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GLOBAL CONVERGENCE AND UNIFORM BOUNDS OF FLUCTUATING PRICES IN A SINGLE COMMODITY MARKET MODEL OF BÉLAIR AND MACKEY

GERGELY RÖST

Bolyai Institute, University of Szeged
Aradi vértanúk tere 1, H-6720 Szeged, Hungary
e-mail: rost@math.u-szeged.hu

Abstract. We extend the analysis of the global dynamics for a special class of single commodity market models introduced by Bélair and Mackey. In particular, we determine uniform asymptotic bounds for the fluctuating price, and give sufficient conditions for the global convergence to the positive equilibrium.

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1. Introduction

Bélair and Mackey introduced a class of integrodifferential equations with delays of the form

$$\frac{dP}{Pdt} = f(D(P_D), S(P_S)),$$

to study the dynamics of price adjustment of a single commodity [1, 2]. Here $P(t)$ is the market price of that particular commodity at time t, f is the relative price change function, D and S refer to the demand and supply functions. Time delays occur due to production lags and storage policies. They determined the stability of equilibria under some conditions, and studied the destabilizing effect of the consumer memory on the equilibrium price. Later several variants of this equation were studied in

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[4, 5, 6, 7]. Farahani and Grove [3] considered the special case

$$\frac{P'(t)}{P(t)} = \frac{a}{b + P^n(t)} - \frac{cP^m(t-\tau)}{d + P^m(t-\tau)} \tag{1.1}$$

with initial condition

$$P(\theta) = \phi(\theta) > 0, \quad \theta \in [-\tau, 0], \quad \phi \in C([-\tau, 0], R)$$

where $a, b, c, d, m, \tau > 0, n \geq 1$, and $C = C([-\tau, 0], R)$ is the Banach space of continuous real valued functions on the interval $[-\tau, 0]$ with the usual supremum norm. They showed in [3] the existence, uniqueness, and boundedness of a positive solution $P(t)$ for all $t \geq -\tau$ from each positive initial function, and provided conditions to ensure that all positive solutions oscillate about a positive equilibrium. By a solution of (1.1) we mean a continuous function $P : [-\tau, \infty) \to R$ which is differentiable on $(0, \infty)$, satisfies (1.1) on $(0, \infty)$, and satisfies the initial condition $P(\theta) = \phi(\theta)$ on $[-\tau, 0]$. In this paper we extend the analysis of Equation (1.1). In particular, we estimate the uniform asymptotic bounds of solutions, and give sufficient conditions for the convergence of $P(t)$ to a positive equilibrium by Lyapunov's direct method.

The economic interpretation of (1.1) is the following (see [2] for a more general case). The price of a single commodity at time t is denoted by $P(t)$, and we assume that the relative variation $\frac{dP}{Pdt}$ of the price is governed by simple demand and supply functions. Demand increases the price while supply decreases. The demand function $\frac{a}{b+P^n}$ is monotone decreasing because higher price leads to less buying, while the supply function $\frac{cP^m}{d+P^m}$ is monotone increasing as industry reacts to higher prices by increasing production. The time delay τ in the supply term is due to the time lag in production, because some time has to elapse after a decision is made to increase production. This time lag can be affected by natural constraints for example in the case of agricultural commodities. On the other hand, the demand term does not have a time lag, since consumers base their buying decisions on the current market price.

2. Explicit bounds and uniform asymptotic bounds of solutions

Let $f(x) = \frac{a}{b+x^n}$ and $g(x) = \frac{cx^m}{d+x^m}$, then from $f(0) = \frac{a}{b} > g(0) = 0$ and

$$0 = \lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x) = c$$

it follows that there exists a positive P_* such that $f(P_*) = g(P_*)$. Given that f is a monotone decreasing and g is a monotone increasing function for $x \geq 0$, this P_* is a unique positive equilibrium of equation (1.1), moreover $f(x) > g(x)$ for $x < P_*$ and $f(x) < g(x)$ for $x > P_*$. Also, $g(x) - f(x)$ is monotone increasing. These monotonicity properties imply the following:

$$\text{if } z > P_*, x > z, y > z \text{ then } g(z) - f(z) < g(x) - f(y). \tag{2.1}$$

Let $Y_t(q) := y(t)$ where $q > 0$ and $y(t)$ is the solution of the initial value problem

$$\begin{cases} y'(t) = \frac{ay(t)}{b+y(t)^n} \\ y(0) = q. \end{cases} \tag{2.2}$$

Equation (2.2) is separable and thus the solution $y(t)$ satisfies

$$n \ln y(t) + \frac{y(t)^n}{b} = \frac{ant}{b} + n\ln(q) + \frac{q^n}{b}.$$

Taking exponential of both sides leads to

$$y(t)^n \exp\left(\frac{y(t)^n}{b}\right) = \exp\left(\frac{ant}{b}\right) q^n \exp\left(\frac{q^n}{b}\right),$$

which gives

$$\frac{y(t)^n}{b} = W\left[\frac{\exp\left(\frac{ant}{b}\right) q^n \exp\left(\frac{q^n}{b}\right)}{b}\right]$$

where W is the Lambert $W(z)$ function (for the history and properties of this function, see [8]), i.e. $W(z)$ is the principal solution of $W(z)e^{W(z)} = z$. Therefore

$$Y_\tau(P_*) = \left(bW\left[e^{\frac{a\tau n}{b}} \frac{P_*^n}{b} e^{\frac{P_*^n}{b}}\right]\right)^{\frac{1}{n}}.$$

As $y(t)$ is increasing, for $q = P_*$ we have the simple estimate $y'(t) \leq \frac{ay(t)}{b+P_*^n}$ which gives

$$Y_\tau(P_*) \leq P_* \exp\left(\frac{\tau a}{b + P_*^n}\right) = P_* e^{\tau f(P_*)}.$$

Introduce the notation P_t for the state of solutions, where $P_t \in C$ is defined by the relation $P_t(\theta) = P(t+\theta)$, $\theta \in [-\tau, 0]$, $t \geq 0$. If we want to emphasize the correspondence of the solution and the state to the initial function $\phi \in C$, then we write $P^\phi(t)$ and P_t^ϕ.

Theorem 2.1. *For any positive solution $P(t)$ of (1.1) we have the following:*

(a)
$$\max_{t\geq 0} P^{\phi}(t) \leq \max\{Y_\tau(P(0)), Y_\tau(P_*)\},$$

(b)
$$\limsup_{t\to\infty} P(t) \leq Y_\tau(P_*),$$

(c)
$$\liminf_{t\to\infty} P(t) \geq f^{-1}(g(Y_\tau(P_*))),$$

(d)
$$\min_{t\geq 0} P^{\phi}(t) \geq \min\{P(0)e^{-\tau c}, P_* e^{-\tau c}\},$$

(e)
$$\liminf_{t\to\infty} P(t) \geq P_* e^{-\tau c}.$$

Proof. Let $\Delta_q := f(P_*+q) - g(P_*+q)$ for $q \in [-P_*, \infty)$. Then clearly $\Delta_q > 0$ if $q < 0$, $\Delta_q < 0$ if $q > 0$, and $\Delta_0 = 0$. For simplicity, we introduce the notation

$$P^\infty := \limsup_{t\to\infty} P(t) \text{ and } P_\infty := \liminf_{t\to\infty} P(t).$$

First we show that $P_\infty \leq P_*$ and $P^\infty \geq P_*$. Suppose $P_\infty > P_*$. Then there is an $\varepsilon > 0$ and a $T > 0$ such that $P(t) > P_* + \varepsilon$ for all $t \geq T$. Then by (2.1),

$$P'(t) < P(t)\Big(f(P_*+\varepsilon) - g(P_*+\varepsilon)\Big) = \Delta_\varepsilon P(t) < 0$$

for all $t \geq T + \tau$. This implies $P(t) < P(T+\tau)e^{\Delta_\varepsilon(t-T-\tau)} \to 0$ as $t \to \infty$, which contradicts $P_\infty > P_* > 0$. Similarly, if $P^\infty < P_*$ then there is some $\varepsilon > 0$ such that

$$P'(t) > P(t)\Big(f(P_*-\varepsilon) - g(P_*-\varepsilon)\Big) = \Delta_{-\varepsilon} P(t) > 0$$

for sufficiently large t, which is a contradiction.

Let $I = [t_1, t_2]$ be an interval such that $P(t) \geq P_*$ for all $t \in I$. We distinguish two cases: if $t_2 - t_1 \leq \tau$, then from $P'(t) \leq \frac{aP(t)}{b+P(t)^n}$ we obtain that for $\theta \in [t_1, t_2]$,

$$P(\theta) \leq Y_{\theta-t_1}(P(t_1)) \leq Y_{t_2-t_1}(P(t_1)) \leq Y_\tau(P(t_1)).$$

If $t_2 - t_1 > \tau$, then for $\theta \in [t_1, t_1+\tau]$ we have $P(\theta) \leq Y_\tau(P(t_1))$ just like before. For $\theta \in (t_1+\tau, t_2]$, both $P(\theta) \geq P_*$ and $P(\theta-\tau) \geq P_*$, thus $P'(\theta) \leq 0$ for all $\theta \in [t_1+\tau, t_2]$, hence $P(\theta) \leq P(t_1+\tau) \leq Y_\tau(P(t_1))$ for all $\theta \in [t_1+\tau, t_2]$. We conclude that

$$\max_{\theta \in I} P(\theta) \leq Y_\tau(P(t_1)). \tag{2.3}$$

Suppose that $P(t_*) > P_*$ for some $t_* \geq 0$. Then there is a maximal interval J_{t_*} such that $t_* \in J_{t_*}$ and $P(t) \geq P_*$ for all $t \in J_{t_*}$, and from the continuity of solutions $J_{t_*} = [t_1, t_2]$ with some $t_1 < t_* < t_2$ or $J_{t_*} = [t_1, \infty)$ with some t_1. If $t_1 \geq 0$, then by the continuity $P(t_1) = P_*$, and from (2.3) we obtain $P(t_*) \leq Y_\tau(P_*)$. If $t_1 < 0$, consider the interval $I = J_{t_*} \cap [0, \infty)$, and from (2.3) we obtain $P(t_*) \leq Y_\tau(P(0))$. Hence we conclude **(a)**.

If $P(0) \leq P_*$, then **(b)** immediately follows. If $P(0) > P_*$, then we consider two cases. If $J_0 \cap [0, \infty) = [0, t_0]$ with $t_0 < \infty$, then for all $t > t_0$ we have $P(t) \leq Y_\tau(P_*)$ and **(b)** follows. If $J_0 \cap [0, \infty) = [0, \infty)$, then $P(t) \geq P_*$ for all $t \geq 0$. In this case $P(t)$ is monotone decreasing for $t > \tau$, thus converges to some limit. Since we have $P_\infty \leq P_*$, this limit has to be P_*, thus from $P_* \leq Y_\tau(P_*)$, **(b)** holds.

To show **(c)**, suppose the contrary, that is $P_\infty < f^{-1}(g(Y_\tau(P_*)))$. Since f is decreasing, we have $f(P_\infty) > g(Y_\tau(P_*))$. By continuity, there is an $\epsilon > 0$ such that $f(P_\infty + \epsilon) > g(Y_\tau(P_*)) + \epsilon)$. From $P^\infty \leq Y_\tau(P_*)$, for this ϵ there is a T such that $P(t) < Y_\tau(P_*) + \epsilon$ for $t > T$, which implies $g(P(t - \tau)) < g(Y_\tau(P_*) + \epsilon)$ for $t > T + \tau$. Note that $P_\infty < f^{-1}(g(Y_\tau(P_*))) \leq P_* \leq P^\infty$ implies that there must exist a $t > T + \tau$ such that $P(t) < P_\infty + \epsilon$ and $P'(t) < 0$. But for such a t, $P'(t) = P(t)(f(P(t)) - g(P(t - \tau))) > 0$, which is a contradiction.

To prove **(d)** we use $f(x) > 0$ and $g(x) < c$ to find the estimate $P'(t) > -cP(t)$. Then we can proceed completely similarly to **(a)**: if $P(t) < P_*$ on some interval, it can decrease only for time τ before $P'(t)$ becomes positive. The proof is analogous to **(a)** hence the details are omitted. Similarly, from **(d)** we can conclude **(e)** the same way as we proved **(b)** from **(a)**. The proof is complete. □

Remark 2.2. *The theorem shows the permanence of* (1.1)*. It depends on the particular parameter values that the estimate for P_∞ in* **(c)** *or in* **(e)** *gives a sharper result. Some comparison is shown in Section 4.*

Remark 2.3. *At the cost of some elaborative calculations, one can sharpen the estimates for P_∞ and P^∞ iteratively by proceeding as follows: suppose we already established $P^\infty \leq q_1 := Y_\tau(P_*)$. Then we can replace the estimate $P'(t) > -cP(t)$ in point* **(c)** *by $P'(t) \geq -g(q_1)P(t)$ for sufficiently large t to obtain $P_\infty \geq p_1 :=$*

$P_* e^{-\tau g(q_1)}$. *Next we use this* p_1 *to improve* $P'(t) \le P(t)f(P(t))$ *in point* **(a)** *by* $P'(t) \le P(t)f(P(t)) - g(p_1)$ *to find a* q_2 *with* $P^\infty \le q_2 < q_1$. *This leads to* $P_\infty \ge p_2 := P_* e^{-\tau g(q_2)} > p_1$ *and so on.*

3. Convergence to the positive equilibrium

Theorem 3.1. *If there is an* $H \in (0, P_*)$ *with*

$$H > \max\{Y_\tau(P_*) - P_*, \min\{P_* - f^{-1}(g(Y_\tau(P_*))), P_* - P_* e^{-\tau c}\}\} \tag{3.1}$$

such that for all $h \in (0, H)$ *the inequalities*

$$f(P_* + h) < g(P_* - h) \text{ and } g(P_* + h) < f(P_* - h) \tag{3.2}$$

hold, then all positive solutions of (1.1) converge to P_*.

Proof. Consider an H that satisfies (3.1). Then by Theorem 2.1, $P^\infty < P_* + H$ and $P_\infty > P_* - H$, so there is a T such that $P(t) \in (P_* - H, P_* + H)$ for all $t > T$. We define the functional $V : C \to R$ by $V(\phi) = \max_{\theta \in [-\tau, 0]} |\phi(\theta) - P_*|$. Then clearly $V(P_t) \in [0, H)$ for $t > T + \tau$.

We show that under condition (3.2), $V(P_t)$ is a nonincreasing function of t for $t > T + \tau$. Suppose the opposite, then there is some $\sigma > T + \tau$ such that $V(P_\sigma) = P(\sigma) - P_*$ and $P'(\sigma) \ge 0$ or $V(P_\sigma) = P_* - P(\sigma)$ and $P'(\sigma) \le 0$. Consider the first possibility, then $P(\sigma) = P_* + V(P_\sigma)$, moreover $P(\sigma - \tau) \ge P_* - V(P_\sigma)$ which implies $g(P(\sigma - \tau)) \ge g(P_* - V(P_\sigma))$. From condition (3.2) we obtain

$$P'(\sigma) \le P(\sigma)\left(f(P_* + V(P_\sigma)) - g(P_* - V(P_\sigma)\right) < 0 \tag{3.3}$$

which is a contradiction. The other case is analogous.

Consider any given positive $\phi \in C$. Since $V(P_t^\phi) \ge 0$, and $V(P_t^\phi)$ is nonincreasing for $t > T + \tau$, it converges to some nonnegative limit c. Define the limit set $\omega(\phi)$ of the solution P_t^ϕ as usual:

$$\omega(\phi) := \{\psi \in C : \text{there is a sequence } t_n \text{ such that } t_n \to \infty \text{ and } P_{t_n}^\phi \to \psi \text{ as } n \to \infty\}$$

The set

$$G = \{\phi \in C : \phi(s) \in [P_* - H, P_* + H], s \in [-\tau, 0],$$

is closed, and contains the ω-limit set of any positive initial function. By the continuity of the functional V, it holds that $V(\psi) = c$ for any $\psi \in \omega(\phi)$. Since $\omega(\phi)$ is positively invariant ([9, Chapter 5.2]), we obtain $V(P_t^\psi) \equiv c$. Because of the argument preceding (3.3), it is not possible that $|P^\psi(t_1) - P_*| < c$ and $|P^\psi(t_2) - P_*| = c$ for some $t_1 < t_2$, hence $V(P_t^\psi) \equiv c$ implies $|P^\psi(t) - P_*| \equiv c$, which is possible only if $c = 0$ and thus $\omega(\phi)$ contains only the constant P_* function. The proof is complete. $\square$

Theorem 3.2. *If $g'(P_*) < -f'(P_*)$ then there is a $\tau_* > 0$ such that all positive solutions $P(t)$ converge to P_* for $\tau < \tau_*$.*

Proof. If $g'(P_*) < -f'(P_*)$, then there is a $\delta > 0$ such that $g'(x) < -f'(x)$ for all $x \in (P_* - \delta, P_* + \delta)$. Given that $Y_\tau(P_*) \to P_*$, $f^{-1}(g(Y_\tau(P_*))) \to P_*$ and $P_* e^{-\tau c} \to P_*$ as $\tau \to 0$, there is a τ^* such that for $\tau < \tau^*$, we have

$$\delta > \max\{Y_\tau(P_*) - P_*, \min\{P_* - f^{-1}(g(Y_\tau(P_*))), P_* - P_* e^{-\tau c}\}\}.$$

Thus (3.1) and (3.2) hold for $\tau < \tau^*$ and $H = \delta$, and we can apply Theorem 3.1. $\square$

Remark 3.3. *Since $f'(x) = -\frac{anx^{n-1}}{(b+x^n)^2}$ and $g'(x) = \frac{cdmx^{m-1}}{(d+x^m)^2}$, evaluating these derivatives at P_* and using $f(P_*) = g(P_*)$, some elementary calculations show that $g'(P_*) < -f'(P_*)$ is equivalent to $adm < cnP_*^{n+m}$. In the special case $a = c$ and $b = d$ we have $P_* = 1$ and this inequality simplifies to $bm < n$.*

Remark 3.4. *The linear variational equation of (1.1) around the equilibrium P_* is*

$$z'(t) = P_* f'(P_*) z(t) - P_* g'(P_*) z(t - \tau).$$

and there is a $\tau^* > 0$ such that P_* is unstable for $\tau > \tau^*$. Since $f'(P_*) < 0$ and $-g'(P_*) < 0$, by applying Theorem 4.7 from [9] we find that if $g'(P_*) < -f'(P_*)$ then P_* is locally asymptotically stable, while if $g'(P_*) > -f'(P_*)$ then there is a $\tau_0 > 0$ such that P_* is stable only for $\tau < \tau_0$, and become unstable for $\tau > \tau_0$.

4. Examples

We already noted that it depends on the parameters whether the estimate **(c)** or **(e)** is better for P_∞ in Theorem 2.1. It may depend also on the delay, as a particular case depicted in Figure 4.1.

The price can either converge or periodically oscillate, as illustrated in Figure 4.2. Recently, Ch. Qian showed in Theorem 2 of [7] that all positive solutions of (1.1) converge to P_* if $\frac{cm}{4d}\tau < 1$. His method is different from what we use in the proof of

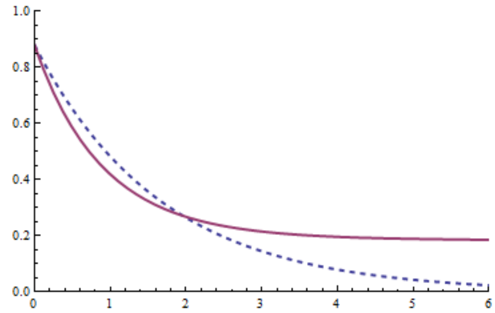

FIGURE 4.1. Bounds for P_∞ as a function of τ in the case $a = 0.2, b = 0.3, c = d = 0.6, m = 10, n = 2$. Solid curve is $f^{-1}(g(Y_\tau(P_*)))$, dashed curve is $P_* e^{-\tau c}$.

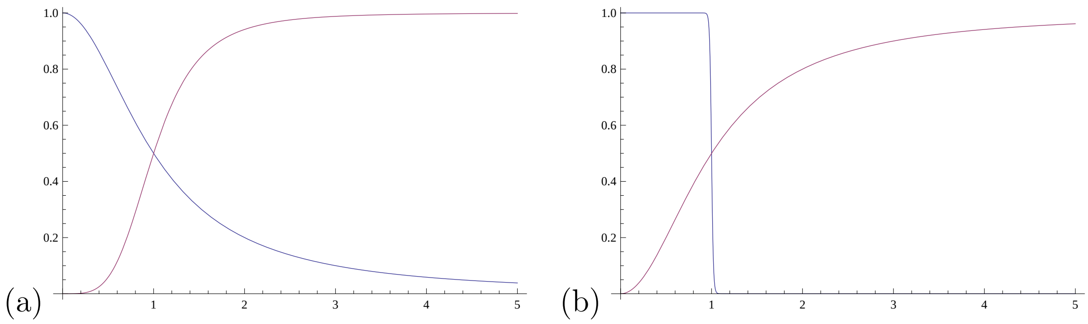

FIGURE 4.2. Graphs of f and g in various situations.

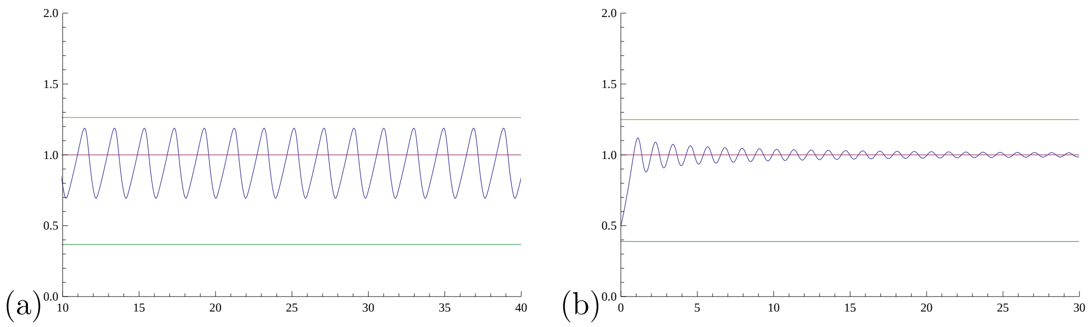

FIGURE 4.3. Examples of price oscillation and price convergence. Horizontal lines indicate the estimates for P^∞, P_∞ from Theorem 2.1 and P_*. The parameter values are (a): $a = c = 2, b = d = 3, m = 20, n = 2, \tau = 0.5$, (b): $a = b = c = d = 3, m = 10, n = 2, \tau = 0.315$.

Theorem 3.1. We finish this paper by showing that Theorem 3.1 includes situations where $\frac{cm}{4d}\tau > 1$, therefore it improves the previous global stability result. Notice that the condition $\frac{cm}{4d}\tau < 1$ is independent from n, while increasing n will help to satisfy the conditions of Theorem 3.1 for global convergence (since by increasing n, the function f converges to a step function) even when $\frac{cm}{4d}\tau > 1$. Consider the following particular example: let $a = b = c = d = 1$, $n = 100$, $m = 2$. Then $P_* = 1$, and $\frac{cm}{4d}\tau \geq 1$ whenever $\tau \geq 2$. See Figure 4.3 (b) for the graphs of f and g in this case. Then one can easily calculate (it is clear from the graphs as well) that (3.2) is satisfied, for example for $H = 0.9$. However, we have $P_\infty \geq e^{-2} > 0.13$ and $P^\infty \leq Y_2(1) < 1.2$ therefore (3.1) also holds and Theorem 3.1 applies, and by continuity the same hold for some $\tau > 2$ as well. Thus, Theorem 3.1 covers global convergence results which are not included in [7].

Furthermore, Theorem 2.1 provides useful information on the asymptotic bounds of the fluctuating price in situations when it does not converge.

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