ defined by $u(k) = f(l_1 k + l_2)$ is in $AA(\mathbb{Z}, V)$. Examples of almost automorphic sequences which are not almost periodic sequences were firstly constructed by Veech [41], the examples are not on the additive group $\mathbb{R}$ but on its discrete subgroup $\mathbb{Z}$. A concrete example of an almost automorphic function, provided later in [11, Theorem 1] by Bochner, is:

$$f(n) = \operatorname{sign}(\cos(n\alpha)), \quad n \in \mathbb{Z}, \, \alpha \in \mathbb{R} - \mathbb{Q}.$$

We denote by $\mathrm{AA}(\mathbb{Z}, V)$ the vectorial space of almost automorphic sequence in $V$. Clearly $\mathrm{AP}(\mathbb{Z}, V) \subset \mathrm{AA}(\mathbb{Z}, V)$ and the norm:

$$\|f\|_\infty := \sup_{k\in\mathbb{Z}} \|f(k)\|_V$$

becomes $\mathrm{AA}(\mathbb{Z}, V)$ into the Banach space.

The following is a fundamental Lemma

**Lemma 1.** *Let* $\mathfrak{B}(V)$ *the Banach space of linear bounded functions of* $V$ *into* $V$ *and* $v \in l_1(\mathbb{Z}, \mathfrak{B}(V))$*, i.e. an operator valued sequence* $v : \mathbb{Z} \to \mathfrak{B}(V)$ *such that*

$$\|v\|_1 := \sum_{k\in\mathbb{Z}} \|v(k)\|_{\mathfrak{B}} < \infty. \tag{2.2}$$

*For* $f \in AA(\mathbb{Z}, V)$ *the convolution sequence defined by*

$$Lf(k) = \sum_{k\in\mathbb{Z}} v(k-l) f(l), \ k \in \mathbb{Z} \tag{2.3}$$

*is also in* $AA(\mathbb{Z}, V)$*. Then, the useful convolutions* $\phi \in AA(\mathbb{Z}, V)$*, where*

$$\phi(k) = \sum_{l=-\infty}^{k} v(k-l) f(l), \ k \in \mathbb{Z}, \ \text{or} \tag{2.4}$$

$$\phi(k) = \sum_{l=k}^{\infty} v(k-l) f(l), \quad k \in \mathbb{Z}. \tag{2.5}$$

*In particular; this is the case for* $A$*,* $P \in \mathfrak{B}(V)$ *and* $v(k) = A^k P$*, when* $\|A\| < 1$*.*

*Proof.* Let $(k_n')$ be an arbitrary sequence of integers numbers. Since $f \in \mathrm{AA}(\mathbb{Z}, V)$, there exists a subsequence $(k_n)$ of $(k_n')$ such that

$$\lim_{n\to\infty} f(k+k_n) = \widetilde{f}(k)$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \widetilde{f}(k-k_n) = f(k)$$

for each $k \in \mathbb{Z}$. As $\|v(l)\| \|f(k-l)\| \le \|v(l)\| \|f\|_\infty$, Lebesgue's dominated convergence theorem, implies

$$\lim_{n\to\infty} \phi(k+k_n) = \sum_{l\in\mathbb{Z}} v(l) \lim_{n\to\infty} f(k+k_n-l) = \sum_{l\in\mathbb{Z}} v(l) \widetilde{f}(k-l) =: \tilde{\phi}(k)$$

In similar way, we prove

$$\lim_{n\to\infty} \tilde{\phi}(k-k_n) = \phi(k),$$

and then $\phi \in \mathrm{AA}(\mathbb{Z}, V)$. □

**Remark 1.** *For* $m$*,* $n \in \mathbb{Z}$ *fixed and* $f \in AA(\mathbb{Z}, V)$*, the sequences*

$$\phi(k) = \sum_{l=n}^{k} v(k-l) f(l) \ \text{ and } \phi(k) = \sum_{l=k}^{m} v(k-l) f(l), \ k \in \mathbb{Z},$$

*are not almost automorphic (they are asymptotically almost automorphic, i.e.* $\phi = \phi_{AA} + c$*, where* $\phi_{AA} \in AA(\mathbb{Z}, V)$ *and* $c \in c_0(\mathbb{Z}, V)$*).*

For applications to nonlinear difference equations the following definition, of almost automorphic sequences depending on one parameter, will be useful.

**Definition 2.** *A function $g:\mathbb{Z}\times V\to V$ is said to be almost automorphic sequence in $k$ for each $x\in V$ if for every sequence of integers numbers $(k_n')$ there exist a subsequence $(k_n)$ such that*

$$\lim_{n\to\infty} g(k+k_n,x) =: \tilde{g}(k,x) \text{ and } \lim_{n\to\infty} \tilde{g}(k-k_n,x) = g(k,x) \tag{2.6}$$

*are well defined for each $k\in\mathbb{Z}$, $x\in V$.*

We will denote $\mathrm{AA}(\mathbb{Z}\times V,V)$ the vectorial space of the almost automorphic sequences in $k\in\mathbb{Z}$ for each $x\in V$.

Important composition results are

**Theorem 2.** *Let $V$, $W$ be Banach spaces, and let $g:V\to W$ is a continuous function, if $L\in AA(\mathbb{Z},\mathbb{C})$ and $\phi\in AA(\mathbb{Z},V)$ then the composite $L(\cdot)g(\phi(\cdot))\in AA(\mathbb{Z},W)$.*

*Proof.* Firstly, if $\phi\in \mathrm{AA}(\mathbb{Z},V)$ then the product $L(\cdot)g(\cdot)\in\mathrm{AA}(\mathbb{Z},V)$. Indeed, given $(k_n')\subset\mathbb{Z}$ it is possible to have a subsequence $(k_n)\subset(k_n')$ such that the translation limits in (2.1) exists for both $L$ and $\phi$ simultaneously. On the other hand $g$ is continuous, we have $\lim_{n\to\infty} g(\phi(k+k_n)) = g(\lim_{n\to\infty}\phi(k+k_n)) = g\left(\tilde{\phi}(k)\right)$. In similar way, we have $\lim_{n\to\infty} g\left(\tilde{\phi}(k-k_n)\right) = g\left(\lim_{n\to\infty}\tilde{\phi}(k-k_n)\right) = g(\phi(k))$, therefore $g\circ f\in\mathrm{AA}(\mathbb{Z},W)$. Finally, $L(\cdot)g(\phi(\cdot))\in\mathrm{AA}(\mathbb{Z},W)$. ☐

**Corollary 1.** *If $A$ is a bounded linear operator on $V$, $L\in AA(\mathbb{Z},\mathbb{C})$ and $\phi\in AA(\mathbb{Z},V)$ then $L(\cdot)A\phi(\cdot)\in AA(\mathbb{Z},V)$.*

**Theorem 3.** *Let $g\in AA(\mathbb{Z}\times V,V)$ and $L\in AA(\mathbb{Z},\mathbb{R}_{\geq 0})$ such that*

$$\|g(k,x)-g(k,y)\| \leq L(k)\|x-y\|,\ k\in\mathbb{Z};\ x,y\in V. \tag{2.7}$$

*Suppose $\phi\in AA(\mathbb{Z},V)$, then $g(\cdot,\phi(\cdot))\in AA(\mathbb{Z},V)$.*

*Proof.* Let $(k_n')$be sequence in $\mathbb{Z}$. Since $L\in\mathrm{AA}(\mathbb{Z},\mathbb{R}_{\geq 0})$, $\phi\in\mathrm{AA}(\mathbb{Z},V)$ and $g\in\mathrm{AA}(\mathbb{Z}\times V,V)$, it is possible to have a subsequence $\{k_n\}\subset\{k_n'\}$ such that the translations limits in (2.6) exists, for every $x\in V$, for the function $g$ and also the translation limits in (2.1) exists for both $L$ and $\phi$ simultaneously (see proof of Theorem 1). Then, applying (2.7) and those limits (2.6) for $g(\cdot,\phi(\cdot))\in\mathrm{AA}(\mathbb{Z},V)$, from

$$\begin{aligned} g(k+k_n,\phi(k+k_n)) - \tilde{g}\left(k,\tilde{\phi}(k)\right) &= g(k+k_n,\phi(k+k_n)) - g\left(k+k_n,\tilde{\phi}(k)\right) \\ &\quad + g\left(k+k_n,\tilde{\phi}(k)\right) - \tilde{g}\left(k,\tilde{\phi}(k)\right), \\ \tilde{g}\left(k-k_n,\tilde{\phi}(k-k_n)\right) - g(k,\phi(k)) &= \tilde{g}\left(k-k_n,\tilde{\phi}(k-k_n)\right) - \tilde{g}(k-k_n,\phi(k)) \\ &\quad + \tilde{g}(k-k_n,\phi(k)) - g(k,\phi(k)). \end{aligned}$$

The conclusion follows. ☐

## 3. Almost Automorphic Solutions of Non-Homogueneous Difference Systems

Difference equations usually describe the evolution of certain phenomena over the course of the time. In this section we deal with those equations known as the first-order difference equations. These equations naturally apply to various fields, like biology (the study of competitive species in population dynamics), physics (the study of motion of interacting bodies), the study of control systems, neurology, and electricity: see [4, 17],[21]-[25],[31]-[40]. Consider the following system of first order linear difference equations

$$y(n+1) = Ay(n) + f(n) \tag{3.1}$$

where $A$ is a complex matrix or, more generally, a bounded linear operator defined on a Banach space $V$ and $f \in \mathrm{AA}(\mathbb{Z}, V)$. We wish to obtain several Massera types theorems under dichotomy conditions. Moreover, the case where the eigenvalues $\lambda$ satisfying $|\lambda| = 1$ is also considered.

**Definition 3.** *We will say that a constant $p \times p$-complex matrix $A$ has a $(\mu_1, \mu_2)$-exponential dichotomy if there exist a projection matrix $P$ which commutes with $A$, constants $k \geq 1$, $\mu_1$, $\mu_2$ with $0 < \mu_1 < 1$, $\mu_2 > 1$ such that*

$$\begin{aligned} \left\|A^{n-k}P\right\| &\leq K\mu_1^{n-k} \ \ for\, k \leq n \\ \left\|A^{n-k}(I-P)\right\| &\leq K\mu_2^{n-k}, \ for\, k > n. \end{aligned}$$

Let $P$ be a projection matrix and define $G$ the Green matrix associate to $P$ by

$$G(n,k) = \begin{cases} G_1(n,k) = A^{n-k}P & \text{for } n \geq k \\ G_2(n,k) = A^{n-k}(I-P) & \text{for } n < k. \end{cases}$$

We have

$$\left\|\sum_{k=-\infty}^{n-1} G_1(n-1,k)\right\| \leq \sum_{k=-\infty}^{n-1} K\mu_1^{n-1-k} = K\sum_{k=0}^{\infty}\mu_1^k = \frac{K}{1-\mu_1}$$

and

$$\left\|\sum_{k=n}^{\infty} G_2(n-1,k)\right\| \leq \frac{K}{\mu_2 - 1}.$$

$$\|G\| := \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{\infty} \|G(n,k)\| \leq K\left(\frac{1}{1-\mu_1} + \frac{1}{\mu_2 - 1}\right). \tag{3.2}$$

**Lemma 2.** *If the constant $p \times p$- matrix $A$ has a $(\mu_1, \mu_2)$-exponential dichotomy and $f \in B(\mathbb{Z}, V^p)$ then the linear non-homogeneous system (3.2) has the unique solution $y \in B(\mathbb{Z}, V^p)$ given by*

(3.3)

$$y(n) = \sum_{k=-\infty}^{\infty} G(n-1,k) f(k) = \sum_{k=-\infty}^{n-1} A^{n-k-1} P f(k) - \sum_{k=n}^{\infty} A^{n-k-1} (I-P) f(k).$$

*Moreover,*

$$\|y\|_\infty \leq \|G\| \, \|f\|_\infty. \tag{3.4}$$

*Proof.* The sequence $y$ given by (3.3) is bounded satisfying (3.4) and (3.1). Indeed

$$\begin{aligned} y(n+1) &= \sum_{k=-\infty}^{\infty} G(n,k) f(k) \\ &= \sum_{k=-\infty}^{n} A^{n-k-1} P f(k) - \sum_{k=n+1}^{\infty} A^{n-k-1} (I-P) f(k) \\ &= APy(n) + Pf(n) + A(I-P) y(n) + (I-P) f(n) \\ &= Ay(n) + f(n). \end{aligned}$$

□

**Theorem 4.** *If the constant $p \times p$ matrix $A$ has a $(\mu_1, \mu_2)$ exponential dichotomy and $f \in AA(\mathbb{Z}, V^p)$, then the solution $y$ in (3.3) is the unique $AA(\mathbb{Z}, V^p)$ of the linear non-homogeneous system (3.1). Moreover,*

$$\|y\|_\infty \leq \|G\| \, \|f\|_\infty.$$

*Proof.* Let $\Gamma f = \Gamma_1 f + \Gamma_2 f$, with

$$(\Gamma_1 f)(n) = \sum_{k=-\infty}^{n-1} G_1(n-1,k) f(k)$$

and

$$(\Gamma_2 f)(n) = -\sum_{k=n}^{\infty} G_1(n-1,k) f(k).$$

We will prove that $\Gamma_1 f$ and $\Gamma_2 f$ belongs to $\mathrm{AA}(\mathbb{Z}, V^p)$.

Let $y = \Gamma_1 f$ and $(\tilde{m}_n)$ a sequence in $\mathbb{Z}$. $f \in \mathrm{AA}(\mathbb{Z}, V^p)$ implies that there exists a subsequence $(m_n) \subset (\tilde{m}_n)$ such that $\tilde{f}(k) = \lim_{n\to\infty} f(k+m_n)$ exists for $k \in \mathbb{Z}$

and $f(k) = \lim_{n\to\infty} \tilde{f}(k - m_n)$ pointwise. Then,

$$\begin{aligned} y(k+m_n) &= \sum_{l=-\infty}^{k+m_n-1} G_1(k+m_n-1,l) f(l) \\ &= \sum_{l=-\infty}^{k+m_n-1} G_1(k+m_n-1,k+m_n-1-l) f(k+m_n-1-l) \\ &= \sum_{l=0}^{\infty} A^l P f(k+m_n-1-l). \end{aligned}$$

Since the $\left(A^l P\right)_{l=0}^{\infty} \in l_1(\mathbb{Z}, \mathfrak{B}(V^p))$, by using Lebesgue's domination theorem $y(k+m_n) \to \tilde{y}(k)$ as $n \to \infty$, where

$$\tilde{y}(k) = \sum_{l=-\infty}^{k-1} G_1(k-1,l) \tilde{f}(l).$$

Similarly,

$$\lim_{n\to\infty} \tilde{y}(l - m_n) = y(l), \ l \in \mathbb{Z}.$$

So, $y \in \mathrm{AA}(\mathbb{Z}, V^p)$. Similarly $\Gamma_2 f \in \mathrm{AA}(\mathbb{Z}, V^p)$ and hence $\Gamma f \in (\mathbb{Z}, V^p)$. $\square$

As a consequence, we have for the scalar abstract case:

$$y(n+1) = \lambda y(n) + f(n) \tag{3.5}$$

**Theorem 5.** *Let $V$ be a Banach space and $f \in AA(\mathbb{Z}, V)$, then there exists a unique solution $y \in AA(\mathbb{Z}, V)$ of (3.5) given by*

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{n-1} \lambda^{n-1-k} f(k), \ \textit{in case } |\lambda| < 1, \ \textit{or} \\ y(n) &= -\sum_{k=n}^{\infty} \lambda^{n-1-k} f(k), \ \textit{in case } |\lambda| > 1. \end{aligned}$$

For $|\lambda| = 1$ we have:

**Theorem 6.** *Let $V$ be a Banach space which does not contain any subspace isomorphic to $c_0$. Let $f \in AA(\mathbb{Z}, V)$ and $|\lambda| = 1$. Then a solution $y$ of (3.5) is bounded if and only if $y \in AA(\mathbb{Z}, V)$. If $F(n) = \sum_{k=0}^{n} \lambda^{-k} f(k)$ is bounded then every solution $y$ of (3.5) $\in AA(\mathbb{Z}, V)$ are given by*

$$y(n) = \lambda^{n-1}(v + F(n-1)), \ n \in \mathbb{Z}.$$

*Proof.* Let $\lambda = e^{i\alpha}$ and $f \in \mathrm{AA}(\mathbb{Z}, V)$ then $\lambda^{-k} f(k) \in \mathrm{AA}(\mathbb{Z}, V)$ then $\lambda^{-k} f(k) \in \mathrm{AA}(\mathbb{Z}, V)$ and by Basit's Theorem $A$, $F \in \mathrm{AA}(\mathbb{Z}, V)$ if and only if it is bounded. So, in this case every solution $y$ of (3.5) is in $\mathrm{AA}(\mathbb{Z}, V)$. $\square$

**Remark 2.** *Even in $V = \mathbb{C}$, a system (1.2) or (3.5) with $f \in B(\mathbb{Z}, V)$ can have no bounded solution as shows:*

$$y(n+1) = \lambda y(n) + c\lambda^n, \ |\lambda| = 1, c \ \text{constant}$$

*with solutions*

$$y(n) = \lambda^{n-1}[v + cn], \ v \in \mathbb{C}.$$

Note that if $|\lambda_i| \neq 1$ $(i = 1, 2, \cdots, p)$ there exists a unique bounded solution, namely that corresponding to (3.6)

$$\begin{aligned} v &= \sum_{k=-\infty}^{-1} \lambda^k f(k), \text{ if } |\lambda_i| > 1, \text{ and} \\ v &= -\sum_{k=0}^{\infty} \lambda^k f(k), \text{ if } |\lambda_i| < 1. \end{aligned}$$

If $A \in \mathfrak{B}(V)$ is a general bounded operator, Lemma 1 implies:

**Theorem 7.** *Let $V$ be a Banach space, and let $A \in \mathfrak{B}(V)$ such that $\|A\| \neq 1$ and $f \in AA(\mathbb{Z}, V)$. Then there is a solution $y \in AA(\mathbb{Z}, V)$ of (3.1) given by:*

$$y(n) = \sum_{k=-\infty}^{n-1} A^{n-1-k} f(k), \ n \in \mathbb{Z}, \text{if } \|A\| < 1,$$

*and*

$$y(n) = -\sum_{k=n}^{\infty} A^{n-1-k} f(k), \ n \in \mathbb{Z}, \text{if } \|A\| > 1.$$

For any constant matrix $A$, there exists a nonsingular matrix $T$ such that $TAT^{-1} = B$ is an upper triangular matrix. This procedure, called "Method of Reduction", was used in the discrete case earlier by Agarwal (cf. [4, Theorem 2.10.1]). In the continuous case, Corduneanu [15, Theorem 6.2.2] used it in the existence of AP $(\mathbb{R}, \mathbb{C}^p)$ solutions and N'Guerekata [30, Remark 6.2.2] with AA $(\mathbb{R}, \mathbb{C}^p)$ solutions. See also [26].

**Theorem 8.** *Suppose $A$ is a constant $p \times p$ complex matrix with eigenvalues $\lambda$ such as $|\lambda| \neq 1$. Then for any function $f \in AA(\mathbb{Z}, V^p)$ there is a unique solution $y \in AA(\mathbb{Z}, V^p)$ of (3.1).*

*Proof.* $f \in \text{AA}(\mathbb{Z}, V^p)$ implies $\overline{f} \in \text{AA}(\mathbb{Z}, V^p)$, $\overline{f} = T^{-1} f$ and $v = T^{-1} y$ satisfy

$$\begin{array}{lcccccccccc} v_1(n+1) & = & \lambda_1 v_1(n) & + & b_{12} v_2(n) & + & \cdots & + & b_{1p} v_p(n) & + & \overline{f}_1(n) \\ v_2(n+1) & = & & & \lambda_2 v_2(n) & + & \cdots & + & b_{2p} v_p(n) & + & \overline{f}_2(n) \\ \cdots & = & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\ v_p(n+1) & = & & & & & & & \lambda_p v_p(n) & + & \overline{f}_p(n). \end{array} \tag{3.6}$$

Theorem 5 implies that the $p$th component $v_p(n)$ of the solution $v(n)$ satisfies an equation as (3.5) and hence any bounded solution $v_p \in \text{AA}(\mathbb{Z}, \mathbb{C}^p)$. Then substituting $v_p(n)$ in the $(p-1)$th equation of (3.6) we obtain again an equation of the form (3.5) for $v_{p-1}(n)$, and so on. The proof is completed. □

Now, we study the case when all the eigenvalues $\{\lambda_i\}_{i=1}^p$ satisfies $|\lambda_i| = 1$. Denote

$$\mathfrak{F}_l(\varphi)(n) = \sum_{k=0}^{n-1} \lambda_l^{-k} \varphi(k), \; n \in \mathbb{Z}.$$

Assume that $v$ satisfies the upper triangular system (3.6). So, by Theorem 7 the $p$-th coordinate $v_p \in \mathrm{AA}(\mathbb{Z}, V)$ and it is given by

$$v_p(n+1) = \lambda_p^n \left[\eta_p + \mathfrak{F}_p\left(\overline{f}_p\right)(n)\right] \tag{3.7}$$

for some $\eta_p \in V$. Replacing this expression in the $(p-1)$th equation in (3.6), we have $v_{p-1} \in \mathrm{AA}(\mathbb{Z}, V)$ and for some $\eta_{p-1} \in V$:

$$v_{p-1}(n) = \lambda_{p-1}^{n-1} \left[\eta_{p-1} + \mathfrak{F}_{p-1}\left(b_{p-1p} v_p + \overline{f}_{p-1}\right)(n)\right].$$

However,

$$\mathfrak{F}_{p-1}(v_p) = \eta_p \mathfrak{F}_{p-1}(\lambda_p) + \mathfrak{F}_{p-1}\left(\mathfrak{F}_p\left(\overline{f}_p\right)\right)$$

and $\mathfrak{F}_{p-1}(\lambda_p) \in \mathbb{B}(\mathbb{Z}, V)$ if and only if $\lambda_p \neq \lambda_{p-1}$. Indeed

$$\mathfrak{F}_{p-1}(\lambda_p)(n) = \sum_{k=0}^{n-1} \lambda_{p-1}^{-k} \lambda_p^{k-1}.$$

Then $\mathfrak{F}_{p-1}(v_p) \in \mathbb{B}(\mathbb{Z}, V)$ if and only if $\eta_p = 0$ and hence

$$v_{p-1}(n) = \lambda_{p-1}^{n-1} \left[\eta_{p-1} + \mathfrak{F}_{p-1}\left(b_{p-1p} \mathfrak{F}_p\left(\overline{f}_p\right) + \overline{f}_{p-1}\right)(n)\right]. \tag{3.8}$$

So, when the eigenvalues $\{\lambda_i\}_{i=1}^p$ of a matrix $A$ satisfy $|\lambda_i| = 1$, $1 \leq i \leq p$ we have

**Theorem 9.** *Let $V$ be a Banach space with does not contain any subspace isomorphic to $c_0$. Let $\{\lambda_i\}_{i=1}^p$ be the eigenvalues of $A$ satisfying $|\lambda_i| = 1$. Then every bounded solution of (3.4) $y \in AA(\mathbb{Z}, V)$. When all these $\lambda_i$ are distinct, these solutions have the form:*

$$y(n) = A^{n-1} \left[v + \sum_{k=0}^{n-1} A^{-k} f(k)\right], \; v \in V^p. \tag{3.9}$$

*In the general case, a formula for the bounded solutions can be also obtained with an infinity of solutions, so much as $V^r$, where $r$ is the number of different eigenvalues $\lambda_i$.*

*Proof.* If $\{\lambda_i\}_{i=1}^p$ are distinct, the transformed system (3.6) is now diagonal and by Theorem 6 and (3.7) we obtain (3.9) . In the general case, we use the previous analysis and the solutions of the form (3.8). □

So, it is possible to combine $|\lambda_i| \neq 1$ and $|\lambda_i| = 1$ without condition on the multiplicity.

**Theorem 10.** *Let $V$ be a Banach space with does not contain any subspace isomorphic to $c_0$ and let $\{\lambda_i\}_{i=1}^p$ the eigenvalues of the $p \times p$ constant matrix $A$. Then every bounded solution $y$ of* (3.4) *satisfies $y \in AA(\mathbb{Z}, V^p)$. Moreover, a formula for the almost automorphic solutions can be explicated with an infinity of solutions so much as $V^r$, where $r$ is the number of different eigenvalues $\lambda_i$ with $\lambda_i = 1$.*

Finally, we can also prove the following result.

**Theorem 11.** *Let $V$ be a Banach space. Suppose $f \in AA(\mathbb{Z}, V)$ and $A = \sum_{k=1}^N \lambda_k P_k$ where the complex numbers $\lambda_k$ are mutually distinct with $|\lambda_k| \neq 1$, and $(P_k)_{1\leq k\leq N}$ forms a complex system $\sum_{k=1}^N P_k = I$ of mutually disjoint projections on $V$. Then the unique bounded solution $y$ of* (3.1) *is in $AA(\mathbb{Z}, V)$.*

*Proof.* Let $k \in \{1, \dots, N\}$ be fixed. By Corollary 1 we have $P_k f \in \mathrm{AA}(\mathbb{Z}, V)$, since $P_k$ is bounded. Applying the projection $P_k$ to (3.1) we obtain

$$P_k y(n+1) = P_k A y(n) + P_k f(n). \tag{3.10}$$

Therefore, by Theorem 8, we get $P_k y \in \mathrm{AA}(\mathbb{Z}, V)$ we conclude that $y(n) = \sum_{k=1}^N P_k y(n) \in \mathrm{AA}(\mathbb{Z}, V)$ as a finite sum of almost automorphic sequences. □

This is an explicit result of the general theorem obtained by Minh et al. [27, Theorem 2.4] for every Banach space.

**Theorem 12.** *Let $V$ be a Banach space that does not contain any subspace isomorphic to $c_0$. Assume that the set formed by $\lambda$ in the spectrum of $A$ with $|\lambda| = 1$ is countable. If $f \in AA(\mathbb{Z}, V)$, then each bounded solution of* (3.5) *$y \in AA(\mathbb{Z}, V)$.*

## 4. Almost Automorphic Solutions of Nonlinear Difference Systems

Now we study the existence of almost automorphic solutions to the equation

$$y(n+1) = Ay(n) + g(n, y(n)), \; n \in \mathbb{Z}, \tag{4.1}$$

where A is a bounded linear operator defined on a Banach space $V$ and $g \in \mathrm{AA}(\mathbb{Z} \times V, V)$.
One of the main results in this section is the following theorem for the quasilinear case:

**Theorem 13.** *Assume that the constant $p \times p$ matrix $A$ has a $(\mu_1, \mu_2)$-exponential dichotomy and $g = g(k, y) \in AA(\mathbb{Z} \times V^p, V^p)$ satisfies the Lipschitz condition*

$$\|g(k, y_1) - g(k, y_2)\| \leq L \|y_1 - y_2\|, \; y_i \in V^p, k \in \mathbb{Z}, i = 1, 2. \tag{4.2}$$

*Then the semilinear system* (4.1) *has a unique solution $y \in AA(\mathbb{Z}, V^p)$ satisfying*

$$y(n) = \sum_{k=-\infty}^{\infty} G(n-1, k)\, g(k, y(k)) \tag{4.3}$$

*if*

$$KL\left(\frac{1}{1-\mu_1}+\frac{1}{\mu_2-1}\right)<1. \tag{4.4}$$

*Proof.* By (3.2) we have

$$\|G\|:=\sup_{n\in\mathbb{Z}}\sum_{k=-\infty}^{\infty}\|G(n,k)\|\leq K\left(\frac{1}{1-\mu_1}+\frac{1}{\mu_2-1}\right). \tag{4.5}$$

For $\phi\in \mathrm{AA}(\mathbb{Z},V^p)$ since $g(k,x)$ satisfies (4.2), we obtain by Theorem 3 that $g(\cdot,\phi(\cdot))\in \mathrm{AA}(\mathbb{Z},V^p)$.
Define the operator $\Gamma:\mathrm{AA}(\mathbb{Z},V^p)\to \mathrm{AA}(\mathbb{Z},V^p)$ by

$$\Gamma(\phi)(n)=\sum_{k=-\infty}^{\infty}G(n-1,k)\,g(k,\phi(k)),\ n\in\mathbb{Z}. \tag{4.6}$$

So $\Gamma$ is well defined thanks to Theorem 4. Now given $\phi_1,\ \phi_2\in \mathrm{AA}(\mathbb{Z},V^p)$, we have

$$\begin{aligned}
\|\Gamma(\phi_1)-\Gamma(\phi_2)\|_\infty &\leq \sup_{n\in\mathbb{Z}}\sum_{k=-\infty}^{\infty}\|G(n-1,k)\|\,\|g(k,\phi_1(k))-g(k,\phi_2(k))\|\\
&\leq \sup_{n\in\mathbb{Z}}\sum_{k=-\infty}^{\infty}\|G(n-1,k)\|\,L\,\|\phi_1(k)-\phi_2(k)\|\\
&\leq L\,\|\phi_1-\phi_2\|_\infty \sup_{n\in\mathbb{Z}}\sum_{k=-\infty}^{\infty}\|G(n-1,k)\|\\
&\leq KL\,\|\phi_1-\phi_2\|_\infty\left(\frac{1}{1-\mu_1}+\frac{1}{\mu_2-1}\right) \qquad (4.7)
\end{aligned}$$

then by (4.4) the function $\Gamma$ is a contraction. Then there exist a unique $y\in \mathrm{AA}(\mathbb{Z},V^p)$ such that $\Gamma y=y$. That is, $y$ satisfies (4.3) and hence $y$ is solution of (4.1). □

Then in the scalar abstract case:

$$y(n+1)=\lambda y(n)+g(n,y(n)),\ n\in\mathbb{Z}. \tag{4.8}$$

**Theorem 14.** *Let $|\lambda|\neq 1$ and $g:\mathbb{Z}\times V\to V$ be almost automorphic in $k$ for each $x\in V$. Suppose that $g$ satisfies the following Lipschitz type condition*

$$\|g(k,y_1)-g(k,y_2)\|\leq L\,\|y_1-y_2\|,\ y_i\in V,\ k\in\mathbb{Z},\ i=1,2. \tag{4.9}$$

*Then (4.8) has a unique solution $y\in AA(\mathbb{Z},V)$ satisfying*
*(i) $y(n)=\sum_{k=-\infty}^{n-1}\lambda^{n-1-k}g(k,y(k))$ in case $|\lambda|<1$, $L<1-|\lambda|$ and*
*(ii) $y(n)=\sum_{k=n}^{\infty}\lambda^{n-1-k}g(k,y(k))$ in case $|\lambda|>1$, $L<|\lambda|-1$ .*

In the particular case $g(k,x)=L(k)\,g_1(x)$ we obtain the following Corollary.

**Corollary 2.** *Let* $|\lambda| \neq 1$. *Suppose* $g_1$ *satisfies a Lipschitz condition*

$$\left\| g_1(x) - g_1(y) \right\| \leq \theta \left\| x - y \right\|, \; x, y \in V. \tag{4.10}$$

*Then for each* $L \in AA(\mathbb{Z}, \mathbb{C})$, (4.1) *has a unique solution* $y \in AA(\mathbb{Z}, V)$ *whenever* $|\lambda| < 1$, $\theta \|L\| < 1 - |\lambda|$ *or* $|\lambda| > 1$, $\theta \|L\| < |\lambda| - 1$.

**Theorem 15.** *Let* $g(k, y) = L(k) g_1(y)$ *satisfying Theorem 3 and assume* $A$ *has a* $(\mu_1, \mu_2)$*-exponential dichotomy and:*

$$\sup_{n \in \mathbb{Z}} \sum_{k=-\infty}^{\infty} \left\| G(n, k) L(k) \right\| < \theta^{-1}.$$

*Then the semilinear system* (4.1) *has a unique solution* $y \in AA(\mathbb{Z}, V)$ *satisfying*

$$y(n) = \sum_{k=-\infty}^{\infty} G(n-1, k) \left[ f(k) + L(k) g_1(y(k)) \right].$$

The case of a bounded operator $A$ can be treated assuming extra conditions on the operator. The proof of the next result follows the same lines of the first part in the proof of Theorem 13, using (3.2).

**Theorem 16.** *Let* $A \in \mathfrak{B}(V)$ *having a* $(\mu_1, \mu_2)$ *exponential dichotomy and* $g \in AA(\mathbb{Z} \times V, V)$ *is such that:*

$$\left\| g(k, x) - g(k, y) \right\| \leq L \left\| x - y \right\|, \; x, y \in V, \; k \in \mathbb{Z}. \tag{4.11}$$

*Then the conclusion of Theorem 13 holds.*

**Corollary 3.** *Let* $A \in \mathfrak{B}(V)$ *with* $\|A\| \neq 1$ *and suppose that* $g \in AA(\mathbb{Z} \times V, V)$ *is such that*

$$\left\| g(k, x) - g(k, y) \right\| \leq L \left\| x - y \right\|, \; x, y \in V, \; k \in \mathbb{Z}.$$

*Then* (4.1) *has a unique solution* $y \in AA(\mathbb{Z}, V)$, *satisfying*

$$y(n) = \sum_{k=-\infty}^{n-1} A^{n-1-k} g(k, y(k)), \text{ if } \|A\| < 1 \text{ and } L < 1 - \|A\|,$$

*and*

$$y(n) = -\sum_{k=n}^{\infty} A^{n-1-k} g(k, y(k)), \text{ if } \|A\| > 1 \text{ and } L < \|A\| - 1.$$

## 5. Applications

Consider the differential equation with piecewise constant argument (1.7), where $A$ and $B$ are constant $p \times p$ complex matrices and $h \in \mathrm{AA}(\mathbb{R}, V^p)$ the solutions are taken continuous. The variation of constants formula gives

$$y(t) = e^{A(t-n)} y(n) + \int_n^t e^{A(t-s)} B y(n)\, ds + \int_n^t e^{A(t-s)} h(s)\, ds,$$

then, if $A^{-1}$ exists,

$$y(n) = \left[e^{A(t-n)} + A^{-1}\left(e^{A(t-n)} - I\right)B\right]y(n) + \int_n^t e^{A(t-s)}h(s)\,ds.$$

So, the continuity condition of $y(t)$ in $t = n+1$ establishes

$$y(n+1) = Cy(n) + f(n)$$

where

$$C = e^A + A^{-1}\left(e^A - I\right)B,\ f(n) = \int_n^{n+1} e^{A(n+1-s)}h(s)\,ds.$$

It is not difficult to show, see [56]

**Lemma 3.** *$h \in AA(\mathbb{R}, \mathbb{C}^p)$ implies $f \in AA(\mathbb{Z}, \mathbb{C}^p)$.*

**Lemma 4.** *$y \in AA(\mathbb{Z}, \mathbb{C}^p)$ if and only if $y \in AA(\mathbb{R}, \mathbb{C}^p)$.*

Then we have:

**Theorem 17.** *Let $A$ and $B$ be constants $p \times p$ complex matrices, $A$ an invertible matrix and $h \in AA(\mathbb{R}, \mathbb{C}^p)$. Then every bounded solution $y$ of system (1.7) is in $AA(\mathbb{R}, \mathbb{C}^p)$. More precisely, $y(n) \in B(\mathbb{Z}, \mathbb{C}^p)$ implies $y \in AA(\mathbb{R}, \mathbb{C}^p)$.*

**Theorem 18.** *For the simplest case $A = 0$:*

$$y'(t) = By([t]) + h([t])$$

*the above conclusion is also true.*

The last result has been studied by Minh-Dat [28] in the abstract situation.

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