Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **85**, 1-20; **http://www.math.u-szeged.hu/ejqtde/**

KRASNOSELSKII'S THEOREM IN GENERALIZED BANACH SPACES AND APPLICATIONS

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Abstract. The purpose of this paper is to extend Krasnoselskii's fixed point theorem to the case of generalized Banach spaces for singlevalued and multivalued operators. As applications, we will give some existence results for abstract system of Fredholm-Volterra type differential equations and inclusions.

Keywords: compact operator, complete generalized metric space, Contraction Principle, fixed point, generalized contraction, generalized Banach space, generalized metric space, integral equation, integral inclusion, iterative method, Krasnoselskii theorem, A-contraction, matrix convergent to zero, multivalued operator, Picard operator, Perov theorem, relatively compact operator, singlevalued operator, Schauder theorem, sum of two operators, vector-valued metric, vector-valued norm, weakly Picard operator, Fredholm-Volterra equation, Fredholm-Volterra inclusion.

2010 MSC: 47H10, 54H25.

1. Introduction

It is well known that Perov (see [20]) extended the classical Banach contraction principle in the setting of spaces endowed with vector-valued metrics (see also Perov and Kibenko [21]). The purpose of this paper is to extend Krasnoselskii's fixed point theorem to the case of generalized Banach spaces for singlevalued and multivalued operators. As applications, we will give some existence results for abstract system of Fredholm-Volterra type differential equations. Perov's theorem and Krasnoselskii's theorem are important abstract tools for the study of differential and integral equation systems. There is a vast literature concerning these two important theorems in nonlinear analysis, see, for example [3], [1], [11], [18], [19], [22], [23], [24], [26], [27], [28], [29], etc. respectively [4], [5], [6], [7], [10], [16], [19], etc.

Recall first some basic results (see [11] and [29]) which are needed for the main results of this paper. Notice that in Precup [29] and Filip-Petruşel [11], are pointed out some advantages of a vector-valued norm with respect to the usual scalar norms.

Definition 1.1. ([20]) Let X be a nonempty set and consider the space $\mathbb{R}^m_+$ endowed with the usual component-wise partial order. The mapping $d : X \times X \to \mathbb{R}^m_+$ which satisfies all the usual axioms of the metric is called a generalized metric in the Perov's sense and (X, d) is called a generalized metric space.

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Let (X,d) be a generalized metric space in Perov's sense. Thus, if $v,r \in \mathbb{R}^m$, $v := (v_1, v_2, \dots, v_m)$ and $r := (r_1, r_2, \dots, r_m)$, then by $v \le r$ we mean $v_i \le r_i$, for each $i \in \{1, 2, \dots, m\}$ and by $v < r$ we mean $v_i < r_i$, for each $i \in \{1, 2, \dots, m\}$. Also, $|v| := (|v_1|, |v_2|, \dots, |v_m|)$.

If $u, v \in \mathbb{R}^m$, with $u := (u_1, u_2, \dots, u_m)$ and $v := (v_1, v_2, \dots, v_m)$, then $\max(u,v) := (\max(u_1, v_1), \dots, \max(u_m, v_m))$. If $c \in \mathbb{R}$, then $v \le c$ means $v_i \le c$, for each $i \in \{1, 2, \dots, m\}$.

For the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^m$.

Notice that the generalized metric space in the sense of Perov is a particular case of Riesz spaces (see [15], [38]) and of, so-called, cone metric spaces (or K-metric space) (see [37], [14]). The advantages of this approach consist in the possibility to obtain some nice properties of the fixed point set and to give several applications.

Let (X,d) be a generalized metric space in Perov's sense. For $r := (r_1, \cdots, r_m) \in \mathbb{R}^m$ with $r_i > 0$ for each $i \in \{1, 2, \cdots, m\}$, we will denote by

$$B(x_0, r) := \{x \in X : d(x_0, x) < r\}$$

the open ball centered in x_0 with radius r and by

$$\tilde{B}(x_0, r) := \{x \in X : d(x_0, x) \le r\}$$

the closed ball centered in x_0 with radius r.

We mention that for generalized metric spaces in Perov's sense, the notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Definition 1.2. A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(A)$ is strictly less than 1. In other words, this means that all the eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{m,m}(\mathbb{R})$ (see [35]).

Definition 1.3. ([33]) Let (X,d) be a generalized metric space and let $f : X \to X$ be an operator. Then, f is called an A-contraction if and only if $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and

$$d(f(x), f(y)) \le A d(x,y), \text{ for any } x, y \in X.$$

Theorem 1.4. *(Perov [20]). Let (X,d) be a complete generalized metric space and $f : X \to X$ be an A-contraction mapping. Then:*

i) there exists a unique fixed point $x^* \in X$ for f and the sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for f (i.e., $x_n := f^n(x_0), n \in \mathbb{N}^*$) is convergent to x^*, for all $x_0 \in X$ and each $n \in \mathbb{N}^*$.

ii) $d(x_n, x^*) \le A^n (I - A)^{-1} d(x_0, x_1)$, for all $n \in \mathbb{N}^*$.

iii) $d(x, x^*) \le (I - A)^{-1} d(x, f(x))$, for all $x \in X$.

The proof of Theorem 1.4 uses some properties of matrices which are convergent to zero.

Lemma 1.5. *(see [2], [35]) Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Then the following statements are equivalent:*

i) A is a matrix convergent to zero;
ii) $A^n \longrightarrow 0$ as $n \to \infty$;
iii) The matrix $I - A$ is non-singular and $(I - A)^{-1} = I + A + \ldots + A^n + \ldots$;
iv) The matrix $I - A$ is non-singular and $(I - A)^{-1}$ has nonnegative elements;
v) $A^n q \longrightarrow 0$ and $qA^n \longrightarrow 0$ as $n \to \infty$, for any $q \in \mathbb{R}^m$.

Remark 1.6. ([29]) Some examples of matrices convergent to zero are:

$$1)\ A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \text{ where } a, b \in \mathbb{R}_+ \text{ and } a + b < 1;$$

$$2)\ A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \text{ where } a, b \in \mathbb{R}_+ \text{ and } a + b < 1;$$

$$3)\ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \text{ where } a, b, c \in \mathbb{R}_+ \text{ and } \max\{a, c\} < 1.$$

In particular, if E is a linear space, then $\|\cdot\| : E \to \mathbb{R}^m_+$ is a vector-valued norm if (in a similar way to the vector-valued metric) it satisfies the classical axioms of a norm. In this case, the pair $(E, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by the norm $\|\cdot\|$ (i.e., $d(x, y) := \|x - y\|$) is complete then the space $(E, \|\cdot\|)$ is called a generalized Banach space.

As a consequence of Perov's Theorem we have the following result.

Theorem 1.7. *Let $(E, \|\cdot\|)$ be a generalized Banach space and $f : E \to E$ be an A-contraction. Then $1_E - f$ is a homeomorfism, i.e., $1_E - f$ is continuous, bijective and its inverse $(1_E - f)^{-1}$ is continuous too.*

Proof. The continuity of $1_E - f$ is obvious, since f is continuous. In order to prove the bijectivity of $1_E - f$, let us consider any $y \in E$ and the equation $(1_E - f)(x) = y$, $x \in E$. If we define the operator $g : E \to E$ by $g(x) := f(x) + y$, then the above equation can be re-written as a fixed point problem for g, i.e., $x = g(x)$. Since f is an A-contraction, we get that that g is an A-contraction too. Hence g has a unique fixed point $x^* \in E$. Thus $1_E - f$ is bijective. The continuity of $(1_E - f)^{-1}$ follows in a similar way to the case of usual Banach space. □

Another consequence of Perov's Theorem is the following local variant which improves Theorem 2.1 in [1].

Theorem 1.8. *Let (X, d) be a complete generalized metric space, let $x_0 \in X \setminus Fix(f)$ and $f : X \to X$ be an A-contraction mapping around x_0. Then there exists $R := (I - A)^{-1} d(x_0, f(x_0))$ such that $\tilde{B}(x_0, R)$ is invariant with respect to f. Morover, in this case f has a unique fixed point in $\tilde{B}(x_0, R)$.*

Proof. Let $x \in \tilde{B}(x_0, R)$. Then we have:

$$d(f(x), x_0) \leq d(f(x), f(x_0)) + d(f(x_0), x_0) \leq Ad(x, x_0) + d(f(x_0), x_0) \leq$$

$$AR + d(f(x_0), x_0) = A(I - A)^{-1} d(x_0, f(x_0)) + d(x_0, f(x_0)) = R.$$

For the second conclusion we apply Perov's Theorem on $\tilde{B}(x_0, R)$. $\square$

For our main results, we also need some concepts in generalized metric spaces (see, for example, [12], [38], [39]).

Definition 1.9. ([38]) Let (X, d) be a generalized metric space. A subset C of X is called compact if every open cover of C has a finite subcover. A set C of a topological space is said to be relatively compact if its closure is compact.

Definition 1.10. ([32]) Let X, Y be two normed generalized spaces, $K \subset X$ and $f : K \to Y$ an operator. Then f is called:

i) compact, if for any bounded subset $A \subset K$ we have that $f(A)$ is relatively compact (or equivalently $\overline{f(A)}$ is compact);
ii) complete continuous, if f is continuous and compact;
iii) with relatively compact range, if f is continuous and $f(K)$ is relatively compact.

We recall now the following Schauder type theorem (see, for example, Theorem (3.2) in [12]).

Theorem 1.11. *Let $(X, \|\cdot\|)$ be a generalized Banach space, let $Y \in P_{cv}(X)$ and $g : Y \to Y$ be a continuous operator with relatively compact range. Then g has at least one fixed point in Y.*

For the multivalued case, in the context of a generalized metric space (X, d), we will use the following notations and definitions.

$P(X)$ - the set of all nonempty subsets of X;

$\mathcal{P}(X) = P(X) \cup \{\emptyset\}$;

$P_{cl}(X)$ - the set of all nonempty closed subsets of X;

$P_{b,cl}(X)$ - the set of all nonempty bounded and closed subsets of X;

If $(X, \|\cdot\|)$ is a generalized normed space, then:

$P_{b,cl,cv}(X)$ - the set of all nonempty bounded, closed and convex subsets of X;

$P_{cp,cv}(X)$ - the set of all nonempty compact and convex subsets of X.

Let (X, d) be a metric space. Then we introduce the following functionals.

$D_d : P(X) \times P(X) \to \mathbb{R}_+, D_d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ - the gap functional;

$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \rho_d(A, B) = \sup\{D(a, B) : a \in A\}$ - the excess functional;

$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, H_d(A, B) = \max\{\rho(A, B), \rho(B, A)\}$ - the Pompeiu-Hausdorff functional.

If (X,d) is a generalized metric space with $d(x,y) := \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_m(x,y) \end{pmatrix}$, then we denote by

$$D(A,B) := \begin{pmatrix} D_{d_1}(A,B) \\ \cdots \\ D_{d_m}(A,B) \end{pmatrix} \text{ the vector gap functional on } P(X),$$

by

$$\rho(A,B) := \begin{pmatrix} \rho_{d_1}(A,B) \\ \cdots \\ \rho_{d_m}(A,B) \end{pmatrix} \text{ the vector excess functional,}$$

and by

$$H(A,B) := \begin{pmatrix} H_{d_1}(A,B) \\ \cdots \\ H_{d_m}(A,B) \end{pmatrix} \text{ the vector Pompeiu-Hausdorff functional.}$$

Notice that, throughout this paper, we will make an identification between row and column vectors in $\mathbb{R}^m$.

We recall the following known result (see for example ([33])).

Lemma 1.12. *Let $(X, \|\cdot\|)$ be a generalized Banach space. Then:*

$$H(Y+Z, Y+W) \leq H(Z,W), \text{ for each } Y,Z,W \in P_b(X).$$

Definition 1.13. ([3]) Let (X,d) be a generalized metric space, $Y \subset X$ and $F : Y \to P(X)$ be a multivalued operator. Then, F is called a multivalued A-contraction if and only if $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and for any $x, y \in Y$ and for each $u \in F(x)$, there exists $v \in T(y)$ such that

$$d(u,v) \leq Ad(x,y).$$

Definition 1.14. ([3]) Let (X,d) be a generalized metric space. Then $F : X \to P(X)$ is a multivalued weakly Picard operator (briefly MWP operator), if for each $x \in X$ and $y \in F(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

i) $x_0 = x$, $x_1 = y$;
ii) $x_{n+1} \in F(x_n)$;
iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point of F.

A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying (i) and (ii) in the above definition is said to be a sequence of successive approximations for F starting from $(x_0, x_1) \in Graph(F)$.

For examples of MWP operators see [31] and [25], while for some fixed point results for multivalued A-contractions, see [25] and [11].

Notice now that using the generalized Pompeiu-Hausdorff functional on $P_{b,cl}(X)$ the concept of multivalued contraction mapping introduced by S.B. Nadler Jr. can be extended to generalized metric spaces in the sense of Perov.

Definition 1.15. ([3]) Let (X,d) be a generalized metric space, $Y \subseteq X$ and let $F : Y \to P_{b,cl}(X)$ be a multivalued operator. Then, F is called a multivalued A-contraction in the sense of Nadler if and only if $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and

$$H(F(x), F(y)) \leq Ad(x,y), \text{ for any } x, y \in Y.$$

Notice that if $F : X \to P_{cl}(X)$ is a multivalued A -contraction in Nadler's sense, then F is a multivalued A-contraction too, but, in general, the reverse implication does not hold.

In the last part of this section, we will present several continuity results for multivalued operators.

If X, Y are two generalized metric spaces, we recall that a multivalued operator $F : X \to P(Y)$ is said to be:

a) lower semi-continuous (briefly l.s.c.) in $x_0 \in X$ if and only if for any open set $U \subset X$ such that $F(x_0) \cap U \neq \emptyset$, there exists a neighborhood V for x_0 such that for any $x \in V$, we have that $F(x) \cap U \neq \emptyset$.

b) upper semi-continuous (briefly u.s.c.) in $x_0 \in X$ if and only if for any open set $U \subset X$ such that $F(x_0) \subset U$ there exists a neighborhood V for x_0 such that for any $x \in V$, we have that $F(x) \subset U$.

c) continuous in $x_0 \in X$ if and only if it is both l.s.c. and u.s.c.

The multivalued operator $F : X \to P(Y)$ is called

a) Hausdorff lower semi-continuous (briefly H-l.s.c.) in $x_0 \in X$ if and only if for any $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}_+^m$ with $\varepsilon_i > 0$ for each $i \in \{1, \cdots, m\}$, there exists $\eta = (\eta_1, \cdots, \eta_m) \in \mathbb{R}_+^m$ with $\eta_i > 0$ for each $i \in \{1, \cdots, m\}$, such that for any $x \in B(x_0, \eta)$, we have $F(x_0) \subset V(F(x); \varepsilon)$, where

$$V(F(x); \varepsilon) = \{x \in X : D(x, F(x)) \leq \varepsilon\}.$$

b) Hausdorff upper semi-continuous (briefly H-u.s.c.) in $x_0 \in X$ if and only if for each $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}_+^m$ with $\varepsilon_i > 0$ there exists $\eta = (\eta_1, \cdots, \eta_m) \in \mathbb{R}_+^m$ with $\eta_i > 0$ for each $i \in \{1, \cdots, m\}$, such that for all $x \in B(x_0; \eta)$ we have $F(x) \subset V(F(x_0); \epsilon)$.

c) Hausdorff continuous (briefly H-c.) n $x_0 \in X$ if and only if it both H-l.s.c. and H-u.s.c.

Notice that, if the multivalued operator $F : X \to P(Y)$ has compact values, then the continuity and the H-continuity of F are equivalent.

Recall also the fact that the image of a compact set through an u.s.c. multivalued operator with compact values is compact too.

2. Main results

In this section, we will prove Krasnoselskii type fixed point theorems in generalized Banach spaces for singlevalued and multivalued operators.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a generalized Banach space and $Y \in P_{cl,cv}(X)$. Assume that the operators $f, g : Y \to X$ satisfies the properties:*

i) f is an A-contraction;

ii) g is continuous;

iii) $g(Y)$ *is relatively compact and* $f(x)+g(y)\in Y$ *for any* $x,y\in Y$.
Then $f+g$ *has a fixed point in* Y.

Proof. We show that for any $x\in Y$, the operator $u_x : Y\to Y$, $u_x(y)=f(y)+g(x)$ is an A-contraction. Notice first that, from the second part of (iii), the operator u_x is well-defined. Next let us observe that

$$\|u_x(y_1)-u_x(y_2)\|=\|f(y_1)-f(y_2)\|\le A\|y_1-y_2\|, \text{ for any } y_1,y_2\in Y.$$

Thus, u_x is an A-contraction. By Theorem 1.4, it follows that there exists a unique $\bar{y}_x\in Y$ such that $f(y_x)+g(x)=\bar{y}_x$. Next we define $c:Y\to Y$, $c(x)=\bar{y}_x$, i.e.,

$$c(x)=f[c(x)]+g(x), \text{ for any } x\in Y. \tag{1}$$

We prove that c is continuous. Indeed, since

$$\begin{aligned}\|c(x)-c(x')\| &= \|f[c(x)]+g(x)-f[c(x')]-g(x')\|\\ &\le \|f[c(x)]-f[c(x')]\|+\|g(x)-g(x')\|\\ &\le A\|c(x)-c(x')\|+\|g(x)-g(x')\|,\end{aligned}$$

we obtain that

$$\|c(x)-c(x')\|\le (I-A)^{-1}\|g(x)-g(x')\|. \tag{2}$$

Thus, by the continuity of g, we have

$$\|c(x)-c(x')\|\xrightarrow{\|\cdot\|}0, \text{ as } x'\xrightarrow{\|\cdot\|}x.$$

Notice now that, from (1) and Theorem 1.7, we have that $c=(1_Y-f)^{-1}\circ g$.

Since $g(Y)$ is relatively compact and c is continuous, we have that $c(Y)$ is relatively compact too and, thus, by Theorem 1.11, there exists $x\in Y$ with $c(x)=x$, i.e., $f(x)+g(x)=x$. Hence, the proof is complete. □

Remark 2.2. For a similar result see Viorel [36].

In the case of multivalued operators, first we give the multivalued form of Theorem 1.4 for multivalued A-contractions in the sense of Nadler which was quoted as an open question in [3].

Lemma 2.3. *Let* (X,d) *be a generalized metric space,* $A,B\subset X$, $q>1$. *Then, for any* $a\in A$, *there exists* $b\in B$ *such that*

$$d(a,b)\le qH(A,B).$$

Proof. Suppose first that $A=B$. Then we can choose $b=a$ such that the property holds. Next, suppose $A\ne B$. Then $H_i(A,B)\ne 0$ for all $i\in\{1,\dots,m\}$. We will prove the conclusion by contradiction. Thus, we suppose that there exists $a\in A$, for any $b\in B$ such that

$$d(a,b)\nleq qH(A,B).$$

It follows that there exists $j\in\{1,\dots,m\}$ such that

$$d_j(a,b)>qH_j(A,B).$$

Passing to $\inf\limits_{b\in B}$, we get the contradiction

$$H_j(A,B) \geq D_j(A,B) \geq qH_j(A,B) > H_j(A,B),$$

which completes the proof. □

Lemma 2.4. *Let (X,d) be a generalized metric space. Then $D(x,A)=0$ if and only if $x\in\bar{A}$.*

Proof. We show that $\bar{A}=\{x\in X \mid D(x,A)=0\}$.
Let $x\in\bar{A}$, equivalent, for any $r\in\mathbb{R}_+^m$ with $r>0$ we have $A\cap B(x,r)\neq\emptyset$, equivalent, for any $r\in\mathbb{R}_+^m$ with $r>0$, there exists $a\in A$ such that $d(x,a)<r$, equivalent, $D(x,A)=0$. □

Lemma 2.5. *Let $A\in\mathcal{M}_{m,m}(\mathbb{R}_+)$ be a matrix convergent to zero. Then, there exists $Q>1$ such that for any $q\in(1,Q)$ we have that qA is convergent to* 0.

Proof. Since A is convergent to zero, we have that the spectral radius $\rho(A)<1$. Next, since $q\rho(A)=\rho(qA)<1$, we can choose $Q:=\frac{1}{\rho(A)}>1$ and hence, the conclusion follows. □

Theorem 2.6. *Let (X,d) be a complete generalized metric space and $F: X\to P_{cl}(X)$ be a multivalued A-contraction in Nadler's sense. Then, for each $x\in X$ and $y\in F(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F starting from $(x,y)\in Graph(F)$ which converge to a fixed point $x^*\in X$ of F and we have the following estimations:*

(a) $d(x_n,x^*)\leq A^n(I-A)^{-1}d(x_0,x_1)$, for any $n\in\mathbb{N}^*$.

(b) $d(x_0,x^*)\leq(I-A)^{-1}d(x_0,x_1)$.

Proof. Let $x_0\in X$ and $x_1\in F(x_0)$. Let $q\in(1,Q)$, where Q is defined by the above lemma. Then, by Lemma 2.3, there exists $x_2\in F(x_1)$ such that

$$d(x_1,x_2)\leq qH(F(x_0),F(x_1))\leq qAd(x_0,x_1).$$

For $x_2\in F(x_1)$, there exists $x_3\in F(x_2)$ such that

$$d(x_2,x_3)\leq qH(F(x_1),F(x_2))\leq qAd(x_1,x_2)\leq(qA)^2 d(x_0,x_1).$$

Inductively, there exists $x_{n+1}\in F(x_n)$ such that

$$d(x_n,x_{n+1})\leq(qA)^n d(x_0,x_1),\text{ for any } n\in\mathbb{N}^*.$$

We have

$$\begin{aligned}
d(x_n,x_{n+p}) &\leq d(x_n,x_{n+1})+\ldots+d(x_{n+p-1},x_{n+p})\\
&\leq (qA)^n d(x_0,x_1)+\ldots+(qA)^{n+p-1}d(x_0,x_1)\\
&= (qA)^n\left[I+qA+\ldots+(qA)^{p-1}\right]d(x_0,x_1)\\
&\leq (qA)^n\left[I+qA+\ldots+(qA)^{p-1}+\ldots\right]d(x_0,x_1)\\
&= (qA)^n(I-qA)^{-1}d(x_0,x_1).
\end{aligned}$$

Thus

(3) $\quad d(x_n, x_{n+p}) \leq (qA)^n (I - qA)^{-1} d(x_0, x_1)$, for $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$.

Letting $n \to \infty$, by Lemma 2.5, it follows that (x_n) is a Cauchy sequence in X. Since X is complete, it follows that there exists $x^* \in X$ such that $x_n \xrightarrow{d} x^*$, $n \to \infty$. Thus,

$$\begin{aligned}
D(x^*, F(x^*)) &= \begin{pmatrix} D_1(x^*, F(x^*)) \\ \cdots \\ D_m(x^*, F(x^*)) \end{pmatrix} \\
&\leq \begin{pmatrix} d_1(x^*, x_{n+1}) + D_1(x_{n+1}, F(x^*)) \\ \cdots \\ d_m(x^*, x_{n+1}) + D_m(x_{n+1}, F(x^*)) \end{pmatrix} \\
&= d(x^*, x_{n+1}) + D(x_{n+1}, F(x^*)) \\
&\leq d(x^*, x_{n+1}) + H(F(x_n), F(x^*)) \\
&\leq d(x^*, x_{n+1}) + Ad(x_n, x^*)
\end{aligned}$$

and letting $n \to \infty$, we get that $D(x^*, F(x^*)) = 0$. By Lemma 2.4, it follows that $x^* \in \overline{F(x^*)}$. Hence, $x^* \in F(x^*)$. Moreover, letting $p \to \infty$ in (3), we obtain

$$d(x_n, x^*) \leq (qA)^n (I - qA)^{-1} d(x_0, x_1), \text{ for any } n \in \mathbb{N}^*.$$

Thus,

$$\begin{aligned}
d(x_0, x^*) &\leq d(x_0, x_1) + d(x_1, x^*) \\
&\leq d(x_0, x_1) + qA(I - qA)^{-1} d(x_0, x_1) \\
&= \left[I + qA(I - qA)^{-1}\right] d(x_0, x_1) \\
&= \left[I + qA(I + qA + \ldots + (qA)^n + \ldots)\right] d(x_0, x_1) \\
&= \left[I + qA + (qA)^2 + \ldots\right] d(x_0, x_1) \\
&= (I - qA)^{-1} d(x_0, x_1)
\end{aligned}$$

and letting $q \searrow 1$, we get that $d(x_0, x^*) \leq (I - A)^{-1} d(x_0, x_1)$. □

A local result in the multivalued case is the following.

Theorem 2.7. *Let (X, d) be a generalized complete metric space, $x_0 \in X \setminus Fix(F)$ and $F : Y \to P_{b,cl}(X)$ be a multivalued A-contraction in the sense of Nadler around x_0. Then, there exists $R := (I - A)^{-1}\delta(x_0, F(x_0))$ such that $\tilde{B}(x_0, R)$ is invariant with respect to F. Moreover, in this case F has at least one fixed point in $\tilde{B}(x_0, R)$.*

Proof. Let $x \in \tilde{B}(x_0, R)$. Then, for any $y \in F(x)$ we have:

$$d(x_0, y) \leq \delta(x_0, F(x_0)) + H(F(x_0), F(x)) \leq \delta(x_0, F(x_0)) + Ad(x_0, x) \leq$$
$$\delta(x_0, F(x_0)) + AR = \delta(x_0, F(x_0)) + A(I - A)^{-1}\delta(x_0, F(x_0)) =$$
$$(I - A)^{-1}\delta(x_0, F(x_0)) = R.$$

This shows that $F(x) \subset \tilde{B}(x_0, R)$. For the second conclusion we apply Theorem 2.6. $\square$

Another useful result is the following data dependence theorem.

Lemma 2.8. *Let (X, d) be a complete generalized metric space and $F_1, F_2 : X \to P_{b,cl}(X)$ be two multivalued A-contractions in Nadler's sense. Then:*

$$\rho(Fix(F_1), Fix(F_2)) \le (I - A)^{-1} \begin{pmatrix} \sup\limits_{x\in X} \rho_{d_1}(F_1(x), F_2(x)) \\ \cdots \\ \sup\limits_{x\in X} \rho_{d_m}(F_1(x), F_2(x)) \end{pmatrix}.$$

Proof. Let $x_0 \in Fix(F_1)$ arbitrary chosen. Then, there exists $f_2^{\infty}(x_0, x_1) \in Fix(F_2)$ such that

$$d[x_0, f_2^{\infty}(x_0, x_1)] \le (I - A)^{-1} d(x_0, x_1), \text{ for any } x_1 \in F_2(x_0).$$

Let $q \in (1, \frac{1}{\rho(A)})$. For $x_0 \in F_1(x_0)$, there exists $x_1 \in F_2(x_0)$ such that

$$d(x_0, x_1) \le q\rho[F_1(x_0), F_2(x_0)].$$

Then, we obtain

$$\begin{aligned} d(x_0, f_2^{\infty}(x_0, x_1)) &\le (I - A)^{-1} q\rho(F_1(x_0), F_2(x_0)) \\ &\le q(I - A)^{-1} \begin{pmatrix} \rho_{d_1}(F_1(x_0), F_2(x_0)) \\ \cdots \\ \rho_{d_m}(F_1(x_0), F_2(x_0)) \end{pmatrix} \\ &\le q(I - A)^{-1} \begin{pmatrix} \sup\limits_{x\in X} \rho_{d_1}(F_1(x_0), F_2(x_0)) \\ \cdots \\ \sup\limits_{x\in X} \rho_{d_m}(F_1(x_0), F_2(x_0)) \end{pmatrix}. \end{aligned}$$

Letting $q \searrow 1$, we get that

$$\rho(Fix(F_1), Fix(F_2)] \le (I - A)^{-1} \begin{pmatrix} \sup\limits_{x\in X} \rho_{d_1}(F_1(x), F_2(x)) \\ \cdots \\ \sup\limits_{x\in X} \rho_{d_m}(F_1(x), F_2(x)) \end{pmatrix},$$

which completes the proof. $\square$

We extend now, to the case of a generalized Banach space, a result given in L. Rybinski [30].

Theorem 2.9. *Let (X, d) be a generalized metric space and Y be a closed subset of a generalized Banach space $(Z, \|\cdot\|)$. Assume that the multivalued operator $F : X \times Y \to P_{cl,cv}(Y)$ satisfies the following conditions:*

i) A is a matrix convergent to zero and

$H(F(x, y_1), F(x, y_2)) \le A\|y_1 - y_2\|$, for each $(x, y_1), (x, y_2) \in X \times Y$;

ii) for every $y \in Y$, $F(\cdot, y)$ is H-l.s.c. on X.

Then there exists a continuous mapping $f : X \times Y \to Y$ such that:

$$f(x,y) \in F(x, f(x,y)), \text{ for each } (x,y) \in X \times Y.$$

Proof. Let us consider the sequence of continuous operators $f_n : X \times Y \to Y$ with property: there exists a matrix convergent to zero $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $M > A$ and $q \in \left(1, \frac{1}{\rho(M)}\right)$ such that for any $(x,y) \in X \times Y$ and for $n = 2, 3, \ldots$ we have

$$1^\circ)\ |f_n(x,y) - f_{n-1}(x,y)| \le qM\, |f_{n-1}(x,y) - f_{n-2}(x,y)|,$$

and for $n \in \mathbb{N}^*$ we have

$$2^\circ)\ D(f_n(x,y), F(x, f_n(x,y))) \le M\, |f_n(x,y) - f_{n-1}(x,y)|.$$

Inductively, we get that

$$|f_n(x,y) - f_{n-1}(x,y)| \le (qM)^{n-1} |f_1(x,y) - f_0(x,y)|,$$

for any $n \in \mathbb{N}^*$. Thus, it is easy to observe that

$$|f_{n+p}(x,y) - f_n(x,y)| \le (qM)^n (I - qM)^{-1} |f_1(x,y) - f_0(x,y)|,$$

for any $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$. Letting $n \to \infty$ it follows that (f_n) is a Cauchy sequence în $X \times Y$ and also, convergent. We denote $f(x,y) = \lim\limits_{n\to\infty} f_n(x,y)$. Thus,

$$\begin{aligned} &D(f(x,y), F(x, f(x,y))) \\ &\qquad \le |f(x,y) - f_n(x,y)| + D(f_n(x,y), F(x, f(x,y))) \\ &\qquad \le |f(x,y) - f_n(x,y)| + M\, |f_n(x,y) - f_{n-1}(x,y)| \end{aligned}$$

and then, $f(x,y) \in F(x, f(x,y))$ for any $(x,y) \in X \times Y$.

Since, for n large enough, the operator f_n is continuous and the operator $(x,y) \to |f_1(x,y) - f_0(x,y)|$ is continuous. Then, by the inequality

$$\begin{aligned} |f(x,y) - f(x_0,y_0)| \le\ & |f(x,y) - f_n(x,y)| + |f_n(x,y) - f_n(x_0,y_0)| \\ & + |f_n(x_0,y_0) - f(x_0,y_0)| \\ \le\ & (qM)^n (I - qM)^{-1} |f_1(x,y) - f_0(x,y)| \\ & + |f_n(x,y) - f_n(x_0,y_0)| \\ & + (qM)^n (I - qM)^{-1} |f_1(x_0,y_0) - f_0(x_0,y_0)|, \end{aligned}$$

we conclude that f is continuous, for any $(x,y) \in X \times Y$.

We suppose that the operators $f_1, \ldots, f_n$ satisfying 1°) and 2°) are defined. We choose a continuous selection f_{n-1} for the multivalued operator F. Let $f_n(x,y) \in F(x, f_{n-1}(x,y))$, then

$$\begin{aligned} D(f_n(x,y), F(x, f_n(x,y))) &\le H(F(x, f_{n-1}(x,y)), F(x, f_n(x,y))) \\ &\le A\, |f_n(x,y) - f_{n-1}(x,y)|, \end{aligned}$$

for any $(x,y) \in X \times Y$. Thus,

$$F(x, f_n(x,y)) \cap \{f_n(x,y) + M\, |f_n(x,y) - f_{n-1}(x,y)|\} \neq \emptyset,$$

for any $(x,y) \in X \times Y$ and the inequality 2°) is satisfied by f_n. Since $F(\cdot, y)$ is H-l.s.c. on X, via Lemma 1 from L. Rybinski [30], we have that the multivalued operator

$$G : (x,y) \to F(x, f_n(x,y)) \cap \{f_n(x,y) + qM\,|f_n(x,y) - f_{n-1}(x,y)|\}$$

is H-l.s.c. and admits a continuous selection. Finally, we get the continuous operator f_{n+1} which satisfies inequalities 1°) and 2°). □

For proving a multivalued version of Krasnoselskii's theorem in generalized Banach spaces we need some auxiliary results.

Lemma 2.10. $(X, \|\cdot\|)$ *be a generalized Banach space. Assume that the operator* $F : X \to P_{b,cl}(X)$ *is a multivalued* A*-contraction in Nadler's sense. Then, the multivalued operator* $1_X - F$ *is continuous with respect to the Hausdorff-Pompeiu generalized metric on* $P_{cl}(X)$*., surjective and* $(1_X - F)^{-1}$ *has closed graph.*

Proof. Since F is an A-contraction, we get immediately get that F is continuous with respect to the Hausdorff-Pompeiu generalized metric on $P_{cl}(X)$. Thus, $1_X - F$ is continuous with respect to the Hausdorff-Pompeiu generalized metric on $P_{cl}(X)$. Let us show now that $1_X - F$ is surjective. For each $y \in X$, we are looking for an element $\bar{x}_y \in X$ such that $(1_Y - F)(\bar{x_y}) = y$. The problem is equivalent with a fixed point problem for the multivalued operator $T(x) = y + F(x)$. Since

$$H(T(x_1), T(x_2)) = H(y+F(x_1), y+F(x_2)) = H(F(x_1), F(x_2)) \leq Ad(x_1, x_2),$$

we get that T is a multivalued A-contraction. Hence, by Theorem 2.6, T has at least one fixed point $\bar{x}_y \in X$. This proves the surjectivity of $1_X - F$. For the last conclusion of this lemma, notice first that $(1_X - F)^{-1} : X \to P(X)$. In order to prove that the graph of $(1_X - F)^{-1}$ is closed, consider a sequence $(y_n)_{n \in \mathbb{N}}$ which converges in X to y and a sequence $x_n \in (1_X - F)^{-1}(y_n)$ which converges in X to x. We will prove that $x \in (1_X - F)^{-1}(y)$. For this purpose, it is enough to prove that $y \in x - F(x)$. Then we have:

$$D(y, x - F(x)) = D(x, y + F(x)) \leq$$

$$d(x, x_n) + D(x_n, y_n + F(x_n)) + H(y_n + F(x_n), y + F(x)) \leq$$

$$d(x, x_n) + H(y_n + F(x_n), y_n + F(x)) + H(y_n + F(x), y + F(x)) =$$

$$d(x, x_n) + H(F(x_n), F(x)) + d(y_n, y) \leq d(x, x_n) + Ad(x_n, x) + d(y_n, y) \to 0,$$

as $n \to +\infty$. □

Recall now a well-known fact, which also takes place in generalized normed spaces.

Lemma 2.11. *Let* X *be a generalized normed space. Then for* $x, y \in X$ *and for* $A \in P_{cl}(X)$ *we have:* $D(x, A + y) = D(y, x - A)$*.*

Another version of the above lemma involves the so-called metrically regularity of a multivalued operator.

Lemma 2.12. *$(X, \|\cdot\|)$ be a generalized Banach space. Assume that the operator $F : X \to P_{b,cl}(X)$ is a multivalued A-contraction in Nadler's sense. Then, the multivalued operator $1_X - F$ is continuous with respect to the Hausdorff-Pompeiu generalized metric on $P_{cl}(X)$, surjective. If additionally, we suppose that $1_X - F$ is metrically regular at each $x \in X$ for $y_0 \in X$, i.e., $(x, y_0) \in Graph(1_X - F)$ and there exists a constant $k > 0$ and neighborhoods U of x and V of y_0 such that*

$$D(u, (1_X - F)^{-1}(v)) \leq kD(v, (1_X - F)(u)), \text{ for all } u \in U \text{ and } v \in V,$$

then $(1_X - F)^{-1}$ is u.s.c. in y_0.

Proof. We will prove the upper semicontinuity of $(1_X - F)^{-1}$ in arbitrary $y_0 \in X$. For this purpose, we have to show that for each $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+$ with $\varepsilon_i > 0$ for every $i \in \{1, \cdots, m\}$ there exists $\eta = (\eta_1, \cdots, \eta_m) \in \mathbb{R}^m_+$ with $\eta_i > 0$ for every $i \in \{1, \cdots, m\}$, such that the following implication holds

$$y \in B(y_0; \eta) \ \Rightarrow \ (1_X - F)^{-1}(y) \subset V((1_X - F)^{-1}(y_0); \epsilon).$$

Let $y \in B(y_0; \eta)$ and $x \in (1_X - F)^{-1}(y)$. We will show that

$$D(x, (1_X - F)^{-1}(y_0)) < \varepsilon.$$

Since $x \in (1_X - F)^{-1}(y)$ we get that $y \in x - F(x)$. Then

$$\begin{aligned} D(x, (1_X - F)^{-1}(y_0)) &\leq kD(y_0, (1_X - F)(x)) \\ &\leq k\left[d(y_0, y) + D(y, x - F(x))\right] \leq k\eta. \end{aligned}$$

If we chose $\eta < \frac{\epsilon}{k}$, then we get the conclusion. $\square$

We will present now a Krasnoselskii type theorem for multivalued operators in generalized Banach spaces.

Theorem 2.13. *Let $(X, \|\cdot\|)$ be a generalized Banach space and $Y \in P_{cp,cv}(X)$. Assume that the operators $F : Y \to P_{b,cl,cv}(X)$, $G : Y \to P_{cp,cv}(X)$ satisfy the properties:*

i) $F(y_1) + G(y_2) \subset Y$, for each $y_1, y_2 \in Y$;
ii) F is a multivalued A-contraction mapping in Nadler's sense;
iii) G is l.s.c and $G(Y)$ is relatively compact.

Then $F + G$ has a fixed point in Y.

Proof. We show that for any $x \in Y$, the operator

$$T_x : Y \to P_{cp,cv}(Y) \quad T_x(y) := F(y) + G(x)$$

is a multivalued A-contraction. We have that

$$\begin{aligned} H(T_x(y_1), T_x(y_2)) &= H(F(y_1) + G(x), F(y_2) + G(x)) \\ &\leq H(F(y_1), F(y_2)) \leq A\|y_1 - y_2\|, \text{ for any } y_1, y_2 \in Y. \end{aligned}$$

Thus, T_x is a multivalued A-contraction. By Theorem 2.6, it follows that for any $x \in Y$ the fixed point set of the multivalued operator T_x, namely

$$Fix(T_x) = \{y \in Y : y \in F(y) + G(x)\}$$

is nonempty and closed. Moreover, since T_x has compact values, by a similar argument to [26] we get that $Fix\,(T_x)$ is compact.

Since, the multivalued operator

$$U : Y \times Y \to P_{cp,cv}(Y), \quad U(x,y) = F(y) + G(x)$$

satisfies the hypothesis of Theorem 2.9, there exists a continuous mapping $u : Y \times Y \to Y$ such that $u(x,y) \in F(u(x,y)) + G(x)$, for each $(x,y) \in Y \times Y$. We consider now the singlevalued operator $c : Y \to Y$, $c(x) = u(x,x)$, for each $x \in Y$. Then $c(x) \in F(c(x)) + G(x)$, for each $x \in Y$ and thus $c(x) \in Fix(T_x)$, for each $x \in Y$. The above relation is equivalent with

$$c(x) \in (1_Y - F)^{-1}(G(x)), \text{ for each } x \in Y.$$

Now, we prove that $c(Y)$ is relatively compact. Notice that, since $G(Y)$ is relatively compact, it is enough to show that the multivalued operator $(1_Y - F)^{-1}$ is u.s.c. and has compact values. The upper semicontinuity follows by Lemma 2.10, by taking into account that Y is compact, while the compactness of the values of $(1_Y - F)^{-1}$ is a consequence of the fact that it has closed values in the compact set Y. Thus, the operator $c : Y \to Y$ satisfies the assumptions of Theorem 1.11. Let $x^* \in Y$ be a fixed point for c. Hence, we have that $x^* = c(x^*) \in F(c(x^*)) + G(x^*) = F(x^*) + G(x^*)$. $\square$

Using an idea of T.A. Burton (see [5]), let us observe that the condition $i)$ in the previous result (Theorem 2.13) can be relaxed as follows.

Theorem 2.14. *Let $(X, \|\cdot\|)$ be a generalized Banach space and $Y \in P_{cp,cv}(X)$. Assume that the operators $F : Y \to P_{b,cl,cv}(X)$, $G : Y \to P_{cp,cv}(X)$ satisfy the properties:*

i) $y \in F(y) + G(x), x \in Y$ then $y \in Y$;

ii) F is a multivalued A-contraction mapping in Nadler's sense;

iii) G is l.s.c and $G(Y)$ is relatively compact.

Then $F + G$ has a fixed point in Y.

Remark 2.15. Let us suppose that the conditions $ii)$ and $iii)$ of Theorem 2.14 holds. If there exists $r \in \mathbb{R}^m_+$ such that for $Y = \{x \in X : \|x\| \le r\}$ we have $G(Y) \subset Y$ and $\|y\| \le D(y, F(y))$, $y \in Y$, then the conclusion of Theorem 2.14 holds.

Indeed, let $y \in F(y) + G(x), x \in Y$. Then there exists $u \in F(y)$ such that $y - u \in G(x), x \in Y$. Since

$$\|y\| \le D(y, F(y)) \le \|y - u\| \le \|G(x)\| \le r$$

we have that $y \in Y$. Hence, the conclusion of Theorem 2.14 holds.

Another Krasnoselskii type fixed point theorem for the sum of two multivalued operator more appropiate for applications is given now below.

Theorem 2.16. *Let $(X, \|\cdot\|)$ be a generalized Banach space and $Y \in P_{b,cl,cv}(X)$. Assume that the operators $F : Y \to P_{b,cl,cv}(X)$, $G : Y \to P_{cp,cv}(X)$ satisfy the properties:*

i) $F(y_1) + G(y_2) \subset Y$, for each $y_1, y_2 \in Y$;

ii) F is a multivalued A-contraction mapping in Nadler's sense;
iii) G is l.s.c and $G(Y)$ is relatively compact;
iv) the multivalued operator $1_Y - F$ is metrically regular on Y.

Then $F+G$ has a fixed point in Y.

Proof. The proof is similar to the proof of Theorem 2.13. The only modification consist in the fact that this time we are using the property of metrically regularity of $1_Y - F$ (instead of the compactness of Y) to get that $c(Y)$ is relatively compact. □

3. An application

It is known that the classical form of Krasnoselskii's Theorem has a lot of interesting applications. See, for example, T.A. Burton [4], [5], [6], [7], L. Collatz [8] A. Petruşel [24], R. Precup-A. Viorel [27], [28], M. Zuluaga [40], etc.

Our purpose is to give some applications of our Krasnoselskii type fixed point theorems in a generalized Banach spaces.

Theorem 3.1. *Let $I=[0,a]$ (with $a>0$) be an interval of the real axis and consider the following system of integral equations*

$$\begin{cases} x_1(t) = \lambda_{11}\int_0^t k_1(t,s,x_1(s),x_2(s))\,ds + \lambda_{12}\int_0^a l_1(t,s,x_1(s),x_2(s))\,ds \\ x_2(t) = \lambda_{21}\int_0^t k_2(t,s,x_1(s),x_2(s))\,ds + \lambda_{22}\int_0^a l_2(t,s,x_1(s),x_2(s))\,ds \end{cases}$$

for $t\in I$, where $\lambda_{ij}\in\mathbb{R}$, for $i,j\in\{1,2\}$.

We assume that:

i) $k_1, l_1 \in C\left(I^2\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n\right)$ and $k_2, l_2 \in C\left(I^2\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^p\right)$;

ii) there exists the matrix $A=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\in M_{2,2}(\mathbb{R}_+)$ such that

$$|k_i(t,s,u_1,u_2)-k_i(t,s,v_1,v_2)|\le a_{i1}|u_1-v_1|+a_{i2}|u_2-v_2|,\ \text{for each}$$
$$(t,s,,u_1,u_2),(t,s,v_1,v_2)\in I^2\times\mathbb{R}^n\times\mathbb{R}^p,\ i\in\{1,2\};$$

iii) $\begin{pmatrix}|\lambda_{12}|\\|\lambda_{22}|\end{pmatrix}\le\begin{pmatrix}\frac{r_1}{2M_{l_1}}\\ \frac{r_2}{2M_{l_2}}\end{pmatrix}$, *where* $M_{l_i}=\max\limits_{t\in[0,a]}\int_0^a |l_i(t,s,x_1(s),x_2(s))|ds,$

for $i\in\{1,2\}$ and $r:=\begin{pmatrix}r_1\\r_2\end{pmatrix}$, with $r_1,r_2>0$;

iv) $\begin{pmatrix}|\lambda_{11}|\\|\lambda_{21}|\end{pmatrix}\le\begin{pmatrix}\frac{r_1}{2a(a_{11}r_1+a_{12}r_2)}\\ \frac{r_2}{2a(a_{21}r_1+a_{22}r_2)}\end{pmatrix}.$

Then, there exists $\left(x_1^0,x_2^0\right)\in C(I,\mathbb{R}^n)\times C(I,\mathbb{R}^p)$ such that the system (3.1) has at least one solution $x^*:=(x_1^*,x_2^*)\in\tilde{B}\left(x_1^0,r_1\right)\times\tilde{B}\left(x_2^0,r_2\right)\subset C(I,\mathbb{R}^n)\times C(I,\mathbb{R}^p)$.

Proof. For the sake of simplicity let us denote $X_1 := \mathbb{R}^n$ and $X_2 := \mathbb{R}^p$. For $i \in \{1,2\}$ and $x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in C(I,X_1) \times C(I,X_2)$, we define

$$\begin{aligned} & f_i, g_i : C(I,X_1) \times C(I,X_2) \to C(I,X_i), \\ & \quad x \longmapsto f_i x, \quad x \longmapsto g_i x, \\ & f_i x(t) := \lambda_{i1} \int_0^t k_i(t,s,x_1(s),x_2(s))\, ds, \text{ for any } t \in I, \\ & g_i x(t) := \lambda_{i2} \int_0^a l_i(t,s,x_1(s),x_2(s))\, ds, \text{ for any } t \in I. \end{aligned}$$

By i), the operators f_i and g_i are well defined, for $i \in \{1,2\}$. Morover, the system 3.1 can be re-written as a fixed point equation of the following form

$$x = (f+g)(x),$$

where $f := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g := \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$. Obviously, $x^* := \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is a solution for our system of integral equations if and only if x^* is a fixed point for the operator $f+g$.

Let us show that f and g satisfies the assumptions of Theorem 2.1. Let $x := (x_1,x_2)$, $y := (y_1,y_2) \in C(I,X_1) \times C(I,X_2)$. We have

$$\begin{aligned} & |f_i(x)(t) - f_i(y)(t)|_{X_i} \\ & \quad \leq |\lambda_{i1}| \int_0^t |k_i(t,s,x_1(s),x_2(s)) - k_i(t,s,y_1(s),y_2(s))|_{X_i}\, ds \\ & \quad \leq |\lambda_{i1}| \int_0^t \left(a_{i1}|x_1(s)-y_1(s)|_{X_1} + a_{i2}|x_2(s)-y_2(s)|_{X_2}\right) ds \\ & \quad = |\lambda_{i1}| \left(a_{i1}\,||x_1-y_1||_{B_1} \int_0^t e^{\tau s} ds + a_{i2}\,||x_2-y_2||_{B_2} \int_0^t e^{\tau s} ds\right) \\ & \quad \leq \frac{|\lambda_{i1}|}{\tau} e^{\tau t} \left(a_{i1}\,||x_1-y_1||_{B_1} + a_{i2}\,||x_2-y_2||_{B_2}\right), \text{ for } i \in \{1,2\}, \end{aligned}$$

where $||u||_B := \begin{pmatrix} ||u_1||_{B_1} \\ ||u_2||_{B_2} \end{pmatrix} = \begin{pmatrix} \sup\limits_{t\in[0,a]} e^{-\tau t}|u_1(t)|_{X_1} \\ \sup\limits_{t\in[0,a]} e^{-\tau t}|u_2(t)|_{X_2} \end{pmatrix}$, $\tau > 0$ denotes the Bielecki-type norm on the generalized Banach space $C(I,X_1) \times C(I,X_2)$. Thus, we obtain that

$$||f_i(x) - f_i(y)||_{B_i} \leq \frac{|\lambda_{i1}|}{\tau} \left(a_{i1}\,||x_1-y_1||_{B_1} + a_{i2}\,||x_2-y_2||_{B_2}\right), \text{ for } i \in \{1,2\}.$$

These inequalities can be written in a vectorial form

$$||f(x) - f(y)||_B \leq M\,||x-y||_B,$$

where

$$M = \left(\frac{|\lambda_{i1}|\, a_{ij}}{\tau}\right)_{i,j=1,2}.$$

Taking τ large enough it follows that the matrix M is convergent to zero and thus, f is an A-contraction. By Theorem 2.6, we have that there exists a unique fixed point $x^0 = \left(x_1^0, x_2^0\right) \in C\left(I, X_1\right) \times C\left(I, X_2\right)$ for $f = (f_1, f_2)$.

Let $Y := \tilde{B}(x_1^0; r_1) \times \tilde{B}(x_2^0; r_2) \subset C\left(I, X_1\right) \times C\left(I, X_2\right)$.

The operator g is continuous and, by a classical argument, we get that $g\left(Y\right)$ is relatively compact.

We will show that we can choose $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ (with $r_1, r_2 > 0$), such that

$$f\left(Y\right) \subset \tilde{B}\left(x_1^0, \frac{r_1}{2}\right) \times \tilde{B}\left(x_2^0, \frac{r_2}{2}\right).$$

Let $x \in Y$, i.e., $(x_1, x_2) \in \tilde{B}(x_1^0; r_1) \times \tilde{B}(x_2^0; r_2)$. We will show that

$$\|f(x) - x^0\|_C := \begin{pmatrix} \|f_1(x) - x_1^0\|_{C_1} \\ \|f_2(x) - x_2^0\|_{C_2} \end{pmatrix} \leq \begin{pmatrix} \frac{r_1}{2} \\ \frac{r_2}{2} \end{pmatrix},$$

where $\|\cdot\|_C$ denotes the Cebîsev norm in the space of continuous function on I.

We have

$$\left|f_1\left(x\right)\left(t\right) - x_1^0\left(t\right)\right|_{X_1} = \left|f_1\left(x\right)\left(t\right) - f_1(x^0)\left(t\right)\right|_{X_1} \leq$$

$$|\lambda_{11}| \int_0^t |k_1\left(t, s, x_1\left(s\right), x_2\left(s\right)\right) ds - k_1\left(t, s, x_1^0\left(s\right), x_2^0\left(s\right)\right)|_{X_1} ds \leq$$

$$\leq |\lambda_{11}| \int_0^t (a_{11}|x_1(s) - x_1^0(s)|_{X_1} + a_{12}|x_2\left(s\right) - x_2^0\left(s\right)|_{X_2}) ds$$

$$\leq |\lambda_{11}| \int_0^t (a_{11}\|x_1 - x_1^0\|_{C_1} + a_{12}\|x_2 - x_2^0\|_{C_2}) ds$$

$$\leq |\lambda_{11}|(a_{11}r_1 + a_{12}r_2)a.$$

Taking $\max\limits_{t\in I}$, we have that

$$\|f_1\left(x\right) - x_1^0\|_{C_1} \leq |\lambda_{11}|a(a_{11}r_1 + a_{12}r_2) \leq \frac{r_1}{2}.$$

In a similar manner, we get

$$\|f_2\left(x\right) - x_2^0\|_{C_2} \leq |\lambda_{21}|a(a_{21}r_1 + a_{22}r_2) \leq \frac{r_2}{2}.$$

Thus, we get

$$\left|\left|f\left(x\right) - x^0\right|\right|_C \leq \begin{pmatrix} \frac{r_1}{2} \\ \frac{r_2}{2} \end{pmatrix}. \tag{4}$$

We will show now that

$$g(Y) \subset \tilde{B}(0; \frac{r_1}{2}) \times \tilde{B}(0; \frac{r_2}{2}),$$

i.e.,

$$\|g(x)\|_C := \begin{pmatrix} \|g_1(x)\|_{C_1} \\ \|g_2(x)\|_{C_2} \end{pmatrix} \leq \begin{pmatrix} \frac{r_1}{2} \\ \frac{r_2}{2} \end{pmatrix}.$$

Indeed, for $x = (x_1, x_2) \in Y$, we have

$$|g_1(x)(t)| \le |\lambda_{12}| \int_0^a |l_1(t, s, x_1(s), x_2(s))| \, ds \le |\lambda_{12}| \, M_{l_1}.$$

Taking $\max\limits_{t \in I}$ and using $iii)$, we have

$$\|g_1(x)\|_{C_1} \le |\lambda_{12}| M_{l_1} \le \frac{r_1}{2}.$$

By a similar approach we get

$$\|g_2(x)\|_{C_2} \le |\lambda_{22}| M_{l_2} \le \frac{r_2}{2}.$$

Thus

$$g(x) \subset \tilde{B}(0, \frac{r_1}{2}) \times \tilde{B}(0, \frac{r_2}{2}), \text{ for each } x \in Y.$$

Then, the operator $f + g$ has the property $(f + g)(Y) \subset Y$. Hence, the conclusion follows by Theorem 2.1. $\square$

Remark 3.2. In a similar way, using a multivalued version of Krasnoselskii's theorem in generalized metric spaces, existence results for the following integral inclusion system in $C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^p)$:

$$\begin{cases} x_1(t) \in \lambda_{11} \int_0^t K_1(t, s, x_1(s), x_2(s)) \, ds + \lambda_{12} \int_0^a L_1(t, s, x_1(s), x_2(s)) \, ds \\ x_2(t) \in \lambda_{21} \int_0^t K_2(t, s, x_1(s), x_2(s)) \, ds + \lambda_{22} \int_0^a L_2(t, s, x_1(s), x_2(s)) \, ds \end{cases}$$

for $t \in I := [0, a]$ (where $\lambda_{ij} \in \mathbb{R}$, $i, j \in \{1, 2\}$) can be given.

Acknowledgement. The first author wishes to thank for the financial support provided from programs co-financed by The Sectoral Operational Programme Human Resources Development, Contract POSDRU/88/1.5/S/60185 - "Innovative Doctoral Studies in a Knowledge Based Society". For the second author, this paper was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0094.

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(Received August 17, 2012)

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